THE EFFECTS OF TWO SIMULATIONS ON CONCEPTIONS OF RATE OF CHANGE

Jenna Tague
California State University at Fresno
jtague@csufresno.edu

Manjula Peter Joseph
Fresno Pacific University
manjula.joseph@fresno.edu

Pingping Zhang
Winona State University
pzhang@winona.edu

The focus of the current proposal is to examine the effect of two dynamic simulations on the participants' conceptions of rate of change. Conceptions of rate of change were measured according to Carlson et al.'s (2002) Mental Actions framework and how the participants related the physical simulations to the graphical representations (Heid, et al., 2006). Results indicate that the simulations increased participants' covariational understanding, but did not help the students create a more accurate understanding of rate of change.

Keywords: Technology, Algebra and Algebraic Thinking

Students who do not understand the concept of rate of change are unlikely to develop a conceptual understanding of algebra (Roschelle, Kaput, & Stroup, 2000) or calculus (Thompson, 2008). It has been suggested that dynamic simulations could help students, particularly in the middle grades, develop a better understanding of rate of change (Rochelle, et al., 2007). However, new simulations are created continuously, and it is unclear how these simulations affect the cognition of the individuals who interact with them. Thus, the focus of the current paper is the following question: How did two dynamic simulations affect middle school, high school, and undergraduate students' understandings of rate of change?

Significance

Past research has shown that visualizations are important in: developing an understanding of rate of change (Roschelle et al, 2007), and the historical mathematical development of rate of change (Struik, 1969). Roschelle, Kaput, and Stroup (2000) propose the inclusion of technology as a necessary aspect of introducing rate of change to students before algebra or calculus.

However, visualizations and simulations change the way that students interact with and develop an understanding of various concepts (Hegadus, 2005). Even when educational experts have designed visualizations, novices notice different features or interpret the features differently than the experts intended (Roschelle, 1991). Further, the introduction of dynamic simulations may result in the *development* of different or undocumented cognitive obstacles. For instance, in a study of preservice teachers' understanding of the definition of limit in interactive geometry environments, Cory and Garofalo (2011) found that their participants became unsure of which variable is dependent on which (amongst N, epsilon, and delta). Because of the structure of the technology, some of the well documented cognitive obstacles disappeared (i.e. what N, epsilon, and delta represent physically), but the misunderstanding of dependence appeared as a new cognitive obstacle.

Other work provides evidence of what tasks or teaching practices would be important in using dynamic simulations as part of rate of change instruction (i.e. Roschelle et al, 2007). However, it does not address how, separate from instruction, simulations may impact an individual's cognition. It is essential to examine how simulations affect the students' conceptions of rate of change. This information will allow for informed implementation of dynamic simulations centered on rate of change into a learning environment and into future research on rate of change.

Background

Development of an understanding of covariation has been linked to an improved understanding of rate of change (Thompson & Thompson, 1996; Confrey & Smith, 1994). An understanding of

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covariation refers to the understanding that as one variable changes continuously, the other dependent variable changes simultaneously. Thompson and Thompson (1996), in documenting a teaching experiment with one 6th grade student, indicated that students tend to initially conceptualize speed as a compound unit called *speed-lengths*: the time it takes to travel a given distance. That is, the participant was only able to understand discrete parts of the variation, rather than describe how continuous variation in time affects the continuous variation in distance. The authors reasoned that school learners first understand "speed as a distance and time as a ratio (total length/speed-length)" (Thompson & Thompson, 1996, p. 3).

In a study of secondary teachers' creation of a graph of a bottle that is similar to the boiling flask shown in Figure 4, the researchers used two lenses to examine the participants' work: "use and coordination of macro-perspective and micro-perspective; and coordination of mathematical entities and their features" (Heid, et al., 2006, p. 4). The study postulated that a central theme in students' reasoning about the bottle problem was the "macro-perspective" (examining the overall view) and the "micro-perspective" (examining a smaller part such as small changes in the height to consider what change in the volume that would cause) (p. 5). A key factor in how successful their participants were was whether or not the participants were conscious of both perspectives and whether or not they could shift between them to overcome obstacles. The second key factor in the participants' success was how the individuals related the mathematical entities (the graph) and the physical entities (the bottle). For example, sometimes participants would be unable to coordinate the mathematical and the physical entities or other times the participants would fixate on a particular connection and use only that connection to generalize.

Thus, the current literature indicates that students will likely have difficulty thinking about how the variables in rate of change tasks are related. In addition to this, participants will struggle to understand when gestalt views of the objects/graphs or piece-wise views of the objects/graphs will be helpful to their reasoning.

Research Design

The current qualitative study included two tasks that were part of a larger effort to document students' conceptions of rate of change (Tague, 2015). Each participant took part in a task-based based interview (Goldin, 2010) lasting, on average, 70 minutes. The goal of the interviews was not to design instruction nor to teach the participants, but instead, to document the participants' conceptions of rate of change before and after use of a dynamic visualization. As such, the interviewer did not push the participants toward a correct solution; however, follow-up questions were asked to clarify the participants' conceptions.

Each interview was video recorded, and transcribed verbatim. The video recordings captured the participants written work, their hand movements, and their interactions with the visualizations. The transcripts included gestures where the participants did not possess the vocabulary to articulate their full understanding of rate of change (Roorda, Vos, & Goedhart, 2009). For example, one of the participants showed, using her hands, that when the bottle narrowed, the graph would increase in slope, by tilting her hands in and then out because she could not articulate the vocabulary for narrow/widen.

The transcripts were then analyzed according to the Mental Action Framework (Carlson, et al., 2002) shown in Figure 1 to determine the level of the participants' understanding of covariation. The framework was developed through studying second year calculus students' understanding of average rate. The authors argued that determining level of understanding of covariation involved examining many mental actions that might be elicited by a task, and that an individual should not attain higher levels of mental actions without mastering the lower levels. We also examined what other features or conceptions were important in completing the task (Heid et al., 2006), in interacting with the

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simulations, and in completing the task a second time. For example, if a participant matched bottles to graphs in the Water Filling task by iconic translation (how closely the shape of the graph matched the physical shape of the bottle) (Monk, 1992), that individual was not actually using any kind of covariation to complete the task.

Mental Action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g. y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the rate of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Construction contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Figure 1. Mental actions and indicators of the covariation framework (Carlson, et al., 2002, p. 357).

Participants

The participants (Table 1) were chosen purposefully to represent students at specific educational levels - before algebra (middle school students), after algebra (high school students and students taking calculus courses), and after calculus (students enrolled in differential equations courses). Algebra (Saldanha & Thompson, 1998) and calculus (Thompson, 2008) have been shown to be key places where a robust understanding of rate of change is necessary.

Middle school and high school participants were recruited through letters sent to parents from teachers in a large professional development program. In the case of the undergraduate students, participants were recruited through Calculus and Differential Equations courses at a large Midwestern University. Participants were chosen from the volunteers to maximize variation amongst the participants. When possible, the participants were chosen from different parts of two Midwestern states, or different courses.

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Table 1: Participants and Pseudonyms

Participant Grade Level	Pseudonym
Middle School – 6^{th} grade	Forrest
Middle School – 8^{th} grade	Amy
High School – Precalculus	Sarah
High School – Precalculus	Kristi
Undergraduate - Calculus	Kyle
Undergraduate - Calculus	Angela
Undergraduate - Calculus	Brian
Undergraduate - Calculus	Amanda

Task Design and Choice of Simulation

Two simulations were used, and in both cases, the participants were asked to complete a task before the simulation, to interact with the simulation, and then to complete the same task again. During the second time through the task, the participants' original work was put away, and they had the option of continuing to use and test options in the simulations while working.

The first dynamic simulation was a Java applet called "The Moving Man" shown in Figure 1 (PhET). The applet has the image of a man that begins in the middle of a horizontal axis. The man can be dragged using the mouse or he can be programed to move in a particular way by choosing an initial position, velocity, and acceleration. If the user moves the man manually, the position, velocity, and acceleration change simultaneously.

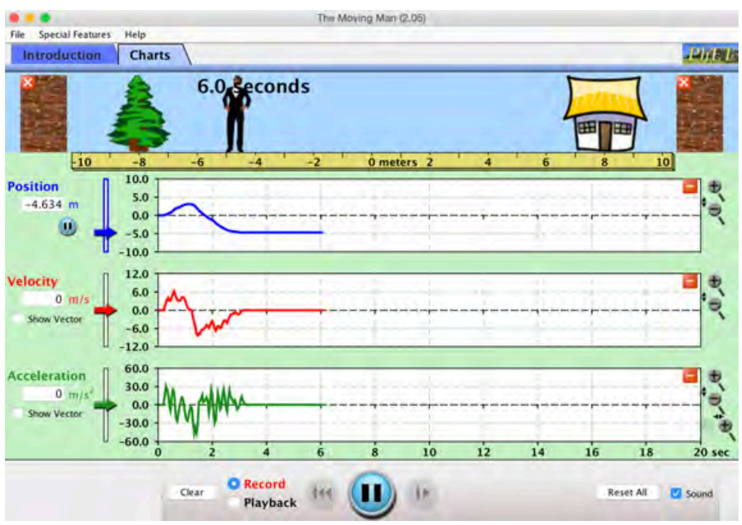

Figure 2. Screenshot of The Moving Man (applet by PhET).

The task associated with this dynamic simulation is the following: *Draw a picture of what you think the position, velocity, and acceleration graphs will look like if the man starts at the tree, realizes he is hungry, and then goes home to eat*. The task was deliberately left vague in order to allow participants to connect with their intuitive knowledge of how people move and how that motion affects their velocity and acceleration. Participants were also asked if they understood the terms position, velocity, and acceleration, and were given explanations if necessary.

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The second dynamic simulation was a screenshot video of an individual playing with Wolfram Alpha's Bottle Filling simulation shown in Figure 3. In the online environment, the user can drag the outside points of the bottle, and then drag the fluid height level up. As the bottle is filled on the left side, a simultaneous graph of volume versus height is created on the right side. The participants could pause the video at any time, drag the action backwards or forwards, and watch as many times as they wanted to while they completed the task for the second time.

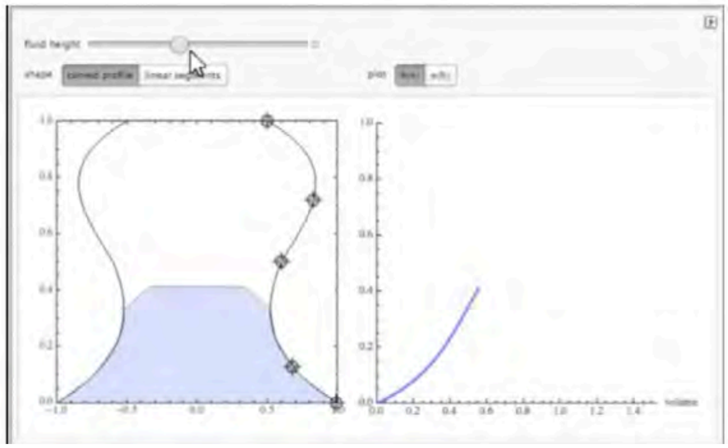

Figure 3. Screenshot of water filling simulation applet (Wolfram Alpha).

The task associated with the water filling simulation stated, "*Imagine filling each of the six bottles below (Figure 4), pouring water in at a constant rate. For each bottle, choose the correct graph, relating the height of the water to the volume of water that's been poured in*" (Annenberg Learner). Note that the graphs of C, G, and H do not match with any of the bottles, but their bottles would look like those shown in Figure 5. After the participants matched the bottles to graphs, we asked them to choose any graph they had leftover and sketch what the associated bottle would be.

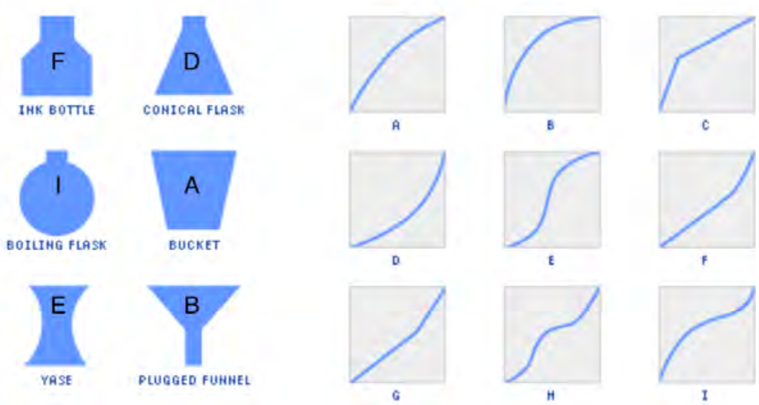

Figure 4. Task associated with the bottle filling simulation with the intended matches marked. (Annenberg Learner).

Figure 5. Bottles matching graphs C, G, and H from Figure 4 (Annenberg Learner).

The two dynamic simulations were chosen purposefully to be accessible, yet challenging to all participants from middle school through differential equations students. Both simulations were also chosen because they represent physical activities that the participants were likely to have experienced. The Moving Man represents a graphing of position, velocity, and acceleration, which is a paradigmatic type of rate of change problem that many individuals come to equate with their

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definition of rate of change (Zandieh, 1997). The bottle filling task is one that has been used by many researchers examining rate of change, and so would allow for comparisons with previous literature (Carlson, et al., 2002; Heid, et al., 2006).

Results and Discussion

Table 2 illustrates the mental actions of the participants associated with covariation before and after interacting with the simulations. As the tables illustrate, the participants generally moved toward a more covariational view of rate of change or maintained their current level. However, more covariational mental actions did not always coordinate with a more accurate physical understanding, as explained further.

Table 2: Mental Actions Before and After the Dynamic Simulations

	Water Filling		Moving Man	
	Before	After	Before	After
Forrest	None	None	MA1	None
Amy	None	None	None	None
Sarah	MA5	MA5	MA5	MA5
Kristi	MA3	MA5	MA1	MA3
Kyle	MA2	MA3	MA3	MA4
Angela	MA4	MA5	MA4	MA5
Amanda	MA5	MA5	MA4	MA5
Brian	MA5	MA5	MA5	MA5

The middle school participants, Forrest and Amy, matched bottles both before and after the simulation, using Monk's (1992) description of *iconic translation*. Monk (1992) described how students sometimes create graphs that replicated the physical features of a problem. For example, when asked to create a rate graph of someone biking across a flat surface and then biking up a hill, students are likely to create a horizontal line attached to a positive sloping line. Forrest and Amy matched the bottles with the graphs based on the physical features of the bottle that matched the physical features of the graphs. For example, they both matched graph D with the vase, and Amy explained, "because I think I was just trying to match the shape of it and not the actual amount of liquid it can be filled with." Further, Amy's explanation indicated that she was not even considering either of the variables involved in the task, and rather looked at the overall shapes to match. Neither one attempted to draw a bottle from one of their leftover graphs.

The rest of the participants were either at the highest covariational understanding of rate of change (for the water filling problem), or moved towards a better understanding (Table 2). Still, as before, improvement in understanding of covariation did not necessarily indicate a more accurate response. For example, Kyle's matches were based on a generalization of one physical feature of the bottles – corners. His reasoning was similar to that of the participants in Heid and colleagues' (2006) study, in that, although he was considering different uniform changes in volume and how that would correlate to height, he based those changes around relating the physical features of corners to physical corners in the graphs. However, he was ranked at MA3 afterwards because he could describe that wider parts of the bottle would results in more volume, but less height whereas before the simulation he could only describe as more water was added, the height would increase.

The Moving Man interaction seemed to have none or a negative effect on the rate of change conceptions of the middle school participants. Forrest actually moved from creating graphs where he considered the change with respect to time to creating graphs with iconic translation (Monk, 1992),

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or using the physical motion of the man to create the shape of the graph. Note, from Figure 6 that all of his graphs were horizontal and the man's movement can only be horizontal. Forrest had no cognitive dissonance about the fact that his graphs differed from those on the simulation. He was insistent that the graphs must look "just like the man moved." In Amy's case, she persisted in creating three discrete points for the graphs: one at (18,8) on the position, one at (18,8) on the velocity, and one at (8,6) for the acceleration. Like Forrest, she was undisturbed by the difference between her graph and the simulation. For the rest of the participants, the simulation moved them towards a more covariational understanding of rate of change. However, all of the other participants also copied what the simulation created, whether or not they understood it. For example, Kristi originally created a linear position function, and after interacting with the simulation, she changed it to curved. When she was asked why, she was unable to provide reasoning until the interviewer asked her what would happen if the acceleration were set to 0.

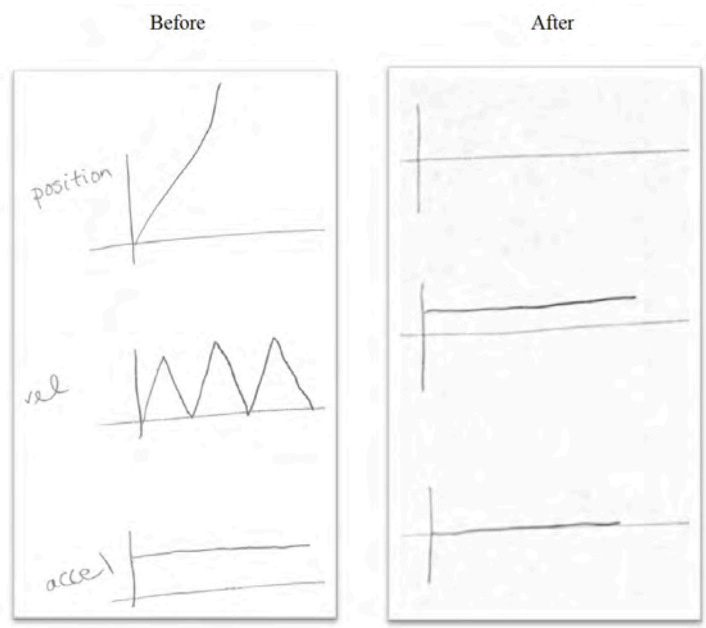

Figure 6. Forrest's graphs before and after the Moving Man simulation.

Conclusion

In summary, the dynamic simulations moved the participants' conceptions of rate of change towards a move covariational understanding. However, a better covariational understanding did not directly mean that they produced a more accurate graph/bottle/etc. It's possible that in addition to developing an understanding of how the variables co-vary, individuals also have prior experiences that cause them to focus only on one aspect of the physical situation or to use iconic translation. A better understanding of the relationship between the prior experiences and how they relate to covariational understanding is necessary to be able to describe fully individuals' understandings of rate of change.

It is clear from the current study, that questioning is essential because exposure to the simulations, in some cases, caused new misconceptions, led to less use of covariational reasoning, or did not address underlying misconceptions. Furthermore, technology is essential in studying understanding of rate of change, but it also transforms understandings, and as such, requires attention to changes caused by the technology, and further study on what kinds of questions and tasks would help students maximize their understandings.

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References

Annenberg Learner. (2015). https://www.learner.org/courses/learningmath/algebra/session1/part_c/filling.html.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352-378.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, 26, 135-164.

Cory, B. L., & Garofalo, J. (2011). Using dynamic sketches to enhance preservice secondary mathematics teachers' understanding of limit of sequences. *Journal for Research in Mathematics Education*, 42(1), 65-96.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In R. Lesh & A. E. Kelly (Eds.), Handbook of research design in mathematics and science education (pp. 517-545). Hillsdale, NJ: Erlbaum.

Hegedus, S. (2005). Dynamic representations: A new perspective on instrumental genesis. In M. Bosch (Ed.), Proceedings of CERME 4, the Fourth Congress of the European Society for Research in Mathematics Education. Barcelona, Spain: Ramon Llull University.

Heid, M. K., Lunt, J., Portnoy, N., & Zembat, I. O. Ways in which prospective secondary mathematics teachers deal with mathematical complexity. Proceedings of the 2006 Psychology of Mathematics Education-North America Conference, Mérida, México, 2006.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.) *The concept of function: Aspects of epistemology and pedagogy, MAA Notes*, Vol. 25 (pp. 175-193). Washington, DE: Mathematical Association of America.

PhET Interactive Simulations (2013). Interactive Online Applet. University of Colorado.

Roorda, G., Vos, P., & Goedhart, M. Derivatives and applications: Development of one student's understanding. Proceedings of the Congress of European Research in Mathematics Education, Lyon, France, 2009.

Roschelle, J. (1991). *Students' construction of qualitative physics knowledge: Learning about velocity and acceleration in a computer microworld*. Doctoral dissertation, University of California, Berkeley.

Roschelle, J., Kaput, J. J., & Stroup, W. (2000). Simcalc: accelerating student engagement with the mathematics of change. In M. J. Jacobsen & R. B. Kozma (Eds.), *Innovations in science and mathematics education: advanced designs for technologies of learning* (pp. 47–75). Hillsdale, NJ: Lawrence Erlbaum.

Roschelle, J., Tatar, D., Shechtman, N., Hegadua, S., Hopkins, B., Knudsen, J., & Singleton, C. (2007). *Scaling up SimCalc project: Can a technology enhanced curriculum improve student learning of important mathematics?* Technical Report 01. SRI International.

Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkins, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 298-303). Columbus, OH.

Struik, D. J. (1969). *A Source Book in Mathematics*. Harvard University Press.

Tague, J. (2015). *Conceptions of rate of change: A cross analysis of modes of knowing and usage among middle, high school and undergraduate students*. (Doctoral dissertation).

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education, (Vol 1, pp. 45-64). Morélia, Mexico.

Thompson, P. W., & Thompson, A. G. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1): 2-24.

Wolfram Alpha. "Filling a Container Defined by a Curve": http://demonstrations.wolfram.com/FillingAContainerDefinedByACurve

Zandieh, M. (1997). The evolution of student understanding of the concept of derivative, Doctoral dissertation, Oregon State University.