DYNAMIC MEASUREMENT: THE CROSSROAD OF AREA AND MULTIPLICATION

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In this exploratory study, our goal was to engage students in dynamic experiences of area as a continuous quantity that can be measured by multiplicatively composing two linear measures (lengths), an approach we refer to as 'dynamic measurement,' or DYME. In this paper, we present the learning trajectory constructed from two cycles of teaching experiments with sixteen third-grade students. We discuss the types of tasks used for developing students' DYME reasoning as well as the forms of DYME reasoning students developed as a result of their engagement with these tasks.

Keywords: Measurement, Technology, Learning Trajectories

Background and Aims

Measurement is defined as assigning a number to a continuous quantity (Clements & Stephan, 2004). In terms of area, several studies focused on using square units to cover surfaces and quantify that covering based on the number of square units needed to cover the surface (e.g. Barrett & Clements, 2003; Battista, Clements, Arnoff, Battista & Borrow, 1998; Clements & Sarama, 2009; Izsak, 2005; Kamii & Kysh, 2006). In these studies, structuring area first involves students counting the individual square units used to cover the surface, then counting unit composites of a row and using repeated addition to find all units, and lastly recognizing that they can count composite units in a row and a column and multiply *rows x columns*. A key structure of this approach is the construction of the grid/array that results when a rectangle is covered with square units (Figure 1a), a difficulty that students face even after extensive covering and tiling activities (Outhred & Mitchelmore, 2000). Ultimately, children must "switch" from the generalization *rows* x *columns* to the area formula of *length* x *width* (Outhred & Mitchelmore, 2000) but this "switch" is not always intuitive and results in students connecting multiplication to area by reciting, not by understanding the formula (Izsak, 2005; Simon & Blume, 1994).

Indeed, to understand how area is generated by multiplying lengths is a very different notion conceptually from the construction of a matrix like shown in Figure 1a. As Piaget argues, "the difference between the two operational mechanisms is the difference between a matrix which is made up of a limited number of elements and one which is thought of as a continuous structure with an infinite number of elements" (p. 350). Thus, "switching" from the notion of counting discrete (discontinuous) squares in rows and columns to the multiplicative relationship of combining two linear continuous measures (lengths) in an area formula can be extremely difficult for students (Baturo & Nason, 1996; Kamii & Kysh, 2006). Piaget et al. (1960) suggested that these difficulties arise because "the child thinks of the area as a space bounded by a line, that is why he cannot understand how lines produce areas" (p. 350). Area measurement involves the coordination of two dimensions (length and width) and is a multiplicative process while covering a surface with unit squares is a one-dimensional process and additive in nature (e.g. Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996; Simon & Blume, 1994).

Consequently, our plan was to test the conjecture that that it was possible for children to experience the two-dimensional multiplicative relationship of area without relying on the switch from the rows by columns structuring; in effect, to provide evidence that students can visualize area as a dynamic continuous structure that can be measured by coordinating two linear measures (lengths). We drew on research on visualizing area as 'sweeping' through the power of motion (Confrey et al. 2012; Lehrer, Slovin, Dougherty, & Zbiek, 2014; Thompson, 2000) for designing

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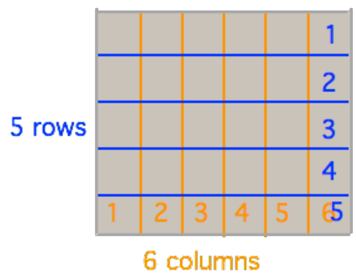
experiences for students to visualize area as a continuous structure dynamically. For instance, as suggested by Confrey et al. (2012), we considered engaging students in dynamic experiences of generating surfaces and visualizing a meaning for area as a ‘sweep’ of a line segment of length *a* over a distance of *b* to produce a rectangle of area *ab* (Figures 1b & c).



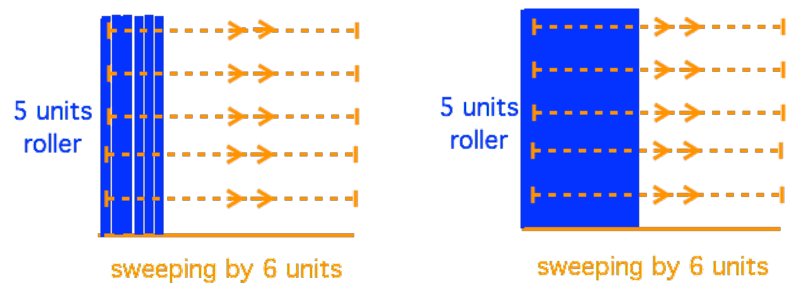


Figure 1a. Area as a discrete structure. **Figures 1b & 1c.** Area as continuous structure.

We distinguish this dynamic continuous approach from other approaches to measurement (e.g. *rows* x *columns* structuring) by referring to it as *Dynamic Measurement* (DYME) (Panorkou & Vishnubhotla, 2017). By looking at the example in Figures 1b & 1c, students visualize area as a continuous dynamic quantity which depends on both the length of the roller and the length of the swipe. This dynamic approach emphasizes the relationship between the boundaries of a shape and the amount of surface that it encloses, so that as the boundaries converge the area approaches zero (Baturo & Nason, 1996).

Although prior work (e.g. Confrey et al., 2012; Lehrer et al., 2014) identified the significance of teaching this dynamic approach, little information exists about how students’ DYME reasoning can be developed. Therefore, our goal was to examine: (1) What kinds of reasoning do students exhibit as they encounter DYME tasks? (2) What are critical aspects of students’ DYME reasoning that constitute increasingly sophisticated ways of understanding area measurement? (3) What kinds of measurement tasks, questioning, and scaffolding help students generalize concepts related to the spatial structuring of DYME?

Methods

Aiming to explore how students’ DYME thinking might be developed and progressed, our attention was drawn to research on learning trajectories (LTs), which have been widely used as an organizing framework for student conceptual growth (e.g. Clements & Sarama, 2009; Barrett et al., 2012; Simon, 1995). The design of the DYME LT followed Simon’s (1995) three components of an LT: a learning goal, a set of learning activities, and a hypothetical learning process. These components were constructed simultaneously during our LT design process; in other words, we defined our learning goals by having some instructional tasks in mind for promoting these goals and also postulated how students’ thinking of DYME may develop when they engage with our specific tasks and goals.

When formulating our initial conjectures about students’ reasoning of DYME, we synthesized existing LTs on length and area measurement (e.g. Clements & Sarama, 2009; Barrett et al., 2012; Confrey et al. 2012), and for each measurement construct we began by asking ourselves, “How can this construct be interpreted/modified/used in terms of DYME?” In contrast to other measurement LTs, the spatial structuring of DYME focuses on visualizing composites of 1-inch paint rollers iteratively dragged over a specific distance to cover a surface. In other words, a surface is described in terms of the number of 1-inch swipes (length) and the distance of each swipe (width). For example, to cover a surface of length 4 cm and width 7 cm, we need 4 swipes of 7 cm. Our design

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promotes students' thinking about commutativity (e.g. 4 swipes of 7 cm is the same as 7 swipes of 4 cm) and also reversibility (e.g. constructing surfaces by iteratively dragging rollers and deconstructing surfaces of length *a* and width *b* by equally splitting the surface into *a* sections of length 1 and width *b* to find area.) The target understanding of DYME involves a dimensional deconstruction (Duval, 2005), in other words analytically breaking down a 2D shape (its area) into its constituent 1D elements (length and width measures) based on relationships. Thus, two quantities (length and width) are coordinated simultaneously when making judgements about size.

In terms of task design, we used dynamic motion and paint rollers to enable students to both visualize area as a continuous quantity and coordinate the two linear measures. Our conjecture was that the paint rollers would act as bridge between the shape, and the number used to describe the size of the shape (Hiebert, 1981). To illustrate this dynamic motion, we used dynamic geometry environments (DGEs) to design our tasks. In addition to the dragging tool, most DGEs have a trace tool, which gives a trace of all the points on line segment (paint roller) following a locus as they move on the screen (see Figures 1b & c). Our conjecture was that the user would associate this discrete trace with the continuous surface formed.

We used a design-based research methodology (Barab & Squire, 2004; Brown, 1992) to develop and refine the LT and the tasks and tools focusing on two cycles of design, enactment, analysis, and redesign. The goal of Cycle 1 was to test our initial conjectures by experiencing students' first hand mathematical learning and how they construct DYME reasoning. We conducted a series of design experiments (DEs) (Cobb et al., 2003) with six pairs of third grade students. For the design of the tasks we used Geometer's Sketchpad (Jackiw, 1995). We had 6-10 sessions of 45-90 minutes with each pair of students. The outcome of Cycle 1 was a significant revision of the LT based on how students interacted with our tasks and tools.

The aim of Cycle 2 was to further demonstrate the feasibility of learning the DYME concepts and evaluate the effectiveness of the revised LT and also to test our tasks and tools with a group of students (Cycle 1 DEs were in pairs). We conducted a DE with a group of four students who participated in a STEM summer camp. Similar to Cycle 1, we had 9 sessions of 45-90 minutes each. For the design of the tasks we switched to the online version of Geogebra because of limitations of Chromebooks for downloading software.

Two stages of analysis occurred with the DE data, on-going analysis following each episode of the experiment (Cobb & Gravemeijer, 2008) and retrospective analysis at the conclusion of all DEs (Cobb et al., 2003). During the ongoing analysis, our initial conjectures evolved and we modified the tasks in light of iterative examinations of changes in students' thinking about area when interacting with the DYME tasks. During the retrospective analysis, we viewed the session videos and other data to create chronological accounts that tracked the forms of reasoning that emerged in the DE, the ways in which they emerged as reorganizations of prior ways of reasoning, and the aspects of tasks and tools that seemed to mediate those changes in reasoning. We drew on these analyses to refine the LT which included both our inferences about students' reasoning and the relationship between students' reasoning and DYME tasks.

A Learning Trajectory for DYME Reasoning

In this paper, we describe the most recent version of the DYME LT which resulted after conducting two cycles of DEs. We present each level of the LT by referring to Simon's three components: learning goals, sample tasks and examples from the student generalizations.

1. Exploring Dimensions and Area as Continuous Quantities

The goal of the first level is for students to build the idea of 2D space by using two linear dimensions (Clements & Stephan, 2004); in other words, visualizing area as a continuous structure

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that can change dynamically. The tasks engage students in coloring surfaces by dragging a given roller in multiple distances (Figure 2) and also matching different-sized rollers with shapes to color them and reason about the paint distance as well as the length of the roller. Students' generalizations include recognizing that the dimensions, length of the roller and the distance dragged, define how big or small a shape is, such as *"the further we drag the roller, the bigger the shape we create" and "the bigger the roller, the bigger the shape we create.* "At the same time, they begin to form relationships between the dynamic action of painting and the dimensions of the shape being painted by recognizing that *"the length of the roller needs to be the same as the height of the rectangle" and "the distance of paint is same as the base of the rectangle."*

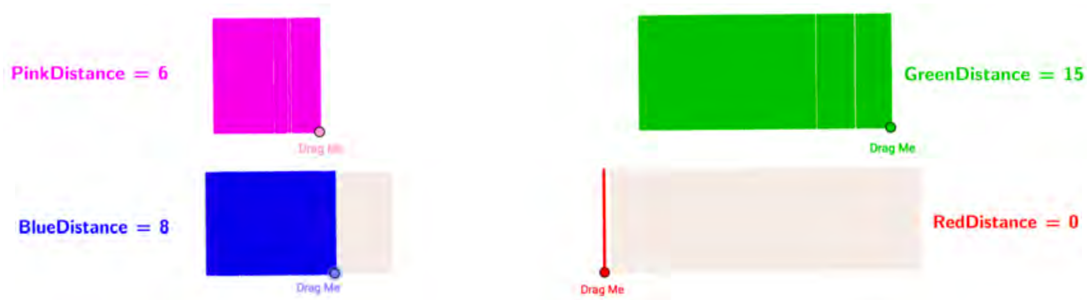

Figure 2. How far did you drag the paint roller to paint each shape?

2. Coordinating Two Dimensions to Compare Area

While in Level 1 students explored each dimension (length and width) independently, the goal of Level 2 is for the students to recognize that the measurement of a surface requires the coordination of two dimensions (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996; Simon & Blume, 1994). The tasks involve asking the students to fit a card into an envelope by modifying the dimensions of the envelope (Figure 3). Students are asked "What do we need to change? What stays the same?" They are asked to first write their predictions and then try it on the screen by modifying the envelope using the dragging tool. The card was movable so students could actually check if it fits in the envelope they created. Their generalizations include recognizing that to compare two shapes, they need to compare both dimensions. They also recognize that if one dimension is same they just have to compare the other dimension, e.g. *"it's bigger because it has the same base but it doesn't have the same height."*

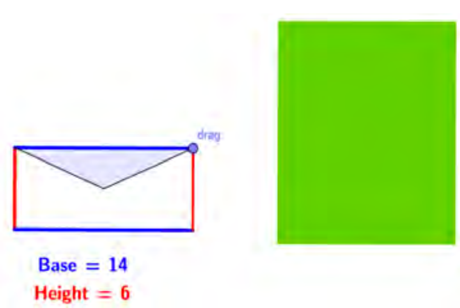

Figure 3. Modify the envelope to fit the size of the card! How big is the envelope you created?

3. Multiplicative Relationship of Length, Width and Area

The goal of this level is for the students to recognize the multiplicative relationship between the two dimensions of a rectangle and its area (Izsak, 2005; Simon & Blume, 1994). To help students identify this relationship, the tasks first involve the use of a 1-inch roller to paint shapes of different lengths and widths and constructing a repeating pattern for covering the shape (Outhred &

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Mitchelmore, 2000; Reynolds & Wheatley, 1996), by considering the distance covered in one swipe with the number of swipes (Figure 4). This experience is critical for generating the use of the multiplicative 'times' language to find the space covered, such as *"this is 30 because the base is 10 and we are going to swipe three times."*

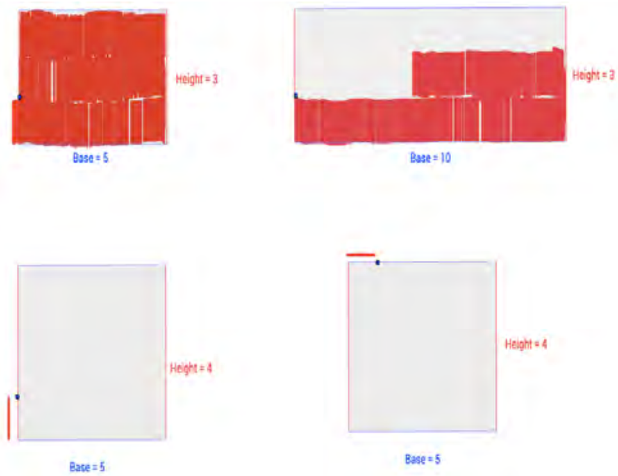

Figure 4. How far did you drag the roller? How many swipes did you need to cover the wall? How much space did you cover?

Central to the construction of area is understanding that the length measure indicates the number of unit lengths that fit along that length (Clements & Stephan, 2004). Consequently, our next goal is for the students to recognize that a roller of size l covers the same area as n rollers of size l/n dragged for the same distance. Our tasks include asking the students to paint the same rectangle first using 1-inch rollers and second by using rollers that are of different sizes and reason about the space covered in both occasions. At this stage, students begin splitting rollers to find the space covered, such as: "*This is a 3-inch roller. But if we cut it into 3 parts and you go across one time it is 4 and then if you go across another time it will be 8 and if you go one more time it will equal 12.*" Gradually, they begin to recognize that the space covered can be found by the length of the roller *times* the base of the rectangle or the distance of swipe. As students recognize that the length of the roller is the same as the height, they also begin using "length of roller" and "height" interchangeably, and this intuitively leads to *height times base*.

4. Multiplicative Coordination of Length and Area

The goal of this level is for students to recognize the effects on the dimensions when the area of a shape is scaled. To explore these effects, our tasks engage students in doubling and tripling areas (Figure 5) and identifying that they can double/triple areas by multiplying only one of the dimensions by the same factor, generalizing that *"to change area we need to change the base or height" or "to double the area we double just the length or just the width."* As a reverse process, we also designed tasks which engage students in doubling, tripling and halving lengths and widths of rectangles and reasoning about how area changes, such as *"since the length is going two times bigger, then the area should go two times bigger."*

As students' multiplicative thinking of area develops further, our next goal is for students to recognize that in order to split area (fractional thinking), they need to split the length or the width. The tasks engage students in creating shapes that have a fraction of an area of another shape. For example, students create a cafeteria which is 1/4 of an 8 by 5 inches garden and argue "*If we split this into four parts, then one of the parts will be the cafeteria. It would be 2 inches [the height of the*

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cafeteria] because the if we use only 1-inch roller it would go 8 times across but if you use 2-inch roller then it would go 1,2,3, and that would go 4 parts."

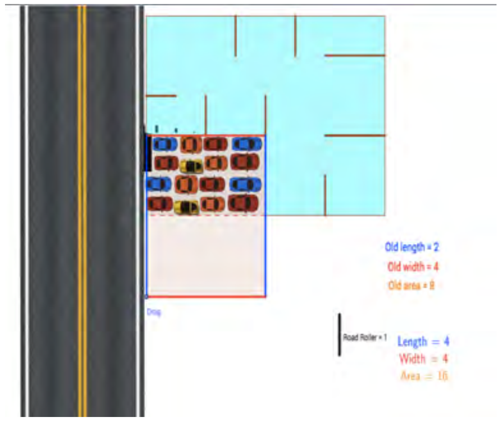

Figure 5. How can you make the parking place twice as big as it is now?

5. Identifying Area as a Multiple of its Dimensions

The goal of this level is for students to recognize area as a multiple of its dimensions and identify factors that give the same area. Our tasks involve asking students to create different rectangles of the same area (e.g. 12 sq. units) (Figure 6). This connects area measurement to geometry and the concept of congruence by recognizing congruent shapes in different orientations (e.g. 2 x 6 or 6 x 2) and describing congruence by using geometric motions such as rotation (Huang & Witz, 2011). It also directly relates to the properties of multiplication (e.g. commutative property) as well as factors and multiples. Students' generalizations include, "*length 4 and width 3 is doing 4 swipes of 3. This is same as two swipes of 6, so length 2 and width 6."* After students create all the rectangles, we ask them which rectangle has more space:

Researcher: Which one has more space?
Student 1: Everything is equal. Everything.
Student 2: The space is the same. The lengths and widths are different.

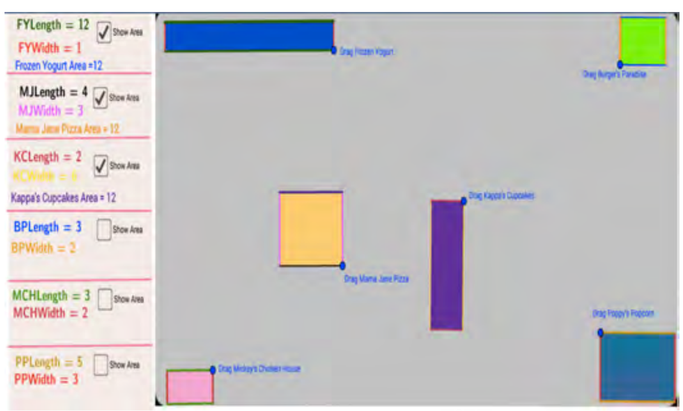

Figure 6. Each store should have an area of 12 sq. meters and different length and width from the other stores.

6. Coordination of Relative Areas

The goal of the final level is for students to coordinate relative areas. The task engages students

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in creating a robot with a fixed area (e.g. no bigger than 190 cm^2) by composing and decomposing shapes of fixed lengths and widths (Figure 7). As part of the task, students need to find the area of each leg (right triangle), by recognizing that if the leg is half the rectangle, then its area is half of the area of the rectangle. For instance, for calculating the area of each blue leg, students argued:

Student 1: You have to do a half of 7 and 2. We do 7 times 2 and a half of that.
Student 2: Both of them are 7 and both of them together are 14.

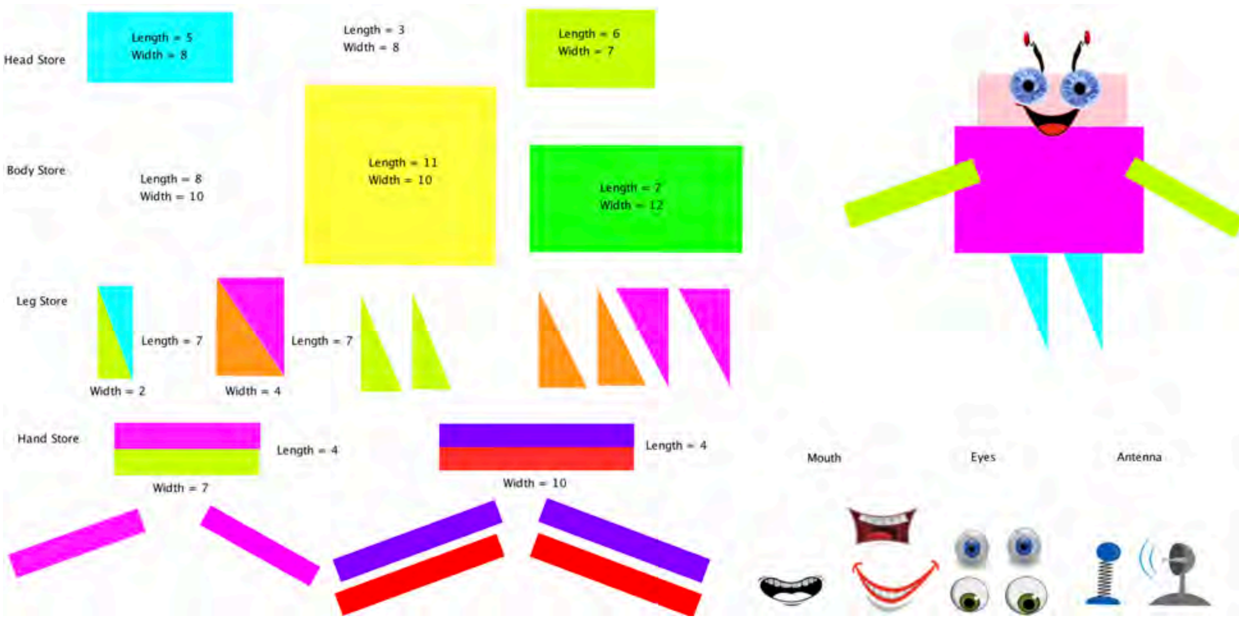

Figure 7. Make the robot as fancy as you like but its area should be no more than 190 sq. inches.

Concluding Remarks

This is an exploratory study examining a dynamic way of learning and teaching measurement. The DE findings show DYME's potential as a route to area measurement that would make the multiplicative relationship of the area formula more intuitive and accessible. The DEs helped in the design of the DYME LT, which is a conjecture of how students' DYME reasoning may evolve in the context of the specific learning activities. The LT shows that DYME lies at the crossroad of multiple mathematical ideas such as multiplication, division, fractions, (shape and unit) transformations, and covariation. Among our future goals is to explore these connections further as well as to examine how the DYME approach could complement the existing *rows* x *columns* structuring approach that is emphasized in research and schools.

Additionally, we are currently preparing a DE for a whole class to evaluate the effectiveness of learning the DYME concepts in a real classroom environment, and designing pre- and post-assessment items to evaluate students' thinking of area (as it develops) and validate the LT. We consider our findings to be very important for initiating a discussion around dynamic measurement and how it can be used for developing a conceptual understanding of area.

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