REFORMULATION OF GEOMETRIC VALIDATIONS CREATED BY STUDENTS, REVEALED WHEN USING THE ACODESA METHODOLOGY

Álvaro Sebastián Bustos Rubilar
Cinvestav
abustos@cinvestav.mx

Gonzalo Zubieta Badillo
Cinvestav
gzubieta@cinvestav.mx

In this article, we report how a geometric task based on the ACODESA methodology (collaborative learning, scientific debate and self-reflection) promotes the reformulation of the students' validations and allows revealing the students' aims in each of the stages of the methodology. To do so, we present the case of a team and, particularly, one of its members who expresses the intention of reformulating validations for a mathematical conjecture besides showing evolution in the form of justifying.

Keywords: Reasoning and Proofs, Geometry and Geometrical and Spatial Thinking

Background and research problem

For some decades, the teaching and learning of proof have been studied by a number of mathematics education researchers to promote their learning (De Villiers, 2010; Hanna, 2000, among others) and identify and classify the procedures provided by the students when developing tasks in which they have to prove (Balacheff, 1987; Bell, 1976). The works by these authors consider that students in general show difficulties when they are asked to prove a mathematical statement or mathematically justify their statements. Proof is probably the only accepted way of validation among mathematics scholars. However, in a context of teaching and learning, students are not necessarily expert in the matter and will not become professional mathematicians (Legrand, 1993). This means that, when asking students to prove, we will probably find that what they consider will be far from what is accepted as proof by a professional or a scholar. Considering that, our research uses the term validation to refer to what a student may provide to justify a mathematical statement. We understand this validation as a dynamic process we expect to evolve according to the context in which the student works.

The aim of this work was to determine how the validation created individually by a student is reformulated and improved when working with tasks based on the ACODESA methodology proposed by Hitt (2007). To do so, we raise the following research question: How does a validation change from its written formulation by a student, its subsequent study by a team and debate by the class, to its final reconstruction in a process of self-reflection?

Theoretical References

In this study, we consider the term validation as a process through which a proposition or mathematical statement is validated, as Balacheff (1987) explains. He considers that proving is the intellectual activity, not entirely explicit, that deals with the manipulation of the given or acquired information to produce new one, aiming to ensure the truth of a proposition. The validation can be expressed in different ways: explanation, argumentation, proof or demonstration. All of them can vary or evolve, according to the context in which the student works. Therefore, as Brousseau (2002, p. 17) states:

> The didactical situation must lead them to evolve, to revise their opinions, to replace their false theory with a true one. This evolution has a dialectic character as well; a hypothesis must be sufficiently accepted—at least provisionally—even to show that it is false.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Accepting a form of validation will depend on the mastery of knowledge each student possesses as well as the social environment and how close he or she is to the validation criteria of the community where the validation occurs. This last element is of great importance since the validity of a proposition must be accepted by the student and his or her social environment. Above all, the validity must be guided by the validation criteria of the discipline—mathematics, in this case. The description of the different forms of validation, according to Balacheff (1987), are the following:

Explanation. Discourse with which the truth of a position or result previously acquired by the speaker is clarified.

Argumentation. Discourse aimed to obtain the listener's consent.

Proof. Explanation accepted by a community that can be rejected by another. It may simultaneously evolve with the advance of the knowledge on which it is based.

Demonstration. It is a series of statements organized according to a well-defined set of rules.

To identify and then categorize the validations produced by the students, we used the typology of levels and types of proof by Balacheff (1987):

Naive empiricism. It occurs when the student asserts the validity of a statement after verifying it in particular cases. The student's resistance to generalization is evident in this type of proof.

Crucial experiment. The student verifies with the least particular example he or she can manage. In this type of proof, the student explicitly generalizes from the example with which the statement is verified.

Generic example. The student provides an example representing the generality; that is, an example that is not considered a particular case but a representative of a type of cases for which the statement is true. In this type of proof, operations and transformations of the mathematical object explain why the statement is valid.

Thought experiment. The student explains the reasons through the analysis of the properties involved in the statement, decontextualizing it and taking it out from a particular representation.

Calculation on statements. Intellectual constructions based on more or less formalized or explicit theories, created in a definition or property. They are based on the transformation of symbolic expressions. This type of proof ranges from the thought experiment to the proof.

Balacheff (1987) groups the types of proofs, described above, in pragmatic and intellectual. On one hand, the naïve empiricism, the crucial experiment and the generic example are pragmatic proofs: they resort to action and concrete examples. On the other hand, the thought experiment and the calculation on statements correspond to intellectual proofs, given that they are supported by the formulation of mathematical properties set in play and the relationship between them.

Methodology

Students of a Master of Educational Mathematics participated in the study for two sessions of two hours each. In each session, two video cameras were used to record an overall view of the classroom and specific moments. Additionally, dialogs between the students were recorded using a voice recorder.

The task implemented was designed and organized according to the principles of the ACODESA methodology (Hitt, 2007), which allows promoting collaborative learning through social interaction and the use of technology. As a result, processes of conjecture, argumentation and validation are created in the classroom (Hitt, 2011; Hitt, Saboya, & Cortés, 2016). The ACODESA methodology has five stages:

1. Individual work. The student develops the task individually using paper and pencil.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

2. Teamwork. The students work in teams of three or four members. Each member presents a solution and justifies the response to the problem in front of his or her peers to create a solution as a team.
3. Debate. Each team presents its proposal in front of all the class. The guidelines to the debate used in the ACODESA methodology must be in accordance with what Legrand (1993) stated.
4. Self-reflection. The students carry out a process of reconstruction of the task. This stage is important because, as Hitt y González-Martín (2014) and Hitt et al. (2016) state, the consensus obtained in the previous stage might be provisional for some students. Therefore, every student has to reconstruct the solution individually using paper and pencil, considering what has been done in previous stages.
5. Institutionalization. The teacher presents the institutional solution to the task in front of the students. To do so, the teacher summarizes what was done in previous stages and highlights the solutions proposed by each team.

The general aim of the task was to create a work environment in which the students could conjecture and then validate such conjectures both individually and in teams. With the activity, we sought to identify the types of validations provided by the students when working on the different stages of the ACODESA methodology. Then, we determined how their validations evolved from the individual formulation to the moment they were shared and discussed during teamwork in a plenary session and up to the moment when they were reconstructed by the student in the self-reflection stage. Due to space limitations, in this article we only report part of the results of the task. The statement in the task and the questions were as follows:

> A parallelogram is known to be a quadrilateral whose opposite sides are parallel. If you choose any given parallelogram and draw the respective diagonals, four triangles will be formed; then,
>
> - What can you say regarding the areas of the four triangles? Justify your response in detail and do not forget to mention which parallelogram you chose.
> - Are your responses above independent from the type of parallelogram you choose? Why? Justify your response in detail.

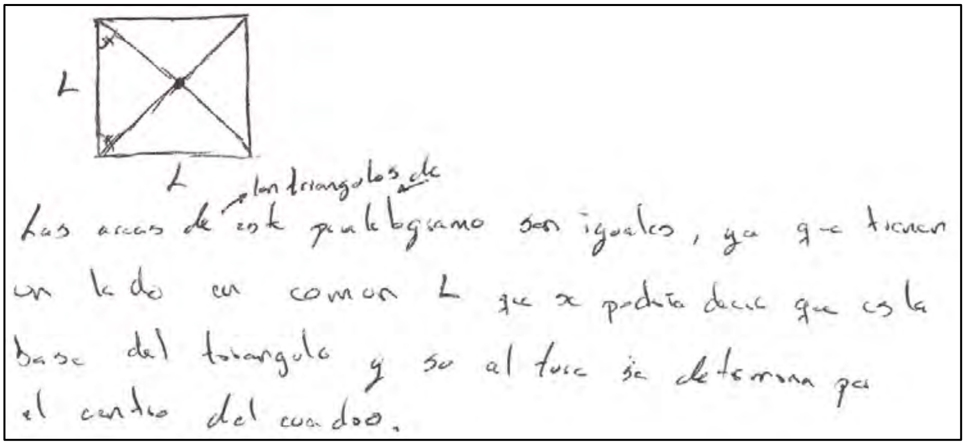
The first question aimed for the students to conjecture and validate for the particular case of a parallelogram. The objective of the second question was to make the students generalize their conjecture and then, validate it. To differentiate between their responses in the different stages, the students were asked to use a black ballpoint pen for the individual work, a red one for the teamwork and a blue one for the debate stage.

Analysis and Result Discussion

In this section, we present the case of Alex, a student whom we considered the most representative of the group in which the task was implemented. The analysis was carried out according to the stages of the ACODESA methodology.

Individual Work

Alex chose a square to formulate the response. To do so, he drew a general representative and assigned a measure L to each side (Figure 1). The student then conjectured that the areas of the four triangles formed randomly when drawing the diagonals were equal.

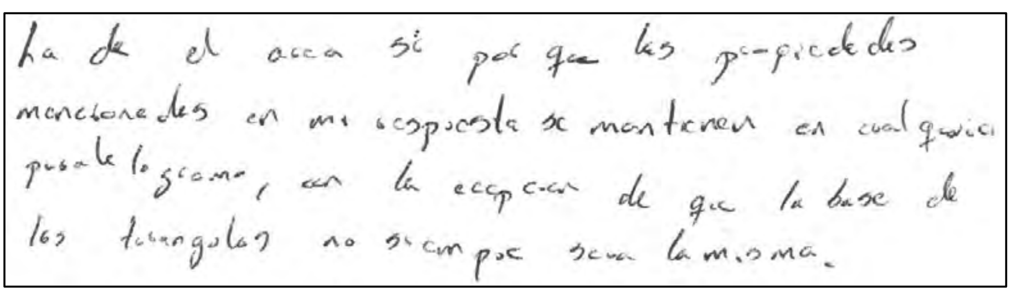


The areas of the triangles of this parallelogram are equal since they have a side L in common, which might be said to be the base of the triangle, and its height is determined by the center of the square.

Figure 1. Particular solution provided by Alex.

The student's strategy was to justify that the areas are equal because the four triangles have the same base and height. In his response, we identify two statements to achieve his objective: (1) the four triangles have equal base because L is a side of the square, and (2) the height of each triangle is determined by the center of the square.

The student justified the first statement through one of the properties of the square (equal sides). For the second statement, related to the congruency of the heights of the four triangles, Alex did not mention nor justified the property that allowed him to say the intersection point of the diagonals was the center of the square. Although his statements are valid, Alex omitted middle justifications (second statement). Regardless, we consider this validation to be a proof corresponding to the incomplete thought experiment, given that the student based the arguments on a general representation of the squares and presented (incomplete) justifications when he applied properties involved in the chosen parallelogram. In the response provided by the student for the second answer, in which he was induced to generalize, Alex claimed that his conjecture (regarding the square case) was independent from the type of parallelogram chosen; that is, the equality of the areas of the four triangles is met for any parallelogram (Figure 2).



Yes to the one of the area because the properties mentioned in my response are maintained in any parallelogram, except that the base of the triangles is not always the same.

Figure 2. Generalization provided by Alex.

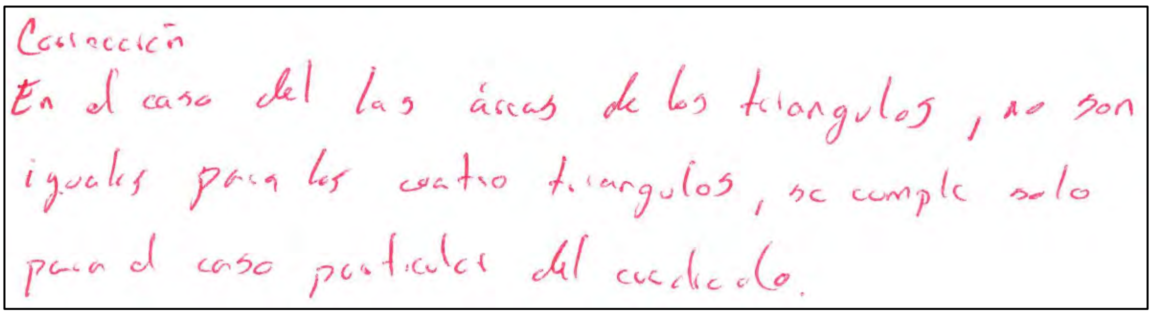
Alex correctly generalized the conjecture since the four triangles will always have the same area. However, when justifying the new conjecture—general conjecture, hereafter—we observe that the

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

student supported his argument on the validation created for the case of the square. In consequence, we infer that the student believed that validation was also true for any parallelogram. Then, we have a generic example-type of proof because the validation of the general conjecture, which refers to all parallelograms, would be supported by the validation Alex created for the square case, which is only a representative of a type of the parallelogram family.

Teamwork

The other two members of the team, S1 and S2, created their own responses from a rectangle and a general parallelogram, respectively. Both students conjectured that only opposite triangles have equal areas and were adamant that the square is a particular case. Unlike Alex, they created thought experiment proofs based on the consistency of triangles. Although Alex conjectured that the four triangles would have equal areas in all parallelograms, he did not say so to his teammates and corrected his conjecture on the answer sheet (Figure 3), but did not create another validation.


Corrección
En el caso del las áreas de los triangulos, no son iguales para los cuatro triangulos, se cumple solo para el caso particular del cuadrado.

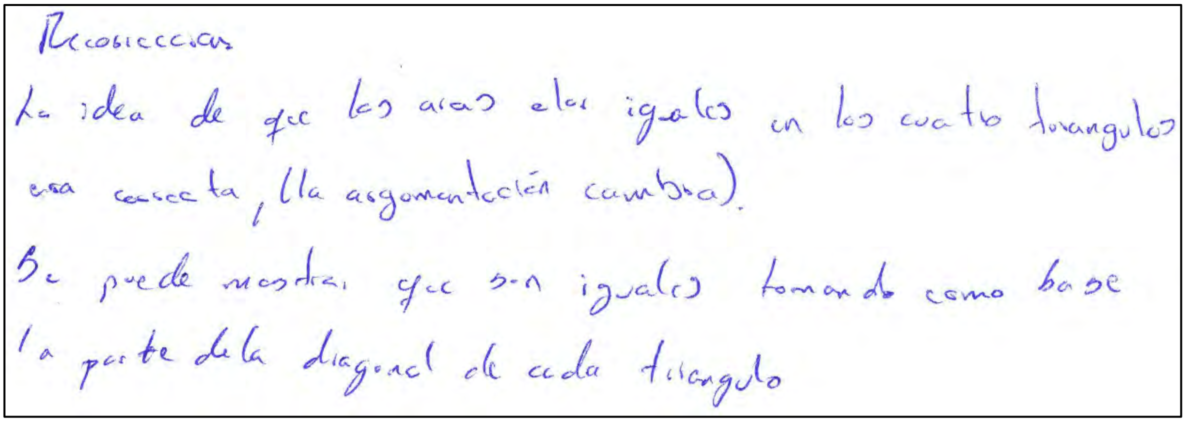
Correction
In the case of the areas of the triangles, they are not equal for the four triangles, it is met only for the particular case of the square.

Figure 3. Correction written by Alex during the teamwork stage.

Debate

During the debate, the students first discussed whether the four triangles or only opposite ones had equal areas. Once the teams presented their responses, the consensus of the debate was that, in any given parallelogram, the four triangles formed would always have the same area. The validation agreed on by all the students was based on congruency of triangles to justify the equality of the opposite triangles and the property of diagonals (the diagonals intersected each other) to justify the equality of the areas of adjacent triangles. After this, Alex corrected his response once more (Figure 4) and went back to his general conjecture—the one he had discarded during teamwork. He then expressed that the argumentation for such conjecture had to be changed. From his response, we infer that, after listening to different responses in this stage, Alex obtained more arguments to create a new validation for his general conjecture, although he only did so in the following stage: self-reflection.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

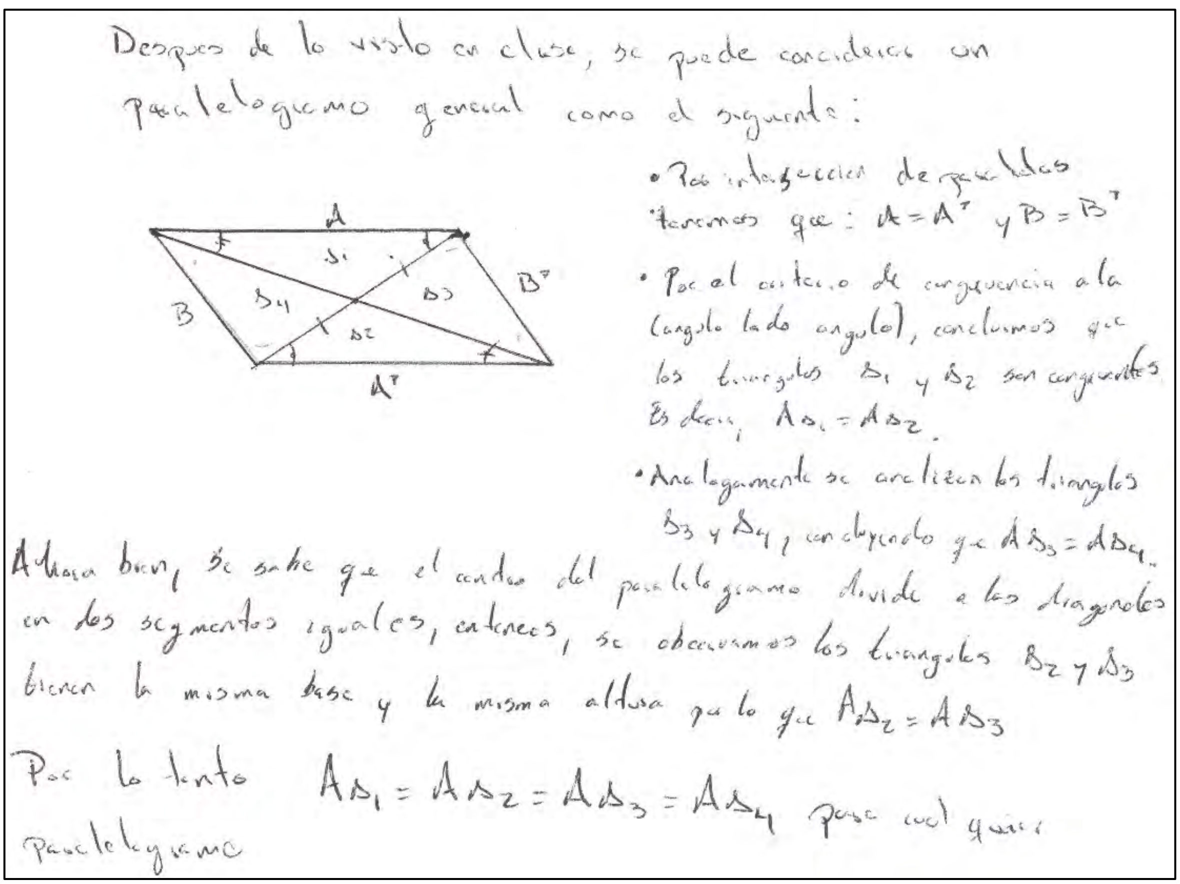


Re-correction
The notion that the areas were equal in the four triangles was correct (the argumentation changes).
We can prove that they are equal based on the diagonal part of each triangle.

Figure 4. Correction created by Alex in the debate stage.

Self-Reflection

In this stage, we observed a more solid validation (Figure 5) than the one created by the student during the individual work stage.



After what we checked in class, a general parallelogram can be considered as follows:

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

- By the intersection of parallels, we have that: $\boldsymbol{A} = \boldsymbol{A}^{\boldsymbol{T}}$ and $\boldsymbol{B} = \boldsymbol{B}^{\boldsymbol{T}}$.
- By the ASA (angle, side, angle) congruency criterion, we conclude that the triangles Δ_1 and Δ_2 are congruent, that is, $\boldsymbol{A}_{\Delta_1} = \boldsymbol{A}_{\Delta_2}$.
- Likewise, triangles Δ_3 and $\Delta_4$are analyzed, concluding that $\boldsymbol{A}_{\Delta_3} = \boldsymbol{A}_{\Delta_4}$.

That said, the center of the parallelogram is known to divide the diagonals in two equal segments, so, if we observe triangles Δ_2 y Δ_3, they have the same base and the same height, then $\boldsymbol{A}_{\Delta_2} = \boldsymbol{A}_{\Delta_3}$.

Therefore, $\boldsymbol{A}_{\Delta_1} = \boldsymbol{A}_{\Delta_2} = \boldsymbol{A}_{\Delta_3} = \boldsymbol{A}_{\Delta_4}$ for any parallelogram.

Figure 5. Validation created by Alex during the self-reflection stage.

Alex used triangle congruency as his first argument to justify the equality of the areas of opposite triangles ($A_1 = A_2$ and $A_3 = A_4$ in the parallelogram in Figure 5); such argument arose during the teamwork stage. The student then used the property of diagonals to justify the equality of the areas of adjacent triangles ($A_2 = A_3$ in the parallelogram in Figure 5); he built this argument during the debate stage. Alex returned to his general conjecture while his validation did not depend on a particular representation anymore. Then, it was a proof of the **type calculation on statements** since Alex based his statements on the definition and properties of the parallelogram.

Conclusions

In the individual work stage, we observed that the student created a thought experiment proof to validate his conjecture regarding the square case. However, when he generalized, his validation became a generic example proof. In the teamwork stage, Alex did not expressly wrote any reformulation to his validations, but did alter his general conjecture (Figure 3). During the debate stage, he went back to his general conjecture and gave indications that the validation had to change.

If we consider that the general conjecture created by the student (the four triangles have equal areas), the validation he created in the individual work stage corresponds to a generic example, a pragmatic proof. In contrast, during the self-reflection stage, the student built an intellectual proof for the same conjecture; it was a calculation on statements and included arguments that arose during the teamwork and debate stages. This revealed a noticeable change between his individual work and self-reflection to validate the same conjecture, given that his validation went from a pragmatic level to an intellectual one. We can credit this to the ACODESA methodology, as indicated by Hitt et al. (2016). An adequate environment was created for the students to conjecture and validate in a context of social interaction.

On the other hand, during the teamwork stage, Alex provisionally rejected his conjecture after listening to his classmates' arguments. This situation probably took place because Alex did not have the necessary arguments in that moment to persuade his team of the veracity of his conjecture. Regardless, this situation was overcome in the debate stage, during which all the students agreed that the four triangles would always have the same area despite the type of parallelogram used. In the previous chapter, we observed how the didactic situation led the students to evolve, as defined by Brousseau (2002), both in the initial conjecture and the arguments used for its validation.

Most of the students rationally justified their assertions, both in the individual and the group stages. Additionally, we observed an environment of discussion around the arguments used to defend the different statements that arose during the development of the task, especially in the debate stage.

References

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics*, *18*(2), 147–176. https://doi.org/10.1007/BF00314724

Bell, W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, *7*(1), 23–40. https://doi.org/10.1007/BF00144356

Brousseau, G. (2002). *Theory of Didactical Situations in Mathematics*. (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds.). New York, Boston, Dordrecht, London, Moscow: Kluwer Academic.

De Villiers, M. (2010). Explanation and proof in mathematics: Philosophical and educational perspectives. *Experimentation and Proof in Mathematics*, 1–294. https://doi.org/10.1007/978-1-4419-0576-5

Hanna, G. (2000). Proof and Its Classroom Role : A Survey. In *Sociedade Portuguesa de Investigação em Educação Matemática – SPIEM –* (pp. 75–104). Retrieved from http://spiem.pt/DOCS/ATAS_ENCONTROS/2000/2000_04_GHanna.pdf

Hitt, F. (2007). Utilisation de calculatrices symboliques dans le cadre d'une méthode d'apprentissage collaboratif, de débat scientifique et d'auto-réflexion. *Environnements Informatisés et Ressources Numériques Pour L'apprentissage. Conception et Usages, Regards Croisés*, 65–88.

Hitt, F. (2011). Construction of mathematical knowledge using graphic calculators (CAS) in the mathematics classroom. *International Journal of Mathematical Education in Science and Technology*, *42*(6), 723–735. https://doi.org/10.1080/0020739X.2011.583364

Hitt, F., & González-Martín, A. (2014). Covariation between variables in a modelling process: The ACODESA (collaborative learning, scientific debate and self-reflection) method. *Educational Studies in Mathematics*, *88*(2), 201–219. https://doi.org/10.1007/s10649-014-9578-7

Hitt, F., Saboya, M., & Cortés, C. (2016). Rupture or continuity: The arithmetico-algebraic thinking as an alternative in a modelling process in a paper and pencil and technology environment. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-016-9717-4

Legrand, M. (1993). Debat scientifique en cours de mathematiques et specificite de l'analyse. *Repères-IREM*, *10*, 123–159. Retrieved from http://www.univ-irem.fr/exemple/reperes/articles/10_article_68.pdf

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.