ROLE OF REPRESENTATION IN PROSPECTIVE TEACHERS' FRACTIONS SCHEMES

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This research report explores relationships between fractions' task representations (discrete, rectangular, or circular) and elementary prospective teachers' (PTs) fractions conceptions. Studies show PTs' conceptions of fractions are centered on a part-whole understanding, which may be problematic when teaching children about improper fractions. We studied PTs' conceptions of fractions using a task-based written assessment. The assessment also included PTs' rankings of task difficulty. We found that PTs' responses involving circles representations aligned best with the empirical trajectory of children's developing understandings of fractions. We discuss implications for supporting PTs in conceptualizing fractions as measures.

Keywords: Rational Numbers, Teacher Education-Preservice, Teacher Knowledge

Introduction

Although the conception of a fraction as a *measure* (Lamon, 2007) is emphasized early in the Common Core State Standards (CCSSM, 2010), many middle grades students and prospective teachers (PTs) remain focused on *part-whole* meanings of fractions (Newton, 2008; Norton & Wilkins, 2010; Olanoff, Lo, & Tobias, 2014). In this research project, we build off the learning trajectory for PTs' understandings of fractions as measures described by Lovin, Stevens, Siegfried, Norton, & Wilkins (2016). Our study aligns well with the PME-NA Conference Theme of "Synergy at the Crossroads" because our approach involves looking across both fraction representation and fraction task structure to better understand the role of each in understanding and supporting PTs' understanding of fractions.

Background and Purpose

Norton and Wilkins (2010) designed written assessments of middle grades students' fractions' schemes using rectangular and circular representations of fractions. A subset of these (rectangular) items were validated via clinical interviews with sixth-grade students (Wilkins, Norton, & Boyce, 2013). Lovin et al. (2016) used the written items to assess the fractions schemes of PTs before and after instruction in mathematics courses for elementary PTs. In examining PTs' written responses, Lovin et al. (2016) noted that PTs often set up proportions and used division to correctly solve fractions tasks in ways that would not be available to elementary or middle grades students. They mentioned that PTs' use of such procedures to find equivalent ratios may have been confounding researchers' assessments of some of the PTs' fractions understandings (potentially producing both false positives and false negatives). This could lead to difficulty in assessing and supporting PTs' learning, particularly for fostering PTs' self-monitoring of their understanding of the mathematical goals and constraints that their prospective students may face.

In this paper, we report on results of modifying Norton and Wilkins' (2010) items and methods to explore relationships between the form of fractions' task representations (discrete, rectangular, or circular) and elementary PTs' fractions conceptions prior to instruction in their college course. Our aim in this study is to understand how PTs' ways of operating with fractions in different representations are connected, so that we, as mathematics educators, can plan to introduce and moderate perturbation (von Glasersfeld, 1995) that will help them to construct more powerful fractions schemes. With this aim, we modified the assessment approach of Norton and Wilkins (2010) and Lovin et al. (2016) to isolate differences in PTs' responses to fractions tasks.

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Theoretical Framework

We adopt a radical constructivist epistemology in modeling PTs' fractions meanings as the product of their organizing mental structures (schemes) to fit their experiences (von Glasersfeld, 1995). The construct of scheme refers generally to the way researchers model how individuals operate mentally in service of a goal. A scheme consists of three parts – recognition of a situation, operations (mental actions), and an expected outcome. Individuals' schemes become established as they become refined and generalized through their use, via processes of assimilation and accommodation (Piaget, 1970). When a scheme is interiorized, the situation, operations, and anticipated result of operating are experienced altogether as a unified and connected structure (a concept) that can itself be operated upon (Piaget, 1970).

Fractions Schemes

We focus on four specific schemes pertaining to fractions identified by Steffe and Olive (2010) – the parts-out-of-wholes fraction scheme (PWS), the partitive unit fraction scheme (PUFS), the general partitive fraction scheme (PFS), and the iterative fraction scheme (IFS). The PWS involves *partitioning* a whole into discrete pieces that can be disembedded (removed from the whole without modifying the whole) and double-counted to form a numerosity of part(s) within a numerosity of a whole. The PUFS builds upon the PWS as the individual conceives of the *size* of a disembedded part and its relation to the size of the whole (i.e., that *iterating* the amount of $1/n$ n times results in the size of 1). The PFS extends this notion to the size of a composite (but proper) fraction. An individual with an iterative fraction scheme (IFS) understands the size of an (im)proper fraction (m/n) as the result of coordinating mental operations to include partitioning the size of '1', disembedding a unit fractional size ($1/n$), and iterating the disembedded fractional unit m times.

In order for an individual's fraction scheme to become interiorized as a fraction concept, his or her fraction scheme must be *reversible*. For instance, an individual with a *reversible* IFS could reverse his or her ways of operating to determine the size of '1' from a given improper fraction size. Reversing the PFS involves forming the size of '1' from a given (composite) proper fraction size, and reversing the PUFS involves forming the size of '1' from a given unit fraction size. Reversing the PWS involves forming the numerosity of the whole from a given proper fraction (e.g., reasoning that if three parts represents the fraction 3/7, then the whole must be 7 parts). In the next section we describe specific examples of these four fractions schemes.

Task Structures

Figure 1 displays four task structures, two involving proper fractions (Task PFS1 and Task PFS2) and two involving improper fractions (Task IFS1 and Task IFS2). Consider that if a task involves a discrete representation for a unit fraction (e.g., a dot or a chip) then forming a size (via a counting measure) is often indistinguishable from forming a numerosity. Thus, Task PFS1 theoretically requires a PWS in the discrete representation and a PFS in the bar and circular representations. In each of the three representations, the correct response is 2/5, but an individual with a PFS might instead respond with a slightly different fraction (such as 4/7 or 3/8) in the bar and circle models if rulers or protractors are not available to make precise measurements.

Task PFS2 theoretically requires a *reversible* PFS in the bar and circle representations. To form the size of a unit fraction from the proper fraction requires intermediately forming the size of the whole. In the dots representation, the task requires a reversible PWS.

Theoretically, Task IFS1 requires a reversible IFS, as it asks for the size of '1' from a given improper size. To form this size, one could partition the given amount into nine equivalent one-fourths and then iterate that amount four times. However, a PT could potentially solve the task in the discrete representation using a ratio understanding. A PT might coordinate partitioning and iterating

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to solve the task in the bar representation, but not the circular representation, because of an established understanding of a circle as necessarily the size of '1'.

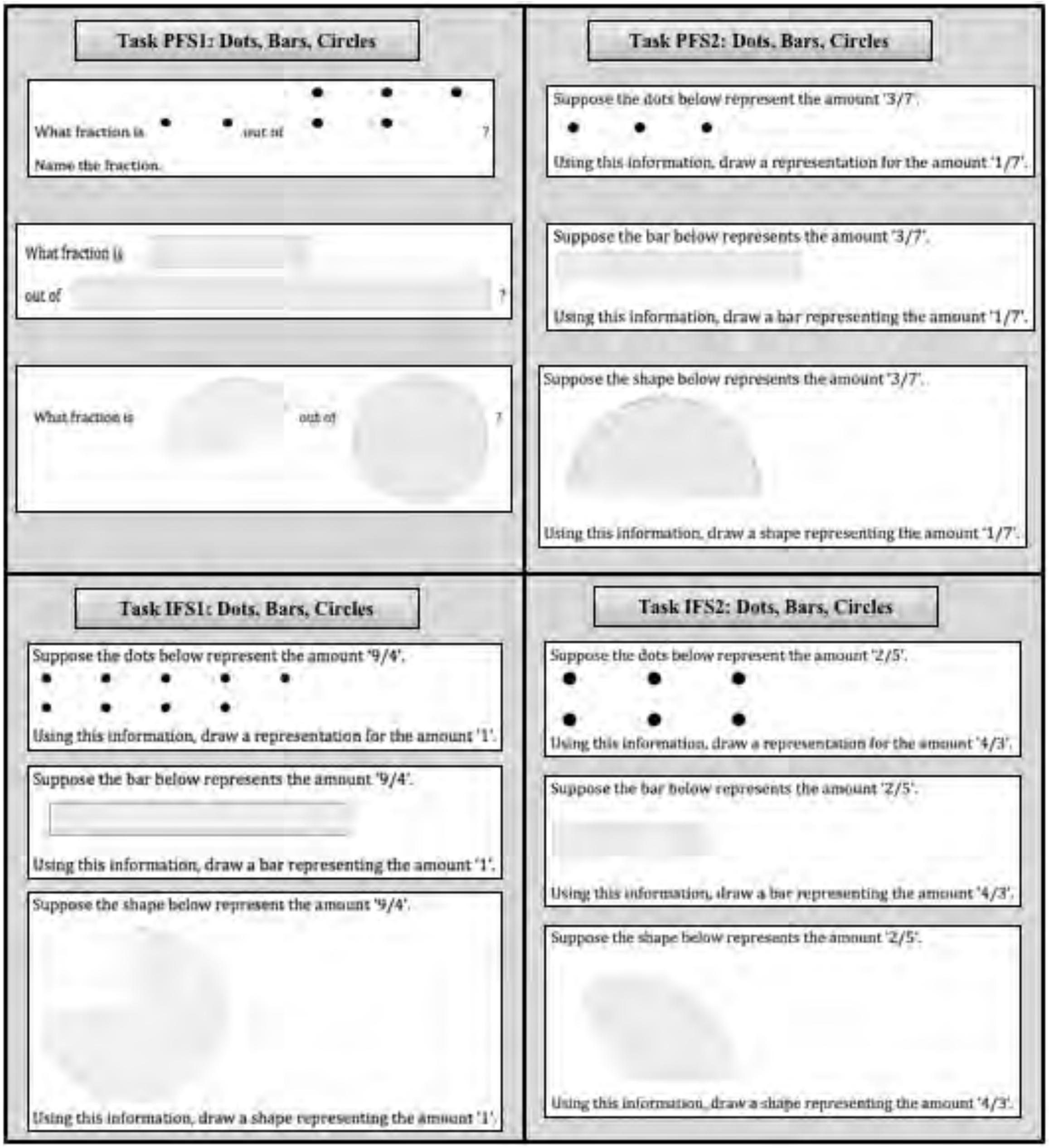

Figure 1. Four fractions task structures with dots, bars, and circles.

Task IFS2 theoretically requires recursive use of an IFS. One could solve the task by first partitioning and iterating to form the size of 1 (using a reversible IFS) and then partitioning and iterating to form an improper fractional size (using an IFS). PTs might instead approach the task by finding a common denominator by which to determine equivalent fractions without forming the size of 1. Lovin et al. (2016) noted that PTs may rely on such procedures when encountering improper fractions because they have yet to coordinate three levels of units: the unit fraction, the improper

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fraction, and the whole–when attempting to iterate a unit fraction beyond the whole, they lose track of the size of the whole (Steffe & Olive, 2010).

Methods

We administered a written assessment to 76 PTs in an Elementary Math Methods course at a U.S. university, prior to class discussion of fractions. To reduce the length of the assessment and to isolate differences in task representation, each of the PTs completed one of three forms. On the first form, there were four tasks with dots followed by four structurally identical tasks with bars. On the second form, the four dots tasks were followed by four circles tasks, and on the third form, the four bars tasks were followed by four circles tasks (see Figure 1). The four tasks in each representation were consistently given in the sequence: IFS1, PFS2, PFS1, IFS2.

These tasks use the same wording and form of items from Norton and Wilkins (2010), and our process for scoring items also followed their approach. First, a graduate assistant blinded and reorganized scanned pages of the assessments so that the two raters (authors) could not identify PTs' names or assessment forms. If there was strong indication that a PT had constructed a scheme, we scored it '1', and if there was strong counter-indication that a PT had constructed a scheme, we scored it '0'.

Figure 2 displays sample responses scored as '0' or '1' for two items in the bars and circles representations. For instance, for Task PFS2, to assign a score of '1', we were looking for evidence that the individual had partitioned the given size into three equally-sized pieces and drawn one of those pieces. Indication that the individual had instead partitioned the given amount into seven pieces would suggest assigning a score of '0' for Task PFS2. Note that we coded Task IFS2 with '1' if the PT determined the (approximately) correct fractional size by first using a procedure to determine equivalent fractions.

We calibrated our scoring by discussing our inferences and interpretations of ten randomly selected responses to each of the 12 items. As we independently coded the remaining responses, we also assigned '0.6' and '0.4' to indicate "leaning" toward indication or counter-indication, respectively. The (linear) kappa scores for the two raters across the 12 items ranged between .44 and 1, with a mean kappa score of .75, suggesting substantial inter-rater agreement (Landis & Koch, 1977). Agreement was strongest for the dots tasks, for which the PTs' responses were less ambiguous (kappa > .9 for each of the four tasks). The bars and circles tasks had the lower kappa values, as we had to make inferences from the PTs' markings about their intent to create a correct fractional size because they were not provided with a ruler or protractor with which to make exact measurements.

After computing a satisfactory kappa, we reconciled our scores. We assigned a reconciled score of '1' if we had each marked either a '1' or a '0.6', and we assigned a reconciled score of '0' if we had each marked either a '0' or a '0.4'. If one rater had marked '0.4' and the other '0.6', then we assigned a reconciled score of '0.5'. For the remaining responses, we returned to the data to decide on a score of either '0' or '1' for each item. Whereas Lovin et al. (2016) further used the sum of four reconciled scores on similar items to assign an overall '1' or '0' regarding an individual's construction of a fractions scheme, our item scoring remained focused at the item level.

We used the Wilcoxon rank sum (Wilcoxon, 1945) to test whether there were significant pairwise differences in mean scores for each item for this group of PTs. The null hypotheses were that there would not be significant pairwise differences in mean scores across task types or representations. We tested for differences in mean scores across task representations (dots, bars, and circles), controlling for the task types (PFS1, PFS2, IFS1, and IFS2), and we also tested for pairwise differences in mean scores across items within each of the three representations.

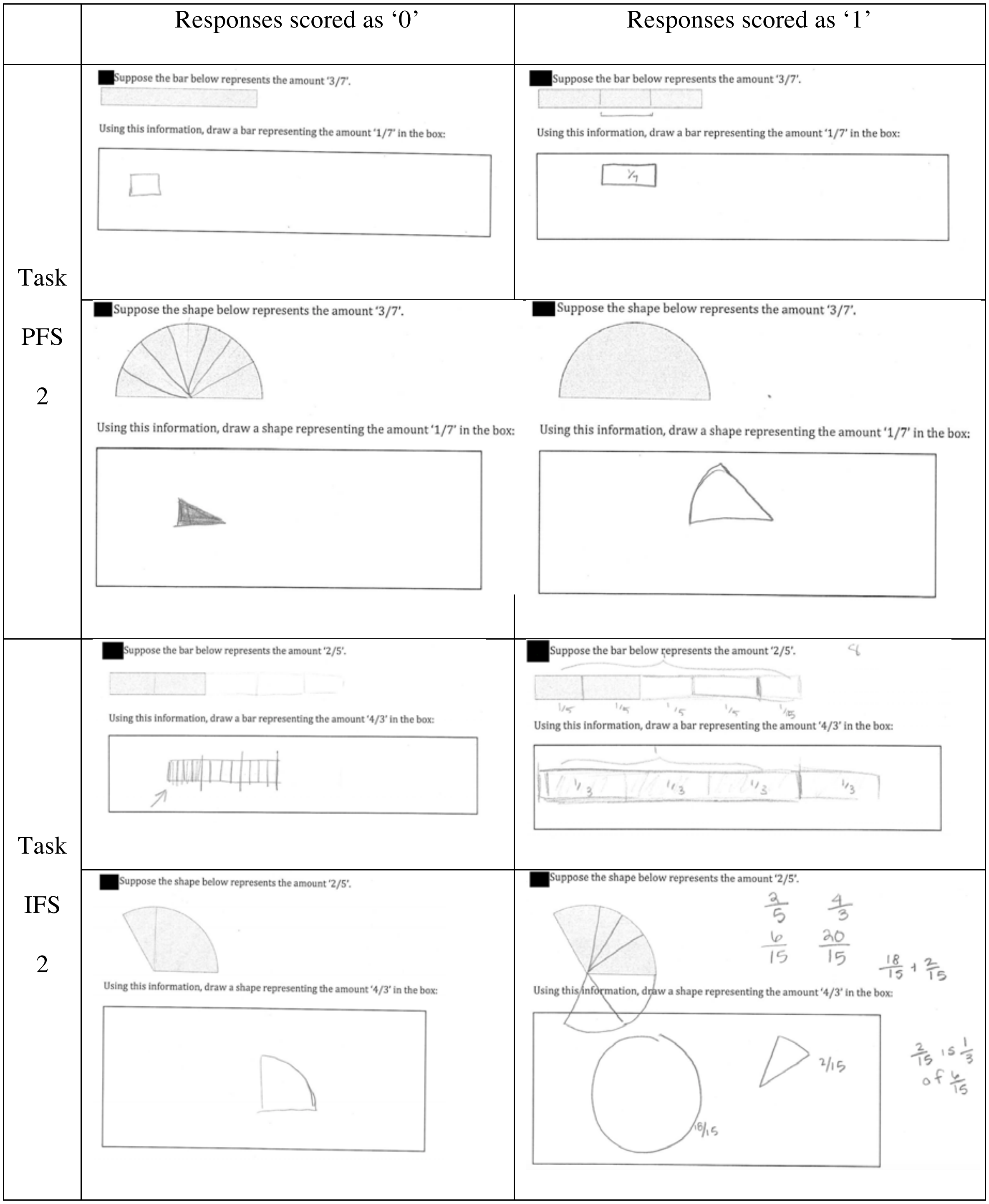

Figure 2. Sample responses coded with '0' or '1' in the bars and circles representations.

PTs' Rankings of Task Demands

Each of the three forms included an identical ninth question, intended to assess PTs experiences of the cognitive demands of their previous eight tasks. The PTs were asked to rank the previous eight tasks they had completed in order from least difficult (1) to most difficult (8) and to then explain their ranking decisions. The instructions emphasized that each of the numbers '1' through '8' were to be used exactly once in the rankings. We discarded PTs' responses to the ninth question in the analysis if they repeated more than one of the numbers 1-8, which resulted in including 71 of the 76 responses. We repeated the Wilcoxon rank sum tests with the PTs' difficulty rankings to test where there were pairwise differences in the PTs' experiences of difficulty across task type or task representation.

Results

Table 1 displays the the mean scores and difficulty rankings for each of the 12 tasks. Within each of the three representation types, the Wilcoxon rank sum test indicated significant differences (at alpha = .05 level) between both the mean performance and difficulty rankings on the two IFS items. Task IFS2 was more difficult than Task IFS1 (and was also more difficult than either PFS1 or PFS2), and this was in concordance with the PTs' rankings of the two items' difficulty. The difference in performance was not significant for the two PFS items across all three representations (p=.500 for dots, p=.355 for bars, and p=.232 for circles). For dots and bars, the ranking of the difficulty of Task PFS2 was significantly greater than Task PFS1; for circles the difference in difficulty was marginally insignificant (p = .012, p=.009, and p=.073, respectively). Across each of the four items, scores on dots tasks were significantly higher than circles tasks. One PT expressed, "The dots make no logical sense to me, and the shapes [bars] are easy until a fraction is more than a whole. What is that supposed to look like?" This response affirms that both task structure and task representation are important considerations for assessing PTs' fractions knowledge.

Table 1: Results Across Task Representation

	PFS1	PFS2	IFS1	IFS2
Dots Mean Score (Mean Dots Difficulty Rank)	.906[a] (2.062)[ab]	.937[ab] (2.917)[b]	.760[b] (3.771)[b]	.344[b] (5.833)[b]
Bars Means Score (Mean Bars Difficulty Rank)	.522[ac] (4.467)[a]	.622[a] (3.333)[c]	.656 (4.467)[c]	.300[c] (6.333)
Circles Mean Score (Mean Circles Difficulty Rank)	.766[c] (4.553)[b]	.681[b] (4.021)[bc]	.457[b] (5.553)[bc]	.128[bc] (6.745)[b]

Notes. Score was from 0 (counter-indication of scheme) to 1 (indication of scheme). Difficulty ranking was from 1 (easiest) to 8 (hardest). Using Wilcoxon signed rank test, with alpha = .05.
[a] Denotes significant difference between dots and bars. [b] Denotes significant difference between dots and circles. [c] Denotes significant difference between bars and circles.

Though Task IFS2 was ranked as the most difficult across all three representations, it was significantly less likely to be answered correctly in the circles representation than in the bars or dots representations. A common explanation for the PTs' assignment of difficulty rankings was that PTs found Task IFS2 "confusing" – particularly in the circles representation. We infer that this was often because the PTs were unfamiliar with a whole circle not representing the size of '1' (see Figure 3 for a sample response).

Supporting this inference, PTs were more likely to correctly answer Task PFS1 (in which the whole circle represented the amount 1) as representing 2/5 in the circles representation than in the bars representation. They more often incorrectly responded with '1/3' or '1/2' in the bars representation. This also explains why the difficulty ranking and score were each lower for Task

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PFS2 in the bars representation than in the circles representation. For instance, one PT mentioned that "we didn't really know what a bar represented, so I just had to divide them up equally and go with it."

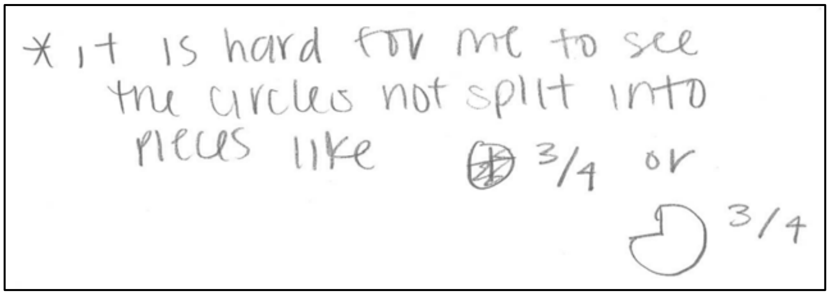

Figure 3. One PT's explanation of her difficulty rankings.

Conclusions

The results of our study generally concur with the learning trajectory described by Norton and Wilkins (2010) and Lovin et al. (2016), in that the PTs in our study were more likely to make sense of PFS tasks than IFS tasks. However, some PTs' familiarity with part-whole interpretations of fractions and proportions resulted in their being able to correctly solve tasks with dots but not structurally identical tasks with bars or circles, even though bars and circles tasks followed the dots tasks in their written assessments. PTs' ranking of task difficulty aligned best with the empirical trajectory of middle grades students' fractions schemes in the circle representation. This suggests that PTs enter their elementary mathematics course well-suited to appreciate the challenges elementary students face in constructing fractions schemes, and that instructors can support PTs by introducing non-standard circular representations of fractions.

We believe more investigation is necessary for the disentanglement of differences between PTs' fractions schemes across representations to inform instructional practice. Anecdotally, we have found in our teaching that focusing on non-standard circular representations early in a course can be frustrating and temporarily reduce some PTs' confidence, but eventually result in their reorganization of their fractions concepts. Future research may investigate the influence of varying the introduction of task structures and representations in elementary math courses. Important considerations include PTs' resilience, motivation, and self-efficacy for teaching mathematics.

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