COLLABORATIVE GESTURES WHEN PROVING GEOMETRIC CONJECTURES

Candace Walkington
Southern Methodist University
cwalkington@smu.edu

Mitchell J. Nathan
University of Wisconsin at Madison
mnathan@wisc.edu

Dawn M. Woods
Southern Methodist University University
dwoods@smu.edu

Research in mathematics education has established that gestures – spontaneous movements of the hand that accompany speech – are important for learning. In the present study, we examine how students use gestures to communicate with each other while proving geometric conjectures, arguing that this communication represents an example of extended cognition. We identify three kinds of "collaborative gestures" – gestures that are physically distributed over multiple learners. Learners make echoing gestures by copying another learner's hand gestures, mirroring gestures by gesturing identically and simultaneously with another learner, and joint gestures where multiple learners collectively make a single gesture of a mathematical object using more than one set of hands. The identification and description of these kinds of collaborative gestures offers insight into how learners build distributed mathematical understanding.

Keywords: Reasoning and Proof, Geometry and Geometrical and Spatial Thinking, Cognition

Introduction

Theories of embodied cognition posit that learners understand ideas, even abstract mathematical ideas, through their bodies and senses (e.g., Lakoff & Núñez, 2000). One important form of embodiment is *gesture* – physical hand movements that people spontaneously formulate to accompany speech. Hostetter and Alibali (2008) argue that gestures are an outgrowth of mental simulations of actions enacted by learners as they think and reason. Considerable research has suggested that gesture production predicts students' learning and performance across a variety of content areas, including mathematics (Goldin-Meadow, 2005; Valenzeno, Alibali, & Klatzky, 2003; Cook, Mitchell, & Goldin-Meadow, 2008).

While the importance of gestures to student learning has been established in a variety of studies, less work has been done detailing how gestures allow for cognition to be physically distributed over multiple learners. Here we focus on how multiple learners use gestures in their interactions with each other, during mathematics classroom learning activities. We argue that these gestures exemplify *extended cognition* (Clark & Chalmers, 1998), the idea that cognitive processes themselves include physical resources beyond the skull. We show evidence of extended mathematical cognition by documenting *collaborative gestures* – gestures made collectively by multiple students as they work together to make sense of mathematical ideas. We discuss the emergence of collaborative gestures in the context of proving geometry conjectures.

Literature Review

Justification and Proof

Justification and proof are central activities in mathematics education (National Council of Teachers of Mathematics, 2000; Yackel & Hanna, 2003). In fact, "proof and proving are fundamental to doing and knowing mathematics; they are the basis of mathematical understanding and essential in developing, establishing, and communicating mathematical knowledge" (Stylianides, 2007, p. 289). Research on mathematicians' proving practices has suggested that proof "is a richly embodied practice that involves inscribing and manipulating notations, interacting with those notations through speech and gesture, and using the body to enact the meanings of mathematical ideas" (Marghetis,

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Edwards, & Núñez, 2014, p. 243). The multimodal nature of proof is also evident for novice students in classroom settings, as students' proofs often take on spontaneous, verbal forms, as opposed to formal, written ones (Healy & Hoyles, 2000), and both teachers and students use gestures as a way to track the development of key ideas when exploring mathematical conjectures (Nathan et al., 2017). Thus, gestures serve as crucial embodied grounding mechanisms for proof-related reasoning in geometry classrooms.

Dynamic Gestures and Dynamic Geometry Systems

One type of gesture identified in prior research as being particularly important is *dynamic gestures* (Göksun, Goldin-Meadow, Newcombe, & Shipley, 2013; Uttal et al., 2012). These are gestures where learners use their bodies, usually their hands and fingers, to physically formulate and then manipulate mathematical entities (see Walkington et al., 2014). For example, when proving that the sum of any two sides of a triangle must be greater than the remaining side, a learner might physically formulate two sides of the triangle with straight hands, and then "collapse" these two sides to show that if the two sides were not larger, the triangle would not be able to close. The presence of dynamic gestures has been associated with more accurate proofs of geometric conjectures, with a medium effect size (Walkington et al., 2014; Nathan & Walkington, in press).

Dynamic gestures allow students to formulate shapes and lines with their bodies in a manner that can be similar to using dynamic geometry software (DGS). DGS allows users to construct, measure, and manipulate objects by dragging and connecting defined objects on a computer screen (Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2004). The direct manipulation of DGS allows users to experiment freely and to have instantaneous interactions with geometric objects and their spatial relations (Marrades & Gutierrez, 2000). Dynamic gestures are limited compared to DGSs in that there is no feedback on whether manipulations are mathematically possible, nor is there exact measurement of geometric objects. However, gestures are highly portable and meaningful to the learner, and are part of the natural way in which human beings communicate, making them a powerful tool for mathematical reasoning. Research has shown that when gesture is facilitated or directed, reasoning is improved (Goldin-Meadow, Cook, & Mitchell, 2009), and when it is inhibited, reasoning is impaired (Hostetter, Alibali, & Kita, 2007). Walkington et al. (2014) found that for geometry conjectures specifically, even the inhibition of sitting in a chair and having a pencil in-hand and paper available reduced the incidence of dynamic gestures, and caused students to formulate correct proofs less often.

Distributed and Extended Cognition

Work in professions involves the coordination of many different inscriptions and representational technologies by differently-positioned actors whose actions occur across a range of social and physical spaces (Goodwin, 1995; Hutchins, 1995). Through joint, coordinated activity, cognition becomes distributed over a patchwork of discontinuous spaces and representational media. In this conceptualization of *distributed cognition*, the environment is used to offload cognitive demands. Theories of *extended cognition* go even further to argue that the social and physical environment of learners is actually constituent of their cognitive system (Clark & Chalmers, 1998). The implication is that cognition, rather than existing in the head of an individual, is distributed over the bodies of multiple learners and the environment around them as they interact. One way in which cognition can be extended across learners is through the use of gestures that extend over multiple persons.

Prior research on students learning origami from instructors has identified *collaborative gestures* as gestures through which a learner interacts with the gestures of a communicative partner (Funiyama, 2000). In the context of this past research, these gestures often involved a learner pointing to or manipulating a teacher's gestures about origami folds. Here we reimagine the idea of

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

collaborative gestures to be relevant to learner-learner interactions around mathematical sense-making, and take such gestures to be a case of extended cognition.

Research Purpose

In the present study, we address the following research question: *What are the ways that team members use collaborative gestures when proving geometric conjectures?* We focus specifically on cases where the physical, gestural activity is distributed over multiple learners, rather than cases of a single student gesturing and another student interpreting that gesture.

Method

Setting and Sample

Eleven undergraduate students enrolled in a teacher education program (ten female and one male) aged 20-22 years voluntarily participated in this 75-minute study. The undergraduates were enrolled in the elementary mathematics method course from a private university situated within a large city in the southwestern United States. Informed consent was obtained from all participants. Sixty-four percent of the participants identified as Caucasian, 18% identified as Asian, and the remaining 18% identified as Latino/a. The undergraduates had already declared a non-education major, but were simultaneously enrolled in a 33-credit hour undergraduate major in education preparing them to pursue teaching careers, work in the social sciences, or informal education paths in non-profit organizations. Students were divided into two groups around two separate gaming systems with each group being video recorded while playing the video game. We focus our analyses on one of the two groups of students, with four females and one male.

Procedure and Measures

The focus of the study was the playing of an educational video game about learning geometry (see Nathan & Walkington, 2017 for more information about the game). Specifically, through the Kinect video game platform, students were prompted to perform specific arm motions and then prove geometry conjectures that were related to those arm motions. While only one participant (the gamer) of each group controlled the Kinect with their body movements, the remaining participants in each group worked collaboratively with the gamer to mathematically prove or disprove the conjectures. The role of the gamer rotated throughout the group so that each participant had the opportunity to perform the directed arm motions and also to take the lead in communicating the proof. In this study, rather than focusing on the directed arm motions that the game directed learners to perform before proving the conjecture, we focus on the hand gestures they spontaneously made while formulating their proofs.

Before playing the video game, students were given a pre-test measuring their knowledge of geometry (basic properties of triangles, circles, and quadrilaterals) and their attitudes towards geometry (items drawn from Linnenbrink-Garcia et al., 2010). Although a detailed analysis of these pre-measures is beyond the scope of this paper, results suggest that the students had neutral or slightly negative attitudes about geometry, rating items like "I enjoy doing geometry and "Geometry is exciting to me" on average between 2 and 3 on a 5-point scale (SDs $\approx$ 1.0). In addition, results suggest that students had somewhat strong knowledge of basic geometric properties (pre-test items included statements like "the angles of a triangle add up to 180 degrees"), scoring an average of 80% (SD = 13%) on the pre-assessment.

Analysis Techniques

The video captured while the participants played the game was transcribed using Transana (Woods & Fassnacht, 2012) in order to integrate text and video data into the analysis. These

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

transcripts and videos were then analyzed to find where the students performed collaborative gestures – gestures that were distributed in some way over multiple individuals. Transcripts from the group formulating proofs for their six conjectures were analyzed using multi-modal analysis (McNeil, 1992) of gestures. Multimodal analysis involves analyzing, interpreting, and reporting the use of gestures in conjunction with speech transcripts, in order to provide the fullest possible picture of learner reasoning. Here we employ a multiple case studies approach (Yin, 1994), since our research goal is to describe phenomena of potential theoretical importance, rather than the manipulation of a relevant behavior. Case study research recognizes that the rich context in which the interactions occur contain many variables interacting simultaneously.

Results

Through a multi-modal analysis of the focal group proving six conjectures, we discovered three types of collaborative gestures. Although we present a single group's activities, these gesture types were also present and important in subsequent work that examined 4 additional classes of students. We give a case for each gesture type. All student names are pseudonyms.

Echoing Gestures

Our first case is taken from the group proving the conjecture, "If you know the measure of all three angles of a triangle, there is only one unique triangle that can be formed with these three angle measurements." Tanya (bottom image, Figure 1) was in front of the game, with the other students, including Karen (top image, left, Figure 1), assisting her in formulating a proof.

1 Tanya: Okay, given that you know the measure of all three angles of a triangle, there is only one unique triangle that can be formed with these three angle measurements.

2 Karen: Because you can always…you have the…you can always scale it once you have those three lengths, like the three angles you scale it.

((Karen makes angles with her index fingers and thumbs and moves them in and out.))

3 Tanya: Okay, so I say false because you have certain angles, but you can bring them in and out to make the side lengths bigger or smaller.

((Tanya moves her hands up and down to show how to make the sides of the triangle bigger or smaller.))

Figure 1. Transcript of echoing gestures.

Once Tanya reads the conjecture (Line 1), Karen explains why the conjecture must be false, and uses a dynamic gesture where she formulates a triangle with her thumb and index fingers, making it grow and shrink (Line 2). Tanya seems to immediately understand and take up this gesture, repeating the gesture herself, and putting Karen's explanation into her own words (Line 3). Tanya and Karen performed echoing gestures, where one person made a dynamic gesture, and then a second person repeated that gesture while making the accompanying verbal reasoning her own. Other literature has identified *gestural catchments* as repeated similar or identical hand gestures used by a single gesturer

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

(usually an instructor) to convey similarity of or highlight important conceptual connections (McNeill & Duncan, 2000). Next, we describe a related use of gesture where one learner echoes and repeats the gestures of another learner.

Mirroring Gestures

Our second case is taken from the group proving the conjecture, "If one angle of a triangle is larger than a second angle, then the side opposite the first angle is longer than the side opposite the second." In this sequence (Figure 2), Haley, shown in the left of the images, works to formulate a proof using gestures. She first draws two angles of a triangle in the air with her fingers, and then points to the angles of the triangle (Line 4). At the same time, Karen (shown on the right, partially cut off) represents a side of the triangle with her arm, interweaving her reasoning ("and the side opposite the first…"; Line 5) into Haley's narrative proof. Haley and Karen perform identical gestures where they form equilateral-like triangles with their thumbs and forefingers (Line 6). Haley then performs a dynamic gesture where she collapses one side of this equilateral-like triangle inwards in order to vary the angle measurements and check how this impacts the side lengths (Line 7). After the transcript ends, they come to a consensus that the conjecture is true, both repeating their prior gestures as they clarify their reasoning.

4 Haley: Okay, so if one angle is larger than the second angle...
((Haley points to a second angle of a triangle that she drew in the air with her finger while Karen connects the first angle with second angle with her forearm.))
5 Karen: ...and the side opposite of the first...
6 Haley: ...and the side opposite of the first angle...
Okay, so one angle of a triangle...
((Haley and Karen both represent a triangle with both index fingers and thumbs.))
7 I am just trying to picture a triangle …is larger than the second angle.
((Haley swings the left side of the triangle to make a right triangle on the upper right side of the triangle.))
8 So it's like that then the side opposite of the first angle is longer than the side. Yeah, I would say it's true because if you're saying if this angle is here then there's more space here then this angle makes a smaller line.
((Haley points to where there is more space in her triangle.))

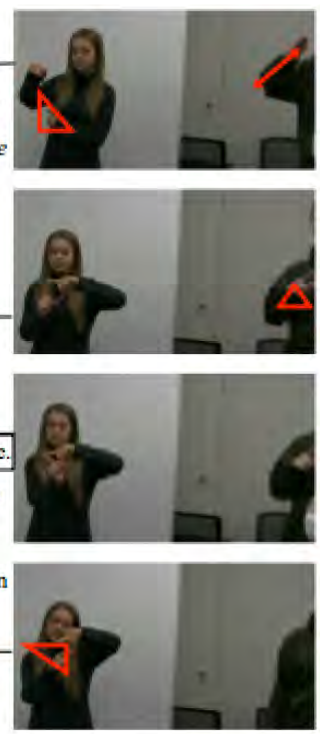


Figure 2. Transcript of mirroring gestures.

Karen and Haley performed *mirroring gestures* as they were gesturing at the same time in response to the same line of reasoning and jointly formulating a mathematical argument. In addition, at times their gestures were structurally identical. Mirroring gestures differ from echoing gestures in that they occur simultaneously – learners are using their bodies in conjunction with each other as they reason together in-the-moment. Echoing gestures, on the other hand, may capture instances where one learner's reasoning is later taken up by another learner, *after* the initial string of reasoning has been communicated and interpreted.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Haley's and Karen's gestures are representing two distinct geometric shapes, with one shape being imagined in the air in front of each of them. In our final case, we observe gestures where two learners operate on a single imagined geometric object using gestures.

Joint Gestures

Our third case is taken from the group proving the conjecture, "The measure of any central angle of a circle is twice the measure of an inscribed angle intersecting the same two endpoints on the circumference." In this sequence (Figure 3), Karen begins by trying to represent both the circle and the angles using her gestures, but struggles to properly represent the conjecture (Line 9). Stephanie misunderstands the reasoning she is communicating using this gesture (Line 10), so Karen seeks a different approach to make her thinking clear to her group. She calls upon Haley to use her hands to make the circle (Line 15), and then Karen layers her hands over Haley's circle to formulate a central angle and then an inscribed angle.

Stephanie, who is controlling the game for this conjecture, then mimics their gesture (Line 20) and agrees with their conclusion that the central angle would be smaller (Line 22). Haley questions their reasoning at two points during the discussion (Lines 19 and 23), but ultimately the group concludes that the central angle is smaller than the inscribed angle (Lines 26-27). This is a common misconception – the central angle is the large angle since it sweeps out more space.

Discussion and Implications

Situated cognition holds that cognitive behavior is embodied, embedded, and extended. An embodied cognition perspective (e.g., Lakoff & Núñez, 2000) focuses on ways body states and body-based resources shape behavior. Embedded and distributed cognition holds that cognition is mediated by the physical and social environment and the environment is used to off-load operations that could otherwise be performed mentally (Hutchins, 1995). Extended cognition takes this further, positing that social actors and the physical environment, in concert with the mind of the one doing the reasoning, constitute the cognitive system (Clark & Chalmers, 1998).

Here we identified three novel ways in which students socially coordinate hand gestures and speech that exemplify extended mathematical cognition. In echoing another's gestures, one learner makes a hand gesture representing a mathematical object, and then another learner repeats it, often making the reasoning it illustrates personally meaningful. In mirroring gestures, two learners simultaneously make the same or similar gestures with each of their set of hands, as a way of following each other's' reasoning in real time. This strategy goes beyond simply observing another's gestures – by making the same gesture, learners may better understand a collaborator's reasoning. Finally, joint gestures illustrate how multiple learners collaboratively build and manipulate mathematical objects that are too complex for one set of hands. Taken together, these findings suggest collaborative gestures have the potential to provide learners with additional tools that facilitate mathematical communication and proof...."

An interesting question for future research is how collaborative gestures influence student learning – our third case shows an ultimately unsuccessful use of collaborative gestures – and whether collaborative gestures are more effective than other tools of extended cognition (e.g., manipulatives, pencil and paper, DGS). We are of the view that there is not one *optimal* tool for learning about geometric properties and conjectures; rather that students need a variety of experiences exploring geometric ideas with different tools for cognition and collaboration. In the present paper, we argue that collaborative gesture should be one element of students' toolboxes as they learn proof in geometry. In this way, we seek to answer the question "***How can we lay the groundwork for future crossroads between theory, research, and practice?***" We use educational research to lay the groundwork for the potential importance of collaborative gestures, connecting our

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

research to theories relating to gesture and extended cognition. By studying these gestures within classrooms where students are engaged in mathematical reasoning, we begin to consider how this research might inform practice.

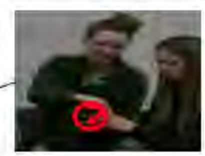
9 Karen: Well, I think of a circle and it's like the inside angle versus the outside angle and I feel like the outside angle is bigger because the inside angle would be like this whereas the outside angle is like this.
((Karen uses her fingers to draw an inside angle.))

10 Stephanie: Yeah, it would make sense. I would guess that it is true.
11 Karen: Oh no, I am saying the opposite.
12 Stephanie: Ok.
13 Karen: Right?
14 Stephanie: Oh no!
15 Karen: So like the top of the circle and like...can you hold your hands up for me? You have like this angle that you are drawing from the center, right?

((Karen has Haley make a circle with her hands and then uses the circle to draw an angle from the center.))
16 Stephanie: Mm mm.
17 Karen: That's going to be smaller than like any angle out here, isn't it?

((Karen shows an angle on the outside of the circle.))
18 Stephanie: Sure.
19 Haley: What do you mean?
20 Karen: Yea, like if we have a circle and we draw like this, that point would be…

((Stephanie is tracing the central angle of her circle, while Karen is tracing the central angle of the circle Haley created with her hands.))
21 Karen: What is it called? Then central angle...
22 Stephanie: The central angle would be smaller.
23 Haley: But it is bigger than what angle?
24 Karen: Then the inscribed angle. And the inscribed angle is what the vertex is on the circumference.



((Karen is pointing to the vertex of an inscribed angle that is on the circumference of Haley's circle.))
25 Stephanie: So the central angle is smaller than the outside angle?
26 Karen: That would make sense to me but...
27 Stephanie: I would think too.

Figure 3. Transcript of joint gestures.

References

Christou, C., Mousoulides, N., Pittalis, M., & Pitta-Pantazi, D. (2004). Proofs through exploration in dynamic geometry environments. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 215–222). Bergen, Norway.

Clark, A., & Chalmers, D. (1998). The extended mind. *Analysis*, 7-19.

Cook, S. W., Mitchell, Z., & Goldin-Meadow, S. (2008). Gesturing makes learning last. *Cognition*, *106*(2), 1047–1058.

Funiyama, N. (2000). Gestural interaction between the instructor and the learner in origami instruction. In D. McNeill *Language and gesture*, pp. 99-117.

Göksun, T., Goldin-Meadow, S., Newcombe, N., & Shipley, T. (2013). Individual differences in mental rotation: What does gesture tell us? *Cognitive processing*, *14*(2), 153–162.
Goldin-Meadow, S. (2005). *Hearing gesture: How our hands help us think*. Cambridge, MA: Harvard Press.
Goodwin, C. (1995). Seeing in depth. *Social Studies of Science*, *25*, 237–274.
Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428.
Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic bulletin & review*, *15*(3), 495–514.
Hostetter, A. B., Alibali, M. W., & Kita, S. (2007). Does sitting on your hands make you bite your tongue? The effects of gesture prohibition on speech during motor descriptions. In *Proceedings of the 29th annual meeting of the Cognitive Science Society* (pp. 1097–1102). Mawhah, NJ: Erlbaum.
Hutchins, E. (1995). *Cognition in the Wild*. Cambridge, MA: MIT Press.
Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.
Linnenbrink-Garcia, L., Durik, A., Conley, A., Barron, K., Tauer, J., Karabenick, S., & Harackiewicz, J. (2010). Measuring situational interest in academic domains. *Educational Psychological Measurement*, *70*, 647-671.
Marghetis, T., Edwards, L. D., & Núñez, R. (2014). More than mere handwaving: Gesture and embodiment in expert mathematical proof. In L. D. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 227–246). Charlotte, NC: Information Age Publishing.
Marrades, R., & Gutierrez, A. (2000). Proofs produced by secondary schools students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*, 88–125.
McNeill, D. (1992). *Hand and Mind: What Gestures Reveal about Thought*. Chicago: Chicago University Press.
McNeill, D., & Duncan, S. D. (2000). Growth points in thinking-for-speaking. In D. McNeill (Ed.). *Language and gesture: Window into thought and action* (pp. 141-161). Cambridge: Cambridge University Press.
Nathan, M. J., & Walkington, C. (2017). Grounded and embodied mathematical cognition: Promoting mathematical insight and proof using action and language. Cognitive Research: Principles and Implications, 2(1), 9.
Nathan, M.J., Wolfgram, M., Srisurichan, R., Walkington, C., & Alibali, M. (2017). Threading Mathematics Through Symbols, Sketches, Software, Silicon and Wood: Integrated STEM Instruction to Produce and Maintain Cohesion. *The Journal of Educational Research, 110*(3)*, 272-293*.
National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.
Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2012). The malleability of spatial skills: A meta-analysis of training studies. *Psychological Bulletin*, *13*(2), 352–402.
Valenzeno, L., Alibali, M. W., & Klatzky, R. (2003). Teachers' gestures facilitate students' learning: A lesson in symmetry. *Contemporary Educational Psychology*, *28*(2), 187–204.
Walkington, C., Boncoddo, R., Williams, C., Nathan, M., Alibali, M., Simon, E., & Pier, E. (2014). Being mathematical relations: Dynamic gestures support mathematical reasoning. In W. Penuel, S. A. Jurow, and K. O'Connor (Eds.), *Learning and Becoming in Practice: Proceedings of the Eleventh International Conference of the Learning Sciences* (pp. 479-486). Boulder, CO: University of Colorado.
Woods, D., and Fassnacht, C., (2012). Transana v2.52. http://transana.org. Madison, WI: The Board of Regents of the University of Wisconsin System.
Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 227–236). Reston, VA: National Council of Teachers of Mathematics.
Yin, R. (1994). *Case study research: Design and methods* (2nd Ed.). Beverly Hills, CA: Sage Publishing.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.