A QUESTIONING FRAMEWORK FOR SUPPORTING FRACTION MULTIPLICATION UNDERSTANDING

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This research examined the role of the teacher in supporting students to make sense of fraction multiplication when using a problem solving approach. Using a qualitative approach, the teaching of four skillful experienced sixth-grade teachers was examined as they implemented a problem-based unit on fraction multiplication. This paper will present a questioning framework used by teachers that supported students' conceptualization in this domain and highlight resulting implications for teacher practice.

Keywords: Number Concepts and Operations; Rational Numbers; Instructional Activities and Practices; Problem Solving

This research is situated in the context of fraction operations with a specific focus on fraction multiplication. It is founded upon arguments that more attention and effort should be paid to unpacking the professional work that teachers do in classrooms (Grossman et al., 2009). The fraction operation research literature has documented that students can invent procedures for operating with fractions (Kamii & Warrington, 1999). In the domain of fraction multiplication, it has been established that students bring initial knowledge that can serve as a starting point for algorithm development (Mack, 2001). In an effort to better understand teacher practice in the area of fraction multiplication the following research questions were posed: What are key conceptual obstacles students encounter when engaged in a problem solving, rather than a procedural approach to understanding fraction × fraction multiplication? How do teachers use questioning and discursive practices to support students to make sense of what fraction × fraction multiplication is an enactment of?

How to engage students in productive mathematical discussions is challenging (Stein, Engle, Smith & Hughes, 2008). One challenge a teacher faces when working to cultivate a teaching practice where mathematics is learned through problem solving, is how to support learners without taking over or reducing the level of mathematical work students should engage in. In my work with teachers, they have commented that they do not know how to guide students when they are struggling. They find themselves explaining what to do rather than redirecting students in a way that helps them think and reason. This research was conducted to inform the creation of professional development materials to use with teachers interested in developing a practice where students are engaged in mathematical reasoning and problem solving as part of learning about fraction operations. The questioning framework for fraction multiplication that is presented here was one tool developed to support practicing teachers.

Theoretical Framework

Gravemeijer and van Galen (2003) emphasize that instead of concretizing algorithms for students, teachers can use an emergent approach where students are positioned to invent algorithms. They describe this guided reinvention process as one that starts with carefully chosen contextual problems where students model a mathematical situation. With this approach, students solve problems through modeling that leads them to reason with numbers in particular ways. Ideas for operating with numbers can emerge from work that focuses on learning to reason with numbers and exploring what is involved when numbers are manipulated in particular ways.

Galindo, E., & Newton, J., (Eds.). (2017). *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

According to Mack (2001), students bring informal knowledge related to partitioning fractions that can support making sense of fraction multiplication. Through modeling students can develop mental images and ideas that will support their understanding of what fraction multiplication is an enactment of. Important areas to develop include fractions as operators, developing meaning for finding *parts of parts of a whole*, and developing flexibility about what is the unit. Flexibility with the unit is especially important because the unit shifts when multiplication is enacted. In additive situations the numbers (ex: 1/2 + 3/4) represent actual quantities such as 3/4 of a pound and 1/2 of a pound. In a multiplication situation one of the numbers represents a quantity. The other number is an operator. For example, when 2/3 × 1/4 is enacted the goal is to determine what 2/3 of the quantity 1/4 is. In a scenario where someone wants to plant 2/3 of 1/4 of a garden with beans there are multiple levels of partitioning taking place. Initially, a whole garden in partitioned so that 1/4 of the whole garden can be represented. Next, one must partition the 1/4 of a garden into thirds and identify 2/3 of the 1/4 in order to know what *part of the part of the whole* garden is planted with beans. Finally, in order to determine how much of the whole garden is used for planting beans, the part of the part of the whole identified for planting beans must be expressed as what fraction of the whole garden is used for beans. While it might be tempting to provide students with the shortcut that "of" means multiply, the actual enactment of multiplication with fractional numbers is much more complex.

Armstrong and Bezuk (1995) offer that in order to make sense of fraction multiplication students need partitioning experiences that lead to the analysis of relationships between partitions and the whole. From an instructional point of view it is important for students to have the opportunity to encounter and make sense of "part of a part of a whole". From a mathematical point of view, students need opportunities to explore through modeling what is happening when the operation of multiplication is enacted. This research aimed to understand the teacher's role in supporting the development of this understanding by engaging students in a problem-solving approach to fraction × fraction multiplication rather than an approach focused on demonstration of a procedure.

Methodology

The setting used for this study were the classrooms of four sixth-grade teachers and their students. Each of the teachers used the Connected Mathematics Project (CMP) II instructional unit *Bits and Pieces II: Using Fraction Operations* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006a) as their primary curriculum. The unit uses a guided-reinvention approach to developing meaning for fraction operations. It allows algorithms to arise through student engagement with both contextual and number-based situations. In this setting assumptions can be made about the tasks used and about the fraction-related concepts that were developed prior to, and during the unit on fraction operations. In the timeline for the sixth graders who were part of this study, students came to the fraction operations unit with previous experiences that supported their understanding of fractions as quantities and their ability to model fractions. Prior to implementing the *Bits and Pieces II* unit, the *Bits and Pieces I: Understanding Fractions, Decimals and Percents* (Lappan, Fey, Fitzgerald, Friel & Phillips, 2006b) unit was also implemented.

The four teachers were skillful experienced teachers. The teachers had between 6 and 16 years of experience teaching with CMP. The researcher had prior opportunities to interact with two of the teachers in their classrooms. These interactions provided information on how the teachers organized their learning environment and engaged students to reason with mathematical ideas. The teachers had a strong understanding of the mathematics they taught and their students as learners of that mathematics. These teachers engaged their students in conversations about their mathematical work as they engaged in problem solving and reasoning. The other two teachers were identified by contacting fellow mathematics educators, researchers, and district-level personnel known by the researcher to have a history of working with teachers in CMP classrooms. They were provided the

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criteria described above and asked to recommend, if they could, a teacher who strongly met all of the criteria. The directions explained that a teacher must meet all criteria and to not make a nomination for the sake of nominating.

This study used a qualitative design. During the teaching of the Bits and Pieces II unit, classroom lessons were videotaped each day during the five to six weeks it took to cover the unit. In addition, teachers wore an audio recorder during each lesson. The video recorder was used to record small group discussions. When visiting the classrooms of the teachers, the researcher engaged in participant observation. This included observing, taking field notes, interacting with students during small group work time, and meeting with the teacher after the lesson to seek their perspective on the lesson and students' mathematical progress. During the part of the instructional unit that focused on fraction × fraction multiplication, the researcher visited each teacher's class during at least one day of the three to four-day lesson sequence. When the unit concluded, the researcher brought all four teachers together to examine selected student work, videos of their teaching, and discuss patterns emerging in the data.

Data analysis was guided by Erickson's (1986) interpretive methods and participant observational fieldwork. The multiple data sources allowed for triangulation. The video and audio data were transcribed and analyzed for emerging themes in relation to the research question. Questions that framed the data analysis included *"What approaches to solving problems emerged in discussions as students shared their reasoning?"*, *"How did teachers respond to students?"*, *"How did teachers direct the mathematical focus of these discussions?"*, and *"What approach did the teacher take when students struggled mathematically?"* When the researcher met with the four teachers, emerging themes along with relevant classroom video clips from lessons were reviewed. It was during this process of data reduction and collaborating with the teachers that the researcher began to identify data that answered the research questions. From this analysis a questioning framework was developed that captured interactions teachers had with students when using an emergent approach to fraction × fraction multiplication. The questioning framework is presented in Figure 1. The questioning framework was linked with issues referred to as "sticky points". The sticky points emerged and became articulated during the researcher's discussions with teachers about their interactions with students. Sticky points are common areas where students struggle mathematically to make sense of the enactment of fraction × fraction multiplication. These are also documented in the literature (ex.: Mack, 2001; Armstrong & Bezuk, 1995). The sticky points are one's that typically or expectedly emerge when instruction uses students' ideas as the starting point. The questions that form the questioning framework were apparent in the classroom teachers' dialogue with students as the teachers attempted to move students through these sticky points toward valid mathematical ideas and understandings.

Results

The CMP curriculum introduces students to fraction multiplication using the context of selling pans of brownies at a school event. A typical problem might ask the following: *What fraction of a pan of brownies will I have if I buy ¾ of a pan that is ½ full.* This context leads to development of an area model. Student were given a labsheet with squares (brownie pans) that they used to model the problem by developing a drawn visual representation of what happens when buying ¾ of ½ of a pan of brownies. Through drawing the students begin to develop a mental image of what fraction multiplication is an enactment of and reason about what each fraction represents in the process. As students draw models to enact and solve the problem there are common sticky spots that arise. The teachers anticipated and used these as opportunities to build new ideas related to understanding fraction multiplication. Figure 1 contains the questioning framework that emerged from the analysis of classroom data described in the methodology section. It is reflective of the mathematical interactions the teachers had with students while working to model and solve brownie pan problems as part of making sense of what fraction × fraction multiplication entailed.

1. What is the problem asking you to do?
2. Tell me about your picture. What does it show?
3. How much are you starting with? How can you show that?
4. What is the problem asking you to find? How can you show that?
5. How much of what you started with do you need?
6. How much of the whole pan do you need?
7. How many pieces are in a whole pan?
8. How many of those pieces do you end up needing?
9. Do you have more than you started with or less? Why does that make sense?
10. What number sentence would write to show what the problem is asking you to do mathematically?

Figure 1. Questioning framework for fraction × fraction multiplication.

As in most any class, students will need support based on where they are in their overall fraction understanding—some more than others. Rather than show students what to do, teachers posed questions to focus their attention on particular mathematical ideas while students worked to develop a picture that modeled what was happening in the brownie pan scenarios they were presented with. See figure 2 as one possible model that a student might develop. Asking students what they are starting with by posing question 3 (also see Fig. 2a), which in the ¾ of ½ scenario is the second fraction ½, and asking students why this makes sense, is used to establish which fraction is the starting quantity. In the problem context, ½ of a pan is an actual quantity. The fraction ¾, which is the operator, is focused on when asking question 4 (also see Fig. 2b). While a teacher could tell students what fraction they should draw first when making their model, the expectation of teachers observed was that students engage in reasoning and problem solving. They wanted students to figure out what they were being asked and work accordingly using what they knew about fractions and partitioning. This is why the teachers asked questions 1 through 4. These questions helped with one of the common sticky points—which fraction do I start with and which fraction am I operating with when modeling a fraction times fraction situation. Often, students want to start with the first fraction written in the problem statement. They want to begin by partitioning and shading the brownie pan to show ¾ of a pan of brownies. However, they need to begin by showing that there is half of a pan of brownies to start with. Questions 3 and 4 when used together draw out or direct reasoning toward what each of the fractions in the multiplication problem represents visually. Students need to understand what each fraction represents—one is a starting quantity and one is an operator. Questions that ask students to read (and reread) the problem context support their ability to process and reason about what the problem is describing as well as asking.

A second sticky point was what represents the unit being partitioned and named. When the problem starts, the unit is the whole brownie pan. You start with half of a whole pan of brownies. When asked to find ¾ of ½ of the pan, the unit or whole that is partitioned is half of the pan. When asked to find ¾ of ½, students may not be aware that there is a shift to a new unit and they mark or partition the full brownie pan. In other words, they find ¾ of the whole pan rather than ¾ of ½ of the pan. Also, while a student may correctly partition the half pan into fourths, they may not be able to articulate why it makes sense to do this. Together questions 3 and 4 draw and focus student reasoning on what each of the fractions represent when the problem is enacted. Question 5 (also see fig. 2b) focuses on what part of the part of the whole do you need. With ¾ of ½ of the pan, question 5 leads students to articulate that they need ¾ of ½ of the whole pan. When students can articulate this, they

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might then realize they need to shade ¾ of ½ of the pan. Or, perhaps a teacher might respond, "If you need ¾ of ½ of the pan, how could you show that in your picture?" At this point most student realize that they need to partition ½ of the pan into four equal parts and shade three of the parts. Question 6 then aims to get students to articulate that they need ¾ of ½ of the whole pan. In some cases this led the teacher to prompt the student to write beside or under their picture "¾ of ½ of a pan". In other words, students were prompted to express and record that the brownie pan scenario could be captured or expressed as "¾ of ½ of a whole pan". This will eventually support question 10 (also fig. 2d) that asks what number sentence could you write to show what the problem is asking you to do mathematically.

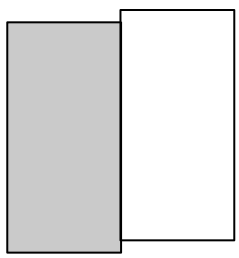
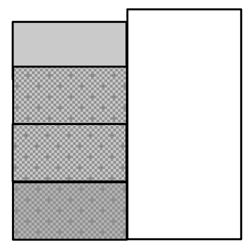
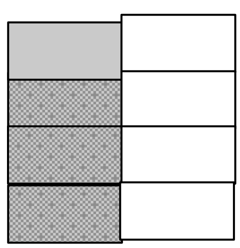
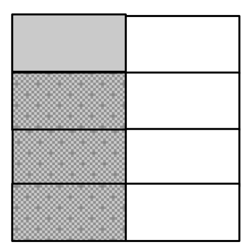
2a.  **Q3: How much are you starting with?** **[½ of a pan of brownies]**	**2b.**  **Q4: What is the problem asking you to find?** **Q5: How much of what you started with do you need?** **[¾ of ½ of the pan of brownies]**
2c.  **Q6: How much of a whole pan do you need?** **Q7: How many pieces are in a whole pan?** **Q8: How many of those pieces do you end up needing?** **[3/8 of the whole pan of brownies; 8; 3]**	**2d.**  **Q10: What number sentence could you write?** **¾ of ½ of a full pan is 3/8** **or** **¾ × ½ = 3/8**

Figure 2. Possible model representing ¾ of ½ of a pan of brownies.

A third sticky point involved expressing the solution based on what the problem is asking. Questions 7 and 8 (also see fig. 2c) are designed to direct attention to what the problem is asking—what part of the part of a whole pan of brownies would get if you bought ¾ of ½ of a pan of brownies. Often students say the solution is ¾. While it is true that ¾ of ½ of the pan is shaded, the solution is expressed as the portion of the original unit or the part of one whole pan. This requires a

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second shifting of units from finding part of ½ of a pan back to finding a part of one whole pan of brownies. The solution 3/8 is the part of the whole pan of brownies that are bought. Question 7 directs students to determine how many 1/8 pieces are in the whole pan. Question 8 focuses on how many 1/8 pieces are being bought. In addition to these questions, teachers might also ask students to reread the problem and to determine what they were asked to find. In the CMP curriculum the brownie pan problems ask, "What fraction of a *whole* pan of brownies is bought?"

As students experience the shifts across units, and because they are developing a visual model to reason with, the teachers focused on what was happening in relation to a fourth sticky point. Drawing from their work with whole numbers, students often think that multiplication leads to a product that is larger than the factors being multiplied together. Question 9 prompts students to look at their brownie pan picture, (or array model when they begin to use symbolic rather context-based problems) and consider that when they multiply by a faction that the solution is less than then what they started with. This sets up an opportunity to discuss what is happening when multiplying fractions and why fraction × fraction multiplication leads a solution where you have less than what you started with.

Returning to question 10, which may not be posed until after modeling and discussion of multiple problems, students are prompted to attach symbolism to the situation and their models. In the data this question led to discussions about why, for example with ¾ of ½, that the solution is expressed in eighths, and why there would be "three" eighths. From discussion of the ideas related to their models and what fraction × fraction multiplication is an enactment of, the algorithm "multiply numerators, multiply denominators" began to emerge. While the initial problems posed used the brownie pan context, students also worked with non-contextual fraction × fraction multiplication problems. While the idea that students could think of multiplication as finding a fraction "of" a fraction was addressed, the teachers focused discussions on finding a "part of a part" and used the brownie context to give meaning to this when instructional tasks shifted from contextual to symbolic. As students responded to questions from the framework they engaged in the thought processes associated with the enactment of fraction × fraction multiplication. The concept of fraction × fraction multiplication as finding a part of a part was supported.

Discussion and Significance

NCTM (2014) argues the importance of letting students engage in productive struggle. Often teachers are concerned that by not demonstrating up front to student what to do to solve a problem it will lead to confusion among students. How to support students and not "tell" is challenging. Providing teachers with problem contexts such as the brownie pan scenario is important in supporting a change in practice. However, a good problem alone does not help teachers develop ways to help students when they are stuck or get students started solving problems without telling them how. Good problem contexts do not ensure a teacher will have a way to support student problem solving. The questioning framework in coordination with "part of a part" problem contexts like the brownie pan problems is a potential tool for helping teachers shift away from a practice based on telling. While it sits outside the scope of research reported here, this framework in conjunction with video case analysis was used with teachers in a professional development setting. These teachers were working to develop their practice to use an emergent approach to fraction operations. Preliminary data analysis suggests that the questioning framework was an important support for teachers working to engage students in fraction multiplication algorithm development based on a problem solving approach.

Just as students benefit from learning to persist, teachers also need to learn to work through students' moments of uncertainty. Making sense of student reasoning while at the same time making instructional decisions about how to reply or what to ask in discussions with students, and how to guide without taking over student thinking is complex (Stein, Engle, Smith & Hughes, 2008). Facilitating discussions with students about what is happening as they work on problems such as those presented here may be daunting to teachers who are new to or working to develop a practice where students engage in problem solving and reasoning. The questioning framework described here, especially when paired with analysis of student work and classroom video cases in a professional development setting, can offer teachers a plan for listening to and responding to their students. It provides a pathway for supporting student reasoning so that teachers do not feel they have to resort to telling.

Acknowledgments

This research is supported by the National Science Foundation under DR K-12 Grant No. 0952661 and the Faculty Early Career Development (CAREER) Program.

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