DISTINGUISHING BETWEEN SCHEMES OF MATHEMATICAL EQUIVALENCE: JOE'S TRANSITION TO ANTICIPATORY QUANTITATIVE RELATIONAL EQUIVALENCE

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This study examined how a child constructed a scheme (abbreviated QRE) for producing mathematical equivalence via operations on composite units between two multiplicative situations consisting of singletons and composite units. Within the context of a teaching experiment, the work of one child, Joe, was analyzed over the course of 14 teaching episodes. Joe made the conceptual advance from a Relational Equivalence Scheme (RE) and absence of a Quantitative Relational Equivalence Scheme (QRE), through the participatory stage, to an anticipatory stage of a QRE scheme. Joe's progression distinguished between creating equivalence with an RE via operations on singletons and a QRE—a conceptual root for fundamental algebraic concepts such as the distributive property, solving linear equations, and a relational understanding of the equal sign.

Keywords: Learning Trajectories (or Progressions), Algebra and Algebraic Thinking

Introduction

This study examines the schemes of mathematical equivalence a child constructed to create equality between two multiplicative situations. Developing an understanding of mathematical equivalence is a foundational idea for algebra (Blanton et al., 2015; Knuth, Stephen, McNeil, & Alibali, 2006; McNeil, Fyfe, & Dunwiddie, 2015). Algebraic concepts such as solving linear equations and a relational understanding of the equal sign are rooted in children's conceptions of equivalence (Kieran, 1981; Knuth et al., 2006; McNeil et al., 2015). It is also well documented, however, children's struggles solving problems that require an understanding of equivalence (Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Knuth et al., 2006; Stephens et al., 2013). This highlights the critical importance of learning what it means for children to come to understand equivalence and how this process occurs.

This study investigated schemes children construct for creating equivalence between two multiplicative compilations of abstract composite units (CUs) (Steffe & Cobb, 1998) (such as 7 baskets with 4 apples in each basket and 13 baskets with 4 apples in each basket). This requires operating on and coordinating two compilations and the difference between them. There are many ways to solve such tasks without concrete 1s present. For the purposes of this study, I will focus on two specific types of solutions that can occur when the CUs differ between the two compilations, but the unit rates are the same (13 CU of 4 and 7 CU of 4). For one solution the child may first find the total 1s in each compilation via multiplication (7•4=28, 13•4=52), subtract to find the difference in 1s (52-28=24), and then operate with the difference in 1s to create equivalence (52-12=28+12). Alternatively, the child may first produce the difference in CUs between the two compilations (13-7=6), and then operate with the difference in CUs to create equivalence (7+3=10, 13-3=10).

Multiplicative reasoning was chosen as the context for this study for two significant reasons. First, research on mathematical equivalence and children's understanding of the equal sign (Falkner et al., 1999; McNeil et al., 2015) has predominately focused on problems that require additive reasoning (e.g. 8+4=__+5). Results in the domain of additive reasoning have pointed to the possibility young children can develop a relational understanding of the equal sign (Baroody & Ginsburg, 1983; Jacobs, Franke, Carpenter, Levi, & Battey, 2007). However, the schemes of mathematical equivalence children develop have not been identified and it is also not known if equivalence schemes evolve as children's reasoning expands. The second reason the use of

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multiplicative situations is so critical is that they allow for differentiation between children's schemes based on the units they operate on and with (Steffe & Cobb, 1988). A child's additive and multiplicative reasoning afford and constrain the algebraic schemes they construct (Steffe, Liss, & Lee, 2014). By including multiplicative situations, how children's constructions of equivalence schemes are afforded and constrained by their current additive and multiplicative reasoning can be studied.

Conceptual Framework

The cognitive foundation for this study was provided by the reflection on activity-effect relationship (*Ref*AER*) framework (Simon, Tzur, Heinz, & Kinzel, 2004; Tzur & Simon, 2004). *Ref*AER* is an elaboration of constructivist scheme theory (Piaget, 1985; von Glasersfeld, 1995) that describes how a learner forms a novel conception through two types of reflections on their mental activity (Simon et al., 2004). *Type-I reflections* are comprised of comparisons between the learner's goal and the effects (as noticed by them) of their mental activity. When newly noticed effects become associated with a particular (mental) activity or activity sequence, the learner can anticipate an effect from that activity and an activity-effect relationship (*AER*) is created. Each time a learner sets a goal, brings forth their mental activity and connects an effect with that activity, a record of experience is formed. *Type-II reflections* consist of comparisons across these records. Through such reflections, a learner can abstract invariants and develop the anticipation of when to call up an *AER*. The stage distinction (Tzur & Simon, 2004) is used to describe the stages that comprise the formation of a new conception. At the participatory stage, a learner has connected a particular mental activity or activity sequence with an effect and has created an *AER*. What they cannot do, until the anticipatory stage, is call up such an *AER* spontaneously and independently when the original activity is not available to them.

The content-specific constructs guiding this study were children's multiplicative reasoning and number schemes. The Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988) and the Generalized Number Sequence (GNS) (Steffe, 1994) were schemes drawn upon to describe children's operating on CUs. In each, the child conceptualizes smaller compilations as embedded within a larger compilation. For example, 7 CUs of 4 (7 baskets of 4 apples) and 6 CUs of 4 (6 baskets of 4 apples) are embedded within 13 CUs of 4 (13 baskets of 4 apples). Additionally, a child with GNS can operate with abstract iterable CUs and can anticipate the multiplicative structure (four 1s distributed over each of the 7 CU) prior to operating with it (Steffe, 1994). The multiplicative schemes included multiplicative double counting (mDC) (Tzur et al., 2013) and Unit Differentiation-and-Selection (UDS) (McClintock, Tzur, Xin, & Si, 2011).

Methodology

This study consisted of 14 teaching episodes taught by the author over 2 months with an 8th grader, Joe. It was conducted as part of a teaching experiment with two pairs of 7th and 8th grade students, designed to develop algebraic reasoning in middle school students. Joe was purposefully selected for this study because he was operating with at least an Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988). McClintock et al. (2011) identified operating with an ENS as key to children's construction of a UDS scheme.

The two primary tasks provided to Joe during the episodes were UDS tasks (McClintock et al., 2011) and QRE tasks. UDS tasks ask the learner to compare two compilations that differ by either the number of CU (3 CU of 5 singletons and 8 CU of 5 singletons) or by the size of the CU (3 CU of 5 singletons and 3 CU of 4 singletons). Once presented with the two compilations, the learner is then asked a succession of questions, "How are these collections similar? How are they different? Who has more cubes and how many more?" (McClintock et al., 2011, p. 166). QRE tasks can follow or be independent of UDS tasks. They also incorporate two compilations that can differ by the quantity or

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size of the CU, but the central question asked to the learner is for them to make the two compilations the same. In some cases, the learner is also asked to build the compilations using Unifix cubes (3 CU of 5 = 3 towers with 5 cubes in each tower, designated $3T_5$) and to demonstrate their solution using the cubes.

Data analysis was conducted in two phases: during the planning and evaluations of teaching sessions and retrospective analysis. In the on-going analysis, critical events such as evidence of current conceptions and the stage of these conceptions were identified and discussed. Through this analysis, appropriate tasks and prompts were then selected for the next episode. The purpose of the tasks and prompts was either to foster development of new conceptions or to test the access a child had to their current conceptions. In the retrospective analysis, previously identified critical video segments were transcribed and analyzed. The children's written work was also analyzed. Of particular importance were segments that enabled inferences about Joe's operating and provided evidence of his transition between stages.

Results

In this section, data supporting Joe's initial relational equivalence (RE) scheme and lack of a quantitative relational equivalence (QRE) scheme are first discussed. Next, analysis of data is provided in support of the assertion that Joe progresses to a QRE scheme. Specifically, I claim Joe transitions into the participatory stage and then anticipatory stage of a QRE scheme.

Relational Equivalence Scheme

During our third session together, Joe and another child, Javier, were given a UDS task (McClintock et al., 2011) followed by a QRE task using the same compilations. Joe was given (for pretend) compilations consisting of 7 towers with 8 cubes in each tower ($7T_8$) and 9 towers with 8 cubes in each tower ($9T_8$), respectively. During the UDS portion of the task, he multiplicatively produced the two totals (7•8=56, 9•8=72) and then operated on 1s (72-56=16) to find that Javier had 16 more cubes than he did. A QRE task then directly followed which asked Joe to create equality between the two compilations. Excerpt 1 contains his explanation of his solution to the QRE task (T=teacher-researcher, J=Joe).

Excerpt 1 (April 17, 2012)

J: I took eight from him and gave me 8.
T: Okay, you took eight from him and gave you eight. Why were they both equal –because they both equaled what?
J: Um. [Looks at his work for a few seconds and then starts to write.] I don't know.
T: That's okay. You just knew that they were 16 apart.
J: Yeah because if I subtracted it, then I could add it, then [inaudible].
T: Okay, so you don't really have to know what the number is. You just know it's equal.
J: [Writes 56 + 8 vertically and then 64 underneath.] It's 64.

The exchange provided evidence that Joe had an anticipatory RE scheme. Joe's goal was to create equivalent quantities of 1s by reducing the difference in 1s between the two compilations to zero. His activity sequence consisted of: select the CUs and unit rate in each compilation and produce two totals of 1s (7•8=56, 8•9=72), subtract the smaller total from the larger total to produce the difference (72-56=16), divide the difference by two to produce half of the difference (16/2=8), dis-embed an amount of 1s equal to half the difference from the larger total and add it the smaller total. Joe relied on his part-to-whole reasoning and his operations with three levels of units (Steffe, 1994). He used part-to-whole reasoning to create a nested relationship between the two totals of 1s. Joe then operated with three levels of units as he transformed the totals. He operated on the three CUs (the

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smaller total, the larger total, and the difference) where the smaller total and difference acted as a second level of units embedded in a third level of units (the larger total). Using his additive operations, he dis-embedded and re-embedded pieces of the difference as he saw fit to bring the two totals into balance.

Evidence of his solution method, designated Halve-the-Difference (HTD), was further supported by his written solution, "To make same u (you) need to take 8 from (Javier) and give them to me." Joe did not need to enact the transformations of the totals to verify that equality had been achieved. He did not produce the 64 until the researcher requested it. Joe anticipated that balance was achieved by his operating (re-distributing the difference to the totals) rather than by the end result of his operating (comparing the two new totals). Joe's RE scheme was at the anticipatory stage. He anticipated creating the embedded structure of 1s and the subsequent operations on 1s to produce equivalence without any prompting in a novel situation.

Fostering Construction of the Participatory Stage of a QRE Scheme

Joe created equivalence, but even when prompted, he did not balance the CUs. I hypothesized he lacked a QRE scheme. To promote Joe's construction of a QRE scheme, he needed to be oriented to operations on CUs. This required Joe to suspend producing the totals multiplicatively in the original compilations and operate with each of them as a collection of CUs. The following three episodes describe how Joe made this transition. In the first episode, our 6th episode together (April 26, 2012), Joe created equivalence via operations on CUs for the first time. The compilations for the task given were Javier scored 4 baskets each game for 5 games and Joe scored 4 baskets each game for 6 games. After finding who scored more baskets and by how many, Joe was given the QRE task, "Can you make them the same?"

Joe enlisted his RE scheme to produce the solution, "Take 2 baskets…" At this point, the teacher-researcher cut Joe off so he did not give the solution away to Javier. Making Joe wait for Javier prompted him to reflect on his operating and produce a second solution. Joe's initial utterance demonstrated he initially created equivalence via the HTD method. He took 2 baskets (half of the difference of 4 baskets) from the total of his compilation (24-2) and added 2 baskets to the total of Javier's compilation (20+2). But after the prompt, he balanced the two compilations and created equivalence by adding one more CU to the CUs in the smaller compilation (5 games with 4 baskets + 1 game with 4 baskets = 6 games with 4 baskets). Evidence of this was provided from his written work that included "6 – 5" and "add 1 game."

Despite Joe's subtraction of 5 from 6, I did not consider him to have purposefully returned to the compilations with the intention to find the difference in CU. Rather, I hypothesized he first reconstituted the difference in 1s as one CU (1 game of 4 baskets) via his segmenting operation (Steffe, 1992). This required Joe to return to the original compilations to retrieve the size of the CUs (4 baskets in each game). It was only after he had produced the CU of 1 game from the 1s in the difference (4 baskets) that he noticed it was the same as the difference in CUs between the original compilations. Joe had abstracted new activity through a Type-I reflection that could be used to produce equivalence, but it was tied to his original goal of operating on the difference as an embedded total of 1s. His current RE scheme still need to be executed before Joe could use CUs to produce equivalence. This indicated Joe still lacked a QRE scheme.

During session 7 on May 8, Joe continued to create equivalence via operating on CUs. I asked Joe to create equality between two compilations: 8 boxes of cookies with 6 cookies in each box (48 cookies) and 7 boxes of cookies with 6 cookies in each box (42 cookies). The purpose of the task was to test the participatory and anticipatory nature of his QRE scheme. If Joe had not moved to the participatory stage of QRE, the task was also designed to foster such a progression by offering him a chance to notice the difference in 1s (6 cookies) was also a difference of 1 CU. Joe produced equivalence by adding 6 to the smaller total of 42 cookies to create totals of 48 cookies. His initial solution demonstrated that he did not have access to operations on CUs in anticipation and was not in the anticipatory stage of a QRE scheme. I then prompted Joe in two ways to orient him to the difference in the CUs.

I prompted Joe by asking him for an alternate solution. Joe again operated on 1s as he found and balanced the two totals by removing 3 cookies from the larger total and adding them to the smaller total (an HTD method). It was not until a second prompt that Joe operated on CUs. When I asked Javier to explain why Joe's method worked, it served as a prompt for Joe who immediately wrote down, "1 box to me." I hypothesized the prompt had re-focused Joe to the original compilations of the task and the relationship between the individual quantities across the two compilations. Excerpt 2 provides evidence of how Joe re-constituted his solution of adding 6 cookies to adding 1 box of cookies to balance the CUs (8 boxes of cookies per person).

Excerpt 2 (May 8, 2012)

J: You can also give me one more box.
T: What do you mean by, "give you another box"?
J: Well, he has 7 boxes. I have 7 and he has 8 boxes and each box has 6 in it, so it would give me one more box.

From his explanation, I inferred Joe was developing a new goal associated with a QRE scheme. Joe reflected on the relationship between the original compilations and the CUs (created through operations on 1s) used to create equivalence. I hypothesized that through a Type-II reflection, Joe was abstracting a new invariant: the embedded nature of the three compilations as collections of CUs. Moreover, Joe was transitioning into the participatory stage of a QRE scheme. My hypothesis was supported during the next task when Joe enlisted a method, designated Distribution-of-Embedded CUs (DECU), which used the embedded structure. The task asked, "Javier buys 7 boxes of cookies with 3 cookies in each box. Joe buys 9 boxes of cookies with 3 cookies in each box. Can you make them the equal?" The left side of Figure 1 shows Joe's initial computations of the totals. At that point, Joe paused for a moment, looked back at the statement of the problem, and then wrote, "add 2 more box(e)s to Javier."

Figure 1. Joe's written work demonstrating his transition to operations on CUs.

Initially calculating the totals served as an internal prompt for his activity associated with operations on CUs. I inferred that a reorganization of Joe's multiplicative and balancing schemes had occurred. Joe's new goal was to produce a difference in CUs and use those CUs to create equivalence. He progressed to a higher level of operating that included operations on CUs, but he could only access this goal through the original context (for him) of two embedded totals of 1s. From Joe's operating, I concluded he was in the participatory stage of a QRE scheme.

Anticipatory Stage of a QRE Scheme

During his 8th session on May 10, I tested Joe's reorganization of his QRE scheme. Joe continued to enlist his HTD method prior to operating on CUs, indicating he was not in the anticipatory stage of a QRE scheme. To help him gain access to his QRE scheme, I re-introduced Unifix cubes to use when he explained his solutions. This gave Joe an opportunity to physically see and mentally reflect on the relationship between the CUs in the original compilations and the

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difference. Moreover, Joe could reflect on how balancing the number of CUs also produced equality of the total 1s because the CUs and total 1s were equal in the two compilations.

When solving a novel task at the beginning of the next episode, Joe independently and spontaneously enlisted operations on CU in anticipation. The task was “Javier buys 19 bags of candy with 6 pieces of candy in each bag. Joe buys 15 bags of candy with 6 pieces of candy in each bag. Make them so you have the same amounts.” Joe was also given the constraint of written work was not allowed. This constraint created a new task for Joe in the sense that it required his QRE scheme. The size of the quantities coupled with the constraint introduced computational complexity for his RE scheme. Joe brought forth his QRE scheme and took 2 bags of candy away from Javier and added 2 bags of candy to his own collection leaving each compilation with 17 (19-2=17 and 15+2=17) bags of candy. Similar to his previous work with 1s, Joe did not produce the 17 bags of candy for each person. He did not have to because his method, designated Halve-The-Difference-CU (HTD-CU), had created equivalence without it.

Joe had balanced two compilations in a similar fashion with 1s when using his RE scheme. His equivalence scheme, in conjunction with his conceptualization of the difference as a qualifier of the nested relationship between the smaller and larger totals of 1s, enabled him to anticipate an even re-distribution of the difference brought the two compilations to balance. Now, as evidenced by the description of his HTD-CU solution in Excerpt 3, Joe had turned his focus to the CUs (4 bags) in the difference.

Excerpt 3 (May 15, 2012)

T: Yeah. It looked like at first you had plus two and minus two.
J: Yeah, I was thinking about bags.
T: You were thinking about bags.
J: Um-hum.
T: Tell me what you mean when you were thinking about bags. You took two bags from?
J: Javier, and [grabs two towers from his 15 towers of 6 and moves them to Javier’s 19 towers of 6] I take these two. And then I give them to me. And then they’d be equal.

Joe enlisted his GNS (Steffe, 1994) as he operated with the four CUs (of size 6 pieces of candy) comprising the difference in a manner similar to his earlier operating with 1s. Joe produced units of units of units as he halved the difference into 2 groups of 2 bags with 6 pieces of candy in each bag. He then operated on the 2 units of units of units by dis-embedding them from the difference and then re-distributing them to the two compilations ($19T_6$ - $2T_6$ and $15T_6 + 2T_6$) to create equivalence. His QRE scheme was now recursively being applied to the 2 CUs in the same way he had operated on 1s previously. Joe operated purposefully only on the CUs throughout the process and did not operate on the 1s in the compilations at all. He anticipated the *AER* (Tzur & Simon, 2004), connecting his mental activity of operating on the embedded the compilations as CUs to the effect of creating equivalence via operations on CUs.

In solving the task, Joe independently and spontaneously created equivalence via the activity sequence (a) differentiate the 1s and unit-rate from the CUs (b), select the CUs and operate on them to find their difference (using subtraction), (c) select the difference in CUs and operate on it to halve it (via division by 2), (d) subtract a quantity of CUs equal to half the difference from the CUs in the larger compilation and add the same quantity to the CUs of the smaller compilation. The key to getting Joe to demonstrate he had an anticipatory scheme was to create a need for him to use it in a novel task. The task was successful because of two aspects: the constraint of not allowing written work and the inclusion of large numbers that made mental computation difficult. Joe’s solution provided evidence he was operating with an anticipatory QRE scheme.

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Discussion

This study provides three contributions. First, this study identifies two schemes for mathematical equivalence. RE and QRE both incorporate additive balancing operations that create equivalence between two multiplicative compilations. When operating with an RE scheme, the learner first multiplicatively produces the totals of 1s from each compilation. They then find the difference in 1s between the totals and create equivalence by operating on the totals with some or all of the 1s in the difference. For a learner with a QRE scheme, operations for creating equivalence with 1s are available, but they instead select CUs for operating. They enlist a method (such as DECU or HTD-CU) that requires operations with three levels of units. The learner produces a difference in CUs between the two compilations and creates equivalence via additive operations on CU. They may re-distribute the CUs in the difference or transform the CU in one of the original compilations to create equivalence. In either case, this study demonstrates how an anticipatory QRE scheme enables a child to relate multiplicative compilations via operations on CUs. Such operations can provide the conceptual roots for fundamental algebraic concepts such as the distributive property of multiplication over addition [19•6-15•6=(19-15)•6] and solving linear equations (8•5=__•10).

Second, differentiating between RE and QRE can help teachers address why their students continue to struggle with understanding the equal sign as a relational symbol (Kieran, 1981). When examining children's understanding of mathematical equivalence, research has focused on children's conceptualization of one side of an equation as "the same as" the other side of the equation (Falkner et al., 1999; Knuth et al., 2006; Stephens et al., 2013). Moreover, because it has been shown that children can use their additive operations to make sense of the equivalence of expressions such as 4+5 and 6+3 (Baroody & Ginsburg, 1983; Stephens et al., 2013), researchers have posited that a relational understanding of the equal sign can be developed as early as first grade. However, this study suggests how children understand "same as" changes as their operations evolve. Children who have constructed a QRE scheme can make sense of equivalence in CUs or 1s, while children with an RE scheme need 1s to conceive of equivalence.

Finally, this study demonstrates the transition from a lack of a QRE scheme to the anticipatory stage of a QRE scheme. This transition is rooted in operations on CUs. The operations on 1s contained in the child's RE scheme must be "lifted" to a higher level where they become operations on collections of CUs. Joe solved the tasks with his current schemes, but through reflection, he made two new abstractions. First, when the compilations differ only in the quantity of CUs, he abstracted that the difference, as a number of CUs, could be operated with to create equivalence between the two compilations. His second abstraction was the embedded relationship between the compilations as collections of CUs. The key to the second abstraction made by Joe and his transition to QRE was his GNS allowed him to anticipate the multiplicative structure of each compilation prior to any operating.

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