SECOND GRADERS' INTEGER ADDITION UNDERSTANDING: LEVERAGING CONTRASTING CASES

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In this study, we explore thirty-two second graders' performance on integer addition problems before and after analyzing contrasting cases involving integers. The students, as part of a larger study, completed a pretest, were randomly assigned to one of three intervention groups, and participated in two small-group sessions, one short whole-class lesson on integer addition, and a posttest. The group interventions differed in terms of which problems students compared in their small-group sessions. Based on students' solution strategies for integer addition problems and their treatment of negative signs, all three groups progressed in solving negative integer addition problems; however, students who initially contrasted adding two positives with adding a negative to a positive showed important differences, which we describe further.

Keywords: Number Concepts and Operations, Cognition, Elementary School Education

Providing students with early access to integer learning is important. Recent standards do not require students to learn integer concepts until sixth grade and integer operations until seventh grade (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010); however, prior standards suggested that students should learn about integers as early as third grade (National Council of Teachers of Mathematics [NCTM], 2000). Moreover, current research indicates that young students can productively learn about and work with integers (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bofferding, 2014). Students who are unfamiliar with negative numbers or who over-rely on whole number concepts, will often ignore the negative signs when solving problems (e.g., -4 + 5 = 9) (Bofferding, 2010; Murray, 1985) or will encounter other obstacles (e.g., thinking addition always results in a larger number) (Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014). Often, students develop incorrect and resistant conceptions, such as the belief that you cannot subtract a larger number from a smaller one (Murray, 1985). It is therefore important to help students notice early on, not only the differences between positive and negative numbers, but how they affect operations.

Contrasting cases can be a powerful instructional tool for helping students focus on important structural features in problems and give students access to new problems or solution methods. In a study on learning algebra equations, students who compared alternative solution methods gained more in procedural knowledge and flexibility than those studying multiple methods sequentially (Rittle-Johnson & Star, 2007). In a subsequent study, students with limited prior knowledge in algebra benefited more from comparing problem types than solution methods (Rittle-Johnson, Star, & Durkin, 2009). In one study, first graders noticed and made use of negative signs more if they were in an instructional condition where they compared problems with and without negative signs (Bofferding, 2014). A feature of the contrasting cases used in Rittle-Johnson and Star's (2007) work is that they involved worked examples, powerful tools for helping students learn new information (Atkinson, Derry, Renkl, & Wortham, 2000).

Some textbooks introduce negative integer addition by first presenting the case of adding two negative numbers (Hake, 2007; Pearson Education, 2014) and then contrasting it with adding two positive numbers through worked examples. Providing students with the opportunity to compare the addition of two positive numbers with that of adding two negative numbers may encourage them to notice that the numeral increases in magnitude just like with positive number addition but with a

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negative sign before the answer. This problem type tends to be one of the easiest for those recently exposed to negatives (Murray, 1985). However, if adding two negatives makes intuitive sense to students, having them see a different problem type first might provide a more productive contrast. Therefore, we investigated the following question: When learning integer addition, what role does the sequence of problem contrasts play in second graders' (a) integer addition performance and (b) the development of solution strategies?

Conceptual Change Framework

Students' initial mental models for number are based on whole number understanding. As students learn about new numbers, they make sense of them in light of their prior understanding (Vosniadou, Vamvakoussi, & Skopeliti, 2008). With negatives, this can lead to a series of transition and synthetic mental models based on how students interpret the order and value of negatives (Bofferding, 2014).

Interpretations of the Minus Sign

Successful integer contrasting cases should help students interpret and make use of the meanings of the minus sign in productive ways. There are three meanings that students might ascribe to the minus signs in arithmetic problems (binary, symmetric, or unary), which play "a major role in the development of understanding and using negative numbers" (Vlassis, 2004, p. 471). The binary interpretation of the minus sign corresponds to the subtraction operation (Vlassis, 2004). Students with only a binary understanding of minus signs might ignore negatives or treat them as subtraction signs (e.g., solving 6 + -8 as 6 – 8 = 0 or 8 – 6 = 2) (Bofferding, 2010). The symmetric or opposite meaning indicates an operation of multiplying by -1 (switching from positive to negative or negative to positive) (Vlassis, 2004). Students rely on the symmetric meaning when they add integers as positive and make the answer negative (e.g., solving -2 + 5 as 2 + 5 and answering -7) (Bofferding, 2010). The unary meaning of the minus sign is that of the negative sign, designating negative numbers. Students with a strong unary understanding will often start at a negative number and count, either solving -2 + 3 by counting incorrectly, "negative three, negative four, negative five," or correctly, "negative one, zero, one" (Bofferding, 2010), depending on their conceptions of addition and integer values. Those with a strong binary and weaker unary understanding may still solve 6 + -8 as 6 – 8 but actually get -2.

Interpretations of Addition

When students add with positive numbers, they learn that counting up corresponds to an increase in a number's magnitude (Vosniadou, Vamvakoussi, & Skopeliti, 2008). However with negative numbers, students need to learn that adding a positive number corresponds to moving right on the number line (or up); whereas, adding a negative number corresponds to moving left on the number line (or down) (Bofferding, 2014). Stranger still, they need to understand that adding a positive or negative could result in either an increase or decrease in magnitude from the initial number (e.g., with -3 + 1, the final answer -2 has a smaller magnitude than -3 but for 3 + 1, the final answer 4 has a larger magnitude than 3). In this paper, we discuss how students' pattern of responses on negative addition problems changed following opportunities to analyze different sets of contrasting problems.

Methods

Participants and Design

Participants included 32 second graders (from a larger study with 109 second graders) from two rural, elementary schools in the Midwest (where 32.2% of students were English-language learners

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and 75.2% qualified for free or reduced-lunch). These students were chosen because when solving the integer addition problems on the pretest, they consistently answered by adding the absolute values of the numbers, ignoring the negative signs (e.g., -4 + 6 = 10 or -1 + -7 = 8) and sometimes answering random positive numbers. After completing a pretest, students were randomly assigned to one of three intervention groups. Students in each group analyzed different sets of contrasting integer addition problems over two sessions, engaged in one integer addition lesson, and finished by taking a posttest.

Data Sources

Pretest and posttest. Although students answered a range of integer problems (e.g., ordering, comparing values, missing addend problems), we focus the present analysis on 14 integer addition problems that students solved on both the pretest and posttest (see Table 1). These were presented to students on one page in their regular classes, and students were asked to solve them as best as they could. Within a few days after students took the pretest, we interviewed 20% of the original sample to learn more about how they solved the problems.

Table 1: Arithmetic Problems Given on the Pretest, Posttest, and Midtest

Negative + Negative	Positive + Negative			Negative + Positive		
-6 + -4	5 + -2	4 + -5	9 + -9*	-5 + 5	-2 + 0	-4 + 6
-1 + -7*	6 + -8	7 + -3	0 + -9	-9 + 2*	-3 + 1	-1 + 8
-4 + -3**, -2 + -2**	4 + -6**	8 + -3**		-1 + 3**	-8 + 8**	

**Indicates a problem also given on the midtest; **Indicates a problem only given on the midtest*

Small-group sessions. As mentioned, students were divided into three intervention groups (sequential, intuitive, and conflicted). Of the 32 students who ignored all negative signs on the pretest, 11 students were from the sequential group, 12 were from the intuitive group, and 9 were from the conflicted group. When studying the contrasts, students worked in small groups of 2-3 students from their same intervention groups. Students in the *sequential group* analyzed each type of addition problem separately and in contrast to similar problems. Students in the *intuitive group* first compared adding two positive numbers versus adding two negative numbers, an intuitive contrast. Students in the *conflicted group* compared adding two positives with adding a negative to a positive in session one (we called this group conflicted because addition usually makes a larger number). During the second sessions, students analyzed the two other types of problems they did not see in their first session (see Table 2).

Each group saw their contrasts within four different illustrated contexts: an inclined number path situated on a hill, a vertical number line showing ants moving below and above ground, positive and negative chips, and a folding number line (see Tsang, Blair, Bofferding, & Schwartz, 2015). Students discussed and wrote about the similarities and differences between the problems and pictures; analyzed incorrect answers based on research of students' common misconceptions (e.g., ignoring the negative sign); and determined how to use the illustrations to correctly solve the problems. At the end of each session, students solved 6 integer addition problems related to the problem types they explore during that session for a total of 12 problems (9 with negatives) across the two session. We refer these collective problems as the midtest in the analysis and results (see Table 1).

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Table 2: Problem Types that each Group Solved During their Two Small-Group Sessions

	Sequential Group	Intuitive Group	Conflicted Group
Session 1	P + P vs. P + P, then N + N vs. N + N	P + P vs. N + N	P + P vs. P + N
Example:	3 + 5 vs. 4 + 4, then -3 + -5 vs. -4 + -4	3 + 5 vs. -3 + -5	3 + 5 vs. 3 + -5
Session 2	N + P vs. N + P, then P + N vs. P + N	N + P vs. P + N	N + N vs. N + P
Example:	-7 + 2 vs. -6 + 1, then 7 + -2 vs. 6 + -1	-7 + 2 vs. 7+-2	-7 + -2 vs. -7 + 2

Note: N = negative integer, P = positive integer

Whole-class instruction. Students participated in a 30-minute lesson focused on solving addition problems using a number path. Adding a negative number corresponded to going down or getting more negative and adding a positive number corresponded to going up the number path or getting more positive, leading to the introduction of additive inverses or "zero pairs" (e.g., -2 + 2). Students then played a card game using one stack of "1" cards, one stack of "-1" cards, and a die, where the goal was to collect cards in order to make zero pairs.

Analysis

To analyze students' integer addition performance, we marked each addition problem as either correct or incorrect and conducted a median test (a nonparametric test used for small sample sizes) on the pretest-posttest gain scores across groups. We did not include midtest results in the median test because the midtest did not have completely identical items.

In order to look for qualitative changes in students' solution strategies, we classified students' solutions to the integer addition problems on the tests according to their treatment of the negative signs and strategies, relying primarily on their response patterns for each negative integer problem type (positive plus positive, positive plus negative, and negative plus positive) and supplemented by interview data. First, we identified students who correctly answered all problems and designated them as *all correct, unary meaning*. If they had one incorrect within a problem type, we identified the type based on the codes and included it with the *all correct* code. For example, all of the students discussed here provided answers on the pretest consistent with ignoring the negative signs and adding the two numerals (e.g., -6 + -4 = 10; 6 + -8 = 14). Given the stability of their response pattern, we considered these students to be using this strategy even if we did not interview them; in some cases we did have interview data to confirm it. On the midtest and posttest, students sometimes provided responses that could have been coded in more than one way (*adds negative sign* vs. *directional* or *subtraction (negative)* vs. *directional)*. In these cases, we used their other responses within the problem type to infer which strategy they used and checked them with the transcripts when available. For example, students could get -6+-4 correct by either knowing that adding a negative means moving to the left on the number line or by adding 6 + 4 and making it negative. If students used the *adds negative sign* strategy on their other problems consistently, we assigned the same code to -6 + -4 = -10 (see Table 3).

Table 3: Strategies Second Graders Used on the Integer Addition Problem Types.

Strategies	Explanation (coded within each problem type)	Example: -4 + 1
Adds Negative Sign[3]	Student answers problems by adding the absolute value of the numbers and then making the answer negative.	4 + 1 = 5, so -5
Subtraction[2] (positive)	Student uses the negative sign as a subtraction sign and gets positive answers.	4 – 1 = 3 or 1 – 4 = 0
Subtraction[2, 3] (negative)	Student uses the negative sign as a subtraction sign and gets negative answers (or makes answers negative).	1 – 4 = -3 or 4 – 1 = 3, so -3
Ignores Negatives	Student ignores the negative signs.	4 + 1 = 5
Negative as Zero[1/2]	Student treats negatives as equivalent to zero.	0 + 1 = 1 or 4 – 4 + 1 = 1
Positive	Student answers with random positive numbers.	-4 + 1 = 10
Negative	Student answers with random negative numbers.	-4 + 1 = -8
Directional[1]	Students' answers are consistent with starting at the negative number and counting up or down the absolute value of the other number but not in a reliable direction.	-4 + 1 = -5 but -9 + 2 = -7
(Deviation)	Deviations to one of the above codes were noted when students' answers to a certain problem did not follow the pattern of the rest of the problems. This occurred particularly with the additive inverse problems (*zero pair*) that were part of instruction, problems where they had to *add zero*, or if a student *skipped* a problem.	

Meaning of minus sign: 1 = unary, 2 = binary, 3 = symmetric

Results

Performance on Integer Addition Problems

Overall, students in the conflicted and intuitive groups spent about 20 minutes in each of their sessions. Students in the sequential group spent about 40 minutes in each of their sessions because each problem type was explored in sequence instead of in comparison to a different problem type. Because the students discussed here ignored negatives on the pretest, they all started with scores of zero. Students in the conflicting contrast group had the highest median gain score from pretest to posttest (11.0) compared to those in the sequential group (8.0) and the intuitive group (4.5), X^2=5.869, p=.053. The conflicted contrasting group had significantly higher median scores than the intuitive contrast group. There was no significant difference between the conflicted and sequential groups; however, the conflicted group made higher gains with an intervention that was half as long as the sequential group. Table 4 presents additional information about the groups' performance on the pretest and posttest, as well as on the midtest. Not only did the conflicted group have the greatest

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median increase from pretest to posttest, but they also had the greatest increase in mean percent correct from pretest to midtest, from midtest to posttest, and from pretest to posttest.

Table 4: Groups' Performance on the Pretest, Midtest, and Posttest Addition Problems

Group	**Pretest (14 items) Mean (SD); %**	**Midtest (9 items) Mean (SD); %**	**Posttest (14 items) Mean (SD); %**
Sequential (n=11)	0 (0); 0%	1.7 (1.8); 19%	7.0 (4.9); 50%
Intuitive (n=12)	0 (0); 0%	1.8 (1.9); 20%	5.6 (4.0); 40%
Conflicted (n=9)	0 (0); 0%	2.3 (2.1); 26%	9.4 (3.6); 67%

Table 5 shows an additional breakdown of the problems by problem type on the midtest and posttest. Recall that the sequential and intuitive groups solved the N+N problem type (N=negative integer) at the end of their first session while students in the conflicted group solved the P+N problem type (P=positive integer). Across the three problem types, 5 out of 11 (45%) sequential group students, 6 out of 12 intuitive group students (50%), and 7 out of 9 conflicted group students (78%) correctly solved at least one problem on the midtest. On average, the groups did the best on problems related to the contrasts they saw in their first session.

Table 5: Students' Percent Correct for Each Problem Type on the Midtest and Posttest

Midtest	**N + N (3 items)** Average correct (% correct); Median	**N + P (3 items)** Average correct (% correct); Median	**P + N (3 items)** Average correct (% correct); Median
Sequential (n=11)	.82 (27%); 0.0	.36 (12%); 0.0	.55 (18%); 0.0
Intuitive (n=12)	**.92 (31%)**; 0.0	.42 (14%); 0.0	.42 (14%); 0.0
Conflicted (n=9)	.78 (26%); 0.0	**.56 (19%)**; 0.0	**1.0 (33%); 1.0**
Posttest	**N + N (2 items)**	**N + P (6 items)**	**P + N (6 items)**
Sequential (n=11)	1.0 (50%); 1.0	3.1 (52%); 4.0	2.9 (48%); 2.0
Intuitive (n=12)	1.1 (54%); 1.5	2.2 (36%); 1.0	2.3 (39%); 1.5
Conflicted (n=9)	**1.2 (61%)**; 1.0	**4.2 (70%); 5.0**	**4.0 (67%); 5.0**

Interpretations of the Negative Sign and Addition

Analysis of students' use of the negative sign and strategies for solving the integer addition problems provides additional insight into differences between the groups. Across the nine negative integer problems on the midtest, the sequential group had 5 out of 11 people (45%) continue to ignore the negatives when solving all of the problems. The intuitive group had 4 out of 12 people (33%) and the conflicted group had only 1 out of 9 people (11%) ignore all of the negatives. On the other hand, students in the conflicted group, who added negatives to positives in their first session, had more people interpret negative signs as subtraction signs (binary meaning) and subtract the numbers. The students in the other groups were more likely to make the answers negative if they did subtract (combination of symmetric and binary meanings) or add a negative sign after adding the

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absolute values (symmetric meaning). Finally, 75% of the students in the intuitive group, 55% of students in the sequential group, and only 22% of students in the conflicted group used a consistent strategy across all of the midtest problems.

On their first session recording sheets, many students in the sequential group ignored negatives. For example, one student ignored the negative signs when copying down two of the problems; another student wrote that the correct answer to -6 + -4 = 10, even though this was identified as wrong on the page. Students in the intuitive group paid much more attention to the negative signs, noticing when problems did or did not have them. Similarly, students in the conflicted group also noticed the negative signs, and, during follow-up interviews, one student indicated that it meant to "go back."

On the posttest, on the problems where students had to add both a positive and negative number, 2 out of 11 students (18%) in the sequential group, 4 out of 12 students (33%) in the intuitive group, and 2 out of 9 students (22%) in the conflicted group had responses consistent with adding the absolute values and making the sum negative (symmetric meaning) for at least one problem type. On the other hand 2 out of 11 students (18%) in the sequential group, 2 out of 12 students (17%) in the intuitive group, and 5 out of 9 students (56%) in the conflicted group correctly answered all questions for each problem type (allowing for one mistake per type), such as N + P. These students got negative answers on problems when they couldn't just add a negative sign, suggesting they had some unary understanding of the minus sign, accepting negatives as numbers in their own right.

Discussion

The results presented here provide insight into how students' strategies and ways of thinking about the negative sign can be influenced when given access to integer problems through contrasting presentations. Although students with limited prior knowledge did well comparing problem types and when presented sequentially as in prior research (Rittle-Johnson, Star, & Durkin, 2009), their intervention took twice as long and the conflicted group still made higher gains. This data suggests students who initially do not acknowledge the negative sign might benefit most from analyzing contrasts where the initial focus is on the result of the operation (Positive + Positive vs. Positive + Negative) as opposed to the change in sign (Positive + Positive vs. Negative + Negative).

Based on the collective students' gains by the midtest, exposure to the contrasting cases helped them change their thinking about the meaning of negatives and their use in addition problems. Further, their differences in strategy use suggest that depending on their contrasts, analyzing the problems gave them differential benefits in interpreting the problems. Students' further gains from the midtest to the posttest suggest that the instruction made additional impact but that the conflicted group benefitted most, especially in terms of changes in their strategies.

Students in the sequential and intuitive groups were more likely to ignore the negative signs, suggesting that they had not developed a usable meaning for the negative signs after analyzing problems that included two negative numbers. Others in these groups were likely to add the absolute values of the numbers and then add the negative sign to the answer, suggesting they interpreted the negative signs as having a symmetric meaning (Vlassis, 2004). However, it is not clear that they understood a negative number as being an opposite of a positive number. A more accurate description may be that they saw the negative sign as a descriptor. By comparison, students in the conflicted group were more likely to interpret the negative sign as indicating a binary operation. Although they initially gave mostly positive answers, a focus on the negative sign as a subtraction sign reflects a shortcut often taught to students (i.e., adding a negative number is the same as subtracting a positive number). Interestingly, by the posttest, the conflicted group did better than the other groups overall, suggesting an improvement in identifying when the minus sign designates a negative number for beginning their count or when it designates an operation.

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