EXPLICATING THE CONCEPT OF CONTRAPOSITIVE EQUIVALENCE

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Abstract: This paper sets forth a concept (Simon, 2017) of contrapositive equivalence and explores some related phenomena of learning through a case study of Hugo's learning in a teaching experiment guiding the reinvention of mathematical logic. Our proposed concept of contrapositive equivalence rests upon set-based meanings for mathematical categories and negation, representing these sets by closed regions in space, and linking conditional truth to a subset relation between these regions in space. Our case study serves to portray that students must construct all of these elements to achieve a sense of necessity in the equivalence. This study thus contributes a set of learning goals for any introductory logic instruction using Euler (or Venn) diagrams, which has been little studied in the mathematics education literature.

Keywords: Reasoning and Proof, Problem Solving

Proof oriented mathematics instruction depends upon mathematical logic to ensure that students learn 1) to interpret mathematical statements the way mathematicians do and 2) to draw inferences that do not violate the mathematical community's norms. Prior research provides ample evidence that students' interpretations of mathematical conditionals (statements of the form "if…, then…") pose a number of difficulties (e.g. Durand-Guerrier, 2003) as does the logical equivalence of contrapositive (CP) statements (Stylianides, Stylianides, & Phillipou, 2004). There is little prior literature on how students are to come to learn CP equivalence (Yopp, 2017) or the meanings by which this can be a logical necessity. This paper seeks to fill this gap through a case study drawn from a larger series of teaching experiments guiding reinvention of mathematical logic through reflective use of mathematical language (Dawkins & Cook, 2016).

Logical Background

The CP of a conditional "If [P], then [Q]" is the conditional "If not [Q], then not [P]." Consider the statement, "If a triangle is obtuse, then it is not acute." Its CP is "If a triangle is acute, then it is not obtuse." Certainly these statements are both true, but how are their truths linked? In our prior studies we observe that students often reason about such statements using examples, properties, or sets (Dawkins & Cook, 2016). We encourage the reader to consider how both statements can be confirmed from the fact that any triangle has exactly one of the properties acute, right, or obtuse (a property-based strategy). Notice that so affirming both conditionals may not reveal the relationship between the two or why all conditionals with that relationship must share a truth-value. For this reason, we propose a distinction between a *CP inference* and *CP equivalence*. A student draws a CP inference when they infer a CP is true from the original conditional or when they use the original conditional to infer not [P] from not [Q] (*modus tollens*). CP equivalence instead entails students constructing a logical equivalence between any conditional and its CP rooted in generalizable meanings for conditional truth and reference.

We present our intended understanding of contrapositive equivalence in terms of Simon's (2017) explication of mathematical concepts. Simon explains, "A mathematical concept is a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (p. 7). This concept thus reflects our understanding of how a student might come to understand the necessity of CP equivalence. Simon clarified that concepts result from reflexive abstraction and thus are anticipations based on the learner's activity. This characterization

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of a concept helps distinguish our work from other relevant studies. Stenning (2002) explored logical reasoning as a broadly assessed without attending to students' meanings for particular concepts or their sense of logical necessity. Stylianides et al. (2004) assessed whether students use a CP equivalence rule they were taught when assessing arguments, and found that students frequently did not apply CP equivalence as intended. They did not study the students' meanings for conditional truth or how they entail CP equivalence's necessity. Hawthorne and Rasmussen (2014) explored students' meanings for elements of formal logic such as truth tables, and found that many learned such formalisms disjoint from their ongoing mathematical activity. They lacked necessity for the learned rules.

We articulate the concept of CP equivalence as follows:

> (Point 1) A mathematical conditional is true whenever the set of objects satisfying the *if* part is a subset of the objects satisfying the *then* part. (Point 2) These two sets can be represented as closed regions in space with points representing the mathematical objects. (Point 3) The negation of a mathematical category refers to the complement set of mathematical objects. (Point 4) Therefore, whenever a conditional is true, its CP must also be true because the complement of the larger region is contained in the complement of the smaller region.

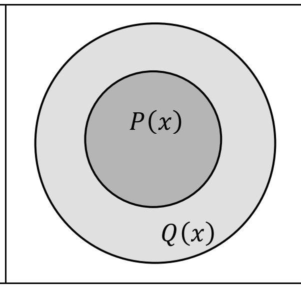
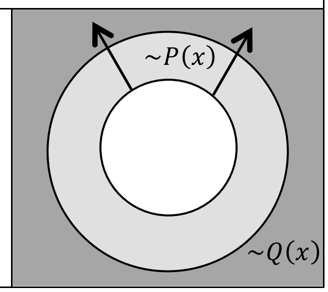
Original conditional		CP conditional	
"If for $x \in S$ $P(x)$, then $Q(x)$." is true whenever $\{x \in S \mid P(x)\} \subset \{x \in S \mid Q(x)\}$		"If for $x \in S$ not $Q(x)$, then not $P(x)$." is thus true because $\{x \in S \mid {\sim}Q(x)\} \subset \{x \in S \mid {\sim}P(x)\}$	

Figure 1. Euler diagrams portraying the subset meaning for a conditional and its CP.

Point 1) We propose the subset meaning for conditional truth as part of Dawkins and Cook's (2016) more general findings that set-based meanings were most propitious for students' reinvention of mathematical logic (over example and property-based meanings). Point 2) Logicians have long represented categories by closed regions (Stenning, 2002), but we find this is not always a natural step for participants in our reinvention. Students often prefer representations that maintain semantic meaning (e.g. the number line, example numerals, example shapes). Representing categories by closed regions reflects an abstract meaning for mathematical definition: any well-defined distinction among mathematical objects. Point 3) Dawkins and Cook (2016) explain that students' interpretations of negative categories do not always correspond to the complement set of examples, but this interpretation is necessary for interpreting CP equivalence diagrammatically. Point 4) CP equivalence is a necessary entailment of the topological relations portrayed in Figure 1. We do not claim that student understanding of CP equivalence must be mediated by visual diagrams (Yopp, 2017), but we anticipate that the necessity of CP equivalence rests upon isomorphic ways of reasoning across semantic content.

The Case of Hugo

In what follows, we shall explore the contours of our concept of CP equivalence through a case study of one student's participation in our guided reinvention teaching experiments. Hugo did not construct the concept of CP equivalence, though his interview partner did. We find Hugo's story of learning helpful because he made clear progress on diagrammatic reasoning about mathematical

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conditionals and in some way exhibited progress regarding each of the first three points of the concept. However, he still clearly lacked a sense of necessity for CP equivalence, though he observed the shared truth-values. We present this case both because it portrays the kinds of activity that we anticipate would foster students' abstraction of CP equivalence and the challenge inherent to such abstractions in mathematical logic. Furthermore, this case allows us to set forth three possible characterizations for logic learning in advanced mathematics, which we observe as an arena in need of clarification and disambiguation.

Methods

The methods of this study mirror those reported elsewhere regarding this series of teaching experiments guiding the reinvention of mathematical logic (Dawkins & Cook, 2016). Each teaching experiment involved pairs of volunteers recruited from Calculus 3 courses at a medium-sized, public university in the Midwestern United States. These students met with a teacher/researcher for 6-11 hour-long sessions. The sequence of activities consisted of presenting students with lists of statements of the same logical form (disjunctions, conditionals, and multiply-quantified) each with varied, familiar mathematical content. Students were asked to:

1. determine whether each was true or false,
2. formulate rules for when statements of the given form were true or false,
3. develop a method for negating statements, and
4. in the case of conditionals, explore the relationship between a conditional and its converse, inverse, and contrapositive.

Table 1: Sample Conditionals that Hugo and Elya Analyzed

1.	If a number is a multiple of 3, then it is a multiple of 4.
2.	If a number is a multiple of 3, then it is a multiple of 6.
3.	If a number is a multiple of 6, then it is a multiple of 3.
4.	If a number is not a multiple of 6, then it is not a multiple of 3.
5.	If a number is not a multiple of 3, then it is not a multiple of 6.
6.	If a triangle is not acute, then it is obtuse.
7.	If a triangle is obtuse, then it is not acute.
8.	If a triangle is not acute, then it is not equilateral.
10.	If a quadrilateral is a rectangle, then it is a parallelogram.
15.	If the sum of two integers x+y is even, then at least one of the numbers x and y is not odd.

The teacher/researcher generally provided minimal direct guidance besides clarifying mathematical facts about the content of each statement (e.g. 1 is not prime, a square is a rectangle), asking students to clarify explanations or compare claims about various statements, and asking partners to respond to one another's reasoning. Based on the earlier findings reported in Dawkins and Cook (2016), the teacher/researcher also explicitly guided Hugo and his partner Elya to focus on the sets of objects making each statement true or false. The interviewer attempted not to introduce any logical formalizations (i.e. notation, terminology, or diagrams) until the students seemed to recognize some relevant pattern or need to express their reasoning. All data was analyzed using the constant comparative method (Strauss & Corbin, 1999).

Results

Hugo and Elya studied conditionals during their third, fourth, and fifth experimental sessions. Elya was absent from the fourth session. During the third session, their initial task was to assign truth-values to (or *assess*) the conditionals. They assigned the same truth-values a mathematician would (*normative* truth-values). They did not exhibit set-based reasoning during this activity; the pair

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relied on examples and properties. For false claims, they recognized what constituted a counter-example to a conditional. For instance, Hugo denied statement #1 with the example 6 and explained this was sufficient for assessment: "So we came up with one case where it's false, so it is false." Hugo affirmed three statements using property-based reasoning. For instance, he reasoned about #8 saying, "not acute would mean either a right triangle or an obtuse triangle. Neither of those can be equilateral, so that would be true." In other cases when Elya affirmed a statement using properties, Hugo introduced examples. Regarding #15, Elya inferred that "one of [x and y] would have to be odd" to have an odd sum. Hugo chose 5 as the sum and considered the possible addends. It is unclear whether Hugo perceived this as a justification or simply an explanation, but it portrays Hugo's overall propensity toward example-based strategies for assessing conditionals even when Elya provided property-based explanations.

Point 1: The Subset Meaning of Conditional Truth

In the last 15 minutes of that session, the interviewer asked Hugo and Elya to consider sets.

I (1): Think about the set of all things that satisfy the *if* part and the set of all things that satisfy the *then* part. And tell me about the relationship between those two […]
H (2): I'd say, if the statement is true then the set for the first part—I'm sorry the set of the second part will be included in the set of the first part.
I (3): Okay. Why do you say that?
H (4): Um, because if we said that it's true then when we pick—something that's true for the first part, then it has to be included in the second part for the whole statement to be true.
I (5): […] So you're saying, if the statement is true then what was the relationship here?
H (6): Then the—*then* will be inside *if*.

Hugo's initial explanation (turn 2) suggests that he had not yet considered the sets of objects referred to by the categories in the given conditionals. It is possible that his reverse subset claim reflects attention to the set of properties in each statement. For instance #3 is equivalent to "If an integer is a multiple of 2 and 3, then it is a multiple of 3." The *then* property is "included" in the *if* property (turn 4), but the subset relation between the sets of integers goes the other way round.

When the interviewer asked Elya, she proposed the normative subset relation that the "*if* has to be in the *then*." She elaborated using statement #3, "all the multiples of 6 are contained in multiples of 3." The interviewer asked Hugo to respond using a particular statement.

H (7): Uh, you wanna talk about number 3. Um in like a circle, and multiples of 3—3,6,9,12. [draws a circle and inside of it writes the numbers he says aloud] Um, multiples of 6 will be included in that circle [draws smaller circle around 6 and 12]. Like 6 and 12 are multiples of 6. So there's an additional circle inside that includes some numbers but does not include others [completes the diagram in Figure 2].
I (8): Okay[…] which are you calling the *if* part and which circle are you calling the *then*?
H (9): The *then* part would be the bigger one [he labels the larger circle]. The inside would be *then*—sorry other way around. *Then* is on the outside. *If* is in [he labels the small circle].

By encircling his short list of examples, Hugo bridged his example-based representation into a set-based representation and acknowledged the normative set-based meaning for conditional truth. In this way, Hugo made initial steps toward the first point in our concept of CP equivalence, though his grasp was at times tenuous through the subsequent interviews.

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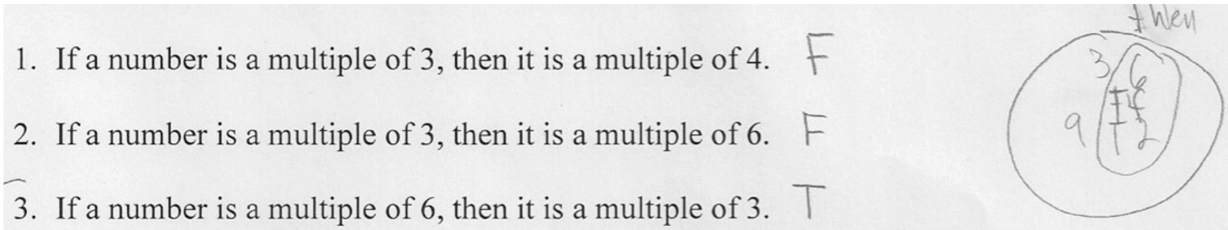

Figure 2: Hugo's first set diagram.

During this third session, Elya and Hugo recognized the syntactic relationships between conditionals traditionally known as *converse* (e.g. #6 and #7) and *inverse* (e.g. #3 and #4). They related CP statements as having undergone both transformations (e.g. #3 and #5, via #4). Using one of their subset diagrams, Elya provided an argument for why the original and CP statements should both be true in a manner compatible with Figure 1. We thus observe that she quickly and easily constructed our concept of CP equivalence from her fluency with set-based meanings and complement operations. Hugo showed little sign of following her reasoning, but he was exposed to a general explanation for why the CP must be true whenever the original statement is.

Point 2: Closed Regions and Their Topology

Elya was absent from the fourth session allowing Hugo to explore his understanding of conditionals and sets independently. Early in the session, Hugo considered statement #6. He appropriately explained that it was false because it failed the normative subset relation, "We said it was false 'cause our first set included… right triangles. So then it was only asking if it was only obtuse. So it could have been a right triangle or obtuse. Not just obtuse." Hugo went on to produce two diagrams to express his understanding of the two sets (Figure 3). The first reflected a traditional Venn diagram arrangement (with "O" standing for "obtuse"). As he unpacked the properties in the statement, he revised his diagram, "This is the *if*, "not acute" we said that could be a 90 triangle or obtuse. And then the *then* was, 'it is obtuse' so I guess that would be—this. So it's a little different than what I originally drew."

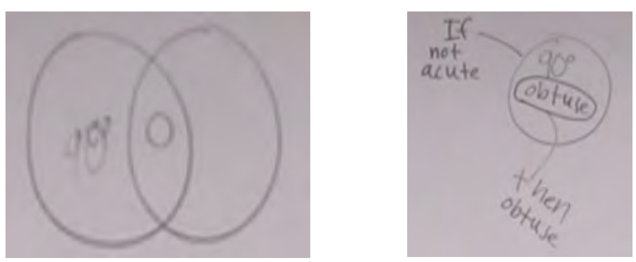

Figure 3: Hugo's two diagrams for statement #6.

Hugo recognized that one region of the Venn diagram did not contain any triangles and represented that in the topology of the two regions in his second, Euler diagram. In this instance, Hugo seemed to clearly make progress regarding the second point in our concept of CP equivalence by using closed regions to represent sets and using their topology to relate those sets.

We conjecture that Hugo's property-based reasoning the previous day influenced his diagram construction. He let the properties stand for the entire category without recourse to representative examples (as in his diagram produced the previous day). It is unclear the extent to which Hugo imagined the curves as encasing the sets of triangles imagined as points or whether they encased the words themselves that stood for the examples. One cause for questioning Hugo's interpretation of the circles and their reference arose when Hugo next considered statement #7. Though statement #7

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contains the same categories, Hugo produced a completely new diagram like the second in Figure 3, except the *if* and *then* labels were reversed. The interviewer asked Hugo to compare the two diagrams, and, upon reflection, he said they were the same. He clearly did not anticipate this relationship.

Point 3: The Negation/Complement Relation

The interviewer invited Hugo to write the inverse statements to both #6 and #7 on the board. He then asked Hugo to assess these statements using the same diagram he produced for #6. Hugo considered #6's inverse, written "If acute, then not obtuse" in the following way:

> So "if acute" then we'd be talking about anything outside of the *if* circle, so everything outside of here, then "it is not obtuse"—right. 'Cause you're not—we're talking about everything except inside this circle. And obtuse is inside the circle. So that'd be true. We said this [statement #6] was false. So it was the opposite, or the negation.

Here, Hugo displayed two novel developments in his thinking. First, he associated the negation of a category with the complement of a closed region. Specifically, the region outside the larger circle represented acute triangles. Once again, we cannot be sure whether this inference was supported by 1) Hugo's knowledge that any triangle is exactly one of acute, right, or obtuse or 2) reasoning about the representational structure of the diagram. In either case, he used the *negation/complement relation*. Secondly, he did not affirm the inverse of #6 by the subset meaning, but rather notes that anything outside the large circle is not inside the small circle ("if *acute*, then not *obtuse*"). We call this the *empty intersection meaning for conditional truth.* This criterion is distinct from the subset meaning Elya used during the previous session, but formally equivalent to it. It depends upon the presence of *not* in the latter half of the conditional.

At the end of the previous quote, Hugo noted that the inverse statements had opposite truth-values, and anticipated this might be the case more generally. To explore this conjecture, the interviewer next asked Hugo about statement #7 and its inverse:

H (10): My guess is that it would be false 'cause it's—it'd be the opposite, but "if not obtuse," so anything that's outside of this little circle—then it is acute. That's not necessarily true because—that would—we still have 90 degree—triangles that are not obtuse but are still not acute. So that would still be false. Or that would be false.

I (11): Okay, now you anticipated it would be false. What was your basis for anticipating that it would be false?

H (12): That one. We took the inverse of this one, we got the opposite—the opposite truth-value.

I (13): […] What about the picture tells me which two [of the four statements] are true? […]

H (14): That if we limited it to this inner circle—the obtuse triangles. Then obviously we would not be talking about if it was outside of the circle. We're only talking about the inside

I (15): What about then this one [inverse of #6]? How can I see it in the picture, this one?

H (16): That if we're talking about anything outside of this circle, the bigger circle here, which would be all the acute. Then that would exclude anything inside the circle—obtuse triangles are only inside the circle. So then we would only be talking about the area out here.

In turn 10, Hugo denied the inverse of #7 because the non-obtuse triangles were not all acute. Hugo associated the negation of *obtuse* with the complement of the smaller region. He identified acute triangles (outside the larger circle) as counterexample to the conditional. Hugo noted that this example also affirmed his conjecture that inverse conditionals have opposite truth-values.

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The interviewer noticed that Hugo was fluidly shifting between subset, empty-intersection, and counterexample meanings to assess the given conditionals. So, in turn 13 he asked Hugo to consider more generally which of the four statements (#6, #7, and their inverses) were true and how the diagram represented this. In turns 14 and 16, Hugo affirmed #7 and its CP both using the empty intersection meaning: the inner circle and outer complement were mutually exclusive. While this observation could provide a general sense of symmetry regarding the truth conditions for some CP statements, there was no evidence of Hugo abstracting this relationship at this point.

Point 4: CP Equivalence

To help Hugo see the syntactic relationship between the true statements he affirmed in the last interchange, the interviewer asked Hugo to specify all of the syntactic relationships among the four statements. Figure 4 shows the results of their discussion (Hugo and his partner used the term "switch" for converses). Using this diagram, Hugo noted that CP statements had the same truth-value: "if we have an *if-then* statement that's true, we take the inverse and the switch […] so far we've proved that it would be true [...] Well I'm observing but I'm trying to articulate why that is." He admitted that this was for him an empirical observation and he could not justify it.

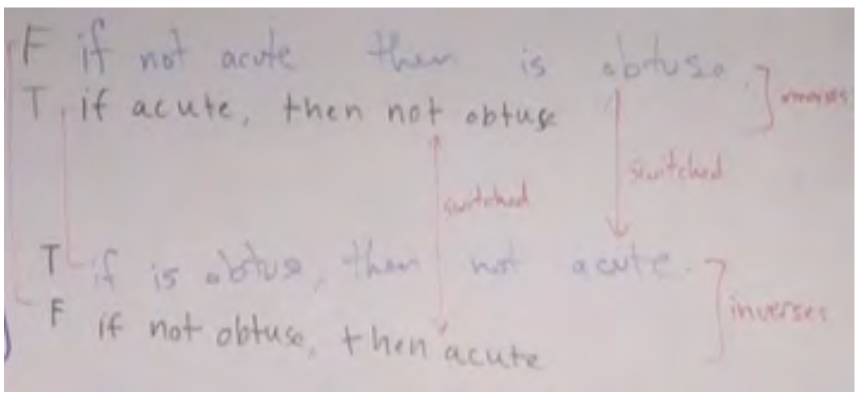

Figure 4: Exploring the syntactic relationship among a conditional quartet.

Not only did Hugo fail to see a general justification for CP equivalence, he was very inconsistent in his use of subset diagrams to assess conditionals about other topics. When asked to discuss the sets associated with statement #10 and its CP later in that session, Hugo drew separate and isolated circles above the words "rectangle" and "parallelogram" in #10. He recognized that all rectangles were parallelograms, but he did not use the topology of the regions to represent this. With prompting, he modified these diagrams to match the previous subset diagrams. Regarding the CP, Hugo began new circles rather than using the complements of the regions drawn for #10. Throughout the rest of that interview and the next, Hugo went on to consider at least three other quartets of conditional, inverse, converse, and contrapositive. Once prompted to produce a subset diagram, he consistently 1) affirmed the base conditional by the subset meaning, 2) denied the inverse and converse by counterexample or by failing to have a subset relation, and 3) affirmed the contrapositive by the empty-intersection meaning. However, he did not routinize creating such diagrams without prompting or begin anticipating the topological relations that would affirm a conditional and its CP. In short he did not construct the concept in such a way as to produce a sense of logical necessity for the shared truth-values.

Conclusions

Our goals in this paper were to 1) set forth our concept (Simon, 2017) of CP equivalence, 2) portray mathematical activity by which this concept could develop, and 3) convey the challenge these abstractions pose through Hugo's learning process. We claim that Hugo made progress regarding each of the first three points in the concept and empirically observed point 4, but did not perceive point 4 as a logical necessity. We presented evidence that Hugo understood that conditionals could be affirmed via a subset relation (Point 1). It is unclear whether this relationship was universal and reversible, or simply a sufficient condition. Hugo was able to represent the relationships between the categories in conditionals using closed regions and their topology (Point 2). At times he used these diagrams flexibly, as when he created the empty intersection meaning. He did not see such diagrams as a universal tool, judging by his alternating strategies and representations. He produced diagrams with different referential structures and often bypassed reasoning with the diagram by resorting to property-based inferences. While we appreciate that Hugo consistently connected his representation to the relevant mathematical categories (cf. Hawthorne & Rasmussen, 2014), he did not consistently use the diagram to draw new inferences about the mathematical categories. Hugo at times associated the negation of a category with the complement of either region in a diagram (Point 3), but he never coordinated two such complement regions simultaneously (as implied in Figure 1). Thus regarding each point of the concept, we see why Hugo's understanding did not support reflexive abstraction.

We intend for this analysis to emphasize the difficulty and nuance involved in constructing logical necessity in diagrammatic reasoning in the course of semantically-rich mathematical activity. We also propose that further literature on logic learning should clearly distinguish the kinds of understanding they intend. We propose three categories. *Reading* involves assessing mathematical statements in normative ways and drawing normative inferences. Hugo did this throughout. *Reflecting* involves finding general representations and criteria for assessing mathematical statements, such as Euler diagrams and the subset criterion. Hugo began this during the study. *Abstracting* involves reflexive abstractions yielding new insights from these representations and criterion. Elya, but not Hugo, displayed this kind of learning regarding CP equivalence. We anticipate that elaboration of these categories will facilitate future investigation.

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