THE PROMISE AND PITFALLS OF MAKING CONNECTIONS IN MATHEMATICS

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Making connections during math instruction is a recommended practice, but may increase the difficulty of the lesson. We used an avatar video instructor to qualitatively examine the role of linking multiple representations for 24 middle school students learning algebra. Students were taught how to solve polynomial multiplication problems, such as (2x + 5)(x + 2), using two representations. Students who viewed an explicit linking episode were more likely to make important connections, but less likely to exhibit problem-solving success than students who did not view the linking episode. Further, the quality of the connections made by the students was negatively related to subsequent problem solving and transfer. Thus, although focusing on connections may support rich understanding, it may decrease learning of solution methods. The results showcase the promise and pitfalls of making connections in mathematics.

Keywords: Problem Solving, Algebra and Algebraic Thinking, Instructional Activities and Practices, Technology

Introduction

Making explicit connections during mathematics learning instruction is a recommended practice (e.g., NCTM, 2000; Pashler et al., 2007). In fact, some researchers have even defined mathematics understanding in terms of the number or kind of connections that have been constructed by the learner (see Crooks & Alibali, 2014). One important type of connection to make is between multiple representations of the same concept or procedure (e.g., the graph of a line and its equation). In the current study, we used an avatar video instructor to examine the role that linking multiple representations during an algebra lesson had on connection-making and problem-solving performance. Our goals were (1) to compare the effects of a lesson that included a linking episode versus a lesson that did not include a linking episode on students' connection-making and problem solving, and (2) to examine how students' connection-making related to subsequent learning and transfer. We selected the domain of algebra because it functions as a "gatekeeper" to future educational opportunities (Moses & Cobb, 2001). Further, algebra is a focal point of reform efforts in mathematics education (e.g., NMAP, 2008).

Theoretical Framework

Mathematical ideas and representations are connected to and build upon other mathematical ideas and representations. The new *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, 2010) is explicit on this point: fundamentally, "mathematics is a connected subject" (p. 5). Understanding these connections is fundamental to having a deep, conceptual understanding of mathematics. Indeed, the notion of connecting mathematical ideas and representations emerges in many of the standards put forth by the National Council of Teachers of Mathematics (NCTM, 2000), one of which is the ability to "translate among mathematical representations" (p. 67).

The current study evaluated the influence of a lesson that explicitly linked multiple representations during instruction. We define *linking episodes* as segments of instruction during which the instructor seeks to make explicit links between ideas or representations (Alibali et al. 2014). For example, imagine instructing students on the concept of mathematical equivalence first

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using a balance scale, then using an equation, and finally by making the correspondences between the balance scale and equation explicit. Establishing the correspondences between the two representations would be considered a linking episode.

On the one hand, linking episodes during instruction should facilitate greater understanding for students because they point out conceptual links among ideas and representations (e.g., Crooks & Alibali, 2013; Rittle-Johnson & Alibali, 1999). For example, Hiebert and colleagues (1997) argue, “we understand something if we see how it is related or connected to other things we know” (p. 4). Further, there are many examples of students benefitting from connections made via a variety of instructional techniques, including direct comparison (Rittle-Johnson, Star, & Durkin, 2009), linking gestures (Alibali et al., 2013) and fading from concrete to abstract representations (Fyfe, McNeil, & Borjas, 2015).

On the other hand, making connections among representations can be cognitively demanding, requiring students to understand each representation as well as their correspondences (e.g., Gick & Holyoak, 1980; Nathan et al., 2011). For novices, this may overload their cognitive resources (e.g., Sweller et al., 1998). Indeed, learning from connections may be difficult for students with low background knowledge (e.g., Clark, Ayres, & Sweller, 2005; Kotovsky & Gentner, 1996). For example, one study found that making connections via comparison was beneficial for advanced students, but not for novices (Rittle-Johnson et al., 2009). Specifically, middle school students who did not know a method for solving the target equations benefitted more from studying two methods sequentially than from comparing two methods directly.

Thus, the inclusion of explicit linking episodes may help students focus on making rich connections between multiple representations. At the same time, it may detract from focusing on learning to work with each individual representation correctly, particularly for novice students.

Current Study

In the current study, we had two specific aims. Our first aim was to compare the effects of a lesson that included a linking episode versus a lesson that did not include a linking episode on students’ connection-making and problem solving. Specifically, middle school students were taught how to solve polynomial multiplication problems, such as $(2x + 5)(x + 2)$, by an avatar instructor using an area-based representation and an equation-based representation (see Figure 1). Students in the *link* condition viewed a subsequent linking episode and students in the *no-link* condition did not. We expected students in the link condition to make more high-quality connections between the two representations than students in the no-link condition, but to have similar problem-solving performance. Our second aim was to examine how the quality of students’ connection-making related to subsequent learning and transfer, regardless of condition. After the initial lesson and assessment, all students were exposed to an instructional linking episode and a posttest. This provided students an opportunity to use the knowledge they acquired from the initial lesson. We expected the quality of students’ connection-making to be positively related to their performance on posttest items that tapped understanding of the links between the representations, but negatively related to their performance on posttest items that tapped understanding of individual representations. This work was part of a larger project that developed a teacher avatar (Anasingaraiu et al., 2016) and is investigating how variations in the avatar's behavior during linking episodes influences student learning. The present study focused on variations in the presence of linking episodes in the avatar lesson.

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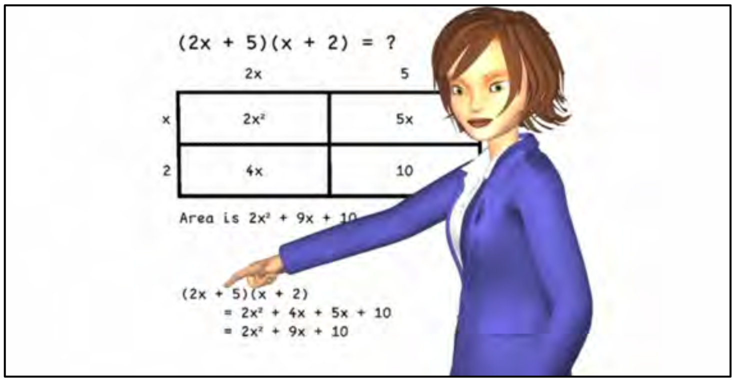


Figure 1. Image of the lesson with the area method (top) and equation method (bottom).

Method

Participants

Participants were 16 seventh-graders and 8 eighth-graders attending one of three middle schools in a mid-sized Midwestern city in the United States. Participants were predominantly White (75% White, 8% Asian, 4% Hispanic, 13% Other) and their mean age was 13.2 years (*min* = 11.5, *max* = 14.2). Sixty-two percent were male. An approved email was sent to all seventh- and eighth-grade students at the schools inviting them to participate in a project that would take place on the university's campus. Each student was compensated $15 for participating.

Design and Procedure

We used a pretest-lesson-posttest design. Each student participated in a single one-on-one session that lasted 45 minutes. Students completed a pretest to assess their background knowledge. Next, they viewed a lesson presented by an avatar video instructor. The lesson focused on multiplying binomials using a target problem: $(2x + 5)(x + 2)$. For the lesson, children were randomly assigned to one of two conditions (Figure 2): *link* ($n = 12$) or *no-link* ($n = 12$).

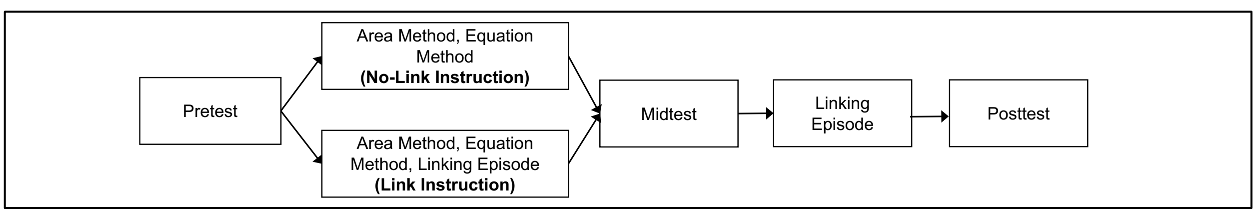


Figure 2. Sequence of activities in the experimental procedure.

The instructor described an area-based method and then described an equation-based method (Figure 1). In the *link* condition, the avatar instructor then provided a linking episode in which she delineated the correspondences between the two representations (e.g., "2x + 5 in the equation corresponds to the length 2x + 5 in the rectangle"). Students then engaged in an explanation of the target problem and solved the items on the midtest. The purpose of the explanation and midtest was to assess differences in learning between students who had viewed a linking episode and those who had not. After the midtest, all students in both conditions viewed the linking episode and completed a posttest. The purpose of the posttest was to evaluate how the quality of students' initial connection-making (as assessed on the explanation and midtest) related to their learning from subsequent instruction. Throughout the session, students were encouraged to think aloud so we could gain a richer account of their thought processes (Ericsson & Simon, 1993).

Materials and Measures

All items on all measures were presented one at a time on an interactive smart board.

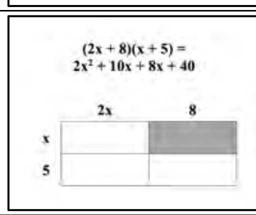
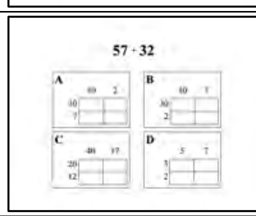
Pretest. The pretest included six items (see Table 1 for examples). The first five items tapped students' background knowledge of operating with variables and calculating area. The sixth item was a target polynomial multiplication problem.

Explanation. After the avatar lesson, students were shown the instructional problem and asked: "Here is the same problem you just learned about. Imagine that another student is seeing this example for the first time. Can you explain how to solve this problem?" Explanations were coded for whether students (1) exhibited a "trouble spot," defined as indicating confusion or displaying incorrect understanding (Alibali et al., 2013), (2) referred to one or both representations, and (3) provided a general solution strategy rather than a step-by-step procedure.

Midtest. The midtest included two items (see Table 1 for examples). The first item was a polynomial multiplication problem. The second item was a linking item.

Posttest. The posttest included seven items (see Table 1 for examples). Two were polynomial multiplication problems. Three were linking items. The final two were transfer items that tapped whether students could apply what they learned about multiplying expressions with variables to multiplying whole numbers. Items were scored as correct or incorrect based on students' written answers and on the verbal think-aloud reports they provided while solving.

Table 1: Example Items Presented on the Pretest, Midtest, and Posttest

Item Type	Example Item	Instructions	Example Responses
Background Knowledge Item (five on pretest)	**X + X**	Simplify the expression.	Correct: 2x Incorrect: 1, 1x, x, 2, x^2
Solve Item (one on pretest, one on midtest, two on posttest)	**(6x + 3)(y + 7)**	Simplify the expression by multiplying the terms 6x plus 3 and y plus 7.	Correct: 6xy + 42x + 3y + 21 Incorrect: 6xy + 10, 21*6xy
Link Item (one on midtest, three on posttest)		Circle the term in the equation that represents the area of the shaded rectangle.	Correct: 8x Incorrect: 40
Transfer Item (two on posttest)		Which area model(s) correspond to the multiplication problem 57 x 32?	Correct: BC Incorrect: Only B, ABC, D

Results

Pretest

Students did moderately well on the five background knowledge items (percent correct ranged from 42% to 88%). However, only one student (out of 24) correctly solved the polynomial multiplication problem: $(x + 2)(x + 1)$. The two most common errors on that problem were to add the two x's and add the two integers to get $2x + 3$, or to combine terms within parentheses to get $2x * 1x$. Conditions were well matched at pretest (M_{link} = 58% vs. $M_{no\text{-}link}$ = 54%). A median split on total percent correct yielded a low-background-knowledge group (n = 12, M = 36%) and a high-background-knowledge group (n = 12, M = 77%). All students but one were unsuccessful on the target problem and were thus novices; the knowledge groups differed in terms of the background knowledge necessary to learn about the target problem.

Explanation

Following the lesson, students were asked to explain how to solve a polynomial multiplication problem. Consider the explanations presented below:

Student 1 in the link condition: "Basically what you would do is multiply each number by every other number that's in the different set. So 2x times x [draws line connecting the 2x and the x in the equation] is $2x^2$ [circles $2x^2$ in the area model]. 2x times 2 [draws line connecting the 2x and the 2 in the equation] is 4x [circles 4x in the area model]. Both of these are one side [circles the 2x and 5 across the top of the area model] so you don't have to multiply these. Then you do 5 times x [draws line connecting the 5 and the x in the equation], which is 5x [circles 5x in area model]. And 5 times 2 [draws line connecting 5 and 2 in equation], which is 10 [circles 10 in area model]. Then you would take all those answers together [circles all four terms in bottom equation] and simplify them. So 4x plus 5x is 9x. Then 10, and $2x^2$."

Student 2 in no-link condition: "So first you would do what's in parentheses…you do 2x plus 5 [points to 2x and 5 in equation], which I think would be 7x [writes 7x under the 2x + 5 in the equation]. Then you do x plus 2, which would be 2x [writes 2x under x + 2 in the equation]. Then you multiply them I think. So, it would be 14x."

Student 1 provides an accurate explanation, mentions a general solution strategy ("multiply each number by every other number that's in the different set"), and refers to both representations. In contrast, Student 2 exhibits a trouble spot (i.e., incorrect understanding), provides only a step-by-step procedure, and relies solely on the equation-based method.

To capture these differences, we created an *explanation quality score*. Explanations received one point for each of the following features: (1) did not contain a trouble spot, (2) offered a general solution method, and (3) referred to both representations. One third of students scored a maximum 3 out of 3, and across all students the average explanation quality score was 1.8 (out of 3; SD = 1.1). This suggests that typical explanations hit about two of the three criteria for being high-quality. It was most common to provide an explanation that was free from trouble spots (19 out of 24 explanations). It was less common to refer to both representations (12 out of 24 explanations) or to offer a general solution method (12 out of 24 explanations).

Students with low background knowledge had difficulty in explaining. Compared to the high-background-knowledge group, they were more likely to exhibit a trouble spot (42% vs. 0%), less likely to state a general solution method (33% vs. 67%), and less likely to refer to both representations (42% vs. 58%). As such, students with low background knowledge had lower quality scores (M = 1.3) than students with high background knowledge (M = 2.3), and there was little variability between the two conditions among low-background-knowledge students.

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However, within the high-background-knowledge group, explanations varied by condition. Compared to students in the *no-link* condition, students in the *link* condition were more likely to provide a general solution method (83% vs. 50%) and more likely to refer to both representations (83% vs. 33%). Indeed, for the high-background knowledge group, students in the *link* condition had higher explanation-quality scores ($M = 2.7$) than students in the *no-link* condition ($M = 1.8$).

Midtest

Over half of the students solved the target polynomial multiplication problem correctly at midtest (54%) and all but three students (88%) solved the linking item correctly. Students with low background knowledge were less likely than their high-background-knowledge peers to correctly solve the multiplication item (33% vs. 75%), and the linking item (75% vs. 100%).

As with explanation quality scores, condition differences were minimal for the low-background-knowledge group. For the high-background-knowledge group, performance on the linking item was at ceiling, but performance on the multiplication item varied. Students in the *link* condition were *less* likely to solve the problem correctly than students in the *no-link* condition (50% vs. 100%). All the high-background-knowledge students who solved the multiplication problem incorrectly also provided explanations focused on both representations, suggesting that a focus on linking potentially interfered with learning at least one method well. Overall, regarding our first research goal, we found that a lesson with a linking episode resulted in higher-quality connection-making among students with sufficient background knowledge, but lower problem-solving success relative to a lesson without a linking episode.

Posttest

The posttest occurred after all students viewed a brief instructional linking episode. It allowed us to evaluate how students' initial connection-making related to subsequent learning and transfer. Overall, performance was moderate on the polynomial multiplication solve items ($M = 60\%$, $SD = 44\%$), high on the three linking items ($M = 88\%$, $SD = 26\%$), and moderate on the two transfer items ($M = 52\%$, $SD = 35\%$). Most students demonstrated some learning by the posttest. At pretest, only one student (4% of the sample) solved a polynomial multiplication problem correctly, but 17 out of the 24 students (71%) solved at least one correctly at posttest.

Recall that students explained a target problem after the initial instruction and received an explanation quality score. These explanation quality scores were related to posttest performance (see Table 2). The correlations in Table 2 suggest that explanation quality scores were positively related to posttest linking scores, weakly related to posttest problem-solving scores, and negatively related to posttest transfer scores. We also examined these associations by splitting students into a high-quality explanation group ($n = 8$, scored 3 out of 3 on explanation quality) and a low-quality explanation group ($n = 16$, scored 0, 1, or 2 out of 3). Among students with high background knowledge, there were clear differences based on explanation quality. Compared to students in the low-quality explanation group, students in the high-quality explanation group had higher posttest link scores (100% vs. 89%), similar posttest solve scores (75% vs. 75%) and lower posttest transfer scores (41% vs. 67%). These differences lend credence to the idea that students who focus on making connections (and therefore have higher-quality explanation scores) do well on items that tap their knowledge of links, but not as well on items that tap their knowledge of the individual solution methods or representations.

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Table 2: Correlations Between Explanation Quality Scores and Posttest Performance

	Whole Sample	Low-Background-Knowledge Group	High-Background-Knowledge Group
Posttest Link Scores	$r_s = .43$	$r_s = .50$	$r_s = .14$
Posttest Solve Scores	$r_s = .22$	$r_s = .29$	$r_s = -.16$
Posttest Transfer Scores	$r_s = -.18$	$r_s = .00$	$r_s = -.42$

To look at this more directly, we made one additional comparison. The explanation quality scores took into account trouble spots, the provision of a general solution method, and references to both representations. To more directly consider connection-making, we compared students who differed only on this last criterion: students who referenced both representations (n = 12) vs. students who referenced only one representation (n = 12). Students who referenced both representations had slightly higher posttest link scores (89% vs. 86%), slightly lower posttest solve scores (58% vs. 63%) and lower posttest transfer scores (41% vs. 63%). These transfer differences were particularly pronounced for students with high background knowledge (36% vs. 80%). Overall, regarding our second research goal, we found that students' initial connection-making was related to their subsequent learning and transfer. Connection-making seemed to support students' understanding of the links between the representations, but not their ability to solve familiar or novel problems about the individual representations.

Discussion

Educational opportunities for all learners expand as we come to understand the conditions under which teachers' connection-making during instruction affects student learning. The current results highlight the promise and pitfalls of including linking episodes during algebra lessons. For middle school students with sufficient background knowledge, a lesson with a linking episode led to higher-quality explanations than a lesson without one. That is, students who saw the linking episode were more likely to provide a general solution method that applied to both representations and to refer to both representations rather than one. This suggests they were developing rich connections necessary for mathematics understanding (Hiebert et al., 1997).

However, students who saw the link were also less likely to solve a target problem correctly than students who did not see the link – potentially because they were focusing on processing the two representations and their correspondences rather than solidifying their knowledge of a correct solution method. Further, regardless of condition, students' engagement in connection-making was related to their learning and transfer from a subsequent instructional episode. Specifically, higher-quality connection-making appeared to be positively related to performance on posttest items that tapped understanding of the links between the representations, but negatively related to performance on posttest items that tapped understanding of individual solution methods. Thus, linking episodes may help students focus on making rich connections, but may also detract from their focusing on learning each solution method correctly, particularly for novice students (see also Clark et al., 2005; Rittle-Johnson et al., 2009). This may represent a trade-off in the development of conceptual versus procedural knowledge (e.g., Crooks & Alibali, 2014). Improvements in understanding conceptual links may come at the expense of improvements in understanding key procedures. Importantly, these results support the recommendation that instruction should include linking episodes that highlight connections among mathematics ideas. We find connection-making can be supported by a video-based avatar and we identify trade-offs between building rich conceptual connections and performing representation-specific solution procedures.

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