AN ANALYSIS OF STUDENTS' MISTAKES ON ROUTINE SLOPE TASKS

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This study extends past research on students' understanding of slope by analyzing college students' mistakes on routine tasks involving slope. We conduct quantitative and qualitative analysis of students' mistakes to extract information regarding slope conceptualizations described in prior research. Results delineate procedural proficiencies and conceptual underpinnings related to various slope conceptualizations that can help both teachers and researchers pinpoint students' understanding and make appropriate instructional decisions to help students advance their understanding.

Keywords: Algebra and Algebraic Thinking

Functions play a crucial role throughout the mathematics curriculum. The concept of slope is critical to the study of linear functions in beginning algebra and extends to describe non-linear functions in advanced algebra (Nagle & Moore-Russo, 2014), the line of best fit in statistics (Casey & Nagle, 2016), and the concept of a derivative in calculus (Stanton & Moore-Russo, 2012). Research has documented students' difficulties with interpreting slope in both functional and physical situations (Simon & Blume, 1994) and with transferring knowledge of slope between problem types (Planinic, Milin-Sipus, Kati, Susac & Ivanjek, 2012). Moore-Russo and her colleagues (Moore-Russo, Conner & Rugg, 2011) have refined and extended the conceptualizations of slope Stump (2001) offered, resulting in 11 conceptualizations which have been documented among secondary and post-secondary students and instructors (Nagle & Moore-Russo, 2013; Nagle, Moore-Russo, Viglietti & Martin, 2013). Procedural knowledge of slope is also important; students need a comprehensive knowledge of a procedure, along with an ability to make critical judgments about which procedure is appropriate for use in a particular situation (National Research Council, 2001).

In the case of slope, procedural knowledge includes familiarity with the symbols typically used in relation to it and the rules used to calculate it (e.g., m, $\frac{rise}{run}$, $\frac{y_2-y_1}{x_2-x_1}$) (Nagle & Moore-Russo, 2013). Conceptual knowledge enables students to make connections between the various notions of slope and to explain why particular procedures for calculating slope work. In a recent study of eleventh grade students' interconnected use of conceptual knowledge and procedural skills in algebra, Egodawatte and Stoilescu (2015) used error analysis to show how prevalent procedural errors sometimes indicated weak conceptual understanding. As described earlier, research has documented students' weak conceptual understanding of slope. However, findings that many students confuse *rise over run* and *run over rise* in the formula for slope and are unsure of the procedure to find a perpendicular line's slope also suggest that students may lack procedural knowledge of slope as well (Stump, 1999).

Since slope is the constant rate of change of two linearly related variables, it is important to consider how students apply covariational reasoning as they conceptualize slope. Described as the "mental coordination of two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen & Hsu, 2002, p. 354), covariational reasoning has been identified as a key prerequisite for advanced mathematical thinking (Carlson, Oehrtman & Engelke, 2010). Carlson and colleagues (2002) describe five developmental stages of covariational reasoning. The first three stages, namely L1 Coordination, L2 Direction, and L3 Quantitative Coordination, are foundational for students' thinking about slope (Casey & Nagle, 2016).

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The Present Study

Past research on slope has described the multitude of ways which students might conceptualize it and described students' limited proficiency. However, these areas of research have not been merged. In particular, past research has not engaged in error analysis of students' solutions on common slope tasks to extract information regarding students' procedural and conceptual knowledge of the various slope conceptualizations. We conduct quantitative analysis of students' solutions to routine slope tasks in order to delineate procedural proficiencies and conceptual underpinnings that can be attributed to those mistakes. We link these to the previously identified slope conceptualizations to provide insight into the procedural and conceptual knowledge underlying each notion of slope. The research questions are:

1. What mistakes did students make when solving the various slope tasks?
2. Which tasks did students have the most trouble with and what mistakes were most prevalent?
3. What do students' mistakes reveal about procedural proficiencies and conceptual understanding of different slope conceptualizations?

Methods

Participants and Assessment

Participants in this study were primarily college freshmen and sophomores at a single four-year college in the Northeastern region of the United States. Seven mathematics instructors representing 13 sections of Quantitative Reasoning (Elementary Algebra), Algebraic Problem Solving (College Algebra or Intermediate Algebra), and Precalculus agreed to administer the slope assessment to their students during class time. The assessment was administered during the second half of the semester, after slope was taught. In all, 256 students completed the assessment with fairly even distribution among the three courses: Quantitative Reasoning (QR, n = 79), Algebraic Problem Solving (APS, n = 94), and Precalculus (Precalc, n = 83). The researchers developed a 15-question assessment containing standard slope questions similar to those that students solved on homework and exams. The 15 questions belonged to six broad categories: (1) write an equation of a line given particular information, (2) write the equation of a line given its graph, (3) write the equation of a line given its graph and interpret in terms of a real problem situation, (4) use a table of values to write a linear equation, (5) determine whether graphs of two equations are parallel, perpendicular, or neither, and (6) sketch a line given particular information. One sample problem from each category, with an actual student response, is provided in Figure 1. The fifteen-item assessment included questions that called on nine of the eleven slope conceptualizations described by Moore-Russo, Connor, and Rugg (2011) as shown in Figure 1. Only the Trigonometric and Calculus conceptions of slope (Moore-Russo et al., 2011) were not reflected in the items included on the assignment.

Data Analysis

Coding began with one researcher grading all responses using a four-point scale: 4 points for a completely correct answer, 3 points for a mostly correct answer, 2 points for a half correct answer, 1 point for a partially (less than half) correct answer, and 0 points for a blank or nonsense answer. Next, the researchers used grounded theory (Glaser & Strauss, 1967) to code students' solutions for mistakes. For every answer that did not receive a perfect score, the researchers analyzed the students' solution to determine what mistake(s) were made. We define a mistake as a wrong action or inaccuracy or lack of action that was demonstrated in the problem solution. We recognize that the same mistake may stem from different sources of misunderstanding and we do not distinguish between these when coding for mistakes. Based on the students' solutions, we generated a list of possible mistakes. When a new solution suggested the need for an additional mistake category, the

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code was added to the list and all responses were revisited in light of the revised list. After generating a list of possible codes, one researcher revisited all student work and completed the coding according to the list of mistakes.

Results

Classifying Mistakes

In total, 18 mistake categories emerged from the grounded theory approach to coding students' solutions on the slope tasks. Table 1 provides a description of all such mistakes and indicates the assessment question(s) on which the mistake was made as well as the frequency of the mistake across all students and questions (n = 3840). Figure 1 illustrates sample responses with the corresponding mistake codes and overall item score (out of 4 points) assigned to the response.

Table 1: Mistake Codes, Related Questions, and Frequency

Code #	Abbreviation code	Description of Mistake	Related Questions	Frequency
1	*NoResponse*	No response or nonsense answer	All questions	496
2	*Arithmetic*	Any type of addition, subtraction, multiplication, or division mistake	All except 14	310
3	*SimpleFraction**	Not changing a fraction to the simplest form	All except 1, 3, 13, 14	128
4	*NoXvariable*	Don't put the x variable after the slope in the equation	All except 6, 11, 12, 13, 14	54
5	*SlopeRunRise*	Calculating a slope as run/rise instead of rise/run	2, 5, 6, 7, 8, 9, 10	57
6	*CoordiPoints*	Calculating $\frac{y_2-y_1}{x_1-x_2}$, hence getting the opposite of the actual slope.	2, 5, 6, 7, 8, 9, 10	17
7	*SubtractCoord*	Calculating $\frac{y_2-x_2}{y_1-x_1}$	2, 8, 9, 10	8
8	*OppSignSlope*	Putting a negative sign for an increasing line's slope or vice versa	5, 6, 7, 8	95
9	*BlockSlope*	Using blocks instead of axis' units to calculate a slope	5, 6	94
10	*MentalAction1*	Does not coordinate the *value* of one variable with changes in the other variable	7, 8	32
11	*MentalAction2*	Does not coordinate the *direction* of change in one variable with changes in the other variable	7, 8	30
12	*MentalAction3*	Does not coordinate the *amount* of change in one variable with changes in the other variable	7, 8	118
13	*CalcYintercept*	Don't know how to calculate the y-intercept with many non-routine points	9, 10	101
14	*NoSlopeInter*	Not revising a standard form to a slope-intercept form when using the coefficient of x as the slope	11, 12	55
15	*GraphOpposite*	Graphing opposite direction with a given slope	13, 14, 15	73
16	*PlotXYchange*	Plotting a point using x-coordinate value as a y-coordinate and vice versa	13, 14, 15	29
17	*NoOppPerp*	Using reciprocal but not opposite slope to apply to the perpendicular line's slope	4	32
18	*NoRecPerp*	Using same slope to apply to the perpendicular line's slope or just put opposite sign	4	29

*We recorded this as a "mistake" to track the frequency of its occurrence, but students were not penalized when a fraction was not written in simplest form.

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Overall Performance on Slope Problems

The mean percentage on the assessment for all 256 students was 65.66%, with APS students scoring highest (66.76%), Precalc students scoring in the middle (65.13%), and QR students scoring lowest (64.92%). A single factor ANOVA showed no significant difference on overall percentage based on the students' course of enrollment [F (2, 253) = 0.15, p = 0.86 >> 0.05]. It is interesting that not only did Precalc students not score significantly higher than students in the more basic Algebraic Problem Solving and QR courses, but they actually scored lower in overall percentage (albeit not statistically significant) compared with the APS students.

Questions with Lowest Average Percentage Scores

Across the 15 questions, the four lowest average percentage scores were Questions 10 (45.4%), 4 (54%), 7 (55%), and 8 (55.5%). Figure 1 illustrates sample responses highlighting typical mistakes for these four questions. Despite being a standard task, students scored the lowest on Question 10. Many students made a mistake when coordinating points in the slope formula, resulting in a positive slope instead of a negative slope. Question 4 had the next lowest average score. The sample response to Question 4 (see Figure 1) illustrates the common mistake of calculating the y-intercept before finding the perpendicular line's slope. Although this solution uses the negative reciprocal slope of -2/3 in the final slope-intercept form of the equation, notice that the original slope of 3/2 was used when calculating the slope-intercept of the perpendicular line. The variable x is also omitted from the slope-intercept form of the equation. Questions 7 and 8 both required students to write an equation (given a graph) and interpret the equation in light of the real world context that was provided. These items, and their common responses, are discussed in the next section.

Covariational Reasoning and Overall Performance

Students' challenges on Questions 7 and 8 generally related to interpreting the equation in terms of the problem situation. The codes *MentalAction1*, *MentalAction2*, and *MentalAction3* emerged from students' difficulties interpreting the slope of this linear equation in context. A code of *MentalAction1* indicated that a student did not demonstrate knowledge of the two covarying quantities (L1 Coordination). This was often seen in responses that considered only a single variable changing. A code of *MentalAction2* indicates that a student *did* demonstrate L1 covariational reasoning but either did not attempt or made errors in L2 Direction covariational reasoning. This generally appeared when students described the direction of change incorrectly (e.g., "the value of the HDTV increases as the number of month increases"). The *MentalAction3* code indicates that a student demonstrated both L1 and L2 covariational reasoning but either did not attempt or made an error when reasoning using L3 Quantitative Coordination covariational reasoning. Generally, this code indicated that a student did not attend to the amount of change (e.g., "the value of the HDTV decreases over time") or did not correctly interpret the slope as a ratio of change in y variable over unit change in x variable. We conducted additional analysis of how students' covariational reasoning levels were related to their overall performance on the slope tasks. Students who exhibited higher levels of covariational reasoning scored higher on the slope assessment as a whole. Demonstrating fluency with L3 covariational reasoning on both Question 7 and 8 was correlated with a higher overall score on the slope assessment (r = 0.294). Fluency with L2 reasoning was also positively correlated with overall score (r = 0.203).

Category 1: Write an equation of a line given particular information.
Question 4. (Slope Conceptualizations: Parametric Coefficient, Determining Property)

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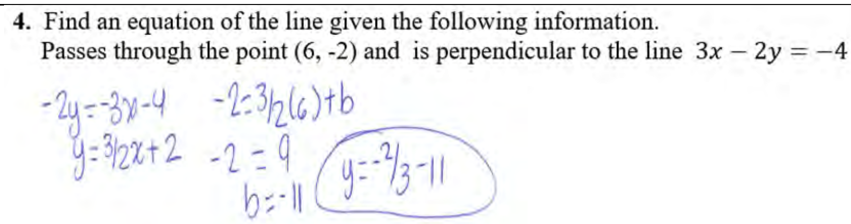


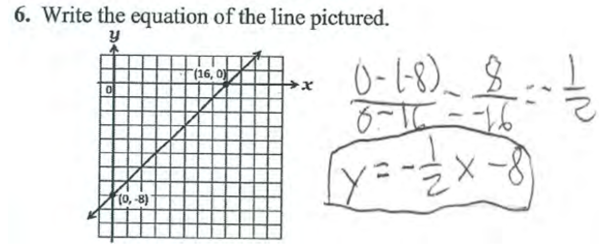
4. Find an equation of the line given the following information.
Passes through the point (6, -2) and is perpendicular to the line $3x - 2y = -4$

$-2y=-3x-4$ $\quad$ $-2=3/2(6)+b$
$y=3/2x+2$ $\quad$ $-2=9$
$b=-11$
$y=-2/3-11$

Response Coding: *NoRecPerp, NoXvariable* (Score 1)

Category 2: Write the equation of a line given its graph.

Question 6. (Slope Conceptualizations: Algebraic Ratio, Geometric Ratio, Parametric Coefficient)



6. Write the equation of the line pictured.

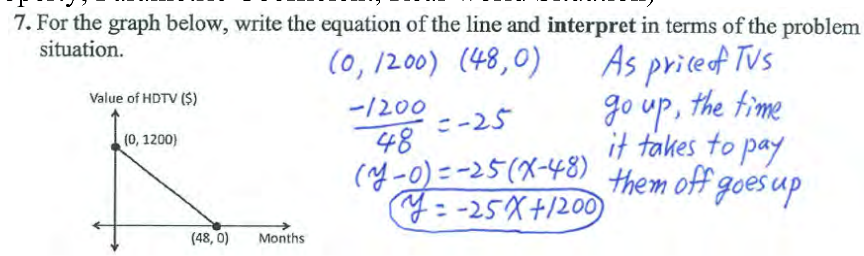
$\frac{0-(-8)}{0-16} = \frac{8}{-16} = -\frac{1}{2}$

$y = -\frac{1}{2}x - 8$

Response Coding: *Arithmetic, CoordiPoints, OppSignSlope* (Score 2)

Category 3: Write the equation of a line given its graph *and* interpret it in the problem situation.

Question 7. (Slope Conceptualizations: Algebraic Ratio, Physical Property, Functional Property, Parametric Coefficient, Real-world Situation)



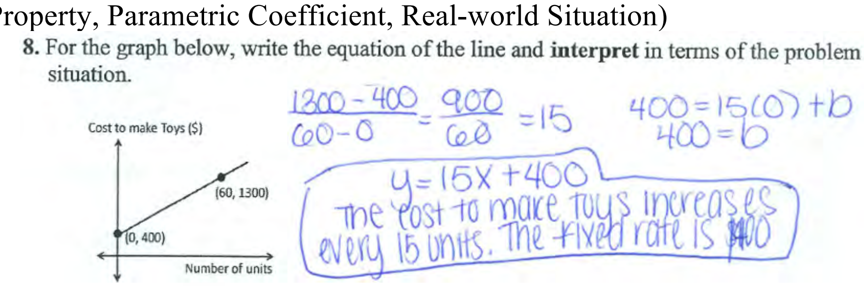
7. For the graph below, write the equation of the line and **interpret** in terms of the problem situation.

(0, 1200) (48, 0)
$\frac{-1200}{48} = -25$
$(y-0) = -25(x-48)$
$y = -25x+1200$

As price of TVs go up, the time it takes to pay them off goes up

Response Coding: *MentalAction2* (Score 2)

Question 8. (Slope Conceptualizations: Algebraic Ratio, Physical Property, Functional Property, Parametric Coefficient, Real-world Situation)



8. For the graph below, write the equation of the line and **interpret** in terms of the problem situation.

$\frac{1300-400}{60-0} = \frac{900}{60} = 15$ $\quad$ $400=15(0)+b$
$400=b$
$y=15x+400$
The cost to make toys increases every 15 units. The fixed rate is $400

Response Coding: *MentalAction3* (Score 3)

Category 4: Use a table of values to write a linear equation.

Question 10. (Slope Conceptualizations: Algebraic Ratio, Parametric Coefficient, Linear Constant)

10. Use the data in the table to write a linear function equation.

x	*y*
-6	22
3	1
-9	29
-12	36

(-6, 22) (3, 1)

$\frac{22-1}{3+6} = \frac{21}{9}$

Response Coding: *CoordiPoints, SimpleFraction* (Score 1)

Category 5: Determine whether graphs of two equations are parallel, perpendicular, or neither.

Question 11. (Slope Conceptualizations: Parametric Coefficient, Determining Property)

11. Determine whether the graphs of the following equations are parallel, perpendicular, or neither.

$2x - y = -6$
$4x - 2y = 12$

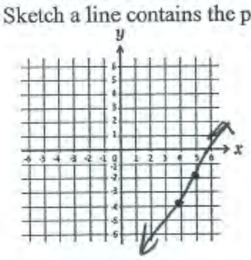
neither

Response Coding: *NoSlopeInter* (Score 0)

Category 6: Sketch a line given information.

Question 13. (Slope Conceptualizations: Geometric Ratio, Behavior Indicator)

13. Sketch a line contains the point (-2, 5) and has slope -2



Response Coding: *GraphOpposite*, *PlotXYchange* (Score 0)

Figure 1. Sample assessment items with anticipated slope conceptualizations, actual student response, and resulting codes.

Discussion

Our study of students' mistakes on routine slope tasks has built on previous literature by analyzing particular mistakes that may hinder students' abilities to reason successfully with the various slope conceptualizations. A total of 18 mistake categories emerged from the grounded theory approach to coding students' solutions. The mistakes indicate that there are many procedural proficiencies required for students to work with the various slope conceptualizations. Arithmetic mistakes were the most widespread mistakes regardless of a student's class of enrollment. These errors carried over into algebraic manipulation with many students making mistakes when adding or subtracting a variable term to the other side of the equation or dividing by the coefficient of the *x*-term when converting from standard to slope-intercept form. This is a reminder that even when a student has a strong conceptual grasp, a lack of procedural proficiency may hinder his or her ability to reason successfully on slope tasks.

Procedural Proficiencies and Conceptual Underpinnings of Slope Conceptualizations

Past research has focused on describing the many different conceptions of slope. Our analysis in this paper does not attempt to distinguish between a student's procedural and conceptual understanding of slope. However, by analyzing the mistakes students made on problems related to each slope conceptualization, we were able to develop a preliminary list of the underlying procedural proficiencies and conceptual underpinnings that may have been linked with the mistakes we saw on the assessment. Next, by linking the mistakes with the slope conceptualizations each problem illustrated, we were able to make a preliminary list of the procedural proficiencies and conceptual underpinnings which may be linked to the various slope conceptualizations (see Table 2). This is an important step which allows teachers and researchers to begin to break down the underlying ideas and practices that are necessary for a student to work fluidly with a particular notion of slope.

Table 2: Procedural Proficiencies and Conceptual Underpinnings for Each Category

Category	Procedural Proficiencies	Conceptual Underpinnings
Geometric ratio	Count "units" for vertical change. Count "units" for horizontal change. Attach a sign to indicate direction (up or right is positive, down or left is negative).	Rise and run are oriented (signed). Units are determined by graph increments (not blocks). The "rise over run" ratio and "run over rise" ratio are reciprocals.
Algebraic ratio	Subtract y-coordinates for change in y. Subtract x-coordinates for change in x.	"Change" is oriented (signed). The "change in y over change in x" and "change in x over change in y" ratios are reciprocals.
Functional property	Interchange the word slope with the phrase "rate of change".	Slope describes the coordinated change of two covarying quantities.
Parametric coefficient	Algebraically manipulate an equation into slope-intercept form or point-slope form. Identify the coefficient m of x.	The coefficient of x reveals different information depending on the form of the linear equation.
Real-world situation	Identify the real-world quantity associated with the input and output variable (using any type of representation).	Interpret change as it relates to a real-world variable (i.e., a decrease in price shows depreciation over time).
Determining property	Calculate the negative reciprocal. Recognize that equal slopes indicate two lines are parallel. Recognize that negative reciprocal slopes indicate two lines are perpendicular.	Slope indicates the number of points shared by two linear relationships and how they intersect (if at all).
Behavior indicator	Visually determine if a line increases/ decreases.	An increasing (decreasing) relationship is one in which the variables change in the same (opposite) direction. MA2: A positive rate of change indicates two variables change in the same direction.
Linear constant	Choose any two points on a graph/in a table when given multiple points.	Slope is independent of the points chosen since the ratio of change between the dependent and independent variables is constant.
Physical property	Visually recognize a line's "steepness".	MA3: The rate of change indicates the amount of change in the dependent variable per unit change in the independent variable.

Future research should analyze the pattern of student mistakes to better understand whether procedural proficiency or conceptual grounding may be the root of the mistake. In particular, a simple isolated incident may mean a student made a procedural slip while repetition of a mistake across problem types and representations may indicate deep-rooted conceptual misunderstandings (Egodawatte & Stoilescu, 2015).

Slope Questions for Instruction

The questions on which students had the most difficulty can also provide important insight for teachers. Results suggest that teachers should consider including tables with x-values that have varying increments and which are non-monotonic. This is supported by students' difficulties with Question 10, a seemingly standard question other than the lack of a pattern in the x-coordinates provided in the table. Students' difficulties with Questions 7 and 8 highlight the need for teachers to

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link the *Algebraic* and *Geometric Ratio* conceptualizations with the *Functional Property* idea of slope as a rate of change of two covarying quantities. Many students struggled on these examples because although they were able to explain that the two variables changed together, many even describing the corresponding directions of change in the variables, they struggled to interpret the slope as the amount of change in the dependent variable per a unit change in the independent variable. Thus, our results remind teachers that L3 covariational reasoning is a conceptual underpinning that helps to link the *Functional Property* conception of slope as the rate of change of two variables with *Behavior Indicator* and *Physical Property* conceptions of slope that focus on the direction and magnitude of change, respectively.

Future Study

Our work has described procedural fluencies and conceptual underpinnings related to nine slope conceptualizations. Future work should repeat error analysis with more diverse pool of students to see whether other mistakes emerge. Future studies could also investigate the patterns of student mistakes over multiple items to analyze whether they indicate procedural errors or more foundational conceptual misunderstandings using the framework we have described.

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