EXAMINING INDIVIDUAL AND COLLECTIVE LEVEL MATHEMATICAL PROGRESS

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A challenge in mathematics education research is to coordinate different analyses to develop a more comprehensive account of teaching and learning. I contribute to these efforts by expanding the constructs in Cobb and Yackel's (1996) interpretive framework that allow for coordinating social and individual perspectives. This expansion involves four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical conceptions that individuals bring to bear in their mathematical work. I illustrate these four constructs for making sense of students' mathematical progress using data from an undergraduate mathematics course in differential equations.

Un reto en la investigación en educación matemática es la coordinación de diferentes análisis para desarrollar una descripción más amplia de la enseñanza y el aprendizaje. Contribuyo a estos esfuerzos mediante la ampliación de los constructos del marco interpretativo de Cobb y Yackel (1996), los cuales permiten la coordinación de las perspectivas social e individual. Esta ampliación involucra cuatro constructos diferentes: las prácticas disciplinares, las prácticas matemáticas del aula, la participación individual en la actividad matemática y las concepciones matemáticas que las personas utilizan en su trabajo matemático. Ejemplifico estos cuatro constructos para dar sentido al progreso matemático de los estudiantes con datos de un curso universitario de ecuaciones diferenciales.

Keywords: Research Methods, Classroom Discourse, Cognition

Recent work in mathematics education research has sought to integrate different theoretical perspectives to develop a more comprehensive account of teaching and learning (Bikner-Ahsbahs & Prediger, 2014; Cobb, 2007; Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Prediger, Bikner-Ahsbahs, & Arzarello, 2008; Rasmussen, Wawro, & Zandieh, 2015; Saxe et al., 2009). An early effort at integrating different theoretical perspectives is Cobb and Yackel's (1996) emergent perspective and accompanying interpretive framework. In this paper I expand the interpretive framework for coordinating social and individual perspectives by offering a set of constructs to examine the mathematical progress of both the collective and the individual. I illustrate these constructs by conducting four parallel analyses and make initial steps toward coordinating across the analyses.

The emergent perspective is a version of social constructivism that coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). A primary assumption from this point of view is that mathematical development is a process of active individual construction and a process of mathematical enculturation (Cobb & Yackel, 1996). The interpretive framework, shown in Figure 1, lays out the constructs in the emergent perspective. The significance of accounting for both individual and collective activity is highlighted by Saxe (2002), who points out that, "individual and collective activities are reciprocally related. Individual activities are constitutive of collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities" (p. 276-277).

My and my colleagues' prior work with the interpretative framework (e.g., Rasmussen, Zandieh, & Wawro, 2009; Stephan & Rasmussen, 2002; Yackel, Rasmussen, & King, 2000) has raised our awareness of the opportunity (and occasional need) to extend the constructs in the interpretative

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framework. In particular, in Rasmussen, Wawro, and Zandieh (2015), we expand the ways to analyze individual and collective mathematical progress. We use the phrase "mathematical progress" instead of "learning" as an umbrella term that admits analyses of collective mathematical development and individual meanings and activity. That is, while it might make sense to speak of individual student learning, it makes less sense to speak of collective learning because this incorrectly implies a deterministic, one size fits all approach. The phrase mathematical progress, on the other hand, offers a way to address both the collective and the individual without suggesting a deterministic stance toward the collective.

Social Perspective	Individual Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Figure 1. The interpretive framework**.**

On the bottom left hand side of the interpretive framework (Figure 1), the construct of classroom mathematical practices is a way to conceptualize the collective mathematical progress of the local classroom community. In particular, such an analysis answers the question: What are the normative ways of reasoning that emerge in a particular classroom? Such normative ways of reasoning are said to be reflexively related to individual students' mathematical conceptions and activity. In prior work that has used the interpretive framework, individual conceptions and activity has been treated as a single construct that frames the ways that individual students participate in classroom mathematical practices (e.g., Bowers, Cobb, & McClain, 1999; Cobb, 1999; Stephan, Cobb, & Gravemeijer, 2003). Such a framing of the individual is, in our view, compatible with what Sfard (1998) refers to as the "participation metaphor" for learning.

In an effort to be more inclusive of a cognitive framing that would posit particular ways that students think about an idea, Rasmussen, Wawro, & Zandieh (2015) split the bottom right hand cell into two constructs, one for individual participation in mathematical activity and one for mathematical conceptions that individual students bring to bear in their mathematical work. With these two constructs for individual progress one now can ask the following two questions: How do individual students contribute to mathematical progress that occurs across small group and whole class settings? And what mathematical meanings do individual students develop and bring to bear in their mathematical work?

Work at the undergraduate level has also highlighted the fact that, in comparison to K-12 students, university mathematics and science majors are more intensely and explicitly participating in the discipline of mathematics. However, the notion of a classroom mathematical practice was never intended to capture the ways in which the emergent, normative ways of reasoning relate to various disciplinary practices (Stephan & Cobb, 2003). In order to more fully account for what often occurs at the undergraduate level, we expand the interpretive framework to explicate how the classroom collective activity reflects and constitutes more general disciplinary practices. Thus there is an additional cell to the bottom left row of the interpretive framework, disciplinary practices. One can now answer the following two questions about collective mathematical progress: What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics? And what are the normative ways of reasoning that emerge in a particular classroom?

To summarize, Figure 2 shows the expansion of the bottom row of the interpretive framework, which now entails four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical meanings.

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Social Perspective		**Individual Perspective**	
Classroom social norms		Beliefs about own role, others' roles, and the general nature of mathematical activity	
Sociomathematical norms		Mathematical beliefs and values	
Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical meanings

Figure 2. Expanded interpretive framework.

The left hand side of the bottom row comprises two different constructs for examining the mathematical progress of the classroom community, while the right hand side comprises two different constructs for examining the mathematical progress of individuals. The contribution that this expansion makes is in providing researchers with a more comprehensive means of bringing together analyses from social and individual perspectives. In particular, the expanded interpretive framework enables a researcher to answer the questions listed in Figure 3.

Disciplinary practices	**Classroom mathematical practices**	**Participation in mathematical activity**	**Mathematical meanings**
What is the collective mathematical progress in terms of the disciplinary practices?	What are the normative ways of reasoning that emerge in a particular classroom?	How do individual students contribute to the collective mathematical progress?	What meanings do individual students develop and bring to bear in their work?

Figure 3. Four constructs for analyzing mathematical progress and respective research questions.

Theoretical and Methodological Background

Classroom mathematical practices

Classroom mathematical practices refer to the normative ways of reasoning that emerge as learners solve problems, explain their thinking, represent their ideas, etc. By normative I mean that there is empirical evidence that an idea or way of reasoning functions as if it is a mathematical truth in the classroom. This means that particular ideas or ways of reasoning are functioning in classroom discourse as if everyone has similar understandings, even though individual differences in understanding may exist. The empirical evidence needed to document normative ways of reasoning is garnered using the approach developed by Rasmussen and Stephan (2008) and furthered by Cole et al. (2012). This approach, which we refer to as the documenting collective activity method, applies Toulmin's (1958) argumentation scheme to document the mathematical progress using three well-developed criteria, all of which involve tracing over time how ideas are used by students. In brief, central to Toulmin's scheme is the core of an argument, which consists of a Claim, Data to support that Claim, and a Warrant that explains the relevance of the Data to the Claim.

Disciplinary practices

Disciplinary practices refer to the ways in which mathematicians typically go about their professional practice. The following disciplinary practices are among those core to the activity of professional mathematicians: defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen, Zandieh, King, & Teppo, 2005). Not all classroom mathematical practices are easily or sensibly characterized in terms of a disciplinary practice. This is because classroom mathematical practices capture the emergent and potentially idiosyncratic collective mathematical progress, whereas a

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disciplinary practice analysis seeks to analyze collective progress as reflecting and embodying core disciplinary practices. In this report I focus on algorithmatizing, the practice of creating and using algorithms. Our method for documenting disciplinary practices builds on prior work that has examined theoremizing, symbolizing, and defining (Rasmussen, Zandieh, King, & Teppo, 2005; Rasmussen, Wawro, & Zandieh, 2015; Zandieh & Rasmussen, 2010).

Mathematical meanings

As students solve problems, explain their thinking, represent their ideas, and make sense of others' ideas, they necessarily bring forth various meanings of the ideas being discussed and potentially modify these meanings (Thompson, 2013). Our analysis of individual student meanings makes use of analyses from prior work that have characterized different ways that students think about the relevant mathematical ideas (e.g., Carlson, Jacobs, Coe, Larsen, & Hu, 2002; Habre, 2000; Harel & Dubinsky, 1992; Thompson, 1994; Trigueros, 2001; Rasmussen, 2001; Zandieh, 2000).

Participation in mathematical activity

This analysis draws on recent work by Krummheuer (2007, 2011), who characterizes individual learning as participation within a mathematics classroom using the constructs of production roles and recipient roles. In the production framing, individual speakers take on various roles, which are dependent on the originality of the content and form of the utterance. The title of author is given when a speaker is responsible for both the content and formulation of an utterance. The title of relayer is assigned when a speaker is not responsible for the originality of either the content nor the formulation of an utterance. A ghostee takes part of the content of a previous utterance and attempts to express a new idea, and a spokesman is one who attempts to express the content of a previous utterance in his/her own words. Within the recipient framing of learning-as-participation, Krummheuer (2011) defines four roles: conversation partner, co-hearer, over-hearer, and eavesdropper. A conversation partner is the listener to whom the speaker seems to allocate the subsequent talking turn. Listeners who are also directly addressed but do not seem to be treated as the next speaker are called co-hearers. Those who seem tolerated by the speaker but do not participate in the conversation are over-hearers, and listeners deliberately excluded by the speaker from conversation are eavesdroppers.

Setting and Participants

I illustrate the four constructs and address the respective research questions from Figure 3 using data from a semester-long classroom teaching experiment (Cobb, 2000) in differential equations conducted at a medium sized public university in the Midwestern United States. I selected a 10-minute small group episode from the second day of class based on its potential to illustrate all four constructs. There were four students in this group, Liz, Deb, Jeff, and Joe (all names are pseudonyms).

There were 29 students in the class. Class met four days per week for 50-minute class sessions for a total of 15 weeks. The classroom had movable small desks that allowed for both lecture and small group work. The classroom teaching experiment was part of a larger design based research project that explored ways of building on students' current ways of reasoning to develop more formal and conventional ways of reasoning (Rasmussen & Kwon, 2007). A goal of the project was to explore the adaptation of the instructional design theory of Realistic Mathematics Education (RME) to the undergraduate level. Central to RME is the design of instructional sequences that challenge learners to organize key subject matter at one level to produce new understanding at a higher level (Freudenthal, 1991). In this process, graphs, algorithms, and definitions become useful tools when students build them from the bottom up through a process of suitably guided reinvention (e.g., Rasmussen & Blumenfeld, 2007; Rasmussen & Marrongelle, 2006; Rasmussen, Zandieh, King, &

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Teppo, 2005).

Results and Discussion

As previously stated, the analysis comes from video recorded work of a small group of four students, Liz, Deb, Jeff, and Joe, on the second day of class. Just prior to the small group work students completed the following task: The previous problem dealt with a complex situation with two interacting species. To develop the ideas and tools that we will need to further analyze complex situations like these, we will simplify the situation by making the following assumptions:

- There is only one species
- The species have been in the lake for some time before what we are calling time t = 0
- The resources (food, land, water, etc.) are unlimited
- The species reproduces continuously

Given these assumptions for a certain lake with fish, sketch three different population versus time graphs (one starting at P = 10, one starting at P = 20, and the third starting at P = 30).

This task was relatively straightforward for students and brought forth an imagery of exponential growth and the graphs they sketched were consistent with this imagery. The instructor then used their graphs as an opportunity to introduce the rate of change equation dP/dt = 3P as a differential equation that was consistent with their graphs. In particular, as P values increase, so does the slope of the graph of P vs. t.

The follow up task, which students worked on for approximately 10 minutes, however, was much more cognitively demanding for students.

Consider the following rate of change equation, where P(t) is the number of rabbits at time t (in years): dP/dt = 3P(t) or in shorthand notation dP/dt = 3P. Suppose that at time t = 0 we have 10 rabbits (think of this as scaled, so we might actually have 1000 or 10,000 rabbits). Figure out a way to use this rate of change equation to approximate the future number of rabbits.

At t = 0.5 and t = 1.
At t = 0.25, t = 0.5, t = 0.75, and t = 1

Be prepared to share the reasoning behind your approach with the rest of the class. Organize your results in tabular and graphical form.

Recall that this is only the second day of class and students have not been introduced to any analytical, numerical, or graphical techniques for analyzing differential equations. In a related analysis, Tabach, Rasmussen, Hershkowitz, and Dreyfus (2015) provide the following a priori analysis of the knowledge elements that we expect students to construct when solving this task:

- Csy – establishing connection between P and dP/dt (if you know P you can find dP/dt)
- Cpit – population iteration (given P and dP/dt at a moment in time allows one to find P at a later time)
- Crit – rate of change iteration (applying Csy at that later time one can find the corresponding dP/dt)
- Cit – Cpit and Crit can be combined into a repeating loop.

To illustrate how these knowledge elements play out in student discourse, consider the following except from Liz which occurred near the end of the 10-minute small group work:

> *Liz*: What I understand is that we found our rate of change initially at time zero and that we are using that to find out what our population is after half a year. If we are expected to grow by 30 rabbits in a year then, in a half a year we grow by 15 rabbits. So we'll have 15, I mean 25 because 15 plus 10 is 25. Then you start over again, so it's kind of like our new initial population. We could label it time equals zero if we wanted to.

An example of Csy occurs when Liz says, "grow by 30 rabbits in a year" because in order to get the value of 30, Liz had to use the initial population value of 10, plug this into the rate of change equation to get 30. Cpit is illustrated by Liz when she says, "in a half year we grow by 15 rabbits. So we'll have 15, I mean 25 because 15 plus 10 is 25." That is, she uses what she knows about the population at the initial time and her knowledge of dP/dt at the initial time to compute how many rabbits there will be half a year later. Finally, when Liz says, "The you start all over again," she demonstrates an understanding that Cpit and Crit can be combined into a repeating loop to compute the population after another half year.

As a reminder, the primary research goal here is to demonstrate an approach for coordinating collective and individual analyses to gain greater explanatory and descriptive power, with the intention to better understand the individual and collective meaning making processes. I will therefore begin with analyzing the collective small group mathematical progress using the previously mentioned documenting collective activity method.

Small group collective mathematical progress

Using the documenting collective activity method I identified the following three ideas that functioned as if shared in this particular small group: dP/dt can be determined from P values (Csy), a value for dP/dt refers to the amount of change over 1 year, Cpit and Crit can be combined into a repeating loop. All three of these findings made use of the second criteria, namely that what was originally a Claim in one argument later functions as Data in a subsequent argument (Rasmussen & Stephan, 2008). In other words, an idea that initially required some form of justification is later used a means to justify new claims. Figure 4 shows the Toulmin analysis for first argument and Figure 5 shows the Toulmin analysis for the fifth argument made in this particular small group.

Data: This is where 10 rabbits at zero

Claim: The initial instantaneous rate of change

Warrant: I would plug in the population of rabbits for P to determine the rate of change

Backing: If we had a graph, its kind of like what we were just talking about, we are trying to determine the rate of change when this time is equal to zero (Liz)

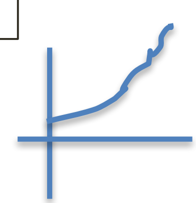

Figure 4. Toulmin analysis for Argument 1.

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Data: We have the 30. Three times ten would give us our rate of change. Say 0.5 years

Claim: which will give us what, the new amount of rabbits plus the old amount

Warrant: Warrant: This is our rate of change. Then we'll take that 0.5 times the rate of change (Deb)

Figure 5. Toulmin analysis for Arguments 5.

We see here in the first argument that Liz makes the Claim that the initial instantaneous rate of change is 30 (note that this is Csy) and then in the fifth argument Deb uses as a fact that the initial rate of change is 30 as Data to support a new claim. Thus per the second criteria it is concluded that one can determine dP/dt from P values functions as if shared in this small group. Full consideration of the data indicate that Liz, Deb, and Jeff (but not Joe) made individual progress compatible with the collective mathematical progress. That is, when a researcher determines that an idea functions as if shared, it does not mean that everyone shares exactly the same way of thinking.

I next step back from the specifics of the Toulmin analysis across the 10-minute episode to highlight overall trends in the discourse. In terms of talk turns, Liz spoke 26 times, Deb spoke 18 times, Jeff 13 times, and Joe 8 times. Thus Liz and Deb were the primary contributors, with Jeff often highlighting a final answer. Overall there were 14 different arguments (à la Toulmin) that consisted of at least Data and Claim. The following table shows the distribution of contributions (some contributions co-constructed).

Table 1: Toulmin argument contributions by student

	Liz	Deb	Jeff	Joe
Data	6	5	1	4
Claim	5	5	5	2
Warrant	2	5	1	0
Backing	2	1	0	0

In light of the collective small group mathematical progress, I next begin to address the following coordination questions: What meanings for dP/dt emerged and who expressed these meanings? What part did these meanings play in the collective mathematical progress? What roles did Liz, Deb, Joe, and Jeff play in all of this? In what ways did students' mathematical work reflect disciplinary practices?

Engaging different meanings for dP/dt

Across the 10-minute episode I identified seven different meanings for dP/dt used by one or more of the fours students. These seven meanings and who within the 10-minute episode engaged these meanings are: as steepness (Liz), ratio (Liz and Jeff), population length (Liz and Deb), tool (Liz), function (Deb), proportion (Deb), and fraction (Jeff). Thus, not only did Liz and Deb have more talk turns, they also engaged more and different meanings for dP/dt compared to Jeff and Joe. I next illustrate how students engaged these meanings, but due to space constraints a complete detailing is not possible.

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17 Liz: So if we have that [initial rate of change is 30], the question is how can we use that to help us figure out the population after a half unit elapsed?
22 Deb: You said the population is 10 right [Liz: Um hmm]. So three times ten would give us our rate of change. Say 0.5 years passes, this is our rate of change. Then we'll take that 0.5 times the rate of change which will give us what [slight pause looks up to Jeff and Joe], the new amount of rabbits plus the old amount of rabbits.

In line 17 we see Liz wonder out loud how knowing that 30 is the initial rate of change can be used to achieve their goal of determining the population a half year later. That is, Liz would like to somehow engage rate of change as a tool to do work for them. Shortly thereafter Deb takes Liz up on how they might use rate of change as a tool and suggests that they could take the 30 and multiply it by 0.5. It is not entirely clear the meaning that Deb is making use of here, but one possibility is that she is engaging in a form of proportional reasoning. Continuing their discussion, we see an important conceptual advance.

25 Liz: So the old amount of rabbits is 10.
26 Deb: Am I making sense?
27 Jeff: I think so, so that would be 25, is that what you're saying?
28: Liz: Okay I think I get what you're saying. Ok, so like we're at time zero and we have 10 rabbits, and supposedly the rate of change, well not supposedly, we're saying that the rate of change is 30 [Jeff: yeah for the] at time zero. So its going to grow at a rate of, I don't know if I'm going to say this right, at 30 rabbits per year? [looks up at Deb]

In line 27 we see Jeff contributing to the discussion by highlighting final computations. Thus, while Jeff is not necessarily leading the intellectual work, he is following along and adding to the discussion. Then in line 28 we see Liz make an interpretation for 30, the initial rate of change, that later serves her and her group well. In particular, she interprets 30 as the amount of rabbits that will accrue over a one year time increment. I refer to such a meaning as a "population length." This is similar to how Thompson documented early meanings of rate as a "speed length" where a student thinks of say, 60 mph, as going 60 miles in one hour. Deb next picks up on what Liz says and then in line 32 Liz returns to how one might use rate of change as a tool for figuring out the number of rabbits a half year later.

29 Deb: Right. [Liz: Ok] So we'll have 30 more rabbits.
32 Liz: And so we're really not figuring out the rate of change we figuring…Well this is the rate of change and we're using the rate of change to figure out the number of rabbits we are going to increase by in half a year.

As the students continue, we see Deb leading the intellectual work of figuring out how to use the 30 to achieve their goal.

38 Deb: This is what I did. First I looked at the fact that this is a rate of change equation. So this is telling me how many rabbits are being produced every year. So If I know 3 times the original population is produced every year, then I have 3 times 10 is produced every year. But I want to know how many is produced in 0.5 years. So I know how many rabbits are produced per year, so if I multiply that by 0.5 then I'll know how many more rabbits have been produced. So I take that new number that I get and add it to the old population.
43 Jeff: I think you can go dp/dt=30, actually your dt will be 0.5, and then you add that to the old and then you do it again for the next one.

In line 38 Deb expresses three different meanings for rate of change. She starts off by saying that the rate of change equation tells one how many rabbits are produced every year. This is similar to a function meaning for rate of change and relates closely to Csy. Given an input you get an output. And the meaning of the output in this case is a population length ("so I know how many rabbits are produced per year"). Deb then engages in some proportional reasoning to determine how many more rabbits there will be in a half year. In line 43 Jeff shows that he is following the discussion and seems to treat dP/dt as a fraction. Soon thereafter Liz recapitulates their line of reasoning as follows, engaging both meaning of rate of change as population length and rate of change as proportion.

> *48 Liz*: Okay, so basically, I get you up into the point where you say you want to put in, what I understand is that we found our rate of change initially at time zero and I understand using that to find out what our population is after half a year. If we are expected to grow by 30 rabbits in a year then, in a half a year we grow by 15 rabbits. So we'll have 15

I now turn to reflecting on the roles these various meanings played in the collective mathematical progress. As we saw, there is a shift in the meaning of dP/dt - from steepness to a "population length" (clearly for Liz and likely for Deb). This shift coincided with "a value for dP/dt refers to the amount of change over 1 year" functioning as if shared AND the initial articulation of how to find the estimate for the population at t = 0.5. In relation to other work, the principle of a form-function-shift (Saxe, 2002) of notations in use is particularly suitable for analyzing the interplay between tool use (in this case dP/dt = 3P) and conceptual development. In particular, the form-function-shift describes the interplay between cultural forms (external representations) and the meanings that develop for structuring and accomplishing specific goals, not unlike what we saw happening with the individual meaning and collective production of meaning.

What roles did Liz, Deb, Jeff, and Joe play in the collective mathematical progress?

Drawing on Krummheuer (2007, 2011), I characterize student participation in the collective mathematical progress in terms of production roles (author, relayer, ghostee, spokesman) and recipient roles (conversation partner, co-hearer, overhearer, eavesdropper). Previously I specified the number of talk turns for each student: Liz 26; Deb 18; Jeff 13; Joe 8. The raw count of co-author shows that there was fairly even distribution (Liz 6/14; Deb 5/14; Jeff 6/14; Joe 4/14). However, a more nuanced look however reveals important differences: Joe offered 2 incorrect arguments, Jeff often revoiced (with and without reformulation), Liz and Deb did the main intellectual lifting (as was evident in the excerpts). For example, Liz was primary author (core of argument) for Csy and as Spokesman for meaning of dP/dt as population length. Deb, on the other hand, was the primary author for Cit The following excerpt provides a snapshot illustration of how the entire 10-minute episode was coded.

> *26 Deb*: [articulates the main iteration idea but without a numerical result – omitted here for space considerations] Am I making sense?
>
> *27 Jeff*: I think so, so that would be 25, is that what you're saying?
>
> *28: Liz*: Okay I think I get what you're saying. Ok, so like we're at time zero and we have 10 rabbits, and supposedly the rate of change, well not supposedly, we're saying that the rate of change is 30 [Jeff: yeah for the] at time zero. So its going to grow at a rate of, I don't know if I'm going to say this right, at 30 rabbits per year? [looks up at Deb]

In line 25 Jeff functions as a relayer as he was not responsible for either the content or the formulation of the idea. In line 28 we see Liz function as spokesman for this is the first time anyone has engaged the meaning of rate of change as population length. As such Liz is responsible for both the content and the formulation.

Regarding recipient design roles, Liz, Deb, and Jeff were for the most part conversation partners and co-hearers. Joe was mostly a co-hearer and at times an over-hearer. While the constructs of production and recipient roles were useful in distinguishing individual differences, I found them to be

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insufficient to account for the different ways these four students participated in mathematical discourse. In particular, my analysis suggested a third role – that of facilitator roles. More specifically, I identified four different ways in which these students facilitated the flow of ideas in their small group. These four roles are:

- Focuser is assigned when a speaker directs attention to a particular mathematical issue
- Elicitor is given when a speaker attempts to bring out another's idea
- Checker is one who seeks agreement or sensibility of an utterance
- Summarizer pulls ideas together

For example, consider the following excerpts:

17 Liz: So if we have that [initial rate of change is 30], the question is how can we use that to help us figure out the population after a half unit elapsed? [32 sec pause, everyone looking down at their papers and making marks]
18 Jeff: So I was just going to say how would we work time into the equation to get the next, uh, population or change in population?
40 Liz: Yeah I get it, do you guys get what Deb is saying?
41 Jeff: Yeah you get 25 and then you get 55.
46 Liz: And the reason for putting in the new population would be what?
48 Liz: Okay, so basically, I get you up into the point where you say you want to put in, what I understand is that …..
53 Deb: Everybody agree?

In line 17 Liz functions as a focuser when she directs her group's attention to how they can use the initial rate of change as a tool for figuring out the population after a half year. In line 18 Jeff also serves the role of focuser when he directs the group's attention to how time gets integrated into their work. In line 40 Liz acts a checker when she queries the group to see if everyone gets what Deb is saying. Similarly Deb acts as a checker in line 53. In line 46 Liz functions as elicitor when she requests the rationale for carrying out a particular mathematical computation. Finally, in line 48 Liz pulls the ideas together and thus functions as a summarizer.

To conclude this section, I reflect on the ways in which students' mathematical work reflects the disciplinary practice of creating and using algorithms, or algorithmatizing.

The disciplinary practice of algorithmatizing

In the analysis previously presented we see the group of four students engaging in the first stage of creating an algorithm. These first steps lay the relational foundation for how to use P values and dP/dt values to approximate a future population value. An expert will recognize students' work as Euler's method, although the students do not as of yet know that what they have produced is in fact related to Euler's method. In the subsequent whole class discussion the different groups in the class discussed their work and together the class created the following algorithm: $P_{next} = P_{now} + \left(\frac{dP}{dt}\right)_{now} \times \Delta t$. The instructor then explained to the class that this algorithm is conventionally know as Euler's method and is an example of a numerical approximation. In subsequent classes students used this algorithm both with and without contexts and investigated the relationship between approximate solutions and exact solutions that are concave up, concave down, and that have a constant rate of change. Students also investigated how approximation graphs with different step sizes compared to each other and even different ways to improve Euler's method. Such mathematical progress reflects the disciplinary practice of creating and using algorithms, or what we refer to algorithmatizing.

More specifically, students' creation of the Euler method algorithm involved the following:

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engaging in goal directed activity, isolating attributes, forming quantities, creating relationships between quantities, and expressing these relationships symbolically. For example, Liz helped her group focus on a specific goal directed activity when she asked her group mates, “So if we have that [initial rate of change is 30], the question is how can we use that to help us figure out the population after a half unit elapsed?” This led to their group to think about 30, the initial rate of change, as the amount of rabbits that would be added in one year. Previously I referred to this meaning as a population length, which is an example of isolating an attribute and forming a quantity for this attribute. The population length was further refined when the group related this quantity to a time interval of a half year. The relationship between change and in time and population length was then further quantified. As pointed to previously, these relationships then formed the basis for Cpit and Crit functioning as if shared in this small group. Expressing these relationships symbolically occurred after the 10-minute small group work analyzed here.

Conclusion

In this section I first consider implications for practice and then implications for research. Regarding instructional design considerations, the analysis presented here raises the possibility of including in the student materials questions that focus student attention on the attributes that Deb and her group found particularly useful. Questions such as the following might be woven into the student materials: What is the initial rate of change? What does this value mean to you? How can you use the 30 to figure out the population after half unit of time? Of course this would have to be done in a way that does not take away from the challenge and cognitive demand of the task. Alternatively, such questions could be folded into instructor resource materials that support mathematics faculty in implementing inquiry-oriented curriculum. Indeed, efforts are underway by Estrella Johnson, Karen Keene, and Christine Larson to create such materials for differential equations, linear algebra, and abstract algebra (see http://times.math.vt.edu/).

Another instructional implication that this analysis raises is the how to help promote productive interactions between small group members. In this particular class the small group analysed worked extremely well together, even on the second day of class. This was largely good fortune. So then what might an instructor do to facilitate more productive interactions in small groups that do not function as well as Liz, Deb, Joe, and Jeff?

I now turn to discussing some implications for research. In addition to using various combinations of the four constructs to more fully account for students’ mathematical progress, there exist multiple ways in which coordination across the four constructs is possible. For instance, one could choose an individual student within the classroom community and trace his/her utterances for the ways in which they contributed to the emergence of various normative ways of reasoning and/or disciplinary practices. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students are that are offering the claims, data, warrants, and backing in the Toulmin analysis used to document normative ways of reasoning. How do those contributions coordinate with those students’ production roles within the individual participation construct? For instance, does a student ever utilize an utterance that a different student authored as data for a new claim that he is authoring, and in what ways may that capture or be distinct from other students’ individual mathematical meanings? One might also imagine ways to coordinate across the two individual constructs as well as across the two collective constructs. For example, how do patterns over time in how student participation in class sessions relate to growth in their mathematical meanings? Are different participation patterns correlated with different mathematical progress trajectories? In what ways are particular classroom mathematical practices consistent (or even inconsistent) with various disciplinary practices? Finally, future research could take up more directly the role of the teacher in relation to the four constructs.

I anticipate that future work will more carefully delineate methodological steps needed to carry out the various ways in which analyses using the different combinations of the four constructs can be coordinated. Indeed, this report is a first step in developing a more robust theoretical-methodological approach to analyzing individual and collective mathematical progress.

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