PSYCHOLOGY IN MATHEMATICS EDUCATION: PAST, PRESENT, AND FUTURE

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Starting with Woodworth and Thorndike's classical experiment published in 1901, major periods in mathematics education throughout 20th century and on into the current century are reviewed in terms of competing epistemological and psychological paradigms that were operating within as well as across the major periods. The periods were marked by attempts to make changes in school mathematics by adherents of the dominant paradigm. Regardless of what paradigm was dominant, the attempts essentially led to major disappointments or failures. What has been common across these attempts is the practice of basing mathematics curricula for children on the first-order mathematical knowledge of adults. I argue that rather than repeat such attempts to make wholesale changes, what is needed is to construct mathematics curricula for children that is based on the mathematics of children. Toward that end, I present several crucial radical constructivist research programs.

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> The accent must be on auto-regulation, on active assimilation – the accent must be on the activity of the subject. Failing this there is no possible didactic or pedagogy which significantly transforms the subject (Piaget, J., 1964).

Mathematics Education—1900-1950

Behaviorism and Faculty Psychology

The classical experiment. The classical experiment by Woodworth & Thorndike (1901) at the beginning of the 20th Century introduced the "scientific movement" in education and it was considered as the death knell of faculty psychology, the doctrine of "mental discipline" (e.g., Whipple, 1930; Thorndike, 1922). In faculty psychology, the mind was viewed as a collection of separate modules or faculties assigned to various mental tasks, such as reason, will, concentration, memory, or language and it was thought that training in one faculty would transfer to another. As a result of their experiment, Woodworth & Thorndike (1901) concluded that, "The improvement in any single mental function rarely brings about equal improvement in any other function, no matter how similar, for the working of every mental function-group is conditioned by the nature of the data in each particular case" (p. 250). The lack of transfer led Thorndike (1903) to develop his theory of identical elements: "The answer which I shall try to defend is that a change in one function alters any other only in so far as the two functions have as factors identical elements" (pp. 80-81). Once this idea was accepted, "arithmetic was on its way to being analyzed into elements so that the stimulus-response theories of Thorndike could be more readily applied" (Van Engen & Gibb, 1956, p. 1).

Cartesian epistemology. There was also a separation or duality between the mind and the body in faculty psychology in that it was thought that mental discipline of the intellect would lead to control of the will and emotions, a duality that has become known as "Descartes error" (Damasio, 1994, pp. 248)—"I think, therefore I am." It is interesting to me that this philosophical rationalism of faculty psychology was regarded as falsified by means of a "crucial experiment" that was conducted in the context of a competing paradigm, empiricism.[i] Although I don't wish to defend faculty psychology, in retrospect I believe that a basic reason why faculty psychology was abandoned transcended Woodworth and Thorndike's classic experiment. In empiricism, the doctrine that the

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world imprints itself on the mind, there is a duality that is similar to the mind-body duality between an endogenic (mind centered) view versus an exogenic (world centered) view (Konold & Johnson, 1991). This mind-reality duality in the main explains why faculty psychology was rejected and why empiricism was so widely embraced. In behaviorism, no explanation of mind was needed nor was it sought so there was already a major conflict in the two views of mind in faculty psychology and in the behaviorism of Woodworth and Thorndike[ii]. That is, there was already a paradigmatic rejection of faculty psychology by the empiricists and the classical experiment corroborated the philosophical rejection. Furthermore, in empiricism, something is true, "only if it corresponds to an independent, 'objective' reality" (von Glasersfeld, 1984. p. 20). So, the idea that the functioning of one faculty would be transferred to the functioning in another faculty would have to be validated by such functioning in objective reality, which is the crux of the classical experiment.

Behaviorism and Progressive Education

Progressive education. Although faculty psychology was abandoned as a psychological rationale in education, there was a competing paradigm to the scientific movement during the period of time that was known as Progressive Education. Under the leadership of John Dewey, the Progressive Educational Association was formed in 1919 and it served as a counterpoint to the scientific movement. Progressive Education promoted the idea of a child-centered education as well as other aspects of education.[iii] As early as 1902 John Dewey wrote;

> Abandon the notion of subject matter as something fixed and ready-made in itself, outside of the child's experience; cease thinking of the child's experience as also something as hard and fast; see it as something fluent, embryonic, vital; and we realize that the child and the curriculum are simply two limits which define a single process. Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction. (Dewey, 1902, p. 11)

This quotation might be interpreted as Dewey introducing a duality between the child and the subject matter. Dewey's (1902) distinction here is the subject matter as known by the scientist and the subject matter as known by the teacher.

> Every subject thus has two aspects: one for the scientist as a scientist; the other for the teacher as a teacher. These two aspects are in no sense opposed or conflicting. But neither are they immediately identical. (p. 22)

For Dewey (1902), subject matter for scientists represented a given body of truths, whereas for the teacher,

> He is concerned not with the subject matter as such, but with the subject matter as a related factor in a total and growing experience. Thus to see it is to psychologize it. (p. 23)

Two concepts of number. Dewey's emphasis on psychologizing subject matter was quite different than that of the behaviorists. The difference is well illustrated in how Dewey and Thorndike regarded number. For McLellan & Dewey (1895),

> Number is not a property of the objects which can be realized through the mere use of the senses, or impressed upon the mind by so-called external energies or attributes…In the simple recognition, for example, of three things as three the following intellectual operations are involved: *The recognition of the three objects as forming one connected whole or group*—that is, there must be a recognition of the *three* things as individuals, and of the *one*, the unity, the whole, made up of the three things. (p. 24)

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So, Dewey was not an empiricist. Recognition is an indication of assimilation, which, for Piaget (1964), is the essential relation involved in learning. Recognizing the three things as individuals is the result of using an operation of the mind, the unitizing operation (von Glasersfeld, 1981), and recognizing the three things as the one, the unity, is the result of using the operation of uniting the three things into a composite unity. Unitizing sensory material from two or more sensory channels into experiential wholes stands in contrast to the assumption that the world imprints itself on the mind, an assumption on which Thorndike's psychology of number was based.

Thorndike (1922) identified three meanings of numbers—the series, collection, and ratio meanings—and he credited McLellan and Dewey for the ratio meaning. However, he made no attempt to engage in an analysis of the operations of the mind that produce these meanings. Of the collection meaning, he wrote:

> Or we may mean by knowledge of the meaning of numbers, knowledge that two fits a collection of two units, that three fits a collection of three units, and so on, each number being a name for a certain sized collection of discrete things, such as apples, pennies, boys, balls, fingers, and the other customary objects of enumeration in the primary school. (pp. 2-3)

As an empiricist, number was taken as a given in reality and imprinted itself on the mind through the senses. Rather than being concerned with the mathematical experience of the child, for Thorndike (1922), "The psychology of the elementary school subjects is concerned with the connections whereby a child is able to respond to the sight of printed words by thoughts of their meanings…" (p. xi).

Thorndike's influence. The influence that Thorndike had in mathematics education is illustrated in the twenty-ninth yearbook of the National Society for the Study of Education.

> Mainly, the main psychological basis is a behavioristic one, viewing skills and habits as fabrics of connections. This is in contrast, on the one hand, to the older structural psychology [faculty psychology][iv] which has still to make direct contributions to classroom procedure, and on the other hand, to the more recent Gestalt psychology, which, though promising, is not yet ready to function as a basis of elementary education. (Knight, 1930, p. 5)

Knight's attempt to separate the behaviorist approach to elementary education and that of the faculty psychologists was spurious because it is difficult to distinguish faculty psychology's educational model (mental discipline) and Knight's development of a behavioristic educational model. In faculty psychology, it was thought that the best way to strengthen the minds of younger students was through drill and repetition of what we might now call the basic skills in order to cultivate the memory[v], which is quite similar to Knight's interpretation of Thorndike's (1922) Psychology of Arithmetic. Thorndike thought that arithmetical knowledge should be treated as an organized interrelated system, whereas his students, of which Knight was one, focused on the mechanics of arithmetic (Van Engen & Gibb, p. 10). Knight also wrote of avoiding progressive education in the same introduction to the yearbook.

> Some readers may feel that the spirit of this Yearbook is too conservative, that it lacks a bold and daring spirit of progressiveness. There has been a conscious attempt to avoid the urging of any point of view not supported by considerable scientific fact. (Knight, 1930, p. 2)

A contentious relationship. The contentious relationship between progressive educators and educators who held the opinion that the function of the school was to train the working class, be they empiricist or faculty psychologists, appeared prior to the publication of the twenty-ninth yearbook. In 1918 Harold Rugg and John Clarke critically analyzed attempts to reconstruct ninth-grade mathematics and presented their own program in the last chapter of their study. "[T]he construction

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of a continuous mathematical course, worked out around two basic principles, one mathematical and the other psychological" (p. 176) was a major component of their program. They did cite a classic textbook series (Wentworth, Smith, & Brown, 1918) as an attempt to reconstruct ninth grade mathematics, but such texts were regarded as coming up short. In a perusal of the cited text I found that basic algebra was as formal, rule bound, and manipulative as one would expect in a text designed to train students in algebra.

The contentious relationship continued on after Rugg and Clarke's 1918 study, this time directed toward Harold Rugg's social study textbooks. Rugg eventually became one of a small group of progressive educators at Teachers College, Columbia University where he published a social study textbook in 1929 from a social-justice perspective titled, "Man and his changing society," that became widely used. Being a social studies textbook, it was appropriate that there was a focus on social problems in the Unites States and the author encouraged students to explore potential solutions. Rugg was eventually accused of socialism and conservative patriotic business groups who did not want school children raising questions about the capitalistic economic system censured his books.

> By the end of the decade Rugg's books and several others were condemned by the American Legion, the Advertising Federation of America, and the New York State Economic Council. In 1940, in a speech to the leaders of the oil industry, H.W. Prentis, the President of the National Association of Manufacturers (NAM), complained that public schools had been invaded by "creeping collectivism" through social science textbooks that undermined youths' beliefs in private enterprise.[vi]

Progressive education was repudiated and, during the decade of the 1950's, it disintegrated as an identifiable movement in education.[vii] Although the movement may have disintegrated, that doesn't mean that the involved principles died with it.

Mathematics Education 1950-1970: The Era of Modern Mathematics

After World War II, wide spread concern for the state of the education of scientists and engineers emerged when compared with that of the Russians. As a result, the mathematics community became integrally involved in the reeducation of college teachers of mathematics (Price, 1988). The concern soon shifted to the education of precollege mathematics (and science) teachers, especially after the Soviet Union launched Sputnik I in October of 1957. Buttressed by the National Science Foundation, a concerted effort was made by several mathematicians to upgrade the precollege mathematics curriculum in order to educate college capable students (CEEB, 1959; Price, 1988). Classical idealism (the doctrine that reality, or reality as we can know it, is fundamentally mental) replaced empiricism as the dominant philosophical position among the reformers and mathematics textbooks were written from the point of view of a mathematician's mathematics (e.g., Allendoerfer & Oakley, 1959; School Mathematics Study Group, 1965).

However, among the curriculum reformers the belief was, and it still is by most contemporary mathematicians, that mathematics is discovered rather than invented by human beings (Stolzenberg, 1984). So, despite a major shift from empiricism to idealism, Cartesian epistemology was still the prevailing epistemology of the curriculum developers and others primarily involved in the modern mathematics movement, including researchers in mathematics education. Behaviorism was rejected and problem solving along with learning by discovery became the major psychological emphases (Pólya, 1945, 1981) for which Wertheimer's[viii] (1945) work on productive thinking served as a basic psychological rationale. Wertheimer considered productive thinking, or the solving of problems, as based on insight and criticized reproductive thinking such as repetition, conditioning, and habits, all of which are emphasized in behaviorism.

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Teaching Modern Mathematics

Interestingly enough, during the modern mathematics movement of the 1960's, mathematics teachers in the main did not change their traditional, behavioristic ways of teaching mathematics. There were at least three reasons for this state of affairs. First, mathematics teachers were not knowledgeable about what was purported to be the psychological emphases of the modern mathematics programs. Institutes for mathematics teachers were held, but the institutes did not offer courses on problem solving or learning by discovery. The primary emphasis in the institutes was on upgrading the mathematical preparation of mathematics teachers.[ix,x] Second, the modern curricula emphasized mathematical structure and the logical, deductive presentation of ideas rather than problem solving and learning by discovery.[xi] There were minimal attempts to psychologize the subject matter in these ways, which was a major oversight because of the influence textbooks have on the classroom teaching of mathematics. Finally, behaviorism is a common sense psychology. Although I would say that few mathematics teachers, including myself, had a working knowledge of Thorndike's psychology of arithmetic or algebra, or of behaviorism more generally, being held accountable for four or five classes of 25-35 students per class can easily lead a teacher to using common sense psychology in teaching without being reflectively aware of doing so. What I mean by a common sense psychology is amply demonstrated in the following citation from an introduction to Thorndike's psychology of algebra.

> Suffice it to say here that it emphasizes the dynamic aspect of the mind as a system of connections between situations and responses; treats learning as the formation of such connections or bonds or elementary habits; and finds that thought and reasoning—the so-called higher powers—are not forces opposing those habits but are those habits organized to work together and selectively. (Thorndike, Cobb, Orleans, Symonds, Wald, & Woodyard, 1926, p. v)

Piaget's Genetic Structures as a Psychological Rationale

It is very interesting that Piaget's genetic structures and stage theory of cognitive development served as a psychological rationale for the modern mathematics programs at the elementary school level (Bruner, 1960). This was primarily due to the logical-mathematical structural emphasis in the modern mathematics programs that left the programs without a psychological rationale. Piaget's *constructivism* did not serve as an epistemological basis for the modern mathematics programs nor was it even emphasized in a conference devoted to Piaget's work and the modern programs that was held at Cornell University in 1964 (cf. Ripple & Rockcastle, 1964). Instead, the interest was in Piaget's stage theory and his formalizations of the thinking of children within the stages as can be seen by Bruner's (1960) citation of Bärbel Inhelder, Piaget's close collaborator, in *The Process of Education*:

> Basic notions in these fields are perfectly accessible to children of seven to ten years of age, provided that they are divorced from their mathematical expressions and studied through material the child can handle himself. (p. 43)

Inhelder's idea was that children in the *concrete operational stage*[xii] were ready to learn, and indeed could learn, "basic notions in these fields". This idea served as the basis of Bruner's (1960) famous concept of the readiness to learn the basic structures of mathematics:

> Any subject can be taught effectively in some intellectually honest form to any child at any stage of development. (p. 33)

Bruner (1960), however, conflated basic structures of mathematics and Piaget's genetic structures when he referred to "less able students":

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> Good teaching that emphasizes the structure of the subject is probably even more valuable for the less able students than for the gifted ones. (p. 9)

By "less able students," I take Bruner as referencing children in Piaget's preoperational stage, children who's thinking was not explained by Piaget's Grouping structures. In this quotation, he seemed caught in Cartesian anxiety.

> [Cartesian anxiety] is an anxiety that permeates all metaphysical and epistemological questions concerning the existence of a stable and reliable rock upon which we secure our thoughts and actions. As Bernstein explains: "Either there is some support for our being, a fixed foundation for our knowledge, or we cannot escape the forces of darkness that envelope us with madness, with intellectual and moral chaos (p. 18)." (Konold & Johnson, 1991, p. 2)

In spite of using Piaget's psychology as a rationale for the emphasis on mathematical structure, Piaget was considered to be an observer rather than a teacher, and the elasticity of the limits of children's minds was not considered as having been established:

> These reformers (and I speak now not only of SMSG) have been so successful in teaching relatively complex ideas to young children, and thus doing considerable violence to some old notions of readiness, that they have become highly optimistic about what mathematics can and should be taught in the early grades. (Kilpatrick, 1964, p. 129)

I had no problem with Kilpatrick's assertion for children who were in Piaget and Inhelder's more advanced concrete operational stage.[xiii] But I did not accept Bruner's famous hypothesis about the readiness to learn for the "less able" children nor did I accept Kilpatrick's assertion for children in Piaget's preoperational stage. Consequently, the way in which Piaget's grouping structures might be relevant in the mathematics education of children became a major problem for me soon after I earned my Ph.D. from the University of Wisconsin in 1966. At that point, research in mathematics education was still based in empiricism and to work scientifically meant to use experimental and statistical methods (Stanley & Campbell, 1963) in the test of hypotheses in a way that was quite similar to Thorndike and Woodworth's classical experiment.

Applying Piaget's Psychology

After joining the Department of Mathematics Education in 1967, I turned to working for a period of approximately eight years in an attempt to reject Bruner's famous hypothesis concerning the readiness to learn mathematics for children who were in Piaget's pre-operational stage. In this effort, I functioned as an experimental researcher with little awareness that Piaget (1980) rejected empiricism.

> Fifty years of experience has taught us that knowledge does not result from a mere recording of observations without a structuring activity on the part of the subject. (Piaget, 1980, p. 23)

My efforts were directed toward *applying* Piaget's psychology in the mathematics education of preoperational children in a "scientific" manner. Although I experimentally rejected Bruner's readiness hypothesis for these children (e.g., Steffe, 1966, 73), the children rather forcefully taught me that I had no insights into the psychology of their mathematical thinking (Steffe, 2012). I considered myself as doing pseudo-science and making only accretional progress if I was making any progress at all. The relationships with the mathematics students that I taught as a mathematics teacher was missing. That is, my contributions to the mathematical thinking and reasoning of the children who were my "subjects" in the experiments was not being realized.

So, rather than rely on Piaget's Grouping structures as a psychology of the child, I returned to my identity as a mathematics teacher and taught two classes of first-grade children over the course of a

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school year so the children could teach me how they think when engaging in mathematical activity (Steffe, Hirstein, & Spikes, 1976).[xiv] The involved children taught me that counting was their primary and spontaneous way of operating in discrete quantitative situations and that counting could have quite different meanings for different children. Piaget had not explained children's counting, so this finding corroborated abandoning attempts to *apply* Piaget's *psychology* in children's mathematical education. It also led to throwing off the straight jacket that controlled experimentation and statistical methodology had on my conception of doing science in mathematics education. In fact, it led to developing the teaching experiment as a method of doing research and using teaching as a method of scientific investigation (Cobb & Steffe, 1983; Steffe, 1983; Steffe & Thompson, 2000b; Steffe & Ulrich, 2013).

The shift to using teaching as a method of scientific investigation was a major shift in doing research and, to my knowledge, at the time it was unprecedented in the United States. I learned later that researchers in the Academy of Pedagogical Sciences in the USSR had already used versions of teaching experiment in their work (Kilpatrick & Wirszup, 1975–1978). Not only did their work provide academic respectability for what then was a major departure in the practice of research in mathematics education in the United States, it was also a departure in the goals of the research. In El'konin's (1967) assessment of Vygotsky's (1978) research, the essential function of a teaching experiment is the production of models of student thinking and changes in it.

> Unfortunately, it is still rare to meet with the interpretation of Vygotsky's research as modeling, rather than empirically studying, developmental processes. (El'konin 1967, p. 36)

So, the new problem that faced me was to construct explanations of the mental processes that are involved in children's counting and, further, to construct explanations of how children might construct those mental processes. I had constructed a typology of the units children create in counting that they taught me. However, I could not explain the processes that are involved in children's construction of these unit types other than Piaget's account of children's construction of what he called arithmetical units (Piaget & Szeminska, 1952). That is*, I realized that it was I who had to construct a psychology of the mathematical children that I taught rather than attempt to apply a psychology that had been constructed for a different purp*ose. That was a major breakthrough in my conception of what it meant to do research in mathematics education.

Mathematics Education 1970-2000: The New Progressive Educators

Interdisciplinary Research on Number

The modern mathematics era ended circa 1970 and behaviorism came roaring back into mathematics education. When von Glasersfeld and I started to work on the project, Interdisciplinary Research on Number (IRON), he had just published his manifesto on radical constructivism (von Glasersfeld, 1974) and it was his intention to start an epistemological revolution that would eliminate the duality between mind and reality in Cartesian epistemology. It was also his intention [and mine] to countermand the stranglehold that behaviorism once again had on mathematics education throughout 1970's and 1980's. Radical constructivism emerged as an epistemology in mathematics as well as in science education (e.g., Driver, 1995) throughout the 1980's and played a role similar to that of progressive education during the first one half of the century. But the role was essentially based on von Glasersfeld's (1989) first principle that, "knowledge is not passively received but actively built up by the cognizing subject" (p. 182) rather than on the research that we were doing in IRON. In fact, I frequently was told that joining radical constructivism was like joining a political party. Few progressive educators appreciated the implications von Glasersfeld's (1989) second principle that, "the function of cognition is adaptive, and serves in the organization of the experiential world, not the discovery of ontological reality" (p. 182), which was the "radical" part of radical constructivism that eliminated the Cartesian dualism between mind and reality (von Glasersfeld, 1974, 1984).

The Standards Movement and the "Math Wars"

Mathematics education was a conceptual wasteland during the 1970's, so it was no surprise that another crisis in education emerged that was marked by the publication of *A Nation at Risk* (National Commission on Excellence in Education, 1983). Influenced by this newly perceived crisis, the constructivist revolution, and the recommendation that problem solving be the focus of school mathematics in the 1980's (National Council of Teachers of Mathematics, 1980), the standards movement in mathematics education officially began in 1989 with the publication of the *Curriculum and Evaluation Standards for School Mathematics* (CESSM; National Council of Teachers of Mathematics, 1989). The influence of Cartesian epistemology was still strong among the progressive educators, so CESSM was a strange mixture of realism and constructivism in spite of the commission claiming a constructivist view of learning, where learning was thought to, "occur through active as well as passive involvement with mathematics" (CESSM, p. 9).

The National Science Foundation funded ten curriculum projects based on the CESSM that were published circa 2000, curricula that unfortunately became known as "constructivist curricula." The publication of these curricula extended the famous "math wars" between conservative mathematicians and progressive mathematics educators that erupted in California (cf. Klien, http://www.csun.edu/~vcmth00m/). The "math wars" had their origin in the 1985 *California Mathematics Framework* (California State Department of Education, 1985). This framework,

> [W]as considered a progressive document—an antecedent of the 1989 NCTM *Standards.* California's professional teacher organization, the California Mathematics Council, was one of the most progressive teacher organizations in the country, and one of the most enthusiastic adopters of the spirit of the 1989 *Standards.* When the next adoption cycle came, the 1992 *California Mathematics Framework* (California State Department of Education, 1992) "pushed the envelope" a good deal further: it emphasized reform, focusing on "mathematical power" and collaborative and independent student work while de-emphasizing traditional skills and algorithms. (Schoenfeld, 2007)

The attempts of the constructivist curricula writers to focus on student work were realized in part through their social agenda, "Mathematics for All," and concomitantly, how they regarded mathematics learning and teaching. In this agenda, it was assumed that all students could learn the mathematics specified in the content standards of CESSM.

> If all students do not have an opportunity to learn this mathematics, we face the danger of creating an intellectual elite and a polarized society. The image of a society in which a few have the mathematical knowledge needed for the control of economic and scientific developments is not consistent either with the values of a just democratic system or with its economic needs. (CESSM, 1989, p. 9)

The social agenda of the writers of the so-called constructivist curricula was based on social constructivism (Bauersfeld, 1995, 1996; Cobb & Yackel, 1996, Voight, 1989). The orientation that shaped the social agenda and the recommendations for teaching is cogently caught in a comment made by Bauersfeld (1995) that, "We can understand the development of mathematizing in the classroom 'as the interactive constitution of a social practice'" (p. 150). This sociological emphasis is compatible with von Glasersfeld's (1989) first principle of radical constructivism if "interactively" is included in "actively." It doesn't, however, take into account von Glasersfeld's (1989) second principle. The reason is that, although interaction is a fundamental assumption in radical constructivism, there are two types of interaction: within subject and between subject interaction (Steffe & Thompson, 2000a). The social constructivists emphasize between subject interaction and make few attempts to model what might go on inside of the heads of children, which is where learning and development take place.

The social agenda served to exacerbate the dissatisfaction the mathematical critics had with the "constructivist" curricula.

> [T]here is a unifying ideology behind "whole math." It is advertised as math for all students, as opposed to only white males. But the word all is a code for minority students and women (though presumably not Asians). In 1996, while he was president of NCTM, Jack Price articulated this view in direct terms on a radio show in San Diego: "What we have now is nostalgia math. It is the mathematics that we have always had, that is good for the most part for the relatively-high socioeconomic anglo male, and that we have a great deal of research that has been done showing that women, for example, and minority groups do not learn the same way. They have the capability, certainly, of learning, but they don't. The teaching strategies that you use with them are different from those that we have been able to use in the past when ... we weren't expected to graduate a lot of people, and most of those who did graduate and go on to college were the anglo males." (Klein, 2000)

Klein went on to say that; "I reject the notion that skin color or gender determines whether students learn inductively as opposed to deductively and whether they should be taught the standard operations of arithmetic and essential components of algebra" (Klein, 2000). So, not only did Klein critique the standards in CESSM and the mathematics that was involved in the "constructivist" curricula, he was also a critic of how teaching was conceptualized and practiced. Essentially, the "math wars" were reminiscent of the contentious relationship between conservative patriotic business groups and progressive educators concerning Rugg's social science textbooks.

Mathematics Education 2000 and Forward: Outcome-Based Education

Klein's rejection of the standards and the social agenda of the constructivist curricula writers foreshadowed the mission of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association for Best Practices and Council of Chief State School Officers, 2010). The release of the CCSSM helped thaw the "math wars" (Lobato, 2014; Norton, 2014) primarily, in my view, because of the presence of more rigorous curriculum standards. We find the following statement in the introduction to CCSSM.

> The standards are designed to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy. The Common Core State Standards provide a consistent, clear understanding of what students are expected to learn, so teachers and parents know what they need to do to help them (CCSSM, 2010, Introduction).

The CCSSM, similar to the CEEB in 1969, was designed primarily for college bound students.[xv] It has carried the emphasis on outcome-based education forward to the present time, whose beginning was marked in mathematics education by the publication of the CESSM in 1989. It might seem surprising that I would say that CESSM ushered in outcome-based education given that it also was an impetus of the constructivist curricula that was so severely criticized in the "math wars". However, one of the main criticisms of the "constructivist" curricula and CESSM by the mathematicians was that the involved standards were weak, not that there were not any standards.

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Outcome-based education is based on Cartesian epistemology with its requirement that something is true only if it corresponds to an independent, objective reality, where the standards constitute that objective reality. The neo-behaviorism of outcome-based education along with the national emphasis on standards-based education by the No Child Left Behind Act of 2001 has had the effect of standardizing precollege mathematics education. For example, students are required to take standardized test throughout their years in school[xvi] and these tests are used in evaluating teachers, a practice that has become known as Value Added Measures [VAM's] of teacher performance. This surge of neo-behaviorism in mathematics education during the first years of the 21st century is exemplified in the report of the National Mathematics Advisory Panel (2008) with its emphasis on *rigorous scientific research*. The research conducted in IRON concerning children's number sequences and fraction schemes and how they are used in the construction of adding, subtracting, multiplying, and dividing schemes that has been published in books and various articles (e.g., Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe & Cobb, 1988; Steffe, 1992; Steffe & Olive, 2010) was not even mentioned in that report. So, obviously, the authors of the report did not consider that research as scientific research if they considered it at all.

Given the ubiquity of the influence of outcome-based education, one might think that there should be another major effort by progressive educators to countermand that influence similar to the era of the modern mathematics programs or to the era of the constructivist curricula. While that may be of critical importance given the current state of mathematics education in precollege education, essentially the attempted wholesale changes in mathematics education that were made following national reports were abandoned after the changes led to major disappointments and failures. If this history can be used to predict what might happen if another round of national reform in mathematics education is attempted, a strong argument can be made that what is needed is to construct mathematics curricula *for* children that is based on the mathematics *of* children rather than continue on with the historical practice of basing mathematics curricula for children on the first-order mathematical knowledge of adults. Simply put, if lasting progress in mathematics education is to be made, researchers must establish the construction of mathematics curricula for children as an academic field. I think of constructing mathematics curricula for children that is based on the mathematics of children as a result of intensive and longish periods of teacher/researcher interactions with children. Toward that end, I present several radical constructivist research programs that are tailored toward constructing mathematics curricula for children that emerge from the work in IRON. Before presenting the programs, I present several basic concepts that I feel will help understand the research programs.

Radical Constructivist Research Programs

Basis Concepts

First- and second-order models. I understand *children's mathematics* as a result of maturation coupled with what children have constructed as a result of interacting in their social-cultural milieu in all of its aspects. The assumption that children construct mathematical knowledge is an assumption of an observer.[xvii] Children's mathematics is thought of as first-order knowledge, which are, "the hypothetical models that the observed subject constructs to order, comprehend, and control his or her own experience (Steffe, et al., 1983, p. xvi). An observer psychologizes children's mathematics by constructing second-order models, which are, "the hypothetical models observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et. al. 1983, p. xvi). The second-order models are referred to as the *mathematics of children* and the children's first-order models are referred to as *children's mathematics.*[xviii] The concept of children's mathematics is based on the belief that mathematics is a product of the functioning of human intelligence (Piaget, 1980). The mathematics of children, which is an explanation of children's mathematics, is a legitimate mathematics to the extent that teachers/researchers can find rational grounds to explain what children say and do.

Epistemological analysis and conceptual analysis. Conceptual analysis is the method by which the second-order models that constitute the mathematics of children are produced. Conceptual analysis is an analysis of mental operations. In explaining conceptual analysis, von Glasersfeld (1995) drew from his experience with Silvio Ceccato's Italian Operational School, whose goal was to, "reduce all linguistic meaning, not to other words, but to 'mental operations'" (p. 6). The main goal of conceptual analysis is defined by a question from Ceccato's group: "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (p. 78). Thompson & Saldanha (2000) reformulated the goal in a way that is more relevant to constructing second-order models of children's language and actions. Their goal is to describe, "conceptual operations that, were people to have them, might result in them thinking the way they evidently do" (p. 315). Although I have extensively engaged in conceptual analysis in the construction of the mathematics of children, I know of no papers that have been written that address the problem of how one might creatively use the analytical tools that are available in radical constructivism in conceptual analysis of children's mathematical concepts and operations.

When conceptual analysis is used in the construction of second-order models, I refer to it as a second-order conceptual analysis. Thompson & Saldanha (2000) included what I refer to as first-order conceptual analysis in their discussion of epistemological analysis, that is, an analysis of one's own mathematical concepts and operations (cf. Thompson, 2008). According to Thompson & Saldanha (2000), epistemological analysis, "is used to model what might be called systems of ideas, like systems of ideas composing concepts of numeration systems, functions and rate of change, or even larger systems like those expressed in quantitative reasoning" (p. 316). First-order conceptual analysis is inextricably involved in second-order conceptual analysis of children's mathematical language and actions. Thompson & Saldanha (2000) also included a teacher/researcher analyzing their own concepts and operations relative to children's concepts and operations in interactive mathematical communication. This kind of analysis involves the teacher/researcher operating as Maturana's (1978) second-order observer; that is, an "observer's ability through second-order consensuality to operate as external to the situation in which he or she is, and thus be observer of his or hers circumstance as an observer (p. 61).

In the following quotation, if "intentionally isomorphic" is interpreted as imputing operations to a mathematically operating child, what I said about making explanations is similar to Maturana's second part of the scientific method.

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed to be intentionally isomorphic to (models of) the systems that generate the observed phenomena. In fact, an explanation is always an intended reproduction or reformulation of a system or phenomenon. (Maturana, 1978. p. 30).

Maturana's second part of the scientific method emphasized second-order conceptual analysis and his first part emphasized first-order conceptual analysis, which was, "observation of a phenomenon that, henceforth, is taken as a problem to be explained" (Maturana, 1978, p. 29). Of the observer, he commented,

> Yet we are seldom aware that an observation is the realization of a series of operations that entail an observer as a system with properties that allow him or her to perform these operations, and, hence, that the properties of the observer, by specifying the operations that he or she can perform determine the observer's domain of possible observations. (Maturana, 1978, p. 30)

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Like Maturana, I take the subject dependent nature of science in mathematics education as a starting point. But I expand on it in two ways. First, the primary reason for engaging children as a teacher/researcher is to allow children to teach one how and in what ways they operate mathematically and, as commented by Thompson & Saldanha, to *create* operations that if a child had those operations, the child would operate as observed. Second, as a teacher/researcher kind of scientist, my *contributions* to children's ways and means of operating mathematically by teaching them is a constitutive part of a conceptual analysis of children's mathematical language and actions. In the words of Steier (1995);

> Approaches to inquiry … have centered on the idea of worlds being constructed … by inquirers who are simultaneously participants in those same worlds. (p. 70)

This understanding of the subject dependent nature of science in mathematics education provides researchers with the power to create images of unrealized possibilities in the mathematics education of children. But these possibilities are subject to the constraints of children as self-organizing systems—the mind organizes the world by organizing itself (Piaget, 1935/71).

Learning and development. A central goal that runs throughout each research program is to learn how to operationalize children's mathematics learning and development as spontaneous processes in mathematics teaching. A virtue of teaching that is focused on constructive itineraries of children's mathematics in which the teacher/researcher is a participant is that it allows the teacher/researcher to become aware of children's constructive processes, which are understood as the construction of schemes and the accommodations that children make in them (cf. von Glasersfeld, 1980). Because of continual interaction with children, a teacher/researcher is likely to observe at least the results of those critical moments when restructuring is indicated by changes in children's operations and anticipation (Tzur, 2014). Major restructuring of mathematical schemes is compatible with a vital part of Vygotsky's (1978) emphasis on studying the influence of learning on development.

Unlike Vygotsky, however, I regard both learning and development in the context of accommodations that children make in their schemes (Steffe, 1991b). But there is a difference in the two kinds of accommodations. Learning is captured by the functional accommodations that occur in a scheme in the context of the scheme being used, whereas development is captured by metamorphic accommodations that occur independently in no particular application of a scheme. A metamorphosis of a scheme is thought to be the result of autoregulation of the process of interiorizing the scheme (cf. Simon, Saldanha, McClintock, Akar, Watanabe, & Zembat, 2010, for a related view).

Learning and development are not spontaneous in the sense that the provocations that occasion them might be intentional on the part of the teacher/researcher. In children's frames of reference, though, the processes involved are essentially outside of their awareness. This is indicated by the observation that what children learn or develop often is not what was intended by the teacher/researcher. It also is indicated when a child learns or develops when a teacher/researcher has no such intention. Even in those cases where children learn what a teacher/researcher might intend, the event that constitutes learning arises not because of the teacher's actions. Rather, teaching actions only occasion children's learning (Kieren, 1994). Learning as well as development arises as an independent contribution of the interacting children. So, although I do not use "spontaneous" in the context of learning and development to indicate the absence of elements with which children interact, I do use the term to refer to the non-causality of teaching actions, to the self-regulation of the children when interacting, to a lack of awareness of the learning process, and to its unpredictability. Because of these factors, I regard learning and development as spontaneous processes in children's frame of reference.

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Trajectories of the constructive activity of children. The construction of trajectories of children's learning and development is one of the most daunting but urgent problems facing mathematics education today. It is also one of the most exciting problems because it is here that we can construct an understanding of how teacher/researchers can profitably affect children's mathematics (Steffe, 2004). By building an understanding of children's mathematical concepts and operations and how a teacher/researcher can engage children to bring forth changes in those concepts and operations, a vision of children's mathematics education can emerge in which children engage in productive mathematical learning and development and teacher/researchers engage in productive mathematical teaching. The principle of self-reflexivity[xix] compels teacher/researchers to consider their own knowledge of children's mathematics, including accommodations in it, as constantly being constructed as they interact with children as the children construct mathematical knowledge. Through the construction of trajectories of children's learning and development that are coproduced by children and teacher/researchers, it is possible to construct trajectories that include an account of teacher/researchers' ways and means of acting and operating relative to children's ways and means of acting and operating (Ellis, 2014). Such an account entails the teacher/researcher operating as a second-order observer.

A trajectory of children's learning and development includes a model of the children's initial concepts and operations, an account of children's constraints and necessary errors, an account of the observable changes in children's concepts and operations as a result of their interactive mathematical activity in situations that are used by a teacher/researcher when interacting with children, an account of the situations relative to a teacher/researcher's models of the involved children's mathematics and the teacher/researcher's goals and intentions, and an account of the involved mathematical interactions. A similar historical account of what transpires in between observed changes is critical not only to understand the changes, but also to provide estimates of the length and the nature of the plateaus in children's mathematical learning and/or development.

Trajectories of the constructive activity of children are third-order models that include the second-order models that constitute the mathematics of children, the first-order models of the teacher/researcher, and relationships between them. In the following research programs that I present, I assume that the models that constitute the mathematics of children produced by IRON will be used at least as starting places in the construction of the trajectories. Because of the nature of the trajectories, I will refer to them as mathematics curricula for children throughout the rest of the paper (Steffe, 2007). Concentrating on constructing mathematics curricula for children does not exclude research programs that center on teacher/researchers working with classroom teachers of children.[xx,xxi] In fact, each stated research program can be reformulated so that it is a research program that involves teacher/researchers working with classroom teachers of children.

The First Research Program

The first research program is to construct mathematics curricula for children who enter their first grade as counters of perceptual unit items over the course of their first eight years in school.[xxii]

The second-order models that were constructed in IRON concerning children's number sequences and how the number sequences are used in the construction of adding, subtracting, multiplying, and dividing schemes have been published in books and various articles (e.g., Steffe, et. al., 1983; Steffe & Cobb, 1988; Steffe, 1992; Steffe & Olive, 2010). Ulrich (2015-16) has published two very readable papers that provide an introduction to the units, schemes, and operations that were constructed in IRON as well as to some of the work that has extended the basic work (e.g., Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009; Hunt, Tzur, &

Westenskow, 2016; Norton & Wilkins, 2013; Tillema, 2013; Ulrich, 2012). To start, I provide a brief summary of the first two stages in the construction of children's number sequences.

The first stage is a sensory-motor or pre-numerical stage that comprises pre-counters, counters of perceptual unit items (CPUI), and counters of figurative unit items (CFUI). Counters of perceptual unit items are restricted to counting items that are in their perceptual field, such as the toys in their toy box, their steps, their heartbeats, or the chimes of a Grandfather clock. For example, an interviewer covered six of nine marbles with his hand and asked Brenda, a six-year-old child, to count all the marbles. Brenda first counted the interviewer's five fingers and then counted the three visible marbles. The interviewer pointed out that he had six marbles beneath his hand and Brenda replied, "I don't see no six!" (Steffe, & Cobb, 1988, p. 23)

Counters of figurative unit items might attempt to count the items in a closed container when told that there are, say, seven items in the container, by touching the container where they believe items might be hidden in synchrony with uttering number words. Because they concentrate on generating images of the items they are counting, they can easily become lost in counting and stop fortuitously. Counting figurative unit items is a step in interiorizing the countable items, which produces abstract unit items (Steffe et.al, 1983). If the child also interiorizes the acts of counting, I mark this monumental event by referring to it as the stage of the initial number sequence (INS). *Spontaneously* counting-on is the indication of the INS (Steffe, & Cobb 1988).

To illustrate some of the constraints that I experienced when teaching CPUI, I recount my experience teaching three such children at the start of their first grade in school. I taught them approximately 60 times in teaching episodes over their first two school years to explore their progress in the construction of counting-on (Steffe, & Cobb, 1988). Although these children also participated in their regular mathematics classrooms, they did not spontaneously count-on in spite of my best efforts to provoke it and, presumably, the best efforts of their teachers. It wasn't until their 3rd Grade that at least one of them had constructed counting-on. Based on my experience in working in teaching experiments and teacher education at UGA and data that were supplied to me by Professor Bob Wright of Southern Cross University, Australia, who started the Mathematical Recovery Program (Wright, Martland, & Stafford, 2000; Wright, Stewart, Stafford, & Cain, 1998), I estimate that 40% of entering first graders in the United States are CPUI. Of this estimate, Professor Wright commented that, "I think that is a good estimate for the number in the perceptual stage or lower, that is the children who can't yet count perceptual items. I think the percentage would be lower in Australia and New Zealand, say about 30%" (Personal Communication).

Of the 40% who enter the 1st Grade as CPUI, I expect that a majority of them to construct counting-on during their 3rd Grade [Wright estimated that from 5 to 8% might not be counting-on by the 3rd Grade]. From that point on the relative percentages are not certain, but because of the length of time and the great difficulties we had in teaching experiments in engendering progress beyond counting on (Biddlecomb, 2002, Hackenberg, 2005; Tillema, 2007), my best estimate is that approximately 30% of the children entering the 6th Grade will be only able to count-on. And those who are at that stage will remain there until their 8th Grade. Wright's estimate was, "that about 30% of kids entering the 6th Grade in the US will only be able to count-on" (Personal Communication).

I consider this program as the most important research program in mathematics education today. My appeal to those who choose to work in such an intractable but crucial research program is to learn how to teach these children in such a way that they do not lose confidence. My practitioner's maxim is that children are never wrong; even children who are CPUI. An adult can easily induce "mistakes" in these children, but my basic and pervasive assumption is that children are rational beings and our responsibility is to find ways of acting and interacting that are not only harmonious with their ways and means of operating, but will also affect them in productive ways. It is crucial to re-establish the NCTM's vision of mathematics for all.

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The Second Research Program

> *The second research program is to construct quantitative mathematics curricula for children who enter their first grade as CFUI or children who can only count-on (1) in the construction of operative measuring schemes, and (2) in the construction of adding and subtracting schemes as reorganizations of their operative measuring schemes during their first two grades in school.*

Children who enter their first grade as CFUI have a quite different constructive trajectory than those who enter as CPUI. It is possible for CFUI to construct the INS by means of a metamorphic accommodation by the end of their first grade in school (Steffe & Cobb, 1988, pp. 308ff). By the end of the second grade, it is possible for their INS to undergo another metamorphic accommodation in the construction of the explicitly nested number sequence (ENS), which is indicated when children spontaneously count-up-to (Steffe, 1992, 94: Steffe & Cobb, 1988).

There are three principal *operations* of the ENS that were not available to children who have constructed only the INS. The first is that units of one have been constructed as *iterable* units; for example, at noon a grandfather clock strikes one twelve times in contrast to simply making 12 chimes. The second is that any initial segment of a (finite) number sequence can be disembedded—"lifted"— from the complete sequence without destroying the sequence.[xxiii] The remainder of the initial segment in the sequence can be also disembedded from the sequence and the numerosity of the remainder can be found by counting its elements starting with "one." This way of counting is referred to as *the recursive property* of the ENS in that children can take the number sequence as its own input (Steffe & Cobb, 1988). That is, children who have constructed the ENS can willfully create their own countable items using elements of their number sequence and count these elements using the same number sequence that was used to create the countable items. It is as if the child has two number sequences "side by side," one to use to create countable items and the other to count the countable items. ENS children have more "mathematical power" than do INS children, to borrow a phrase from the California Mathematics Framework. So, there are three distinct stages in children's construction of their number sequences entering their first grade in school; CPUI, CFUI and the INS, and the ENS. There is a more advanced number sequence that only rarely can be observed that is referred to as the generalized number sequence (GNS; Ulrich, 2014, 2016)

My best estimate is that children who enter their first grade as CFUI or who can only count-on comprise 45% of the first-grade population. Table 1 contains my best estimates of the percent of children who enter their first grade in each of the three number sequence types. The question of whether stage shifts can be engendered by means of specialized interactions has been

Table 1: Number Sequence Type Across Grades for Children Who Enter their First Grade Counting-on (INS) or as CFUI.

Grade/N Seq.	CFUI or INS	ENS	GNS
First	≈ 45 Percent	≈ 10 to 15 Percent	≈ 0 to 5 Percent
Second	≈ 30 Percent	≈ 25 to 30 Percent	≈ 0 to 5 Percent
Third	≈ 5 Percent	≈ 45 to 50 Percent	≈ 0 to 10 Percent

worked on by Norton and Boyce with an *eleven-year old* child (2015). These authors did demonstrate that by working intensively with the child individually in 14 teaching sessions, he did make progress in reasoning from one level of units (INS) to two levels of units (ENS). The authors note, however, that,

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> Cody did not seem able to coordinate units in continuous contexts in the same way he could in discrete contexts…We conjecture that that limitation is due to the lack of physical referents for the embedded units within composite units that are continuous. For example, a tablespoon contains three teaspoons, but these three units are not as evident within the tablespoon as they would be with three chips within a cup. Rather, students have to produce the units within a continuous composite unit through some kind of segmenting or partitioning activity (Steffe, 1991a), which involves breaking down the composite unit. (Norton & Boyce, 2015, p. 229)

Children who have constructed the ENS and, hence, two levels of units, do use their number concepts spontaneously in partitioning continuous units. So, there is always an issue of the generality of the learning process when the situations used in the teaching experiment are with only one type of quantity. According to some authors, a fundamental question that pervades mathematics education today is whether mathematical thinking begins with counting or with comparisons of quantity (Sophian, 2007). Based on the work of Davydov (1975) and influenced by Doughtery (2004), Sophian (2007) commented that, "The most fundamental idea I have derived from those papers is the idea that mathematical thinking begins, not with counting, but with comparisons between quantities, in particular the identification of equality and inequality relationships" (p. xiv). This notion of quantity is based on Davydov's (1975) formal definition that a quantity is any set for the elements of which criteria of comparison have been established. However, establishing the quantitative property of a composite unit called its numerosity and the quantitative property of a continuous item called its length precedes a need for comparing the numerosity of two collections or the length of two continuous items (Steffe, 1991a). So, it's not a matter that mathematics begins with comparisons between quantities be they discrete or continuous. Rather, one might say that mathematics begins with establishing the quantitative properties of objects (Steffe, 1991a). This fits with Thompson's (1994) notion of a quantity as, "composed of an object, a quality of that object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 184). This idea of quantity, both discrete and continuous, leads to the following reorganization hypothesis.

> *Reorganization Hypothesis*: Operative measuring schemes and their use in constructing adding and subtracting schemes can emerge as reorganizations of children's INS.[xxiv]

In this hypothesis, the main goal is for children to use their INS in measuring activity in order to transform the measuring activity, such as described in CCSSM standard 1.MD.2 stated below, into operative measuring schemes and to what Thompson, Carlson, Byerley, & Hatfield, (2014) referred to as additive measurement.

> Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.

It is important to note that this CCSSM standard is written in such a way that emphasizes the activity of measuring. After actually measuring linear objects to establish how to measure and the units used in measuring, INS children can engage in *operational* measuring activity such as finding the length of a 64-inch string after it is increased by seven inches. If operational measuring is generalized across other quantities such as time, money, temperature, weight, etc., children can construct operational measuring schemes that they could use as if they were using the INS in discrete quantitative situations. They could also be asked to find, say, how many tablespoons of powder could be made from nine teaspoons of powder to engender the construction of composite units—or units of units—which, at this point, I consider as essential in engendering a metamorphosis of the "INS measuring schemes." Furthermore, in the case of discrete quantity, *children construct adding and subtracting schemes as reorganizations of their number sequences* (Steffe, 2003). So, by the children using their INS in the construction of operative measuring schemes, they can in turn use their measuring schemes in the construction of operative adding and subtracting schemes across different quantitative contexts. My hypothesis is that if a stage shift is observed from an INS to an ENS measuring scheme in the case of one type of quantity, a corresponding stage shift will be observed in all of the measuring schemes that the INS was used in establishing. Such a constructive generalization would lead to considerable mathematical power of the children, to borrow a phrase from the California standards.

For children who are CFUI, engaging in measuring activity that includes counting activity extends the goals, situations, activities, and results of their figurative counting schemes. Similar to the INS children who use their counting schemes in measuring activity, the effects of the CFUI using their figurative counting schemes in measuring activity is yet to be determined. Still, it is possible that their measuring activity could serve in engendering metamorphic accommodations like that which produces the INS (cf. Steffe, & Cobb, 1988, pp. 306 ff) if for no other reason than a teacher/researcher could capitalize on children's need to measure things in such a way that provokes monitoring re-presentations of measuring activity.

The Third Research Program

> *The third research program is to construct quantitative mathematics curricula for ENS*[xxv] *children in the construction of extensive quantitative measuring schemes and their use in constructing adding, subtracting, multiplying, dividing, and numeration schemes in which strategic reasoning and relationships between quantities are of primary importance.*

I agree with Smith & Thompson (2007) that an emphasis on quantitative reasoning needs to begin early on in children's mathematics education and that building quantitative reasoning skills for the majority of students is not a one or two-year program. Their paper concerned how a shift in current school curricula could emphasize quantitative reasoning, whereas my emphasis is on constructing a quantitative mathematics for children based on abstractions from actually teaching children to establish learning trajectories in the sense that Ellis (2014) explained. In this context, it is critical to understand what schemes can be considered as extensive quantitative schemes, which I refer to as genuine measuring schemes. Rather than think of extensive quantities as substances as would be the case when considering 5/4 as referring to a point on the number line, von Glasersfeld & Richards (1983) pointed out that Gauss focused on extensive quantities as relations.

> To forestall the idea that the extensive quantities he is referring to are a matter of inches or degrees, Gauss hastens to add that mathematics does not deal with quantities as such, but rather with relations between quantities. These relations he calls "arithmetical" and in arithmetic, he explains, quantities are always defined by how many times a known quantity (the unit), or an aliquot part of it, must be repeated in order to obtain a quantity equal to the one that is to be defined, and that is to say, one expresses it by means of a number" (pp. 58-59).

The ENS is the first numerical counting scheme that qualifies as an extensive quantitative scheme in that any number such as 50 can be conceived of as one fifty times as well as 50 ones. The other operations of the ENS are also critical in constituting this scheme as an extensive discrete quantitative measuring scheme. So, by viewing the construction of measuring schemes more generally as reorganizations of the operations that produce the ENS, the hypothesis is that the measuring schemes will emerge as extensive quantitative schemes.

The Fourth Research Program

> *The fourth research program is to construct quantitative mathematics curricula for children in (1) the construction of quantitative measuring schemes as reorganizations of their fraction schemes,*[xxvi] *and (2) the construction of multiplicative and additive measuring schemes as reorganizations of their fraction schemes.*

A reorganization hypothesis that was fundamental in the work of IRON that centered on children's construction of fraction schemes was that *children's fraction schemes can emerge as accommodations in their numerical counting schemes.* The fraction schemes that emerged were of a different genre than the number sequences that were used in their construction primarily because children used their number sequences (or concepts) in partitioning in their construction of fraction schemes.[xxvii] Two basic fraction schemes that emerged were the partitive and the iterative fraction schemes.

The partitive fraction scheme. When ENS children use their number concepts in partitioning, they establish an equi-partitioning scheme (Steffe & Olive, 2010, p. 75ff). For example, when the number concept five is used in partitioning a candy bar, say, an estimate can be made of where to mark off one of five equal parts. Once a mark is made, the child can disembed the marked part (mentally or physically), use it in iterating to make five equal parts, and mentally compare the five parts to the original bar to test if the five parts together are equivalent to the original bar. If a child considers that the disembedded part is one out of five equal parts, or a fifth of the candy bar, this produces the first genuine fraction scheme that is referred to as the partitive fraction scheme (PFS; Tzur, 1999).

The iterative fraction scheme and fractional numbers. For children who have constructed the ENS and the PFS, it would seem that the CCSSM Standard 4.a under Number and Operations—Fractions would be appropriate for these children.

> Understand a fraction *a*/*b* as a multiple of 1/*b*. For example, use a visual fraction model to represent 5/4 as the product 5×(1/4), recording the conclusion by the equation 5/4 = 5×(1/4).

This standard was meant to illustrate how *multiplying a fraction by a whole number* might be modeled by a mathematics teacher in a straightforward way. But it doesn't explain the operations that are involved in children constructing fractions as *fractional numbers*. There is a scheme in the fractional knowledge of children, the *iterative fraction scheme* (IFS), where the fraction 5/4 is constituted as a fractional number; as five times one fourth of the candy bar (Steffe, & Olive, 2010, p. 333ff). The structure of the "candy bar" produced consists of a unit of units of units. That is, as a composite unit containing a composite unit comprised by 4/4 of the candy bar and one more partitive unit fraction. Once constructed, children can use the scheme to produce fractional connected number sequences {1/4, 2/4, 3/4, 4/4, 5/4, 6/4, …} that are constructive generalizations of their explicitly nested number sequence (Steffe, & Olive, 2010, p. 333ff) . This is the first fraction scheme that can be judged as an *extensive quantitative scheme*. The PFS constructed using the ENS is still constrained to the fractional whole. The construction of fractional numbers is not in the zone of potential construction of the children who have constructed the PFS in any short-term sense because it involves a stage shift from two to three levels of units coordination.

The splitting scheme. The splitting scheme, which is a reorganization of the equi-partitioning scheme, is used in the construction of fractional numbers. The splitting scheme is indicated when children can mentally produce a hypothetical stick that can be iterated seven times when given a stick and told that the given stick is seven times longer than their stick and are asked to make their stick. After the splitting scheme is constructed, if a child mentally splits a stick into, say, 48 parts, the child knows that one of the parts would be one forty-eighth of the whole stick because

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the whole stick is 48 times as long as the part. The result of the scheme is an inverse multiplicative relation between the part and the partitioned whole in the sense that Gauss specified extensive quantitative relations (cf. also Thompson, & Saldana, 2003).

Assessments of fifth through eighth grade children. With this brief introduction to the PFS and the IFS, I now turn to assessments of fifth, sixth, seventh, and eighth grade children concerning these schemes. Norton & Wilkins (2009) found that only 34% of the fifth graders and 35% of the sixth graders in their sample could engage in splitting, which is an indication of the presence of the operations that produce three levels of units[xxviii]. Of those same children, only 14% and 20%, respectively, provided some indication of having constructed the iterative fraction scheme.[xxix] In other assessments, Norton & Wilkins (2010) found that only 13% of their seventh grade sample and 19% of their eighth grade sample could produce the fractional whole when given, say, a stick partitioned into three parts and told that it was three sevenths of a candy bar and asked to draw the whole candy bar, which I consider as an assessment of fractional numbers.[xxx] In their earlier study Norton & Wilkins (2009) reported similar percentages for their fifth and sixth grade samples (14% and 18%). These data are consistent with an analysis of the percentages of children at one, two, and three levels of units that I present in Table 2 in which Norton's and Wilkins' data are included.

Table 2: Estimated Percent of Children at Each Level of Units by Grade

Grade/Level	One Level	Two Levels	Three Levels	IFS
Third	45	45	10	
Fifth	35	40	25 (34%)[NW]	(14%)[NW]
Sixth	30	30	40 (35%)[NW]	(18%)[NW]
Seventh				(13%)[NW]
Eighth				(18%)[NW]

It is especially disconcerting that only approximately 15.5% of Norton & Wilkins' seventh and eighth grade sample indicated that they had constructed a fraction as a multiplicative concept. It's disconcerting because, based on my own estimates, at least 40% of this sample should be able to construct a fraction as a multiplicative concept; that is, they should have been able to construct the IFS. But this expectation is tempered by the realization that the children in the fractions project constructed the iterative fraction scheme by working with us in teaching experiments. The fraction standards of the CCSSM are stated by grade level and as such underestimate what children who have constructed three levels of units can accomplish. On the other hand, children who have constructed two levels of units are constrained to constructing the PFS, a scheme that children use to construct proper fractions. What this means is that approximately 45% of the third-grade population, 40% of the fourth grade population, and 30% of the sixth grade population are able to construct partitive fractions, but not fractional numbers. When combined with the children who have constructed only one level of units throughout these three grade levels, we see that approximately only 15% of the third graders, 25% of the fourth graders, and 40 percent of the sixth graders will be able to construct the IFS and engage in producing fractional numbers.

Recommendations of the NMAP. Children's construction of fractions as well as the teaching of fractions must be changed. In the report of the National Mathematics Advisory Panel (2008), the following comment was made.

> Difficulty with learning fractions is pervasive, and is an obstacle to further progress in mathematics and other domains dependent on mathematics including algebra. … Conceptual and procedural knowledge about fractions with magnitudes less than 1 do not necessarily transfer to fractions with magnitudes greater than 1. Therefore, understanding of fractions with magnitudes in each range needs to be taught directly and the relation between them discussed. (p. 28)

Apparently, the authors of this report believed that fractions (proper and improper) can be taught directly to children regardless of the levels of units the children have constructed. The report of the panel, as I interpret it, exemplifies an empiricist as well as a neo-behavioristic agenda in the teaching of mathematics in precollege education that harks back to Thorndike's influence on the teaching of mathematics during the first one-half of the last century. Still, I do agree with the writers of the report concerning the pervasive difficulty that the learning of fractions presents to schoolboys and schoolgirls and also to the pervasive difficulty that the teaching of fractions presents to their mathematics teachers. Resorting to direct teaching in an attempt, for example, to raise children who have constructed only the PFS to the IFS could be interpreted as a more or less empirical enterprise and as generating a whole industry of empirical research on mathematical learning, to paraphrase Michael Cole's (2004) comments concerning the training studies of the 1960's that were conducted to prove Piaget wrong. In contrast, for the children who have constructed at least the partitive fraction scheme, my hypothesis is that quantitative measuring schemes can emerge as reorganizations of children's fraction schemes.

This hypothesis is similar to the hypotheses in the second and third research programs that additive measuring schemes can be constructed as reorganizations of children's number sequences. It is quite different, however, in that partitioning is a fundamental operation in the construction of the measuring schemes, which opens the way for children to construct measuring schemes involving two levels of units; for example, meters and centimeters, minutes and seconds, pounds and ounces, weeks and days, etc. Measuring systems in multiple levels of units might still be problematic. It is especially crucial to investigate possible changes that indicate *fundamental* transitions between reasoning with two levels of units and three levels of units induced in the construction of quantitative measuring schemes and their use in the construction of multiplicative and additive measuring schemes.

The Fifth Research Program

> *The fifth research program is to construct quantitative mathematics curricula for children in their construction of the rational numbers of arithmetic and the rational numbers, and the schemes and operations entailed in and by these constructions.*

Fractional numbers are a major achievement of children who can use three levels of units as assimilating operations, but fractional numbers are not equivalent to the Rational Numbers of Arithmetic nor to the Rational Numbers. Constructing the rational numbers of arithmetic involves the operations that generate the generalized number sequence (cf. Ulrich, 2014, p. 256). To exemplify those operations, an eight-year old child, Nathan, was presented with copies of a string of three toys and a string of four toys and asked to make 24 toys. Nathan reasoned out loud as follows,

> Three and four is seven; three sevens is 21, so three more to make 24. That's four threes and three fours! (Steffe & Olive, 2010, p. 278)

In solving the task, Nathan integrated a unit of three and a unit of four into a unit of seven, iterated the unit of seven three times to produce 21, increased 21 by three to produce 24, disunited 21 into three threes and three fours, integrated the additional three with the three threes, and produced four threes and three fours. These operations are operations of a GNS. In a GNS, any composite unit can be taken as the basic unit of the sequence in such a way that the composite unit implies the sequence just as the unit of one implies the ENS. Similar to the ENS, in the GNS a child can establish two number sequences “side by side”, a sequence of units of three and a sequence of units of four and combine the basic units of each sequence together to produce another sequence of units of seven. What this amounts to is the coordination of two three-levels of unit structures.

The rational numbers of arithmetic can be regarded as those operations that can be used to transform a given fraction into another given fraction; that is, the operations that are involved in quotitive fraction division. Quotitive fraction division involves the coordination of two three-levels of units structures; units within units within units. For example, consider a case where a child is given a segment that is said to be 1/5 of a unit segment and another segment that is said to be 1/3 of the same unit segment, and asked to use the 1/3-segment to produce the 1/5-segment. If the child partitions the 1/3-segment into five parts, takes one of these parts as a 1/15-segment and iterates this segment three times to produce the 1/5-segment, and if the child abstracts the operations as 3/5 of 1/3, then 3/5 is referred to as a rational number of arithmetic. After operating, I would also want to know if the child knows that 3/5 of the 1/3-segment is the 1/5 segment without actually taking 3/5 of the 1/3-segment. I would also want to know if the child can engage in reciprocal reasoning and understand that 5/3 of the 1/5-segment is the 1/3-segment (Hackenberg, 2010, 2014; Thompson & Saldanha, 2003; Thomson, et. al., 2014).[xxxi] The child is aware of the operations needed, not only to reconstruct the unit whole from any one of its parts, as in the case of fractional numbers, but also to produce any fraction of the unit whole starting with any other fraction, which are the operations involved in quotitive fraction division. (cf. Olive, 1999, for an interpretation of the schemes and operations involved in the production of the rational numbers of arithmetic). My hypothesis is that construction of the rational numbers of arithmetic entails a metamorphic accommodation relative to fractional numbers, and learning how to engender this accommodation and the constructive possibilities it entails is included in the first part of the fifth research program.

One might think that the distinction between the rational numbers of arithmetic and the rational numbers is “simply” that the latter involve negative as well as positive rational numbers of arithmetic. But that is not the case at all. My hypothesis is that a scheme of recursive distributive partitioning operations is involved in constructing rational numbers. In general, distributive partitioning operations are those operations that allow a student to share n units among m people and interpret one share as *n*/*m* of one unit and as 1/*m* of all *n* units (Liss, 2015; Steffe, Liss, & Lee, 2014; Lamon, 1996). Distributive partition operations are involved in what Thompson et al. (2014) referred to as “Wildi Magnitudes”. The power of Wildi’s definition of magnitude is that it makes explicit the fact that, “the magnitude of a quantity is invariant with respect to a change of unit” (Thompson, et. al., 2014, p. 4). So, if a quantity measures 22 inches, and if there are 12 inches/foot, then the quantity also measures 22inches/(12 inches/foot), whose transformation into 22*(1/12 foot) or 22/12 feet involves rational number of arithmetic operations. It also involves use of a scheme of recursive distributive partitioning operations because, according to Thompson (2014), “When a person *anticipates* that any measurement of Q[xxxii] with respect to an appropriate unit can be expressed in *any other* (emphases added) appropriate unit by some conversion without changing Q’s magnitude, she possesses Wilde’s meaning of magnitude” (p. 4)

When the scheme of recursive distributive partitioning operations can be used to produce what I would consider an equivalence class of fractional numbers, I would judge that the child has constructed a rational number.[xxxiii] I hypothesize that the construction of the rational numbers constitutes a stage shift relative to the rational numbers of arithmetic, and learning how to engender this stage shift and the constructive possibilities it entails is included in the second part of the fifth research program. The scheme of recursive distributive partitioning operations that is involved in the construction of rational numbers is also involved in the construction of intensive quantity (Liss,

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2015; Steffe, et al., 2014). The main difference is that intensive quantity involves relative magnitude, which means that a quantity is measured using a quantity of a different nature (Thompson, et al., 2014). In the case of rational numbers, a quantity is measured using a unit quantity of the same nature as the quantity to be measured.

The Sixth Research Program

> *The sixth research program is to construct quantitative mathematics curricula for children in their construction of integers and rational numbers as measures of change in an unsigned quantity, where "unsigned" refers to the magnitude of the quantity, and operations with them.*

Based on work by Thompson & Dreyfus (1988), Ulrich (2014) defined an integer as a measure of change in an unsigned quantity, where "unsigned" refers to the magnitude of the quantity. Concerning integer addition, Ulrich (2014) commented that,

> Unlike in unsigned addition, in which the second addend can have a different quality than the first addend, the addends in this case need to be of the same type in the mind of the student. Depending on the relative magnitudes, the sum could be a subsequence of either addend. … I hypothesize that a student will need to have constructed the GNS in order to conceptualize addition in this way, precisely because both addends need to be reified composite units (which seems to correspond to iterability and the ability to disembed while maintaining a nested relationship) so that the sum can be disembedded from either addend (p. 256).

Ulrich's hypothesis concerning the operations that are needed to construct integer addition leaves open the question of the operations that are needed to construct the concept of an integer other than her comment concerning "reified composite units." I interpret the meaning of a "reified composite unit" in terms of Thompson's (1994) hypothesis that, "an integer is a reflectively abstracted constant numerical difference" (p. 192). So, Ulrich's hypothesis concerning the operations needed to construct integer addition also pertains to the construction of the concept of integers. Although it might seem unusual that the operations needed to construct integers are two steps beyond the operations that are needed to construct the natural numbers of the ENS as extensive quantities, all of the operations of the ENS have to be reorganized and extended to produce an integer as a difference of two such natural numbers. That is, as a reflectively abstracted concept, an integer is the difference of any two signed quantities *a and b,* denoted by $a - b$, such that $a - b$ is a constant number of units between a and b in the direction from b to a. This concept of an integer is crucial in algebraic reasoning and should not be finessed by using the sum of a and the additive inverse of b as the definition of a difference $a - b$ like it is done in CCSSM.

I extend this way of regarding integers to the construction of signed rational numbers, where rational numbers are regarded as magnitudes in the way that I regard them in the above text. Based on my experience teaching middle school children in teaching experiments as well as teaching prospective middle school mathematics teachers, finding sums and differences of signed quantities whose magnitudes are rational numbers will require at least a constructive generalization of integer operations. Furthermore, although the product and quotients of signed quantities are rarely considered in studies of children's mathematics, they are fundamental as preparation for more general algebraic reasoning and involve constructive generalizations of rational number of arithmetic operations. Constructive trajectories also need to be established in which students establish the laws of signs for products as a logical necessity as well as patterns of reasoning that might be recognized as distributive, associative, and commutative reasoning.

Finally, because of the preponderance of children who are yet to construct the rational numbers of arithmetic or even fractional numbers in the middle school and beyond, it is essential to explore

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what a quantitative mathematics curricula involving signed quantities might look like for children who have constructed only three levels of units. This problem is especially acute for children who have constructed only two levels of units.

The Seventh Research Program

The seventh research program is to construct quantitative algebraic curricula for children in the construction of basic algebraic knowing.

The first aspect of the program is to learn the operations that are involved in children's construction of combinatorial reasoning. My hypothesis is that the concept of natural number variable is essential. Even children who can reason with three levels of units make extensive lists when finding the possible outcomes of two or more events that occur together rather than reason with compositions of natural number variables (Panapoi, 2013). Further, my hypothesis is that the multiplicative principal of combinatoiral reasoning and the dimensionality involved in spatial coordinate systems (Lee, 2017) both involve recursively coordinating two three levels of units structures. Lockwood (2015), in her work with college students, and Panapoi (2013) and Tillema (2007, 2013, 2014), in their work with middle grade students of differing levels of units, have made substantial progress in this program. But extensions of their work are needed to establish mathematics curricula for children involving combinatorial reasoning across differing levels of units.

The second part of this research program is to extend the fifth and sixth research programs to working with operations on quantities of unknown measurements, which could be considered as "generalized arithmetic." An extensive quantitative unknown refers to the potential result of measuring a *fixed* but unknown extensive quantity before actually measuring it (Liss, 2015, p.30). An intensive quantitative unknown refers to the potential result of enacting the operations that produce a fixed but unknown equivalent ratio. The production of such a ratio implies the availability of the operations needed to produce an equivalent ratio and, thus, a proportional relationship (Liss, 2015, pp. 31-32). Hackenberg (2005, 2010, 2013, 2014), Hackenberg & Tillema (2009), Hackenberg & Lee (2015), and Liss (2015) have made substantial progress in this program by working with students of differening levels of units. An extension of this work is needed so that quantitative algebraic curricula for children are established across differing levels of units.

The third part of this research program is highly related to the second part. It is to construct quantitative algebraic curricula for children concerning the construction of the basic rate scheme and its use in the construction of linear functions. Given two co-varying quantities, I consider a rate as the result of enacting the operations that produce a ratio equivalent to a unit ratio at any but no particular time (Steffe, et al., 2014, p. 52). The basic rate scheme can be considered as a metamorphosis of intensive quantitative unknowns and proportional reasoning. One might consider the result of enacting a rate formally as an equivalence class of ratios, but that doesn't say anything about the involved metamorphic accommodation that produces rate. Toward that end, Thompson's (1994) commented that, "A rate is a reflectively abstracted constant ratio, in the same sense that an integer is a reflectively abstracted constant numerical difference" (p. 192). Although I agree with this way of thinking about a rate, it too doesn't specify the operations that children use to produce the reflective abstraction. There are various studies that contribute to understanding such mental operations (Ellis, Özgüra, Kulowa, Williams, & Amidonba, 2015; Hackenberg, 2010; Hackenberg & Lee, 2014; Johnson, 2012, 2014; Liss, 2015; Moore, 2014, Thompson, 1994; Tillema, 2013). But how teacher/researchers might provoke such a reflective abstraction is a fundamental problem in establishing quantitative algebraic curricula for children across differing levels of units.

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Endnotes

[i] I surmise that, in part, it was because of what was considered as sufficient to falsify a theory during that period of time. According to Lakatos (1970) all justificationists, "whether the intellectualists and empiricists, agreed that a 'hard fact' may disprove a universal theory" (p. 94).

[ii] Thorndike considered himself a connectionist, which I regard as a form of behaviorism, but not radical behaviorism.

[iii] There was also an emphasis on social interaction, active citizen participation in all spheres of life, and democratization of public education.

[iv] Comment in brackets is added to the quotation.

[v] (http://www.eds-resources.com/facultytheory.htm)

[vi] (http://schugurensky.faculty.asu.edu/moments/1938rugg.html)

[vii] (http://www.uvm.edu/~dewey/articles/proged.htm)

[viii] Wertheimer was one of the three founders of Gestalt psychology along with Kurt Koffka and Wolfgang Köhler.

[ix] I attended a sequential summer institute for secondary school mathematics teachers during the summers of 1961, 62, and 63 at Kansas State Teachers College, Emporia, Kansas. There were no courses on teaching via problem solving that emphasized discovery learning by students although we did solve a lot of mathematical problems!

[x] James W. Wilson offered a course on problem solving for MEd and Ph.D. students at the University of Georgia for many years.

[xi] There were modern programs that did emphasize experiential learning of mathematics (Davis, 1990).

[xii] Piaget's grouping structures served as an abstracted model of the reasoning of children in what Piaget called the concrete operational stage.

[xiii] Piaget thought that the construction of the length unit was more advanced than the construction of the arithmetical unit.

[xiv] I am indebted to Dr. Larry Hatfield for his colleagueship and insight that led us to teach 1st and 2nd grade children in order to learn children's thinking.

[xv] A mathematician writer of the content standards told me that the standards are designed so that students can take college mathematics courses.

[xvi] In some cases, students can opt out of taking these tests.

[xvii] In constructivist research, Maturana's concept of the observer is essential. According to Maturana (1978), "Everything said is said by an observer to another observer who can be himself or herself" (p. 31).

[xviii] "Students" can be substituted for "children". I use "children" throughout the paper to be consistent.

[xix] Self-reflexivity involves applying one's epistemological tenets first and foremost to oneself.

[xx] Cf. Student-Adaptive Pedagogy for Elementary Teachers: Promoting Multiplicative and Fractional Reasoning to Improve Students' Preparedness for Middle School Mathematics, Dr. Ron Tzur, Principal Investigator.

[xxi] Cf. AIMS Center for Math and Science Education

[xxii] Cf. the work of Dr. Robert Wright's US Math Recovery Council.

[xxiii] Twenty-nine, say, can be disembedded from fifty while leaving it "in" fifty.

[xxiv] In stating this hypothesis, I assume that in the case of continuous quantity, children will primarily use units like inches, pounds, etc., in segmenting.

[xxv] This research program is not restricted to six-year-old children.

[xxvi] Cf. Hackenberg, Norton, & Wright (2016) for an excellent start on this problem.

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[xxvii] A number concept such as five is a composite unit containing five arithmetical unit items containing records of counting “1, 2, 3, 4, 5.”

[xxviii] Hackenberg (2007) found that some children who constructed only two levels of units could engage in splitting.

[xxix] These authors referred to this scheme as the generalized measurement scheme for fractions (GMSF).

[xxx] These authors referred to this scheme as the measurement scheme for proper fractions (MSPF).

[xxxi] Reciprocal reasoning of the kind Thompson, et al. (2014) identified involves coordinating two three-levels of units structures.

[xxxii] Q is taken as 22 inches in length.

[xxxiii] I did not observe a child construct what might be called an equivalence class of fractions even in the case of the GNS children (Steffe, & Olive, p. 337ff)

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