THE CHANGING EXPECTATIONS FOR THE READING OF GEOMETRIC DIAGRAMS

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Students studying geometry at the secondary level are expected to read diagrams in different ways than those in elementary school. In this paper, we present an analysis of the changes in diagrammatic expectations by comparing the geometric diagrams found in Grade 1 U.S. textbooks with those in U.S. high school geometry textbooks. This work included developing and using a coding scheme that recognizes dimensions of reading a diagram geometrically, including the type of object represented, use of deduction, use of mental redrawing, interpretation of markings, and the necessity of the diagram. The way in which elementary and secondary students are expected to interpret diagrams was shown to change along several of these dimensions, posing potential learning barriers for students. We end our paper with a discussion of what our results mean for the learning of geometry.

Keywords: Curriculum, Geometry and Geometrical and Spatial Thinking, Elementary School Education, High School Education

An identical task with the same geometric diagram can be found at different grade levels with different expectations for interpreting the diagram (Dietiker & Brakoniecki, 2014). For example, in elementary school, a diagram of a quadrilateral with 4 apparent right angles is supposed to be identified as a rectangle, whereas in high school, the same diagram is expected to be interpreted as a quadrilateral that is not necessarily a rectangle. How are students expected to read information from geometric diagrams in mathematical tasks, specifically those found in textbooks? And how might these expectations change? In a study of the expectations of textbooks with respect to how students read geometric diagrams, Dietiker and Brakoniecki (2014) expand on Pimm's (1995) notion of *reading geometrically* and propose multiple dimensions of reading geometric diagrams. These dimensions, gleaned from analyzing the geometric tasks in multiple elementary and secondary textbooks (including traditional and reform curricula from multiple countries), represent distinct aspects of geometric diagrams that students are expected to pay attention to and interpret as they negotiate the meanings of mathematical tasks.

In this present paper, we report on our continuing analysis of textbooks to reveal how the expectation of diagrammatic reading changes as students progress through school. In particular, we compared the geometric diagrams found in Grade 1 U.S. textbooks with the diagrams of U.S. high school geometry textbooks in order to learn how different the expectations are. This work included developing and using a coding scheme that recognizes the dimensions of reading geometrically, which are described in detail in this paper.

We end our paper with a discussion of what our results mean for the mathematical learning of geometry. With evidence that students are expected to develop sophisticated ways to negotiate meaning from diagrams, we argue that within each of these dimensions, educators can craft opportunities for students to develop strategies for reading geometric diagrams to ease the transition from elementary to secondary school.

Framework

This study examines the mathematical content with regard to geometric diagrams within the *textbook curriculum*. The textbook curriculum is specifically limited to comprehensive written curricular materials that are published for use by teachers and students. Although the textbook curriculum has an impact on curriculum as enacted in classrooms, this analysis is limited to the content as it is interpreted by readers (i.e., the researchers) of the texts. For this study, *problems* include all textbook prompts (whether interrogatives or not), such as tasks, activities, and questions for which an expected response from a student is provided in the teacher edition, although withheld from students. Thus, worked examples (i.e., tasks that are completely solved within the student text materials) are not framed as problems.

An *expectation* of a problem is framed as a limiting condition with regard to a student's response of a question or task. For example, if an assumption from a diagram is necessary (such as interpreting an unmarked angle in a geometric diagram as a right angle) to get the expected answer provided in the textbook, then we argue that this assumption is a *diagrammatic expectation*. In any geometric diagram, there are many potential assumptions that could be made. We limit our definition of expectations to those that are required based on the given answers in the teacher textbook.

Methods

In order to learn how the expectations for reading geometric diagrams differ from elementary to high school, the teacher and student materials from four U.S. textbook series were selected for analysis, including two from first grade and two from high school. These grade levels were selected in order to demonstrate the change in expectations of diagram interpretation that students experience. The two elementary textbooks include the University of Illinois at Chicago's *Math Trailblazers* (2008, "MT") and the University of Chicago's *Everyday Mathematics* (2007, "EM"). Within these textbooks, we considered diagrams in the problems in all chapters focused on geometry, including topics such as shapes, volume, and symmetry. The two high school textbooks include the *CME Project Geometry* (2009, "CME") and *Prentice Hall Mathematics Geometry* (2004, "PH"). In these textbooks, we analyzed all diagrams for problems and questions in Chapter 1 in order to learn about the assumed expectations of geometric reading at the start of a formal geometry course in high school. In all textbook portions that were analyzed, we eliminated from analysis any problems for which the teacher edition listed an incorrect answer.

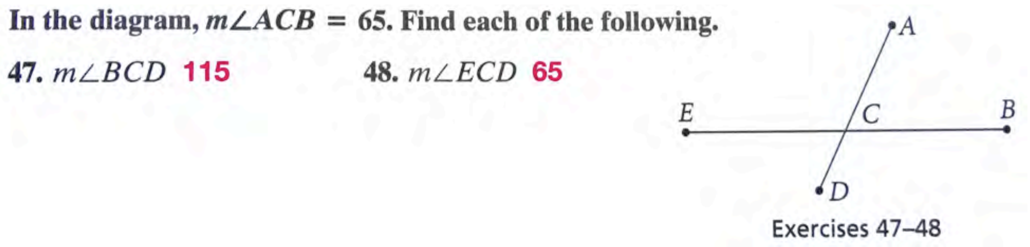
Due to the fact that the purpose of this work is to establish the expectations for how students interact with diagrams, only the diagrams that are part of a student task were analyzed. This does not include diagrams included in exposition or in worked examples, as students do not have to interpret or make decisions about these diagrams. Additionally, because we did not have the assessment materials for all four curricula, we restricted our analysis to lesson materials focused on learning new content.

Methods of Analysis

To describe the reading expectations of the geometric diagrams, we developed five overarching codes. The first describes how the reader is expected to interpret the diagram *as* something (e.g., a real life object or a representation of a set of objects). When analyzing our interpretation of geometric diagrams *as* something, we recognized multiple distinguishable characteristics that became sub-codes. Some tasks included geometric diagrams that were meant to be interpreted *as drawn* (as indicated in the task statement and answer). For example, a task that asks a reader to measure the diagram to make statements about the geometric object is expecting this reader to interact with the diagram as the geometric object under study. Another example is a task in which a student is expected to indicate (by drawing) a line of symmetry for a geometric object depicted in a diagram.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

In analyzing those tasks that require a reader to interpret the geometric diagram as drawn, we recognized that some require an assumption of either a metrical or topological relationship condition by the reader. For these tasks, there is a positive consequence for making assumptions based on the diagram, and having skepticism toward the diagram is disadvantageous. For example, in the task shown in Figure 1, a reader needs to assume that points E, C, and B are collinear in order to get these answers (displayed in pink) correct.



In the diagram, $m\angle ACB = 65$. Find each of the following.

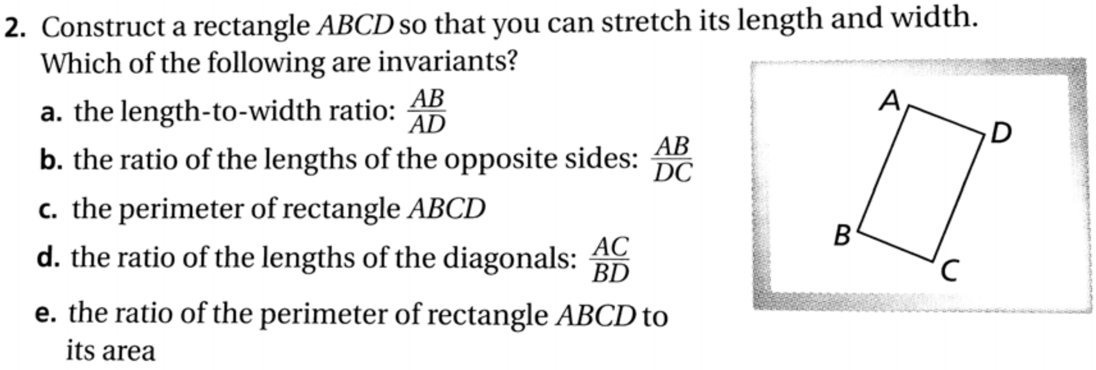
47. $m\angle BCD$ **115** **48.** $m\angle ECD$ **65**

Figure 1. An example of *As Drawn With Necessary Assumption* from PH (2004, P. 31).

However, not all geometric diagrams are positioned by the text to be taken as drawn. Others are positioned in such a way that they are *representations* of an abstract geometric object (or a set of objects) and thus, a student is expected to not make assumptions of the geometric object based on the diagram. In these cases, a reader may be expected to read the diagram as a representation of a particular geometric abstraction when given a diagram that is not necessarily accurate. For example, for the diagram in Figure 2, which accompanies a prompt for students to determine the largest rectangle in the image, a reader would have a negative consequence if they assumed the angles of the rectangular characters were as depicted (which are not drawn as right angles because of the 3D orientation).

Figure 2. Example of *Representing with Assumptions* from MT (2008, p. 187).

Reading as *representing* also includes geometric diagrams that represent multiple geometric objects (read as a generality). That is, in some tasks, a reader is expected to recognize that a geometric diagram is a single representation of a *multiplicity*. Along with these, we note that some of these require a reader to make at least one additional assumption. The geometric diagram in the task in Figure 3 is an example of a diagram representing a multiplicity since a reader is expected to interpret the diagram as one of many.



2. Construct a rectangle *ABCD* so that you can stretch its length and width. Which of the following are invariants?

a. the length-to-width ratio: $\frac{AB}{AD}$

b. the ratio of the lengths of the opposite sides: $\frac{AB}{DC}$

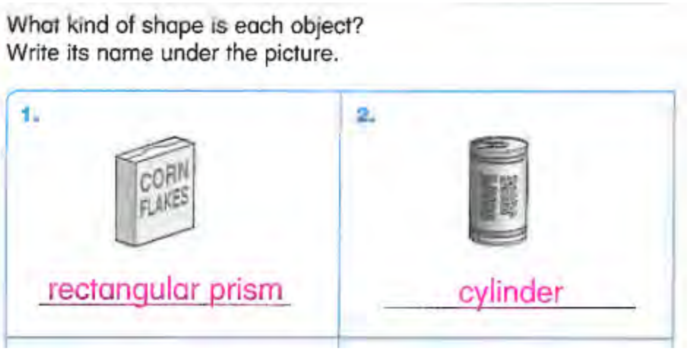
c. the perimeter of rectangle *ABCD*

d. the ratio of the lengths of the diagonals: $\frac{AC}{BD}$

e. the ratio of the perimeter of rectangle *ABCD* to its area

Figure 3. Example of *Representing as Multiple* from CME (2009, p. 54).

Other tasks do not explicitly include what we commonly consider to be geometric diagrams, but instead include an image of a real world object (such as the soda can in Figure 4) with the expectation that it will be *interpreted as a geometric object* (i.e., a cylinder).



What kind of shape is each object?
Write its name under the picture.

1. rectangular prism

2. cylinder

Figure 4. Example of *Real World Object* from EM (2007, p. 147).

Beyond the representations of geometric objects, we identified some geometric diagrams that are *not* representations of geometric objects. Some of these contain information that renders a geometric object as impossible or contradictory. For example, if a diagram of a triangle were marked with angle measures that do not sum to 180°, we interpreted that diagram as a *misrepresentation*. In addition, we coded geometric diagrams for which there is no expectation that a reader interprets the objects as geometric as *non-geometric*. For example, in a pattern problem with a string of triangles and squares, the students are not expected to interpret the objects as geometric. In fact, the use of the geometric shapes could easily be replaced with diagrams of flowers and firetrucks with no effect on the task.

The sub-codes for distinguishing how an object in a problem is to be interpreted and their interrelationships are represented in Figure 5.

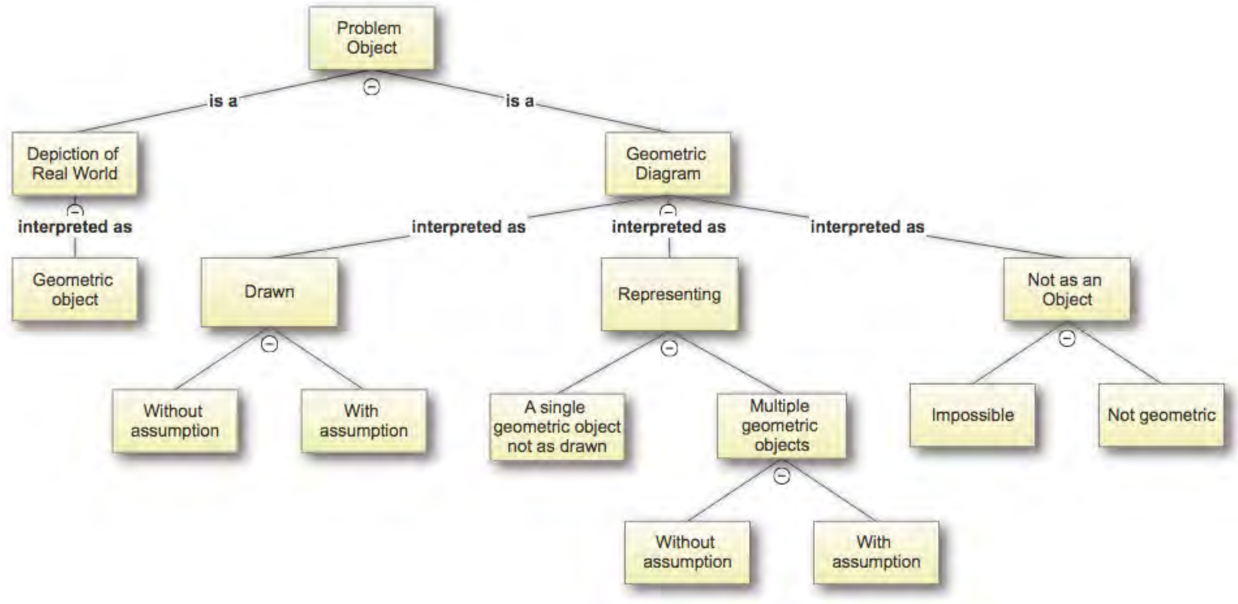


Figure 5. Diagram of sub-codes and their interrelationships for *Interpreting as*.

In addition to coding for the interpretation of the diagram, we coded four other dimensions of reading geometrically: whether *deductive reasoning* from the diagram is required to solve the task, whether the reader needs to *mentally redraw* the diagram to answer the task, whether the reader needs to *interpret conventional mathematical markings* to solve the task (e.g., reading the labels for the points in Figure 1), and whether *reading the diagram* is necessary to answer the task (e.g., the diagram in Figure 3 is supplementary while that in Figure 1 is necessary to solve the task).

Using this coding scheme, the three researchers analyzed each diagram from the selected portions of textbooks for the expectations of reading geometrically. These researchers include two mathematics educators and one doctoral student, of which two have high school teaching experience and one has extensive textbook design experience. Each code represents a consensus of all three researchers.

We suspected that there was a relationship between the intended grade of the textbook (elementary or secondary) and the various categories described above (the expected interpretation of the geometric object, whether deduction was necessary, whether mental redrawing was necessary, whether markings needed to be interpreted, and whether the diagram was necessary to the problem at all). To test the grade level's independence on each of these categories, we performed a Fisher's Exact Test (Fisher, 1922) between each grade level category and each of the above listed task categories to test the hypothesis that each of these categories was independent of the grade level of the textbook. Observed differences were statistically significant for $p < 0.01$.

Findings

The frequency of each type of diagrammatic expectation for textbooks of each grade level is reflected in Table 1.

When comparing how the textbooks expect students to *interpret as*, there was a statistically significant difference between the distribution of categories for elementary and high school texts ($p<0.001$). This enables us to assume that there is some dependence between the grade level of the textbook and the method of interpreting the diagram as an object. While the Fisher Exactness Tests indicates a likely dependence between categories, it does not identify specifically where the dependence exists. Thus, what follows is a summary of the more striking differences found in our coding results, highlighting where these differences likely exist.

Table 1: Frequency of Geometric Diagram Expectations

Task Expectation	Sub-Code	Elementary (n=61)	Secondary (n=156)	Fisher Exact P
Interpreting as	Real world as geometric object	5 (8.2%)	10 (6.4%)	0.000*
	Drawn	10 (16.4%)	51 (32.7%)	
	Drawn with assumption	36 (59.0%)	35 (22.4%)	
	Representation of single object	1 (1.7%)	4 (2.6%)	
	Representation of multiple	0 (0.0%)	45 (28.9%)	
	Representation with assumption	1 (1.7%)	6 (3.9%)	
	Impossible/contradictory	0 (0.0%)	1 (0.6%)	
	Non-geometric	8 (13.1%)	4 (2.6%)	
Using deduction	Required	0 (0.0%)	7 (4.5%)	0.195
	Not required	61 (100.0%)	149 (95.5%)	
Mentally redrawing	Required	0 (0.0%)	16 (10.3%)	0.007*
	Not required	61 (100.0%)	140 (89.7%)	
Interpreting conventional markings	Necessary	0 (0.0%)	81 (51.9%)	0.000*
	Supplementary	0 (0.0%)	26 (16.7%)	
	No markings	61 (100.0%)	49 (31.4%)	
Reading the diagram	Necessary	59 (96.7%)	123 (78.9%)	
	Supplementary	2 (3.3%)	33 (21.2%)	0.001*

Note. *Significant to $p < .01$.

Among the 61 diagrams of the elementary school textbooks and the 156 diagrams in the secondary textbooks, the most commonly expected interpretation of elementary textbook diagrams was *as drawn with assumptions*, with 59% of the diagrams in elementary. This means that a majority of the diagrams in elementary textbooks require students to interact with the diagram as the geometric object and that the student needs to make assumptions about measurements (such as a perceived right angle or a relationship between lengths) or properties (such as whether sides are parallel) based on how the diagram looks. Interestingly, high school textbooks also contain diagrams with this expectation, although they occur less frequently (22.4%). Instead, the most common expectation in the secondary diagrams is to interpret a diagram *as drawn* but without assumptions, which occurs in 32.7% of that grade level's diagrams, in contrast to 16.4%, as found in elementary textbooks.

Another noticeable difference between the grade levels' expected interpretations of diagrams was found for diagrams that *represent multiple* objects. No elementary school problems required students to interpret a diagram as a representation of multiple objects. In contrast, this was the second most frequently expected interpretation of the secondary diagrams, required for 45 (28.9%) of them. This suggests that high school texts expect students to know how to interpret a geometric object in a diagram as a general (rather than particular) representation at the start of a formal geometry course.

The geometric diagrams that *did not require the interpretation as a geometric object* were more found more often in elementary textbooks (13%) than secondary textbooks (3%). In addition, although there were relatively few instances of interpreting a diagram as *a single representation* overall, with only 5 diagrams in total, this occurred more often in the secondary textbooks (4). Although we expected to find more instances of diagrams depicting *real world* objects as geometric objects in elementary textbooks, the frequency of these diagrams was surprisingly similar in both grade levels (8.2% in elementary, compared to 6.4% in secondary).

Among the other categories of analysis, several also showed significant differences between elementary and secondary problems. A statistically significant difference was found when considering whether or not the diagrams needed to be *mentally redrawn* to solve the task ($p<0.01$). In the elementary textbooks, this expectation was not found. However, of the diagrams in the secondary textbooks, approximately 10% required a reader to visually manipulate a geometric object in order to solve the problem. Examples of these problems included tasks that require students to visualize what would happen to a geometry object if a vertex were dragged or how a diagram might change if a particular edge varied in length.

Another statistically significant difference was found when we compared the need to *interpret markings* of elementary diagrams versus those in high school diagrams ($p<0.001$). In the two elementary school textbooks, not a single diagram included any markings (right angle, congruent segment length, point name marking, etc.). This is in contrast to the high school textbooks' diagrams, of which almost two-thirds (68.6%) contain conventional markings. Of these, the majority required the interpretation of markings to solve the task (75.7% of those with markings, or 51.9% of all secondary diagrams). The remaining 16.7% of the secondary diagrams that contained conventional markings included a text prompt that supplied the information conveyed by these markings, rendering the markings in the diagram supplementary.

Lastly, we found a statistically significant difference ($p<0.001$) between the grade levels as to whether a student is expected to *read a diagram*. The diagrams in the elementary textbooks were almost always necessary to solve the task (96.7% of the time), in contrast to the high school texts which more frequently included diagrams that were supplementary to the task (21.2% of the time).

In contrast to the statistical differences described above, there was not a significant difference between the elementary and secondary diagrams regarding *using deduction* to solve a problem based on a diagram. None of the diagrams in the elementary textbooks require deduction and less than 5% of the diagrams in the high school tasks do so. In the elementary texts and opening chapters of the high school texts, it is almost never necessary for students to deduce a piece of information about a diagram which then needs to be used to learn additional information about that same geometric object.

Discussion and Implications

In this paper, we provide evidence that at some point in the transition from elementary school to the beginning of high school, there is a shift in expectations of how students are expected to read diagrams. As they start school, students are typically expected to make geometric assumptions based on how a diagram appears without being explicitly told about relationships that are necessary to solve a problem. By the time these students enter high school, they are expected to be able to reason about an object using only the information they are explicitly told and to not make assumptions based on how a diagram appears. This change in geometric interpretation of a diagram is consistent with van Hiele's (1959) description of sophistication of geometric understanding; younger children are expected to interpret geometric diagrams as a whole and only later begin to recognize the properties of geometric objects and their interrelationships. If a student does not recognize that a right angle is a property of a square, for instance, then marking right angles of diagrams of squares is pointless. Thus, it is sensible that textbooks for young children would contain the expectation that geometric diagrams be interpreted based on features as drawn (i.e., it looks like a square, therefore it must be a square).

However, we found it surprising that high school students are still expected to interpret geometric diagrams as drawn. Since high school geometry includes formal proofs, for which students are typically expected to reason only from given statements, it appears that students are expected to recognize and distinguish when they are able to make assumptions based on a diagram and when they are not. Even when students are not expected to assume metrical properties (e.g., an angle is a right angle just because it looks like a right angle), the students are expected to assume topological properties (e.g., if it looks like the figure is closed, it is). We wonder how students learn to distinguish when it is “okay” to make assumptions from diagrams and when it is not.

Interestingly, there was one aspect of reasoning with geometric diagrams that was not shown to be statistically significantly different from elementary to high school, which was whether diagrams required deductive reasoning. We expect that had we analyzed subsequent chapters in the high school textbooks, especially chapters in which students are asked to prove properties of geometric figures, that there would be more diagrams that require students to deduce new information about a geometric object from a diagram. Thus, based on this analysis, this shift may occur within the geometry course in high school.

Among all these dimensions in which reasoning about diagrams is expected to change, we wonder how aware curriculum authors and teachers are of these changes, and in what ways (if at all) these changes are communicated to students. We suspect that some of students’ difficulty with geometry may be at least partly due to an inability to successfully navigate the implicit expectations of reading of geometric diagrams and we believe that helping students recognize the multiple roles that diagrams can play in geometry and mathematics is critical for their success.

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