CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2015 ANNUAL MEETING /
RENCONTRE ANNUELLE 2015

Université de Moncton
June 5 – June 9, 2015

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PROCEEDINGS OF THE 2015 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2015 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

39th Annual Meeting
Université de Moncton
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CONTENTS / TABLES DES MATIÈRES

TOPIC SESSIONS / SÉANCES THÉMATIQUES

NEW PHD REPORTS / PRÉSENTATIONS DE THÈSES DE DOCTORAT

AD HOC SESSIONS / SÉANCES AD HOC

MATHEMATICS GALLERY / GALLÉRIE MATHÉMATIQUE

APPENDICES / ANNEXES

INTRODUCTION

Olive Chapman – President, CMESG/GCEDM
University of Calgary

The 39th annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'étude en didactique des mathématiques [CMESG/GCEDM-2015] was another highly memorable learning and social event!

Our hosts at the Université de Moncton made sure we were well fed, entertained, and accommodated. The excursion to the Bay of Fundy and the Hopewell Rocks allowed us to enjoy this natural wonder as we basked in the simple beauty of this land and seascape. The conference dinner at the Plage Parlee Beach (which allowed some of us to also enjoy the ocean while getting our feet wet and sandy on the beach) was positively beyond expectation. Thanks to our colleagues Dr. Manon LeBlanc and Dr. Viktor Freiman for their time commitment and thoughtful planning and hosting of the conference. Manon's approach to encourage participants to engage with both languages throughout the conference was very successful in creating a welcomed bilingual atmosphere. Thanks to the other members of the organizing team and volunteers during the conference for their valuable contribution to the planning and smooth running of the conference: undergraduate and graduate students (who did a tremendous amount of work both before and during the meeting) Janelle Cormier, Michèle Cyr (better known as Michèle Hébert during the conference… she got married this summer!), Caitlin Furlong, Lise-Sara Rousselle Breau (better known as Lise-Sara Rousselle during the conference… she also got married this summer!) and Mylène Savoie. Graduate student Roman Chukalovskyy also helped with errands during the conference. In addition, thanks for the financial support (gifts for the participants) of the Faculty of Education, the *Librairie acadienne*, and the town of Moncton. In general, thanks to the Faculty of Education, the Faculty of Sciences, the associate vice-president of academics and faculty affairs (Jean-François Richard), the Dean of the Faculty of Education (Marianne Cormier) and the Université de Moncton (Moncton campus) for their support. Finally, from our hosts, Manon and Viktor: "Thanks to the participants, who gave life to this meeting. Thanks for the energy, the positive comments, the smiles, and all that you did to make this meeting truly bilingual!"

I also acknowledge the CMESG/GCEDM-2015 executive for organizing another stimulating program with topics relevant to our membership of mathematicians, mathematics teacher educators and mathematics education researchers. On behalf of the executive, thanks to the two plenary speakers Dr. Éric Roditi for engaging us in his work on the diversity, variability and commonalities among teaching practices and Dr. Deborah Hughes Hallett for engaging us in the importance of enabling postsecondary students to see connections between different parts of mathematics and between mathematics and other fields. Thanks also to the leaders of the five Working Groups; the presenters of the five Topic Sessions; the nine new-PhD presenters; the Ad Hoc and Math Gallery Walk presenters; the presenters of the Panel (Drs. Frédéric Gourdeau, Peter Taylor, Ralph Mason, and Elaine Simmt) for an entertaining debate on "Should we continue to teach fractions in school?"; and all of the participants for making the 2015 meeting a meaningful and worthwhile experience.

This publication of the proceedings of the conference offers readers the opportunity to learn about some of the mathematics education research and interests of our community. It is hoped that it also provides a means for participants to further reflect on and build on their experiences at the conference and for others to share in and be inspired by the work of the mathematics education community in Canada.

Horaire

Vendredi 5 juin	Samedi 6 juin	Dimanche 7 juin	Lundi 8 juin	Mardi 9 juin
	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Panel**
	10:15 – 10:45 Pause café	10:15 – 10:45 Pause café	10:15 – 10:45 Pause café	10:15 – 10:30 Pause café
				10:30 – 11:00 **Séances *ad hoc***
	10:45 – 12:15 **Groupes de travail**	10:45 – 12:15 **Groupes de travail**	10:45 – 12:15 **Groupes de travail**	11:10 – 11:40 **N^{lles} thèses (4)**
				11:45 – 12:30 **Séance de clôture**
	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	
	13:30 – 14:30 **Café dessert et Gallérie mathématique**	13:30 – 14:20 **Séances *ad hoc***	13:30 – 14:00 **Petits groupes**	
14:30 – 18:45 **Inscription** 15:30 – 16:20 **Amis de FLM**	14:40 – 15:10 **Petits groupes**	14:30 – 15:30 **Plénière 2**	14:10 – 15:10 **Plénière 2 : Séance questions**	
	15:15 – 16:15 **Plénière 1 : Séance questions**		15:20 – 16:10 **Séances Thématiques**	
	16:15 – 16:35 Pause café	16:00 – 22:00 **Excursion** *Hopewell Rocks* *Magnetic Hill Winery* **Souper**	16:10 – 16:30 Pause café	
17:00 – 18:45 **Souper**	16:40 – 17:10 **N^{lles} thèses (1)** 17:15 – 17:45 **N^{lles} thèses (2)** 17:50 – 18:20 **N^{lles} thèses (3)**		16:40 – 18:05 **Assemblée générale annuelle**	
18:45 – 19:30 **Séance d'ouverture** 19:30 – 20:30 **Plénière 1** 20:30 – 23:30 **Réception**	18:30 – ? **Souper libre**		18:30 – 22:00 **Souper** *Plage Parlee*	

Schedule

Friday June 5	Saturday June 6	Sunday June 7	Monday June 8	Tuesday June 9
	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Panel**
	10:15 – 10:45 Break	10:15 – 10:45 Break	10:15 – 10:45 Break	10:15 – 10:30 Break 10:30 – 11:00 ***Ad hoc* Session**
	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	11:10 – 11:40 **New PhDs (4)** 11:45 – 12:30 **Closing Session**
	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	
	13:30 – 14:30 **Dessert, Coffee and Mathematics Gallery**	13:30 – 14:20 ***Ad Hoc* Session**	13:30 – 14:00 **Small Groups**	
14:30 – 18:45 **Registration** 15:30 – 16:20 **Friends of FLM**	14:40 – 15:10 **Small Groups** 15:15 – 16:15 **Plenary 1 Discussion** 16:15 – 16:35 Break	14:30 – 15:30 **Plenary 2**	14:10 – 15:10 **Plenary 2 Discussion** 15:20 – 16:10 **Topic Sessions** 16:10 – 16:30 Break	
17:00 – 18:45 **Dinner**	16:40 – 17:10 **New PhDs (1)** 17:15 – 17:45 **New PhDs (2)** 17:50 – 18:20 **New PhDs (3)**	16:00 – 22:00 **Excursion** *Hopewell Rocks* *Magnetic Hill Winery* **Dinner**	16:40 – 18:05 **Annual General Meeting**	
18:45 – 19:30 **Opening Session** 19:30 – 20:30 **Plenary 1** 20:30 – 23:30 **Reception**	18:30 – ? **Dinner on your own**		18:30 – 22:00 **Dinner** *Parlee Beach*	

Plenary Lectures

Conférences plénières

DIVERSITÉ, VARIABILITÉ ET CONVERGENCE DES PRATIQUES ENSEIGNANTES

DIVERSITY, VARIABILITY, AND COMMONALITIES AMONG TEACHING PRACTICES

Éric Roditi

Sorbonne Paris Cité
Université de Paris Descartes
Laboratoire EDA

Les recherches présentées en conférence plénière de cette 39e rencontre du Groupe canadien d'études en didactique des mathématiques ont déjà donné lieu à diverses publications, en français et en anglais (Roditi, 2003, 2005, 2008, 2009a, 2013a). Elles proposent de s'interroger sur les pratiques enseignantes, ces dernières conditionnent en effet l'enseignement des mathématiques de manière significative. Les questions à l'origine de ces recherches portent sur les contraintes institutionnelles et les normes professionnelles qui tendent à homogénéiser les pratiques des enseignants. Ces derniers investissent néanmoins des marges de manœuvre, produisant ainsi des différences interpersonnelles entre les pratiques. Les activités des élèves en classe sont également influencées par l'enseignant et son action en classe.

En étudiant l'exemple de l'enseignement français de la multiplication des nombres décimaux dans la première année de l'enseignement secondaire (11 ans), nous nous interrogeons sur la régularité et la variabilité des pratiques des enseignants de mathématiques. En référence à la « *double approche didactique et ergonomique des pratiques d'enseignement des mathématiques* » développée par Robert et Rogalski (2002) et Roditi (2013b), il s'agit de comprendre ce que fait l'enseignant, tant d'après ses objectifs quant aux apprentissages des élèves que par le fait qu'il cherche aussi à répondre à des impératifs professionnels.

Les pratiques de quatre enseignants ont été analysées. Les régularités de leurs pratiques permettront d'élucider les contraintes qui s'exercent sur elles, en amont de la classe comme en classe. Leur diversité, malgré leur variabilité, conduiront à mettre au jour une forme de cohérence qui tiennent à la personnalité de l'enseignant, à ses connaissances ou ses croyances vis-à-vis de des mathématiques ou de l'enseignement des mathématiques.

UNE RECHERCHE DIDACTIQUE POUR ALLER AU CŒUR DES PRATIQUES

Après avoir précisé la problématique de la recherche, nous présentons la méthodologie mise en œuvre que nous justifions par la « double approche » qui distingue des composantes institutionnelle, sociale et personnelle des pratiques des enseignants dont les finalités visent à la fois l'apprentissage des mathématiques par les élèves et l'apprentissage des normes et valeurs sociales qui garantissent un bon fonctionnement du groupe classe.

REGULARITÉ ET DIVERSITÉ DES PRATIQUES ENSEIGNANTES

L'interprétation de la régularité et de la diversité des pratiques enseignantes soulève trois groupes de questions que nous détaillons en lien avec les moyens mis en œuvre pour les traiter.

Le premier groupe de questions porte sur les enseignements possibles et sur les choix des enseignants observés par rapport à ces possibles, en référence aux composantes institutionnelle et sociale des pratiques. Après avoir évalué les enjeux de la multiplication des décimaux, nous en recherchons les transpositions didactiques possibles à la lumière des nombreuses publications qui abordent ce sujet. Puis nous comparons les scénarios élaborés par les enseignants à ceux qui ont précédemment déterminés.

En référence à la composante personnelle, le deuxième groupe de questions porte sur le déroulement des enseignements, et plus particulièrement sur le travail des élèves en fonction de celui de leur enseignant. Nous cherchons à mettre en rapport les tâches prévues et les activités effectives des élèves, et nous étudions parallèlement les interactions en classe ainsi que les aides que les enseignants apportent aux élèves pendant la réalisation des tâches proposées.

Le troisième groupe de questions porte sur les contraintes et les marges de manœuvre ainsi que sur la cohérence des pratiques. Par une étude des textes officiels, nous spécifions les contraintes de l'institution scolaire qui fixe la durée d'enseignement et les savoirs à transmettre. Par des entretiens avec les enseignants, nous tentons aussi d'évaluer le poids des contraintes issues des attentes diverses des professionnels de l'établissement scolaire, comme de l'exercice même du métier, en classe, avec les élèves. Nous cherchons également à mettre les marges de manœuvres investies par les enseignants en regard de ces contraintes. Enfin, entre contraintes et marges de manœuvre, se pose la question de la cohérence des choix des enseignants. Nous cherchons à savoir comment elle se manifeste en croisant les choix effectués par les enseignants depuis la préparation des cours jusqu'au déroulement en classe.

LE CORPUS : DES SOURCES PUBLIÉES ET DES OBSERVATIONS DE PRATIQUES

La détermination des scénarios possibles d'enseignement a été menée par le croisement d'études de la notion mathématique, des prescriptions institutionnelles, et des conditions liées aux connaissances antérieures des élèves comme aux difficultés connues d'apprentissage de cette notion. Ces études reposent sur des sources publiées que sont les programmes, les manuels, les évaluations des compétences des élèves, les publications à l'intention des enseignants ainsi que les travaux de recherche en didactique des mathématiques.

Pour l'analyse de la notion enseignée, nous nous sommes référés à la théorie des champs conceptuels (Vergnaud, 1990) pour ce qui concerne le « sens » de la multiplication. Nous avons également tiré parti des recherches en didactique des mathématiques sur l'enseignement des nombres décimaux et celui de la multiplication : Brousseau (1997, 1998) et Douady et Perrin-Glorian (1986) ; Vergnaud, Rouchier, et Ricco (1979) ; Vergnaud (1981, 1983), Rogalski (1985) et Butlen (1985). Les enseignements possibles et observés sont ainsi analysés suivant différents critères : les situations multiplicatives, les propriétés de l'opération et les techniques de calculs, les écritures multiplicatives et leurs transformations, ainsi que les liens qu'entretiennent les situations, les propriétés et les écritures.

Les enseignements possibles ont été déterminés à partir de diverses publications qui traduisent l'état de la transposition didactique : programmes scolaires, brochures et manuels à l'intention des enseignants, propositions d'enseignement présentes dans les recherches en didactique. Nous avons analysé aussi les résultats à différentes évaluations afin de mieux connaître les difficultés d'apprentissage des élèves auxquelles les enseignants sont confrontés et dont ils peuvent tenir compte lorsqu'ils programment leur enseignement.

Les enseignants dont les pratiques ont été observées ont été choisis de manière à pouvoir travailler la question de la diversité interindividuelle des pratiques. Ainsi, toutes les variables qui concernent l'enseignement ont été fixées, sauf celles qui sont liées à l'enseignant en tant que personne. La notion enseignée est commune, les classes sont globalement de même niveau de performance scolaire, de même effectif et disposent d'horaires identiques, le manuel scolaire utilisé en classe est identique et les enseignants sont tous expérimentés. Afin de neutraliser aussi le facteur temps, chaque enseignant a été observé durant toutes les séances consacrées à la multiplication des décimaux. Le terme « séquence » désignera l'ensemble de ces séances.

Les observables définis pour le recueil des données sur les programmations des enseignants et sur les déroulements effectifs des séances ne sont ni trop fins, pour ne pas masquer les régularités, ni trop grossier, pour ne pas écraser les différences.

OBSERVABLES POUR L'ANALYSE DES SCENARIOS D'ENSEIGNEMENT

La programmation – ou scénario – de l'enseignement est distingué de son déroulement effectif avec les élèves. Trois observables ont permis l'analyse des scénarios proposés par les auteurs des publications ou mis au point par les enseignants observés : 1° le champ mathématique ; 2° la stratégie d'enseignement ; 3° les tâches mathématiques proposées aux élèves. Le *champ mathématique* recouvre les notions, les situations, les représentations symboliques et leurs transformations éventuelles, les propriétés et les théorèmes. La *stratégie d'enseignement* est l'organisation des contenus mathématiques selon un itinéraire déterminé pour des raisons cognitives ou mathématiques. Ainsi, certains enseignants commencent par exposer le savoir avant de donner aux élèves des problèmes mathématiques à résoudre alors que d'autres procèdent inversement. Enfin, les *tâches mathématiques* sont analysées selon des critères mathématiques et didactiques présentés plus loin.

OBSERVABLES POUR L'ANALYSE DES DEROULEMENTS DES ENSEIGNEMENTS

Pour l'analyse des déroulements des séances d'enseignement, trois observables ont été définis : les activités effectives des élèves, les aides que les enseignants apportent aux élèves, et la chronologie de l'enseignement.

Une tâche étant proposée à la classe, l'activité potentielle est ce que l'élève doit faire pour réaliser cette tâche, l'activité réelle est ce que l'élève a fait pour accomplir la tâche, et l'activité effective est la reconstitution par le chercheur de ce qu'aurait pu être l'activité réelle, en fonction de l'activité potentielle et des productions recueillies, notamment ce qui est dit par l'élève.

Voici trois exemples de tâches et de l'activité potentielle correspondant, sachant que toutes les trois peuvent conduire à la même activité effective : déterminer le produit de deux nombres décimaux à l'aide d'une calculatrice.

- **Tâche n°1** : calcule 3,14 × 2,08. Activité potentielle : appliquer la technique opératoire pour calculer le produit de deux décimaux ;
- **Tâche n°2** : Vrai ou Faux ? 3,14 × 3 = 9,43. Activité potentielle : déterminer le dernier chiffre du produit de deux décimaux ;
- **Tâche n°3** : Place la virgule au résultat 3,4 × 2,5 = 85. Activité potentielle : déterminer un ordre de grandeur du produit de deux décimaux.

Les aides apportées aux élèves par les enseignants observés sont essentiellement des aides procédurales répondant à ce que nous appelons des incidents didactiques ; en conséquence, ces aides ont été assimilées à des modes de gestion des incidents. Les incidents considérés ici ne sont pas des manquements à l'ordre scolaire, mais des interventions en décalage négatif par rapport aux réponses correctes possibles. Quatre types d'incident majeurs ont été repérés : les questions, les erreurs, les réponses incomplètes et les silences (lorsque un élève interrogé se tait alors que l'enseignant attend une réponse). Voici pour illustrer les incidents les plus fréquents, des exemples relatifs à la tâche n°4 : « *Place la virgule manquante dans l'égalité* 1,35 × 42 = 5,67 ».

- **Question. Raphaël :** « *Madame, a-t-on le droit de dire qu'il ne manque pas de virgule* ? » Manifestement, Raphaël compte les décimales. Sa question montre un décalage négatif par rapport à l'activité qui mène à la réponse exacte.
- **Erreur. Maud :** « *Pour placer la virgule, j'ai ajouté un zéro. J'ai écrit* : 1,35 × 0,42 = 5,67 ». L'erreur de Maud est certainement héritée de l'addition des décimaux.
- **Réponse incomplète.** Si Maud avait dit seulement « *Pour placer la virgule, j'ai ajouté un zéro* » sa réponse incomplète aurait constitué un incident. La classe aurait pu alors se demander si Maud pensait à 0,42, à 4,02, à 4,20 ou à 42,0, toutes ces réponses correspondant à des démarches possibles.

La gestion d'un incident est l'intervention de l'enseignant consécutive à cet incident. Les modes de gestion des incidents observés dans les séquences des enseignants qui ont participé à la recherche ont été classés en deux groupes suivant qu'ils tendent ou non à relancer le travail de réalisation de la tâche par les élèves. On comprend que l'accueil des incidents par l'enseignant, et la gestion qu'il en a, sont des facteurs qui influent sur le travail des élèves et, nous en faisons l'hypothèse, sur leur apprentissage.

Les séances ont été décomposées en épisodes caractérisés par le but spécifique que l'enseignant veut atteindre, cela permet de restituer une forme de chronologie à la séquence. Au niveau global, cette chronologie permet d'analyser l'organisation des moments de l'apprentissage et la dynamique entre le cours et la résolution de problèmes. À un niveau local, cette chronologie nourrit l'analyse de la gestion des incidents, notamment quant à l'influence du temps qui passe, sur ces interactions entre l'enseignant et les élèves.

DES SCÉNARIOS POSSIBLES AUX SCÉNARIOS RÉELS

À partir des publications dont les références ont été données précédemment, nous avons identifié les enseignements possibles de la multiplication des nombres décimaux. Par une évaluation des contraintes qui pèsent sur cet enseignement, et l'étude des manuels scolaires, nous avons déterminé les scénarios réalisables.

DES SCENARIOS POSSIBLES AUX SCENARIOS RÉALISABLES

À la lecture des publications, les enseignements de la multiplication des décimaux se différencient par la représentation des nombres décimaux et par des choix didactiques globaux. Trois types de scénarios possibles se distinguent en fonction de l'itinéraire cognitif programmé pour les élèves. Dans les scénarios du premier type, la technique opératoire est d'abord exposée par l'enseignant, puis elle est appliquée par les élèves pour calculer des produits éventuellement issus de problèmes où la multiplication est contextualisée. Avec un scénario du deuxième type, l'enseignant propose d'abord un problème en introduction, la technique opératoire est élaborée seulement partiellement par les élèves et/ou sans lien avec le problème précédent, puis elle est appliquée. Dans le troisième type de scénario, des problèmes issus de situations multiplicatives sont proposés aux élèves, leur résolution conduit à l'élaboration de la technique opératoire qui sera institutionnalisée et réinvestie dans de nouveaux problèmes.

Les manuels scolaires proposent tous des scénarios des deux premiers types, les décimaux y sont considérés indépendamment des fractions. Les propriétés algébriques de l'opération sur lesquelles repose la technique opératoire restent toujours implicites. L'étude de situations multiplicatives est globalement délaissée : la multiplication est toujours décontextualisée sauf quand les problèmes portent sur des calculs de prix. À l'opposé, les ouvrages à l'intention des enseignants et les recherches en didactique des mathématiques envisagent seulement des scénarios du troisième type. L'analyse montre aussi que leurs auteurs relient les écritures fractionnaires et les écritures décimales, mais ne tissent pas toujours de lien entre le sens de la multiplication et la technique opératoire.

LE POIDS DES CONTRAINTES

Pour concevoir un scénario d'enseignement, les enseignants utilisent les sources publiées, leurs connaissances mathématiques et leur expérience professionnelle qui les amènent à tenir compte de certaines contraintes dont les plus importantes sont les exigences du programme officiel, les connaissances réelles des élèves auxquels ils s'adressent, et les difficultés connues de l'apprentissage de la notion enseignée.

En France, les fractions sont introduites dès l'école élémentaire, mais ne sont pas approfondies à ce niveau d'enseignement où ce sont plutôt les nombres décimaux qui sont approfondis. Les nombres décimaux restent néanmoins, pour certains élèves, deux entiers séparés par une virgule, ces entiers ayant éventuellement des statuts différents. Les pourcentages d'erreurs qui correspondent à cette conception varient en effet entre 10% et 50% suivant les questions posées. On trouve peu de situations multiplicatives dans les évaluations ; les seules qui sont évaluées sont les situations d'isomorphisme de grandeurs et de calcul de l'aire d'un rectangle. De tels constats ne peuvent être sans conséquence sur les choix d'un enseignant dont la tâche est importante (compléter l'acquisition de la notion de nombre décimal, élargir le sens de la multiplication et enseigner une technique opératoire sur laquelle de nombreux élèves trébuchent) alors que le temps de l'enseignement est compté (on peut estimer une durée de 4 à 6 heures pour une séquence portant sur la multiplication des nombres décimaux, y compris la résolution de problèmes issus de situations multiplicatives).

Il est peu probable alors qu'un enseignant élabore un scénario où la multiplication est contextualisée, où les fractions et les décimaux sont reliées, et où les élèves construiront et justifieront la technique opératoire à partir d'une situation multiplicative.

LES SCÉNARIOS DES ENSEIGNANTS SONT GLOBALEMENT CONVERGENTS

Les séquences de quatre enseignants ont été comparées depuis l'élaboration du scénario jusqu'à son déroulement en classe. Ces enseignants sont désignés par des noms de mathématiciens afin de les distinguer et de les reconnaître au long des analyses : Mesdames Germain, Agnesi et Theano et Monsieur Bombelli.

Analyse du champ mathématique

Par comparaison aux enseignements proposés dans les publications, les choix des enseignants sont convergents : les scénarios sont tous du 1er ou du 2e type, et les nombres décimaux sont toujours traités indépendamment des fractions.

Le champ mathématique est composé des contenus étudiés : techniques de calcul, propriétés de l'opération, représentations symboliques des nombres, situations multiplicatives. Le tableau 1 résume l'analyse comparative des champs mathématiques.

Contenus mathématiques	Mme Germain	M. Bombelli	Mme Agnesi	Mme Theano
Technique et propriétés				
Technique opératoire	♦	♦	♦	♦
Justification de la technique	♦	♦	♦	♦
Calcul mental ou approché	♦	♦	♦	♦
Multiplication par zéro ou par un				
Propriétés algébriques de l'opération	♦	♦	♦	
Effet de la multiplication sur l'ordre	♦		♦	
Multiplication par un facteur inférieur à un	♦	♦	♦	
Représentation des décimaux				
Notation décimale	♦	♦	♦	♦
Notation fractionnaire des décimaux				
Écriture avec unité de mesure			♦	
Situations multiplicatives				
Isomorphisme de grandeurs	♦	♦	♦	♦
Produit de mesure				
Opérateur sur une mesure				
Composition d'opérateurs				

Tableau 1. Champs mathématiques des séquences observées.

Ainsi, tous les enseignants ont enseigné la technique opératoire, l'ont justifiée et ont proposé à leurs élèves d'autres méthodes de calcul, comme le calcul mental, réfléchi ou approché. De même, tous les enseignants ont traité le cas de la multiplication par un facteur inférieur à un. Ce cas particulier est important car il remet en cause le fait que la multiplication agrandisse : cette propriété, héritée du travail sur les entiers, est une source de nombreuses difficultés. Unanimité aussi des enseignants pour ne pas traiter la multiplication par 0 ou 1. L'unanimité disparaît en revanche pour les propriétés algébriques de la multiplication et son effet sur l'ordre. En ce qui concerne les représentations symboliques, tous les enseignants ont repris la signification de l'écriture décimale, mais aucun n'a fait le lien avec l'écriture fractionnaire. Madame Agnesi est la seule à proposer des liens entre l'écriture décimale et les changements d'unité de mesure. L'absence d'étude des situations multiplicatives fait l'unanimité complète des enseignants. Les seuls problèmes dans lesquels la multiplication des décimaux est contextualisée sont des problèmes de prix traités dans le cadre numérique. Aucune autre situation n'est étudiée, aucun autre cadre n'est convoqué.

Analyse de la stratégie d'enseignement

Une certaine unité se dégage quant aux stratégies d'enseignement, notamment en ce qui concerne la construction du nouveau savoir : pas de situation a-didactique, pas de changement de cadre, pas de dialectique outil/objet. Malgré cette convergence au niveau global, on constate des dynamiques différentes entre le cours et les exercices, exercices qui sont parfois des problèmes qui visent l'introduction du nouveau savoir.

Ainsi Madame Germain pose-t-elle d'entrée de jeu la question à ses élèves : « Comment calculer le produit de deux décimaux ? » Elle les laisse produire des règles efficaces sur certains cas particuliers. À la fin de la séquence seulement, toutes ces règles étudiées conduiront les élèves à l'élaboration de la technique usuelle. Monsieur Bombelli, contrairement à sa collègue, commence par exposer la technique opératoire qu'il justifie à l'aide d'opérateurs multiplicatifs, puis il propose des exercices d'application. Madame Agnesi commence par des problèmes de prix de marchandises, les produits de facteurs décimaux peuvent se calculer en effectuant des conversions. Les exemples ainsi fournis permettent d'induire la technique opératoire. Le reste de la séquence est consacré à des exercices d'application et à l'examen systématique des propriétés de la multiplication. Madame Theano introduit le calcul du produit de deux décimaux en utilisant les ordres de grandeurs, la méthode permet d'induire la technique opératoire. Les élèves contrôlent leurs conjectures avec la calculatrice. Suivent des exercices d'application et de calcul mental dont certains viennent interroger la technique opératoire.

Il apparaît finalement une grande homogénéité quant aux contenus enseignés et une certaine diversité quant à aux dynamiques entre la construction du nouveau savoir et sa mise en œuvre pour résoudre des problèmes. Qu'en est-il alors précisément des tâches mathématiques proposées aux élèves ?

Analyse des tâches mathématiques proposées aux élèves

Nous avons, d'une part, analysé les tâches proposées par les enseignants pour introduire le savoir nouveau, et, d'autre part, celles dont les activités potentielles visent son apprentissage par des applications (détermination du produit de deux décimaux), des questionnements plus théoriques ou la résolution de problème issus de situations multiplicatives. Le tableau 2 résume les résultats obtenus.

Tâches mathématiques	**Mme Germain**	**M. Bombelli**	**Mme Agnesi**	**Mme Theano**
Introduction du savoir nouveau				
Situation a-didactique				
Cadres mobilisés	numérique	numérique	numérique	numérique
La multiplication est un objet de savoir	♦	♦	♦	♦
La multiplication est un outil				
Situation multiplicative			♦	
Activités potentielles				
Détermination d'un produit	**75%**	**71%**	**50%**	**64%**
Par calcul posé ou instrumenté	*17%*	*14%*	*17%*	*09%*
Par calcul mental, réfléchi ou approché	*58%*	*57%*	*33%*	*55%*
Questionnement théorique	**25%**	**29%**	**33%**	**18%**
Situation multiplicative	**00%**	**00%**	**17%**	**18%**

Tableau 2. Tâches proposées aux élèves en fonction des activités attendues.

L'analyse des tâches proposées aux élèves confirme les choix communs des enseignants concernant l'introduction du nouveau savoir. On remarque également une certaine homogénéité concernant les exercices proposés aux élèves. Une grande partie d'entre eux (50 à 75%) induit une activité potentielle de calcul du produit de deux décimaux, mais les applications techniques (9 à 17%) sont très minoritaires devant le calcul mental, réfléchi ou approché (33 à 58%). Les autres exercices conduisent à des questionnements théoriques (18 à 33%) ou à la résolution de problèmes issus de situations multiplicatives (0 à 18%).

Les projets des enseignants observés sont finalement convergents quant aux contenus abordés et aux tâches prescrites, mais ils se distinguent en partie par la stratégie d'enseignement. L'analyse des déroulements permettra de conclure quant aux activités effectives des élèves.

DES DÉROULEMENTS QUI SIGNENT DES PRATIQUES PERSONNELLES

L'étude des déroulements repose sur l'analyse des activités effectives des élèves et celle des aides apportées par les enseignants. Précisons que les séquences observées ont duré de 2h30 à 5h00, évaluation non comprise. La durée estimée à partir des textes officiels a été donc respectée.

DES ACTIVITÉS EFFECTIVES PLUS VARIÉES QUE LES ACTIVITÉS POTENTIELLES

Le passage des activités potentielles aux activités effectives demande quelques explications méthodologiques. Lorsqu'une tâche pose une difficulté aux élèves, la tâche n°4 « Place la virgule manquante dans l'égalité $1,35 \times 42 = 5,67$ » par exemple, les enseignants peuvent proposer des aides qui conduisent les élèves à des activités différentes. L'enseignant qui demande de poser l'opération $1,35 \times 42$ provoquera une activité technique à l'issue de laquelle les élèves obtiendront le produit 56,70 ; il pourra alors relancer l'activité des élèves qui mèneront cette fois un calcul raisonné pour en déduire que $1,35 \times 4,2 = 5,67$. Mais un enseignant qui demanderait à ses élèves de penser aux ordres de grandeurs provoquerait une activité toute autre. Ces exemples permettent de comprendre que les activités effectives relevées dans les déroulements des séances observées sont à la fois plus nombreuses et différentes des activités potentielles repérées par l'analyse des tâches. Et c'est bien l'effet enseignant sur cette transformation que nous cherchons mettre au jour et à interpréter ici. Les résultats obtenus sont indiqués dans le tableau 3.

Activités potentielles et effectives	**Mme Germain**	**M. Bombelli**	**Mme Agnesi**	**Mme Theano**
Activités potentielles				
Calcul posé ou instrumenté	17%	14%	17%	09%
Calcul mental, réfléchi ou approché	58%	57%	33%	55%
Questionnement théorique	25%	29%	33%	18%
Problème (multiplication contextualisée)	00%	00%	17%	18%
Activités effectives				
Calcul posé ou instrumenté	09%	62%	27%	40%
Calcul mental, réfléchi ou approché	58%	25%	50%	44%
Questionnement théorique	33%	13%	16%	13%
Problème (multiplication contextualisée)	00%	00%	07%	03%

Tableau 3. Répartition des activités potentielles et des activités effectives.

Le tableau révèle des différences, pour chaque enseignant entre le scénario et son déroulement, et entre les déroulements des quatre enseignants. La référence à la stratégie d'enseignement permet d'interpréter ces résultats. La stratégie de Madame Germain est de laisser les élèves élaborer des règles pour calculer certains produits, ces règles devant les conduire à la technique opératoire. Les exercices techniques sont souvent enrichis de questions complémentaires favorisant les démarches raisonnées ou le questionnement des élèves. La stratégie de Monsieur Bombelli est, au contraire, de présenter la technique opératoire et de la faire appliquer ; cet enseignant renforce les activités de calcul posé au détriment des activités de calcul mental, réfléchi ou approché et des questionnements théoriques. Madame Agnesi a choisi d'introduire la technique opératoire par des problèmes de prix de marchandises ; cela conduit ses élèves à des activités de calcul réfléchi. Enfin, Madame Theano demande à ses élèves de placer la virgule en déterminant l'ordre de grandeur du produit, puis de contrôler le résultat avec la calculatrice, ce qui engendre des activités de calcul approché et instrumenté.

Pour conclure l'analyse, on retiendra que les activités effectives des élèves témoignent d'une plus grande variété que les activités potentielles ne le laissaient prévoir. Le travail de l'enseignant en classe apparaît alors déterminant sur l'activité des élèves : pendant le déroulement, il modifie sensiblement les tâches proposées conformément à sa stratégie d'enseignement.

LES INCIDENTS DIDACTIQUES ET LES AIDES DES ENSEIGNANTS

Pour rendre compte des interventions des élèves en classe et de leur gestion par les enseignants, nous avons relevé les incidents didactiques survenus en classe et les aides apportées par les enseignants en réponse à ces incidents.

Le nombre d'incidents par heure de cours varie suivant les enseignants. Ils sont pour tous très nombreux : un incident toutes les trois minutes en moyenne dans la séquence de Monsieur Bombelli qui en rencontre le moins, et le double dans celle de Madame Agnesi qui en rencontre le plus. Leur répartition est indiquée dans le tableau 4. On constate trois valeurs qui s'écartent sensiblement des valeurs moyennes et qui sont indiquée en gras dans le tableau 4. Ces valeurs sont sources d'interprétations des pratiques des enseignants.

Incidents didactiques	Ensemble	Mme Germain	M. Bombelli	Mme Agnesi	Mme Theano
Erreur	25%	27%	28%	21%	26%
Question	18%	16%	**32%**	15%	20%
Réponse incomplète	38%	36%	**16%**	**49%**	36%
Silence	09%	12%	08%	06%	07%
Autres	10%	08%	16%	09%	11%

Tableau 4. Répartition des incidents dans les séquences observées.

Dans la classe de Monsieur Bombelli, les questions sont nombreuses, alors que chez Madame Agnesi, ce sont les réponses incomplètes. Cette différence correspond à une divergence pédagogique : Madame Agnesi valorise l'expression des élèves alors que Monsieur Bombelli exige des réponses abouties, aussi, quand ses élèves ne sont pas sûrs d'eux, au lieu de répondre de façon incomplète, ils préfèrent le questionner.

L'étude des aides apportées par les enseignants en réponse aux incidents qui surviennent dans l'enseignement est également révélatrice de leur pratique et des conséquences sur l'activité des élèves. Le tableau 5 montre, pour chaque enseignant, la répartition des modes de gestion des incidents, entre ceux qui relancent l'activité des élèves et ceux qui ne la relancent pas.

Gestion des incidents	Mme Germain	M. Bombelli	Mme Agnesi	Mme Theano
Relance l'activité des élèves	72%	21%	42%	50%
Ne relance pas l'activité des élèves	28%	79%	58%	50%

Tableau 5. Gestion des incidents par les enseignants.

La gestion de Madame Germain relance l'activité des élèves dans plus de 70% des cas ; à l'opposé Monsieur Bombelli, près de 80 fois sur 100, préfère ne pas la relancer et finit par réaliser lui-même la tâche proposée. Entre ces deux pôles, se situent les gestions de Mesdames Agnesi et Theano. Le mode de gestion des incidents apparaît donc comme un aspect personnel des pratiques enseignantes avec un effet important sur le déroulement des cours.

Finalement, les analyses montrent que les scénarios d'enseignement sont globalement contraints, notamment par des facteurs institutionnels, mais aussi qu'il reste certaines marges de manœuvre qui sont investies par les enseignants tant pour l'élaboration d'un itinéraire cognitif pour les élèves que pour la gestion des interactions en classe, et cela conformément à leurs conceptions de l'enseignement et de l'apprentissage.

DISCUSSION DES RESULTATS SUR LES PRATIQUES DES ENSEIGNANTS

L'approche ergonomique, en considérant les pratiques des enseignants comme étant à la fois personnelles et partie prenante d'un milieu professionnel, permet d'avancer quelques hypothèses pour discuter les résultats obtenus précédemment quant à la convergence des pratiques au niveau global, et quant à leur diversité à des niveaux plus locaux.

HYPOTHÈSES POUR L'INTERPRÉTATION DES RESULTATS

Chaque fois que les enseignants ont effectué des choix convergents pour élaborer leur projet, nous nous sommes demandé à quelles nécessités professionnelles cela répondait. Les analyses et les entretiens avec les enseignants ont permis de formuler quelques hypothèses, énoncées sous forme de principes car tout se passe comme si des principes de nécessité professionnelle étaient respectés.

Les enseignants observés ont respecté le contenu du programme et aussi le rythme. Ils répondent ainsi à un « *principe de conformité aux programmes officiels* » qui leur assure une légitimité professionnelle face aux élèves et à leurs parents, aux collègues qui prennent en charge les élèves l'année suivante, et aux inspecteurs enfin qui sont garants de la mise en œuvre des consignes ministérielles.

Deux autres principes permettent de mieux comprendre les choix convergents des enseignants pour délimiter le champ des contenus mathématiques enseignés. Le « *principe d'efficacité pédagogique* » traduit le fait que les enseignants n'abordent pas les contenus mathématiques avec lesquels les élèves éprouvent des difficultés lorsqu'ils ne sont pas indispensables à la séquence. Ainsi, les fractions et les aires de rectangles ont été écartées. En outre, le « *principe de clôture du champ mathématique* » conduit les enseignants à ne pas traiter les contenus trop directement liés à ceux qui ont été écartés. Ainsi, les objets mathématiques qui restent dans le champ de la séquence sont reliés entre eux mais ne dépendent pas (ou peu) de ceux qui n'y ont pas été intégrés. De tels principes sont surprenants puisqu'ils conduisent apparemment à écarter de l'enseignement ce qui pose le plus de difficultés aux élèves ! En fait, plus précisément, ces deux principes conduisent à hiérarchiser les contenus et à éviter ceux qui risquent de poser des difficultés telles que l'enseignant ne pourra pas les traiter, sauf à être dévié de l'itinéraire prévu et à risquer une confusion peu propice à l'apprentissage. Il garantit ainsi une ligne directrice forte qui lui permet de rester dans ce que Rogalski (2003) appelle « *l'enveloppe des trajectoires acceptables du déroulement* ».

Enfin, le « *principe de nécessité de succès d'étape* » explique aussi que les enseignants segmentent leur enseignement de manière à mettre régulièrement l'élève en activité d'application de ce qui vient d'être enseigné. Ne disposant d'aucun modèle complet de la dynamique des apprentissages, ils utiliseraient les tâches techniques simples et isolées pour évaluer l'impact de leur enseignement au fur et à mesure du déroulement.

QUELLE COHÉRENCE DES PRATIQUES D'ENSEIGNEMENT ?

Le constat simultané de convergence et de diversité des pratiques enseignantes pose la question de leur cohérence, pour chaque enseignant. Des analyses de chaque séquence, effectuées en croisant les différents résultats recueillis, ont permis de repérer des niveaux de cohérence des pratiques. Une telle affirmation peut sembler imprudente en s'appuyant sur seulement quatre exemples de pratiques, mais nous choisissons de les indiquer car les recherches menées dans cette lignée de travaux et rapportées dans différents ouvrages confirment ces résultats (Vandebrouck, 2008, 2013).

On a remarqué souvent l'opposition entre la séquence de Monsieur Bombelli et celle de Madame Germain. Ce qui oppose ces enseignants semble tenir à leur conception de la classe : un lieu d'exposition et d'application du savoir pour Monsieur Bombelli ou bien, pour Madame Germain, un lieu de construction du savoir par les élèves. Chaque conception donne une cohérence aux pratiques. Dans une classe conçue comme un lieu d'exposition et d'application du savoir, l'exposition des savoirs a lieu très tôt, les activités effectives sont surtout des applications, les incidents sont plutôt des questions ou des erreurs, et leur gestion relance rarement l'activité des élèves afin que l'enseignant puisse, tant que nécessaire, montrer l'exemple. Dans une classe conçue comme un lieu de construction du savoir, le savoir est institutionnalisé assez tard, les activités de recherche dominent, et la gestion des incidents relance l'activité des élèves.

La pratique de Madame Agnesi se distingue de ces deux pôles. Cette enseignante souhaite que ses élèves s'expriment facilement, elle cherche à en impliquer le maximum dans le déroulement des séances et favoriser ainsi leur activité. Ses conceptions de l'enseignement et de l'apprentissage font qu'elle attend de sa classe qu'elle soit d'abord un lieu d'échange entre l'enseignant et les élèves. Et les élèves répondent à son attente : le nombre d'incidents didactiques est important et particulièrement celui des réponses incomplètes dont la fréquence est nettement supérieure à celle qui est constatée dans les autres classes.

Avant de conclure, signalons que la recherche montre aussi la variabilité de la pratique de chaque enseignant qui, malgré les contraintes et ses conceptions qui organisent son enseignement, adapte, à chaque instant, ses interventions au déroulement de la classe. Un résultat concerne en particulier l'effet de la pression du temps sur les pratiques de certains enseignants. Les conceptions de la classe de Mesdames Germain et Agnesi, comme un lieu de construction du savoir ou comme un lieu d'échange, demandent de laisser du temps aux élèves. Pour respecter néanmoins le rythme qu'impose le principe de conformité aux programmes officiels, une fois passée la première moitié de la séquence, les enseignantes se sont retrouvées contraintes d'adopter une gestion plus fermée des interactions avec leurs élèves.

CONCLUSION

Cette étude des pratiques des enseignants de mathématiques est une étude de type clinique. Les résultats portent sur le travail de quatre enseignants seulement, ce qui en limite la portée. Néanmoins, ces résultats ne sont pas infirmés par de nombreuses autres recherches sur les pratiques enseignantes.

Les régularités constatées montrent que l'institution scolaire, en fixant le savoir à enseigner et la durée de l'enseignement, contraint les pratiques enseignantes, depuis la préparation des cours jusqu'à leur déroulement en classe avec les élèves. D'autres recherches montrent que c'est parfois le vide des programmes concernant un contenu d'enseignement qui contraint les pratiques des enseignants : la recherche que nous avons menée sur l'histogramme (Roditi, 2009b) en est un exemple. En outre, les conditions d'exercice du métier conduisent les enseignants à partager quelques principes généraux, et par conséquent des choix globalement analogues quant aux contenus et à l'organisation adoptée pour les transmettre. Ces invariants délimitent une enveloppe au sein de laquelle s'inscrivent les enseignements observés, et qui ne contient pas tous les scénarios envisageables *a priori* avec seulement des critères liés à l'apprentissage des élèves. De tels résultats intéressent la formation des enseignants.

Néanmoins les pratiques sont variées, les enseignants investissent les marges de manœuvre qui existent par-delà les contraintes, et la palette des différences constatées couvre aussi bien les activités induites en classe chez les élèves que les aides proposées par les enseignants. La diversité observée s'explique par la composante personnelle des pratiques, dont le rapport avec les conceptions de l'enseignement et de l'apprentissage a été montré ici. Cette recherche a finalement permis de dégager des éléments qui tiennent aux personnes et qui expliquent la diversité des pratiques enseignantes. Elle a montré aussi que certains éléments sont partagés par les enseignants, que cela homogénéise leurs pratiques, ces éléments tiennent bien sûr aux contraintes institutionnelles, mais aussi, plus largement, au métier.

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CONNECTIONS: MATHEMATICAL, INTERDISCIPLINARY, ELECTRONIC, AND PERSONAL

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ABSTRACT

How can creating connections improve our students' learning? We argue that students' appreciation of intellectual connections profoundly affects their grasp of the material. We argue that personal connections have a major effect on their confidence and motivation. We consider how the landscape will change as courses move online. We analyze a range of different types of connection, characterizing each and identifying its challenges and benefits, and discuss how each might be created.[1]

WHY CONNECTIONS?

Learning involves making connections: Connections between ideas, connections between fields, and connections between people:

> *Stunning new research on the brain by neuroscientists is adding a new dimension to our knowledge about learning that reinforces our previously tentative conclusions from cognitive psychology. This research provides growing evidence that **learning is about making connections**.* (Cross, 1999, p. 5)

Expert teachers at every level advocate helping students make connections between ideas. According to The Public Broadcasting Service (PBS), "Understanding what [mathematical] symbols represent in the physical world is important to how well and how easily a child will remember a concept" (WGBH, 2002, para. 4). *You*Cubed at Stanford describes mathematics as "about connections and communicating" (Boaler, 2016, p. 6) and Harvard's Katherine Merseth's advice on teaching and learning includes, "If you can hook a new understanding to an existing one, it will stick better" (Walsh, 2016, para. 6).

In this paper we consider what it takes to build different types of connections and the challenges and benefits each brings.

MATHEMATICAL CONNECTIONS

The most salient connections are those between different areas of mathematics. For us, as mathematicians, these connections are the life-blood of mathematics—they create the depth of understanding that allows us to analyze problems from a flexible point of view. Insight flows

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[1] Based on abstract for the KHDM Conference, Hannover, Germany, December 2015.

from the power to match the point of view to the task at hand. However, our students do not always think the same way. For them, connections may be an added burden rather than a welcome tool.

For example, mathematicians readily see the differences in algebraic structure between the equations $(x-1)^2+4=0$ and $(x-1)^2-4=0$, making it immediately clear why one has real solutions and the other does not. Many students, especially at the time they are learning to solve quadratic equations, find seeing such structure hard—and not worth the trouble. They prefer a single algorithm that does not require special insight in special cases. To analyze the solutions of these two equations, many students prefer to expand the squared term, use the quadratic formula, and look at the sign under the square root. This opens the door wide to computational mistakes. But from the student's point of view, the expanding method is safer because it always works. The need to use different insights in different problems is alarming—students wonder how they should know what to do?

Later, in linear algebra, many students struggle to see a set of polynomials as a vector space. The students' mental images of polynomial curves clash with their mental images of vectors as arrows. Students find it difficult to see the connection between the graphical representation and the properties that define a vector space.

In order to address students' gaps in understanding, we need to understand why so many students do not see connections. Student feedback makes it clear that their vision of mathematics is often not the same as ours. For example, first-year Harvard students in calculus agreed[2] with the statements that "*A well-written problem makes it clear what method to use to solve it*" and "*If you can't do a homework problem, you should be able to find a worked example in the text that will show you how*" (Hughes Hallett, 1990). These Harvard student responses suggest that for them, as for many other students, mathematics is a set of procedures—solving problems on a pre-determined list of types. They see their role as students as learning to execute these procedures accurately, not as understanding the connections between them. This belief is unfortunately reinforced by the fact that the tests they take can often be done entirely procedurally, even if they were not intended that way. Thus we play a role in solidifying the idea that math is a set of procedures rather that an interconnected set of ideas.

While there is general acknowledgement that procedures *are* indeed important, they cannot be learned effectively in isolation: "To develop procedural fluency, students need experience in integrating concepts and procedures and building on familiar procedures as they create their own informal strategies and procedures" (NCTM, 2014, NCTM Position section). Students who try and learn mathematics entirely as procedures are generally not successful; they misapply what they have learned and they forget quickly. Thus, even to achieve computational fluency, students need to understand some concepts and see some connections. The question is how to accomplish this?

Research shows us that "as students learn a discipline, their knowledge of the structural relationships among parts of the discipline becomes more like that of experts" (Schoenfeld, 1985, p. 245). In other words, some understanding of connections occurs naturally as students learn more of a subject. However experience suggests that different students gain insight at very different rates. Some students are content with procedural knowledge for a long time. Others, like my statistics student who did not calculate until she saw exactly why an algorithm worked, can't do calculations until they understand the connections.

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[2] Scoring over 4 out of 5 on a Likert scale.

How does this help us in teaching? The fact that most students gain insight in stages suggests that we should not expect an understanding of connections right away. But that does not mean that we should abandon the effort. Far too often we teach a topic, test procedurally so as not to discourage those who are just learning, and move on to the next topic before students have had time to build connections. Our syllabus needs to provide time for students to master the ideas at a deeper level so they start to see the connections.

> **Challenge:** How can we increase students' appreciation of the connections between different areas of mathematics?

The answer lies in the activities students do. Active learning is the most effective way to deepen understanding (Mazur, 2014; Prince, 2004), so problems are our central tool for this endeavor. The way in which we give these problems is also important: research at Harvard and at Cornell suggests that probing questions are much more effective when they are followed by a period of discussion in which students reflect on why solution strategies work and think about other strategies that might work (Terrell, 2005).

Problems involving multiple representations (the *Rule of Four*) are particularly useful to deepen student understanding. For example, the following problems directly probe students' ability to see connections in calculus (see Figure 1).

The first problem pushes students to think about concavity in terms of the link between graphs and function values, rather than in terms of a mechanical procedure of calculating second derivatives.

24. Each of the functions in Table 1.11 decreases, but each decreases in a different way. Which of the graphs in Figure 1.35 best fits each function?

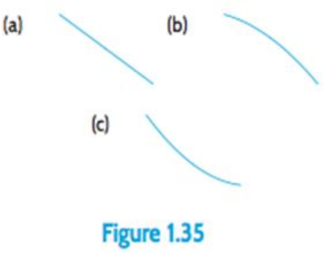

Figure 1.35

Table 1.11

x	$f(x)$	$g(x)$	$h(x)$
1	100	22.0	9.3
2	90	21.4	9.1
3	81	20.8	8.8
4	73	20.2	8.4
5	66	19.6	7.9
6	60	19.0	7.3

Figure 1. Connections in calculus (Hughes Hallett, Gleason, & McCallum, 2013, p. 18).

Being able to make the connection between graphs, formulas, and calculations helps students use mathematics in all the natural and social sciences. The following two problems focus on this link (see Figure 2).

43. The graphs in Figure 5.59 represent the velocity, v, of a particle moving along the x-axis for time $0 \leq t \leq 5$. The vertical scales of all graphs are the same. Identify the graph showing which particle:

(a) Has a constant acceleration.
(b) Ends up farthest to the left of where it started.
(c) Ends up the farthest from its starting point.
(d) Experiences the greatest initial acceleration.
(e) Has the greatest average velocity.
(f) Has the greatest average acceleration.

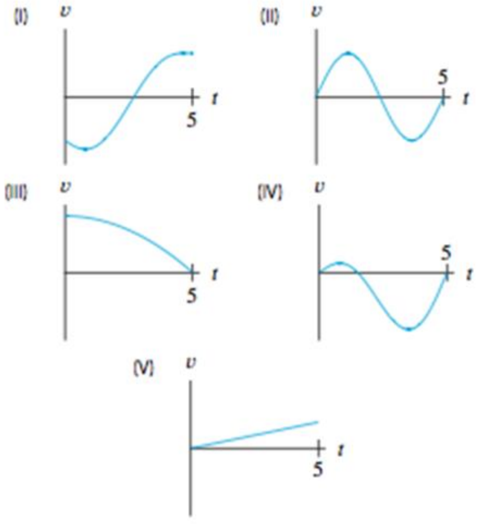

Figure 5.59

30. A warehouse charges its customers \$5 per day for every 10 cubic feet of space used for storage. Figure 5.48 records the storage used by one company over a month. How much will the company have to pay?

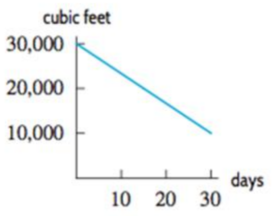

Figure 5.48

Figure 2. Connections between graphs, formulas and calculations
(Hughes Hallett, Gleason, & McCallum, 2013, p. 297 and p. 296, respectively).

The connection between words and symbols is central to understanding, and we often take this link for granted. Asking students what their calculations represent, we may be surprised to find that they do not know—and more surprised to find that they have not thought about it. The following problems are designed to focus on the link between symbols and interpretation (see Figure 3).

1. If $f(t)$ is measured in dollars per year and t is measured in years, what are the units of $\int_a^b f(t)\,dt$?

2. If $f(t)$ is measured in meters/second^2 and t is measured in seconds, what are the units of $\int_a^b f(t)\,dt$?

Figure 3. Connections between symbols and interpretation.

A similar but harder set of problems, challenging for almost all students, follows (see Figure 4).

In Problems 52–54, list the expressions (a)–(c) in order from smallest to largest, where $r(t)$ is the hourly rate that an animal burns calories and $R(t)$ is the total number of calories burned since time $t = 0$. Assume $r(t) > 0$ and $r'(t) < 0$ for $0 \le t \le 12$.

52. Letting $t_0 = 0, t_{100} = 12$:

(a) $\sum_{i=0}^{99} r(t_i)\,\Delta t$ **(b)** $\sum_{i=1}^{100} r(t_i)\,\Delta t$ **(c)** $\int_0^{12} r(t)\,dt$

53. **(a)** $R(10)$ **(b)** $R(12)$ **(c)** $R(10)+r(10)\cdot 2$

54. **(a)** $\int_5^8 r(t)\,dt$ **(b)** $\int_8^{11} r(t)\,dt$ **(c)** $R(12)-R(9)$

Figure 4. Further connections between symbols and interpretation.

The fact that students may learn to calculate without knowing what they are calculating should worry us. The consequence of our retreating into procedural problems is that we leave the impression that they are *all* of mathematics. Nothing could be further from the truth!

EVALUATION

Evaluation of classes taught using the *Rule of Four* has shown that this is a successful technique. Students at Baylor University taught by this method scored higher on a final exam than students taught in the traditional way (Tidmore, 1994). Since the traditional instructors had written the final exam, this study provided good evidence of the effectiveness of multiple representations.

In 2008, the University of Michigan's calculus program gave all its students the Calculus Concept Inventory (CCI), which tests for gains in conceptual understanding. The CCI reports class gains on a scale of 0 to 1. Prior to Michigan, most gains had been between 0.14 and 0.20. Michigan's 51 sections had an average of 0.35 and ten sections over 0.40 (Rhea, 2008).

Their success is attributed to their use of challenging problems, multiple representations, and an active learning environment.

The contribution of an active learning environment is also seen in the evaluation of teaching with *ConcepTests* (clicker questions): students taught with these questions scored significantly higher on conceptual understanding and did not lose ground on computation (Pilzer, 2000).

INTERDISCIPLINARY CONNECTIONS

For the majority of students taking mathematics in the service of another field, interdisciplinary connections provide essential motivation and inspiration. In an Indian school where over 80% had failed mathematics, students reported in interviews "they are not interested in studying Mathematics" (p. 5) because they perceive the subject as "too far from life to catch [their] interest" (p. 3) (George & Thomaskutty, 2007).

Unlike mathematicians, who are often attracted to mathematics for its beauty, many students are unmoved by its beauty. They want to know how mathematics is used—how it will contribute to their impact as an engineer, a doctor, a linguist, or an economist, or whatever profession they choose. Knowing the uses of mathematics gets many students actively engaged in learning the subject; otherwise they may be inclined simply to survive it.

Many of the recent advances in research are also interdisciplinary. The National Academies (2004) report that, "[a]dvances in science and engineering increasingly require the collaboration of scholars from various fields" (para. 1). Finance, biology, economics, education, and the law all use mathematics increasingly frequently and now depend heavily on data. Our teaching will similarly benefit from interdisciplinary input.

Interdisciplinary links in mathematics courses are not sugar-coating—their role is not to make an unpalatable subject palatable. Their role is to enable students to develop a deeper understanding of both mathematics and the other field. To ensure that this occurs, the applications shown have to be authentic. Contrived examples simply reinforce students' notions that mathematics is not actually useful.

> **Challenge:** How do we create meaningful interdisciplinary connections to other fields? Especially with fields that we have never studied ourselves?

In order to make these connections, we first need to learn how mathematics is used by other disciplines. Informal meetings with colleagues, textbooks from other departments, and former students are all good sources of information. Co-teaching, the use of joint assignments, and applied projects within a mathematics course can all be vehicles for cooperation. Making this link for our students is an intellectual challenge, but an important component of our teaching. Our effort to understand is greatly appreciated, not only by students but also by faculty of other departments, as evidenced by the enthusiastic response to the meetings described in the Mathematical Association of America (MAA) report, *Voices of the Partner Disciplines* (Ganter & Barker, 2004).

The uses of mathematics in other fields can be demanding—and surprisingly different from what we teach. For example, economists' use of mathematics relies heavily on graphical and intuitive reasoning and requires an understanding of the connections between the two. The following diagram is used to argue that the minimum average cost occurs when average and marginal costs are equal (see Figure 5).

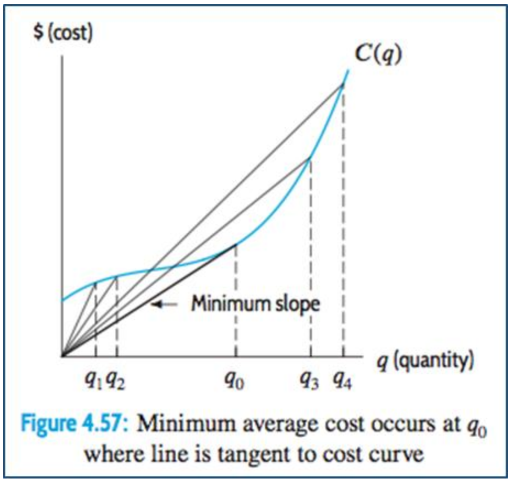

Figure 5. Minimum average cost (Hughes Hallett, Gleason, Flath, & Lock, 2014, p. 204).

Many disciplines use non-linear scales that are unfamiliar to our students. For example, the following graph was given to illustrate the growth of the US Consumer Price Index (CPI) over the past century (see Figure 6). Economics students are expected to conclude—without the help of the fitted line and its equation—that inflation had averaged about 3.25% annually over this period.

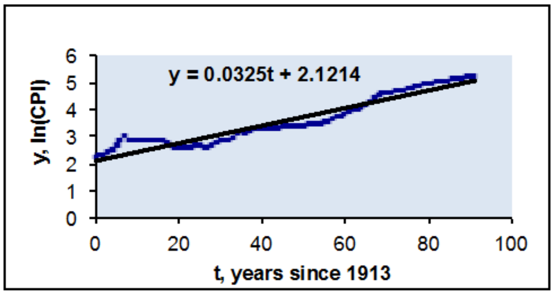

Figure 6. The US Consumer Price Index.

Students in biology are expected to recognize that if $V_{\max}$ and K_M are constants, the Michaelis-Menten equation governing a chemical reaction,

$$V_0 = \frac{V_{\max}[S]_0}{K_M + [S]_0},$$

is linearized by a reciprocal plot of $1/V_0$ against $1/S_0$, which rewrites the equation in the form

$$\frac{1}{V_0} = \frac{K_M}{V_{\max}} \cdot \frac{1}{[S]_0} + \frac{1}{V_{\max}}$$

Both the CPI and Michaelis-Menten examples use only topics that are taught in high school, yet they are hard even for students who have had calculus because they use logarithms and reciprocals in unfamiliar ways.

Links between fields are easier in some parts of mathematics than others. With data readily available, statistics is an excellent course to showcase interdisciplinary connections. In its *Guidelines for Assessment and Instruction in Statistics Education* (GAISE), the American Statistical Society recommends that statistics courses use real data wherever possible, (Garfield et al., 2005). Since data often sheds light onto issues of importance to students, our statistics courses benefit greatly from including such data.

For example, the *Black Lives Matter* movement, which campaigns against violence towards black people, has brought attention to racial profiling.[3] The police data for many cities is available on the web, so we can look at the numbers of people stopped, frisked, and arrested. The 2013 New York City stop-and-frisk data in Table 1 below provides a poignant setting to understand conditional probability, Bayes' Theorem, and statistical significance (NYCLU, 2014).

People Stopped	White	Black	Latino	Total
Frisked	9,729	63,998	31,824	105,551
Not Frisked	11,101	40,451	23,106	74,658
Total	20,830	104,449	54,930	180,209

Table 1. NYC stop-and-frisk data.

The discussion of significance leads directly into the role of confounding variables. Since the sample was not random, what might have affected the data, besides race? Possibly region; the New York Police Department reports show that 911 calls are not evenly distributed through the city. (Bratton, 2015)

What is the impact of including such examples in a course? Do they distract from the mathematics? Consider the enthusiastic reaction of the student who took a statistics course based on such examples, saying, "*This is the first math course I have taken that was about anything*". It is sobering to realize that this student had previously taken calculus—at a very good institution—but apparently not found it meaningful.

Real examples can engage students who otherwise find mathematics disconnected from their experiences and interests. To be successful with them, we have to start where such students are and make the case that the mathematics we are teaching can benefit them.

ELECTRONIC CONNECTIONS

With the rapid growth in online courses, we need to be able to adapt our efforts to make connections. In the curricula arena, it is not hard to imagine the transition. Mathematical and interdisciplinary materials can be either paper or electronic. Interdisciplinary videos are likely to be an improvement over paper materials. The transition to online is likely to occur naturally with time.

Online grading is already well-established and fairly robust. *WebWork* and *WebAssign* already carry out a great deal of the day-to-day grading in US mathematics departments and are surprisingly helpful. When the grading of verbal answers and explanations becomes possible, it will be a huge boost for electronic courses.

However security, which is vital for credit courses, although not for MOOCs, is currently a gaping hole. The current online proctoring options lag far behind what is needed.

[3] http://blacklivesmatter.com/

Challenge: How will the technology adapt to provide security for credit courses?

Meeting this challenge is unlikely to involve us directly, except as users. But as companies become increasingly interested in providing online training and MOOCs become essential gateways rather than luxuries, the pressure to improve security will likely produce results. As a community we need to be ready with requests that can shape the services offered into ones we can use.

An online medium has huge potential. Instead of spending time presenting material that students can read or learn from a video, we will be able to focus on teaching.

Challenge: Before a paradigm is thrust on us, the community of mathematicians should shape the electronic connection they want.

Office hours may involve stitching together existing software or the next generation offspring—for example, *Facetime*, *Skype*, screen-sharing software—to enable us to see into the minds of our students in the same way as we can during an office visit.

PERSONAL CONNECTIONS

The most undervalued tool in our teaching arsenal is our ability to forge personal relationships with our students. Asked to recall a good teacher, most people point to someone who believed in them—not a teacher who did a brilliant job of presenting a theorem.

Technical teaching skill is important—but it is only part of the story. Equal enthusiasm for the material and for the students is necessary because "[p]eople don't care how much you know until they know how much you care" (Merseth, as cited by Walsh, 2016, para. 5).

Yet there are often difficulties: culture, language, or interest may be barriers, and time is always short. In addition, the enrollment is often huge in introductory courses where these connections may be the most important, making any kind of human contact difficult. We may never meet online students face-to-face.

Challenge: How do we bridge the gaps and structure beneficial relationships with students?

Challenge: How will we interact with online students?

The answer includes listening as much as talking, restricting advice until it is asked for, and knowing who would benefit from our support and who would not. Refer elsewhere the ones who would benefit from other advice. Some students do not want anything from a course except the material, but others are questioning the direction of their lives and greatly value our support or critique. The moments spent talking outside class may be some of the most influential teaching moments we have.

Many students attribute their later success to someone who took the time to listen and who believed in them even when they no longer believed in themselves. This is the most important contribution we can make.

The power of having a teacher who believes in you is priceless.

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Working Groups

Groupes de travail

TASK DESIGN AND PROBLEM POSING

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France Caron	Stephen MacGregor	Sophie Pinard
Bernardo Galvao-Sousa	Asia Matthews	Elaine Simmt
Jennifer Holm	Erin Murray	Peter Taylor

Working group, for to
Task design; problem posing.
Focus on structure.

Les structures et la restructuration ont, à travers notre expérience, joué un rôle important dans la conception, le développement et la mise en œuvre de taches d'apprentissage et de problèmes riches en mathématiques et en enseignement des mathématiques. Lorsque nous référons à la (aux) structure(s), nous entendons :

- Structures mathématiques — telles que les structures algébriques de groupes, anneaux ou corps commutatif, ainsi que des structures comme définition-théorème-preuve ou problèmes-raisonnements-relations (Watson & Mason, 1998);
- Structures pédagogiques — telles que la contextualisation, l'accompagnement, des dispositions environnementales, ou des structures et restructurations cognitives.

As in a haiku,
structures in mathematics
can inform purpose —

imagination;
expressions of ideas;
interpretation;

with each intention
nuances and emphases
give way to newness.

In this working group, we explored questions and issues around structure and (re)structuring in task design and problem posing. Our investigations were guided by several questions, and in what follows, we share participants' reflections on issues such as, intentional ambiguity, diversity of structures, influences on task design, and the emergence/imposition/construction of structure.

Les questions suivantes guideront notre réflexion:

- Qu'est-ce qui constitue un bon problème pour l'enseignement? À quoi peut-on reconnaitre un bon problème? (*What constitutes a good problem for teaching? How might that be recognized?*)

- Quel est le rôle de la structure dans les connaissances disciplinaires de l'enseignant? (*What is the role(s) of structure within a teacher's disciplinary knowledge?*)

- Est-ce que concevoir une tâche pourrait être un exemple de résolution d'un problème riche? (*Is designing a task an example of solving a rich problem?*)

- Comment peut-on construire des tâches, des problèmes ou des recherches pour attirer l'attention sur la structure mathématique? Si les problèmes sont abstraits? Si les problèmes sont dans un contexte de la vie courante? S'ils sont pertinents socialement? (*How can we structure tasks, problems or investigations to draw attention toward mathematical structure? If the problems are abstract? "Everyday"? Socially relevant?*)

COMMENT PENSEZ-VOUS À LA STRUCTURE MATHÉMATIQUE ET AUX STRUCTURES POUR L'ENSEIGNEMENT DES MATHÉMATIQUES?

HOW DO YOU THINK OF MATHEMATICAL STRUCTURE AND STRUCTURES FOR MATHEMATICS TEACHING?

Richard: I see structure in mathematics. That is, the structure is something I perceive; it is not independent of me (even if it feels like it is).

Peter T.: Mathematics is the study of structure.

Elaine: There are multiple domains within which we can observe for structures, e.g., ways of thinking, mathematics as a discipline, phenomena that are modelled with mathematics, tools that are used, the knowledge of others. Each of these are implicated in mathematical activity actions/interactions/knowing/knowledge.

Asia: I see mathematics as both the formal system, in which there are formally identified and described structures (groups, relationships, systems, categories, notations…) and a mental activity, in which structuring is performed (visualizing, organizing, representing, symbolizing, modeling).

Susan: My attention has been drawn to the many layers of structure that can be considered when designing tasks. There are the mathematical structures, both those you intend the students to encounter, and those they may unexpectedly introduce (or stumble upon). There are the structures that they will use to help them make sense of the problem and the structures that reveal connections to other mathematical structures. Then there are the pedagogical structures, including the structuring of the question(s), the information given, the timing; also attending to the social structures (pairs, groups, jigsaw…) and the physical structures (vertical surfaces, round tables…). All with the goal to ensure that ultimately the mathematical structures are revealed/reconstructed/embodied by our students.

EXEMPLES DE STRUCTURES?

Elaine: Ways of thinking – inductive reasoning and the value of being systematic
Mathematics as a discipline – our activity is structured by the norms/conventions/practices of those that engage in mathematics
Phenomena – these constrain our activity/actions/interactions/knowing/knowledge
Tasks – different tools (mental, physical, digital) impact the mathematics we create and our ways of knowing.

Asia: Within the formal system, as mathematicians we identify separations of fields based on the structures that are recognized between objects/situations. There is a curriculum design issue here. The identification of these divisions right off the bat is not necessarily a good way of drawing out the kind of awareness and attention to structure that should/would/could be the study of mathematics. It could be that this focused attention to a mathematical field of structures leads to a somewhat fragmented view of maths as a whole.

AU SUJET DE LA STRUCTURE DES TÂCHES MATHÉMATIQUES — QUE CONSTITUE UN BON PROBLÈME POUR L'ENSEIGNEMENT?

ON STRUCTURING MATHEMATICAL TASKS — WHAT CONSTITUTES A GOOD PROBLEM FOR TEACHING?

Deborah: Low floor, high ceiling, rich content, intentional ambiguity, liberating constraints. There is more (much more!) to engaging into a task than the math content it contains (or the content "discovered" by solving the task). For example, there's also the diverse math structures one can choose to engage into a task.

Jennifer: I appreciate the task "structure" provided by Peter of motivating question, access to lots of data, and liberating constraints. It gave me something to hold onto as I plan tasks…. Intentional ambiguity is important since it more mimics problem solving in real life. In the end, rich tasks are not just real contexts but should use real life skills to solve (not "school" math vs. "real" math).

France: The notion of "liberating constraints", to close (a little) or structure a task that would be too open has been awoken by being here. Language structure, imposing ambiguity, e.g., what is a hypothesis? Fairness as a strong motivator in engaging students in structuring a situation, in looking for a structure in a situation. Is *fair trade* fair?

Peter L.: Structure of tasks based on – math as a noun (content), math as a verb (thinking/action). What has awakened? An awareness that I have an organizational framework for classifying tasks that I use: [ones that] always work, work when the population is right, or work as long as I do the right things as a teacher. I have not analysed qualities of these. There has been some discussion of "structure" that might help me with this.

STRUCTURING TASKS: TWO EXAMPLES

Questions around imposing extracting/revealing/injecting structure into mathematical tasks were explored in the context of different mathematics problems and contexts. We discuss two examples here. Two additional problems can be found in Appendix 1.

FAIRNESS AND FAIR TRADE

We explored a context problem that looked at fair trade through a mathematical lens, which served as fodder for critiquing and developing a structural approach to designing mathematical explorations that are framed in issues of social justice.

Discussions included the motivating and de-motivating factors of the task—fairness and fair trade are compelling, but you need to know a lot of background information before being able to address problems. Text-based questions can become de-coding exercises, videos can offer an alternative; caution must be taken so that the framing of the problem invites inquiry without creating potential biases. There is a sense that students may need to become aware that there are questions to pose about such issues (and that these questions may not have answers).

Peter L. summarized an approach to framing such tasks: 1) motivating question, 2) data to share, 3) liberating constraints. Asia and Erin discussed different possible contexts that address fairness: cancer treatment and deciding on an order of people to treat, election data and counting votes in different ways. Questions were raised about how we can think about (school) curricula such that problems like this could fit in a mathematics class, as well as how to tame the scope of a problem that might easily get away from the learners exploring it.

HEXOMINOES

Some games and/or problems offered contexts for doing mathematics with a conscious view toward sharpening awareness of the place of structure. Conditions and materials contributed to our discussion of structure. Presentation implicitly draws some towards while pushing others away from engagement. The example shared here uses a definition as a starting point. Games may use a set of rules.

We will define a *hexomino* as a figure consisting of six interlocking blocks that can be placed flat on a table so that each block has a base on the table. The challenge is to create a complete set of hexominoes. Note that hexominoes that are rotations or reflections of others are not considered to be distinct.

On définit un hexomino *comme une figure formée de 6 blocs qui s'emboîtent et qui peuvent être placés à plat sur une table de façon à ce que chaque bloc ait une base sur la table. Le défi consiste à créer un ensemble complet d'hexominos. Il est à noter que les hexominos qui représentent une rotation ou une réflexion des autres hexominos ne sont pas considérés comme étant distincts.*

The richness in this challenge was associated with the structure of solution. The statement of the task combined with prior mathematical experiences to impose a sense of finiteness. It is clear that there is an attainable answer that will consist of a collection of arrangements. The problem solver is challenged to impose an organizational system, as randomly making figures is likely to pose its own problems with distinctness. When are we done? How is it that we know that there are no other possible hexominoes to be found? Essentially the recreational mathematical challenge brought us to proof.

AU SUJET DE LA STRUCTURE DES TÂCHES MATHÉMATIQUES - QUELLES QUESTIONS PERSISTENT?

ON STRUCTURING MATHEMATICAL TASKS – WHAT QUESTIONS REMAIN?

Sandy: The piece that interests me that previously I have never thought about is the structuring of thinking or structuring of the resulting math, that the person interacting with the task does, and whether this structuring is conscious or unconscious? I wonder if those who are strong at math are more conscious of this, which is part of their success, than those who struggle with mathematical tasks?

Jimmy: How do the structures of mathematics influence our task design? What informs the decisions when choosing a task?

Deborah: Isn't there room/shouldn't I make room for my students to do a task, for example, for the sake of learning the diversity of structures (e.g., solving in two different ways)? To engage into a task "just to" organize, find patterns, notate, etc.?? Content as a tool, instead of end result (though both are not mutually exclusive)?

Minnie: I think we can both design a task based on specific mathematics we want to highlight, or take a task and push specific mathematics out of the task. I am not certain which is easier or which makes more sense, but both are probably rather do-able.

Peter T.: Thoughts about the need to widen my scope of what "math class" ought to be—of what kind of experience my students should be having—both in learning and in assessment. And I look at ideas through a lens of structure.

Richard: In a teaching/learning situation, a negotiation happens between participants (e.g., teacher/student) in which the difference in perceived mathematical structure encounter each other and adapt to each other. How can this negotiation be organized so that it is productive?

Jennifer: I felt our group discussion about how we structure a problem solving method is deeply personal. As teachers we need to set up tasks that allow students to use multiple structures/representations to allow them to access these personal structures as well as stretch their repertoire. A point I have been pondering since day 1: my group raised the point that they enjoyed the hexagon problem up until they had to "do the math" to solve. Was it because they could already complete the formulas? How does this impact us as classroom teachers?

France: Can the use of problems considered to be isomorphic contribute to developing a sense of the value of structure? Or is using different structures to approach the same problem a more profitable approach? How can you make students want to consider a different structure when the absorbing one (typically reduced to its techniques/ processes) seems to be doing the job?

Anon: I find myself questioning how I might initiate the designing of a task; whether effective task design should begin with big ideas/curriculum expectations in mind, or whether these measurements should be attached to something engaging and relevant. I have also found myself questioning the mathematical processes I ask students to perform in the tasks I design. Is the way in which I unpack and approach the mathematics subjectively affecting how I structure my tasks and the mathematics I expect from my students?

MOT DE LA FIN

Not surprisingly the spirit of a CMESG working group and the subjects of task design and problem posing combined to produce more questions than answers. Questions raised remain unresolved collectively in that no pat answer will suffice. Rather the discussions and activities allowed us to delve deeper into our own individual understandings of structure, task design, and more. Perhaps the idea of *awakening* best reflects this experience.

The ideas were neither novel nor new, but instead the concentrated attention facilitated an awakening within each of us to something we were not so keenly aware of before, or possibly an entirely new idea emerging from participation in the working group. As with the report, there is no way of "recapturing the moment", so to speak. One participant, Sophie, aptly captures this with her closing reflection:

> The attempt to re-pack my luggage in anticipation for the flight home was severely uncomfortable. How can I repack all this into a tight kernel when I feel so many discussions remain unresolved?… Thinking of the end brings me back to the beginning, what was at the kernel of our task?

Fittingly we close with a question for you to consider, thus, highlighting the open nature of the topic as biases, experiences, and contexts figure into evolving personal responses.

> As we do mathematics how is it that we consciously unpack the math, and in unpacking mathematics, how are we doing mathematics?
>
> *Comme nous le faisons mathématiques, comment est que nous déballons consciemment le calcul, le mathématiques et le déballage comment allons-nous faire de mathématiques?*

Acknowledgments: We would like to express appreciation to the volunteers, particularly Caitlin Furlong, who provided translations to problems and activities.

APPENDIX A: STARTING POINTS

Many problems and activities were shared including some that were submitted to a Dropbox account. (People interested in seeing other contributions are welcome to contact the working group leaders for access.) The examples below were contributed by France Caron and Peter Liljedahl respectively.

LE RUBAN ENROULÉ

Un long ruban flexible, non extensible et très mince, est enroulé serré autour d'une bobine circulaire. La longueur du ruban est de 1 km et son épaisseur est de 0,1 mm.

Sachant que le rayon de la bobine vide est de 10 cm, déterminer approximativement le rayon (en cm) de la bobine pleine contenant le ruban enroulé (L'Association mathématique du Québec, 1979).

CHAIN LINKS

You are back-packing through Europe. You have one month left until your flight home, but you have run out of money. However, you have a 50 link gold chain and you have found a hotel that is willing to accept one link per night for payment of room and board. However, the manager wants payment every day and he is willing to help you out by cutting links for you. The problem is that he wants one gold link payment for every link he cuts. What is the most number of links that you will have left when you fly home?

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INDIGENOUS WAYS OF KNOWING IN MATHEMATICS

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Roman Chukalovskgy	Limin Jao	

INTRODUCTION

As we planned for this group we acknowledged that there are many ways that this working group might 'work' on these ideas. Over the three days, this group engaged in a variety of experiences to explore what is meant by *Indigenous knowledges* and how we hold onto these ideas in relationship with what we've come to know as 'mathematics' or 'mathematics education' or 'mathematics teacher education'.

STARTING EACH DAY IN A GOOD WAY

As we planned, we imagined that we would be able to have an Elder alongside. However, this did not occur. However, Florence brought along a bundle of sage from the Indigenous Teaching and Learning Gardens at the University of Alberta. The sage was a reminder of the sacred medicines and the importance of the conversations that we were about to embark upon. The sage was gift from the place we now call Alberta. The sage bundle provided us with a reminder of our connectedness to the land each morning as we started with a sharing circle. The sharing circle began each of the three days so that we could each acknowledge our heart and mind.

The teachings that Florence has been given around the importance of the circle as a place of beginning is that individuals acknowledge their own heart and mind as they enter into the conversation. In the circle, each person is invited to describe "*what is on the heart*" and "*what is in the mind*" as they hold the sage bundle. Once this is shared, it is now a part of the 'collective' or the 'community' and we can now 'hear each other'.

DOING BIRCH BARK BITING

The second part of the first meeting was dedicated to experiencing birch bark biting first hand. Lisa shared her story of coming to birch bark biting following a conversation with a Mi'kmaw elder who described it as a common pastime when she was young and encouraged Lisa to learn more about it. Lisa shared that while following up on the conversation she came across an article that demonstrated that birch bark biting was indeed a historical part of the Mi'kmaw community:

> *That she was "the last one that can do it" was the same phrase echoed in 1993 by Margaret Johnson, an Eskasoni Micmac elder from Cape Breton. Continuing research has revealed that two other Micmac women – including Johnson's sister on another reserve – can also do it.* (Oberholtzer & Smith, 1995, p. 307)

Lisa had known both Margaret, who was affectionately known as Dr. Granny, and her sister, Caroline Gould, who had resided in the community where Lisa had taught and often visited the school. Unfortunately both women had already passed away by the time Lisa began researching this practice.

Birch bark biting involves folding thin pieces of bark and biting shapes into the bark to create designs. The act of folding the bark presents an opportunity to think about fractions, angles, shape, and symmetry. Creating the designs draws in geometric reasoning and visualization of geometric shapes. However, on this day we began in much the same way Lisa has explained she begins with students—showing photos of birch bark biting and asking the question, "*How do you make this?*"

After some quick tips on how to fold the bark, we first practiced on waxed paper which enabled people to get a sense of folding and biting before moving on to using real birch bark. Fold lines are easily seen on the waxed paper (see Figure 1). After a few tries with waxed paper, we were ready to work with the real birch bark. The bark had to be peeled into single layers which were very thin and often very delicate.

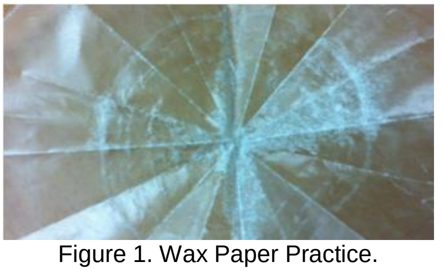

Figure 1. Wax Paper Practice.

Through doing the birch bark biting many questions and observations emerged. The pictures tell the story of the birch bark biting. See Figures 2 and 3.

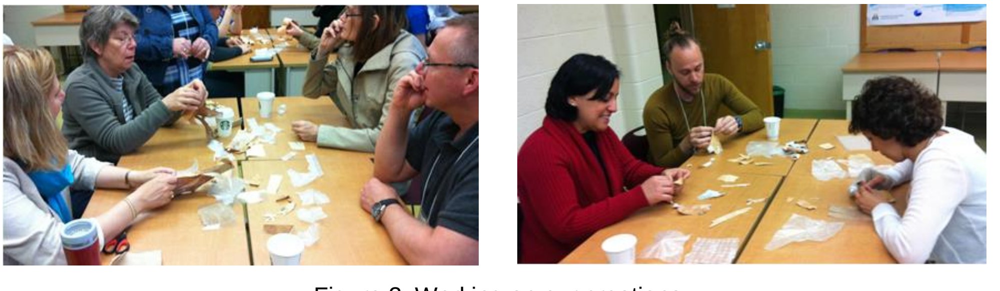

Figure 2. Working on our creations.

Figure 3. Some of our work.

EXPLORING ALTERNATE WAYS OF DOING MATHEMATICS

On day 2, we reconnected in circle, sharing some thoughts from the previous day and were excited to continue learning. As a way of considering how mathematics learning can emerge through centring Indigenous knowledge, Lisa shared videos from two projects that were part of the Show Me Your Math (SMYM) program in Nova Scotia public and Mi'kmaw schools. These SMYM projects were inspired by Doolittle's (2006) idea of pulling in mathematics by beginning in aspects of community culture where the already present, inherent ways of reasoning within the culture can help students to make sense of the 'school-based' concepts of mathematics in the curriculum. One goal of this work is to have teachers and students learning alongside one another as they explore practices that are relevant to the community. As such, these projects have been called *Mawkinamasultinej!* Let's Learn Together! as a way of emphasising this focus on learning together. As a group we watched the videos from the eel project done by a grade 4/5 class in a public school called #KataqProject using the Mi'kmaw word *kataq* meaning eels (see Figure 4). We also watched *Sap to Syrup*, a project about maple syrup making done by a grade 5/6 class in a Mi'kmaw school. The links to these videos are given below.

> #KataqProject Link:
> https://www.youtube.com/playlist?list=PLJzemBBS0KqjfxhC8sdlwImxjgOAdi8xZ
>
> Sap to Syrup Link:
> https://drive.google.com/a/pictoulandingschool.ca/file/d/0B9xZFTcbPGMienpJekVNZU1Bb3c/view?usp=sharing

These videos provided examples of culturally-based inquiry that can be connected to the school-based mathematics curriculum. Teachers involved in these projects have shared that the outcome connections easily emerged as they engaged in these projects.

Others in the group shared videos and presentations about their work in communities. Cynthia described her work in British Columbia, Annie in Northern Quebec, Bev in Northern Ontario, Elysée in Southern Quebec, and Florence in Alberta.

Figure 4. The #KataqProject.

Our discussions of how mathematics might emerge in different contexts led us to discuss how we do mathematics differently in different contexts, including using different algorithms. Our diverse group had diverse approaches to doing a standard mathematics problem, which helped to prompt discussions about these varied approaches and the implication for teaching and learning.

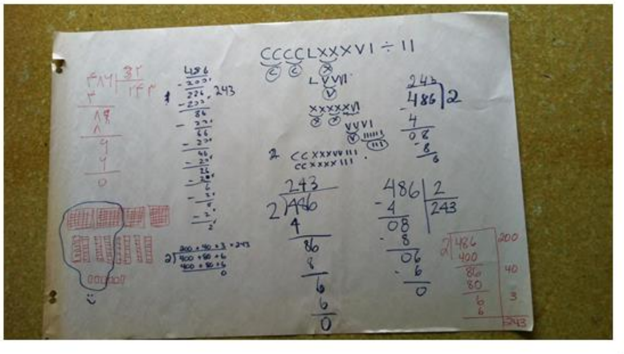

Figure 5. Working Group recording of varied approaches to division.

SHARING OUR WORDS

Those of us who had extensive experience working in Indigenous contexts were able to share some words we had been given that we felt helped us to conceptualize some of the ideas we were discussing. Lisa shared *mawikinutimatimk* meaning "coming together to work together", a word use to describe a way of working together to solve a problem or discuss an issue, and *mawkina'masultinej* meaning "let's learn together", a word that has been used to describe the inquiry projects that have become part of the Show Me Your Math program. In a similar vein, Florence shared the Cree word *Miyowichitowin* meaning "coming together to learn and live 'in a good way'" From her experience working in Fort Francis, Bev share the word *Gaa-maamawi-asigagindaasoyang*' which was developed by Ojibwe language teacher, Jason Jones, in February of 2015 to describe mathematics. It translates as "We are the ones doing the math together" and has a sense of the ongoing nature of this work. The components of the word are identified below:

- *maamawi* – together
- *asig* – gathering
- *agind* – read or count/put it through thought
- *aaso* – performing useful action
- *Gaa* and *yang* are used to put the word into a noun

These words were used to guide our thinking and inform our discussions.

KEY LEARNINGS

On day 3, our focus turned to discussions again in circle and in small groups to begin to synthesize our learning. Some key learnings are described below.

RELATIONSHIPS/RELATIONALITY

We discussed the idea that 'being together with' or 'building with' or 'learning together with' community is essential in this work. Trusting relationships are at the heart of such decolonizing endeavours. In particular, if we are not members of an Indigenous community, we must acknowledge that we have as much to learn as we have to teach; we are not the experts. We must work to build community and develop trust. We need to be aware of who we are working with and honour what the community identifies as needs. We must honour elder knowledge and value the contributions of these knowledge holders. This requires that we listen with sincerity and openness, in a true spirit of learning.

PLACE

Connecting to place and learning from place were key ideas that emerged in our discussions as well. We discussed questions of how we learn to connect to place, especially if we are outsiders to that place, and in so doing begin to learn from place. We discussed how connections to place might help us to generate different ways of knowing and different ways of relating to one another and to the place itself.

RECLAIMING / RECONNECTING / REVITALIZING

The SMYM program, the Indigenous teaching and learning gardens at the University of Alberta, Cynthia's work in Haida Gwaii, Bev's work in Fort Francis, Elysée's work in Québec, and Annie's work in Northern Québec, all provided examples of how mathematical inquiries can provide opportunities to centre Indigenous knowledge as a place of learning. Such centring often generates an opportunity for communities to reconnect with knowledge that has been nearly forgotten, taken away by forced assimilation and colonization. When such reclaiming or reconnecting happens, this often inspires community members to share stories almost forgotten or recall memories not yet shared with the next generation. This allows students to see that mathematical thinking has always been a part of community knowledge though it may not be articulated in the same way as it is in mathematics textbooks.

MATHEMATICS IN RELATION

Our discussions also focused on the nature of mathematics itself. We considered how we might bring Western and Indigenous mathematics and knowledge systems into conversation with one another.

FINAL THOUGHTS: MATHEMATICS EDUCATION FOR RECONCILIATION

As we reflected on our discussions and the ideas that had emerged in these conversations, we turned our attention to the recently released report of the Truth and Reconciliation Commission of Canada that identified 94 *Calls to Action* (www.trc.ca).

FROM THE CALLS TO ACTION OF THE TRC:

> 62. *We call upon the federal, provincial, and territorial governments, in consultation and collaboration with Survivors, Aboriginal peoples and educators to:*
>
> *i. Make age-appropriate curriculum on residential schools, Treaties, and Aboriginal peoples' historical and contemporary contributions to Canada a mandatory education requirement for Kindergarten to Grade Twelve students.*
>
> *ii. Provide the necessary funding to post-secondary institutions to educate teachers on how to integrate Indigenous knowledge and teaching methods into classrooms.*
>
> *iii. Provide the necessary funding to Aboriginal schools to utilize Indigenous knowledge and teaching methods in classrooms.*
>
> *iv. Establish senior-level positions in government at the assistant deputy minister level or higher dedicated to Aboriginal content in education.*
>
> 63. *We call upon the Council of Ministers of Education, Canada to maintain an annual commitment to Aboriginal education issues including:*
>
> *i. Developing and implementing Kindergarten to Grade Twelve curriculum and learning resources on Aboriginal peoples in Canadian history, and the history and legacy of residential schools.*
>
> *ii. Sharing information and best practices on teaching curriculum related to residential schools and Aboriginal history.*
>
> *iii. Building student capacity for intercultural understanding, empathy, and mutual respect.*
>
> *iv. Identifying teacher-training needs relating to the above.* (TRC, 2015, p. 7)

We asked ourselves, what is the role of mathematics education, mathematics educators, and mathematics in responding to the TRC *Calls to Action*? We left our working group with this enduring question as we move towards teaching mathematics in a good way and developing research with communities.

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THEORETICAL FRAMEWORKS IN MATHEMATICS EDUCATION RESEARCH

LES CADRES THÉORIQUES DANS LA RECHERCHE EN DIDACTIQUE DES MATHÉMATIQUES

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(Texte en français suit.)

INTRODUCTION

A working group devoted to theoretical frameworks in mathematics education in this year's CMESG conference can be seen as a symptom of our domain's perennial preoccupation with the theme of theories. Different aspects of this theme—diversity of theories, networking of theories, or theoretical frameworks, perspectives, and approaches—are discussed in mathematics education journals (Reflections, 2002), special issues of journals (Sriraman & Kaiser, 2006), books (Bauersfeld, 1988; Bikner-Ahsbahs & Artigue, 2014; Sriraman & English, 2010), book reviews (Leikin & Zazkis, 2012), and conferences. There have been discussion groups and topic groups on the theme at the quadrennial ICME congresses, at least since 2004: *Discussion Group (DG) 10: Different perspectives, positions and approaches in mathematics education research* at ICME 10 in 2004 in Copenhagen, Denmark (*Discussion Group 10*, 2004); *DG 13: Challenges posed by different perspectives, positions and approaches in mathematics education research* at ICME 11 in 2008 in Monterrey, Mexico (Prediger & Trouche, 2008); and *Topic Study Group (TSG) 37: Theoretical issues in mathematics education* at ICME 12 in 2012 in Seoul, South Korea (Bikner-Ahsbahs &

Clarke, 2012). *TSG 51: Diversity of theories in mathematics education* is being prepared for ICME 13 in 2016 to be held in Hamburg, Germany. The theme has also appeared in the annual Psychology of Mathematics Education conferences. The 29th PME in 2005, in Melbourne, Australia held a Research Forum (RF04) on the theme of "Theories in Mathematics Education" (English & Sriraman, 2005); the above mentioned book edited by Bharath Sriraman and Lyn English (2010) was one of the outcomes of this forum.

One could expect a theme so thoroughly studied and discussed at so many different fora to be terminally exhausted and worn out. Yet mathematics educators never seem to tire of it and our group attracted a fair number of participants. This interest in theoretical frameworks among mathematics educators may be fuelled by both practical and intellectual needs. It is often argued that if one wants to publish an article about one's research in a mathematics education research journal, one must make explicit one's theoretical framework and show awareness of the existence of alternative approaches. For example, "Advice to prospective authors" in *Educational Studies in Mathematics*, published in the paper version of the first issue of each volume, includes the following recommendations:

> *The journal seeks to publish articles that are clearly educational studies in mathematics, make original and substantial contributions to the field, are accessible and interesting to an international and diverse readership, provide a well-founded and cogently argued analysis on the basis of an explicit* theoretical *and methodological* framework*, and take appropriate account of the previous scholarly work on the addressed issues…[i]n treating a particular area or aspect of mathematics, a submission should show* critical awareness of other possible approaches. (Advice, 2011, pp. vii-viii) (our emphasis)

Guidelines for submitting a Research Report of an empirical study at a PME conference make this point as well:

> *Reports of empirical studies should contain, at minimum, the following:*
>
> - *a statement regarding the focus of the submitted paper;*
> - *the study's theoretical framework;*
> - *references to the related literature;*
> - *an indication of and justification for the study's methodology; and*
> - *a sample of the data and the results….* (PME, 2015)

Applying for a grant likewise requires including an explicit description of the theoretical framework underpinning the planned research. For example, in our Canadian Social Sciences and Humanities Research Council (SSHRC) granting agency, the "Detailed Description" of the proposed research in an Insight Grant application must contain three broad sections: a section stating the objectives of the research, a section describing the context (including literature review and theoretical approach), and a section on methodology (SSHRC, 2015).

Note that in the "Advice for Prospective Authors" and the SSHRC Insight Grant instructions (SSHRC, 2015), the theoretical framework or approach description is not treated as a stand-alone piece but is mentioned as part of a larger section. Theoretical and methodological issues are intertwined in the "Advice"; in the SSHRC Insight Grant instructions, theoretical approach and literature review constitute elements of "the context". In the French language guidelines for researchers, the literature review, the methodology and the theoretical framework are sometimes presented as different aspects of a definition of the "*problématique*" of a research proposal (see, e.g., Université de Genève, 2015), closely related to and supporting the process of defining the general and specific research questions.

It is this particular point of view that formed the backbone of our (the leaders') thinking about the organization of the Working Group. Starting from the assumption that theoretical frameworks in mathematics education do not exist in a void but make sense in and derive their meaning from the context of a whole social-educational project or a research *problématique*, our intention was to discuss theoretical frameworks in relation with problems in mathematics education. Thus two dual questions were proposed in the leaders' introduction to the group during the first session, in English and French:

- What are the major problems in mathematics education and what theoretical frameworks could help understand them and, if possible, solve them? / *Quels sont les "grands" problèmes en didactique des mathématiques et quels sont les cadres théoriques qui peuvent aider à les comprendre et, possiblement, les résoudre?*
- What are the major theoretical frameworks in mathematics education and what problems have they served to understand and/or solve? / *Quels sont les cadres théoriques majeurs en didactique des mathématiques et quels problèmes ont-ils permis de comprendre et/ou de résoudre?*

The word *problématique* required some explanation. In the first session it was defined briefly as an intertwined system of research questions, theories and methodologies. In a *problématique*, research questions are taken together with their interpretations which are seen as dependent on a theoretical and/or philosophical perspective and the methodological tools used as means of treating the questions. Participants kept wondering about it, however, perhaps because this definition implies the impossibility of treating research questions and theoretical frameworks as independent entities, while the discussion questions, as formulated above, appear to suggest otherwise: that it is possible to name a problem in mathematics education that everybody will understand in the same way and then choose an appropriate theoretical framework from amongst a list of possible ones to deal with it. Such tensions are inevitable, because by using nouns as names of concepts, our language conveys the illusion that those nouns refer to distinct objects and so that the concepts represent disjoint or independent categories. The use of the verb *understand* in the proposed discussion questions tempers this independence somewhat: the way a problem is understood depends on the choice of the theoretical perspective; what is considered a solution to the problem from the perspective of one theoretical framework may not be a solution from the point of view of another. The idea is schematized in Figure 1.

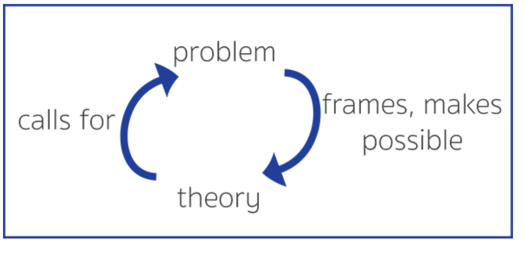

Figure 1. The need of a problem to develop a theory, and the need of a theory to address it.

We think that this principle of interdependence between problem and theoretical framework applies also to the problem of theoretical frameworks in mathematics education. Yet, in our (the leaders') preparatory readings on theoretical frameworks prior to the conference, we were intrigued by their often-found a-theoretical character for discussing the role of theoretical frameworks in research. Arguing for their importance, most authors did not ground their discussion in a theoretical framework themselves (a notable exception being Mason and Waywood, 1996). In a way, we could say that our entry into the preparations of this working group has not been a-theoretical: we chose to anchor the discussions about the problem of theoretical frameworks in mathematics education in the problem of the interrelations between theoretical frameworks and problems in mathematics education. This may sound somewhat circular, but it is a *creative circle* as Varela (1984) would say.

In this vein, prior to and at the conference, we started questioning the meaning of the term *theoretical framework* itself as well as the notion of *problem in mathematics education*. We asked, in English and in French:

- What is a theoretical framework, particularly in mathematics education? / *Qu'est-ce qu'un cadre théorique, particulièrement en didactique des mathématiques?*
- How different is it from a "practical framework", a "conceptual framework", an "experimental framework", a "framework of data analysis", etc.? / *Quelles sont les différences entre un "cadre pratique", un "cadre conceptuel", un "cadre expérimental", un "cadre d'analyse", etc.?*
- What is a problem in mathematics education? Is it different from a problem *of* mathematics education? / Qu'est-ce qu'un problème en didactique des mathématiques? Est-ce différent d'un problème d'enseignement des mathématiques?
- What would it mean to understand a problem in/of mathematics education? / Que signifierait de comprendre un problème en didactique des mathématiques ou d'enseignement des mathématiques?
- Is it possible to "solve" a problem in/of mathematics education? In what sense? / *Est-ce possible de "résoudre" un tel problème? De quelles façons?*

The discussions in the group were conducted both in French and in English, with translations on demand and when necessary and useful. Accordingly, this report will be written in two versions, English and French.

In this report, we present some details of the Working Group's activities during the three days. We close with a summary of the participants' written reflections on the outcomes of the work in the group for them.

DAY 1 ACTIVITIES

In a CMESG working group, Day 1 usually starts with participants presenting themselves. People say their names and affiliations, and sometimes they are asked to say why they chose to take part in this particular working group. Our dual "problem – theory" perspective led us to ask participants to present themselves by describing the problems they have been trying to address in their practice as researchers, teachers, or teacher educators and saying if there is a theory or a theoretical, practical, or conceptual framework that helps them to deal with, or cope with or solve these problems. This was the first activity of Day 1:

ACTIVITY 1.1

a. Briefly formulate or describe one or two problems that you are trying to address in your practice as a researcher/teacher/teacher educator. / *Formulez ou décrivez brièvement un ou deux problèmes que vous essayez d'aborder dans votre pratique de chercheur(e), enseignant(e) à l'école ou en formation des enseignants.*

b. Is there a theory or a theoretical/practical/conceptual framework that helps you to understand / explain / cope with / deal with / solve these problems? Name or describe it (them). / *Pour comprendre / expliquer / traiter / résoudre ces problèmes, avez-vous recours à une théorie ou à un cadre théorique / pratique / conceptuel? Nommez ou décrivez le (la, les).*

The problems that the participants formulated and theories they named reflected the interdisciplinary nature of mathematics education. There were socio-cultural-political problems, driven by *positioning theory* (Harré & Van Langenhove, 1999; Herbel-Eisenmann, Wagner, Johnson, Heejoo, & Figueras, 2015), *commognition theory* (Sfard, 2010), the *Anthropological Theory of the Didactic* (ATD) (Chevallard, 1992, 1999), *activity theory* (Engeström, Miettinen, & Punamäki, 1999), *theories of complexity* in Davis' version, (Davis & Simmt, 2003; Davis & Sumara, 2006) and Wenger's *communities of practice* (Wenger, 1998). Here is a sample of such problems, mentioned by participants:

> How is mathematics constructed / positioned in school? The problem is that it might be constructed as a series of expert-objective approaches to unabashedly simplified human problems. – David Wagner
>
> How do social relationships in learning emerge? – Tanya Noble
>
> Can school produce something else besides students – i.e., good subjects of the school institution? Can it produce citizens, for example? – Sophie René de Cotret
>
> Why does the applicability of mathematics not entail mathematics application: Why are people not using the knowledge learned at school outside of school even in situations where it would be useful to do so? – Sophie René de Cotret, (see René de Cotret & Larose, 2007)
>
> To study teaching practices in relation with students' learning: in particular, to relate a teacher's practices addressed to the whole class or to groups of students or to individual students and the students' math tasks-related actions; to study how these relations vary depending on the socio-economical backgrounds of the students. – Eric Roditi
>
> People use different ways to solve the same problem: different ways of doing mathematics. What does this depend on? What are these different ways based on? How to describe these different ways of doing so that they can be communicated or learned? – Eric Roditi
>
> To understand teacher blogging: Why and how are math teachers blogging in relation to their practice? What valuable characteristics are occasioned by the mathematics teacher blogosphere? – Judy Larsen

Some problems were posed from the perspective of linguistics. *Systemic functional linguistics* was proposed as a useful framework to study relational dynamics in mathematics teaching and learning by helping make the interactions that appear normal to appear strange (David Wagner). *Corpus linguistics techniques* (Biber, Conrad, & Reppen, 1998) were proposed to be used to study the oral and written discourses used in graduate level mathematics courses: to identify the key features of these structures and the inter-discourse dynamics and to analyze the function of the moves in the dynamics (Andrew Hare).

The above-mentioned problems aimed at understanding the reality of mathematics education or the way it is perceived. But participants also formulated problems aimed at changing this reality, by design, based on, e.g., the *"design-experiment" methodology* (Cobb, diSessa, Lehrer, & Schauble, 2003), *Theory of Didactic Situations* (TDS) (Brousseau, 1997), or *instrumentation theory* (Artigue, 2002). The latter theory was proposed as useful in investigating how we can prepare undergraduate mathematics students to develop fluency in programming: what curriculum, what pedagogy? (Chantal Buteau). A combination of a general *theory of experience* (Dewey, 1997/1938) and ATD (Chevallard, 1992, 1999) was proposed as a framework for studying a more general aspect of university mathematics education:

> How to engage university mathematics students in 'authentic' mathematical practices while abiding by institutional rules? What tasks, what classroom

> environment, what interventions, what classroom (and extended-classroom) activities? What are the conflicts or issues that emerge in a university mathematics course when the teacher is trying to engage students in behaviours characteristic of an institution—the mathematics of professional mathematicians—of which they are not the subjects. – Nadia Hardy

Design projects generally appear to require recourse to several theories, some grown in mathematics education, others in general education or in cognitive science (Stavy & Babai, 2010), as in the problem proposed by Elena Polotskaia:

> How can we develop efficient education – learning? What resources (tasks, materials, teachers), what knowledge processes (brain development, brain functioning), what research methodologies, what empirical evidence are needed for this development?

But participants viewed these design projects as producing not only curricular products (task sequences, lessons, reading materials, etc.) but also descriptive results in the form of specific theories of learning particular mathematical content—as is, in fact, expected of *design experiments* in mathematics education (Cobb et al., 2003).

Also related to technology in mathematics education, was a sketch of a whole field of research on the issues of design and study of the implementation of enriching learning and teaching by mathematical experiences with technology, raised by Viktor Freiman. This broad field of research draws on a variety of theories or concepts, from very general educational theories, e.g., *connectivist* (Siemens, 2005) or *constructionist views of learning* (Papert & Harel, 1991), and concepts, e.g., *learning environments* (Brown, 1992; Collins, Brown, & Holum, 1991), to concepts specific to learning with technology, e.g., the *concept of digital literacy* (Gilster, 1997), or *digital competence* (Ferrari, Punie, & Redecker, 2012). Part of this field of research was a more specific *problématique* related to preparing teachers to teach probability with technology (Mathieu Thibault).

Responses to this first activity sometimes contained "meta-level" remarks on methodological issues in research with respect to theoretical frameworks:

> Some theoretical frameworks make abstraction from the (social-cultural-institutional) contexts of mathematical – educational interactions. But removal of context in research impacts data rendering it un-generalizable. – Tanya Noble

> Dilemma: Staying within a single framework may impoverish data analysis (reductionism), but using more than one may be criticized as eclecticism. – Andrew Hare

> Some of the theories we borrow from other domains (sociology, linguistics) are weak in their conceptualization of learning (especially mathematics) so their usefulness to mathematics education is limited. – David Wagner

A first draft of the above summary of participants' responses was done by the co-leaders of the WG in the evening of the first day, based on the collected written material. It was presented to the group in the next day's session where it sparked a lively discussion.

In the next activity, participants were led to reflect on a collection of problems and questions. Some of these problems have been posed by well-known mathematics educators in response to explicit requests to identify and formulate the major problems in (or of) mathematics education. Such was the request that Hans Freudenthal was faced with when invited to give a plenary talk at ICME 4 in Berkeley, in 1980 (Freudenthal, 1981). Freudenthal chose to focus on what he considered to be the major problems *of* mathematics education not as a research domain but as a social project. A few years later, in 1983, David Wheeler, father and first editor of *For the Learning of Mathematics* (FLM), wrote to about 60 mathematics education researchers in several countries with a request to identify some of the major *research* problems in mathematics education. He published a selection of the responses he received in *FLM*, Volume 4 (1984) under the title “Research Problems in Mathematics Education – I” (Wheeler et al., 1984) and “… – II” (Confrey, Bishop, Fischbein, Kuijk, & Vergnaud, 1984). In compiling a collection of problems of/in mathematics education, we drew on these sources, sometimes quoting them almost word for word, sometimes rephrasing the problems somewhat. We also used a few other sources, e.g., Eisenhart (1991), and added some questions we thought of ourselves. We did not give references for these problems to the participants in the WG to prevent biasing their choices or raising too much “curiosity”!

We divided the problems into three sets. Problems in set A addressed instructional issues mainly (*problèmes d’enseignement ou de didactique des mathématiques*). Problems in set B were centred on the psychology of mathematics learning. Set C contained questions about the cultural, political, economic and affective aspects of mathematics education (these categories were not given to the participants, either, for the same above-mentioned reasons). Participants chose the set of problems they wanted to think and talk about and then discussed the problems in small groups with other participants who had chosen the same set of problems.

ACTIVITY 1.2

Choose one of the three sets of problems A, B, or C for discussion in small groups (see Appendix 1).

Suggestions of questions to guide the discussions: For each problem, consider these questions:

- Does the problem make sense to you?
- How could it be tackled?
- Is there a theory or a theoretical / practical / conceptual framework that could help understand / explain / cope with / deal with / solve the problems?
- Why have you chosen this particular set of problems?

Please write the main ideas that have emerged in your group on poster-sized paper and stick it to the wall.

At the end of the Day 1 session, representatives of the small groups presented summaries of their discussions using posters and plenary discussion took place.

DAY 2 ACTIVITIES

The first part of the Day 2 session was devoted, as mentioned above, to a plenary discussion of the participants’ responses to Activity 1.1.

In the second part of the session an exercise in applying three theoretical concepts to analyzing data was proposed. The concepts were all models of specific patterns of teacher-student interactions in the context of solving a mathematical problem: the *Topaze effect* (Brousseau, 1997), the *funnel pattern* (Bauersfeld, 1988), and the *semiotic bundle* (Arzarello, 2006).

The definitions of the concepts in the activity, as well as the transcripts used, were not taken from those original sources, however, but from a chapter of the book on the networking of theories (Bikner-Ahsbahs, Artigue, & Haspekian, 2014) whose authors felt compelled to refine and disambiguate the original descriptions of the concepts in order to decide if a teaching episode exhibited any one of these *patterns of interaction*.

> ACTIVITY 2
>
> You are given definitions of the *Topaze effect*, the *funnel pattern* and the *semiotic bundle* and examples of transcripts of teacher-student interaction. (See Appendix 2)
>
> In small groups of 3-4 people, please
>
> - discuss if the transcripts can be qualified as representing the *Topaze effect*, the *funnel pattern* or the *semiotic bundle* or some other phenomena;
> - construct examples of each of the three phenomena based on your own or observed experience of teaching mathematics.
>
> Please summarize the results of your small group work on posters.

Participants continued discussing in small groups until the end of the Day 2 session. Presentations of summaries of the small group discussions, using posters and plenary discussion of Activity 2 took place the next day.

DAY 3 ACTIVITIES

After discussions of Activity 2, the co-leaders presented a brief overview of the theoretical distinctions related to the theme of the WG and distributed a handout with longer quotes about these distinctions among the participants. The presentation was interactive, with frequent interventions of the participants. Information given in the presentation is summarized in Appendix 3.

The last activity in the WG was an individual reflection that participants were asked to write and share orally with us in view of writing the report for the proceedings of the conference. The participants had to reflect on the following:

> Have you changed your mind regarding theoretical frameworks in mathematics education as a result of your participation in this WG? If yes, how? If not, why? / *Avez-vous changé ce que vous pensez des cadres théoriques en didactique des mathématiques à la suite de votre participation dans le GT? Si oui, comment? Si non, pourquoi?*

The overarching comment present in the participants' responses was that they enjoyed the WG's work and the open discussions between experienced and less experienced researchers. Through their responses, and appreciative comments, the participants expressed the diverse nature of the experiences lived in the group. These experiences obviously varied from one participant to another, but we have attempted to outline wider themes of interest present in their responses (and apologies for other wider themes we have surely missed):

- There are participants who asserted that they have not changed their views or learned new ideas from the WG, but that the WG's work and the sharing of ideas brought them to deepen, challenge, nuance, and make more explicit their understanding of theoretical frameworks in mathematics education.
- There are participants who questioned the significance of the various distinctions made between various sorts of theories and frameworks (see e.g. Appendix 3), but at the same time highlighted the significance of exploring those distinctions and making sense of them to learn more about theoretical frameworks and what we can do with those as researchers.
- There are other participants who stressed the importance of the emphasis placed in the discussions of the WG on thinking of theories in a dynamical fashion, that is, of theories not as static knowledge but as evolving ways of making sense that grow and get transformed through their usage (somehow in an iterative manner).

- There are a number of participants who flagged the importance, not necessarily of knowing all theories and their differences, but of engaging in discussions as researchers about what we know and the various actions we undertake in relation to theories in our work, to attempt to make that explicit and to communicate and listen to others about it.

Hence, as the above summary makes clear and as we have mentioned at the beginning of the report, this topic of theoretical frameworks in mathematics education research is very alive and of interest in our research community; it is our contention that much more ink will be spilled in the future over this rich matter.

CONCLUSIONS

Overall, the work led participants to question and raise questions about a significant number of elements in relation to research endeavors and theoretical frameworks. For example, several participants expressed the need to clarify the differences between a "question", especially a "research question", and a "problem" in or of mathematics education. The meaning of other methodological terms was questioned as well: "theory", "*problématique*", "framework", "concept", "construct", etc. Through this questioning, we were often, if not always, led back to an important aspect, which is that each problem presupposes a theory to become a problem in itself, and that each theory presupposes some sort of a foundation in itself. But what would such a foundation be founded on? This question reminded us of a well-known fable cited by Clifford Geertz in his *Interpretation of Cultures*.

> *There is an Indian story – at least I heard it as an Indian story – about an Englishman who, having been told that the world rested on a platform which rested on the back of an elephant which rested in turn on the back of a turtle, asked (perhaps he was an ethnographer; it is the way they behave), what did the turtle rest on? Another turtle. And that turtle? "Ah, Sahib, after that it is turtles all the way down."* (Geertz, 1973, pp. 28-29)

So that's how it is with theoretical frameworks: "it's turtles all the way down"!

[Note: Appendices and References follow the French version.]

♦♦♦♦♦♦♦♦♦♦

INTRODUCTION

La présence d'un groupe de travail centré sur les cadres théoriques en didactique des mathématiques à la conférence du GCEDM de cette année peut être vue comme un symptôme d'une préoccupation continue de notre domaine au sujet des théories. Divers aspects de ce thème—diversité des théories, réseautage de théories ou cadres théoriques, perspectives et approches—ont été discutés dans les revues en didactique des mathématiques (Reflections, 2002), numéros thématiques de revues (Sriraman & Kaiser, 2006), livres (Bauersfeld, 1988; Bikner-Ahsbahs & Artigue, 2014; Sriraman & English, 2010;), recensions (Leikin & Zazkis, 2012) et conférences. Il y a aussi eu des groupes de discussion et des groupes thématiques sur ce thème au congrès quadriennal ICME depuis au moins 2004 : *Groupe de discussion (DG) 10 : Different perspectives, positions and approaches in mathematics education research* à ICME 10 en 2004 à Copenhague, Danemark (*Discussion Group 10*, 2004); *DG 13 : Challenges posed by different perspectives, positions and approaches in mathematics education research* à ICME 11 en 2008 à Monterrey, Mexique (Prediger & Trouche, 2008);

Groupe thématique (TSG) 37 : Theoretical issues in mathematics education à ICME 12 en 2012 à Seoul, Corée du Sud (Bikner-Ahsbahs & Clarke, 2012). *TSG 51 : Diversity of theories in mathematics education* est en préparation pour ICME 13 en 2016 qui se tiendra à Hambourg en Allemagne. Ce thème est aussi présent lors des conférences annuelles du groupe international *Psychology of Mathematics Education* (PME). La 29^{e} conférence de PME, en 2005 à Melbourne, Australie, a tenu un forum de recherche (RF04) sur le thème de *Theories in Mathematics Education* (English & Sriraman, 2005); le livre mentionné ci-haut, édité par Bharath Sriraman et Lyn English (2010), est l'un des produits de ce forum.

On pourrait s'attendre à ce qu'un thème si bien étudié et discuté à l'intérieur de tant de forums différents soit déjà pas mal épuisé et usé. Et pourtant, les didacticiens des mathématiques ne semblent pas s'en lasser et notre groupe a attiré un bon nombre de participants. Il se peut que cet engouement pour les cadres théoriques parmi les didacticiens des mathématiques soit alimenté par des intentions autant pratiques que théoriques. Souvent, les revues en didactique des mathématiques recommandent (voire exigent) l'explicitation du cadre théorique sous-jacent dans la recherche présentée dans l'article et une discussion critique de son rapport aux approches alternatives. Par exemple, dans l'« *Advice for prospective authors* » de *Educational Studies in Mathematics,* publié en version papier du premier numéro de chaque volume, on retrouve les recommandations suivantes :

> *The journal seeks to publish articles that are clearly educational studies in mathematics, make original and substantial contributions to the field, are accessible and interesting to an international and diverse readership, provide a well-founded and cogently argued analysis on the basis of an explicit* theoretical *and methodological* framework, *and take appropriate account of the previous scholarly work on the addressed issues…. [i]n treating a particular area or aspect of mathematics, a submission should show* critical awareness of other possible approaches. (Advice, 2011, pp. vii-viii) (our emphasis)

Les directives pour soumettre un *Research Report* d'une étude empirique à la conférence PME en font également état :

> *Reports of empirical studies should contain, at minimum, the following:*
>
> - *a statement regarding the focus of the submitted paper;*
> - *the study's theoretical framework;*
> - *references to the related literature;*
> - *an indication of and justification for the study's methodology; and*
> - *a sample of the data and the results….* (PME, 2015)

Les demandes de subvention requièrent aussi une description explicite du cadre théorique sous-jacent à la recherche proposée. Par exemple, au Canada, pour les demandes de subvention au Conseil de recherches en sciences humaines (CRSH), la « Description Détaillée » de la recherche proposée dans l'application à une « subvention Savoir » doit contenir trois grandes sections : une discutant les objectifs de la recherche, une décrivant le contexte (qui inclut la recension des écrits et l'approche théorique), et une section sur la méthodologie (SSHRC, 2015).

Il est à noter que dans l'« *Advice for prospective authors* » de ESM et dans les instructions du CRSH pour les subventions Savoir (SSHRC, 2015), le cadre théorique et la description de l'approche ne sont pas traités comme des parties isolées, mais comme faisant partie d'une section plus large. Les questions théoriques et méthodologiques sont étroitement liées dans l' « *Advice* »; dans les instructions du CRSH pour les subventions Savoir l'approche théorique et la revue de littérature constituent des éléments du *contexte*. Dans les instructions en français pour les chercheurs, la recension des écrits, la méthodologie et le cadre théorique sont parfois présentés comment étant différents aspects de la *problématique* d'une recherche (voir, par exemple, Université de Genève, 2015), étroitement en lien avec le processus de définition des questions générales et spécifiques de recherche.

C'est ce point de vue particulier qui a formé la pierre angulaire de l'approche des coresponsables pour l'organisation de ce Groupe de Travail. Partant de l'hypothèse que les cadres théoriques en didactique des mathématiques n'existent pas en isolement, mais puisent leur sens dans le contexte d'un projet socio-éducatif ou d'une problématique de recherche, notre intention était de discuter les cadres théoriques en relation avec les problèmes et questions de recherche sur l'enseignement des mathématiques. Ainsi, durant la première séance, deux groupes de questions, en français et en anglais, ont été proposés par les coresponsables lors de l'introduction au groupe :

- *What are the major problems in mathematics education and what theoretical frameworks could help understand them and, if possible, solve them?* / Quels sont les « grands » problèmes en didactique des mathématiques et quels sont les cadres théoriques qui peuvent aider à les comprendre et, possiblement, les résoudre?
- *What are the major theoretical frameworks in mathematics education and what problems have they served to understand and/or solve?* / Quels sont les cadres théoriques majeurs en didactique des mathématiques et quels problèmes ont-ils permis de comprendre et/ou de résoudre?

La notion de *problématique* a nécessité quelques explications. Dans la première séance, elle a été brièvement définie comme un système imbriquant des questions de recherche, des aspects théoriques et des dimensions méthodologiques. Dans une problématique, les questions de recherche sont nécessairement ancrées dans une interprétation spécifique ; leur sens est relatif à la perspective théorique et/ou philosophique et aux outils méthodologiques utilisés pour les aborder et les traiter. Les participants du Groupe de Travail continuaient toutefois de s'interroger à propos du sens de cette notion, probablement parce que cette définition implique l'impossibilité de traiter des questions de recherche et des cadres théoriques de façon indépendante, alors que les questions de démarrage du Groupe de Travail (telles que formulées ci-dessus) semblent suggérer le contraire. Elles semblent supposer qu'il est possible de formuler un problème en didactique des mathématiques que tout le monde comprendra de la même manière, et qu'il est alors possible de choisir un cadre théorique approprié à ce problème parmi une liste de cadres possibles. Ces tensions sont inévitables, parce qu'en utilisant des noms pour nommer des concepts, notre langue donne l'illusion que ces noms réfèrent à des objets distincts et qu'alors les concepts représentent aussi des catégories disjointes ou indépendantes. L'usage du mot *comprendre* dans les questions de démarrage tempère quelque peu cette indépendance : la manière dont un problème est compris dépend de la perspective théorique dans laquelle il est ancré ; ce qui est considéré comme une solution au problème de la perspective d'un cadre théorique peut ne pas en être une du point de vue d'un autre. Cette idée est schématisée dans la Figure 1.

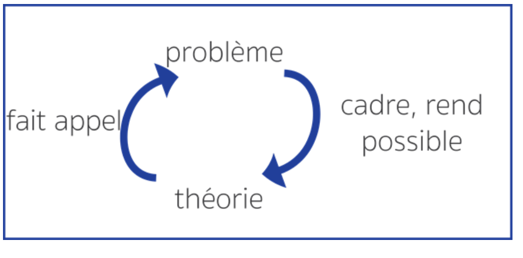

Figure 1. On a besoin d'un problème pour développer une théorie, et l'on a besoin d'une théorie pour l'aborder.

Nous croyons que ce principe d'interdépendance entre un problème et un cadre théorique s'applique aussi au problème des cadres théoriques en didactique des mathématiques. Ainsi, dans nos (coresponsables) lectures préparatoires sur les cadres théoriques pour la conférence, nous avons été intrigués par le caractère souvent a-théorique des discussions du rôle des cadres théoriques dans les recherches. Discutant de leur importance, la plupart des auteurs n'ancraient pas eux-mêmes leurs discussions dans un cadre théorique (une exception étant le chapitre de Mason et Waywood, 1996). D'une certaine façon, on peut dire que notre entrée dans la préparation de ce groupe de travail n'a pas été a-théorique : nous avons choisi d'ancrer les discussions sur les cadres théoriques en didactique des mathématiques dans le problème de l'interrelation entre les cadres théoriques et les problèmes en didactique des mathématiques. Cela peut sembler quelque peu circulaire, mais nous avons espéré que, comme dirait Varela (1984), ce serait un *cercle créatif*.

Dans cette lignée, avant et à la conférence, nous avons commencé à nous poser des questions au sujet du sens des termes *cadre théorique* et *problème en didactique des mathématiques* eux-mêmes. Nous avons demandé, en français et en anglais :

- Qu'est-ce qu'un cadre théorique, particulièrement en didactique des mathématiques? / *What is a theoretical framework, particularly in mathematics education?*
- *How different is it from a "practical framework", a "conceptual framework", an "experimental framework", a "framework of data analysis", etc.?* / Quelles sont les différences entre un « cadre pratique », un « cadre conceptuel », un « cadre expérimental », un « cadre d'analyse », etc.?
- Qu'est-ce qu'un problème en didactique des mathématiques? Est-ce différent d'un problème d'enseignement des mathématiques? / *What is a problem in mathematics education? Is it different from a problem* of *mathematics education?*
- *What would it mean to understand a problem in/of mathematics education?* / Que signifierait-il de comprendre un problème en didactique des mathématiques ou d'enseignement des mathématiques?
- Est-ce possible de « résoudre » un tel problème? De quelles façons? / *Is it possible to "solve" a problem in/of mathematics education? In what sense?*

Les discussions dans le groupe ont été conduites autant en français qu'en anglais, avec des traductions sur demande lorsque cela était nécessaire et utile. Par conséquent, le rapport est disponible aussi bien en anglais qu'en français.

Dans ce rapport, nous présentons quelques activités du Groupe de Travail au cours des trois jours de sessions (3 heures chaque jour). Nous terminons par un résumé des réflexions écrites des participants concernant les retombées de ce Groupe de Travail pour/sur eux.

ACTIVITÉS JOUR 1

Au GCEDM, la première journée d'un Groupe de Travail débute habituellement par la présentation des participants. Les personnes disent comment ils s'appellent et à quelle institution ils appartiennent. A l'occasion, on leur demande pourquoi ils ont choisi de participer dans ce Groupe de Travail particulier. Notre perspective « problème – théorie » nous a amené à demander aux participants de se présenter par les problèmes qu'ils tentent d'aborder dans leur pratique de chercheurs, d'enseignants ou de formateurs d'enseignants, en plus d'expliquer quelle théorie ou quels cadres théoriques, conceptuels ou pratiques les aidait à aborder ces problèmes, à leur faire face ou à les résoudre. Voici la description des activités de la première journée :

ACTIVITÉ 1.1

a. *Briefly formulate or describe one or two problems that you are trying to address in your practice as a researcher/teacher/teacher educator.* / Formulez ou décrivez brièvement un ou deux problèmes que vous essayez d'aborder dans votre pratique de chercheur(e), enseignant(e) à l'école ou en formation des enseignants.

b. *Is there a theory or a theoretical/practical/conceptual framework that helps you to understand / explain / cope with / deal with / solve these problems? Name or describe it (them).* / Pour comprendre / expliquer / traiter / résoudre ces problèmes, avez-vous recours à une théorie ou à un cadre théorique / pratique / conceptuel? Nommez ou décrivez le (la, les).

Les problèmes que les participants ont formulés et les théories qu'ils ont nommées reflétaient la nature interdisciplinaire de la didactique des mathématiques. Il y avait des problèmes socio-culturels et politiques guidés par *la théorie du positioning* (Harré & Van Langenhove, 1999; Herbel-Eisenmann, Wagner, Johnson, Heejoo, & Figueras, 2015), *la théorie de la commognition* (Sfard, 2010), *la théorie anthropologique du didactique* (TAD) (Chevallard, 1992, 1999), *la théorie de l'activité* (Engeström, Miettinen, & Punamäki, 1999), *la théorie de la complexité* dans la version développée par Davis (Davis & Simmt, 2003; Davis & Sumara, 2006) et *les communautés de pratique* de Wenger (1998). Voici un échantillon des problèmes mentionnés par les participants :

> Comment les mathématiques sont-elles construites/« positionnées » à l'école? Le problème est qu'elles pourraient être construites comme une série d'approches expert-objectif pour simplifier sans scrupule les problèmes humains. – David Wagner
>
> Comment les relations sociales émergent-elles durant l'apprentissage? – Tanya Noble
>
> L'école peut-elle produire autre chose que des élèves – i.e., de bons sujets de l'institution qu'est l'école? Est-ce qu'elle peut par exemple produire des citoyens? – Sophie René de Cotret
>
> Pourquoi l'applicabilité des mathématiques n'entraine-t-elle pas l'application réelle des mathématiques ? Pourquoi les gens n'utilisent-ils pas les connaissances apprises à l'école en dehors de l'école, et ce, même dans des situations où il serait utile de le faire? – Sophie René de Cotret, voir (René de Cotret & Larose, 2007)
>
> Étudier les pratiques des enseignants en lien avec l'apprentissage des élèves : en particulier, pour décrire des pratiques des enseignantes qui s'adressent à toute la classe ou à des groupes d'élèves ou à des élèves individuels et les actions reliées aux tâches mathématiques des élèves ; pour étudier comment ces relations varient dépendamment du contexte socio-économique des élèves. – Éric Roditi
>
> Il y a différentes manières de résoudre le même problème : différentes manières de faire des mathématiques. De quoi cela dépend-il? Sur quoi sont basées ces différentes manières? Comment décrire ces différentes manières de faire afin qu'elles soient communiquées ou apprises? – Éric Roditi
>
> Comprendre les pratiques de « *blogging* » des enseignants : Pourquoi et comment les enseignants de mathématiques utilisent les blogs en relation avec leur pratique? Quelles dimensions significatives sont occasionnées par la blogosphère des enseignants de mathématiques? – Judy Larsen

Certains problèmes ont été posés depuis la perspective linguistique. *La linguistique fonctionnelle systémique* a été proposée comme un cadre utile pour étudier les dynamiques relationnelles dans l'enseignement et l'apprentissage des mathématiques, en rendant étranges les interactions dites « normales » (David Wagner). *Les techniques de corpus linguistics* (Biber, Conrad, & Reppen, 1998) ont été proposées pour étudier les discours oraux et écrits dans les cours gradués en mathématiques : pour identifier les principales caractéristiques de ces structures et les dynamiques inter-discours et pour analyser la fonction des mouvements dans ces dynamiques (Andrew Hare).

Les problèmes mentionnés ci-dessus ont pour but de comprendre la réalité de l'enseignement des mathématiques ou la manière dont il est perçu. Mais les participants ont aussi formulés des problèmes ayant pour but de changer cette réalité, à travers *la méthodologie du design-experiment* (Cobb, diSessa, Lehrer, & Schauble, 2003), *la théorie des situations didactiques* (TSD) (Brousseau, 1997) ou *la théorie de l'instrumentation* (Artigue, 2002). Cette dernière a été suggérée utile pour étudier l'initiation des étudiants inscrits au baccalauréat en mathématiques à la programmation en informatique : quel curriculum, quelle pédagogie? (Chantal Buteau). Une combinaison de *la théorie générale de l'expérience* (Dewey, 1938/1997) et de la TAD (Chevallard, 1992, 1999) a été proposée en tant que cadre permettant d'étudier les aspects plus généraux de la formation universitaire en mathématiques:

> Comment insérer les étudiants universitaires en mathématiques dans une pratique mathématique « authentique » tout en respectant les règles institutionnelles? Quelles tâches, quel environnement de classe, quelles interventions, quelles activités en classe (et hors classe)? Quels sont les conflits ou problèmes qui émergent d'un cours de mathématiques universitaire lorsque l'enseignant cherche à engager les étudiants dans des comportements caractéristiques d'une institution dont ils ne sont pas les sujets – à savoir l'institution des mathématiques de mathématiciens professionnels ?
> – Nadia Hardy

La conception de projets du type « *design experiments* » requière généralement le recours à plusieurs théories, certaines formées en didactique des mathématiques, d'autres en éducation ou en sciences cognitives (Stavy & Babai, 2010). C'est le cas du problème proposé par Elena Polotskaia :

> Comment peut-on développer une formation – un apprentissage – efficace? Quelles ressources (tâches, matériels, enseignants), quels processus (développement du cerveau, fonctionnement du cerveau), quelles méthodologies de recherche, quelles preuves empiriques sont nécessaires pour ce développement?

Les participants ont vu ce genre de projets comme aboutissant non seulement aux produits curriculaires (séquences de tâches, leçons, matériel de lecture, etc.), mais aussi comme donnant des résultats descriptifs sous la forme de théories spécifiques de l'apprentissage d'un contenu mathématique particulier, en accord avec la vision de Cobb et al. (2003) des *design-experiments* en didactique des mathématiques.

En rapport avec l'informatique en didactique des mathématiques, Viktor Freiman a tracé les grandes lignes d'un champ de recherches sur les problèmes de conception et d'étude de l'implémentation d'un enseignement et apprentissage enrichis par les expériences mathématiques avec des outils informatiques. Ce vaste champ de recherches repose sur toute une panoplie de théories et concepts, partant de théories éducatives très générales, par exemple, *les théories connectivistes* (Siemens, 2005) ou *constructionnistes de l'apprentissage* (Papert & Harel, 1991), et de concepts, par exemple, *environnements d'apprentissage* (Brown, 1992; Collins, Brown, & Holum, 1991) et allant jusqu'aux concepts spécifiques d'apprentissage dans les environnements informatiques, par exemple, *le concept d'alphabétisation digitale* (Gilster, 1997) ou de *compétence digitale* (Ferrari, Punie, & Redecker, 2012). Il y avait aussi une problématique plus spécifique liée à la préparation des enseignants à l'enseignement des probabilités dans un environnement informatique (Mathieu Thibault).

Les réponses à cette première activité contenaient parfois des remarques à un niveau plus « méta » sur les problèmes méthodologiques en recherche en relation avec les cadres théoriques :

> Certains cadres théoriques font abstraction du contexte (social-culturel-institutionnel) des interactions mathématiques et éducationnelles. Toutefois, le retrait du contexte en recherche a un effet sur les données, les rendant non généralisables. – Tanya Noble
>
> Dilemme: Se placer dans un cadre unique peut appauvrir l'analyse des données (réductionnisme), mais utiliser plus d'un cadre peut être vu comme étant éclectique. – Andrew Hare
>
> Plusieurs des théories que nous empruntons aux autres domaines (sociologie, linguistique) sont faibles dans leur conceptualisation de l'apprentissage (spécialement des mathématiques), alors leur utilité à la didactique des mathématiques est limitée. – David Wagner

Une première esquisse du sommaire de ces réponses a été réalisée par les coresponsables du Groupe de Travail après la première journée, sur la base du matériel reçu. Cette esquisse a été présentée aux participants lors de la session suivante, où elle a provoqué une vive discussion.

Dans l'activité suivante, les participants ont été amenés à réfléchir sur un groupe de problèmes et de questions. Quelques-uns de ces problèmes ont été posés par des didacticiens de mathématiques bien connus en réponse à une demande explicite d'identifier et de formuler des problèmes majeurs de (ou dans la) didactique des mathématiques. Telle était la demande faite à Hans Freudenthal, alors qu'il a été invité à offrir une conférence à ICME 4 à Berkeley en 1980 (Freudenthal, 1981). Freudenthal a choisi de se centrer sur ce qu'il considérait être les problèmes majeurs *de* l'éducation mathématique, non pas comme domaine de recherche, mais en tant que projet social. Quelques années plus tard, en 1983, David Wheeler, le fondateur et premier éditeur de la revue *For the Learning of Mathematics* (FLM), a écrit à près de 60 chercheurs en didactique des mathématiques dans plusieurs pays en leur demandant d'identifier les problèmes de *recherche* majeurs en didactique des mathématiques. Il a publié une sélection des réponses obtenues dans le volume 4 de FLM (1984) sous le titre « *Research Problems in Mathematics Education – I* » (Wheeler et al., 1984) et « *... II* » (Confrey, Bishop, Fischbein, Kuijk, & Vergnaud, 1984). En compilant la collection de problèmes de/dans la didactique des mathématiques, nous nous sommes inspirés de ces sources, parfois en les citant mot à mot, parfois en les reformulant. Nous avons aussi utilisé d'autres sources, par exemple Eisenhart (1991), et ajouté quelques questions auxquelles nous avons nous-mêmes pensé. Nous n'avons pas donné aux participants les références à ces problèmes afin de ne pas biaiser leurs choix ou susciter trop de « curiosité »!

Nous avons divisé les problèmes en trois sous-ensembles. Les problèmes du sous-ensemble A abordent principalement des problèmes d'enseignement ou de didactique des mathématiques. Les problèmes du sous-ensemble B sont centrés sur la psychologie de l'apprentissage des mathématiques. Le sous-ensemble C contient des questions à propos des aspects culturels, politiques, économiques ou affectifs de la didactique des mathématiques (ces catégories n'étaient pas données aux participants pour les mêmes raisons que celles mentionnés plus haut). Les participants choisissaient le sous-ensemble de problèmes auquel ils souhaitent réfléchir et discuter, puis discutaient de ces problèmes en petits groupes avec d'autres participants ayant choisi le même ensemble de problèmes.

ACTIVITÉ 1.2

Choisir un des trois sous-ensembles de problèmes A, B, C pour une discussion en petits groupes (voir Appendice 1).

Suggestions de questions pour guider vos discussions : Pour chaque problème, considérez ces questions :

- Est-ce que le problème a du sens pour vous?
- Comment peut-il être abordé?
- Y a-t-il une théorie ou un cadre théorique / pratique / conceptuel qui pourrait aider à comprendre / expliquer / faire face / traiter / résoudre les problèmes?
- Pourquoi avez-vous choisi ce sous-ensemble de problèmes particulier?

S.v.p. écrire les idées principales qui émergent de votre groupe sur une affiche et la coller au mur.

À la fin de la séance de la première journée, des représentants de chaque groupe ont présenté un résumé de leurs discussions en utilisant leurs affiches, et des discussions plénières ont eu lieu.

ACTIVITÉ JOUR 2

La première partie de la séance de la seconde journée a été consacrée, tel que mentionné ci-dessus, à une discussion plénière des réponses des participants à l'Activité 1.1.

Dans la seconde partie de la rencontre, un exercice d'application de trois concepts théoriques pour analyser des données a été proposé. Les concepts étaient tous des modélisations de patterns d'interaction entre enseignant et élèves en contexte de résolution de problèmes en mathématiques : *l'effet Topaze* (Brousseau, 1997), *le funnel pattern* (*effet entonnoir*) (Bauersfeld, 1988), et *le semiotic bundle* (Arzarello, 2006).

Les définitions des concepts dans l'activité, tout autant que les verbatims utilisés, n'ont pas été pris de ces sources originales, mais plutôt à partir d'un chapitre du livre sur le réseautage de théories (Bikner-Ahsbahs, Artigue, & Haspekian, 2014). Dans le projet de réseautage que raconte ce livre, les participants se sont sentis obligés de raffiner et lever l'ambigüité sur les descriptions originales de ces concepts pour pouvoir décider si un épisode d'enseignement représentait oui ou non un *pattern d'interactions* donné.

ACTIVITÉ 2

Voici les définitions de *l'effet Topaze*, du *funnel pattern* et du *semiotic bundle*, ainsi que des verbatims d'interactions entre enseignant et élèves. (Voir Appendice 2).

En petits groupes de 3-4 personnes, s.v.p.,

- discutez si le verbatim peut être qualifié comme représentant *l'effet Topaze, le funnel pattern* ou *le semiotic bundle* ou un autre phénomène;
- construisez des exemples de chacun de ces trois phénomènes à partir de votre propre expérience ou une expérience observée d'enseignement des mathématiques.

S.v.p. résumez les résultats de votre travail sur une affiche.

Les participants ont continué leurs discussions en petit groupes jusqu'à la fin de la deuxième rencontre. Les présentations des résumés des discussions de chacun des sous-groupes, à l'aide d'affiches et d'une discussion plénière de l'activité 2 ont eu lieu le lendemain lors de la 3e séance.

ACTIVITÉ JOUR 3

Après les discussions de l'Activité 2, les coresponsables ont présenté un bref aperçu de diverses distinctions théoriques reliées au thème du GT et ont distribué des notes avec des citations plus longues à propos de ces distinctions. La présentation a été interactive, avec de fréquentes interventions de la part des participants. Les informations données dans la présentation sont résumées dans l'Appendice 3.

La dernière activité du GT a été la production d'une réflexion individuelle par chaque participant. Individuellement, les participants devaient réfléchir et répondre aux questions suivantes par écrit:

> *Have you changed your mind regarding theoretical frameworks in mathematics education as a result of your participation in this WG? If yes, how? If not, why?* / Avez-vous changé d'idée à propos des cadres théoriques en didactique des mathématiques à la suite de votre participation dans le GT? Si oui, comment? Si non, pourquoi?

Par la suite, les participants ont partagé leurs réponses oralement entre eux. Le commentaire le plus présent dans les réponses écrites des participants a été qu'ils ont apprécié le GT et les discussions libres et ouvertes entre les chercheurs expérimentés et ceux avec moins d'expérience en recherche. Au travers de leurs réponses, et de leurs commentaires d'appréciation, les participants ont exprimés la nature diverse de leurs expériences vécues dans le groupe. Ces expériences variaient évidemment d'un participant à l'autre, mais nous avons tenté de définir des thèmes d'intérêt plus larges présents dans leurs réponses écrites et orales (et nos excuses pour les thèmes que nous avons surement manqués).

- Certains participants ont affirmé ne pas avoir changé de vision ni appris de nouvelles idées lors du GT, mais que leur travail dans le GT et le partage des idées les a amenés à approfondir, défier, nuancer et rendre plus explicites leurs compréhensions des cadres théoriques en didactique des mathématiques.
- Certains participants ont questionné l'importance des distinctions faites entre les diverses sortes de théories et cadres (voir p.ex. Appendice 3), mais en même temps ils ont souligné l'importance d'explorer ces distinctions et de leur donner un sens afin d'en apprendre plus à propos des cadres théoriques et de ce que l'on peut faire avec eux en tant que chercheurs.
- Certains autres participants ont souligné l'importance de l'accent mis dans les discussions du GT sur l'aspect dynamique des théories, soit de voir les théories non pas comme des connaissances statiques, mais comme des manières évolutives de donner du sens, qui grandissent et se transforment à travers leurs usages (de manière itérative).
- Certains participants ont aussi soulevé l'importance de ne pas nécessairement connaître toutes les théories et leurs différences, mais plutôt d'engager des discussions en tant que chercheurs à propos de ce que l'on connaît sur elles, et les actions diverses à prendre en lien avec les théories dans nos travaux, dans le but de rendre le tout plus explicite et de communiquer et entendre les autres à ce propos.

Ainsi, comme le résumé ci-dessus l'illustre, et tel que mentionné au début de ce rapport, le thème des cadres théoriques dans les recherche en didactique des mathématiques est bien vivant et est d'intérêt pour notre communauté de recherche ; encore beaucoup d'encre sera versé à l'avenir sur cette riche thématique.

CONCLUSION

En somme, le GT a mené les participants à questionner et soulever des questions à propos d'un nombre significatif d'éléments en lien avec le travail de recherche et les cadres théoriques. Par exemple, plusieurs participants ont exprimé le besoin de clarifier les différences entre une « question », plus spécifiquement une « question de recherche », et un « problème » d'enseignement des mathématiques ou en didactique des mathématiques. Le sens d'autres termes méthodologiques a aussi été questionné : « théorie », « problématique », « cadre », « construit », etc. À travers ce questionnement, nous avons été souvent, sinon tout le temps, ramenés à un aspect important : chaque problème présuppose une théorie pour devenir un problème en soi et chaque théorie présuppose aussi un fondement qui lui est propre. Mais, la question est aussi de se demander sur quoi peut bien reposer un tel fondement. Cette question nous rappelle une fable bien connue citée par Clifford Geertz dans son *Interpretation of Cultures*.

> *There is an Indian story – at least I heard it as an Indian story – about an Englishman who, having been told that the world rested on a platform which rested on the back of an elephant which rested in turn on the back of a turtle, asked (perhaps he was an ethnographer; it is the way they behave), what did the turtle rest on? Another turtle. And that turtle? "Ah, Sahib, after that it is turtles all the way down."* (Geertz, 1973, pp. 28-29)

Voilà donc ce qui en est avec les cadres théoriques: « *it's turtles all the way down!* »

APPENDICE 1 / APPENDIX 1: SETS OF PROBLEMS FOR ACTIVITY 1.2

SET A

A-1 What is the best way for students to learn and appreciate mathematics? (Eisenhart, 1991)

A-2 What are the factors on which responses to questions such as, "how best to introduce the notion of variable?", "how best to prepare textbooks?"; "how best to organize a classroom?" depend? (Wheeler, et al., 1984)

A-3 What techniques are available for making mathematics "accessible" without "explanation"? (Wheeler, et al., 1984)

A-4 How to develop suitable contexts in order to teach mathematizing? (Freudenthal, 1981)

A-5 In introducing mathematical ideas, teachers try to help students make sense of them, develop an "insight" into these ideas. When these ideas are then institutionalized into a conventional terminology, notation, definitions, and techniques of solving typical problems, the insight risks to be lost or forgotten. How to keep open the sources of insight during the teaching and learning of mathematics, and how to stimulate the retention of the insight? (Freudenthal, 1981)

A-6 A teacher asks a question or poses a problem to a student. The student fails to produce the expected answer. What does / can the teacher do in this situation?

Example of a problem:
A jar can be filled to capacity with either 8 short glasses or 6 tall glasses of water.
What fraction of the capacity of the short glass is the capacity of the tall glass?

Example of an answer to this problem

S – short glass, T – tall glass

Given: 8 S = 6 T

So: S is 8/6 of T

How would you react to this answer?

SET B

B-1 What are the constructive mechanisms of mathematics/particular mathematical notions and how do they function in knowledge building? Example: Counting is a constructive mechanism of whole number knowledge building. What are the constructive mechanisms of rational number knowledge building? To what extent are computer procedures constructive mechanisms for persons developing or executing them in a mathematical context? (Wheeler, et al., 1984)

B-2 What outcome measures measure understanding? (Confrey, et al., 1984)

B-3 What are the sources of students' difficulties in [a particular area of mathematics: fractions, geometry, algebra, etc.]?

B-4 What in the school situation could have contributed to students' behaviors such as: not being able to solve non-routine problems; having weak strategies for solving problems; focusing on generating answers, manipulating blindly; perceiving math as an alien formal system on which they have no claim, and to their various misconceptions or alternative conceptions of mathematical concepts? (Confrey, et al., 1984)

B-5 In mathematical problem solving in a school situation, what are the control processes at work used by students? (Wheeler, et al., 1984)

B-6 In a school situation, mathematical proof appears more often as a discourse to reproduce than as an instrument of control of the validity of one's solution. The problem is to design teaching situations in which mathematical proof would function as an instrument of control in problem solving in the social situation of a classroom debate. (Wheeler, et al., 1984)

B-7 How does mathematical ability manifest itself; how does it develop? Is it innate or does it develop by means of appropriate education? (Wheeler, et al., 1984)

SET C

C-1 What is the purpose and nature of mathematics education in different cultures? How do teachers perceive their role (in different cultures)? What happens / can be done if the teacher of mathematics and his or her students do not share the same views on the purpose and nature of mathematics education? (Wheeler, et al., 1984)

C-2 How is students' motivation to learn mathematics affected by economic, political and social changes? (Wheeler, et al., 1984)

C-3 What is the emotional content of mathematics? "The impact of a triangle on a circle is no less dramatic than that of the hand of Michelangelo's Adam reaching out to God's." (This quote is attributed to V. Kandinsky by D. Tahta in (Wheeler, et al., 1984). The image below is a reproduction of a picture by V. Kandinsky.)

Composition VIII (Kandinsky, 1923)

APPENDICE 2 / APPENDIX 2: DEFINITIONS AND TRANSCRIPTS FOR ACTIVITY 2

DEFINITIONS OF THREE DIDACTIC PHENOMENA

<u>Topaze effect</u>

a) The teacher has a precise expectation in terms of students' answers.
b) There is a substantial difference between the students' initial productions and utterances and these expectations.
c) One can observe a succession of questions or dialogue piloted by the teacher for obtaining the expected answer, drastically reducing the mathematical meaning of it.
d) When the expected answer is produced, the teacher tries to maintain the fiction that the answer is really significant and that the didactic contract has not been broken. (Bikner-Ahsbahs, et al., 2014, p. 204)

<u>Semiotic bundle</u>

a) The teacher has a precise expectation regarding the mathematical reference in students' answers.
b) The teacher is observing that the students' initial semiotic productions are close to the expected mathematical productions: this proximity is an indicator for the teacher that the student is entering a zone of proximal development for the concept at stake.
c) There is a teacher-student interaction where typically the teacher 'lends' the student the right words and/or signs to express verbally and/or symbolically what he judges to be on the way to the right answer considering the students' semiotic productions. To stress the correctness of the students' answers, he echoes students' non-verbal productions: hence he takes the responsibility of a multi-modal production, which supports the students to formulate what they were grasping in a fuzzy and imprecise language in more proper mathematical language. In doing so, the mathematical meaning of the student's answer increases. The students are put in the condition of being able to express it in the shared scientific language.
d) When the expected answer is produced, the teacher underlines that the answer is really significant and that the didactic contract has been fulfilled. (Bikner-Ahsbahs, et al., 2014, p. 208)

<u>Funnel pattern</u>

a) The student does not recognize the mathematical operation or is not able to draw an adequate conclusion.
b) The teacher asks an additional question but gets a false answer or does not get any answer.
c) The teacher continues his effort to get at least a part of the expected answer. Understanding is no longer approached.
d) Missing the expected answer, the teacher tends to narrow his efforts, aiming at just what is expected just being said, no matter who says it. Self-determined behavior of the students decreases and at the same time the situation becomes more and more emotionalized.
e) The process is finished as soon as the answer occurs, no matter whether the student or the teacher has produced it. (Bikner-Ahsbahs, et al., 2014, p. 211)

EXAMPLES OF TEACHER-STUDENT INTERACTIONS

Example 1: Adding one-digit numbers in elementary school

Teacher: Nine plus seven?

Student: 14

Teacher: Okay. 7 plus 7 equals 14. 8 plus 7 is just adding one more to 14 which makes [*voice slightly rising*]

Student: 15

Teacher: And 9 is one more than 8. So 15 plus one more is [*voice rising*]

Student: 16

Teacher: That's it!

Source: (Bauersfeld, 1988)

Example 2: Subtracting two-digit numbers in elementary school

Teacher: Solve the problem: 66 – 28 = ___ [*writes it on the board*]

[Children work at their desks]

Teacher: Johnny, could you tell the others how you solved the problem?

Johnny: [*Writing on the overhead projector*] We put the 28 under the 66. [*As he talks, he writes 66 – 28 in a vertical format*] And we took away… we… I took… the 6 and 8 off. And we said there was 60 and 20 there. [*He puts his finger on the 60 and then on the 40*] And if you take away 20 from 60, it's 40. [*Holds up his fist*] And you still have to take away the 8. So we took… there's 46 left over. If you take that 6 back, and take away that 6 [*points to the 6 in 46*] and that's… uhm… back to 40 and you still have to take away 2, so 39 [*he holds up a finger*] then 38. [*He writes 38*]

Teacher: Any questions?

Mathew: I don't understand.

Teacher: Johnny, could you re-explain?

Johnny: We put the 66 under the 28. Then we took off the 6 and the 8, and if you take away 20 plus 60, it's 40. And if you put the 6 back on and the 8, we have 46. Then we take away that 6, now you have 40 back and you still have to take away 2.

Elisabeth: But why did you take the 6 and the 8 off?

Johnny: It was more easier.

Teacher: [*Looking around and deciding that the others still may not understand what he did*] OK, could you write down beside it what you did? Maybe that would help us see it. Instead of 66 minus 28, what did you do?

Johnny: 60 take away 20 equals [he writes 60 – 20 vertically, and looks at the teacher]

Teacher: Would you write what you get? [*He writes 40 under 20*] OK, what did you do next?

Johnny: Then we put the 6 back on. Then it equaled 46. [*He writes +6 next to 40*] And you still have to take away 8, so you have 40 back. And… uhm… if you take away 2 – you have to take away 2 more, so we got 38.

Teacher: [*To the class*] Makes sense? He said [*coming to the front of the class to use the overhead projector*] I have, let's put the 46 up here [*she writes 46 at the top*] That's what he has and then he said I've got to go back to 40. Okay, why did you go back to 40?

Johnny: 'Cause we took away that 6, 'cause you have to take away 8 and you still have to take away 2 more.

Teacher: You understand how he did that?

Class: Yeah.

Teacher: [*Long pause*] Very interesting way to do that. Thank you.

Source: (Wood, 1998)

<u>Example 3: An interaction on division of fractions (quantity by a number)</u>

Prior to the following episode, children were looking at fractions as parts of areas of circles.

Teacher: If we have 20 kids and 25 apples, how are we going to split it up so everyone gets the same amount… equal pieces?

Student: One

Teacher: We give everybody one and what are we going to do with the other 5 apples?

Student: Throw them away.

Teacher: We're going to throw them away. Well, we can throw them away but that's kind of wasteful.

Student: Split them in half.

Teacher: [*Speaking simultaneously*] What are we going to do? Split them in half.

Students: Split them in fourths, split them in fourths.

Matt: Split them in 20 more pieces

Teacher: Right. Split them in 20 more pieces. So sometimes a fraction is not only a whole thing or a group it has… it has extra pieces.

Alan: You split the apples into 5 pieces… because…

Bke: No, into fourths.

Teacher: Wait a minute. Shhh [*to the rest of the class*] OK

Bke: Five apples: 5 times 4 is 20.

Teacher: 5 times 4

Bke: It would be fourths. Split the apples into fourths.

Teacher: We would split the apples into fourths. And so how much would everybody get?

Bke: One and a fourth.

Teacher: One and a fourth apples. Or we would get 5/4ths. [*She circles 5/4 on the overhead projector*] Wouldn't we?

Source: (Voigt, 1998)

<u>Example 4: An arithmetic introduction of the square root</u>

Teacher: Take out your rough paper. I am going to ask you three questions, which I will put up on the board…. 1. Are there any numbers whose square is -1? 2. Can two different numbers have the same square? 3. Are any of the following numbers square integers? 40, 9, -16, 0, 25/4, 1, 400, 10^5, 121, 0.04, 9^{10}. Ok, go!

Teacher: Let's check what you have done. Can you find any number whose square is "minus one"?

Michael: Yes. If you take the negative square…

Teacher: Michael, how do you mean? The negative square, the square of a negative number? Let's hear Michael's explanation.

Michael: You take one, you put minus…

Teacher: I take one and how do I write it? I put minus one [*writes* $(-1)^2$]… and this is…?

Michael: No!!!

Teacher: Come and write it for us.

Michael: [*Writes on the board*] $-(-1)^2 = -1$

Teacher: Lise, so what is the answer to Question 1?

Lise: Well, it's No!

Teacher: It's no! Why?

Lise: The square of a negative number is always positive!

Teacher: The square of a negative number is always positive. And the square of a positive number, then?

Lise: Positive.

Teacher: It's always positive. How come the square of a negative number is always positive, Adhil!

Adhil: It is the property of multiplication of negative numbers.

Teacher: Yes, go ahead, explain!

Adhil: If we multiply two negative numbers the result will be positive.

Student: Why?

Teacher: If we multiply two negative numbers the result is always a positive number. Thus the square of a negative number is a positive number and the square of any number is a positive number. Write it in your notebooks. Is everybody sure of that? OK.

Teacher: [*Writes on the board*] "The square of -5 is 25, the square of 5 is 25."

Student: No! That's wrong!

Many students: The square of -5 is -25!

Teacher: Oh, you don't agree?

Student: If you put -5 without brackets you get -25.

Teacher: I did not put brackets. You want me to put brackets here [*points at* -5]? [*Writes, "The square of (-5) is 25, the square of 5 is 25."*] Are you satisfied now?

Student: Yes!

Source: (Comiti & Grenier, 1995)

APPENDICE 3 / APPENDIX 3: DEFINITIONS AND DISTINCTIONS THAT HAVE BEEN MADE BY MATHEMATICS EDUCATORS AND THEORETICIANS OF EDUCATION REGARDING THEORIES AND FRAMEWORKS

BROAD DISTINCTIONS

Theory levels

- Grand theories; intermediate frameworks; design tools
- Theory vs theoretical framework

Kinds of frameworks used in research

- Theoretical
- Practical
- Conceptual

Kinds of knowledge structures

- Horizontal
- Vertical

FINER DISTINCTIONS

Distinction between grand theories, intermediate frameworks and design tools

In an analysis of the processes and products of design-based research, Ruthven, Laborde, Leach, and Tiberghien (2009) distinguish among grand theories, intermediate design frameworks, and design tools.

They consider the theory of didactic situations as an example of an *intermediate design framework*.

Grand theories

The process of didactical design is informed by the professional knowledge of the designers and by other kinds of knowledge, including *grand theories* which are theories general in scope and abstract in form, e.g., theories of human development and learning, epistemology of mathematics, and theories of the process of instruction.

Intermediate frameworks

Intermediate frameworks are more specific to objects of the design. Such frameworks extract, coordinate, and contextualize those aspects of several grand theories that are pertinent to developing, analyzing, and evaluating teaching designs.

Design tools

Ruthven et al. (2009) describe design tools on the example of the *Theory of Didactic Situations* (Brousseau, 1997):

> *Over the course of its development, TDS has drawn on a wide range of theoretical ideas [particularly by] Piaget's theorization of cognitive development as a process of constructive adaptation and… Bachelard's theorization of knowledge growth as encountering intrinsic obstacles.…TDS developed … as an intermediate framework in which components appropriated from grand theories were combined, elaborated and refined to produce an explicit didactical apparatus for the design of teaching sequences, an apparatus that – although TDS itself does not use such a term – includes identifiable design tools [such as "adidactical situation" or "didactical variables"].* (Ruthven, et al., 2009, p. 330)

There is a similar distinction made by Assude, Boero, Herbst, Lerman, and Radford (2008, p. 341). In a sense, it is more general because not restricted to design-based research:

- grand theories (e.g., what is the mathematics education field?)
- middle-range theories (e.g., what is classroom mathematics instruction?),
- local theories (e.g., what levels of development exist in students' learning of fractions?)

Distinction between theoretical framework and theory

The ATD (*Anthropological Theory of the Didactic*) proposes to analyze any practice by identifying

- the types of tasks it has been developed to solve,
- the techniques used to accomplish these tasks,
- the system of principles and rules that justify the techniques (technology),
- and a theory that justifies and connects the different elements of the technology.

The result of this analysis is a *praxeology*. We propose that the difference between a theoretical framework and a theory in the practice of research in mathematics education is the difference between the *technology* and the *theory* in an ATD model of this practice.

Example of works representing a theoretical framework and a theory in the practice of a researcher

Let us take an example from the work of Alan H. Schoenfeld on problem solving. In (Schoenfeld, 1992), the author distinguishes between a *framework* for understanding mathematical cognition, and a *theory* of mathematical thinking.

The *framework* is a description of an "overarching structure for the examination of mathematical thinking" (p. 335) claiming that "the fundamental aspects of thinking mathematically" (p. 335) include core knowledge, problem solving strategies (heuristics), effective use of one's resources, having a mathematical perspective, and engagement in mathematical practices.

In the 1992 paper, only an overview of various theories of mathematical thinking is given. It has taken the author many years to develop such a theory himself (Schoenfeld, 2011). The author describes the theory by the questions it is supposed to answer and the claims it makes:

> *Here are the core questions I address. Suppose a person is engaged in some well-practiced activity. What determines what that person does, on a moment-by-moment basis, as he or she engages in that activity? What resources does the person draw upon, and why? What shapes the choices the person makes? What accounts for the effectiveness or lack of effectiveness of that person's efforts? The main claim in the book is that what people do is a function of their resources (their knowledge, in the context of available material and other resources), goals (the conscious or unconscious aims they are trying to achieve) and orientations (their beliefs, values, biases, dispositions, etc.). I argue that if enough is known, in detail, about a person's orientations, goals, and resources, that person's actions can be explained at both macro and micro levels. That is, they can be explained not only in broad terms, but also on a moment-by-moment basis.* (Schoenfeld, 2011, p. ii)

Distinctions between theoretical, practical and conceptual frameworks

We found these distinctions in a PME-NA 1991 plenary address of Eisenhart (1991).

Theoretical framework

A *theoretical framework* is a structure that guides research by relying on a formal theory; that is, the framework is constructed by using an established, coherent explanation of certain phenomena and relationships, e.g., Piagetian theory of conservation, Vygotsky's theory of socio-historical constructivism, or Newell and Simon's theory of human problem-solving. (Eisenhart, 1991)

Practical framework

A *practical framework*… guides research by using 'what works' in the experience or exercise of doing something by those directly involved in it, e.g., in the case of educational research: by using 'what works' in teaching, administering, trying to change schools, being the helpful parent of a school-aged child, as a 'kernel' idea or action that, if extended to other teachers, etc., could help to alleviate some educational problem. (Eisenhart, 1991)

Conceptual framework

> *A conceptual framework is a skeletal structure of justification…. A conceptual framework is an argument including different points of view and culminating in a series of reasons for adopting some points – i.e., some ideas or concepts – and not others. The adopted ideas or concepts then serve as guides: to collecting data in a particular study, and/or ways in which the data from a particular study will be analyzed and explained.* (Eisenhart, 1991)

<u>Horizontal and vertical knowledge structures</u>

[We did not have the time to discuss this Bernsteinian distinction in the WG, but we include it here in the Report, because it served as a basis of Steve Lerman's contribution to a book on theories in mathematics education and is therefore relevant to the theme of our WG.]

Lerman (2010) draws on Basil Bernstein's distinction between *horizontal* and *vertical knowledge structures* to defend the view that the diversity of theories in mathematics education (and lack of a single, unified theory) is not necessarily bad for the development of the domain. A further useful distinction is between discourses with strong grammar and discourses with weak grammar. [Relevant quotations are included in the "Handout" section of this Appendix.]

> *Bernstein's… notion 'verticality' describes the extent to which a discourse grows by progressive integration of previous theories, or the insertion of a new discourse alongside existing discourses and, to some, extent, incommensurable with them…. Bernstein offers science as an example of a vertical knowledge structure and interestingly, both mathematics and education (and sociology) as examples of horizontal knowledge structures. Science generally grows by new theories incorporating previous ones (as in Newtonian mechanics and relativity) or by revolutions, whereas new mathematical theories tend to be new domains with their own language and theorems that don't replace other theorems…. He uses a further distinction that enables us to separate mathematics from education: the former has a strong grammar, the latter a weak grammar, i.e., with a conceptual syntax not capable of generating unambiguous empirical descriptions…. It will be obvious that linear algebra and string theory have much tighter and specific concepts and hierarchies of concepts than radical constructivism or embodied cognition. Adler and Davis (2006) point out that a major obstacle in the development of accepted knowledge in mathematics for teaching may well be the strength of the grammar of the former and the weakness of the latter. Whilst we can specify accepted knowledge in mathematics, what constitutes knowledge about teaching is always disputed.* (Lerman, 2010, pp. 101-102)

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EARLY YEARS TEACHING, LEARNING, AND RESEARCH: TENSIONS IN ADULT-CHILD INTERACTIONS AROUND MATHEMATICS

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INTRODUCTION

Adults are often seen as indispensable to the learning of young children. Whether they are teachers, parents, or older siblings, adults interact with young children in ways that, intentionally or not, impact their mathematical development (Aubrey, Bottle, & Godfrey, 2003; Walkerdine, 1988). Discussions in the research community highlight the complexity of the mathematical conversations adults have with children in and out of school (Anderson, Anderson, & Thauberger, 2008; Cobb, Yackel, & McClain, 2000). Indeed, what impacts a child's thinking in mathematics is a complex interplay of a number of factors, including the goals and features of the conversation, the context in which the interaction takes place, and the cognitive and affective characteristics of the child (Hiebert & Grouws, 2007; Lampert & Cobb, 2003; Lobato, Clarke, & Ellis, 2005; Osana et al., 2012).

In this Working Group, we studied ways adults can, and do, engage young children in mathematical conversations in prior-to-school and school environments, such as the home and early elementary (K-3) classrooms. We used the *dance of agency* (Boaler, 2003; Pickering, 1995) as an overarching metaphor to guide our examination of adult-child interactions. In our discussions, we explored the interchange (or dance) in conversations where informal mathematics emerged as well as in formalized or standard interactions about mathematics. Although our discussions took a number of different turns, our primary focus was to understand if and how agency shifts from one entity (i.e., the child, the activity, the adult) to the other(s), and to problematize its role in the typification of mathematical events.

PARTICIPANTS

Twelve participants attended all three days of the Working Group. The group was diverse with respect to background and research interests. Some participants were faculty members and others were graduate students, all representing a number of different universities across Canada. Research interests were wide-ranging, and included mathematical language and discourse, task design, preschool mathematics, secondary teacher development, mathematical argumentation and proof, games, parent-teacher communication, student motivation, instructional strategies, symbolization, and special needs.

DAY 1

INTRODUCTION AND OVERVIEW OF WORKING GROUP ACTIVITIES

The first day of the Working Group began with general introductions from the leaders and the participants. The leaders then launched into a brief overview of the goals of the Working Group. One of the primary goals was to provide the participants an opportunity to share ideas with each other about the types of mathematics that emerges when adults and children interact, whether in settings that are deliberately about mathematics (intentional) or in settings where the mathematics was decidedly unintentional. The leaders then introduced the participants to the notion of *dance* as a lens to examine adult-child interactions.

When describing mathematicians' work, Pickering (1995) described the interchange and tensions between what he called *human agency* and the *agency of the discipline*. We set this as the backdrop for the Working Group discussions: What would the 'dance' look like between adult and child in (a) a 'school mathematics' interaction between teacher and a second-grader, and (b) an out-of-school setting in which an adult family member interacts with a young child? At the end of the overview, we presented the structure of the Working Group to the participants, which consisted of a focus on the elementary setting on Day 1, a focus on the preschool setting on Day 2, and a synthesis of both settings and wrap-up on Day 3. The activities were designed so that the participants had the opportunity to engage in whole- and small-group viewing and discussions of four video clips of adult-child interactions.

INTRODUCTION TO THE ELEMENTARY SETTING

At the elementary level, there is often a tension between the teacher as *provider of information* and the student as *constructor of knowledge*. This tension can often be seen in the complex interaction between the teacher and his or her students as they engage with mathematical ideas in the classroom. In their reformulation of *telling*, Lobato et al. (2005) argued that providing information to students, whether through telling or questioning, is most appropriate when the teacher is aware of how students are thinking about the task. Teachers learn about their students' thinking through elicitation, defined as a process of asking questions specifically crafted to assess how they are thinking about the target concepts.

A tension presents itself, however, between the teacher and students as agents of the mathematics that is taking place. In conceptualizing agency in adult-child interactions about mathematics, Boaler (2003) described the interplay, or dance, between students' own constructions of meaning and the standard rules and procedures they already know. In the elementary portion of the Working Group, we extended the dance metaphor as a lens through which to examine the interchange between teacher and student as they discuss the meaning of the equal sign (=). In the conversations captured on a video segment we shared, one sees the child's intuitive mathematics in her contributions as well as the formalized or standard ways of doing and thinking about mathematics visible in the teacher's.

On Day 1, the participants viewed two videos in which second-graders interacted one-on-one with an adult on the meaning of the equal sign. There are several accounts in the literature of children holding deep-seated misconceptions about the equal sign (Jones & Pratt, 2006; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; McNeil & Alibali, 2005; Sherman & Bisanz, 2009). Although it is a symbol that signifies a relation between two equal quantities, many children view the equal sign instead as a signal to *do something*, such as adding all the terms to the left of the equal sign and placing the sum somewhere to the right. Children who have such misconceptions have been said to hold an *operator* view of the equal sign (e.g., Carpenter, Franke, & Levi, 2003), and researchers have found that this view explains the difficulty children have solving non-canonical (i.e., atypical) equations, such as $3 + 1 + 12 = 9 + \square$ (Baroody & Ginsburg, 1983; Falkner, Levi, & Carpenter, 1999; McNeil & Alibali, 2005; Seo & Ginsburg, 2003). In contrast, children who understand that the equal sign indicates that the amounts on each side are the same are said to hold a *relational* view (Empson, Levi, & Carpenter, 2011; McNeil, 2014; Sherman & Bisanz, 2009).

Carpenter et al. (2003) argued that carefully orchestrated discourse can support students' conceptual understanding of the equal sign. They describe an instructional approach, which we call here the *inquiry approach*, that relies on interpreting students' reasoning about the equal sign and presenting targeted follow-up problems to challenge their thinking. In one variation of the approach, the teacher begins the lesson by presenting an equation to the students (such as a True-False number sentence, e.g., 8 = 5 + 3) and asks them to comment on whether the number sentence is true or false. By inviting students to share their thinking, the teacher can extricate the faulty rules they use to make their decisions (e.g., "*It's false because you can't have two numbers on the right side of the equal sign.*"). Upon hearing such a rule, through a sequence of probes if necessary, the teacher can challenge it by writing "8 = 8" and again ask the students if the sentence is true or false. If students then respond with, "*that is false because there must be two numbers on the left side of the equal sign*," the teacher could challenge this rule by presenting "5 + 3 = 5 + 3". Each follow-up equation suggested by Carpenter et al. (2003) 'confronts' the student by presenting a situation that in one way concedes the child's rule, but at the same time is non-canonical, and in this way further challenges the student to reconfigure his or her thinking about the equal sign.

THE MATTHEW VIDEO

The Matthew video was a 10-minute clip of Matthew, a second-grade student, interacting with an adult on the meaning of the equal sign. An excerpt of Matthew's work on a prior assessment of his conceptions of the equal sign is presented in Figure 1. As is clear from his work, Matthew understands the meaning of the symbol in certain contexts, but he holds clear misconceptions in others (Seo & Ginsburg, 2003). The video shows the adult engaging Matthew in the *inquiry approach* advocated by Carpenter et al. (2003), eliciting his thinking frequently and providing hints where she thought it would support his reconceptualization of the equal sign. At the beginning of the conversation, Matthew struggles with making meaning of non-canonical equations, but there are signs at the end of the video that he is beginning to understand the notion of substitution (Jones, Inglis, Gilmore, & Dowens, 2012). For the equation 2 + 3 = 4 + 1, for example, Matthew understands that this is a true number sentence because "2 + 3" is the same as 5 but is written differently.

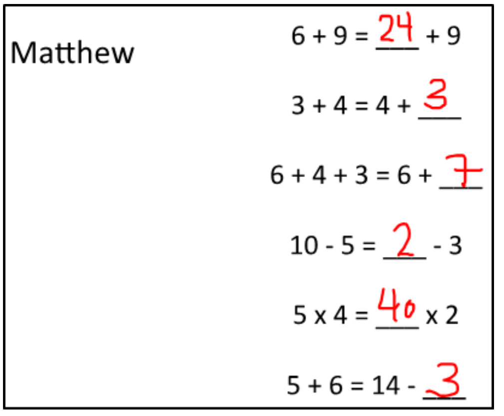

Figure 1. Matthew's work on a test of equivalence.

<u>Themes emerging from the Matthew video</u>

After viewing the video of Matthew, a few Working Group participants questioned whether some of the prompts offered by the teacher were necessary, particularly because, as the interview with Matthew progressed, he appeared to be actively constructing meaning through the interactions that were taking place. One participant proposed, however, that it was difficult to determine whether or not Matthew was actually constructing 'accurate' (in a mathematical sense) conceptions of the equal sign through his conversation with the teacher: Was it possible that his learning (defined in a broad sense) was superficial and that he had not yet internalized the targeted meaning of the symbol?

Another major theme that emerged from the discussion regarded the didactic contract between Matthew and his teacher. Some participants viewed Matthew's responses to his teacher's questions as trying to 'figure out' the rules of this game. It was clear to them that Matthew was cognizant that the nature of this interaction was different from those that take place in math class and that he was trying to understand what the goal was in answering these odd questions. In contrast, others interpreted Matthew's responses as similar to those that take place in class: The teacher holds the 'power', which includes trying to trick students with clever questions.

Another theme that emerged from the discussion was the nature and purpose of the teacher's switch in instructional approach mid-way through the conversation. During the first two minutes of the video, the interaction between the teacher and student resembled a dance, in the sense that the teacher was reluctant to provide too much information to Matthew about the meaning of the equal sign, and instead asked questions and generated follow-up equations to prompt him to further reflect on his thinking. At about the 1:40 mark of the video, however, the Working Group participants noticed a change in the teacher's instructional approach. The change marked a shift from a dance (one in which there was an almost equal give-and-take between Matthew and his teacher) to a more directed teaching style that involved the adult asking questions to focus Matthew's attention to specific relationships between the numbers of both sides of the equal sign.

THE BRENDA VIDEO

Similar to the Matthew video, the Brenda video was a 10-minute clip of a second-grader interacting with her teacher on a number of canonical and non-canonical number sentences. The goal of the discussion was also to support the development of Brenda's understanding of the equal sign. As can be seen from Brenda's written work in Figure 2, she not only had difficulty with the meaning of the equal sign, but also struggled to find effective strategies for single-digit addition. These issues were also evidenced in the video on numerous occasions. The teacher's stance at the start of the interview was based on the inquiry approach advocated by Carpenter et al. (2003), but the nature of the conversation shifted rapidly to one that the Working Group participants considerably more teacher-directed. The teacher turned to highlighting, in an increasingly direct fashion, the contradictions that emerged in Brenda's reasoning from equation to equation. By drawing attention to the contradictions, the teacher's goal was to help Brenda restructure her operational 'adding' schema (McNeil, 2014) into a more relational one. At a few points during the conversation, particularly nearing the end of the clip, there were some clues that Brenda was beginning to appropriate a relational view of the equal sign, but her newly constructed understanding appeared to be quite tentative. When presented with a new equation, for example, Brenda reverted to her original operational views to judge its 'truth'. In other cases, Brenda may have been simply repeating the terminology and phrasing used by her teacher but without the underlying conceptual understanding.

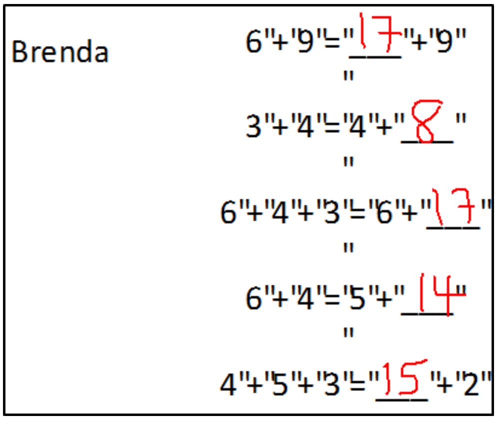

Figure 2. Brenda's work on a test of equivalence.

<u>Themes emerging from the Brenda video</u>

After viewing the Brenda video, the participants were all in agreement that the teacher's approach in this conversation was considerably more directive than her approach with Matthew (it was the same teacher in both videos). The participants also discussed the differences in the students' responses during the conversations, noting that Brenda was more focused on the superficial conventions of mathematics (such as symbolic syntax) than Matthew was.

Finally, the meaning of the word *true* in the context of the two conversations came up in the discussions about both videos. Some participants wondered whether the student interpreted the mathematical meaning of the word *true* (i.e., "*Is* 8 + 4 = 12 *a true number sentence?*") in the same way as the teacher. Did the student interpret the word *true* to mean "*for real*", or "*this is in a format that is acceptable to me*"? On the other hand, perhaps Brenda's use of the word *weird* was intended to mean "*false*". Did *true* mean "*right*" and *false* mean "*wrong*"? This led to the question: Was the children's use of the words *true* and *false* related to their understanding of the equal sign symbol or the everyday meanings of the words? Further, the question arose whether the use of the words in this context reflected a legitimate syntax structure in school mathematics or even a legitimate instance of a school mathematics problem.

The discussion about the meaning of the word *true* led to a more general exchange among Working Group participants on the language used by both teacher and students in the videos. Some participants questioned how the students interpreted the meaning of the word *same*, and in fact, they considered whether Matthew and Brenda held different meanings for the word and whether their interpretation was different also from the teacher's. For example, Brenda may have interpreted the meaning of the word *same* to mean "*the number has the same digits*" or "*each side of the equal sign has the same numbers*". One participant suggested that the teacher and the students were using different codes, and for each to understand each other, the teacher needed to "*switch codes*". One example of code switching was when the teacher abandoned her use of the term *equation* and used *number sentence* instead. As another example, the teacher also began couching the conversation in the context of addition, which was more accessible to Brenda. Once the teacher switched codes, at least in Brenda's case, then the dynamic of the conversation changed and Brenda appeared more able to engage with the targeted learning objective.

DAY 2

LOOKING BACK: WORD ASSOCIATION CONTINUUM

We began Day 2 with participants generating words and phrases they associated with the interviews of the second-graders and placing them on a negative to positive continuum according to connotations the words conveyed. As seen in Figure 3, a majority of the words and phrases fell around the middle of the continuum, with more appearing on the negative side. In addition, similar terms evoked different connotations. For example "*rules of the game*" and "*you're tricking me*", both seemed to associate the interviews with emphasizing conventions (rules), but on the continuum, they sat on opposite sides.

NEGATIVE (-)

choppy…**pulling**…twisting her words…limiting vocabulary…outcome based (teaching)

debugging…symbolic argument…learning?

rules of the game…discourse…not teaching…co-construction…struggle… partially formed meaning making…positioning in relation to student…confusion…

researcher-teacher tension…tension between school and math norms…power…authority… positioning…didactic contract…tone

---CENTER---

eliciting a response…eureka!...articulating thoughts…fluid…grammatical analysis of sentences

considerate…comparing to previous experience…"**you're trying to trick me**"…

eliciting…superficial

nudging…tone…prompting…gesture…nudge

POSITIVE (+)

Figure 3. Word-associations along a negative to positive continuum.

In the discussion that ensued, one participant explained that Matthew cast *being tricked* positively, as it seemed to signal that he realized they were playing a game. Another participant clarified that what the students were able to do in response to 'the trick' was very positive. Yet, in contrast, the more explicit direction and guidance around the *rules of the game* in the Brenda video left a negative feeling. This resonated with others, who associated *pulling* [Brenda video] and *nudging* [Matthew video] with the teacher-child dynamics, and while *nudge* was positive, *pulling* was negative. It appears then, that Brenda's video evoked more negative connotations than did Matthew's. A participant responded to this tension by referring to theories of cognitive dissonance and indicated that "*if you want kids to change their schemas, rather than just add on to it, they need to experience sufficient discomfort*" which in turn may suggest "*they need to be 'tricked'*". If so, can it be consistently positive, and how would that be achieved?

As the discussion continued, questions arose regarding Brenda's mathematical understandings and its impact on the adult-child interactions. What did the interviewer know about the children and how did this affect her intervention? Participants believed the interviewer herself felt a tension between her roles as teacher and as researcher. It appeared unclear whether the goal was to find out what the students already knew or to push their understanding. Two participants argued that Brenda is not less facile, explaining that her "*ability to count doesn't tell me anything about her understanding of the equal sign*" and "*her reason for answering five plus three equals six may have been a matter of social anxiety while working with an adult interviewer*". Hence, did the 'dance' (i.e., interview dynamics) change because of a perception of the child as less able, or was the child positioned as less able because the dance changed?

INTRODUCTION TO THE PRIOR-TO-SCHOOL CONTEXTS

In prior-to-school contexts, many young children engage with mathematics within everyday conversations, many of which—although not all—are associated with joint parent-child activity. In her seminal work, Walkerdine (1988) analyzed the audiotaped conversations of 30 mothers and their 4-year-old daughters, collected for a language project by Tizard, and audio recordings of 6 preschoolers (3 girls, 3 boys) in their homes to determine the types of mathematics present. Based on her findings, she characterized tasks in which these parents and children engaged as *instrumental* or *pedagogical*.

> *Instrumental referred to tasks in which the main focus and goal was a practical accomplishment and in which numbers were an incidental feature of the task* […] *In the pedagogic tasks* […] *numbers were the explicit focus of the task,* […] *predominantly the teaching and practice of counting.* (Walkerdine, 1988, p. 81)

More recently, Aubrey et al. (2003) carried out non-participant observations (one-hour video tapings) of mother-child dyads once every four months over several years, from which they reported two illustrative cases. They remarked that "the two parental styles bore resemblance to Walkerdine's (1988) typifications …" (p. 102) in that Child H and her mother experienced mathematics through everyday and play activities, and Child L and her mother engaged in mathematics largely through games and puzzles with an explicit pedagogical focus. Because the teachers in that study described both children as average to above average, however, neither parental style was considered more effective.

In contrast, Anderson and Anderson's (2014) longitudinal study of six middle-class families suggests that parental styles fall along a continuum from *pedagogical* to *instrumental*, rather than the dichotomy previously noted. In this study, using a best fit, each video-recorded parent-child activity was identified according to the perceived goal (i.e., math is core, math occupies a major portion, math holds an equal part, math is minor, or math is incidental). Accordingly, findings indicated that one of the families engaged in all pedagogical (math is core) activities, another engaged in mostly instrumental (or incidental) mathematics, while the other four families engaged in a mixture of activities along the continuum. Although evidence continues to build that suggests much of young children's exposure to math prior to school does not occur during explicitly didactic interactions (Benigno & Ellis, 2008), many seem to dismiss or undervalue incidental math, with varied attempts to help (tell) parents and early childhood educators how to make the mathematics more explicit or pedagogical. Yet, since much of the activity in which we inevitably engage children has mathematics ‘present’, is it not probable that such incidental mathematics is equally important in children’s sense making?

On Days 2 and 3, Working Group D participants viewed two videos in which a preschool child interacted with an adult family member while engaging in an everyday activity chosen by the mother. These two videotaped conversations, along with 11 others, had been categorized as incidental mathematics in Anderson and Anderson’s (2014) study. In one clip, the research assistant (RA) recorded the interactions between a mother and her daughter while the daughter played with a water sprinkler on a hot sunny day. In the second, a mother recorded interactions between a grandmother and her granddaughters as they looked through family photos. In the following sections, a brief description and short excerpt from the verbatim transcript (see Figures 4 and 5) precede our discussions of them.

THE LAWN SPRINKLER VIDEO

As the RA records, the female preschooler turns on the outdoor water faucet. The child approaches the water sprinkler on the front lawn, talking about what it is doing. After she returns to the tap, adjusting the water level to “*super high*” and then to “*low*”, she again approaches and backs away from the sprinkler multiple times. After a brief episode where the child and mother attend to making “*soup*” in a bucket of water, the preschooler rides her bike for a short while, after which the child’s attention returns to the water-sprinkler. Now dressed in a swimsuit, the child runs through the water, jumps over the sprinkler, holds her head in the stream of the water, and when desired, asks her mother to regulate the water for her (see Figure 4). At times the child is seen lifting her foot to assess the height of the water; on other occasions she seems to watch where the water lands.

89	**Child:**	[*cautiously steps towards sprinkler, gets wet and runs off*] Can you turn the sprinkler a little bit high?
90	**Mother:**	High?
91	**Child:**	Lower.
98	**Mother:**	How low, this low? [*adjusts sprinkler’s water level (height)*]
99	**Child:**	Yeah that’s how low.

Figure 4. Mother and daughter conversation during water-sprinkler play.

<u>Themes emerging from lawn sprinkler video</u>

After viewing the video of this mother and daughter, along with the accompanying transcript, the participants shared various ideas regarding the definition, value, nature, and prevalence of the perceived incidental math that had occurred. For instance, questions arose about intentionality with respect to incidental mathematics. Does incidental mathematics have anything to do with intentionality? Does incidental necessarily mean less intentional? How is intentionality being framed? Is it about the adult being intentional? How do we view/observe a child’s intention and how do we interpret it?

The group also raised questions about *finding math*, after the fact, in such parent-child activity. For instance, what is the purpose of identifying the math, and what do we *do* with the math we find? In addition to listing the math we saw, we noticed many points where the parent could have ‘taken up’ mathematics (i.e., mathematizing the moments during play). As one participant indicated, parents do not always understand how to do that (e.g., “*Show me how high you want the sprinkler, 10 inches, 12 inches?*”) but, she argued, the incidental dialogue and learning has implications for the child upon arrival at school. There was recognition that such outcomes rely on parents becoming cognizant of what is happening in front of them and nudging the child. This in turn led to comments about parent perceptions and attitudes, noting that, "*when you say* math*, parents think 'school' math*," and, "*when parents see math—is it fun or not?*" This then led us to question: How does one get parents to mathematize the elements of play without taking the play away from the child? How do parents come to recognize math in the moment and what types of conversations would help them to do so? Considering the challenges inherent in supporting teacher candidates in noticing everyday mathematics, how can one help parents to become more pedagogical in these moments? Thus, rather than "*telling parents how to do it* [incidental math] *differently, is there a way to document these incidental math moments to determine if, and what, this instrumental math activity is contributing to young children's early mathematics learning?*"

On this note, we turned our attention to classroom implications for our findings regarding the math shared by parent-child dyads. In the video, the child was engaging with an understanding that she wanted a certain height without the parent intervening with "*how high?*" How is that child making sense of such a mathematical concept? How does this experience help her in making sense of measurement? How do children hold on to that prior-to-school experience as a springboard for their subsequent in-school experiences? If one were to look at how early mathematics parallels early literacy learning, are there lessons to be learned that might drive a shift in pedagogy? More generally, however, the question remains: How/what might mathematics teachers in the early-years classroom (and beyond) learn from children's everyday experiences of mathematics?

DAY 3

We began Day 3 by viewing the second preschool video (see Figure 5).

THE FAMILY PHOTOS VIDEO

As her mother video-records, a preschool girl and her grandmother look at family photos together. In the last quarter, her older female sibling joins them, and the grandmother interacts with both granddaughters as they view and comment on the photos. In the video, the grandmother is seated on the edge of a bed and the preschooler is alternately standing or seated next to her. The photos have been printed on 11"x17" sheets of paper, resembling intact photo album pages. The grandmother holds the large sheets, resting them partially on her lap to allow her to gesture at times towards specific photos. The child leans into and over the sheets at times, specifically pointing at people or objects in individual photos. The child on occasion turns, as if to view her mother, or include her in the conversation, even though she is positioned behind the camera.

	Child:	Hey, what is that doggy?
18	**Grandma:**	That was a very big doggy that she got for [*inaudible*].
	Child:	Why?
20	**Grandma:**	It was such a big doggy that you could sit on it.
	Child:	Yeah, and get a ride, get a ride.
86	**Grandma:**	Who is that?
	Child:	I don't know.
96	**Grandma:**	No, I think that might be [*child's name*].
	Child:	No that is not me.
	Mother:	That is you.

Figure 5. Grandmother-granddaughter conversation about family photo.

Themes emerging from the family photos video

After viewing the video, our discussions once again gravitated towards the mathematics we noticed and additional reflections around incidental mathematics in prior-to-school contexts. In this discussion, defining incidental mathematics gave way to voicing connections. While noting that the passage of time is a challenging concept for children, the grandmother and her 4-year-old granddaughter easily engaged with it. One participant connected what she saw with Dewey's notion that incidents are either *educative* (growth) or *mis-educative* (hampers growth), explaining that this was an educative moment. Another participant drew us back to the dance metaphor and posited a "*winding pedagogy*" to describe the teaching and learning seen in the video. Others toyed with defining the math they were witnessing in the videos as "*embedded for these children*", perhaps a kind of embodied mathematics. Another participant spoke to connections with the Pirie-Kieren model, suggesting that in this video, there are "*probably a lot of images being formed about number in terms of relationships and they are very different images than what students might see in school; there are some really big ideas here that kids are being exposed to; all of this is very rich in terms of primitive knowing*." Is it possible, then, that finding and elucidating such connections serves to further validate incidental mathematics? Or, as it was also suggested, is operationalizing incidental mathematics a key way forward in future research with young children?

When we turned our attention to the nature of the mathematics, we were noticing that mostly relational comparisons (e.g., ages, times, years, young/old, *just* two, *how* big) were involved. One participant spoke specifically to the conversations around photos capturing "*an interesting possibility with numbers related to time and space*" (e.g., "*she was six years old*" connects past with present and "*there*" with "*here*"). This was further elaborated by another, who noted: "*I like the 'math' that's in here that isn't like 'early school math'; it's this bigger thing of time and place and space; they may not understand it now and it may not get built on, but it's a very rich concept to be exposed to.*" This discussion of the 'big ideas' segued into a brief conversation about the intricacies within the conversations on the video—for instance, how looking at photos entails looking at scaled representations of people as they were in the past and, discerning their size, through what is known about them in the present in proportional, rather than literal ways. How does a preschool child come to understand these multiple representations at play? Another participant pointed to the complexity and nuances of language when we consider the 'messiness' of terms used in everyday conversations. For example, the same word in this scenario was used for multiple attributes (e.g., *little* referred to size (*little* ≠ *big*), to height (*little* = *short*), to age (*little* = *young*), and so on). For some, the complexity of the mathematics shared was possible because of the relationships between the adult and the child (e.g., the preschooler trusted her grandmother and believed what she was saying). And, unlike the same-age classroom structure, the family context allowed this preschooler to be included in other complex conversations (e.g., grandmother—older sibling—preschooler). And so, we wondered how were the personal relations, the mathematical complexity, and the incidental nature of these early years' experiences interrelated?

When our discussion turned to the relation between prior-to- and in-school mathematical experiences, the family photos video provoked reflection on the relationship between preschool experiences and children's achievement. As noted, research on what parents are doing in the home that is mathematical (e.g., LeFevre et al., 2009) shows that math talk positively impacts children's later achievement. Likewise, in classrooms that have more math language, children are advantaged in terms of their learning in the long run. Therefore, if parents were having more explicit math conversations, would these children achieve even greater success in school? For, as was recognized by at least one participant, students who begin school with the necessary conceptual prerequisites tend to be successful throughout their schooling and those without tend not to catch up. Interestingly, this latter point was attributed to a belief that the kindergarten curriculum is not ambitious enough. Indeed, seeing the big ideas with which the preschoolers in the two videos engaged magnified the underestimation of children's capacity within the early-years curriculum. For instance, one participant felt this grandmother was laying a deep foundation for typical 'age' problems (e.g., John is 2 years older than Mary; Mary is 4 years younger than Joe…), which are typically assigned to older students. Another indicated that subitizing is part of the kindergarten curriculum, but can be emphasized much earlier, elaborating that we need to be more aspiring. Others extrapolated that teachers and parents need to structure activities that afford the opportunity for mathematics to emerge—but not force it—seemingly conceding that incidental mathematics is beneficial.

JUXTAPOSING THE ELEMENTARY AND PRESCHOOL SETTINGS

Following the break on Day 3, we returned to the four videos and, in small groups, Working Group members viewed the school interviews again, but this time searching for incidental mathematics in the school-based videos and intentional mathematics in the pre-school videos. Seeking out the incidental in the Brenda video, for example, some participants noticed that the interviewer had unintentionally dismissed Brenda's attempt to share her own idea. While looking for intention in the family photos video, some participants posited that the grandmother had intentionally commented on size. As we shared such examples, we began to question where and with whom the intentionality lies. Is the intentionality in the learner or in the adult co-creating the moment? Was the discussion about intentional mathematics, intentional mathematics teaching, or intentional teaching? For example, the idea to let the child explore might have been intentional teaching. Specifically, was the mother's decision not to change the height of the water intentional, with the goal of letting her daughter own the moment? What appears to be unintentional about the second-grade conversations is what the students might internalize about the meaning of the equal sign without direct intervention on the part of the teacher. How then, as observers, do we ascribe intentionality, or the lack thereof, to a situation? For instance, since the grandmother appears not to exploit the mathematical potential in the discussion with her granddaughter, is this evidence that the grandmother did not have that intentionality to capitalize on the mathematics? Or is it possible she recognizes that to insert a mathematical question into the middle of their conversation about the photos would alter the richness and naturalness of the conversation, instead capitalizing on the math only to the extent to which it is evident?

CONCLUSION

As the Working Group drew to a close, we asked our participants to share one or two main *residues* (Hiebert et al., 1997) they would take away. With respect to the preschool discussions, one residue noted was that "*there are far too many things that are too delicate and special to be plucked or mathematized.*" Similarly, "*my job as an educator is to create the opportunities for the incidental—there is intention in creating the opportunity, but whether the child takes up that opportunity is incidental.*" From a different perspective, it was noted that, "*as a parent, whenever we do things with our children intentionally, it becomes very stressful. How can we do these intentional things without that tension?*" It was suggested that sharing exemplars of incidental mathematics with parents would inspire them to look at the incidental in their lives, and maybe become models for more intentional conversations. Another participant commented that "*if we tell them* [parents] *to do it, it no longer becomes incidental and becomes intentional—but in telling teachers about the incidental mathematics, can they then use it as groundwork or reference for their intentional mathematics?*" There was even a suggestion that we might think more about "*how to make intentional activity more like the non-intentional.*"

Another residue concerned itself with what seemed to be an emphasis on children's display of knowledge in the early years, with verbal interactions between teacher and child often about *show me/tell me*; yet there is more than that—there is the nervousness, body language, and gesture that signal a different perspective. Within the preschool conversations, there was a sense that the mathematics content resided in the play more so than the words spoken. Thus, it was concluded that relationships (both personal and mathematical) took on importance in both sets of conversations. For example, when an interviewer's response unintentionally leaves the child feeling less capable in the mathematics, the relationship between teacher and child applies less to the mathematical content than to the development of the child's identity.

LOOKING BACK AND FORWARD

As we reconsider our conversations over the three days through the writing of this report, we recognize that one outcome of the Working Group was a recognition that attributes of intentional and incidental mathematics conversations need further consideration, examination, and interrogation. Further research is needed to better understand how intentional mathematics activity can be made more fluid and natural, more like the unintentional: how we might bring intention to the incidental, and how we might build the intentional from the incidental. A key theme that appeared to resonate with participants in Working Group D was the valuing, and validating, of the incidental mathematics in children's lives, as foundational alongside the pedagogical tasks they encounter, so that one does not fall victim to privileging the intentional in ways that we once privileged 'direct teaching'.

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INNOVATIONS IN TERTIARY MATHEMATICS TEACHING, LEARNING AND RESEARCH

INNOVATIONS AU POST-SECONDAIRE POUR L'ENSEIGNEMENT, L'APPRENTISSAGE ET LA RECHERCHE

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(Texte en français suit.)

INTRODUCTION

> *Innovation is everywhere. In the world of goods (technology) certainly, but also in the realm of words: innovation is discussed in the scientific and technical literature, in social sciences like history, sociology, management and economics, and in the humanities and arts. Innovation is also a central idea in the popular imaginary, in the media, in public policy and is part of everybody's vocabulary. Briefly stated, innovation has become the emblem of the modern society, a panacea for resolving many problems, and a phenomenon to be studied.* (Godin, 2008, p. 5)

Our aim was to examine specific innovations in tertiary mathematics education and research, while examining innovation as a phenomenon as described by Godin (2008, 2014), with a focus on the tertiary education context. Despites its ubiquity, and perhaps because of the fuzziness of the concept, participants in the group were much more interested in playing with and examining innovations themselves along with their mathematical and pedagogical aspects, and thus the discussion of these will form the bulk of this report.

A few themes that can be gleaned from the three days:

1. Historical perspectives can make the mathematics of seemingly magical algorithms and procedures explicit. They can provide innovative ways of coping with a given mathematical topic or content in the classroom, by recalling what were the problems that gave rise to the development of these maths, and by proposing methods that are not necessarily the most mathematically efficient but are pedagogically enlightening.
2. Many innovations that were of particular interest to the group are ones that help move students between ways of representing mathematical concepts (i.e. from any one of symbolic, contextual, linguistic, concrete, and pictorial to one of the others).
3. Assessments (formative and summative) present areas of mathematics education rich in a potential for innovation both in form and content (e.g. 'good' problems).

There was general agreement that within higher education, innovations can be classified as *systemic* (i.e. related to the institutions within which the education takes place; i.e. funding models, accountability, modes of delivery—online vs. face to face, etc.) or *classroom-/subject-/content-based*.

Our group focussed predominantly on classroom-based innovations in math education which we classified as follows (note that these classifications are not necessarily mutually exclusive). Innovations could be

- content based (to improve the teaching of a particular mathematical topic, skill or concept),
- curricular (e.g. addition or removal of topics—like calculations—to/from the curriculum),
- pedagogical (focussed on classroom methods—the how—e.g. technologies like clickers, active learning, etc.),
- conceptual (a reconceptualization of various aspects of mathematics education has an effect on content, curriculum and pedagogy).

Innovations can be motivated by

- content – teachers recognize that a particular approach is not working,
- institutions – through imposition or encouragement,
- from faculty initiative or driven by external circumstances.

Innovations that were discussed ranged from small 'i' innovations (small steps) or capital 'I' Innovations (possibly revolutionary). There was also a clear recognition that what could be an innovation for one person—e.g. the student—may not be one for another—e.g. the professor. There was also a strong sense that innovations often were disruptive or uncomfortable/risky, and with that respect they must have been (concretely) implemented or have to be implemented before being considered.

DAY 1

The focus was on content innovations with some attempt at gaining an understanding of what makes an activity innovative.

CONTENT DRIVEN INNOVATION: AN ALGORITHM FOR ADDING AND SUBTRACTING INFINITE REPEATING DECIMAL EXPANSIONS

Denis Tanguay introduced this classroom innovation. The general idea is that infinite decimal expansions, even when periodic, are not conceptualized as numbers, and are not (well) *objectified* by students. They have a fuzzy status in student's minds, something closer to a process than to an object, this being linked to the obstacle of passing from *potential infinity* to *actual infinity*. Indeed, in most of the curricula, for instance in the French curriculum as Eric Roditi described it in this year's CMESG opening plenary talk, decimal numbers/expansions and fractions are dealt with as if in two independent worlds. Decimal numbers appear essentially when *measurements* are involved. They are then necessarily finite, and if not they will be made so by considering approximations.

To give infinite repeating decimal expansions a true 'number' (or object) status, and to allow students to have a true 'number sense' about them, teachers together with students have first to constitute them into a true *semiotic system*, i.e. a system of representations that allows inner processing and computations, without resorting to transformation into representations from another register. This hypothesis is from Laurent Vivier (2012). It, of course, would have repercussions not only on the conceptualization of rational numbers but also of real numbers, repeating decimal infinite expansions being an unavoidable step before considering general infinite expansions.

GOING BEYOND THE CURRICULUM – THE MATHEMATICAL SOPHISTICATION INSTRUMENT

Taras Gula introduced the *Mathematical Sophistication Instrument* (MSI) (Szydlik, Kuennen, & Seaman, 2009). The MSI was introduced with very little background except that it was developed as a tool to help future elementary (pre-service?) math teachers develop mathematical sophistication and improve their teaching of mathematics.

> [*The MSI is a 25 question*] *paper-and-pencil instrument to measure the mathematical sophistication of prospective elementary teachers. We call an individual mathematically sophisticated if her mathematical values and ways of knowing are aligned with those of the mathematical community based on nine interwoven traits involving patterns, structures, conjectures, definitions, examples and models, relationships, arguments, language, and notation.* (Szydlik et al., 2009, p. 1)

The following is a sample question from the MSI (Szydlik et al., 2010):

3 + 7 = ____ + 8 = ____. What numbers go in the blanks?

a) 10 and 18 respectively.
b) 2 and 10 respectively.
c) Both of the above options work.
d) None of the above options works.

Elicited discussion: Is it an innovation? If so, how and if not why not? The types of questions in the MSI were not considered innovative by many of the participants. Though there was recognition that the MSI did get at mathematical thinking, the questions themselves were not sufficiently new to qualify as innovative. The tone of the discussion changed once the research and thinking underlying the MSI was introduced—specifically the nine interwoven traits that comprised mathematical sophistication mentioned in the quotation above. It was generally agreed that the conceptualization of Mathematical Sophistication was innovative and helped make the MSI itself much more likely to be considered innovative.

The evaluation of the MSI as innovative helped focus attention on what helped make an activity, or an idea, innovative. Below are some of the comments that arose from that discussion.

- Innovation must not be mindless and must be of interest—must not have inherent weaknesses.
- Innovations must be evaluated within the context within which they are created and may be implemented.
- It is critical to consider necessary and sufficient conditions for innovation.
- Innovations must stimulate curiosity and questioning.
- Motivation—what is the motivation of the innovator/teacher—and to what extent does his/her innovation align itself with the motivations of the student/learner?
- Constraints on innovations were discussed—the biggest being time and its management, with the possible impact on, and hindrance to the 'math content' (its tools, procedures, organization, etc.)

DAY 2

THE BEAUTY OF HISTORY

Paul Deguire started the second day by showing us how a historical perspective can make mathematics of seemingly magical algorithms and procedures beautiful in their simplicity. His report on the presentation is presented in the Appendix to this report.

FLASHCARDS AS AN EFFICIENT FORM OF COMMUNICATION

Patrick Reynolds' presentation was practical: how to replace clickers with flashcards—a low tech innovation. It was agreed that the key to this innovation was that it was fostered by faculty, and informally evaluated, on an ongoing basis.

> **Patrick:** *As part of my attempt to adopt a 'student-centred' or 'active-learning' approach, I have found brief multiple choice questions to be generally quite effective and engaging. These are sometimes called* ConcepTests*, or 'clicker questions' (where clickers are used).*
>
> *After experimenting with three 'technologies' for administering such questions, I have found flashcards (elaborated upon below) to be optimal in terms of cost and ease-of-use. I initially used 'clickers' and later, services that allow students to 'text' responses using their phones, thus decreasing the cost considerably. With each of these two technologies, however, there is a cost to students, and without assigning participation marks, it is challenging to achieve high participation rates. Clickers are easy to use but expensive; phones are cheaper but the technological problems were annoying. Furthermore, designing questions is dependent on the particular software, which can be quite difficult to use.* LaTeX *and graphs may require extra 'fiddling', for instance.*
>
> *Inspired by a colleague in physics, I moved to using flashcards: each student receives a set of 5 different-colored cards and raises a card (or cards!) according to their answer. The benefits to this system include that:*
>
> - *I can easily see who is participating, and can prompt those who are not to join in.*
> - *I can bring extra cards in case anyone 'forgot' theirs.*
> - *Students can register multiple responses (holding up two or more cards).*
> - *I've found it easier to get higher participation rates than any other system (considerably so!).*
> - *No technological difficulties.*
> - *Easier to create questions (see below).*

I personally find it easier to create questions, even 'on the fly' during class, as there is no dependence on any particular software. I've found I'm able to be more creative with the types of questions I can ask as well—for instance, I might have two options, "concave up, concave down", and then draw different graphs, which allows a rapid sequence of questions that I would find cumbersome to implement with the other technologies. The main drawback is the lack of anonymity. While this does not affect participation rates in my experience, it may prompt some dishonesty in responses, as everyone can see the other answers. I am hoping to find cards that are white on one side to mitigate this effect somewhat in large lecture halls.

SOLVING PROBLEMS THROUGH SHIFTING REPRESENTATIONS

Richard Hoshino presented a classroom experience where students develop the key problem-solving strategy of "finding isomorphisms", i.e. recognizing when a challenging problem in one context can be reformulated as a simpler equivalent problem in another context. He rooted his ideas in the work of Richard Lesh (Lesh, Post, & Behr, 1987). The key to this 'simple' innovation is to use contextually-rich applied problems in courses that motivate a key topic or concept, and highlight its application to a real-world scenario.

Richard: *To illustrate this innovation with a specific example, here is a problem used as part of a Mathematical Problem-Solving course, to start the second day of class:*

Problem #1

Several students on campus have formed various clubs, based on academic subjects that most interest them. The clubs consist of the following students:

Astronomy Club:	Graeme, Linda Joe	Biology Club:	Linda, Di
Calculus Club:	Joe, Caitlin	Dance Club:	Joe, Fiona, Andrew
Economics Club:	Caitlin, Graeme, Di	Food Studies Club:	Bronwyn, Fiona, Graeme
Geology Club:	Bronwyn, Andrew		

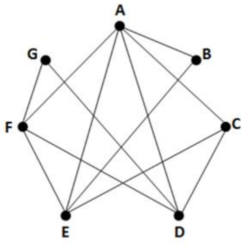
Each of the seven clubs wants to have an hour-long meeting on Friday afternoon; each person in the club must be present for the meeting. Class ends at noon, and the eight students want to get their club meetings over with as soon as possible. What is the earliest possible time at which all eight students can complete each of their hour-long meetings?

Many students find the correct answer of 3:00 pm through trial and error. A common approach is to make an 8×7 table with students' names in the rows and clubs in the columns, to see where the possible 'conflicts' arise. But overall, the students are unsatisfied with this solution, as making a 56-element table is tedious and lengthy. They realize that all that is required is to determine which clubs have conflicts—e.g. Astronomy and Biology cannot meet at the same time because one individual belongs to both clubs: what matters is that there is an individual belonging to both clubs, not who that individual is.

Through this process, the instructor is able to motivate the key idea of shifting the representation mode to pictorial by solving scheduling problems using graph theory, *to show that the above scheduling question can be solved by creating a 'conflict graph' on seven vertices (representing the seven clubs denoted by the letters A, B, C, D, E, F, G), where two vertices are joined by an edge if and only if some individual belongs to both clubs, and would therefore have a conflict if both clubs scheduled their meetings at the same time. The students then discover that the seven-club scheduling problem is completely equivalent to the following question, which is both simpler and equivalent!*

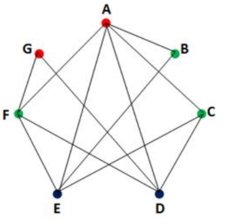
Restatement of Problem #1

We say that a graph is *k*-colourable if each of its vertices can be coloured with one of *k* colours so that there is no edge connecting two vertices of the same colour. Show that the following graph, with 7 vertices and 12 edges, is 3-colourable but not 2-colourable.



Now the problem is much easier to solve. For example, since AEF is a triangle (representing the three different clubs Graeme is in), each of these three points must be assigned different colours; thus a 2-colouring is impossible.

To generate a 3-colouring in this 'conflict graph', we let A be red, E be blue, and F be green. Then, B and C must be green (since they are both adjacent to the red vertex A and the blue vertex E), which in turn forces D to be blue and G to be red. Therefore, we have found a valid 3-colouring:



From the definition of a proper 'colouring', no edge connects two vertices of the same colour. Thus, we are guaranteed a solution to the seven-club scheduling problem by simply assigning time slots to the three colours: Red ↔ 12 pm – 1 pm, Blue ↔ 1 pm – 2 pm, and Green ↔ 2 pm – 3 pm. Indeed, we can quickly verify that this is the exact same solution as what was given earlier.

***Epilogue:** As teachers of mathematics, let us bring real-life problem-solving into the classroom, to enable and empower students to make connections between abstract mathematical theory and applied practice. The benefits of this approach are enormous.*

Recently, I was asked to consult on an employee timetabling project for the Britannia Mine Museum, a national historic site in my hometown of Squamish, British Columbia. The museum, which had previously been the location of the world's largest copper mine, is now an award-winning museum, attracting 70,000 visitors each year. Along with a 21-year-old undergraduate student, we consulted the Britannia Mine Museum on this project; the 21-year-old undergraduate student immediately *realized that the museum's scheduling conundrum could be reformulated as a graph theory problem, where the key was to take the 6×4 = 24 possible tours, represent each tour by a vertex, and draw an edge between two vertices if and only if the corresponding tours had a 'conflict', i.e. if they would be in the same location in one or more time slots.*

From there, a simple computer program determined how the conflict graph could be coloured to ensure zero conflicts, no matter which set of programs were selected by the school groups. We then provided a master Excel *sheet to the museum, for every possible scenario, which they could consult to determine which combination of tours would be optimal for that particular day. Thus, a 30-minute process that an employee called the bane of her existence reduced to a simple 15-second check on an* Excel *sheet.*

Needless to say, this 21-year-old undergraduate felt so empowered to be able to make this contribution to a local non-profit organization in our community, and have her idea implemented in such a practical way. This is just one of many benefits of teaching applied problem-solving in our classroom, which then enables students to go out, solve real-life problems, and be the change they wish to see in this world.

TEACHING MATHEMATICS WITHOUT A TEXT

Deborah Hughes-Hallett presented the idea of a math course without textbooks—this is being piloted at *Phillips Exeter Academy* (a private secondary school in the US) which teaches its math courses without textbooks, instead using sets of problems. Students attempt the problems before class; during class solutions are put on the whiteboard, and general results extracted. The *Phillips Exeter Teaching Materials* and their Math main page can be found at

http://www.exeter.edu/academics/72_6539.aspx and
http://www.exeter.edu/academics/72_6532.aspx

WHY ARE WE STILL TEACHING CALCULATION BY HAND AND OTHER 'SACRED COWS'

A vibrant discussion ensued after watching Conrad Wolfram's Ted-talk: http://www.ted.com/talks/conrad_wolfram_teaching_kids_real_math_with_computers?language=en

His thesis in a nutshell: *part of maths we teach—calculation by hand—isn't just tedious, it's mostly irrelevant to real mathematics and the real world*. He presents his radical idea: teaching mathematics to kids through computer programming. The video and thesis triggered discussion about many other 'sacred cows', including group work (teamwork in the real world rarely looks like the group work we expect students to engage in inside the classroom) and flipped classrooms (e.g. Lage, Platt, & Tregalia, 2000), were judged by some group members not to be innovative at all, as it is not unusual for post-secondary teachers to expect students to come to class prepared by having done previously assigned readings. The pressure on faculty (institutional pressure) to innovate came up, and some of the participants came up with a way for faculty to easily raise their innovation score—the back-flipped (or perhaps post-flipped?) classroom. Yes, you guessed it—the lecture. ☺

Is it all just hype? At the end of the day we examined the HYPE cycle, which maps out a model for how innovations progress through time—due to space constraints we will ask you to *Google* it: you won't regret it.

DAY 3

USING TECHNOLOGY IN AN INNOVATIVE WAY

Denis Tanguay presented a classroom activity he and his research group have experimented with in classrooms with children between 11- and 13-years old (Tanguay, Venant, Saboya, & Geeraerts, 2013). He suggests that using technology in the classroom, e. g. dynamic geometry software packages such as *GeoGebra*, calls for pedagogical and epistemological reflections and adaptations, of both the contents and the way they are taught. But the tools and devices made available by the software must also frequently require some adaptation, and it may go as far as doing some previous programming, as is the case with the following activity.

It consists in asking students, who are paired at the computers, to construct as many regular polygons as possible, starting from a *GeoGebra* file 'instrumentalized' as follows. On the screen appear an isosceles triangle ΔACB ($AC = CB$) and two sliders α and n. The slider α varies between 0 and 180 and allows changing the degree measurement of $\angle ACB$. The slider n varies between 1 and 360. It allows spreading out n triangles arranged in a fan, the first one being ΔACB, the other ones being its images after rotations about center C through angles measuring respectively α, 2α, 3α, … $(n - 1)\alpha$. The idea is of course that ΔACB and the 'last' triangle of the fan make a 'well-closed' polygon, without gap or overlap. (See Figures 1 and 2).

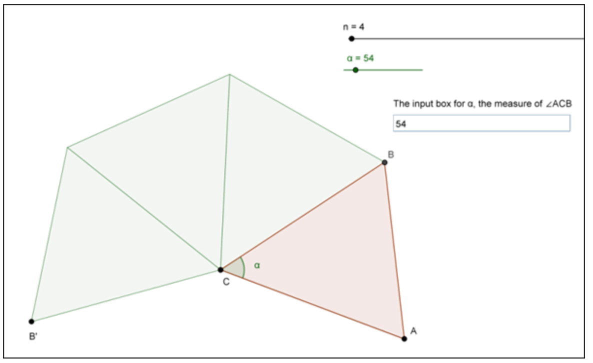


Figure 1. *The GeoGebra* screen given at the start.

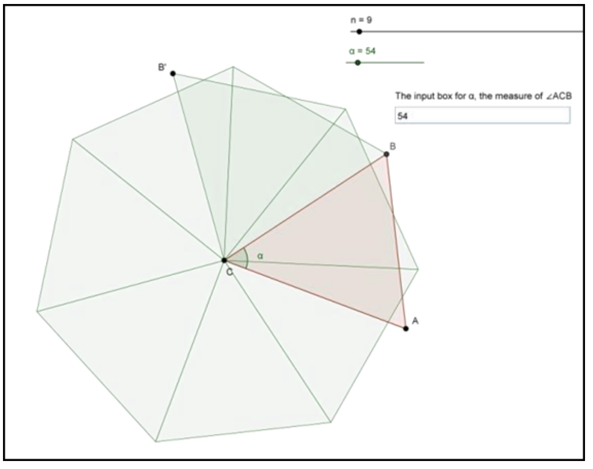


Figure 2. Increasing the number of triangles with the slider *n*.

Contrary to what one may think, it is not mainly a geometrical activity, but rather an arithmetical activity! Indeed the first session consists in producing the list of all the divisors of 360, each divisor being a possible value for α (or for *n*, no matter!). So this is the goal of the first classroom discussion and 'wrap up', together with the issues: "*Why do we have this symmetry in the list? Would it be the case for the list of divisors of any other integer? Is it true that any integer has always an even number of divisors?*"

Then the second session deals with the construction of the heptagon, the regular polygon having 7 sides. The interesting feature is that once the students have entered an approximation of 360/7 into the 'α box', you then use the zoom device of *GeoGebra* to show that the polygon they just obtained is NOT well closed (you may have to 'zoom in' quite a lot!). So then, what would be the α value that would make the heptagon tightly closed? The discussion then elicited between the students (and the teacher) are fairly interesting and rich, for sure…

TWO STAGE ASSESSMENTS

Kseniya Garaschuk introduced all of us to 2-stage assessments. These are common practice at UBC, but new to the other participants in the group.

> Stage 1: A standard formal assessment that students complete working alone.
>
> Stage 2: After students turn in their individual 'test', small groups solve identical problems during the remainder of the examination time. Group questions are

identical to the individual (+/– one or two "difficult" questions). Students work on group problems collaboratively and have to reach consensus, each group having only one answer sheet at its disposal!

Example: http://www.cwsei.ubc.ca/resources/SEI_video.html

All options of assessment types, question types, and *group* versus *individual* exam similarity have been used.

Results:

Math 200 Multivariate Calculus: Pre-final exam two-stage review – 18 multiple choice questions, 67 students, 25 groups

24% average improvement between individual and group effort (40% → 64%); 19 groups did better than individual top score; 4 groups did the same as individual top score, while 2 groups did worse.

Math 256 Differential equations: Last day of class, final exam review – 12 multiple choice questions, 30 minute individual work, 20 minute group work; 92 students, groups of ~3

14% average improvement; 62 students improved; 21 did the same, 11 did worse.

FOSTERING A SENSE OF COMMUNITY OUTSIDE OF THE CLASSROOM

Carly Rozins: *If you have ever googled a math problem, you have probably, perhaps unknowingly, stumbled upon the stack exchange* (www.stackexchange.com). Stack Exchange *is an online community of question askers and question answerers. On the* Stack Exchange*, you can find answers to an unimaginable number of math (and other) questions. For years, the* Stack Exchange *has been providing me with solutions to problems, but not until recently have I become a member of the community and really started learning from it.*

Once you create an account on the website, you can start posting and answering questions. At first your privileges are limited. For example, you cannot upload images or comment on people's questions. The website awards 'points' for community approved solutions to questions. These points, the satisfaction in being able to provide an answer to a question, and the learning thus elicited, is what motivates me to participate in this community. Each time you successfully answer a question, edit a question, ask a good question..., you are awarded points or badges. I have found myself spending hours of my free time working out solutions to other people's questions.

This type of system is already being incorporated into the classroom. Many universities have online forums where students can ask and answer questions. Classroom forums lack some very important features that are found on the Stack Exchange*. Classroom forums lack*

- *the massive volume of experts found on the stack exchange;*
- *the points (badges and other honors that are fun to collect);*
- *the interaction (on the* Stack Exchange *you can comment on people's questions and even edit them);*
- *profile pages, which keep track of all of your progress (questions asked, answered, points, profile picture and specialties...).*

In my opinion, providing a solution to a posed problem is one of the best ways to become an expert at the subject, and the Stack Exchange *can foster that community outside the classroom.*

THE PRODUCT OF OUR WORKING GROUP: CRITERIA TO ADOPT AN INNOVATION

Though we did not develop a conceptualization of innovation, nor examine the phenomenon as thoroughly as some may have liked, we did spend the last part of Day 3 creating a set of criteria that get at this goal of ours, indirectly. Throughout the three days, one of the tackled themes was the interplay between applied ('real' world) mathematics and mathematical theory. This theme may have elicited the following reflection: as the group discussed and thought about innovations in tertiary math education, their relevance and mathematical 'heft' appear to us to be related to their potential *in the classroom*, without a doubt, but also *in the world beyond the classroom*; 'beyond' with respect to space and contexts, but also with respect to time, on a long term basis. The following are a set of criteria which were developed by the group:

- not too far from professor's comfort zone, but…with an element of discomfort or disruption—if not, it's no more an innovation!
- simple things that we can do to be more efficient or effective as teachers—many in the group were not interested in grand innovations (revolutions)
- expectation of improvement (and ability to identify it); innovations must serve the classroom
- experience and confidence of the teacher implementing the innovation—some teachers are bolder than others and more willing to take risks
- is it safe, doable, trustworthy, for students, for administration, for ourselves teachers? (for instance will the assessment duties 'explode' with that innovation?)
- what are the stakes for the student? On what aspects has it (mainly) its effects?
- adaptability—innovations designed for one setting may not transfer well to others—e.g. student expectations at community colleges are different than in universities
- institutional constraints—related to safety. How supportive is the institution of the project/professor? What constraints are imposed on the professor?
- What is the interest of the institution, colleagues, students? How do they evaluate the innovation?

[Note: The Appendix and References follow the French version.]

♦♦♦♦♦♦♦♦♦♦

INTRODUCTION

> *L'innovation est partout. Certainement dans le monde de la production (technologique), mais aussi dans les domaines où le langage est central : on discute d'innovation dans la littérature scientifique et technique, dans les sciences sociales comme en histoire, en sociologie, en management et en économie, en arts et lettres. L'innovation est aussi une idée centrale dans l'imagination populaire, les médias, les politiques publiques, aussi fait-elle partie du vocabulaire de tout un chacun. En bref, l'innovation est maintenant emblématique de nos sociétés modernes où elle est une panacée pour résoudre de nombreux problèmes, et donc en soi un phénomène à étudier.* (Godin, 2008, p. 5, notre traduction)

Nous avions pour but d'examiner des innovations spécifiques en enseignement des mathématiques postsecondaires et dans la recherche liée à ce domaine, et de réfléchir en parallèle à l'innovation en tant que phénomène tel que le décrit Godin (2008, 2014), en nous focalisant sur l'enseignement postsecondaire. Malgré son omniprésence, et peut-être à cause du flou qui entoure ce concept, les participants du groupe se sont vite montrés plus intéressés à examiner les innovations en elles-mêmes et à « jouer » avec leurs aspects mathématiques et pédagogiques, de telle sorte que ce sont les discussions là-dessus qui seront au cœur du rapport.

Quelques thèmes que l'on peut glaner dans ces trois journées de travail :

1. Une perspective historique peut aider à expliquer les mathématiques qu'il y a dans des algorithmes et procédures apparaissant de prime abord comme « magiques ». L'histoire permet de proposer des avenues innovantes pour traiter en classe d'un sujet ou d'un contenu mathématique donné, en rappelant les problèmes qui avaient suscité le développement de ces mathématiques, et en proposant des méthodes qui ne sont pas nécessairement les plus efficaces mathématiquement mais qui sont éclairantes pédagogiquement.
2. Plusieurs innovations qui ont particulièrement intéressé le groupe sont celles qui aident les étudiants à naviguer entre les différentes façons de se représenter les concepts mathématiques : du symbolique, du contextuel, du langagier, du concret, du schématique ou de la visualisation à n'importe quel autre de ces modes d'appréhension.
3. L'évaluation, aussi bien formative que sommative, délimite des zones de recherche en enseignement des mathématiques qui sont riches en possibilité pour de l'innovation, autant du point de vue des contenus que de la forme; par exemple trouver de « bons » problèmes.

Nous sommes globalement tombés d'accord qu'au niveau postsecondaire, les innovations pouvaient être catégorisées comme *systémiques* (c'est-à-dire relatives aux institutions qui encadrent l'enseignement, mettant en cause par exemple les dispositifs de financement ou d'aménagement, l'évaluation et l'imputabilité, les modes de transmission—à distance, en présence ou autrement, etc.) ou comme *orientées vers la classe, les sujets, les contenus*.

C'est plutôt à cette seconde catégorie que notre groupe a principalement consacré ses discussions, et cette catégorie a elle-même donné lieu à une sous-catégorisation. Les innovations destinées à la classe de mathématiques peuvent en effet être rangées dans les catégories, non mutuellement exclusives, suivantes : innovations orientées vers…

- un contenu (améliorer l'enseignement d'un concept, d'un sujet, d'une habileté mathématique spécifique),
- le curriculum (par exemple, l'adjonction, le retrait ou la réorganisation de sujets, de procédures—par exemple tel ou tel type de calculs—du curriculum),

- la pédagogie (avec focalisation sur les méthodes d'enseignement—le comment—par exemple l'utilisation des technologies comme les *clickers* (tests ou questions gérés en temps réel), l'apprentissage actif, etc.,
- la conceptualisation (une « reconceptualisation » d'aspects variés de l'enseignement mathématique peut avoir des effets sur les contenus, le curriculum, la pédagogie).

Les innovations peuvent par ailleurs être motivées par

- les contenus—quand par exemple les enseignants prennent conscience qu'une approche spécifique ne fonctionne pas,
- les institutions—celles-ci peuvent alors les imposer ou simplement les encourager,
- des initiatives prises librement ou provoquées par des circonstances extérieures.

Les innovations dont nous avons discuté allaient des innovations avec un 'i' minuscule (de petits pas) aux Innovations avec un 'I' majuscule (les grands bouleversements). Nous avons aussi clairement reconnu que ce qui est une innovation pour un individu—par exemple un étudiant—peut ne pas l'être pour un autre—par exemple son professeur. A également été mise en avant notre préoccupation à l'effet que les innovations sont souvent déstabilisantes, parfois risquées, et peuvent engendrer des malaises; à ce titre, elles doivent avoir été concrètement expérimentées ou devront l'être avant que leur implémentation systématique soit sérieusement considérée.

JOUR 1

Nous avons porté notre attention sur des innovations relatives à des contenus, tout en cherchant à mieux comprendre ce qui fait qu'une activité donnée est innovante.

INNOVATION À PROPOS D'UN CONTENU : UN ALGORITHME POUR ADDITIONNER OU SOUSTRAIRE DES EXPANSIONS DÉCIMALES PÉRIODIQUES

Denis Tanguay a d'abord fait vivre l'innovation sous forme « d'atelier ». L'idée générale est que les expansions décimales infinies, même quand elles sont périodiques, ne sont pas conceptualisées comme des nombres, ne sont pas (bien) *réifiées* par les élèves. Elles ont un statut flou dans l'entendement de l'élève, plus proche de celui de processus que de celui d'objet, ceci étant lié à l'obstacle du passage de *l'infini potentiel* à *l'infini actuel*. De fait, dans la plupart des curriculums, comme par exemple dans le curriculum français tel qu'il a été décrit par Éric Roditi dans la conférence plénière de la rencontre 2015 du GCEDM, les expansions/nombres décimaux et les fractions sont travaillés comme s'ils vivaient dans deux mondes séparés. Les expansions décimales apparaissent essentiellement quand des *mesures* sont impliquées. Elles sont alors nécessairement finies, et quand ce n'est pas le cas on s'y ramène en considérant leurs approximations.

Pour donner aux expansions décimales périodiques un vrai statut de 'nombre' (ou d'objet) et pour permettre aux étudiants de développer un vrai 'sens du nombre' autour de ces objets, les enseignants doivent d'abord les constituer en véritable *système sémiotique*, c'est-à-dire en registre de représentations qui permet des calculs et traitements sans avoir à recourir à des transformations dans des représentations d'un autre registre. Cette hypothèse est celle de Laurent Vivier (2012). Le travail en classe des modes et règles de calcul discutés dans le groupe aurait des répercussions sur la conceptualisation non seulement des rationnels, mais aussi des nombres réels, les expansions infinies périodiques étant un passage obligé avant de considérer les expansions infinies dans toute leur généralité.

ALLER AU-DELÀ DU CURRICULUM – UN INSTRUMENT DE MESURE DE LA SOPHISTICATION MATHÉMATIQUE

Taras Gula nous a présenté le « Mathematical Sophistication Instrument » (MSI) (Szydlik, Kuennen, & Seaman, 2009), un outil pour mesurer la sophistication mathématique. Il a peu parlé de l'arrière-plan contextuel qui a prévalu à la conception du MSI, excepté pour dire qu'il s'agit d'un outil qui a été élaboré pour aider les futurs enseignants de l'élémentaire à évaluer et développer leur propre sophistication mathématique, et à ainsi améliorer leur enseignement.

> *Le MSI est un « instrument papier-crayon » composé de 25 questions qui permettent de mesurer la sophistication mathématique de futurs enseignants du primaire. Nous disons qu'un individu est « mathématiquement sophistiqué » si ses valeurs mathématiques et ses façons d'apprendre sont en phases avec celles de la communauté mathématique, ceci sur la base de neuf éléments caractéristiques inter-reliés : patterns, structures, conjectures, définitions, exemples et modèles, relations, arguments, langage et notations.* (Szydlik et al., 2009, p. 1, notre traduction)

Un exemple de questions extrait du MSI (Szydlik et al., 2010) :

3 + 7 = ____ + 8 = ____. Quels nombres vont dans les blancs ?

a) 10 et 18 respectivement.
b) 2 et 10 respectivement.
c) Les deux options ci-dessus sont valides.
d) Aucune des deux options ci-dessus n'est valide.

La discussion provoquée : s'agit-il d'une innovation ? Si oui, en quoi et si non, pourquoi pas ? Les questions dans le MSI n'ont pas été considérées comme innovantes par plusieurs des participants. Malgré que nous ayons reconnu l'adéquation du MSI pour pointer certains aspects de la pensée mathématique, les questions en elles-mêmes n'ont pas été évaluées comme suffisamment neuves pour être qualifiées d'innovantes. La tournure de la discussion a changé au moment où la recherche et les réflexions qui sous-tendent le MSI ont été introduites—en particulier les neuf éléments caractéristiques inter-reliés qui circonscrivent la sophistication mathématique et qui sont cités ci-dessus. Nous avons généralement convenu que l'idée de conceptualiser la sophistication mathématique est innovante, et nous ont fait pencher vers une évaluation du MSI, pour ce qu'il aspire à être, comme une innovation.

Cette appréciation du MSI nous a aidés à focaliser notre attention sur ce qui contribue à faire d'une activité ou d'une idée une innovation. Ci-dessous, des commentaires qui ont émergé de cette discussion.

- L'innovation doit être réfléchie et intéressante—elle ne doit pas comporter de faiblesse intrinsèque.
- Les innovations doivent être évaluées dans le contexte qui les a fait naître, et implémentées en tenant compte de ces mêmes contextes.
- Il est primordial de prendre en considération les conditions nécessaires et suffisantes qui prévalent.
- Les innovations doivent stimuler la curiosité et le questionnement.
- La motivation : quelle est la motivation de l'innovateur/enseignant ; et dans quelle mesure son innovation s'aligne-t-elle avec les motivations de l'étudiant/apprenant ?
- Les contraintes à l'innovation ont fait l'objet de discussions—la plus grande contrainte étant celle du temps ; celle-ci étant par ailleurs liée à la question d'une possible entrave vis à vis les contenus mathématiques (leurs outils, leurs procédures, etc.)

JOUR 2

LA BEAUTÉ DE L'HISTOIRE

Paul Deguire a ouvert la seconde journée en nous montrant comment une perspective historique peut faire ressortir les beautés et simplicité des mathématiques dans des algorithmes et processus apparemment « magiques ». On trouvera le compte rendu de sa présentation à l'appendice au présent rapport.

LES « FLASHCARDS » COMME FORME EFFICACE DE COMMUNICATION

La présentation de Patrick Reynolds se voulait résolument pratique : comment remplacer les *clickers* par des *flashcards*—une innovation peu exigeante technologiquement. Les *clickers* sont des télécommandes actionnées par les étudiants en classe, qui permettent de gérer des tests ou des questions-sondages en temps réel. Il s'agirait de les remplacer par des cartes de support visuel (en carton, de couleur ou non !). Nous étions d'accord que l'atout de cette innovation était d'emporter l'assentiment des directions facultaires et de permettre ajustements et évaluation informelle au jour le jour.

> **Patrick :** *Avec le souci d'adopter des approches d'apprentissage actif, centrées sur l'étudiant, j'estime que les questions brèves à choix multiples sont de fait, en général, plutôt efficaces et stimulantes. On les appelle parfois des* ConcepTests *(tests conceptuels), ou des « questions à clickers », là où les clickers sont utilisés.*
>
> *Après avoir expérimenté avec trois « technologies » pour gérer de telles questions, j'ai trouvé que les* flashcards *(décrites ci-dessous) sont optimum du point de vue coût et facilité d'utilisation. J'utilisais auparavant les* clickers *et plus tard des ressources qui permettent aux étudiants de « texter » leurs réponses sur leur téléphone portable, ce qui réduit considérablement les coûts. Toutefois avec chacune de ces deux technologies, il y a un coût pour les étudiants, et sans l'attribution de points de participation, il est difficile d'atteindre des taux de participation élevés. Les clickers sont faciles d'utilisation mais ils coûtent cher, les téléphones sont moins coûteux mais ils causent des problèmes technologiques qui dérangent. De plus, les questions à élaborer vont dépendre du logiciel particulier à utiliser, ce qui limite et complique les possibilités.* LaTeX*, graphes et graphiques peuvent par exemple nécessiter de la « cuisine technique » additionnelle.*
>
> *Inspiré par un collègue de physique, je suis passé à l'utilisation de* flashcards *: chaque étudiant reçoit un ensemble de 5 cartes et lève une carte (ou* des *cartes !) selon la réponse qu'il veut donner. Les bénéfices de ce système sont les suivants :*
>
> - *Je vois d'un coup d'œil qui participe, et je peux encourager ceux qui ne s'y mettent pas.*
> - *J'apporte des cartes supplémentaires pour ceux qui auraient « oublié » les leurs.*
> - *Les étudiants peuvent donner plus d'une réponse (en levant plus d'une carte).*
> - *J'ai obtenu plus facilement (considérablement plus) qu'avec n'importe quel autre système de hauts taux de participation.*
> - *Pas de difficulté technologique.*
> - *Il est plus facile de créer des questions (voir ci-dessous).*
>
> *Je trouve pour ma part plus facile de créer des questions, et même de les improviser en cours, parce qu'il n'y a pas de dépendance à quelque logiciel spécifique. Je me suis également trouvé plus créatif dans les types de questions demandées, par exemple je peux donner deux options, « incurvé vers le haut ou vers le bas ? », et là je trace deux graphiques différents, ce qui permet des séquences rapides de questions qui auraient été laborieuses à implémenter avec d'autres technologies. Le*

principal inconvénient est le manque d'anonymat. Bien que cela n'affecte pas les taux de participation selon mon expérience, cela peut provoquer des réponses de mauvaise foi, puisque chacun voit les réponses des autres. J'espère trouver des cartes qui sont blanches d'un côté pour contrer en partie cet effet dans les grandes salles de cours.

RÉSOUDRE DES PROBLÈMES EN CHANGEANT LES REPRÉSENTATIONS

Richard Hoshino a présenté une expérience de classe où les étudiants développaient une stratégie clé de résolution de problèmes en « trouvant des isomorphismes », c'est-à-dire en repérant quand un problème représentant un défi dans un contexte peut être reformulé dans un autre contexte en un problème plus simple équivalent. Ses idées sont enracinées dans les travaux de Richard Lesh (Lesh, Post, & Behr, 1987). La clé de cette innovation « simple » est d'utiliser des problèmes d'application contextuellement riches, dans des cours autour d'un sujet ou concept clé, et qui mettent l'accent sur ses applications à des scénarios du monde réel.

Richard : *Pour illustrer cette innovation avec un exemple spécifique, voici un problème utilisé dans un cours de résolution de problèmes mathématiques, et qui amorce le 2e jour de classe :*

Problem #1

Des étudiants sur le campus ont formé des clubs variés, sur la base des sujets scolaires qui les intéressent le plus. Ces clubs ont pour membres les étudiants suivants :

Club d'astronomie:	Graeme, Linda Joe	Club de biologie:	Linda, Di
Club de calcul diff. et intégral:	Joe, Caitlin	Club de sciences économiques:	Caitlin, Graeme, Di
Club de dance:	Joe, Fiona, Andrew	Club de nutrition:	Bronwyn, Fiona, Graeme
Club de géologie:	Bronwyn, Andrew		

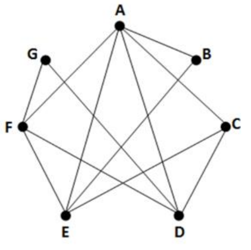
Chacun des sept clubs veut avoir une rencontre d'une heure le vendredi après-midi ; tous les membres du club doivent y être présents. Les classes se terminent à midi, et les huit étudiants veulent avoir terminé leur rencontre au club le plus tôt possible. Quelle est l'heure la plus tôt à laquelle les huit étudiants peuvent avoir complété toutes leurs réunions ?

Plusieurs étudiants ont trouvé la réponse correcte de 15:00 par essais-erreurs. Une approche usitée est de faire un tableau 8×7 avec les noms des étudiants sur les lignes et les clubs sur les colonnes, pour voir où les éventuels « conflits » se présentent. Mais globalement, les étudiants ne sont pas satisfaits de cette solution, vu que faire un tableau de 56 éléments est long et fastidieux. Ils ont bien vu qu'il suffit de déterminer quels clubs sont en conflit—par exemple Astronomie et Biologie ne peuvent avoir leur réunion en même temps parce qu'un étudiant doit être présent aux deux : ce qui importe, c'est qu'un individu soit membre des deux clubs, et qui il est importe peu.

À travers ce processus, l'enseignant est capable de motiver l'idée clé du changement de mode de représentation pour le diagramme en résolvant des problèmes d'horaire à l'aide de la Théorie des graphes. *Il montre ainsi que le problème d'horaire ci-dessus peut être résolu en créant un « graphe des conflits » à sept sommets (représentant les sept clubs par les lettres A, B, C, D, E, F, G), où deux sommets sont joints par une arête si et seulement si ils ont un membre en commun, et seront donc en conflit s'ils planifient une réunion au même moment. Les étudiants découvrent alors que le problème de l'horaire des sept clubs est complètement équivalent à la question plus simple suivante !*

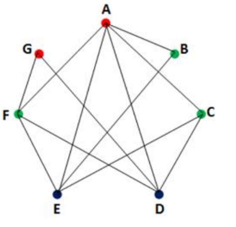
Reformulation du problème #1

Nous disons qu'un graphe est *k*-colorable si chaque sommet peut être coloré avec une parmi *k* couleurs, de sorte qu'il n'y ait pas d'arête qui lie 2 sommets de même couleur. Montrez que le graphe suivant, à 7 sommets et 12 arêtes, est 3-colorable mais n'est pas 2-colorable.



Maintenant, le problème est beaucoup plus facile à résoudre. Puisque AEF est un triangle (représentant les 3 clubs dont Graeme est membre), on doit assigner à ces 3 points 3 couleurs différentes : une 2-coloration est donc impossible..

Pour générer une 3-coloration dans ce « graphe des conflits », on colore A en rouge, E en bleu et F en vert. Alors B et C doivent être en vert (puisque les deux sont adjacents au sommet rouge A et au sommet bleu E), ce qui en retour force D à être bleu et G à être rouge. Donc, nous avons trouvé une 3-coloration valide :



Une solution au problème de l'horaire des sept clubs est donc garantie simplement par l'assignation d'une plage horaire distincte à chacune des trois couleurs : Rouge ↔ 12:00-13:00, Bleu ↔ 13:00-14:00 et Vert ↔ 14:00-15:00. On vérifie aisément que cette solution fonctionne.

Épilogue : *Comme professeurs de mathématiques, cherchons à proposer en classe de la résolution de problèmes de la vie réelle, permettant ainsi aux étudiants de faire des liens entre les théories mathématiques abstraites et leurs applications pratiques. Les bénéfices d'une telle approche sont énormes.*

Récemment, moi et une étudiante de 21 ans encore au 1er cycle avons été consultés à propos d'une grille horaire des employés du Britannia Mine Museum*, un site national historique à Squamish, en Colombie Britannique. Le musée, où était dans le passé située la plus grosse mine de cuivre au monde, a gagné des prix et attire jusqu'à 70 000 visiteurs chaque année. L'étudiante s'est immédiatement rendue compte que le casse-tête des horaires du musée pouvait être reformulé en un problème de théorie des graphes, où la clé consistait à faire des 6×4 = 24 visites possibles les sommets d'un graphe, et à tracer une arête entre deux sommets du graphe si et seulement si les visites étaient en conflit, c'est-à-dire si elles allaient se dérouler au même endroit durant au moins une des plages horaires.*

À partir de cela, un simple programme informatique a déterminé comment le graphe des conflits pouvait être coloré pour s'assurer qu'il n'y ait pas de conflit, peu importe quel ensemble de programmes devaient être choisi par les groupes scolaires. Nous avons donc fourni au musée une feuille Excel *maître, que les responsables peuvent consulter pour tous les scénarios possibles afin de déterminer une combinaison de visites optimale pour chaque jour en particulier. Ainsi, un procédé de 30 minutes qui était la croix et la bannière d'un employé se trouvait-il réduit à une simple vérification de 15 secondes sur une feuille* Excel.

Inutile de dire à quel point cette étudiante de 21 ans au 1er cycle s'est sentie fière de sa contribution à une organisation sans but lucratif de notre communauté, et de ce que son idée soit mise en œuvre de si pratique façon. Ceci n'est qu'un bénéfice de l'enseignement de la résolution de problèmes d'applications, qui permet alors à nos étudiants de résoudre des problèmes de la vie courante hors des murs de la classe, et de susciter dans notre monde les changements qui leur tiennent à cœur.

ENSEIGNER LES MATHÉMATIQUES SANS TEXTE

Deborah Hughes-Hallett a présenté l'idée d'un cours de maths sans manuel ni texte—un projet piloté par la *Phillips Exeter Academy* (une école secondaire privée aux États-Unis), où les cours sont donnés sans manuel, et font plutôt appel à des ensembles de problèmes. Les élèves s'essaient sur les problèmes avant les cours ; en classe les solutions sont inscrites au tableau et les résultats généraux en sont extraits. Voir les sites ci-dessous pour plus de détails :

> http://www.exeter.edu/academics/72_6539.aspx et
> http://www.exeter.edu/academics/72_6532.aspx

POURQUOI ENSEIGNONS-NOUS ENCORE LE CALCUL À LA MAIN ET AUTRES « VACHES SACRÉES »

Une discussion animée a suivi le visionnement d'une conférence donnée par Conrad Wolfram : http://www.ted.com/talks/conrad_wolfram_teaching_kids_real_math_with_computers?language=en

Sa thèse en deux mots : *une partie des maths que nous enseignons—le calcul à la main—n'est pas seulement fastidieuse, elle n'est surtout pas pertinente pour les « vraies » mathématiques et le monde réel*. Il présente sa thèse radicale : enseigner les mathématiques aux jeunes enfants à travers la programmation informatique. La vidéo et la thèse ont déclenché une discussion à propos d'autres « vaches sacrées », incluant le travail en groupes (le travail en groupes dans la réalité ressemble rarement à ce que nous attendons qu'il soit en classe) et la classe inversée (voir par example Lage, Platt, & Tregalia, 2000). Certaines de ces propositions ont été jugées par certains dans le groupe comme n'étant pas du tout des innovations, puisqu'il n'est par exemple pas du tout inusité pour les enseignants du post-secondaire d'attendre de leurs étudiants qu'ils arrivent aux cours préparés via des lectures préalablement assignées. Les pressions institutionnelles pour innover ont alors été objet de débat, et l'on a suggéré un moyen facile de paraître plus novateur aux yeux des directions (facultaires, départementales ou autres) : *la classe doublement inversée*. Oui, vous avez bien deviné, le bon vieux cours….☺

Et au fait, si tout cela n'était simplement que du faux-semblant ? À la fin de la journée, nous nous sommes penchés sur le « HYPE cycle », qui fournit un modèle de la progression des processus d'innovation à travers le temps. À cause des contraintes d'espace, nous prions le lecteur de rechercher la ressource sur Google : il ne le regrettera pas.

JOUR 3

UTILISER LA TECHNOLOGIE DE FAÇON INNOVANTE

Denis Tanguay a présenté une activité que son groupe de recherche et lui ont expérimentée dans des classes avec des élèves de 11 à 13 ans (Tanguay, Venant, Saboya, & Geeraerts, 2013). Il suggère qu'avoir recours aux technologies en classe, par exemple à des logiciels de géométrie dynamique tels *GeoGebra*, demande des réflexions et adaptations pédagogiques et épistémologiques, tant des contenus que des façons dont ils seront enseignés. Mais les outils et les dispositifs accessibles par le logiciel demandent eux aussi fréquemment d'être adaptés, et cela peut aller jusqu'à nécessiter de la programmation préalable, comme c'est le cas pour la présente activité.

Elle consiste à demander aux élèves, en équipes de deux aux postes informatiques, de construire autant de polygones réguliers que possible à partir d'un fichier *GeoGebra* « instrumentalisé » comme suit. À l'écran apparaît un triangle isocèle ΔACB ($AC = CB$) et deux curseurs, α et *n*. Le curseur α varie entre 0 et 180 et permet de changer la mesure en degrés de Δ*ACB*. Le curseur *n* varie entre 1 et 360. Il permet de déployer *n* triangles formant éventail, le premier triangle étant Δ*ACB*, les autres étant ceux obtenus par les rotations de centre *C* et d'angles mesurant respectivement α, 2α, 3α, … (n – 1)α. L'idée est bien sûr que Δ*ACB* et le « dernier » triangle de l'éventail forment un polygone « bien fermé », sans trou ni chevauchement. (Voir Figures 1 et 2).

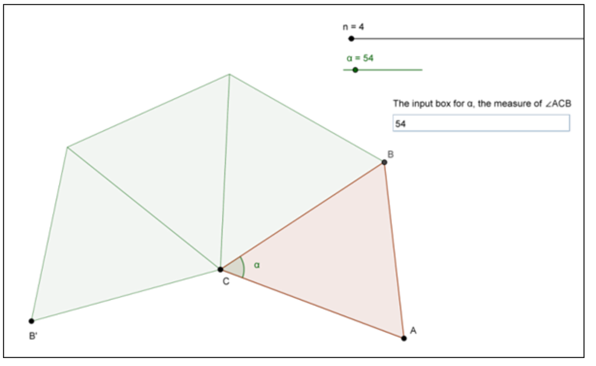


Figure 1. L'écran *GeoGebra* donné au depart.

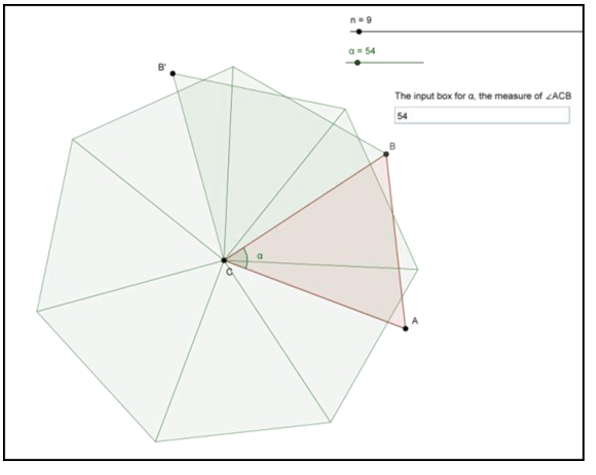


Figure 2. On augmente le nombre de triangles en déplaçant le curseur *n*.

Contrairement à ce qu'on pourrait penser de prime abord, ce n'est pas principalement une activité de géométrie, mais plutôt une activité arithmétique ! En effet la première période consiste à produire la liste de tous les diviseurs de 360, chaque diviseur étant une valeur possible pour α (ou pour *n*, peu importe !) Donc il s'agit là de la cible du premier bilan-discussion en grand groupe, avec également la question « *Pourquoi y a-t-il cette symétrie dans la liste ? Est-ce que ce serait le cas de la liste des diviseurs de n'importe quel autre entier ? Est-il vrai que tout entier a toujours un nombre pair de diviseurs ?* »

Dans la 2^{e} période, on s'attaque à la construction de l'heptagone, le polygone régulier avec sept côtés. La particularité intéressante ici, c'est qu'une fois que les étudiants ont tapé une approximation de 360/7 dans la « boîte α », l'enseignant utilise alors la fonction *zoom* de *GeoGebra* pour montrer que le polygone obtenu N'EST PAS bien fermé (il faut peut-être zoomer passablement !). Mais alors, quelle devrait être la valeur de α pour qu'on obtienne un polygone bien fermé ? La discussion provoquée entre élèves (et enseignant) est alors à coup sûr riche et intéressante…

UNE ÉVALUATION EN DEUX ÉTAPES

Kseniya Garaschuk nous a tous initiés aux *évaluations en deux étapes*. Elles sont pratiques communes à UBC, et étaient pourtant nouvelles pour les autres participants du groupe.

> Étape 1 : une évaluation formelle standard que les étudiants complètent seuls.
>
> Étape 2 : après le travail individuel, les étudiants résolvent des problèmes identiques en petits groupes pour la durée du reste de l'examen. Les questions de groupe sont identiques aux questions individuelles (plus une ou deux questions plus « difficiles »). Les étudiants font du travail collaboratif et doivent arriver à un consensus, chaque groupe n'ayant qu'une feuille de réponse à sa disposition.
>
> Exemple : http://www.cwsei.ubc.ca/resources/SEI_video.html

Toutes les options des types d'évaluation, des types de questions et des combinaisons groupe versus individuel dans des examens semblables ont été utilisées.

<u>Résultats</u>

Math 200 Multivariate Calculus : examen pré-final en deux étapes – 18 questions à choix multiples, 67 étudiants, 25 groupes

Une amélioration moyenne de 24% entre l'individuel et les efforts de groupe (40% → 64%) ; 19 groupes ont fait mieux que le score maximum individuel ; 4 groupes ont fait aussi bien, et 2 groupes ont fait pire.

Math 256 Differential equations : dernier jour de classe, examen final – 12 questions à choix multiples, 30 minutes de travail individuel, 20 minutes de travail de groupe ; 92 étudiants, groupes, la plupart de 3 étudiants.

Une amélioration moyenne de 14% ; 62 étudiants ont amélioré les performances, 21 ont fait aussi bien, 11 ont fait pire.

ENCOURAGER UN SENS COMMUNAUTAIRE EN DEHORS DE LA CLASSE

> ***Carly Rozins :*** *Si vous avez déjà cherché un problème mathématique sur* Google, *alors vous êtes probablement tombé, peut-être sans le savoir, sur le* Stack Exchange *: www.stackexchange.com. Il s'agit d'une communauté en ligne dont les membres posent des questions ou répondent à des questions. Sur le* Stack Exchange, *vous pouvez trouver les réponses à un nombre inimaginable de questions mathématiques (ou autres). Depuis des années, le* Stack Exchange *m'a fourni des solutions à des problèmes, mais ce n'est que récemment que je suis devenue membre de la communauté et que j'ai véritablement commencé à y apprendre quelque chose.*
>
> *Une fois que vous avez créé un compte sur le site, vous pouvez commencer à afficher des questions ou à répondre. Au début vos privilèges sont limités. Par exemple, vous ne pouvez télécharger des images ou commenter les questions des autres. Le site web attribue des « points » pour des solutions approuvées par la communauté. Ces points, la satisfaction d'avoir pu fournir une réponse à une question, et l'apprentissage ainsi suscité sont ce qui m'a motivée à participer à la communauté. À chaque fois que vous répondez avec succès à une question, que vous révisez une réponse, que vous posez une bonne question…, on vous octroie des points ou des badges. Je me suis surprise à passer des heures de mes temps libres à travailler sur les solutions des questions des autres.*

> *Ce type de système est déjà intégré à des classes. Plusieurs universités ont des forums en ligne où les étudiants peuvent poser des questions ou y répondre. Mais il manque aux forums de classes des caractéristiques qu'on trouve dans le* Stack Exchange*. Aux forums de classes manquent :*
>
> - *le volume important de réponses expertes qu'on trouve dans le* Stack Exchange *;*
> - *les points, badges ou autres récompenses qui sont amusants à cumuler;*
> - *l'interaction. Dans le* Stack Exchange*, on peut commenter les questions des autres et on peut même les réviser.*
> - *Des pages « profil », qui permettent de garder la trace de votre progression (les questions demandées, les réponses, les points, votre portrait, vos spécialités...).*
>
> *Selon mon opinion, fournir une solution à un problème posé est une des meilleures façons de devenir un expert dans un sujet, et le* Stack Exchange *permet de stimuler la communauté en dehors de la classe.*

LE RÉSULTAT DU TRAVAIL DE NOTRE GROUPE : DES CRITÈRES POUR ADOPTER UNE INNOVATION

Bien que nous n'ayons pas développé une conceptualisation de l'innovation, ni examiné le phénomène aussi profondément que certains l'auraient souhaité, nous avons tout de même passé la dernière partie du jour 3 à élaborer un ensemble de critères qui atteignait indirectement ces buts qui étaient les nôtres. À travers ces 3 jours, un des thèmes abordés a été celui de l'interaction entre les mathématiques appliquées (au monde « réel ») et les mathématiques plus théoriques. Peut-être ce thème a-t-il suscité la réflexion suivante : à mesure que nous discutions en groupe à propos des innovations au post-secondaire, leur pertinence et leur « poids » mathématiques nous sont apparus certes reliés à leurs potentialités *en classe*, mais aussi dans le monde *au-delà de la classe*, au-delà par rapport à la fois à l'espace et aux contextes, mais aussi par rapport au temps, à plus longue échéance. Voici donc cet ensemble de critères développés par le groupe.

- On ne doit pas être trop loin de la zone de confort de l'enseignant, mais... il doit bien y avoir un élément d'inconfort ou de déstabilisation sans quoi ça n'est plus une innovation !
- Des choses simples que nous pouvons faire pour être plus efficaces comme enseignants—plusieurs dans le groupe n'étaient pas intéressés par de grands bouleversements (ou révolutions).
- On s'attend à des améliorations (et on doit être capables de les repérer, de les identifier) ; les innovations doivent servir la classe.
- L'expérience et la confiance de l'enseignant qui implémente l'innovation joue pour beaucoup : certains enseignants sont plus audacieux et plus prêts à prendre des risques.
- Est-ce sûr, faisable, fiable, pour les étudiants, pour l'administration, pour nous autres enseignants ? (Par exemple, est-ce que les tâches d'évaluation vont « exploser » avec cette innovation ?)
- Quels sont les enjeux pour les étudiants ? Sur quels aspects l'innovation a-t-elle principalement ses effets ?
- Adaptabilité—des innovations calibrées pour un contexte peuvent ne pas se transférer bien à d'autres contextes. Par exemple, les attentes d'étudiants de collège sont différentes de celles des étudiants universitaires.
- Contraintes institutionnelles—sous l'angle de la sûreté, de la faisabilité. Dans quelle mesure l'institution est-elle prête à supporter le projet, l'enseignant ? Quelles contraintes sont imposées à l'enseignant ?

- Quel intérêt y trouvent l'institution, les collègues, les étudiants ? Comment évaluent-ils l'innovation ?

APPENDIX – ELEMENTS OF THE HISTORICAL PRESENTATION FROM PAUL DEGUIRE

FROM THE *ADEQUALITY* OF FERMAT TO INFINITESIMALS TO NON-STANDARD ANALYSIS

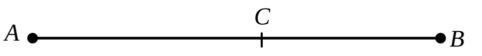
The use of infinitely small quantities viewed as an innovation

Pierre de Fermat was a French mathematician of the 17th century. Although he worked full time as a lawyer at the parliament of Toulouse, he found enough time to do mathematics in quantity, in diversity and in quality.

Fermat used the word *adégalité* to mean a sufficiently close equality, or a satisfying equality, in a sense that mathematicians would only understand centuries later. Fermat said that he took the word from Diopohantus' *Arithmetica*. This notion of *adégalité*, that we will translate by *adequality*, helps compare things that are infinitely close to each other and then derive results from this adequality, for instance finding an extremum or the slope of the tangent line to a curve. This was done in the 1630s, 30 years before Christian Huygens, presenting the work of Fermat at the French Academy, created the expression *infinitely small* (1667). This was 50 years before the first publications on the new calculus by Newton and Leibniz. This was more than 100 years before Euler defined what a function was and 150 years before Lagrange introduced the term *derivative* and spoke about the *derivative function*. Fermat therefore never spoke about derivatives or functions. He spoke about *slope of a tangent*, he spoke about *curves* and he solved optimisation problems that were not expressed in terms of functions, although this is what we would do today.

The use of Fermat's *adequality* is in fact the use of infinitely small quantities in order to find solutions to these new types of problems. Infinitely small quantities were used before Fermat to calculate areas and volumes. Archimedes used it to calculate the volume of the sphere for instance. After the invention of analytic geometry in the early 17th century, separately done by René Descartes and Fermat, the study of curves became an important occupation of the mathematicians of the time and determining their extremums, or their tangent lines were among the most important questions to solve. Fermat was the first to use the notion of infinitesimals to solve these problems and all mathematicians coming after him before the introduction of a correct definition for the limit used these infinitely small quantities.

Let us present the first problem solved by Fermat using the notion of adequality. A segment AB is given.



The question is where should we put the point C on AB such that the rectangle on AC and CB has a maximal area? If we suppose that the length of AB is a and the length of AC is b, Fermat was looking for the maximum of the expression $b(a – b)$. Suppose that C has been properly located and that our maximum is $b(a – b)$. Oresme (14th century) and Kepler (early 17th century) both observed that near a maximum, the value of such expressions changes very little. Fermat was aware of that.

In modern language, we know this as a consequence of the fact that the derivative in the neighborhood of a point where a maximum is obtained is very small.

Fermat supposed that e is a very small quantity so that $(b + e)(a – (b + e))$ would then be very close to $b(a – b)$. He wrote: $(b + e)(a – (b + e)) \sim b(a – b)$, the symbol ~ meaning that both quantities are adequal.

Therefore, $ab – b^2 – be + ae – be – e^2 \sim ab – b^2$

- **First step of the method:** simplification. Eliminate the common terms (which are those where e is absent). Fermat was left with $ae \sim 2be + e^2$
- **Second step:** division by e. So that $a \sim 2b + e$
- **Third step:** Let $e = 0$. He could conclude that $a = 2b$.

Therefore Fermat correctly observed that the point C has to be in the middle of the line AB and that the rectangle with perimeter $2a$ with the largest area is a square.

This method works well. Similar use of infinitely small quantities, such as the e in the work of Fermat, worked well during the birth, infancy and development of the calculus, that is for more than 200 years. It was used by Fermat and by later mathematicians not only to find extrema but to calculate derivatives and their applications.

But even an average modern student would question the method by asking: *how can you divide by e if e is equal to 0?* The legitimate objection was made by other mathematicians and by philosophers. But soon mathematicians realized that the methods, or variants of it, did work and provide a lot of fantastic new results not obtainable otherwise. **So the innovation stayed** even if nobody knew why it worked. The physical world agreed with the new calculus as was quickly shown after the work of Newton, and at that time, the agreement of nature was sufficient for mathematicians to go on.

In modern times we would justify Fermat's methods by saying that if f is a function giving the value of the area, then what Fermat did is similar to showing that $\frac{f(b+e)-f(b)}{e}=0$ when e tends to zero, that is the derivative of f is 0 at b which is the case when a maximum is obtained at that point. We know what a limit is and we can provide rigorous analytical arguments.

The modern student who objects to this method should take a look at what he is doing himself to calculate certain limits.

Suppose he wants to evaluate $\lim_{x\to 1}\frac{(x^2-1)}{(x^2+x-2)}$.

This is an indeterminate form of the type $\frac{0}{0}$. It requires some work to lift the indetermination. The student would first factor and write

$$\lim_{x\to 1}\frac{(x^2-1)}{(x^2+x-2)}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x+2)} \quad \text{(Step 1: algebraic manipulation)}$$

Then he would divide by $(x – 1)$ to obtain $\lim_{x\to 1}\frac{(x+1)}{(x+2)}$ (Step 2: division by $x – 1$)

And finally he would put $x = 1$ to conclude $\lim_{x\to 1}\frac{(x^2-1)}{(x^2+x-2)}=\frac{2}{3}$ (Step 3: $x - 1 = 0$).

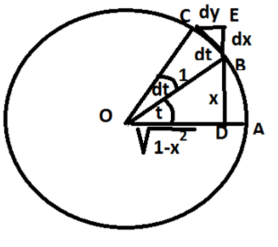
He is in fact doing the exact same thing that Fermat was doing nearly four centuries ago. And if the student is not a math student and never properly studies the notion of limit, he knows no more than Fermat knew at the time.

The next two examples, from Gottfried Wilhelm Leibniz and Guillaume François Antoine, Marquis de l'Hôpital, give an idea of how the notion of infinitely small was used in the first years of the new calculus.

<u>Leibniz and the differential triangle</u>

In the following diagram the circle is of radius 1, the angle is t, dt is a small increment of t and $x = \sin(t)$. Leibniz works with the small triangle *BEC* whose sides are differentials and therefore infinitely small quantities. Therefore he can consider the side *BC*, tangent to the circle at *B*, to be equal to the little arc *dt*. We know that this is only a good approximation, but the smaller *dt* is, the better the approximation is, as

$$\lim_{dt\to 0}\frac{\tan(dt)}{dt}=1\,.$$



All quotients of pairs of sides in this triangle are indeterminate of the form $\frac{0}{0}$. The opponents of the new methods said that these quotients had no meaning. Leibniz had no problem because this infinitely small triangle *BEC* is similar to the finite triangle *ODB*, therefore each quotient of sides of the infinitesimal triangle is equal to a well-defined quotient of sides of a usual finite triangle. By the principle of continuity, things in nature don't change in jumps, so that while reducing the sides of the triangle continuously, the value of these quotients remains.

For example, the indeterminate quotient $\frac{dx}{dt}$ is equal to the well-defined finite quotient $\frac{\sqrt{1-x^2}}{1}$.

Because $x=\sin(t)$, then $\sqrt{1-x^2}=\cos(t)$ and therefore, $\frac{d(\sin(t))}{dt}=\cos(t)$. The formula is well known today, and we see that Leibniz was easily able to obtain such formulas.

This is only the first part of a calculation by Leibniz that led to the differential equation $\frac{d^2x}{dt^2} = -x$ from which he was able to find the infinite McLaurin series for $x = \sin(t)$. This example of calculation with infinitely small quantities and infinitesimal triangles shows that by using the ordinary laws of geometry and some easy calculations, the use of this innovation, the infinitesimals, led to the development of an entirely new calculus that had important applications. Nobody was able then to define properly what an infinitely small quantity was; nobody would be able to for a long time, but the new calculus had so much success in applications, notably to the real world, that in spite of criticism by some mathematicians or philosophers, nothing stopped its evolution. For most of the next century it would be made more precise, expanded and applied, although the basic notion of limit was at best a good intuition and the notion of infinitesimals had no solid ground to stand on.

The first differential calculus book

One of the first followers of Leibniz, himself an important contributor to the new calculus, was Jean Bernouilli. The Marquis de l'Hôpital, a French nobleman interested in mathematics, learned the new calculus from Bernouilli and wrote the first calculus text book in 1696, *L'analyse des infiniments petits pour l'intelligence des lignes courbes*. The well-known rule of l'Hôpital appears in print for the first time in this book, explaining its name, but it was part of what l'Hôpital learned from Bernouilli.

In this text l'Hôpital is interested with differences between variable quantities, for instance if x is augmented to $x + dx$, dx is the difference (we would say the differential). For most applications these differences have to be considered to be infinitely small. The rules of derivation are easy to follow (although we wouldn't speak about derivatives for another century). For instance, to find the difference of a product xy, l'Hôpital simply calculated $d(xy) = (x + dx)(y + dy) – xy$.

Therefore, he had $d(xy) = xdy + ydx + dxdy$. Because $dxdy$ was negligible compared to the other two terms of the right side, he concluded $d(xy) = xdy + ydx$.

From this he was able to easily find the formula for the difference of a quotient. Suppose $z = \frac{y}{x}$. Therefore $xz = y$, $d(xz) = dy$, $zdx + xdz = dy$, $\frac{y}{x}dx + xdz = dy$ and finally $dz = d(\frac{y}{x}) = \frac{xdy - ydx}{x^2}$. A very simple way to find the formula to differentiate a quotient.

The modern and final judgment about infinitely small quantities.

In the 19th century, Augustin-Louis Cauchy would define properly the notions of limit and convergence and he had a definition for *infinitely small*, as well, that he frequently used. Following the developments of mathematics in the second half of the 19th century, notably the creation of mathematical logic, a mathematical definition of the limit was provided by Weierstrass. The use of quantifiers made it possible to say mathematically what Cauchy was saying in French. The mathematical definition could be worked with in full rigor; the mathematics were on solid ground again. The use of infinitely small quantities became useless and was somewhat forgotten.

Then, in 1960, Abraham Robinson wrote about a new notion: *Non-Standard Analysis*. Non-Standard Analysis is based on an extension of the reals called *hyperreals*. Around each real number x you have a family of numbers that differs from x by an infinitely small quantity. All *hyperreal* numbers in this family are of the form $x + x'$ where x' is infinitely small. Zero being itself an infinitely small quantity, x itself is of the same form as $x = x + 0$. Ordinary real numbers are called standard. If $x + x'$ is a hyperreal number, with x standard and x' infinitely small, we say that x is the standard part of $x + x'$.

In Non-Standard Analysis, you have axioms and theorems about non-standard objects, infinitely small or infinitely large numbers, for instance. Infinitely small quantities, except 0, are all non-standard and any positive infinitely small quantity is strictly smaller than any positive standard real number. Similarly, all positive infinitely large quantities are non-standard and are strictly larger than any positive standard real number.

Using Non-Standard Analysis, all the results from the standard (ordinary) analysis can be obtained, often quite easily as the notion of convergence is greatly simplified. But as no important new results about the standard real numbers have been obtained this way, mathematicians continue to use classical standard analysis. Nevertheless, Non-Standard Analysis is sound and it shows that the infinitely small quantities of Fermat, Leibniz and Cauchy could be used with rigor. In fact with Non-Standard Analysis, Robinson precisely showed that the language of the infinitesimals is fully compatible with mathematical rigor.

Paul Deguire,15 mai 2015

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Topic Sessions

Séances thématiques

SOME THOUGHTS ON MATHEMATICS AS THE ALIEN WORD

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INTRODUCTION

Learning mathematics is sometimes equated with learning a language. In *Speaking Mathematically*, David Pimm (1987) quotes Spike Milligan (a British humourist) as saying "*Mathematics? I speak it like a native*"[1], an allusion to the popular image of mathematics as being a foreign language (to everyone, presumably—there are, in fact, no native speakers). I don't actually agree with the idea that mathematics is a language in this sense. Pimm prefers to treat the claim as a metaphor. Thinking of learning a language as a metaphor for learning mathematics highlights certain aspects of the latter: there is, for example, a particular vocabulary, grammar and set of symbols to be mastered. Most of all, however, the metaphor is really about the feeling of many students that there is something 'other' about mathematics, something not of 'my' language, something 'alien'. The metaphor implies that learning mathematics is about learning not just an alien way of speaking, but an alien way of thinking. My purpose in this topic session was to explore this idea, drawing particularly on Bakhtin's (1981) writing about language and more specifically on his notion of the *alien word*.

To begin, I offered the following problem (from Hess-Green, Heyd-Metzuyanim, & Hazzan, 2015), which I invite you to work on for a few minutes.

> Prove that for each number
>
> $$n! + 2, n! + 3, \dots n! + n$$
>
> there is a prime divisor that does not divide any other number from this set.

In the session, participants worked on the problem individually, in pairs or in small groups. Once some initial ideas and strategies had emerged, I asked the participants to reflect on different ways in which language was relevant to their work on the problem. Some different responses to this question included:

- The vocabulary and symbols used in the problem;
- The structure of the problem, which is quite particular to mathematics;
- Individual and collective processes of interpreting the text;
- Participants' interaction with each other as they worked on the problem;
- The use of multiple languages (French, English, Russian, etc.) and ways of talking;

[1] The quote is from an episode of the 1950s BBC radio comedy series *The Goon Show*.

- Societal discourses about the learning and teaching of mathematics (e.g. algebraic symbols are scary; knowing mathematical vocabulary makes you clever, this is a nice problem).

I chose this problem because its presentation is particularly ‘mathematical’. In order to make sense of it—to understand what it is asking and to develop some strategies to work on it—familiarity with the words, symbols, discourses and genres of mathematics is necessary. You need to know what ‘prove’ means, what ‘!’ indicates, what a ‘prime divisor’ is, what ‘this set’ refers to, and you need to be able to see the logical structure expressed by ‘a prime divisor that does not divide any other number from this set’. But how, as mathematicians and mathematics educators, do we come to know these things? How do we come to be able to read a problem like this and make sense of it? There can only be one answer to these questions: we learn by interacting with others: our teachers, our colleagues, the authors of mathematical texts or textbooks. So what is this process like?

A BAKHTINIAN VIEW OF LANGUAGE

For the Russian literary theorist Mikhail Bakhtin (1981), language is not simply a means of transmitting ideas from one person to another. Language is a part of how we see the world—and we learn it from others: “The ideological becoming of a human being…is the process of selectively assimilating the words of others” (p. 341). For me, this statement says that we learn to make sense of the world by taking on the words of others—what Bakhtin calls the *alien word*—and turning them to our own purposes. In particular, we learn how to make sense of mathematics by assimilating the discourse of mathematics.

The process of assimilation is not straightforward; our words are never entirely our own: “Language is not a neutral medium that passes freely and easily into the private property of the speaker’s intentions; it is populated—overpopulated—with the intentions of others” (Bakhtin, 1981, p. 294). That is, when we use others’ words for our own purposes, they carry the traces of meanings from their previous history of use. For example, at different ages, children will mean different things when they use or see the word ‘prove’; as mathematicians and mathematics educators, however, we will hear the meaning most common in the discourse of mathematics, even if this meaning is less salient (or even invisible) to a particular student. In learning mathematics, then, students must grapple with the discourses of mathematics as they experience them and learn to use them to express their mathematical thinking in recognisable and productive ways. Even as they do so, every utterance will partly reflect (possibly unintentionally) ‘other’ ideologies or worldviews about proof, about mathematics and so on. These different worldviews are in constant dialogue within each utterance.

Otherness is significant in another way: whatever we say is a response to previous utterances:

> any utterance is understood to be a response, one turn in an ever-unfolding chain of utterances, which “cannot fail to be oriented toward the ‘already uttered,’ the ‘already known,’ the ‘common opinion’ and so forth” (Bakhtin, 1981, p. 279).

As a response, each utterance is to some extent shaped or influenced by preceding utterances: a response speaks to someone—an other—about something. Our assumptions about our addressee, for example, influence how we formulate what we say. And each utterance is heard as contributing to what has gone before. In human interaction, such as in a mathematics classroom, there is a constant interchange of utterances/responses (each utterance is also a response). Meaning arises out of this interchange. Indeed, without otherness, there can be no meaning: only an empty uniformity.

In summary (inasmuch as it is possible to summarise Bakhtin's ideas), otherness is central to Bakhtin's thought in several respects:

- We must use others' words and discourses;
- These words partly reflect others' 'intentions' and voices;
- These words partly reflect others' ideologies or worldviews;
- Our utterances are always addressed to others, a response to others' words;
- Without otherness, there is no dialogue, no meaning.

Having presented these ideas, in the rest of the session I offered three examples from my own research, as a way to explore further what they might mean for the learning of mathematics.

EXAMPLE 1: DOUBLING

The first example came from research conducted in an elementary school in London, UK (Barwell, 2012). In the following exchange from a grade 1 class, K, an ESL student and refugee, and his classmate Steven, are working with a teacher, T2. As part of a lesson focused on doubling, they need to work out how many cars are double eight cars. (Square brackets represent overlapping speech.)

T2:	Okay now, four cars. D'you know what you've done look here. 'Kay it's eight cars and it should be double eight and you've halved it, you've made half of eight and it must be double eight, what's double eight?
K:	umm
T2:	eight plus eight
K:	two
T2:	eight and eight together
K:	seven!
T2:	what's eight, and another eight
Ste:	I know
T2:	eight plus eight
K:	two!
T2:	[no
Ste:	[sixteen
T2:	sixteen
K:	oh
T2:	so it should be sixteen cars

In this exchange, the role of the other for the students is crucial. The alien word is represented by the word *double*, which K must interpret and act on: his task is to assimilate the word *double*. In the exchange, there is a dialogue between various possible meanings for *double* as indicated by the various responses. K's responses do not coincide with the intentions carried by the word *double* as used by the teacher and, more generally, in mathematics. The exchange shows different worldviews coming into contact, with the pressure of standard mathematical usage (what Bakhtin would call a *centripetal force*) encountering K's non-standard responses to the term (what Bakhtin would call a *centrifugal force*). The teacher is also encountering an other, in this case, K (and Stephen). Between these two encounters, K responds, in some ways appropriately (he's offering numbers) and in some ways not (his answers don't correspond to what is expected). It is through the encounter with the other that meanings for *double* emerge.

EXAMPLE 2: UNE CLASSE D'ACCUEIL

The second example is from a *classe d'accueil* for grades 5 and 6 in an elementary school in the province of Québec (Barwell, 2013).[2] The students are all learning French as a second language. They have been learning about polygons and have participated in the following activities:

- *Des différentes manières de classer les élèves dans la salle, par ex. « les jeans » vs. « non-jeans »*
- *Classer des formes régulières et irrégulières dans deux groupes distincts et discussion des critères*
- *Déduction d'une définition de « polygone » à partir de quelques exemples*
- *Classer les figures sur la feuille de polygone ou de non-polygone*
- *Discussion des réponses*
- *Récapitulation : un élève doit définir un polygone et un non-polygone.*

The students have thus encountered various informal and formal ways of talking about polygons, including their own formulations, those of the teacher, and a written definition given on a worksheet:

> Un polygone est une ligne brisée fermée, tracée sur une surface plane.

In one exchange, the students E53 and E54 are asked by the teacher (Ens) to give definitions of a polygon and a non-polygon.

E53: Un polygone c'est comme forme qu'il a des lignes droites, et il n'y a pas de trous

Ens: Ok lignes droites fermées, si je te demande c'est quoi un non-polygone, [E54]

E54: Non-polygone c'est comme ah il y a comme il y a un trou dans le carré ou les lignes sont c-courbes

Ens: Donc il y a une ligne courbe ou une ligne qui n'est pas fermée

This brief exchange captures a moment in the process of assimilating the language of polygons. E53 offers a definition in accented, non-standard French. He uses a mixture of standard mathematical words (*forme*, *ligne droite*) and informal words (*trou*). He has heard from the mouth of his teacher, his classmates, and the worksheet. His 'struggle' to 'force' these alien words to conform to his intentions is almost tangible. This struggle arises, moreover, in response to and addressed to his teacher. He must try to express this definition in a particular way *for his teacher*, not for himself. The teacher, in turn, responds by reformulating E53's words, transforming *trou* into *fermées*. A similar pattern arises in the next turns, in which E54 offers a definition of non-polygon and the teacher reformulates it, transforming *un trou dans le carré* into *une ligne qui n'est pas fermée*. In each pair of turns, meaning arises for both the teacher and the student through the differences in the words they use in their responses to each other. There is an expectation that the students will move towards speaking more like the teacher.

EXAMPLE 3: THE TULIP PROBLEM

The final example comes from research conducted in a grade 5-6 class in an elementary school in the province of Québec (Barwell, 2014).[2] The children were from Cree communities in the James Bay area, and had migrated, often temporarily, to the city. They spoke Cree at home. On one occasion, I worked with two students on a 'situational problem'. The problem included a lengthy introduction describing the history of the Ottawa tulip festival. On the next

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[2] Examples 2 and 3 come from research funded by the Social Sciences and Humanities Research Council of Canada, grant number 410-2008-0544. See the related publications for more details.

page, it was explained that the students were to imagine they were gardeners. Their task was to complete an arrangement of tulips, shown as a diagram. The arrangement formed a geometric series using three different colours. I asked the students to read the introduction and then we discussed what it meant:

RB:	okay, so what's it about?
Curtis:	it's about, world's biggest flower- I don't know
RB:	Ottawa's biggest
Curtis:	tu- lip festival
RB:	tulip festival, do you know any of those? Do you know what a tulip is?
	[hm]
Curtis:	[flower]
RB:	flower right, have you ever seen a tulip?
	[…]
Ben:	(…) it's white
RB:	they are lots of different colors white ones red ones
Curtis:	like a rose?
RB:	yellow ones say again
Curtis:	rose
RB:	no it's a bit different from a rose, roses yeah, tulips just come up in the spring and have a nice flower for about two weeks, then they are finished, there we go, let me see your picture
	[*laughter*]

In this exchange a very concrete kind of otherness is apparent: the problem is about tulips but the students have little experience of tulips—they do not grow in James Bay. To make sense of the problem and, in particular, of what is required of them, the students must engage with this alien word. This otherness is not simply a question of not being familiar with a type of flower; it includes implicit worldviews about the kinds of experiences elementary school children in Québec should have.

Once we moved on to the arithmetic problem, the two students had no trouble finding a solution. They found it more challenging, however, to 'show their work':

RB:	so (.) that's a good beginning (.) but you need to explain like the calculations that you did (.) you need to say what kind of calculations you did
Curtis:	times
RB:	yup but precisely what did you times what did you add
Curtis:	I timesed seven (.) times seven (.) six times (.)
RB:	right right
Curtis:	seven plus that's it
RB:	so like when you worked out for purple
Curtis:	I did five times five
RB:	uhum
Curtis:	plus one
RB:	right so I would write purple and then exactly what you just said

As with the previous example, there is a tangible sense of struggle as Curtis grapples with an other's words in order to express himself. Part of the struggle is, of course, that he must express himself in particular ways, as a requirement of the situational problem, enforced by me. He must find a way to use the words of the problem, and the worldview that goes with them, to formulate an acceptable response (where 'acceptable' is constrained by particular discourses of school mathematics).

THE ALIEN WORD IN THESE EXAMPLES

The three examples presented above show how, in learning mathematics, learners must use others' words and discourses. The students are clearly in the process of learning to use words that are not their own, such as *polygon*, *tulip*, *times*. They are also in the process of learning to use features of mathematical discourse, such as, for example, responding to closed arithmetic questions (example 1), interpreting and producing definitions (example 2), and giving mathematical explanations (example 3).

The examples also illustrate how utterances are always addressed to others, are always a response to others' words. Words and meanings emerge through this interaction, such as, for example, the relation between *trou* and *fermé*. Indeed, without otherness, there is no dialogue, no meaning. Learning what a polygon is, for example, requires dialogue between multiple voices; the dialogue between *trou*, *fermé*, *brisé* etc. is the source of meanings about polygons.

On the other hand, in all three examples, it can be seen that the expectation is for students to adapt to the other: they must, in some sense, *become* the other. By this I mean that the students must adopt the language of the other (e.g. *fermé*) with its associated ideologies, and leave their own words (e.g. *trou*) to one side. Various mechanisms are used in socialising students into using 'correct' mathematical language, including repetition, reading aloud, revoicing, and reformulation of students' utterances. These mechanisms are really all ways to in some sense impose (centripetally) a particular way of talking, and so a particular world view. As the students come to use more mathematical language, they are also coming to conform to a particular, 'standard' mathematical worldview. This worldview may marginalise alternative ones.

CONCLUDING QUESTIONS

The thinking I presented in the topic session is ongoing. I chose not to present a definitive conclusion. Instead, I first invited participants to think back to their own work on the opening prime divisor problem. To make sense of this problem, many of the participants felt they were encountering the alien word! They had to make sense of the symbols, the words and the underlying logical relations. They also had to communicate with each other, to try to express their half-formed ideas for an other, and make sense of an other's ideas. And they had to think about proof, a specific mathematical genre with its own requirements. Through these various encounters with the alien word, some mathematical meaning emerged.

The metaphor of learning mathematics as learning a language perhaps has more to it than I first thought. Learning a language certainly involves a kind of ideological becoming. It is not enough to learn the words of a language; one must learn how to speak them. Learning to speak mathematics may be straightforward for some, more challenging for others. There remains much to understand. I ended with further questions:

- What is the process of *selective assimilation* like in learning mathematics?
- What is excluded?
- What is it like for our students to become 'other'?
- What is it like to want students to become mathematically other?
- Are we happy to do this?
- What responsibility do we have?

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INTERACTION BETWEEN A MATHEMATICS DEPARTMENT IN A UNIVERSITY AND THE SCHOOL SYSTEM

INTERACTION ENTRE UN DÉPARTEMENT DE MATHÉMATIQUES UNIVERSITAIRE ET LE SYSTÈME SCOLAIRE

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ABSTRACT / RÉSUMÉ

For historical reasons, there is little mathematical tradition in French-speaking New Brunswick, or more precisely among Acadians. But things are slowly changing due to the interaction between the *Département de mathématiques et de statistique* of the Université de Moncton and the local school system. Various mathematical activities are organised by professional mathematicians inside the school system (high schools and elementary schools). A special day dedicated for mathematics was introduced in 2013 and takes place in February each year. A group of professionals in mathematical education has been organised to help improve the mathematical experience in the school system. In this talk, I will discuss these activities and explain our main objectives.

Pour des raisons historiques, il n'y a pas de tradition mathématique au Nouveau-Brunswick francophone, chez les acadiens. Toutefois, les choses changent peu à peu en raison de l'interaction entre le département de mathématiques et de statistique de l'Université de Moncton et le système scolaire. Plusieurs activités mathématiques sont organisées par des mathématiciens professionnels à l'intérieur du système scolaire, dans les écoles primaires aussi bien que secondaires. Une journée spéciale dédiée à célébrer les mathématiques a lieu annuellement, en février, depuis 2013. Un groupe de professionnels spécialistes de l'enseignement des mathématiques a été mis sur pied pour enrichir l'expérience mathématique dans le système scolaire. Dans cette présentation nous discuterons de ces activités et nous expliquerons quels sont nos objectifs.

INTRODUCTION

Among the universities in Atlantic Canada, l'Université de Moncton is a young one. Before its creation in 1963, access to higher education in French New Brunswick was very limited and mathematics or statistics were not among the options. Even when the university was created, its Faculty of Sciences didn't include a Department of Mathematics. Soon enough someone realised that just to cover the mathematical needs of students in other fields of study, for instance honours students in physics, there was almost enough material for a major program in mathematics. A few courses were then added and a major program was created. The Department of Physics became the Department of Physics and Mathematics. French-speaking students in New Brunswick had access for the first time in the early 70s to mathematical education leading to a university degree in mathematics. No honours degree (still none today), no masters degree program (not until 2000), but at least mathematics existed. Young students with a love for mathematics could come to l'Université de Moncton and learn mathematics.

Since then, many young Acadians have been studying mathematics, in French, without leaving the province. Some of them have continued after Moncton, all the way to a PhD in mathematics (to our knowledge, none before the mid 80s). Others have become high school teachers and are now in the French school system of the province. They are in a good position to reverse the historical trend in French New Brunswick, to create a mathematical tradition and to lead students towards university programs in mathematics or statistics. But they need help and the purpose of this presentation is to present an experience underway that has the objective of helping the mathematics teachers from kindergarten to the end of high school to improve the mathematics experience in and out of their classrooms, with the help of specialists from the university. New Brunswick is a small province and French New Brunswick is only 30% of it. We may not be strong enough to create a mathematical association, or it might simply be too early to try. We have to work on more feasible options.

In 2009 when I was chairman of the *Département de mathématiques et de statistique* of the Université de Moncton, I organised the first annual Day of Reflection about Mathematics in the French NB school system. This attracted people from all levels of mathematical teaching. Other similar days were organised yearly until 2012 when we went a step further. In Fall 2012 we created the *Groupe d'Action pour les Mathématiques en Acadie* (GAMA), a group of people committed to creating a bridge between mathematicians and didacticians in the university and teachers in the school system. Presenting the objectives of GAMA and its early work are the main objects of this presentation.

HISTORY

Before the creation of GAMA, a few things were done.

Starting in the mid 80s, first at the University of New Brunswick but soon after at the Université de Moncton, a provincial mathematics competition was organised for students from grades 7, 8 and 9. The competition is bilingual, organised jointly by the two universities, and students come to the university to write the competition exam. Roughly 1200 students, year after year, come to this competition that has become a beloved tradition among the best students of the province in middle-school mathematics.

With the financial help of the ACFAS (*Association Canadienne Française pour l'Avancement des Sciences*) professors from the Université de Moncton, were able to travel to high schools anywhere in the province to present various talks and similar activities to high school students. I started to travel within New Brunswick to present talks on a variety of mathematical topics in 1989. As the ACFAS has stopped contributions for these talks, it is now difficult to visit far away schools, so that only high schools not too far from Moncton can easily be visited.

Since 2009, I have been organising a mathematical club in a local elementary school. Only selected students take part in this activity as the purpose is not only to present them with new material but mostly to give them the chance to actually work on this new material. The size of the group is important, as is a positive attitude towards mathematics from all participants.

WHY IS MATHEMATICS SPECIAL? HOW DOESTHIS AFFECT TEACHING MATHEMATICS?

Students do mathematics from kindergarten to grade 12. They do far less science at school. Nevertheless, it is easier for them to explain what an astrophysicist does than what a mathematician does. The main reason is not that mathematics is difficult to master, other sciences are difficult as well. The problem lies elsewhere. Two questions are at the heart of the problem: *Why is mathematics useful*? and *Why are we all studying mathematics*? These are asked frequently by students in the school system, most of the time placing their teachers in an uncomfortable position. Often the answers provided by teachers are not satisfactory and the students move a bit away from mathematics each time.

Mathematics is not straightforward. Everybody understands why medical doctors are important in a society. What medical doctors do is pretty straightforward and nobody has any problem understanding the usefulness of medicine. On the other hand, mathematics is extremely useful because it is abstract. Mathematics has an existence of its own, it is independent of its applications. Because it is abstract, it has a wide range of applications, but the original mathematical idea is always quite different from the application so that non-mathematicians often make no connection between the mathematical idea and the application. In a sense, the extraordinary usefulness of mathematics makes it hard to explain the usefulness of mathematics. It has a wide range of applications because it is abstract. Mathematical ideas are abstractions that can connect to many concrete ideas. But its usefulness is not easy to see because non-mathematicians have problems with abstraction.

How does this affect the teaching of mathematics in schools? Most important applications require using advanced non-elementary mathematics. The usefulness of what we learn at school is not readily seen; examples of applications are often too simple and not attractive. Often when a student asks, "*Why are we learning this?*" the most direct answer that the teacher can provide is "*Because it will be on the test*". Of course even the worst teacher knows better than that, but many teachers who have no real answer can only stress the importance of the upcoming test.

As an easy example, if students knew that intersecting spheres (finding a common solution of a family of quadratic equations) is at the heart of the inside work of a GPS (Global Positioning System), this abstract work would become more pleasant. Another example: knowing that matrices are at the heart of the best search engines on the internet makes matrices more sexy.

EXAMPLES OF THE ABSTRACT NATURE OF MATHEMATICS

The number π comes from geometry. It became important when Greek mathematicians learned three things. First, in all circles, the ratio of the circumference to the diameter is always equal to the same number. Second, in all circles, the ratio of the area to the square of the radius is always equal to the same number. Third, those two numbers are the same. Not only are two very distinct ratios constant, the constant is the same for both. This constant has to be important.

Nowadays this constant π appears in many different areas of mathematics. Here are a few examples:

1. $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots)$

   This formula comes from the calculus (calculus of power series). It has been discovered independently by different mathematicians, from India (12^{th} century) to Western Europe (17^{th} century).

2. $\varsigma(2) = \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots$

   This equality is a particular case of the well-known Riemann hypothesis, maybe the most important unsolved problem in mathematics today. Among other applications, if proven, the Riemann hypothesis would provide information on the distribution of prime numbers.

3. $\frac{\pi}{2} = \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \cdot \ldots$

   Contrary to the previous formulas, there is an easy link between the origin of the number π (area of a circle of radius one) and this formula. It was obtained by François Viète using trigonometry.

These are but a few of the various places in mathematics where the number π pops out, sometimes rather unexpectedly. Each of these presents possible applications inside as well as outside of mathematics. In a sense, all these are connected by the number π. There might be something very deep related to this number, but at the same time something very scary for those who meet one of these formulas and have no clue about their origin nor about why the number π belongs to these formulas. Why π? We are not speaking about circles, so why π?

WHAT SHOULD WE DO IN SCHOOLS TO IMPROVE THE TEACHING OF MATHEMATICS?

Even if maths started with the study of concrete things, like measuring the earth (geometry) or commerce, it has now achieved such a high level of abstraction that it seems to be totally disconnected from the real world. Abstraction makes maths hard to study and not attractive and pushes students to ask themselves, *why should they study maths*?

But abstraction is also the primary quality of maths. Maths does not accept to be confined to particular cases; it applies everywhere. Abstraction is at the heart of the connections between things that do not seem to be connected. Just think about the number π.

The abstract nature of mathematics helps understand why mathematics has applications to almost all domains, from physical sciences to life sciences to social sciences and commerce. It appears everywhere. Every one of us should have a decent knowledge of mathematics.

But mathematics is not only abstract, it is rigorous. You need a sharp mind to do maths. You need rigor and patience. You have to make an effort. Students want to know what they will get for their effort. Being ready for next week's exam is not a satisfactory answer.

> Abstraction and rigor will be more readily accepted by students who understand the usefulness of mathematics, the usefulness of what they do.

This being said, explaining that mathematics is universal and applies everywhere is one important thing, but we also need to show that mathematics can be exciting and attractive. We have to convince students that mathematics can be fun, that mathematics can be beautiful.

Abstraction and rigor will be more readily accepted by students who have fun doing mathematics.

This is not an easy task and there are no magic formulas that will turn every math teacher into someone who will be able to convince his or her students that mathematics is important and that mathematics is fun. But we have to give it a try. The more who succeed, the better. Providing teachers with tools to help them do these two things can only improve the mathematics experience in schools:

1. Math is important; it applies everywhere
2. Maths is fun and beautiful.

This is the goal of GAMA, the *Groupe d'Action pour les Mathématiques en Acadie*.

GAMA

GAMA has been created to

1. Improve and develop math life in NB schools at all levels.
    - Help the students develop a more positive attitude toward mathematics (usefulness and fun).
    - Create fun activities for students / provide tools for teachers.

2. Establish (and promote) a dialogue between mathematicians and didacticians at the Université de Moncton and the French schools and colleges in New Brunswick.
    - University teachers can use their expertise in various way to help school teachers: conferences / workshops / math clubs / written documents or web pages filled with math stuff.

3. Work on the image of mathematics in the public at large.
    - Via public activities like talks opened to a large public or via newspapers, for instance.

A few things that have been done and had success include:

a) A few years ago, just before GAMA, we asked students from all school levels what a day without maths would be like. We received answers from students from grade 3 to grade 12. All very clearly showed that mathematics was everywhere and that a day without maths would be a day where nothing would work properly. (The youngest said that you could no longer buy anything because there would be no prices; the oldest created catastrophic scenarios from plane crashes to the end of our civilisation).

   It seems therefore that when you ask students about the importance of mathematics, outside of the classroom in a more fun context, they pretty much have a good idea. We could build on this idea to help develop a positive attitude toward mathematics. But we don't know if the answers reflected the opinion of those who already like mathematics and are good at it, of if they reflected the opinion of the majority.

b) We started a mathematics club in a local elementary school 6 years ago. We work on puzzles (students love it) or on fun mathematics like magic squares. We also teach them useful things like the binary system (and we use it to have fun doing multiplications the old Egyptian way). We have a meeting every two weeks.

c) We present talks in high schools about a variety of mathematical topics, often using an historical approach. Most students seem to appreciate the human effect of the historical approach: mathematics was done by real human beings, very often working on concrete and understandable problems.

d) GAMA created *la fête des mathématiques*. Once a year, in February, most French schools in the province organise mathematical activities, either using material provided by GAMA or using their own material. This date is now well fixed in the school calendar and many students wait for this date with great anticipation. Each year has its own theme. Copernicus (and astronomy and the calendar) in 2013 (it was held on February 19th, 2013; Copernicus was born 540 years before on February 19th, 1473). In 2014, a year of Winter Olympics, the theme was the Olympic games. In February 2015, the theme was cryptography. Finally the theme in 2016 is leap years (with problems related to various calendars).

e) Together with Manon LeBlanc (also a founding member of GAMA), professor in the *Faculté des sciences de l'éducation* at the Université de Moncton and former student of the *Département de mathématiques et de statistique* of the Université de Moncton, I went to the annual meeting of the French school teachers of the province (in Tracadie, September 1-2, 2015). We met more than 120 teachers, from elementary and high schools. We explained GAMA to them and received their feedback. GAMA is not well known yet, but we are working on it.

f) In an effort to make our group a little more official, we decided to have a competition among students in the province in order to find a logo for GAMA. We received many good submissions. The competition had one fine result: some people that never heard of GAMA now know what GAMA is. Another result is our new logo, shown at the end of this document.

THE FUTURE OF GAMA

Our first goal is to get better known. Teachers from all French schools in the province have to know that we exist and can help them.

We are a small group with almost no financing. We all have full-time jobs and therefore don not have much time to put towards GAMA. But nothing is impossible and these limitations mean only that we will not grow as fast as we wish, but we will grow.

1. Using our logo in all our activities and communication will put us firmly on the map. People will recognize and remember us more easily.
2. Of course we can't go in person to each school of the province to present talks, animate mathematical activities or supervise a math club. But we will visit as many schools as we can and we will provide tools for the elementary and high school teachers to organize activities of their own, including math clubs.
3. Every written document that we create will remain, so that we can only grow. Actually, we can be found on the web page of the *Département de mathématiques et de statistique* where some information about GAMA can be found, including material for *la fête des mathématiques*. We intend to have our own web page in the near future. We also plan on a page that will be partially interactive (with maths problems for instance). The web page will also be a place to organise all the documents that we will create, from the material for *la fête des mathématiques* to examples of activities for

math clubs: everything will be available and ready to use. Our web site will contain links to other useful and fun sites, including sites with mathematical videos. We have already made some videos ourselves. For high school students, to have an activity centered on watching a mathematical video done by a university teacher can compensate for the fact that the university teacher can't visit all schools. Virtual visits are better than no visit.

4. We intend to organize frequent competitions with modest prizes for the winners. Some of these could be connected with activities during *la fête des mathématiques*, but not necessarly. These competitions could be of all kinds, from problem solving to deeper reflections (like thinking about a day without math).

5. Mathematics is important in many jobs and professions, sometimes unexpectedly. You don't have to be a mathematician to use math on a daily basis. For instance, carpentry and woodwork are often heavily dependent on geometry. We plan to prepare a document where the mathematical needs of many kinds of jobs and professions will be described. This will help teachers answer the question: "*Why are we studying mathematics?*"

6. In the world of mathematics education in French New Brunswick, GAMA is good news. Other good news has also occurred in recent years. The most important is the creation, by Donald Violette, teacher in the *Département de mathématiques et de statistique* at the Université de Moncton, of mathematics camps for good students, selected through participating and winning math competitions created especially for this purpose. Math camps have been created for grade 12, grade 8 and grade 5 students, each level having its own competition, open to all French schools of the province. The purpose of these camps is more specific than the purpose of GAMA. GAMA works on improving the mathematical experience in the school system. The mathematics camps acknowledge the best maths students and provide them with a few days of real math immersion. GAMA and the mathematics camps complement each other. Their creation at about the same time by different people shows that the lack of mathematical tradition in French New Brunswick will soon be a thing of the past.

CONCLUSION

GAMA is a young ongoing project. It is too early to speak about success or lack of success. But so far we have received a strong welcome by teachers at all levels. They believe that we will be able to provide them with help that has been long needed. We will try to do just that, but without the active participation from some teachers already in the school system, nothing will be easy. Our success might very well be proportional to the number of school teachers involved.

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Groupe d'Action pour les
MATHÉMATIQUES
EN ACADIE

THANKS

This text does not require a bibliography. Specific mathematical ideas presented are part of the common knowledge of many mathematicians and didacticians. Some of these ideas have therefore been presented before by various authors in a variety of situations and contexts. I would like to thank all those anonymous authors or thinkers whose ideas have been reflected in this text.

I also would like to thank the other persons involved in the creation of GAMA, namely Manon LeBlanc, Viktor Freiman, Laurie Landry and Diane Dupuis.

PERMANENT MEMBERS OF GAMA

- **Paul Deguire** (President)
  Département de mathématiques et de statistique, Université de Moncton.
- **Manon LeBlanc** (Vice President)
  Département d'enseignement au primaire et de psychopédagogie, Université de Moncton
- **Laurie Landry**
  Agent pédagogique provincial en mathématiques
  Direction des programmes d'études et de l'évaluation
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- **Ahcène Brahmi**
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We have contacts in some high schools and some elementary schools. We expect in the future to have more permanent members from various levels of the school system.

BUILDING THINKING CLASSROOMS: CONDITIONS FOR PROBLEM SOLVING[1]

Peter Liljedahl
Simon Fraser University

In this session, I first introduce the notion of a thinking classroom *and then present the results of over ten years of research done on the development and maintenance of thinking classrooms. Using a narrative style, I tell the story of how a series of failed experiences in promoting problem solving in the classroom led first to the notion of a* thinking classroom *and then to a research project designed to find ways to help teachers build such a classroom. Results indicate that there are a number of relatively easy-to-implement teaching practices that can bypass the normative behaviours of many classrooms and begin the process of developing a thinking classroom.*

MOTIVATION

My work on this paper began over 10 years ago when I was observing a grade 7/8 teacher introducing problem solving into her teaching for the first time. Problem solving was something that, at the time, was becoming more and more prominent in the BC curriculum, and Ms. Ahn was interested in incorporating it into her classroom. Despite her best intentions the results were abysmal. The students gave up almost as soon as the problem was presented to them and they resisted her efforts and encouragement to persist. After three days of constant struggle, Ms. Ahn and I both agreed that it was time to abandon these efforts. Wanting to better understand why our well-intentioned efforts had failed, I decided to observe Ms. Ahn teach her class using her regular style of instruction.

That the students were lacking in effort was immediately obvious, but what took time to manifest was the realization that what was missing in this classroom was that the students were not being asked to think. More alarming was that Ms. Ahn's teaching was predicated on an assumption that the students either could not, or would not, think. The classroom norms that had been established in Ms. Ahn's class had resulted in, what I now refer to as, a *non-thinking classroom*. Once I realized this, I proceeded to visit other mathematics classes—first in the same school and then in other schools. In each class I saw the same basic behaviour—an assumption, implicit in the teacher's practice, that the students either could not, or would not think. Under such conditions it was unreasonable to expect that students were going to spontaneously persist through a problem-solving encounter.

[1] An extended version of this article, including research on teacher uptake, can be found in Liljedahl (in press).

What was missing for these students, and their teachers, was a central focus in mathematics on thinking. The realization that this was absent in so many classrooms that I visited motivated me to find a way to build, within these same classrooms, a culture of thinking, both for the student and the teachers. I wanted to build, what I now call, a *thinking classroom*—a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together, and constructing knowledge and understanding through activity and discussion. A classroom where thinking was assumed to be possible and was expected in every activity. Such a classroom will intersect with research on mathematical thinking (Mason, Burton, & Stacey, 1982) and classroom norms (Yackel & Rasmussen, 2002). It will also intersect with notions of a *didactic contract* (Brousseau, 1984), the emerging understandings of *studenting* (Fenstermacher, 1986, 1994; Liljedahl & Allan, 2013a, 2013b), *knowledge for teaching* (Hill, Ball, & Schilling, 2008; Shulman, 1986), and *activity theory* (Engeström, Miettinen, & Punamäki, 1999).

My early efforts to do so involved a series of three workshops designed to help teachers implement problem solving in their classroom. The results of these workshops were mixed. Some teachers reported that they saw great enthusiasm in their students, while others reported experiences similar to those I had observed in Ms. Ahn's class. Further probing revealed that teachers who reported that their students loved what I was offering tended to have practices that already involved some level of problem solving. It also revealed that those teachers who reported that their student gave up easily or resisted their efforts had practices mostly devoid of problem solving. The experiences that that the teachers were having implementing problem solving in the classroom were being filtered through their already existing classroom norms (Yackel & Rasmussen, 2002). If there was already a culture of thinking and problem solving in the classroom then the teachers were reporting success. If the culture was one of direct instruction and individual work then, although some students were able to rise to the task, the majority of the class was unable to do much with the problems.

These latter classroom norms are a difficult thing to bypass (Yackel & Rasmussen, 2002), even when a teacher is motivated to do so. The teachers that attended these workshops wanted to change their practice, at least to some degree, but their initial efforts to do so were not rewarded by comparable changes in their students' problem-solving behaviour. Quite the opposite, many of the teachers I was working with were met with resistance and complaints when they tried to make changes to their practice.

From these experiences I realized that if I wanted to build thinking classrooms—to help teachers to change their classrooms into *thinking* classrooms—I needed a set of tools that would allow me, and participating teachers, to bypass any existing classroom norms. These tools needed to be easy to adopt and have the ability to provide the space for students to engage in problem solving unencumbered by their rehearsed tendencies and approaches when in their mathematics classroom.

This realization moved me to begin a program of research that would explore the elements of thinking classrooms. I wanted to find a collection of teacher practices that had the ability to break students out of their classroom normative behaviour—practices that could be used not only by myself as a visiting teacher, but also by the classroom teacher that had previously entrenched the classroom norms that now needed to be broken.

GENERAL METHODOLOGY

The research to find the elements and teaching practices that foster, sustain, and impede thinking classrooms has been going on for over ten years. Using a framework of noticing (Mason, 2002)[2], I initially explored my own teaching, as well as the practices of more than forty classroom mathematics teachers. From this emerged a set of nine elements that permeate mathematics classroom practice—elements that account for most of whether or not a classroom is a thinking or a non-thinking classroom. These nine elements of mathematics teaching became the focus of my research. They are:

1. the type of tasks used, and when and how they are used;
2. the way in which tasks are given to students;
3. how groups are formed, both in general and when students work on tasks;
4. student work space while they work on tasks;
5. room organization, both in general and when students work on tasks;
6. how questions are answered when students are working on tasks;
7. the ways in which hints and extensions are used while students work on tasks;
8. when and how a teacher *levels*[3] their classroom during or after tasks;
9. and assessment, both in general and when students work on tasks.

Research into each of these was done using design-based methods (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003) within both my own teaching practice as well as the practices of a number of teachers participating in a variety of professional development opportunities. This approach allowed me to vary the teaching around each of the elements, either independently or jointly, and to measure the effectiveness of that method for building and/or maintaining a thinking classroom. Results fed recursively back into teaching practice, each time leading either to refining or abandoning what was done in the previous iteration.

This method, although fruitful in the end, presented two challenges. The first had to do with the measurement of effectiveness. To do this I used what I came to call *proxies for engagement*—observable and measurable (either qualitatively or quantitatively) student behaviours. At first this included only behaviours that fit the *a priori* definition of a thinking classroom. As the research progressed, however, the list of these proxies grew and changed depending on the element being studied and teaching method being used.

The second challenge had to do with the shift in practice needed when it was determined that a particular teaching method needed to be abandoned. Early results indicated that small shifts in practice, in these circumstances, did little to shift the behaviours of the class as a whole. Larger, more substantial shifts were needed. These were sometimes difficult to conceptualize. In the end, a contrarian approach was adopted. That is, when a teaching method around a specific element needed to be abandoned, the new approach to be adopted was, as much as possible, the exact opposite to the practice that had shown to be ineffective for building or maintaining a thinking classroom. When sitting showed to be ineffective, we tried making the students stand. When leveling to the top failed we tried levelling to the bottom. When answering questions proved to be ineffective we stopped answering questions. Each of these approaches needed further refinement through the iterative design-based research approach, but it gave good starting points for this process.

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[2] At the time I was only informed by Mason (2002); since then I have been informed by an increasing body of literature on noticing.

[3] *Levelling* (Schoenfeld, 1985) is a term given to the act of closing of, or interrupting, students' work on tasks for the purposes of bringing the whole of the class (usually) up to certain level of understanding. It is most commonly seen when a teacher ends students work on a task by showing how to solve the task.

FINDINGS

In what follows I present, in brief, the results of the research done on each of the nine elements and discuss how all nine elements hold together as a framework to build and maintain thinking classrooms. All of this research is informed dually by data and analysis that looks both on the effect on students and the uptake by teachers.

1. THE TYPE OF TASKS USED, AND WHEN AND HOW THEY ARE USED

Lessons need to begin with good problem solving tasks. At the early stages of building a thinking classroom these tasks need to be highly engaging collaborative tasks, usually non-curricular, that drive students to want to talk with each other as they try to solve them (Liljedahl, 2008). Once a thinking classroom is established the need for problems to be inherently engaging diminishes. As a result, the problems shift to towards curricular mathematics (Schoenfeld, 1985) that can be linked to the curriculum content to be 'taught' that day and permeate the entirety of the lesson.

2. THE WAY IN WHICH TASKS ARE GIVEN TO STUDENTS

Tasks need to be given orally. If there are data or diagrams needed, these can be provided on paper, but the instructions pertaining to the activity of the task need to be given orally. This very quickly drives the groups to discuss what is being asked rather than trying to decode instructions on a page.

3. HOW GROUPS ARE FORMED, BOTH IN GENERAL AND WHEN STUDENTS WORK ON TASKS

Grouping needs to be done frequently through visible randomizations (Liljedahl, 2014). Ideally, at the beginning of every class a visibly random method is used to assign students to a group of 2-4 for the duration of that class. These groups will work together on any assigned problem solving tasks, sit together or stand together during any group or whole class discussions.

4. STUDENT WORK SPACE WHILE THEY WORK ON TASKS

Groups of students need to work on vertical non-permanent surfaces such as whiteboards, blackboards, or windows. This will make visible all work being done, not just to the teacher, but to the groups doing the work. To facilitate discussion there should be only one felt pen or piece of chalk per group.

5. ROOM ORGANIZATION, BOTH IN GENERAL AND WHEN STUDENTS WORK ON TASKS

The classroom needs to be de-fronted. The teacher must let go of one wall of the classroom as being the designated teaching space that all desks are oriented towards. The teacher needs to address the class from a variety of locations within the room and, as much as possible, use all four walls of the classroom. It is best if desks are placed in a random configuration around the room.

6. HOW QUESTIONS ARE ANSWERED WHEN STUDENTS ARE WORKING ON TASKS

Students only ask three types of questions: (1) proximity questions—asked when the teacher is close; (2) stop thinking questions—most often of the form "*Is this right?*"; and (3) keep thinking questions—questions that students ask so they can get back to work. Only the third of these types should be answered. The first two need to be acknowledged, but not answered.

7. THE WAYS IN WHICH HINTS AND EXTENSIONS ARE USED WHILE STUDENTS WORK ON TASKS

Once a thinking classroom is established, it needs to be nurtured. This is done primarily through how hints and extensions are given to groups as they work on tasks. *Flow* (Csíkszentmihályi, 1990) is a good framework for thinking about this. Hints and extensions need to be given so as to keep students in a perfect balance between the challenge of the current task and their abilities in working on it. If their ability is too high the risk is they get bored. If the challenge is too great the risk is they become frustrated (Liljedahl, 2016).

8. WHEN AND HOW A TEACHER LEVELS THEIR CLASSROOM DURING OR AFTER TASKS

Levelling needs be done at the bottom. When every group has passed a minimum threshold, the teacher needs to engage in discussion about the experience and understanding the whole class now shares. This should involve a reification and formalization of the work done by the groups and often constitutes the 'lesson' for that particular class.

9. ASSESSMENT, BOTH IN GENERAL AND WHEN STUDENTS WORK ON TASKS

Assessment in a thinking classroom needs to be mostly about the involvement of students in the learning process through efforts to communicate with them where they are and where they are going in their learning. It needs to honour the activities of a thinking classroom through a focus on the processes of learning more so than the products, and it needs to include both group work and individual work.

DISCUSSION

However, this research also showed that these are not all equally impactful or purposeful in the building and maintenance of a thinking classroom. Some of these are blunt instruments capable of leveraging significant changes while others are more refined, used for the fine-tuning and maintenance of a thinking classrooms. Some are necessary precursors to others. Some are easier to implement by teachers than others, while others are more nuanced, requiring great attention and more practice as a teacher. And some are better received by students than others. From the whole of these results emerged a three-tier hierarchy that represents not only the bluntness and ease of implementation, but also an ideal chronology of implementation (see Table 1).

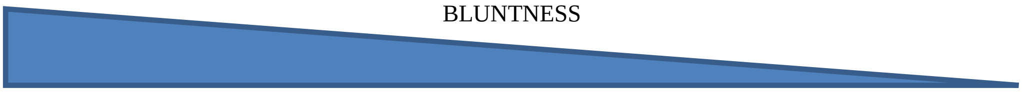
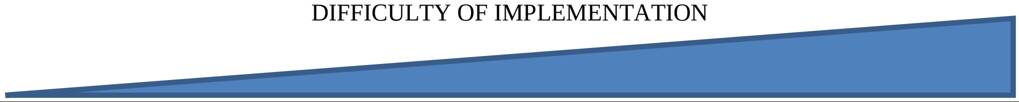
STAGE ONE	**STAGE TWO**	**STAGE THREE**
• begin lessons with problem-solving tasks • vertical non-permanent surfaces • visibly random groups	• oral instructions • defronting the room • answering questions	• levelling • assessment • managing flow
BLUNTNESS 		
DIFFICULTY OF IMPLEMENTATION 		

Table 1. Nine elements as chronologically implemented.

These results are not definitive, exhaustive, or unique. The teaching methods that emerged as effective for each of these elements emerged as a result of an *a priori* commitment to make change in a contrarian fashion. This continued until positive effects began to emerge, at which point refinements were recursively explored. It is possible that a different approach to the research would have yielded different methods. Different methods could, likewise, emerge a different set of stages optimal for the development of thinking classrooms.

CONCLUSIONS

The main goal of this research is about finding ways to build thinking classrooms. One of the sub-goals of this work on building thinking classrooms was to develop methods that not only fostered thinking and collaboration, but also bypassed any classroom norms that would potentially inhibit this from happening. Using the methods in stage one while solving problems, either together or separately, was almost universally successful. They worked for any grade, in any class, and for any teacher. As such, it can be said that these methods succeeded in bypassing whatever norms existed in the over six hundred classrooms in which these methods were tried. Further, they not only bypassed the norms for the students, but also the norms of the teachers implementing them. So different were these methods from the existing practices of the teachers participating in the research that they were left with what I have come to call *first person vicarious experiences*. They are first person because they are living the lesson and observing the results created by their own hands. But the methods are not their own. There has been no time to assimilate them into their own repertoire of practice or into the schema of how they construct meaningful practice. As such, they experienced a different way in which their classroom could look and how their students could behave. They experienced, thorough these other*ly* methods an other*ly* classroom—a thinking classroom.

The results of this research sound extra-ordinary. In many ways they are. It would be tempting to try to attribute these to some special quality of the professional development setting or skill of the facilitator. But these are not the source of these remarkable results. The results, I believe, lie not in what is new, but what is not old. The classroom norms that permeate classrooms in North America, and around the world, are so robust, so entrenched, that they transcend the particular classrooms and have become institutional norms (Liu & Liljedahl, 2012). What the methods presented here offer is a violent break from these institutional norms.

By constructing a thinking classroom, problem solving becomes not only a means, but also an end. A thinking classroom is shot through with rich problems. Implementation of each of the aforementioned methods associated with the nine elements and three stages relies on the ubiquitous use of problem solving. But at the same time, it also creates a classroom conducive to the collaborative solving of problems.

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GROWING MATHEMATICAL UNDERSTANDING AND FOLDING BACK: THE INDIVIDUAL, THE COLLECTIVE, THE TEACHER

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INTRODUCTION

In 1996, Susan Pirie noted that

> *we can assess with ease the "what" but not the "how" of the learning taking place as students struggle to construct a path to understanding. We have standardized and accepted ways to photograph their arrival, but not the means to film their journey. Yet, if we seek effective teaching we need ways of recognizing the paths that the students are laying down and a willingness to explore with them and to implicate ourselves in their constructing.* (Pirie, 1996, p. xv)

Almost twenty years later, this comment still remains as relevant as ever and in this paper, through a particular focus on the notion of *folding back* (Martin, 1999, 2001, 2008), I will present some of my research into how we might usefully describe and theorize the growth of mathematical understanding—initially at the level of the individual, more recently at that of the collective—and also consider the complex place of the teacher within the process of laying down paths of understanding.

THE PIRIE-KIEREN THEORY FOR THE DYNAMICAL GROWTH OF MATHEMATICAL UNDERSTANDING

The Pirie-Kieren theory contains eight potential layers-of-action for describing the growth of understanding of a specific person and for a specified topic or concept. These layers are named *Primitive Knowing*, *Image Making*, *Image Having*, *Property Noticing*, *Formalising*, *Observing*, *Structuring* and *Inventising*. Eight nested circles provide a two-dimensional diagrammatic representation or model of the theory. This nesting illustrates the fact that growth in understanding need be neither linear nor mono-directional. In addition, each layer contains all previous layers and is included in all subsequent layers, to emphasize the embedded nature of mathematical understanding. Although the rings of the model grow outward toward the more abstract and general, growth in understanding is not seen to happen in such a linear way. Instead, growth occurs through a continual movement back and forth through the layers of knowing. Using the model the growth of understanding of a learner, for a particular mathematical concept, can be mapped out. The model is illustrated in Figure 1 together with a diagrammatic representation of a hypothetical pathway of the plotted growth of understanding of a particular mathematical concept, for a particular learner, over a particular period of time.

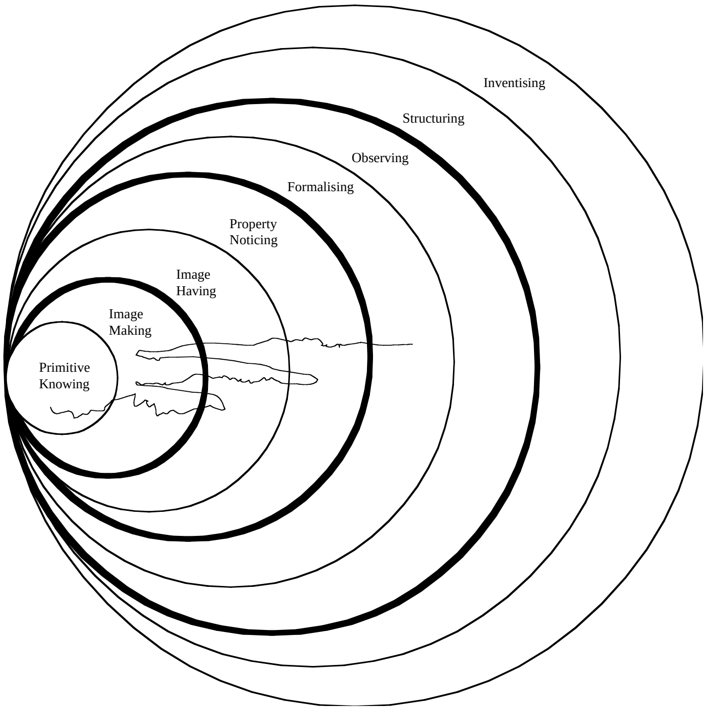

Figure 1. The Pirie-Kieren Model for the Dynamical Growth of Mathematical Understanding.

The definitions of the various layers of understanding actions have been fully set out in earlier work by Pirie and Kieren (see for example Pirie & Kieren, 1992, 1994) and thus here only brief definitions of the layers relevant to the data discussed in this paper are provided. At *Image Making* the learner is engaging in activities aimed at helping him or her to developing particular representations for the topic and mathematical idea; to get an idea of what the concept is about. These images need not only be visual or pictorial in nature, and they can be ideas expressed in language or in action. By the *Image Having* stage the learner is no longer tied to an activity, he or she is now able to carry a mental plan for these activities with them and use it accordingly. This frees the mathematical activity of the learner from the need for particular actions or examples.

FOLDING BACK AND THE GROWTH OF MATHEMATICAL UNDERSTANDING

Pirie and Kieren (1991) define *folding back* as

> *A person functioning at an outer level of understanding when challenged may invoke or fold back to inner, perhaps more specific local or intuitive understandings. This returned to inner level activity is not the same as the original activity at that level. It is now stimulated and guided by outer level knowing. The metaphor of folding back is intended to carry with it notions of superimposing ones current understanding on an earlier understanding, and the idea that understanding is somehow 'thicker' when inner levels are revisited. This folding back allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding.* (p. 172)

This definition suggests then that an individual will, when faced with a problem at any layer that is not immediately solvable, return to an inner layer of understanding. This shift is illustrated on Figure 1 by the hypothetical traced pathway, which illustrates two phases of folding back, the first from *Image Having* to *Image Making* and the second from *Property Noticing* to *Image Making*. The result of folding back is, ideally, that the learner is able to extend their current inadequate and incomplete understanding by reflecting on and then reorganizing their earlier constructs for the concept, or even to generate and create new images, should his or her existing constructs be insufficient to solve the problem.

However, in this process of reorganization the learner now possesses a degree of self-awareness about his or her understanding, informed by the operations at the higher level. Thus, the inner layer activity is not identical to that originally performed, and the learner is effectively building a 'thicker' understanding at the inner layer to support and extend his or her understanding at the outer layer that they subsequently return to. It is the fact that the outer layer understandings are available to support and inform the inner layer actions, which give rise to the metaphor of *folding* and *thickening*. Although a learner may well fold back and be acting in a less formal, more specific way, these inner layer actions are not identical to those performed previously.

Martin (1999, 2008) developed a *framework for folding back* that identified three key phases to any act of folding back—source, form and outcome. These described, in an observational tool, the stimulus for a learner to fold back, the kinds of actions engaged with at the inner layer, and finally the effect that folding back had on his or her continued pathway of understanding. This framework substantially elaborated the existing definition of folding back and offered more detailed ways to consider the role of, and interaction with others, in the process. The framework is presented below in Figure 2.

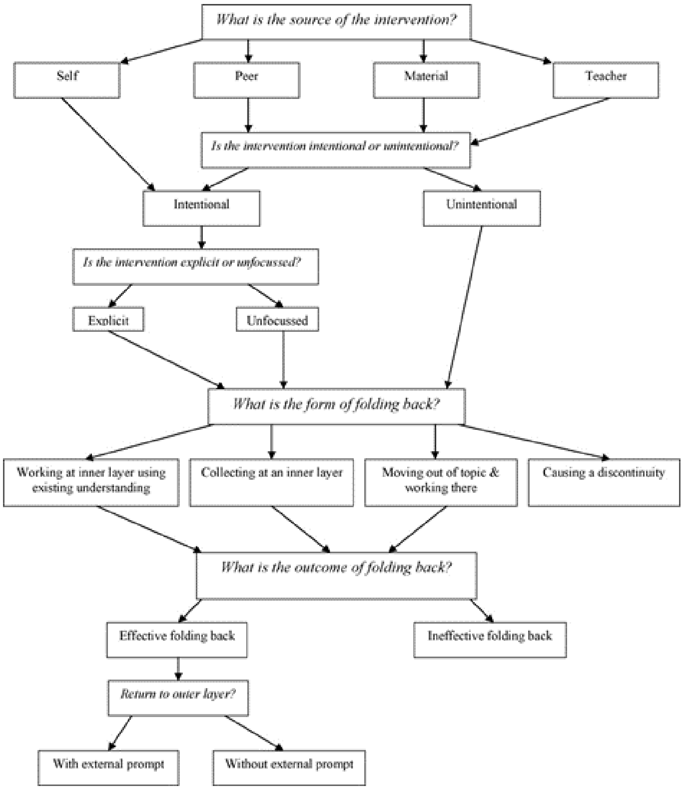

Figure 2. The framework for folding back.

COLLECTIVE IMAGE MAKING, COLLECTIVE IMAGE HAVING AND COLLECTIVE FOLDING BACK

While Pirie and Kieren's work offers a powerful theoretical framework for observing the growth of mathematical understanding, their model is still primarily one for the growth of personal understanding. Nevertheless, the modes of understanding can, with elaboration, also offer a powerful language through which to talk about the growth of collective mathematical understanding. This extension to the theory is extensively presented in Martin and Towers (2015) and Towers and Martin (2014). In particular, and through drawing on ideas from improvisational theory, these papers extend the notions of *Image Making*, *Image Having*, *Property Noticing* and *Folding Back* to incorporate collective action. Again, only brief descriptors of these elaborations, relevant to the data extracts subsequently discussed are provided here.

In *Collective Image Making* (CIM) no single learner (working as a part of a group) recognizes or is able to identify or engage in an appropriate action necessary for the making of a useful and appropriate image for the mathematical concept. Instead, what is seen in CIM is the offering, by individuals, of partial fragments of ideas and understandings, which are then picked up, elaborated, and acted on by others in the group. This process of interweaving individual contributions to create and make a coherent shared idea or representation is what gives the image making its collective nature.

In *Collective Image Having* (CIH), the group is at a point where they have a useable and workable idea—and nothing new is being introduced. A collectively chosen focus has been followed and built upon by the interweaving contributions of the group members and now emerges as something useable by the group in the context of the task at hand. What is significant here is that this image is not one that is held only by one individual, but one that, through collectively building on an idea, is distributed and shared across the group. There is a sense that everyone is 'on board' with the emerging direction and that no explanations of why particular actions are being taken are necessary. Typically, as we observe the shift from CIM to CIH, we see participants shifting from working on how they might solve the problem (CIM) to actually solving it using their image (CIH).

In *Collective Folding Back* (CFB) there are points in the mathematical activity of the group where an existing understanding, for example an image, is sometimes no longer viable or useable (often because it is too local or specific in nature). Thus, the group needs to return to CIM and to remake, rework, or rebuild their image. To do this, there needs to be agreement around the need to fold back and also as to what the new image making actions will involve. This willingness and capacity to build on a better idea and to alter the current way of acting is a collective process.

To illustrate CFB, consider the data from a problem-solving session involving three students, aged between nine and ten years old, in which they were working on the well-known 'painted cube' problem. The students were given a printed sheet with the following question, and had available a large number of small interlocking cubes. "*Imagine a large cube made up from 27 small cubes. Imagine dipping the large cube into a pot of yellow paint so the whole outer surface is covered, and then breaking the cube up into its small cubes. How many of the small cubes will have yellow paint on their faces? Will they all look the same? Now imagine doing the same with other cubes made up from small cubes. What can you say about the number of small cubes with yellow paint on*?"

The students begin the task by building a 3x3x3 cube from the smaller cubes. They decide to only build one model and actually all contribute to the physical making of the model. As they are constructing this they also start to collectively hypothesize about the solution to the problem, with individual students offering thoughts or ideas around how many cubes will have a particular number of sides painted. These ideas are reacted to, built upon, and acted on through ongoing interaction. For example as they are building the model (but before it is complete) one student says, "*there are going to be four with three painted faces*" and this is picked up by the others in the group. It is pointed out there are more than four corners, an answer of six is briefly suggested, but the group then quickly agree on eight (and verify with reference to their model). Here, they are making an image that 'whatever size the larger cube there will always be eight small cubes with three painted faces'.

The group reaches a point in their working where they are confident that they have correctly determined a solution for the 3x3x3 cube. They are then posed the question of what would happen if the cube were larger. Here they articulate a number of components of their image, including that there will always be eight small cubes with three sides painted, and that for the other numbers in a larger cube they simply need to multiply their solutions from the 3x3x3 case by some appropriate number. At this point they do not see the need to build any further physical models, as they believe they can simply predict and calculate answers from what they already know. That is, they have an image they can use without recourse to specific actions (e.g., building a 4x4x4 cube and counting).

Although the group believe that their collective image is a correct one, the observer knows it is not generalizable and that, although they are correct to be seeking a pattern, their current thinking will not give correct solutions for larger cubes. She therefore asks the group to build a larger cube "*to show me*" and they start to do this, still confident that their predictions will work. However, having built the larger cube (again something done by all three students) and then trying to apply their rules, they come to realize they are not correct. They state that, "*it doesn't make sense*" and then one student comments, "*let's count*". At this point their actions shift from being a process of using and demonstrating their image (working at CIH) to needing to rework it (thus folding back to CIM). In once again counting numbers of small cubes they have collectively folded back to work with a specific case in order to then be able to say something more general (which would be evidenced through returning to CIH). When the suggestion is made to count cubes, the group collectively sees it as appropriate—there is a sense that this is an appropriate way forward as their image is no longer viable—and all participate in the counting process. What makes this new act of CIM different (thicker) from that when working with the 3x3x3 cube initially is that they are now purposefully looking for a pattern; they are looking not to discard totally their existing images, but to modify these, through finding which elements of their more general ideas are valid and which require modifying. The group engages in a collective process to which they all contribute and it this shared action that makes folding back effective for them and enables the continued growth of their mathematical understanding.

FOLDING BACK AS A PEDAGOGICAL TOOL

While the notion of *folding back* as defined by Pirie and Kieren (1994) and elaborated by Martin (1999, 2008) speaks to the place of the teacher in the process (e.g. as a source of folding back) this is only in relation to the observed actions of a learner or group of learners. However, the potential of folding back as a pedagogical tool, that could inform the planning and teaching of lessons, was never explicitly considered. The NCTM *Learning Principle* (NCTM, 2000), that focuses on the importance of conceptual understanding, states "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (p. 20). However, it does not speak directly to specific pedagogical actions or teaching strategies that might occasion such building to occur, and it is the identification of these on which a current research project focuses. In particular the research considers how the deliberate and purposeful teaching action of encouraging *folding back* with its associated cognitive act of *thickening* (Martin, 2008) can offer a way for students to engage with specific ‘problematic met-befores’—prior knowings that may hinder understanding within a new context (McGowan & Tall, 2010)—and so promote understandings that are both more general in nature and conceptually connected.

Again, to illustrate the potential of folding back as a pedagogical tool, consider the case of Mort, an experienced teacher who, at the time of the study, was teaching in a private high school. As a part of the project he participated in a planning meeting with the research team, where potential lesson structures and plans were considered with a particular focus on where students might need to, or be encouraged to, fold back. Two designed lessons, which focused on the concept of vectors in three-dimensional space, were then taught, with two different classes, and these were observed and video-recorded by two members of the research team.

Mort introduces the content of the lesson working from a smart board at the front of the classroom. Firstly, he explicitly tells the students that he wants them to focus on making a connection between today’s work and what they did in year nine. However, he also sets this within the new context of three-dimensional space, setting out clearly that the purpose of what they will be doing is to develop three forms of an equation of a line (referring here to vector, parametric and symmetric forms) and that they will also be expected to draw on what they have done so far in the course. The students work with Mort in reviewing the $y = mx + b$ equation form and, more significantly for what will follow, the *meaning* of the different elements—in particular the notion that m represents slope. Mort asks “*what does the slope tell me*?” and follows up on a student’s offering of “*direction*” and begins to explore the idea of steepness and then the need for a point to actually locate “*where the line is*”. He consolidates this to summarize that slope gives a sense of direction and the point defines the location of the line. He then asks “*why does the idea of slope the way that we’ve defined it, the way that we understand it, not work for what we want to do in* $\mathbb{R}^3$?” A student offers the answer of the same slope having different directions in $\mathbb{R}^3$, and then provides a visual explanation of this using a pencil—something that Mort mimics with a metre stick. He then turns to the formula for slope (rise over run) and asks “*Anywhere in there, did we say z*?” before going on to say that slope is a planar concept and cannot be extended to $\mathbb{R}^3$. He asks “*what else can I use to define that direction*?” and a student offers the idea of a vector. This is taken up by Mort who then works for the rest of the lesson on using vectors to define a line, firstly in $\mathbb{R}^2$ and then in $\mathbb{R}^3$.

In considering these actions through the lens of folding back we see that at the start of the lesson the students are positioned by Mort to make an image for the equation(s) of a line in $\mathbb{R}^3$. However, in order to do this he explicitly prepares them to fold back to their existing understanding of equations of lines and work with these in the new context of $\mathbb{R}^3$. He encourages the students to collect various ‘met-befores’ (e.g. slope, $y = mx + b$, etc.) related to the equation of a line in $\mathbb{R}^2$, and he reminds them that they know a lot about this. However, this is not simply a process of revisiting these existing understandings, but of also considering which of these will be useful in $\mathbb{R}^3$ and in the developing of the vector equation. He works with the students to recall their images and understandings for equation of a line in $\mathbb{R}^2$, both in terms of the equation and also what this means in terms of actual lines and how they are defined. Mort works with, and problematizes, this existing image in $\mathbb{R}^3$. He is deliberately and clearly explicit that their existing understanding for the equation of a line in $\mathbb{R}^2$ cannot simply be transferred to $\mathbb{R}^3$. He takes the met-before of slope and sets out that it “*falls apart in* $\mathbb{R}^3$”. However, he does not simply tell the students this and move on, but instead wants to explore why the idea of slope “*in the way that we’ve defined it*” is no longer valid. In introducing vectors, and working with these initially at $\mathbb{R}^2$, Mort is thickening their understandings for an equation of a line; in order to later extend these more generally to be useable in $\mathbb{R}^3$.

While McGowen and Tall (2010) highlight the need to identify and make explicit *met-befores* they do not offer particular descriptions of what 'addressing' met-befores might mean, or of how teachers might facilitate this. Folding back, with its explicit consideration of not only the importance of earlier understandings, but also of the need to actively work on and thicken these, in the kinds of ways illustrated by the teaching actions of Mort, offers a useful theoretical and practical way to more fully detail how met-befores can be addressed in the mathematics classroom. Folding back is a cognitive action that a teacher can explicitly encourage in their students through deliberate pedagogical actions and one that can enable learners to address problematic met-befores, and not merely overcome these, but also to work *on* and *with* them, through thickening, to enable a connected, deeper understanding.

CONCLUDING COMMENTS

The Pirie-Kieren theory remains one of the most detailed frameworks for considering and describing the growth of mathematical understanding. While its original purpose, to characterize the growth of understanding of an individual learner, still remains important and valid, the elaborations discussed in this paper enhance the power and potential of the theory to more fully account for collective as well as individual action, and also to more fully account for the role of the teacher. Folding back offers a metaphorical language for both researchers and teachers to consider the place of prior knowings in the emergence of new understandings and to continue to see the growth of mathematical understanding as a complex, non-linear, recursive phenomenon.

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DE LA « CLOCHETTE DE VIGILANCE » AU CHANGEMENT DE POSTURE : QUELQUES QUESTIONS ISSUES DE LA DIDACTIQUE DES MATHÉMATIQUES ET ÉTUDIÉES DEPUIS LA DIDACTIQUE DU SENS COMMUN[1]

FROM THE USE OF AN 'ALERT BELL' TO A CHANGE OF POSTURE: SOME QUESTIONS EMERGING FROM THE DIDACTICS OF MATHEMATICS BEING STUDIED BY THE DIDACTICS OF COMMON SENSE[1]

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One of the problems faced by education consists of finding ways to ensure that students are able to export the knowledge learned in school to everyday situations. On several occasions we have observed, however, that knowledge gained in school, in particular mathematical knowledge, was not used outside the context in which it was learned when it would have been useful to do so. Instead, it was replaced with common sense, thus sometimes leading to inadequate solutions.

*The didactics of common sense was developed to study such a phenomenon of "non-usage" of knowledge learned. Among other things, this didactics looks at the importation of school knowledge, from the common sense perspective, thus reversing and complementing the usual process of exportation. A first question looking at "non-usage": how to prevent, when presenting a solution to a problem, the exclamation, "*I knew it but I did not think of it!*"? One possible answer: the acquisition for oneself of an 'alert bell'. This is to get common sense to slow down one's spontaneous action to eventually turn to scientific knowledge, such as mathematical knowledge. This didactics also studies the dynamics between common-sense knowledge and scientific knowledge in order to understand how each one can meaningfully contribute to solving a problem. As a result of this work, it appears that the heart of this issue was not so much in the dynamic relationship between two types of knowledge, but rather a matter of the posture taken by the resolver. From what perspective does he apprehend the problem, that of student, child, athlete, scout, or other? Each of these postures can lead to a different solution and, if it were possible, a worthwhile goal would be to ensure that the student voluntarily navigates through each of these postures. This would perhaps give him access to a variety of solutions between which he could choose the most appropriate one in a given context.*

[1] Cette recherche a bénéficié d'une subvention CRSH-IDR et CRSH ordinaire.

In the topic session, I discussed the evolution of the main concerns of the didactics of common sense, as well as the means used to study them. Although this didactics is not specific to mathematical problems, they have served us well as a field of investigation. I did also invite the participants to answer an online questionnaire to see if they were victims of cognitive illusions. Interested parties can also access it (www.didacttic.info) by entering in the Répondants *section* Prénom: Répondant, Nom*: A (or B if A is already taken, or C, or D, etc.) and* Mot de Passe: gcedm*.*

♦♦♦♦♦♦♦♦♦♦

Un des problèmes auxquels est confronté le monde de l'éducation consiste à trouver des moyens de faire en sorte que les élèves puissent exporter des savoirs appris à l'école dans la sphère du quotidien. À différentes occasions, nous avons cependant observé que des savoirs scolaires appris, notamment des savoirs mathématiques, n'étaient pas utilisés en dehors de leur contexte d'apprentissage alors qu'il aurait été pertinent qu'ils le soient, cédant vraisemblablement le pas au savoir de sens commun et conduisant parfois à des solutions inadéquates. C'est pour étudier ce phénomène de « non usage » d'un savoir appris que s'est développée la didactique du sens commun.

Au cours de la séance thématique, j'ai présenté brièvement l'évolution des principales préoccupations de cette didactique de même que les moyens mis en oeuvre pour les étudier. J'en ferai état dans ce texte.

La présentation a débuté par une invitation lancée aux participants à répondre, sur le site du Sens commun (www.didacttic.info), à quelques questions afin de voir s'ils étaient victimes d'illusions cognitives. Le lecteur intéressé pourra s'y rendre aussi en entrant, dans la section Répondants, le Prénom : Répondant, le Nom : A (ou B si le A est déjà pris, ou C, ou D, etc.) et le Mot de Passe : gcedm.

UN PHÉNOMÈNE DE NON-USAGE D'UN SAVOIR APPRIS : « JE LE SAVAIS MAIS JE N'Y AI PAS PENSÉ! »

Vous est-il déjà arrivé de produire une solution à un problème qui vous semblait satisfaisante, bien qu'elle ne le soit pas, puis, en voyant la « bonne » solution de vous exclamer : « *Je le savais, mais je n'y ai pas pensé* ! » ? Une telle exclamation témoigne de la surprise, et peut-être aussi de la déception, du sujet qui prend conscience qu'il n'a pas usé des moyens dont pourtant il disposait pour produire une solution adéquate. La didactique du sens commun fait l'étude de ce phénomène qu'elle nomme, *phénomène de non-usage d'un savoir appris*.

Le problème suivant est un bon candidat pour déclencher un tel phénomène : *Un bâton de baseball et une balle coûtent 1,10$ au total. Le bâton coûte 1$ de plus que la balle. Combien coûte la balle?* (Kahneman, 2011, p. 44). En effet, parmi 88 élèves âgés de 13 à 18 ans auxquels nous avons soumis ce problème, 57 ont fourni une mauvaise réponse tout en indiquant qu'ils étaient certains de leur réponse (René de Cotret & Larose, 2006). Au moment de revoir le problème avec les élèves, certains se sont exprimés en ces termes : « *Je le savais mais je n'y pas pensé* ! ». Ce commentaire témoigne de ce que ces élèves ne sont pas ignorants des savoirs utiles à la résolution du problème bien qu'ils n'en aient pourtant pas fait usage.

D'autres observations de ce phénomène de non-usage (René de Cotret, 2011) nous ont convaincue qu'il ne s'agissait pas d'un phénomène isolé et qu'il méritait une investigation. La première question qui s'est alors posée est celle-ci : Pourquoi n'utilise-t-on pas, dans la vie quotidienne, un savoir scolaire appris alors qu'il serait pourtant utile et pertinent de le faire? Nous avons choisi d'examiner cette question par l'entrée des savoirs en supposant qu'il existait un savoir « substitut » au savoir scolaire qui était convoqué lorsque le phénomène de non-usage se manifestait. Notre hypothèse de travail est que ce savoir substitut est un savoir de sens commun, celui-ci pouvant être défini comme « … un savoir intuitif et immédiat sur ce qui est raisonnable de faire, un savoir qui est culturellement acquis au cours de l'éducation ou de la pratique quotidienne » (Gueorguieva, 2002, p. 2).[2]

DU PHÉNOMÈNE AU PROBLÈME

Le fait de mettre en évidence le phénomène de non-usage, admettant que le sens commun puisse l'emporter sur le savoir scolaire appris, ne signifie pas pour autant qu'il y a problème de savoirs. Toutefois, si on se réfère à ce que préconise le ministère de l'éducation des loisirs et du sport du Québec (MELS), il semble que oui. On peut en effet lire dans le Programme de formation de l'école québécoise :

> *L'école* […] *n'est pas sa propre finalité et doit en conséquence préparer à la vie à l'extérieur de ses murs. Le décloisonnement entre l'école et son environnement encourage l'élève à entreprendre une démarche de réflexion sur l'utilité et l'applicabilité de tel ou tel apprentissage dans différents contextes. Cette réflexion est à son tour susceptible d'accroître sa capacité à transférer ses acquis dans des situations qui sont nouvelles pour lui et pour lesquelles il n'a pas encore réalisé d'apprentissages spécifiques.* (MELS, 2006, p.11)

Il semble donc que l'utilisation du savoir scolaire en dehors de son contexte d'apprentissage soit une finalité de l'école. Ainsi, la visée de l'éducation serait notamment de faire en sorte que les savoirs scolaires que les élèves apprennent soient « exportés » dans leur quotidien.

UNE « CLOCHETTE DE VIGILANCE »

Nous proposons, par la didactique du sens commun, une position réciproque et complémentaire à celle de l'exportation en cherchant plutôt à faire en sorte que, dans son action de tous les jours, le sujet fasse appel au savoir scolaire, en d'autres termes qu'il importe le savoir scolaire dans la sphère du quotidien. Ainsi, nous avons développé une stratégie visant à ce que les élèves développent ce que nous avons appelé une « clochette de vigilance », soit une attitude qui conduit à questionner sa réponse spontanée et, éventuellement, à susciter le désir d'en examiner d'autres. Par cette stratégie, nous cherchons à ce que les élèves suspendent leur action immédiate pour en questionner la validité et, le cas échéant, convoquent des savoirs scolaires qui seraient plus utiles à la résolution du problème.

La stratégie élaborée pour amener les élèves à développer cette clochette repose sur deux leviers, un levier didactique et un levier social. Le levier didactique vise tout d'abord l'apprentissage d'un nouveau savoir : il existe des illusions cognitives, étant entendu « qu'une illusion cognitive est au raisonnement rationnel ce qu'une illusion d'optique est à la perception » (Piattelli-Palmarini, 1994, p. 184). Il vise par la suite la prise de conscience d'être victime de telles illusions et la conviction qu'il pourrait en être autrement. Pour amener les élèves à réaliser qu'ils peuvent être victimes d'illusions cognitives, nous tentons de prendre explicitement leur sens commun en défaut. Pour ce faire, nous leur soumettons divers problèmes (des pièges et des leurres, cf. René de Cotret, LeBlanc, et Larose (2011) auxquels ils doivent répondre en indiquant un degré de certitude (Leclercq, 2003; Leclercq, Denis, Jans, Poumay, & Gilles, 1998). Le lecteur aura pu expérimenté quelques-uns de ces problèmes s'il a répondu, tel qu'il était invité à le faire en début de texte, aux questions du site du sens

[2] Pour une description plus élaborée de cette assise théorique du sens commun, voir René de Cotret & Larose (2006) et René de Cotret (2011).

commun. Le levier social, quant à lui, vise à transformer cette prise de conscience collective en savoir de sens commun.

En admettant, qu'à l'issue de cette stratégie, la clochette arrive à freiner l'action initiale du sujet pour la remettre en question, rien n'assure pour autant que cela engage la recherche d'une autre solution, d'où l'importance, dans un second temps, de favoriser une telle recherche.

UN CHANGEMENT DE POSTURE : « *À QUOI D'AUTRE POURRAIS-JE BIEN PENSER*? »

Le moyen que nous avons retenu pour encourager la recherche d'autres solutions à un problème s'appuie sur le changement de posture. Cette proposition découle d'une révision de notre hypothèse de travail. En effet, parallèlement à la mise en œuvre de la stratégie décrite ci-dessus, nous avons aussi étudié la dynamique présumée entre les savoirs scolaires et les savoirs de sens commun (René de Cotret, Vincent, & Larose, 2008). Les résultats de ces études ont conduit à poser une nouvelle hypothèse selon laquelle le phénomène de non-usage ne relèverait pas tant d'une mainmise du savoir de sens commun sur le savoir scolaire, mais plutôt de l'adoption de différentes postures par le sujet en fonction de ce à quoi il est sensible au moment où il résout le problème.

Nous avons en effet constaté que, selon le contexte dans lequel une question est posée à une personne, les réponses varient. Dans la foulée des travaux de DeBlois & Squalli (2002), qui ont examiné différentes postures depuis lesquelles de futurs maîtres analysent des erreurs d'élèves, nous avons défini la posture comme étant « … le rôle depuis lequel le sujet aborde une question, ce rôle sollicitant alors un certain ensemble de connaissances et pouvant changer selon le contexte pour une même question » (René de Cotret, 2014, p. 407).

Pour illustrer l'influence possible de la posture adoptée sur la solution à un problème, voyons un exemple à partir du problème suivant tiré d'un manuel de mathématiques du secondaire.

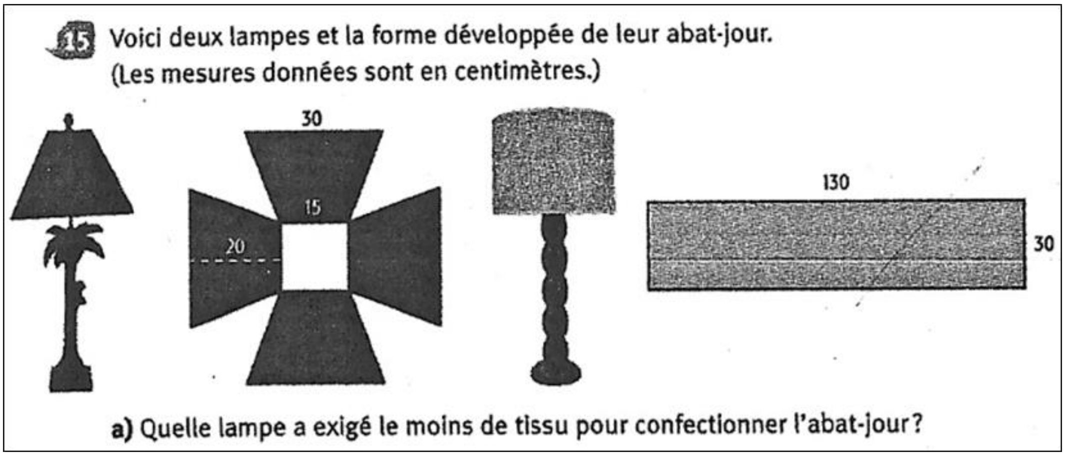

Figure 1. Problème tiré de *Perspectives, 1er cycle du secondaire, Manuel 1B*, 2004, p. 201.[3]

La posture attendue pour résoudre ce problème, si l'on se fie au guide du maître, est celle de l'élève qui étudie les aires de solides. On s'attend à ce qu'il calcule, dans un premier temps, l'aire des quatre trapèzes qui forment le premier abat-jour et celle du rectangle du deuxième abat-jour puis, dans un second temps, qu'il compare ces deux aires afin de conclure que la

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[3] Reproduit avec l'autorisation de l'éditeur, Éditions Grand Duc.

lampe qui exige le moins de tissu est celle dont l'aire de l'abat-jour est la plus petite. La solution sera tout autre si on adopte une autre posture, par exemple celle de la couturière qui doit effectivement confectionner les abat-jours. Depuis cette posture, plusieurs questions se posent afin de fournir une réponse réaliste. Il faudrait, par exemple, savoir si le tissu utilisé comporte un motif, ce qui aura une influence sur la quantité de tissu nécessaire pour permettre qu'il soit à l'endroit sur chacun des pans de l'abat-jour-jour. La largeur du rouleau de tissu est aussi importante. Si elle est plus petite que la largeur du développement de l'abat-jour de la seconde lampe (soit 130 cm) il faudra acheter une longueur d'au moins 130 cm de tissu, tandis que, dans le cas contraire, un peu plus de 30 cm suffiront. Aussi, selon que le tissu s'effiloche ou non une fois coupé, il faudra prévoir un peu plus pour faire les ourlets nécessaires sur chacun des côtés de la pièce de tissu. Le fait que le tissu soit extensible ou non viendra aussi modifier la quantité de tissu nécessaire. On pourrait imaginer encore d'autres variables importantes à prendre en considération, mais celles évoquées suffisent à montrer qu'un même problème, selon la posture adoptée pour son traitement, donne lieu à des questions et à des solutions différentes. Et, ce qui nous importe, c'est que la production et l'étude de ces différentes solutions pourraient constituer une richesse pour l'élève. En effet, si on pouvait l'encourager à chercher délibérément une variété de façons de considérer un problème, selon différentes postures, cela contribuerait à enrichir l'ensemble de solutions qu'il peut produire et, ce faisant, lui permettrait peut-être de convoquer une plus vaste gamme de savoirs.

UNE LIGUE D'IMPROVISATION MATHÉMATIQUE ? !

La question est maintenant de savoir comment amener ou encourager l'élève à passer volontairement d'une posture à une autre. Comment faire en sorte que le sujet modifie sa posture de façon délibérée de manière à ce qu'il puisse prendre la décision la plus efficace pour traiter le problème posé? Peut-on imaginer une façon de favoriser le passage volontaire d'une posture à une autre dans le traitement d'un problème, de manière à ce que l'élève puisse produire des solutions selon différents points de vue, puis en évaluer la pertinence. En d'autres termes, serait-il possible d'entraîner les élèves à changer de posture lors de la résolution d'un problème—passant par exemple de la posture d'élève à celle de couturière dans le problème précédent—de manière à ce qu'ils élaborent quelques solutions différentes puis en évaluent la validité en fonction des besoins de la posture adoptée.

Le moyen d'entraînement que nous souhaitons mettre à l'essai est *Une ligue d'improvisation en maths!* Ce jeu, inspiré de la Ligue Nationale d'Improvisation (LNI) (www.lni.ca), consisterait à demander aux élèves de résoudre un problème « *à la façon de…* ». De même qu'à la LNI, il pourrait y avoir des improvisations comparées, des improvisations mixtes mettant en scène des joueurs de deux équipes adverses et, bien sûr, des improvisations à la façon de… son ami, du professeur, de son père, etc. Un tel jeu a pour but de forcer le changement de posture faisant en sorte que les élèves puissent réaliser, d'une part, qu'il leur est possible d'effectuer un tel changement et, d'autre part, qu'il pourrait être intéressant de changer volontairement de posture afin d'examiner un problème depuis plusieurs points de vue et, en conséquence, de produire un ensemble de solutions, déjouant peut-être ainsi une potentielle illusion cognitive …

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Présentations de thèses de doctorat

SECONDARY SCHOOL MATHEMATICS TEACHER CANDIDATES' RESEARCH, PEDAGOGICAL, AND CONTENT KNOWLEDGE

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ABSTRACT

University-based initial teacher education aims at instilling in teacher candidates the idea of the interconnectedness of the knowledge of content, pedagogy and educational research by allowing meaningful interaction between teacher candidates and teacher educators. The theory-practice divide—the disconnectedness between teacher candidates' university coursework experiences and their school practicum experiences—is presented in the literature as a barrier to achieving this goal.

This mixed methods research study re-conceptualizes the theory-practice divide from a problem into an opportunity. Secondary school teacher candidates can use contradictions and tensions, surrounding the theory-practice divide, for synthesizing diverse perspectives on the knowledge of content, pedagogy and educational research and for integrating the emerged balanced perspective in their practice teaching.

The study examined secondary school teacher candidates' perspectives on the interaction of their knowledge of content, pedagogy and educational research in practice teaching as well as factors contributing to these perspectives. The study found that participants' different perspectives on these three types of knowledge were associated with the different levels of their reform-mindedness, defined as the propensity towards reform ideals in mathematics education and measured by the attitudes and beliefs survey. The low, medium and high reform-minded participants placed as the first priority pedagogical knowledge, content knowledge and educational research knowledge, respectively.

INTRODUCTION

In its *Accord on Initial Teacher Education*, the Association of Canadian Deans of Education (ACDE) states that the reason for initial teacher education to be university-based is to "allow the meaningful interaction of student-teachers with research-oriented faculty and to promote awareness of the interconnected nature of theory, research, and practice in the profession" (ACDE, 2005, p. 2). One of the principles of initial teacher education in Canada is the development of teacher candidates' "pedagogical knowledge and academic content knowledge as well as an introduction to research and scholarship in education" (ACDE, 2005, p. 2). Understanding the nature and processes of educational research is the "key to better conceptualizing, describing, and documenting its contributions" to teaching practice (ACDE, 2010, p. 2).

An effective initial teacher education program provides opportunities for teacher candidates to collaborate with teacher educators in “interweaving theory, research, and practice” and supports “a research disposition and climate that recognizes a range of knowledge and perspectives” (ACDE, 2005, p. 4). To fulfill these ACDE recommendations, it is crucial to add teacher candidates’ voices to the scholarly discourse on the role of their knowledge of subject-matter content, pedagogy and educational research in the process of learning to teach within Canadian initial teacher education programs. The purpose of this study is to understand more deeply the secondary school mathematics teacher candidates’ perspectives on interactions between their knowledge of content, pedagogy and educational research during teaching practicum at schools, as well as to identify factors influencing these perspectives.

THEORETICAL FRAMEWORK

As part of the theoretical framework of this study, I developed the Research Pedagogical and Content Knowledge conceptual framework (RPACK) and ‘a teacher candidate’s reform mindedness in mathematics education’ theoretical construct.

RESEARCH PEDAGOGICAL AND CONTENT KNOWLEDGE CONCEPTUAL FRAMEWORK

The Research Pedagogical and Content Knowledge (RPACK) conceptual framework is designed to describe how teacher candidates synthesize their knowledge of content, pedagogy and educational research and integrate them into practice teaching secondary school mathematics. The RPACK conceptual framework can be visualized as a Venn diagram (Figure 1). In the process of learning, teacher candidates interact with social others (e.g., teacher educators, other teacher candidates, school students) and the environment. During the process of this interaction, teacher candidates build their knowledge of content, pedagogy and educational research as a tool for teaching.

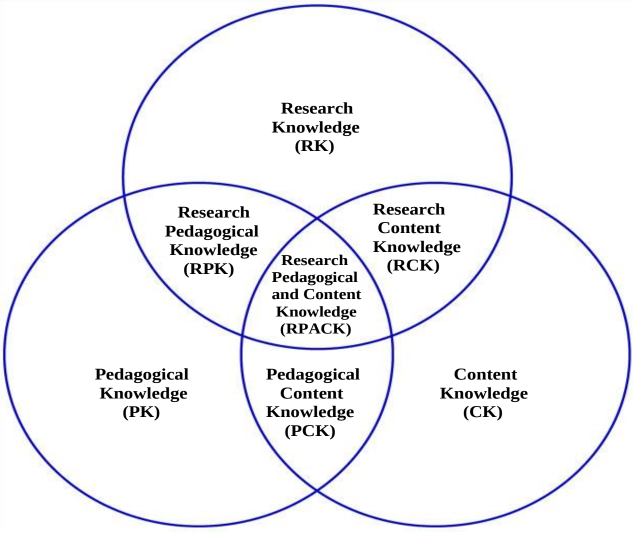

Figure 1. The RPACK framework and its knowledge components.

In Figure 1, Pedagogical Content Knowledge (PCK) is the knowledge of content specific pedagogy (Shulman, 1986, 1987). Research Pedagogical Knowledge (RPK) is the knowledge of how educational research can support pedagogical goals. Research Content Knowledge (RCK) is the knowledge of content specific educational research. Finally, the synergy and synthesis of all three types of knowledge produces teacher candidates' RPACK as a tool for teaching.

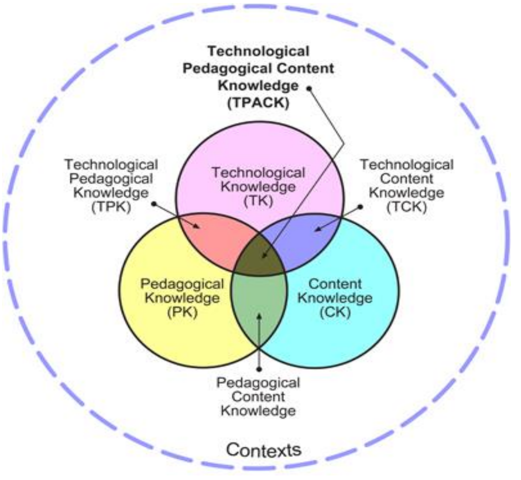

Figure 2. The TPACK framework and its knowledge components, adapted from Koehler and Mishra (2009, p. 63).

The three-component RPACK's frame structure was adapted from Koehler and Mishra's (2009) *Technological Pedagogical and Content Knowledge* (TPACK) conceptual framework visualized as a Venn diagram (Figure 2). Both RPACK and TPACK draw on Shulman's construct of *Pedagogical Content Knowledge* (PCK), which is often used in research on teacher knowledge. Koehler and Mishra (2006, 2009) and Niess (2005) used Shulman's PCK construct in developing TPACK. The essence of TPACK is in integrating the three types of knowledge (technology, pedagogy, and content) in teaching. In Figure 2, *Pedagogical Content Knowledge* (PCK) is the knowledge of a content specific pedagogy (Shulman, 1986, 1987). *Technological Pedagogical Knowledge* (TPK) is the knowledge of how technology can support pedagogical goals. *Technological Content Knowledge* (TCK) is the knowledge of how a subject matter is transformed by the application of technology. Finally, the integration of all three types of knowledge in teaching produces *Technological Pedagogical and Content Knowledge* (TPACK).

TEACHER CANDIDATES' REFORM MINDEDNESS IN MATHEMATICS EDUCATION

I developed the *reform mindedness* construct within the context of the mathematics education reform in North America. In 1983, *A Nation at Risk* and *Educating Americans for the 21st Century* reports argued for higher expectations in mathematics achievements for students from all social, ethnic and economic backgrounds, not for elite students only (Romberg, 1997). The National Council of Teachers of Mathematics (NCTM) responded to the reports by publishing the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 2000). This publication triggered a wave of empirical studies on the impact of the implementation of the NCTM *Standards* in school mathematics (Ross, Hogaboam-Gray, McDougall, & Le Sage, 2003). Based on their synthesis of 154 empirical studies conducted from 1993 to 2000 on the implementation of the *Standards*, Ross, Hogaboam-Gray, McDougall, and Bruce (2002) and Ross, McDougall, and Hogaboam-Gray (2002) generated a nine-dimension framework of standards-based teaching in mathematics.

This framework included the following dimensions: (1) program scope, (2) student tasks, (3) constructivist approach, (4) teacher's role, (5) manipulatives and tools, (6) student-student interaction, (7) student assessment, (8) teacher's conceptions of math as a discipline, and (9) student confidence. For each dimension, Ross et al. (2003) developed one or more statements reflecting the essence of the dimension. These statements became the items of the questionnaire in the survey. For example, dimension 4 (teacher's role) was connected to the following two statements (questionnaire items 5 and 17): "*I often learn from my students during math time because my students come up with ingenious ways of solving problems that I have never thought of*" (item 5); and "*I teach students how to explain their mathematical ideas*" (item 17). The questionnaire contained twenty 6-point Likert-type items, ranging from *Strongly Agree* to *Strongly Disagree*.

The construct 'a teacher candidate's reform mindedness in mathematics education' is defined by this 20-item survey's questionnaire. This construct is operationalized by the levels of reform mindedness. Each level is identified by the survey score as its numerical characteristic. The higher survey score corresponds to the higher level of reform mindedness. In this study, the construct 'a teacher candidate's reform mindedness in mathematics education' with its numerical characteristic is viewed as the independent variable. The dependent variable is the ranking given by teacher candidates' to the three types of knowledge—(1) content (mathematics), (2) pedagogy and (3) educational research—within the RPACK conceptual framework. In this study, the reform-mindedness construct is used to identify three cases of participants (low, medium and high reform-minded teacher candidates) depending on participants' survey scores.

METHODOLOGY

This is a developmental sequential two-phase mixed methods research study. Mertens (2003) and Punch (2005) suggest that one of the viable reasons for engaging in mixed methods research is for the purpose of using quantitative data to inform further qualitative study that enables the extension of findings of the former. The sole purpose of the initial quantitative phase in a mixed methods research study can be to obtain some numerical characteristics of individuals to "guide the purposeful sampling of participants for a primarily qualitative study" (Creswell, Plano Clark, Gutmann, & Hanson, 2003, p. 227). A mixed methods study in which the results of one method are used to inform the development of another "represents long-standing inquiry practice in multiple methodological traditions, now brought under the mixed methods umbrella" (Greene, 2007, p. 126). The present study utilizes a developmental design that uses the results of one method "to inform the development of the other method, where development is broadly construed to include sampling and implementation, as well as actual instrument construction" (Greene, 2007, p. 102). The design of this developmental mixed methods research study has a sequential two-phase structure (Creswell & Plano Clark, 2007).

In Phase 1, the quantitative phase of the study, I obtained the numerical characteristics of participants to guide purposeful sampling in Phase 2, the qualitative phase that was a case study. Purposeful sampling is the research term used for qualitative sampling (Creswell, 2012). Several qualitative sampling strategies exist. I use the *maximum variation cases sampling strategy* (Creswell, 2012; Patton, 2002). In this strategy, the researcher samples cases that "differ on some characteristic" (Creswell, 2012, p. 208). In my study, this strategy can be viewed as a combination of extreme case sampling and typical case sampling. Extreme case sampling is a form of purposeful sampling in which the researcher studies "an outlier case or one that displays extreme characteristics" (Creswell, 2012, p. 208). The purpose of typical case sampling is to "describe and illustrate what is typical to those unfamiliar with the setting—not to make generalized statements about the experiences of all participants" (Patton, 2002, p. 236). Typical cases can be selected using "survey data, a demographic analysis of averages, or other statistical data that provide a normal distribution of characteristics from which to identify 'average-like' cases" (Patton, 2002, p. 236).

In Phase 2, I have an instrumental case study (Stake, 2005) with the three cases. In an instrumental case study, "the researcher focuses on an issue" (Creswell, 2013, p. 99). In an instrumental case, the issue is central, and the case is secondary (Stake, 2005). In my thesis, the issue is the reform-mindedness of teacher candidates in mathematics education. The level of the reform-mindedness is described by a certain numerical parameter that is measured by a survey. The cases are bounded by this parameter that stays within the fixed 1-120 range of the scores according to the design of the survey. The system is a case when it "can be bounded or described within certain parameters" (Creswell, 2013, p. 98).

MAJOR FINDINGS

1. Participants' different perspectives on their knowledge of content, pedagogy and educational research, viewed through the theoretical lens of the RPACK conceptual framework, were associated with the different levels of their reform mindedness in mathematics education as measured by the attitudes and beliefs survey. The three sets of participants (low, medium and high reform-minded teacher candidates) were identified.
2. The low reform-minded participant placed pedagogical knowledge as the first priority (Figure 3). He placed content knowledge and educational research knowledge as the second and the third priorities, respectively.

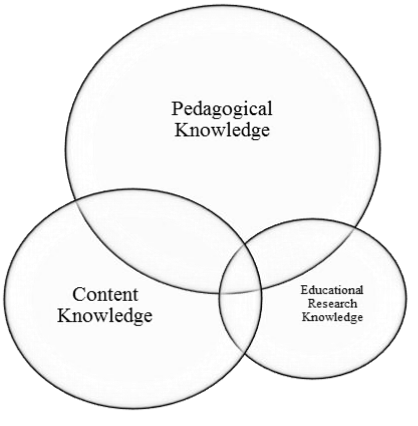

Figure 3. The teacher candidates' RPACK model in Case 1.

3. The medium reform minded participants placed content knowledge as the first priority (Figure 4). They placed pedagogical knowledge and educational research knowledge as the second and the third priorities, respectively.

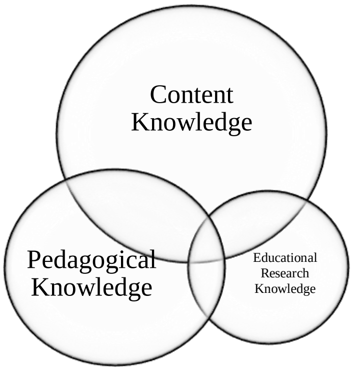

Figure 4. The teacher candidates' RPACK model in Case 2.

4. The high reform-minded participants placed educational research knowledge as the first priority (Figure 5). They viewed the knowledge of content and pedagogy as influenced by educational research knowledge. The high reform minded participants emphasized the importance of synthesizing all three types of knowledge.

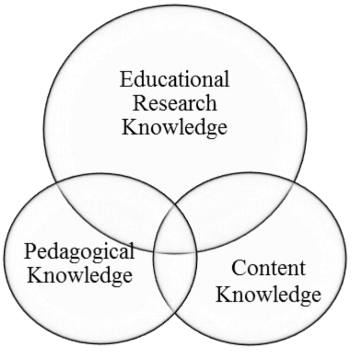

Figure 5. The teacher candidates' RPACK model in Case 3.

CONCLUSION

The findings of this study suggest that the idea of the interconnectedness of the knowledge of content, pedagogy, and educational research can be instilled in teacher candidates by developing their reform-mindedness in mathematics education through the re-conceptualization of the theory-practice divide from a problem into an opportunity. Teacher educators should encourage secondary school mathematics teacher candidates to reflect on contradictions and tensions, surrounding the theory-practice divide, and to synthesize diverse perspectives on the knowledge of content, pedagogy and educational research for integrating the emerged balanced perspective in their practice teaching.

Teacher educators should capitalize on teacher candidates' prior disciplinary research knowledge and experiences to help them to spark their interest in reading original educational research papers and to become the informed consumers of educational research. Teacher educators should encourage teacher candidates to share and compare their different educational backgrounds and worldviews with the intention to help them to be better prepared for teaching in culturally and socially diverse secondary school classrooms.

This research is the first step in the program of studies on teacher candidates' experiences in the process of learning to teach mathematics curriculum to school students of different ages in various types of initial teacher education programs. A valuable contribution to the literature on initial teacher education can be produced by a future study that focuses on the evolution of teacher candidates' perspectives on their knowledge of content, pedagogy and educational research and factors influencing this evolution; in this future study, participants should be surveyed and interviewed at least at three points in time: at the beginning and the end of the initial teacher education program, as well as at the end of the first year of participants' teaching at schools.

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ÉTUDE DES SIGNIFICATIONS DE LA MULTIPLICATION POUR DIFFÉRENTS ENSEMBLES DE NOMBRES DANS UN CONTEXTE DE GÉOMÉTRISATION… ET UN APERÇU DE QUELQUES RÉFLEXIONS D'ORDRE ÉPISTÉMOLOGIQUE

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INTRODUCTION

Dans cette communication je donne un aperçu du travail de recherche théorique et expérimentale réalisé dans le cadre de ma thèse de doctorat ainsi que de certaines réflexions d'ordre épistémologique sur le travail mathématique des élèves. Mon travail de thèse porte sur une étude de la multiplication pour différents ensembles de nombres dans un contexte de géométrisation et de médiations sémiotique et sociale (Radford, 2004; Sfard, 2008). Dans ce contexte, nous avons voulu déterminer et analyser le travail mathématique des élèves pour ainsi rendre compte de leur parcours tout au long d'une séquence d'apprentissage proposée par l'enseignant. Plus spécifiquement, nous cherchions à identifier et à différencier des interactions entre les composantes de l'espace de travail mathématique personnel des élèves rendant compte d'une compréhension géométrique de la multiplication pour différents ensembles de nombres. Les analyses de ce que nous avons appelé des *parcours d'individus*, ont été effectuées sous le regard de l'approche théorique des Espaces de Travail Mathématique (Kuzniak, 2011, 2014) et des éléments associés à la Médiation Sémiotique (Bartolini Bussi & Mariotti, 2008). Le terme *individu* représente un groupe d'élèves. Il a été utilisé puisque nous avons donné à chaque groupe le statut d'une entité. Ainsi, les parcours de chaque entité avaient pour but de rendre compte d'expérience de médiation, de collaboration et d'une appropriation spécifique de l'espace de travail mathématique qui leur a été proposé.

Dans un premier temps, je présente dans cet article une brève synthèse de l'articulation théorique développée dans le cadre de ma thèse. Dans un deuxième temps, je donne à titre d'exemple, des extraits et des analyses d'un des parcours étudiés. Enfin, ouvrant sur une réflexion d'ordre épistémologique sur le travail mathématique des élèves, je propose de porter une attention particulière à l'espace de travail mathématique *approprié* par l'élève, appropriation qui résulte du pouvoir générateur des mathématiques et qui rend compte d'une multiplicité d'actions qui révèlent une réinvention permanente de l'espace de travail mathématique.

POURQUOI PARLER D'UN ESPACE DE TRAVAIL MATHÉMATIQUE ?

L'approche théorique des Espaces de Travail Mathématique (ETM) (Kuzniak, 2011) m'a particulièrement intéressée puisqu'elle cherche à étudier des éléments épistémologiques et cognitifs impliqués dans tout travail mathématique.[1] Dans ce cadre, le travail mathématique est organisé et d'une certaine façon contextualisé par l'interaction de plusieurs composantes qui en sont représentatives. Pour fonctionner, les ETM supposent une mise en réseau de ses composantes à partir de différentes genèses qui peuvent être de nature *sémiotique*, *instrumentale* et *discursive*.[2] Cette mise en réseau se produit grâce à l'appropriation que l'élève fera de l'ETM qui lui a été proposé par l'enseignant. De ce fait, il existe une diversité d'espaces de travail dans une classe de mathématiques, à l'intérieur desquels l'articulation des composantes peut être de nature différente. D'un point de vue classique, le départ de l'articulation de composantes dans un ETM se produirait au niveau épistémologique. Notre analyse de l'approche théorique de l'Espace de Travail Mathématique nous a permis de proposer une entrée par le plan cognitif, laquelle enrichirait le regard en ce qu'elle permet une ouverture à des nouvelles composantes. Ces nouvelles composantes favoriseraient une lecture alternative du travail mathématique des élèves quand ce travail porte particulièrement sur un processus de recherche mathématique. Je me réfère, par exemple, à la mise en relation entre la visualisation en géométrie d'une notion mathématique et sa représentation non géométrique. La visualisation pourrait ainsi être vue comme le résultat d'une interprétation métaphorique du concept mathématique en jeu (Lakoff & Núñez, 2000), laquelle donne du sens à d'autres représentations du même concept. De ce fait, les métaphores viennent aussi habiter les ETM des élèves comme des points de référence (Kilhamn, 2011), et elles peuvent jouer un rôle important dans la mise en fonctionnement des autres composantes à travers lesquelles les élèves font la rencontre des mathématiques.

L'ESPACE DE TRAVAIL MATHÉMATIQUE ET LA MÉDIATION SÉMIOTIQUE

Selon Vygotsky, dont la théorie socioculturelle est un des fondements de la Théorie de la Médiation Sémiotique (Bartolini Bussi & Mariotti, 2008), il y a une relation inhérente entre l'activité interne et l'activité externe des individus. L'internalisation de fonctions par lesquelles l'activité—en contexte social—se réalise permet de caractériser l'apprentissage (Wertsch & Addison Stone, 1995). Les signes, en tant que médiateurs dans le processus complexe visant la formation d'un concept (Vygotsky, 1934/1997) acquièrent un statut privilégié puisqu'ils favorisent l'interaction entre différents éléments lors de la rencontre des mathématiques, par exemple, notamment dans le cas de notre recherche, entre la représentation et la visualisation d'une notion mathématique. Dans ce cadre, Noss et Hoyles (1996) discutent l'idée de « *signes situés* », lesquels deviendraient constitutifs du processus de conceptualisation des mathématiques vécu par les élèves. Ces signes peuvent ainsi devenir des médiateurs visuels, « *defined as providers of the images with which discursants identify the object of the talk and coordinate their communication* » (Sfard, 2008, p. 147). Le rôle essentiel ainsi attribué à la médiation sémiotique, au niveau épistémologique et didactique, nous a conduits à situer l'Espace de Travail Mathématique dans un contexte socioculturel de l'apprentissage. Dans ce contexte, les dimensions sociale et sémiotique s'articulent dans la zone de développement proximale définie par Vygotsky (1934/1997). C'est ainsi qu'un autre regard théorique est venu compléter l'étude de certains enjeux propres au travail mathématique des élèves lors qu'ils résolvent de problèmes en contexte de collaboration. Nous avons rendu explicite la présence d'un processus de médiation sémiotique dans un contexte d'interactions sociales, au cœur de l'Espace de Travail Mathématique.

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[1] Pour plus de références voir Kuzniak (2004 – 2014)

[2] Sur ce lien vous trouverez le schéma représentant un Espace de Travail Mathématique (Kuzniak, 2014) https://drive.google.com/file/d/0B2PIBsYMh2gCSHhKNVJuYWJSYVE/view?usp=sharing

Le processus de médiation sémiotique viendrait faire émerger les différentes genèses qui articulent les deux plans de l'ETM, favorisant, par exemple, la mise en relation entre représentation et visualisation ou entre métaphores et référentiel théorique. En conséquence, la présence explicite d'un processus de médiation sémiotique dans un contexte d'interactions sociales a élargi les critères de ce que nous voulions observer et analyser dans le travail mathématique des élèves. Ainsi, des analyses *a priori* de l'ETM à proposer et aussi *a posteriori* de l'activité mathématique effective des élèves—i.e. leur appropriation de l'ETM proposé—nous permettaient de déterminer de quelle façon ces médiations peuvent favoriser l'évolution partagée de connaissances, lorsque plusieurs élèves résolvent un problème mathématique (Bartolini Bussi & Mariotti, 2008).

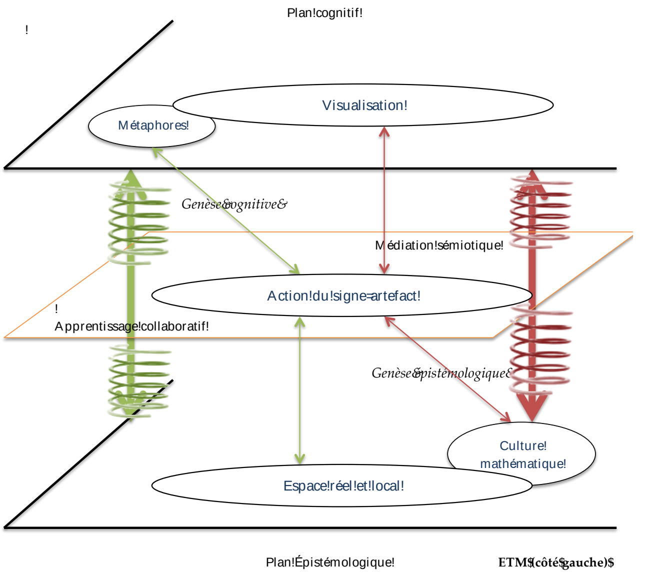

Figure 1. Articulation des plans cognitif et épistémologique de l'ETM et la présence d'un processus de médiation sémiotique et sociale.

UNE PHASE EXPÉRIMENTALE POUR ÉTUDIER L'ESPACE DE TRAVAIL MATHÉMATIQUE EN CONTEXTE DE MÉDIATION SÉMIOTIQUE

La phase expérimentale de ma thèse consiste à la mise à l'épreuve d'une séquence d'apprentissage conçue pour étudier—dans un contexte de médiations sémiotique et sociale—la construction géométrique de la multiplication pour différents ensembles de nombres. Dans le cadre de cette communication je ne vais que présenter un aperçu de l'analyse d'une partie de cette séquence.[3] L'expérimentation a été mise en place en Ile de France dans quatre classes de Terminale S. Les élèves ont eu à résoudre une suite de cinq questions proposant une approche géométrique de la multiplication de nombres réels et complexes. Dans un premier temps, les élèves effectuent la construction géométrique du produit de deux nombres réels dans le cadre proposé par Descartes dans sa *Géométrie*. Puis, il leur était demandé de trouver une relation entre des points donnés dans le plan et la multiplication de deux nombres complexes. En s'appuyant sur ces activités, notre but était de décrire et de caractériser des parcours d'individus dans l'ETM en précisant notamment leurs possibilités de rencontrer des

[3] Sur ce lien vous trouverez la totalité de la séquence de tâches proposée aux élèves https://drive.google.com/file/d/0B2PIBsYMh2gCMGlkU0lKbXVyOUE/view?usp=sharing

significations géométriques de la multiplication à partir de l'analyse de la représentation géométrique de la multiplication de Descartes. Pour décrire le rôle de la géométrisation dans l'approche de la multiplication chez les élèves, nous avons étudié leur manière de résoudre des problèmes de construction géométrique mettant en jeu la multiplication des nombres réels et des nombres complexes. Petit à petit nous avons été amenés à chercher et à déterminer les interactions entre les plans cognitif et épistémologique de l'ETM approprié par les élèves, pour ainsi rendre compte de la rencontre de la multiplication géométrique de deux nombres complexes.

L'ESPACE DE TRAVAIL MATHÉMATIQUE APPROPRIÉ PAR LES ÉLÈVES

Une des tâches posées aux élèves dans cette séquence consiste à lire la multiplication de deux nombres complexes construite géométriquement (tâche 4.a – Annexe 2). La tâche suivante leur demande de construire le produit de deux autres nombres complexes (tâche 4.b – Annexe 2). L'objectif spécifique de ces tâches était de mettre en relation les propriétés de la multiplication de deux nombres complexes et la représentation géométrique qui résulte de leur mise en œuvre. Les élèves ont les traces numériques et algébriques de ces propriétés puisqu'ils les ont déjà dégagées dans la question précédente. Plus tôt dans la séance, les élèves avaient vérifié que la multiplication géométrique pour différents ensembles de nombres pouvait être prouvée par l'existence d'une relation de proportionnalité entre les facteurs, l'unité et le produit (ils font référence à cela en parlant de « Thalès »). Ainsi, la recherche d'une mise en relation entre le produit complexe et une situation de proportionnalité—géométriquement représentée—favoriserait non seulement la rencontre de la signification géométrique de ce produit mais viendrait aussi la valider. Les élèves reçoivent de temps en temps quelques informations de la part de l'enseignante, ce sont surtout des interventions qui éclairent certaines questions qui pourraient « bloquer » leur travail.

Figure 2. Groupe A.

Dans ce contexte, l'activité mathématique des élèves résultant de leur appropriation (*interprétation et participation volontaire*) de l'Espace de Travail Mathématique proposé, rend compte de plusieurs interactions par lesquelles différentes notions mathématiques se mélangent, se combinent ou semblent même s'opposer. Les élèves articulent la représentation géométrique de la multiplication de deux nombres complexes et les propriétés exprimant des relations entre les facteurs et le produit en proposant différentes façons de faire. Leur travail se déroule autour des allers-retours entre l'énoncé de la question, une figure, des discours, la représentation géométrique de deux nombres complexes, les représentations numériques ou algébriques des propriétés du produit des nombres complexes et les éléments de base sur lesquels ils font leur construction, soit, la représentation géométrique de deux nombres complexes. L'appropriation de l'ETM par les élèves les conduisant vers la construction recherchée met en évidence des gestes qui articulent, *comme dans une sorte de bricolage,* le numérique et le géométrique : serait-il possible pour les élèves d'établir des relations entre le pôle propriétés de la multiplication de nombres complexes et les éléments de réponse concernant les transformations géométriques ? Comment les propriétés de la multiplication de nombres complexes ont influencé le travail des individus surtout face aux questions portant sur les significations géométriques de la multiplication et sur les constructions ? La visualisation de la multiplication comme une transformation dans le plan, a-t-elle émergée dans l'Espace de Travail Mathématique approprié par les élèves ? Et que peut-on dire sur la médiation résultant du travail collaboratif ? Notre travail a permis d'avancer dans la recherche d'éléments de réponse à certaines de nos questions même si ce travail n'est pas encore été terminé. Pour rendre compte d'éléments du déroulement de l'une de nos séances expérimentales, je vais donner quelques exemples du travail mathématique d'un groupe d'élèves—groupe A—portant sur le processus de construction géométrique de la multiplication de deux nombres complexes. Je présente des transcriptions et des analyses de certains de leurs échanges verbaux ainsi que certaines de leurs productions écrites.

GROUPE A

Les échanges entre les élèves (A, B, C, D, E) semblent cumuler des éléments qui pourraient ou non devenir les fondements du travail mathématique demandé et donc de la construction évoquée. La tâche semble très complexe. L'extrait suivant en donne un aperçu :

Extrait 1

A: En fait, il faut construire le... produit de ça.

A: En multipliant l'un et en ajoutant l'autre (*rires*).

B: En fait, c'est simple. Ah, mais je sais, j'ai trouvé. (*Ils lisent et relisent la question, puis ils reviennent en arrière pour regarder leur réponse à la question précédente*).

C: Attendez, dans quel sens déjà on donne les [...]

B: On représente sur le graphique, alors... Il faut construire la représentation, tu vois ?

A: (*C fait une construction où le produit ne correspond pas au produit de modules car la longueur est trop courte. La position n'est pas correcte non plus car la droite où le produit a été placé ne correspond pas à une addition des angles des facteurs. Cette droite correspond à celle qui a été placé par B et que nous décrivons par la suite.*) Mais c'est tout petit !

B: Déjà voilà, c'est ça (*B trace une droite qui correspond à un élargissement dans le sens négatif du vecteur OA. Cette droite correspond exactement à la droite portant le produit dans la représentation géométrique donnée dans la question 4.a ; En même temps, C réécrit en langage algébrique les propriétés de la multiplication de nombres complexes : somme d'angles et produit de modules. Puis tous les autres vont tracer la même droite que B*).

P: (*L'enseignante fait une remarque à toute la classe car plusieurs groupes étaient restés très longtemps bloqués dans la dernière petite question de la question 4.a*.) Vous m'écoutez deux secondes ? […] On vous demande juste les normes, entre elles, entre les normes des facteurs et les normes du produit, et entre les angles des facteurs et l'angle du produit. Et après c'est tout.

Le travail des élèves ici est très local et l'enseignante essaie de mieux cibler la question pour qu'elle devienne plus accessible aux élèves. Leurs questions se centrent sur *la forme*, par exemple *nommer ou non un point, comment le faire* ou *où le placer*. Néanmoins des aides et des commentaires sont bien partagés entre les élèves qui, petit à petit, modifient leurs constructions géométriques. Ils font un pas puis un autre en faisant avancer leur recherche même si, à plusieurs reprises, leurs réponses semblent découpées. Pour avancer, nous les voyons en quelque sorte retourner sans cesse aux points de départ du problème ainsi qu'aux idées mathématiques déjà travaillées : faire un graphique, trouver un produit, tracer une droite, etc. Nous pourrions dire qu'ils se trouvent dans ce que Nishida (1870-1945) appelle *un fait* : l'union de pensée et perception dans le processus de retourner aux fondements infinis des mathématiques, toujours en mouvement (Dalissier, 2009). Le travail mathématique des élèves semble se situer dans un contexte de perception pure, dans la recherche d'une intuition qui accompagne toujours le travail mathématique des chercheurs. Les élèves mobilisent des outils, réfléchissent, se questionnent, effacent, recommencent. Ils font intervenir des mathématiques *déjà là*, c'est-à-dire celles qui font partie de leur histoire et qu'ils font *émerger dans leur travail présent*. Dans ce contexte, à plusieurs reprises tout au long du travail réalisé pendant la séance, certains élèves cherchent des modèles qui pourraient expliquer ce qui se passe, d'autres font de conjectures à partir des éléments présents. Ils changent aussi de direction, ils reviennent sur leur pas, leurs mesures, les nombres choisis : ils refont, ils font ensemble. Même s'ils reviennent, s'ils répètent des actions ou s'ils reprennent des modèles, ils sont au cœur de la réinvention. Il s'agit d'une réinvention partagée, une réinvention en collaboration. Tout au début de la séquence, le théorème de Thalès n'était qu'un outil pour la preuve de la multiplication de Descartes. Plus tard, les composantes de l'icône le représentant paraissent intégrées au discours mathématique associé à la construction géométrique de la multiplication. De ce fait, nous avons notamment observé l'intention de prendre en compte les droites parallèles constituant l'icône du théorème de Thalès : d'après leur discours, les élèves avaient l'intention de prendre en compte « la propriété de Thalès ». De cette manière, l'évolution de notre signe-artefact s'est bien manifestée dans le discours d'un des élèves ; néanmoins, elle reste limitée et même, à certains moments, bloquée à cause de la forte influence d'une pensée habituée à la mise en place de recettes—en termes plus élégants, de techniques—ou encore, au besoin de suivre un bon modèle :

<u>Extrait 2</u>

C: Parallèles...

C: On a des parallèles ici.

A: Il faut faire la parallèle ici, non ?

C: Je montre la parallèle, là, non ?

D: C'est la parallèle. Sinon on ne pourra pas utiliser [...]

D: En traçant la droite, et en traçant la parallèle ici... (*D signale dans sa feuille, avec une équerre, deux droites parallèles : l'une d'entre elles corresponde à la droite passant par les points U et A. Sa parallèle passe par le point B et coupe OA permettant, de cette façon, de trouver « le produit » cherché*).

Il y a bien du bricolage, des choses trouvées, du partage des idées, mais il est impossible de savoir si elles sont pertinentes ou non avant d'avoir terminé la construction demandée (Barrera Curin & Maheux, 2014). L'œuvre finale apparaît donc comme le résultat de ce partage et de la médiation résultant de leurs échanges. Les élèves cherchent non seulement la réponse à l'énoncé mais aussi « ce qu'il faut faire », résultant du contrat didactique de la classe (Brousseau, 1998). De ce fait, ce qui a motivé les élèves à tracer des parallèles reste encore très flou, car l'intérêt de les utiliser pourrait juste correspondre au fait que *c'est ce qu'il faut faire*. Ainsi, nos interprétations de cette évolution restent au niveau de conjectures puisque le discours des élèves doit nécessairement être interprété à l'aide d'une forme d'intervention qui puisse encourager sa verbalisation (Radford, 2003) et qui leur permette de développer leur discours ! Par contre, ce développement n'a pas pu se produire de façon optimale à l'intérieur des échanges que la conception de notre séance expérimentale permettait entre les élèves. Nous avons à ajouter que la place que nous avons donnée à la médiation de l'enseignant était limitée (je reviendrai un peu plus tard sur cet aspect). Si nous prêtons attention aux commentaires de l'élève D (extrait 3), nous observons que celui-ci insiste sur ce besoin de construire en respectant les « propriétés de Thalès » mais ses commentaires se perdent dans des discussions dans lesquelles ses idées ne peuvent pas se développer, parce que non prises en compte par ses camarades et non guidées par l'enseignant, alors absent :

<u>Extrait 3</u>

D: Il faut s'aider des angles afin de trouver là où il fallait tourner la droite.

B: Excuse...est-ce qu'on peut m'expliquer encore ?

D: En fait, on repart à zéro !

D: Madame P nous a conseillé de construire la droite, z", avec l'histoire des angles.

E: Tu as un rapporteur ? (*A ce moment ils se concentrent en mesurer les angles (au rapporteur) et les modules (à la règle graduée)*).

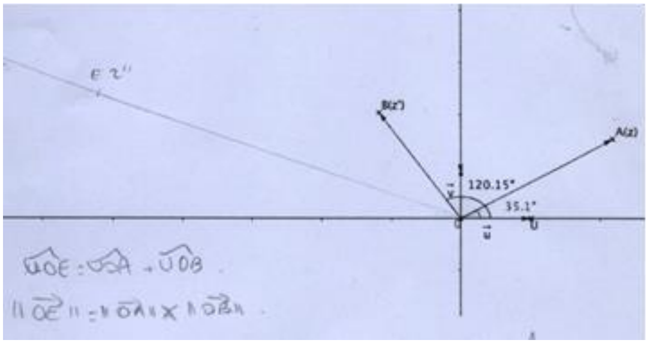

Figure 3. Construction géométrique du produit de deux nombres complexes réalisé par le groupe A.

Ils effacent leur construction avec les parallèles et ils trouvent, après beaucoup de réflexion et des calculs, la position de la droite où il faudra finalement placer le produit résultant de l'addition d'angles et du produit de modules en centimètres. Même en connaissant la mesure résultant de la somme des angles des facteurs ils n'arrivent pas facilement à placer le produit.

D: Ça me parait vraiment étrange que l'on peut pas arriver à la propriété de Thalès.

B: C'est pas forcement que j'ai utilisé Thalès à chaque annexe.

D: Mais justement on doit vérifier les mêmes propriétés que... là bas

B: Non mais c'est que tu évolues. Au début on disait Thalès, après on a parlé de normes, après on a parlé de...pour arriver aux nombres complexes [...]

Les propositions verbales et les gestes non verbaux des élèves ne rendent pas nécessairement compte d'un débat rationnel ou de la poursuite d'un chemin préétabli qui se limiterait à la recherche d'une solution particulière. À certains moments de leur débat, la discussion semble ne pas évoluer et plusieurs propositions restent sans réponse. Finalement, et suite à d'autres échanges très intéressants (Barrera Curin, 2013), les élèves ne concluent que sur une mise en œuvre littérale des propriétés géométriques du produit complexe. La complexité des processus cognitifs impliqués dans notre séquence ainsi que la diversité d'appropriations de l'Espace de Travail Mathématique au milieu d'un travail collaboratif, nous ont conduits à des réflexions portant sur des changements de variables liés à l'importance d'une médiation et d'une orchestration plus active de l'enseignant (Bartolini Bussi & Mariotti, 2008). Nous nous sommes aussi beaucoup questionnés sur la pertinence des rétroactions du *milieu* (Margolinas, 1997) que nous avions prévues lors de la conception de l'ETM à proposer.

DISCUSSION

Peut-on identifier et différencier des interactions entre les composantes de l'espace de travail mathématique personnel des élèves rendant compte d'une *compréhension* géométrique de la multiplication pour différents ensembles de nombres ? Cette question, portant directement sur le travail mathématique des élèves et sur le regard théorique sous lequel ce travail a été étudié, correspond à notre deuxième question de recherche. Une séance non-traditionnelle d'apprentissage a été conçue spécifiquement pour répondre à nos intentions et à nos fins didactiques, lesquelles portaient sur la mise en relation entre les composantes d'un espace de travail mathématique grâce à l'action de genèses sémiotiques et à la médiation sémiotique d'un signe-artefact dans un contexte de médiation sociale. Comme je l'ai déjà mentionné, nous avons intégré explicitement à l'ETM des intermédiaires, des signes médiateurs. Ainsi l'ETM a été lui-même intégré à un processus socioculturel de l'apprentissage (Vygotsky (1934/1997). Dans ce contexte, et de façon plus spécifique, notre objectif était d'étudier si les représentations géométriques, données ou demandées, reconnues comme signes-artefacts, peuvent renvoyer à un signifié mathématique précis (Falcade, 2006), dans notre cas, la multiplication.

Nous avons pris conscience des difficultés des élèves à transposer leurs connaissances à des situations différentes, leur résistance à un nouveau discours théorique, ainsi que leur attachement à la résolution de problèmes mettant en place des techniques qui suivent un modèle déjà connu. La diversité d'espaces de travail mathématiques appropriés de manières très différentes nous a permis de déterminer des parcours d'élèves dans lesquels des entrées cognitives se sont produites. Nos intentions didactiques nous ont conduits à la conception de séances expérimentales donnant priorité à la médiation entre paires mais une place limitée aux interventions de l'enseignant. Je ne peux que souligner l'importance de concevoir des situations—pour des classes ordinaires—qui permettent aux élèves de s'exprimer, de développer leurs idées et de répondre à leurs questionnements sous une médiation plus active de l'enseignant.

Par ailleurs, mes analyses du travail mathématique effectif des élèves, réalisées un an et demi après la soutenance de ma thèse—et partiellement articulées aux anciennes analyses dans le cadre de cet article—m'ont conduite à des réflexions me poussant à, d'une part, veiller à ne pas restreindre le travail mathématique des élèves à un regard tourné complétement vers des *a priori* et, d'autre part, à sortir d'une épistémologie réaliste des mathématiques au sein de laquelle les mathématiques existeraient en dehors de celui qui les fait. Le travail mathématique des élèves résulte d'une appropriation du travail mathématique proposé. C'est ce mot *appropriation* ce que je souhaite questionner. L'analyse initiale des processus de résolution des tâches proposées m'a permis d'ébaucher ce processus d'appropriation mais, notamment dans le cadre de ma thèse, je suis surtout allée chercher si les élèves arriveraient ou non à l'état optimal du processus d'apprentissage attendu : la *construction* de la notion mathématique en jeu à travers l'accès à une signification visée et dont l'émergence (*son* émergence) était le but à atteindre à la fin du processus. Néanmoins, en observant plus en détails les échanges des élèves, je me suis rendue compte que me limiter à un tel regard peut être très restrictif. En effet, la lecture faite de l'espace de travail mathématique approprié par les élèves—sous l'angle précisé précédemment, c'est-à-dire consistant à le voir comme une sorte de transposition de l'ETM proposé—ne peut que partiellement tenir compte de l'expérience des élèves ou du contexte dans lequel se réalise la rencontre des mathématiques : un contexte de travail collaboratif, complexe et très riche dans lequel chacune des réflexions exprimées, chaque mot partagé, chaque question, mérite une attention plus profonde.

Ces réflexions font émerger une question élémentaire mais assez complexe : comment formuler les liens entre l'ETM de l'enseignant (ou du chercheur) et celui des élèves en tenant compte de la diversité et de la plasticité propres à tout travail mathématique en classe ? L'idée de *mouvement* (Châtelet, 1993) dans et entre les composantes attribuées à l'ETM nous permet de mieux approcher le travail résultant du pouvoir générateur des mathématiques et ses répercussions dans le travail mathématique des élèves. La mise en valeur des *cheminements* des élèves dans un tel espace, jamais fixe, nous permettrait de mieux apprécier sa réinvention grâce aux interactions et aux appropriations de la tâche proposée. Dans ce contexte, l'histoire de chacun des élèves fait *émerger dans le présent* (Varela, Thompson, & Rosch, 1993) une diversité d'espaces de travail mathématiques, lesquels se manifestent comme un *devenir* en faisant *fleurir* ou *exploser* (Châtelet, 1993) les idées mathématiques en jeu.

Mon travail de thèse a rendu compte de mes premiers pas dans l'étude du travail mathématique des élèves. À partir de là plusieurs questionnements ont émergé. Aujourd'hui, c'est bien sur ces questionnements que je travaille tout en sachant que ce ne sont pas non plus les dernières questions autour desquelles j'aimerais développer mon travail de recherche. Comme je l'ai dit dans ma communication orale, il ne s'agit pas d'établir de vérités didactiques, ni de limiter l'activité mathématique des élèves à l'apprentissage de significations préfixées. Que ces apprentissages se produisent, que des gestes soient retrouvés dans le *faire* des élèves, que des modèles soient cherchés en faisant émerger des invariants dans les conduites d'élèves dans des contextes différents, tout cela, ne peut résulter d'une transposition d'un espace de travail mathématique. Les appropriations des ETM proposés par l'enseignant, si elles se produisent, doivent être vues comme fondamentalement propres à un contexte particulier, à une histoire personnelle et encore à une expérience de rencontre (Barrera Curin & Maheux, 2014). En conséquence, ces réflexions me conduisent à enrichir mes questionnements par rapport à la façon d'approcher les mathématiques et la façon dont j'aimerais les voir vivre en contexte scolaire. La beauté toujours présente dans la possibilité de création que nous donnent les mathématiques elles-mêmes, et la beauté d'observer le *faire* des élèves au cœur d'un processus de médiations sémiotique et sociale—par exemple dans la mise en relation entre nombres et géométrie—constituent toujours le cœur de ma recherche.

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THE ROLE OF DRAWING IN ARITHMETIC WORD PROBLEM SOLVING: MICROANALYSIS OF THE SUBJECT/MATERIAL DIALECTIC

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Arithmetic word problem solving remains problematic for primary school students. Research in this field emphasizes the importance of the sense of the problem. In my doctoral thesis, I conducted an experiment with six regular third-grade students: Joelle, Ruth, Esther, Rosetta, Joshua and Jeanne. They solved three arithmetic word problems in a familiar context similar to their usual class. During the experiment, Jeanne used drawing significantly more than the other participants for solving the third problem. I propose that drawings are used as a personal material. I used the conceptual framework of the subject/material dialectic and three instruments of data collection: a semi-structured interview, direct observation, and a worksheet to analyse the behaviour of Jeanne. I did a microanalysis of data collected. The main finding shows that during the experiment, the drawings used explain an evolution of the thinking of Jeanne from an idiosyncratic form of material to a conventional form. An unconscious relationship with the pictorial material explains this kind of response. Understanding comes from grasping something in the sense of physical object.

INTRODUCTION

Arithmetic word problems continue to challenge students at elementary school. Research shows that the formulation of a response involves an understanding of the text (e.g., Bautista, Mitchelmore, & Mulligan, 2009). Understanding the text refers to different aspects of the problem, for example, the meaning of the words (Vergnaud, 2008), the context (Brousseau, 2004) and the tools used (Rabardel, 2005). Beyond those aspects, students' responses usually present words, symbols and drawings (Richard, 2004). Limiting this paper to the behaviour of Jeanne during the solving of the third problem, the purpose of this paper is to explain the role of drawing in arithmetic word problem solving. First, I will present a brief review of literature related to the use of drawing in arithmetic problem solving. Second, I will explain the conceptual framework of the subject/material dialectic. Third, I will describe the methodology. The fourth section will present the findings and a discussion. A paragraph of conclusion will end the paper.

REVIEW OF LITERATURE

Research shows that students of primary schools use different ways to solve arithmetic word problems (e.g., Bautista et al., 2009; De Corte & Verschaffel, 1987; Vergnaud, 2008). They can focus on a particular process (Kintsch & Greeno, 1985) or/and on a representation of the problem. Also, they can concentrate on the meaning of words (Vergnaud, 2008) for example, the words *won* and *lost* in an arithmetic word problem can mean respectively *addition* and *subtraction*. Generally, when a primary school student solves an arithmetic word problem, they use a cognitive approach. This approach can involve the use of words, processes and representations of the problem (Bautista et al., 2009; Vergnaud, 2008). They can also use a sensory approach using physical materials (e.g., marbles) and/or pictorial materials (e.g., drawing). Beyond the difference between a cognitive approach (not visible) and a physical approach (visible), when a student solves an arithmetic problem, they make sense of the text before formulating a response. The advantage of the physical approach is it allows the researcher to see what the student does, then the realized activity. But, what they do is not a gauge of what they think while they are solving the problem. This paper adopts the physical approach because it is more helpful for understanding the behaviour of the student. Manipulatives, gestures, ideas expressed and traits like drawing are sensory. The link between the understanding of the text of the arithmetic word problem and the drawing used as response can be more evident. Some studies highlight some links. Richard (1984) proposes that when a student draws, the act of drawing describes a selection of relevant elements of the problem for example, the context, the mathematical concept and the words. The Ontario Numeracy and Literacy Secretary (Secrétariat de la numératie et de la littératie, 2008) presents the drawing like a model that enhances mental calculation. It is a matter here of a cognitivist explanation of the use of drawing, then a cognitive approach. However, Radford (2013) postulates another point of view. According to him, learning is a reflection of a physical object upon the environment, a social act. Then, the drawing becomes a reflected object (Radford, 2013). In this case, the drawing can be used as an appropriate visualization method (Csíkos, Szitányi, & Kelemen, 2012). According to me, the reflection or the visualization method is not static but it evolves because the problem does not say to draw and the student does not end the solving with the drawing. In this case, Hughes (1986) argues that young children (3-7 years old) range from a more or less clear and personal form to a very clear and conventional form. Hughes (1986) identified four different levels of representation:

- **First:** Idiosyncratic forms, no direct correspondence with the quantity (for example, a house) (p. 56);
- **Second:** Pictographic forms, the drawing has an analogic relationship with the quantity. It is the most personalised (for example, three marbles represent the quantity of three) (p. 57);
- **Third:** Iconic forms, the drawing is relatively symbolic (for example, a mark representing the quantity of ten) (p. 58);
- **Fourth:** Conventional symbols, the least personalised, the sense that symbols have (for example the symbol 6 (six)) (p. 63).

In brief, students solve an arithmetic word problem using drawings from which they make sense and address a mathematical response with mathematical symbols. But how the student links the understanding of the problem and the drawing as a response to the problem is not clear in the literature yet. Considering the drawing that a student uses to produce their response as material, I propose that the subject/material dialectic according to the relationship with a material explains the evolution of the thinking of the student. Then, my question of research is: What is the role of drawing in arithmetic word problem solving according to the subject/material dialectic? The following paragraph will present the theoretical framework of the subject/material dialectic.

SUBJECT/MATERIAL DIALECTIC

Ricard (1999) suggests that dialectic is a structuring of knowledge, a way to make sense. Hegel (1977) argues that dialectic is the overcoming of the contradiction related to the appropriation of a physical object (see also Marx, 1973). A physical object is at the centre of the sense made. Hegel (1977), a philosopher, presented a social contradiction through the materialism dialectic. He explained the materialism dialectic as the struggle to appropriate physical objects (e.g., car, house). This struggle generates social classes. According to him, the idea of appropriation of physical objects starts a kind of internal dynamism. However, according to another point of view, Douady (1986), a didactician of mathematics, proposes that dialectic is a relationship between thought and knowledge. According to her, dialectics refers to a change in the tools used. Also, every field of knowledge refers to the use of specific tools within a frame. The appropriate use of tools from one field to another involves a change of framework and the building of a new knowledge. It is a matter of frameset.

"Actually, we can construct mathematical knowledge by playing a tool-object dialectic game within appropriate frameworks, with problems meeting some conditions" (Douady, 1986, p. 9). Among these conditions, Douady suggests the availability of mathematical tools, such as mathematical language, to solve a given mathematical problem. For Douady, the changes of direction of thought are at the centre of mathematical activity. The main idea of the tool/object dialectic is to lead students to think like a mathematician during a mathematical activity. I accept Douady's view. However, it is important to understand how students make sense of knowledge to be able to effectively lead them to think like a mathematician.

Beyond the idea of contradiction and the idea of change, the concept of dialectic suggests two key ideas. On one hand, it is a dynamic process of making sense. On the other hand, knowledge and thinking are at the centre of making sense. Following these ideas, this study aims to understand the sense made when a student uses drawing as material in solving an arithmetic word problem. The relationship with the material used seems to be at the centre of the sense made beyond its use. In this case, the sensory approach characterised by the subject/material dialectic and the methodology of microanalysis highlight well the students' behaviour.

MICROANALYSIS

This methodology is used to analyze gesture, traits and words during an activity. It offers three instruments of data collection: direct observation, semi-structured interview, and worksheet. The following paragraph presents those instruments.

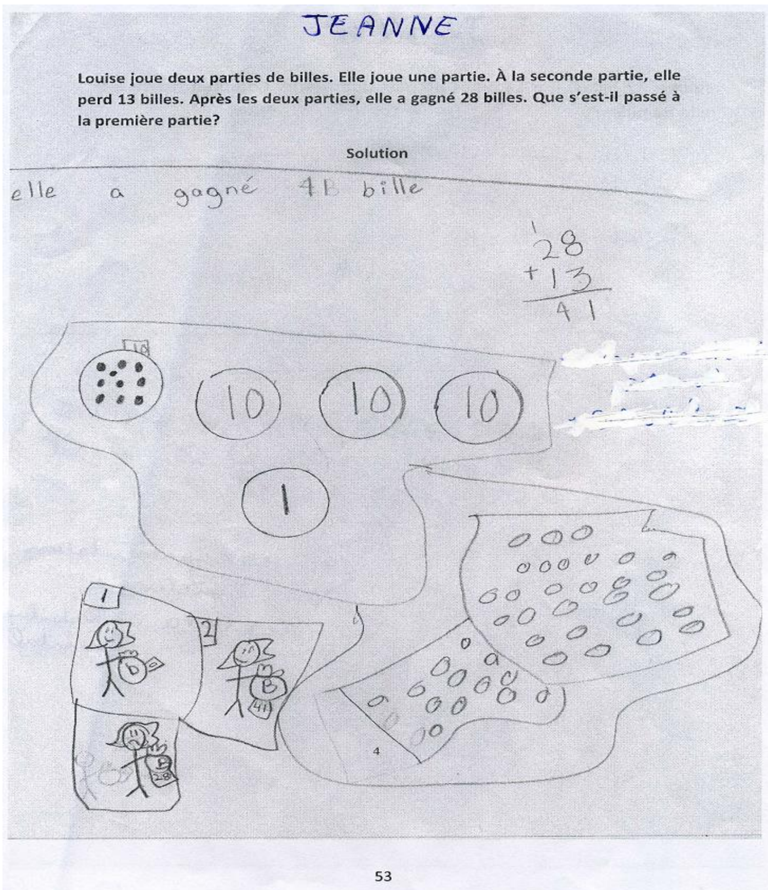
INSTRUMENTS

Direct observation means attentively looking at behaviour without participating. The goal is to select and/or to record gestures, drawing and/or words. It is important to note what Jeanne does to solve the problem. For example, she could think aloud, manipulate physical objects, write words and symbols, draw pictures or use any other strategy.

The *semi-structured interview* included open-ended questions. I previously conducted a pilot study with regular third-grade students to shape the questions. The semi-structured interview is reproduced verbatim. It includes all words and silences.

The *worksheet* is the text of the mathematical problem. The written response contains writings. In this study, the writings include drawings, words, images and symbolic notations of numbers (see the problem and writings of Jeanne in Figure 1, below).

Indeed, this is an arithmetic word problem of additive type (Depaepe, De Corte, & Verschaffel, 2010), which involves numbers less than 100 (Ministère de l'éducation de l'Ontario (MEO), 2007; Ministère de l'Éducation, du Loisir et du Sport du Québec (MELS), 2009), common words and a familiar situation (Depaepe et al., 2010; Kintsch & Greeno, 1985). However, the unknown first part of the problem involves a higher level of difficulty than other problems with a known first part. It is a matter of a regular problem for third graders.



A type of change problem
Louise plays two games of marbles. She plays a first game. In the second game, she loses 13 marbles. After both games, she has won 28 marbles. What happened during the first game?

Figure 1. Copy of Jeanne's third problem.

THE PROCESS OF DATA COLLECTION AND ANALYSIS

In a context similar to their regular class with chips (tokens) available, I proposed three identical arithmetic word problems to each participant. I gave three sheets of paper to each participant. Each sheet had a problem. I used a high definition camera to observe their behaviour during the whole experiment. I slowly reviewed the recordings with him or her and I conducted a semi-structured interview with each participant.

ANALYSIS

I conducted an analysis of data produced with each participant. I linked responses, gestures and writings, which make sense related to the use of a material. In general, the microanalysis followed three recursive steps:

1. Watching the recording activities (writings) slowly to note gestures, glances and words.
2. Establishing a chronologic link between the writings and the behaviours (gestures) from the beginning to the end of the problem solving.
3. Watching the semi-structured interview slowly and linking writings, gestures and verbatim (responses).

The findings with each instrument present an aspect of participants' behaviour while solving arithmetic word problems. This paper presents only the behaviour of Jeanne while solving the third arithmetic problem.

Direct observation

Jeanne turned the second sheet and began the solving of the third problem. First, she drew 15 balls and 28 balls in three sections. Then, she circled 10 points, three times the number 10 and one time the number 1. She finished the solving by doing an addition (28 + 13 = 41) and writing the response: "*She won 41 balls.*" The following paragraph presents a section of the relevant semi-structured interview.

Semi-structured interview

Researcher: Tell me, why do you prefer to draw marbles instead of using chips?

Jeanne: Well no, it's because sometimes I make calculation mistakes but when I draw I do not.

Researcher: I saw in your logs, there is a part that you hatched and a portion on which you had a stroke.

Jeanne: When I drew in, it is 15 balls Mary and I crossed it was to count as it should (...) what remains

Researcher: But, is that you would not have been able to take and use as tokens?

Jeanne: No!

Researcher: Why?

Jeanne: Yes (....) I've always found it long.

Researcher: OK! What does it make you draw the thing?

Jeanne: It's as if it's true for me.

Researcher: Does that mean that when you finally solve the problem (...) you end up like this and you are not checked your result?

Jeanne: Sure, but I checked in my head.

Worksheets

Below, in Figures 2 through 5, is a presentation of the chronological development of Jeanne's worksheet during the third arithmetic problem solving (see Figure 1 for the full page of work).

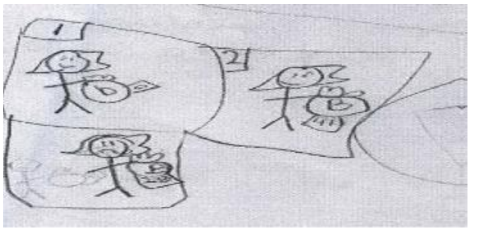


Figure 2. Jeanne’s third problem—idiosyncratic forms.

In Figure 2, this form has, in defined areas, the temporal unfolding of the situation described in the statement. In the center of a space, there is a character. Each character has a part of the game space and the characters represent the whole context of the situation described in the problem. Collections in areas are not determined and/or calculated yet.

Secondly, the pictographic form establishes an analog correspondence image/unit that seems to explain the content areas of the idiosyncratic form. The following figure (Figure 3), another section of Jeanne’s work during Activity 3 shows an example of pictographic forms.

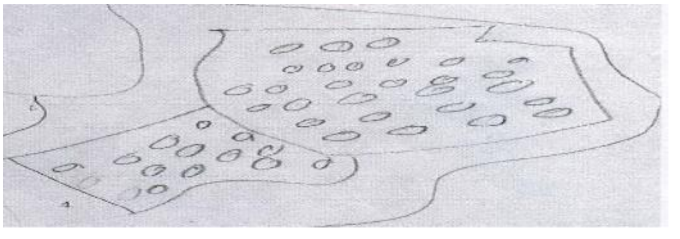


Figure 3. Jeanne’s third problem—pictographic forms.

In this figure, each frame contains a set of ball: 13 in one and 28 in the other. Together the sets total 41 balls, which were earned in the first game.

The iconic form used in some cultures to represent the cardinality has a regularity based on the shape, position or colour (base 10 block). Each collection is represented by a direct correspondence form/group of units. This form seems to represent an early symbolization by the group of drawing objects and represents a saving of graphs and time. The following section examines Jeanne’s next steps, a representation at once of iconic and symbolic forms.

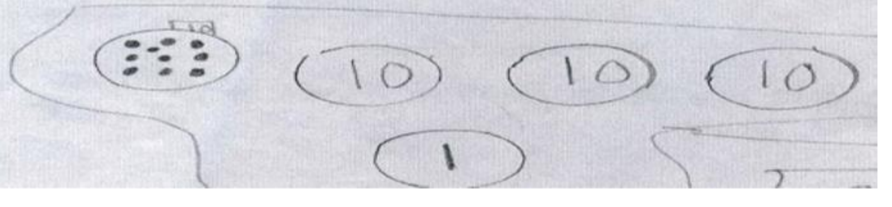


Figure 4. Jeanne’s third problem—iconic and symbolic forms.

Four circles have ten balls and one circle has one ball, which is the last drawn. The beads are counted one by one, except in the first circle. In the following three circles, the symbol 10 appears to replace 10 drawn balls. The symbol “1” in the last circle seems to represent the uniqueness of correspondence symbol/drawing, paving the way for a group, so saving space and writing.

Thirdly, conventional symbolic form has neat graphs juxtaposed with the value a graph represents on the one hand, following a data collection. On the other hand, in the decimal numbering system, this value may be larger or smaller depending on its position in the ordered sequence of graphs. I propose that this form completes the process of symbolization and economy of writing. It has at the same time the advantage of a more universal collection. As such, it is conventional, as in the following figure (Figure 5) derived from Jeanne's activity writings.

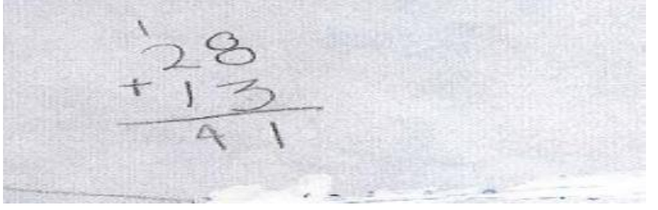


Figure 5. Jeanne's third problem—symbolic or conventional forms.

The addition retainer 28 and 13 is raised to the vertical and totals 41. This corresponds to 41 balls of the first part.

The four figurative forms represent Jeanne's evolution of thought from a personal figurative form to a conventional or symbolic form. These forms show a dynamic construction of meaning with the use of her own material: the drawing. Beyond given words and numbers, mathematical propositions reflect those four figurative forms of representation.

First, idiosyncratic form is a drawing that does not allow one to link with any cardinality due to the lack of regularity. When Hughes conducted a study with children from ages 3 to 7 years, the idiosyncratic form was a scribble, which did not allow one to infer a pattern. For older children such as Jeanne in this study, the individualized representation of the problem can link some regularity such as houses and/or boats and/or spaces and/or characters that populate her imagination. Older children will not tend to produce scribbles in this situation but rather individualized forms.

The challenge of learning is significant for a student and speaks to all stakeholders in education. Improvement of education involves a better understanding of learning. The paradigm of the teacher as a transmitter of knowledge has been shaken by the teacher as a builder of conditions of situations of learning. In this research, I argue that the subject/material dialectic explains the construction of meaning of mathematical knowledge. The construction of meaning becomes visible through the manifestation of the behaviour.

From all previous considerations, two key points challenge me.

- Students who solve arithmetic word problems use their own material. The thought of the student evolves. The subject/material dialectic explains the evolution.
- The evolution is shown through characteristic gesture related to the material used. This gesture reflects two different relationships with a material: a pictorial and a symbolic relationship. The use of a material does not depend on an available material but on a relationship with a material. Chips were available and visible but Jeanne did not use them.

On the theoretical side, I highlighted an explanation of the role of materials in arithmetic problem solving: the relationship of the student with a material does not seem to be necessarily focused on a physical object but on the use of any object. It is a matter of using their own material. On a practical level, the number of materials used is not the gauge of success but rather their relationship to the material. Word arithmetic problems refer to the set *N* as a sum of units. It is in any case the unit of something.

CONCLUSION

These various drawings show us Jeanne's evolving interpretation of this arithmetic word problem. This interpretation can be seen as a selection of relevant elements, an appropriate visualization method and the presentation of different figurative forms and a reflection. Behind all those interpretations of writings, the main question was: "*How does the student build their response to an arithmetic word problem?*" I propose that the student uses first their own material and second that their thinking evolves from an idiosyncratic form to a conventional form to communicate their response which is the usual form of mathematical language. The microanalysis of subject/material dialectic explains the process of arithmetic word problem solving through the material used: the drawing. The role of this material is to facilitate one's own representation of the problem, because to understand one must understand something in the sense of an object. It is matter of using their own object.

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HISTOIRE DES MATHÉMATIQUES DANS LA FORMATION DES ENSEIGNANTS DU SECONDAIRE : UNE NARRATION POLYPHONIQUE SUR LA FRAGILITÉ, L'ADVERSITÉ ET L'EMPATHIE – RÉSUMÉ DE THÈSE

HISTORY OF MATHEMATICS IN A PRESERVICE TEACHER'S TRAINING CONTEXT: A POLYPHONIC NARRATION ON FRAGILITY, ADVERSITY AND EMPATHY – THESIS SUMMARY

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[English translation follows.]

HISTOIRE ET FORMATION À L'ENSEIGNEMENT DES MATHÉMATIQUES

Depuis plusieurs décennies, de nombreux chercheurs ont exploré les apports de l'étude de l'histoire des mathématiques dans la formation des enseignants. En parallèle, la présence de l'histoire des mathématiques a augmenté considérablement dans les milieux de formation à l'enseignement (Barbin, 2006; Fasanelli et al., 2000). Du côté de la recherche, plusieurs travaux soulignent l'importance de l'histoire. Une idée récurrente est celle du dépaysement épistémologique (Barbin, 1997, 2006; Jahnke et al., 2000). En effet, les chercheurs soulignent que l'histoire des mathématiques dépayse et bouscule les perspectives coutumières des étudiants sur la discipline en mettant en évidence sa dimension historico-culturelle. Globalement, l'étude de l'histoire amènerait un regard critique sur l'aspect social et culturel des mathématiques et pousserait les futurs enseignants à reconsidérer leur rapport à la discipline. Cela dit, bien que commentées par de nombreuses études, ces considérations sur le dépaysement épistémologique ne semblent pas encore avoir fait l'objet de recherches systématiques de terrain qui donneraient véritablement la voix aux acteurs des milieux de formation (Guillemette, 2011; Jankvist, 2007; Siu, 2007).

APPROFONDISSEMENTS CONCEPTUELS

Avant tout, trois points de vue sur le dépaysement épistémologique sont présentés dans la thèse : celui de *l'épistémologie historique* (Barbin, 1997, 2006, 2012); de l'*humanisme* (Bidwell, 1993; Brown, 1996; Fried, 2007); et des *approches socioculturelles en éducation mathématique* (Radford, 2012; Radford, Furinghetti, & Katz, 2007; Roth & Radford, 2011). Ce dernier point de vue est approfondi par une analyse de la théorie de l'*objectivation* (Radford, 2011; 2012; 2013), une théorie émergente en éducation mathématique.

Dans cette perspective, le sens particulier attribué aux objets mathématiques est circonscrit aux limites de notre propre expérience. Cette limite ne peut être franchie que par la rencontre avec une forme étrangère de compréhension, car « meaning only reveals its depths once it has encountered and come into contact with another, foreign meaning: they engage in a kind of dialogue, which surmounts the closedness and one-sidedness of these particular meanings » (Bakhtine, 1986, cité dans Radford et al., 2007, p. 108). Dans ce sens, l'histoire des mathématiques est « a place to enter into a dialogue with others, and with the historical conceptual products produced by the cognitive activity of those who have preceded us in the always changing life of cultures » (Radford et al., 2007, p. 109).

L'histoire apparait donc comme un lieu rendant possibles l'introspection, la confrontation et la réflexion critique autour de ses propres conceptions et connaissances (Radford et al., 2000). Or, notons que le regard est ici porté non pas sur un individu rencontrant des possibilités d'émancipation personnelles, mais vers la possibilité pour les apprenants de découvrir de nouvelles manières d'être-en-mathématiques, d'ouvrir, avec les autres, l'espace des possibles de l'activité mathématique (Guillemette, sous presse). En effet, la classe de mathématique est ici perçue comme un espace communautaire, politique et éthique, ouvert à la nouveauté et à la subversion (Radford, 2006, 2008, 2011).

Habitée par les questions que soulèvent ces positions et appuyée conceptuellement par la théorie de l'objectivation, cette étude s'est donnée pour objectif de décrire le dépaysement épistémologique vécu par les futurs enseignants de mathématiques du secondaire dans le cadre d'activités de formation où intervient l'histoire des mathématiques.

POSITIONS, APPROCHES ET JUSTIFICATIONS

Pour ce faire, une approche phénoménologique a été adoptée et adaptée à la perspective *bakhtinienne* (associée aux thèses du philosophe Mikhaïl Bakhtine) que porte la théorie de l'objectivation. Concernant l'approche phénoménologique (Meyor, 2007; Van Manen, 1990, 1994), elle vise à décrire le vécu intime et subjectif des participants et à expliciter le sens de leurs expériences. Quant à la perspective *bakhtinienne* (Bakhtine, 1929/1977, 1963/1998, 1986), elle souligne qu'une œuvre scientifique ou littéraire se doit d'être « polyphonique », c'est-à-dire d'offrir une pluralité de discours et de compréhensions du monde, afin que la réalité perde de son statisme et de son naturalisme. Habitée par cette perspective compréhensive et critique, cette thèse propose une description du vécu du dépaysement épistémologique qui prend la forme d'une narration polyphonique.

Globalement, l'étude emprunte à la démarche phénoménologique en science humaine le regard exploratoire sur l'objet de recherche et l'attitude d'ouverture face aux participants, lesquels sont perçus dans leur existence concrète. Elle s'inspire de plusieurs études phénoménologiques (p. ex., Deschamps, 1987; Lamarre, 2004) afin de structurer les phases d'analyses de données. Dans un second temps, sont empruntés à la perspective dialogique bakhtinienne, les concepts de dialogisme, de polyphonie et de discours indirect libre. Ceux-ci ont permis de dégager les moyens nécessaires à l'élaboration d'une description du vécu du dépaysement épistémologique en phase avec le point de vue socioculturel que porte la théorie de l'objectivation, et qui puisse nous inclure comme formateur/chercheur. Cette description répond d'ailleurs à une conception de l'apprentissage qui stipule que savoir est nécessairement manière d'être-avec-les-autres. Le tout est chapeauté par les concepts de la théorie de l'objectivation qui participent *a priori* de notre propre discours sur le dépaysement épistémologique et de notre propre orientation appréciative concernant l'éducation mathématique.

OPÉRATIONNALISATION ET TRAITEMENTS

La sélection des participants de l'étude a été faite parmi les futurs enseignants du secondaire inscrits à un cours d'histoire des mathématiques offert à l'Université du Québec à Montréal. Sept activités de lecture de textes historiques ont été vécues en classe (A'hmosè : *Papyrus de Rhind* – problème 24, Euclide : *Les Éléments* – proposition 14 – Livre 2, Archimède : *La quadrature de la parabole*, Al-Khwarizmi : *Abrégé du calcul par la restauration et la comparaison* – types 4 et 5, Nicolas Chuquet : *Tripartys en sciences des nombres* – problème 166, Gilles Personne de Roberval : *Observations sur la composition des mouvements et sur le moyen de trouver les touchantes des lignes courbes* – Problème 1, Pierre de Fermat : *Méthode pour la recherche du minimum et du maximum* – problème 1 à 5). Six étudiants ont été recrutés.

Des captations vidéo des activités de classe, des entretiens individuels et un entretien de groupe ont été réalisés et ont fourni les données de l'étude. Pour les captations vidéo, une analyse séance par séance a permis de décrire les processus d'apprentissage ayant eu cours en classe. Pour les entretiens individuels, inspirée par les procédures de plusieurs chercheurs phénoménologues en sciences humaines (p. ex., Deschamps, 1993; Lamarre, 2004), le traitement et l'analyse des données ont mené à l'obtention d'une description spécifique du vécu du dépaysement épistémologique pour chaque participant. La narration polyphonique a ensuite été construite à partir d'extraits de l'entretien de groupe et peaufinée à partir des phases précédentes d'analyse.

DISCUSSIONS

La description obtenue fournit plusieurs regards, lesquels, mis en tensions, sont porteurs d'un discours fécond sur le vécu du dépaysement épistémologique. Spécifiquement, l'étude montre que celui-ci amène; la perception des mathématiques comme fragiles, débutantes et précaires, le vécu d'une forte adversité dans l'interprétation des textes et le déploiement d'une empathie envers l'auteur. La narration polyphonique suggère que cette empathie se déploie aussi vers la classe de mathématique des futurs enseignants. Ces derniers semblent insuffler une attention et une valorisation plus grande envers les raisonnements marginaux de leurs élèves, ainsi que la créativité et le risque dans l'activité mathématique.

Enfin, retournant aux thèses de Levinas (1971/2010, 1979/2011) et Bakhtine (1986/2003), penseurs centraux de la théorie de l'objectivation, une courte réflexion fondamentale sur l'empathie a été élaborée en guise de conclusion. Dans cette perspective, les investigations phénoménologiques, lesquels ont été soutenues et dynamisées par le déploiement de la narration polyphonique comme appareillage discursif original, invitent à penser que ces activités de formation, par le biais du dépaysement épistémologique qu'elles suscitent, supportent une éducation mathématique non violente.

La thèse est disponible en ligne à cette adresse : http://www.archipel.uqam.ca/7164/1/D-2838.pdf

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HISTORY AND TEACHER TRAINING IN MATHEMATICS

For decades, many researchers have explored the contribution of the study of the history of mathematics in teacher education. Concurrently, the presence of history of mathematics has increased considerably in teaching-learning environments (Barbin, 2006; Fasanelli et al., 2000). In terms of research, several studies highlight the importance of history for preservice teachers. A recurring idea is that of *dépaysement épistémologique* (epistemological disorientation) (Barbin, 1997, 2006; Jahnke et al., 2000). Indeed, the researchers point out that the history of mathematics tackles students' common perspectives on the discipline by highlighting its historical and cultural dimensions. Overall, the study of history brings a critical look at the social and cultural aspects of mathematics and pushes future teachers to reconsider their relationship to the discipline. Having said that, having been addressed by numerous theoretical studies, considerations of this *dépaysement épistémologique* do not yet appear to have been the object of systematic empirical research that truly gives voice to the different actors in training environments (Guillemette, 2011; Jankvist, 2007; Siu, 2007).

GOING DEEPER

Three different perspectives on *dépaysement épistémologique* are presented in the thesis: that of *historical epistemology* (Barbin, 1997, 2006, 2012); *humanism* (Bidwell, 1993; Brown, 1996; Fried, 2007); and sociocultural approaches in mathematics education (Radford, 2012; Radford, Furinghetti, & Katz, 2007; Roth & Radford, 2011). The last one is further discussed by providing an analysis of some concepts of the theory of *objectification* (Radford, 2011, 2012, 2013), an emerging theory in mathematics education.

With this perspective, meanings attributed to mathematical objects are confined within our own experience. This limit can be exceeded only by an encounter with a foreign form of understanding, because "meaning only reveals its depths once it has encountered and come into contact with another, foreign meaning: they engage in a kind of dialogue, which surmounts the closedness and one-sidedness of these particular meanings" (Bakhtine, 1986, quoted in Radford et al., 2007, p. 108). In this sense, history of mathematics is "a place to enter into a dialogue with others, and with the historical conceptual products produced by the cognitive activity of those who have preceded us in the always changing life of cultures" (Radford et al., 2007, p. 109).

Thus, history appears as a place, making possible introspection, confrontation and critical reflection about one's own ideas and knowledge (Radford et al., 2000). However, the focus here is not on one individual meeting personal emancipation possibilities, but on the possibility for learners to discover together new ways of being-in-mathematics and to open, with others, the possible realms of mathematical activity (Guillemette, in press). In fact, the mathematics classroom is seen here as a community space, as well as a political and ethical place open to novelty and subversion (Radford, 2006, 2008, 2011).

Inhabited by the issues raised by these positions and conceptually supported by the theory of objectification, this study has set the objective to describe the *dépaysement épistémologique* lived by the future mathematics teachers within the context of training activities where history of mathematics, particularly the reading of historical texts, comes into action.

POSITIONS, APPROACHES AND JUSTIFICATIONS

To do this, a phenomenological approach was adopted and adapted to Bakhtin's perspective (associated with the works of philosopher Mikhail Bakhtin) that carries the theory of objectification. Concerning the phenomenological approach (Meyor, 2007; Van Manen, 1990, 1994), it aims to describe the intimate and subjective experience of the participants and to clarify the meaning of their experiences. Concerning Bakhtin's perspective (1929/1977, 1963/1998, 1986), he claimed that a scientific or literary work must look for *polyphonic* aspects, that is to say to provide a plurality of discourse and understandings of the world. In a polyphonic work, reality loses its static facet and its naturalism. Inhabited by these comprehensive and critical elements, the thesis proposes a description of the lived experience of *dépaysement épistémologique* that takes the form of a polyphonic narration.

Overall, the study borrows from the phenomenological approach to human sciences the exploratory look and the attitude of openness to the participants lived experience, which are seen in their concrete existence. It takes into consideration a few phenomenological studies (e.g., Deschamps, 1987; Lamarre, 2004) in order to structure data analysis phases. Secondly, borrowed from Bakhtin's dialogical perspective are the concepts of dialogism, polyphony, and free indirect discourse. These provide the conceptual and methodological elements needed to develop a description of the *dépaysement épistémologique* which is in step with the socio-cultural point of view carried by the theory of objectification, and which could include us as trainer/researcher. This description also responds to a conception of learning which states that knowing is necessarily a way of being-with-others. Everything is overseen by the concepts of the theory of objectification involved *a priori* in our own discourse on the *dépaysement épistémologique* and our own perspective on mathematics education.

OPERATIONALIZATION AND TREATMENTS

The selection of participants was made from future secondary school teachers taking part in a history of mathematics course offered at the Université du Québec à Montréal. Seven reading activities of historical texts were proposed in class (A'hmosè: *Rhind papyrus* – problem 24, Euclid: *Elements* – proposition 14 - book 2, Archimedes: *The quadrature of the parabola*, Al-Khwarizmi: *The Compendious Book on Calculation by Completion and Balancing* – types 4 and 5, Nicolas Chuquet: *Tripartys en sciences des nombres* – problem 166, Gilles Personne de Roberval: *Observations sur la composition des mouvements et sur le moyen de trouver les touchantes des lignes courbes* – problem 1, Pierre de Fermat: *Méthode pour la recherche du minimum et du maximum* – problems 1 to 5). Six students were recruited as participants for the study.

Video recordings of classroom activities, individual interviews and group interview were conducted and provided the study data. For video recordings, a session by session analysis allowed description of the learning process in the classroom. For individual interviews, inspired by the phenomenological procedures developed by several researchers (e.g., Deschamps, 1993; Lamarre, 2004), processing and analysis of data led to specific descriptions of the experience of *dépaysement épistémologique* for each participant. The polyphonic narration was then constructed from excerpts of the interview group and refined with results from previous analysis phases.

DISCUSSION

The description obtained provides multiple looks which, put in tension, are carriers of a fruitful discourse on the experience of *dépaysement épistémologique*. Specifically, the study shows that it brings: the perception of mathematics as fragile and precarious; the experience of adversity in the interpretation of texts; and the deployment of empathy toward the author. The polyphonic narration suggests that this empathy also spreads to the mathematical classroom of future teachers. They seem to take attention and greater appreciation towards marginal reasoning of their students, as well as creativity and risk in mathematical activity.

Finally, returning to the ideas of Levinas (1971/2010, 1979/2011) and Bakhtin (1986/2003), central thinkers of the theory of objectification, a short fundamental reflection on empathy was developed in conclusion. In this perspective, the phenomenological investigations, which were supported and boosted by the deployment of the polyphonic narration, invite us to think that these training activities, through the *dépaysement épistémologique* they generate, support a non-violent mathematics education.

The thesis is available online at: http://www.archipel.uqam.ca/7164/1/D-2838.pdf

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IMPROVING MATHEMATICS TEACHING THROUGH PROFESSIONAL LEARNING GROUPS

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INTRODUCTION

In order to teach mathematics well, teachers must have a specialised knowledge of the content (Silverman & Thompson, 2008) and believe in effective teaching methods (Philipp, 2007). Ball, Hill, and Bass (2005) call for the need for teachers who understand mathematics and the curriculum to be teaching it, so ensuring appropriate professional development is especially important because high-quality professional development is needed in order to have high-quality, effective teachers in the field (Gojmerac & Cherubini, 2012). This need for professional development is compounded by the fact that research has indicated that teaching mathematics effectively may require teachers to use pedagogy that they have never experienced for themselves (McNeal & Simon, 2000). In order to support changes in the mathematics teaching practices of in-service teachers, professional learning groups are one possible way of addressing teacher needs. The research described herein addresses a call by Johnson (2009) that professional learning groups need to be investigated further to ensure their effectiveness as professional development. A long-term case study of one professional learning group of grade 6 to 10 mathematics teachers in Ontario was conducted to explore how the discussions provided the needed support for mathematics teachers in using research-based pedagogy in their classrooms.

LITERATURE

The literature that lends itself to this research study is research of effective mathematics pedagogy, the specialised knowledge of mathematics for teaching, teachers' beliefs in mathematics teaching and learning, and finally professional learning group research. A brief summary of the salient points of each as they pertain to this study is presented here. To begin, the National Council of Teachers of Mathematics (2000) as well as the Ontario Ministry of Education (2005) advocate for the use of problem-solving, manipulatives, models, and explorations as components in every mathematics classroom. For a teacher to use these types of methods, the teacher must believe that they are the most effective for student learning, as well as understand mathematics deeply enough to be able to use them effectively in a classroom. Wilkins (2008) notes that teacher beliefs are the strongest influence on personal practice. Other studies have shown how a teacher who believes that procedures and rules are the best ways to learn mathematics has difficulties when making changes to their practices (i.e., Cross, 2009).

Not only must teachers believe that these research-based pedagogies are appropriate for teaching, they need to have the knowledge to enact them. Silverman and Thompson (2008) claim that mathematics knowledge for teaching only becomes the knowledge needed by teachers when understandings of content are linked to pedagogical knowledge. An example of this intersection with pedagogical knowledge is "understanding why a student may arrive at a particular answer or knowing different instructional approaches for demonstrating a mathematical concept" (Chamberlin, Farmer, & Novak, 2008, p. 441). Teachers would also need to know how to base mathematical lessons on the knowledge students already possess in order to bring students toward the lesson goals (Baumert et al., 2010; Silverman & Thompson, 2008). As such, the understandings needed by teachers are complex and varied.

The Ontario Ministry of Education (2007) has advocated for the use of professional learning groups for professional development and has defined this type of collaboration as "a group of people who are motivated by a vision of learning and who support one another toward that end" (p. 1). Hord and Sommers (2008) note that there are eight characteristics that define professional learning groups: shared beliefs, values, and vision; shared and supportive leadership; collective learning and its application; supportive conditions; and shared personal practice. DuFour and Eaker (1998) add that professional learning groups should take similar traits to action research and note three important characteristics: action orientation and experimentation, continuous improvement, and results orientation.

The work of Kajander and Mason (2007) highlighted how professional learning groups can be vastly different. Their research focused on two contrasting professional learning groups and their discussions. They found that one group used the meeting time to create new, better tests; whereas, the other group focused on examining practices of the teachers toward creating opportunities for students to deeply understand mathematics. This study raised an important question about who defines 'success' in a professional learning group. Since a professional learning group should be a personal journey, the voices of the teachers in the group and their opinions of the professional learning need to be taken into account in research.

METHODS AND DATA SOURCES

My research was a qualitative, narrative case study that focused on the discussions of, as well as the benefits of or problems with, a mathematics professional learning group in northwestern Ontario. This group was composed of both elementary (grades 6 to 8) and secondary (grades 9 and 10) teachers. My focus question for this research was "*What are the conditions of a professional learning group of intermediate mathematics educators that improve their teaching practices?*" To begin to answer this question, my goal was to record the plurality of voices of the different members of the professional learning group. Professional learning group characteristics provided by the research literature were examined in relation to this case study in order to determine how such groups could be developed in mathematics. Conversations about beliefs and knowledge were also analysed, in order to provide an understanding of how the group focused on mathematics teaching and learning. Narrative inquiry was also used to provide in-depth descriptions of five of the teachers, chosen in order to show a range of the members within the group.

Field notes, meeting recordings, and interviews comprised the data collection for this portion of the study. Field notes and meeting recordings were collected over the entire three years of the study. Although all the teachers were observed and contacted for an interview, only five of the teachers were focused on for the narrative study. Interviews were conducted at the end of the final year with four secondary teachers and five elementary teachers agreeing to participate in the semi-structured interviews. The meeting recordings were partially transcribed in order to be analysed. Following the creation of partial transcripts for each meeting, the entire body of data was read and coded based on the research questions that I set out at the beginning of my research. These codes could then be explored in further detail as each question was being examined. The recordings were then listened to in their entirety to identify themes in the conversations. Field notes of activities within the meetings were collected as well as a collection of the artefacts from the meetings. Interviews were fully transcribed and then analysed in a similar manner as the meeting transcripts in looking for themes.

FINDINGS

In order to illustrate what was seen during the professional learning group, I now present two of the cases within the study: Blaine and Wesley. These two cases particularly illustrate how different teachers can be within a single professional learning group, and highlight some of the considerations that need to be made in order for this type of professional learning to be effective for supporting changes in mathematics teaching practices of all the teachers involved.

BLAINE

Blaine taught grade 6/7 during the time he was a part of the professional learning group. He had been teaching for 15-19 years at the time of the study. In addition to his Bachelor of Education, Blaine had both a Bachelor of Arts and a Bachelor of Physical and Health Education. Blaine claimed he was the "*parasite*" of the group because his knowledge of teaching mathematics was so weak that he could not participate, only gain knowledge. He noted that all of the good lessons he used in his classroom came from the grade 8 teacher at his school (who was also part of the professional learning group). He rarely participated in meetings. When he did participate, his conversations showed how much he valued student exploration, and he wanted his students to understand mathematics. For example, he discussed how he had changed the way he discussed finding volume of right prisms to finding the area of the base and multiplying it by the height. He found that this discussion really had meaning for his students and eliminated misunderstandings. Blaine felt that his "*knowledge of teaching mathematics is so shallow that anything is going to help*". He found that meetings where the teachers discussed student work would allow him to gain insights into the practices of his peers. Blaine acknowledged that he gained a lot of new skills and ideas that he used in his own classroom from the conversations in the professional learning group.

WESLEY

Wesley was an experienced secondary teacher, who had been teaching more than 30 years at the time of the study. In addition to his Bachelor of Education, Wesley also had a Bachelor of Mathematics and additional qualifications in mathematics. Wesley felt that he often was "*thinking down*" in his conversations with the other teachers because his own mathematics knowledge was so much higher. He also felt that the purpose of the professional learning groups was for the elementary teachers to make changes to better support what is being done in secondary school. Wesley shared a lot during the meetings, and his conversations showed how much he valued rules and procedures for mathematics. For example, he brought in a page of rules that he gives to his grade 10 Applied students to learn algebra because "*at this point in time in grade 10, we probably have zero time to go ahead and talk about, bring out the picture of the teeter totter*" for studying algebra. Wesley felt the professional learning groups were too structured and did not allow enough for personal freedom in discussions (he was the only teacher to feel this way).

CONCLUSIONS

In the end, all of the teachers began the professional learning groups at different places in their beliefs and knowledge, and all ended at different points. Blaine already believed that mathematics should be taught through engaging explorations to lead to deep understandings, and the professional learning groups helped him gain the knowledge he needed to achieve that goal. Wesley believed that mathematics is about rules and procedures that need to be taught and memorized, but one professional learning group experience led him to question if there was something different he should try to get more from his mathematics lessons. During the course of the meetings, the group attended a presentation by David Stocker about using social justice lessons in the classroom to teach mathematics. For Wesley, this was a turning point. As a result he began to consider changing the types of problems he used in the classroom to "*leave the kids with something they're going to remember*". He attempted using problem solving in his classroom with a problem inspired by the talk, but it was met with failure. He believed his students were simply too low to benefit from problem solving but vowed to try again.

These two teachers illustrated how although the teachers engaged in the same activities and conversations, they had different ideas from the conversations and enacted them in different ways in their classrooms. This led me to question, "*What is it about a professional learning group that needs to be considered to allow for changes and growth to be made?*" In order to further understand the phenomenon that was being presented through the professional learning group, I created a model to define the characteristics that influenced the group.

MODEL OF PROFESSIONAL LEARNING

In order to describe each section of the model, I will now detail how it was presented in this research study. The model begins with ensuring the supportive conditions of the professional learning group are met. Supportive conditions include being given the time to meet and a space to have conversations. For the teachers in this study, it also included the freedom to make their own decisions about what needed to be discussed. Once the supportive conditions have been met, then the professional learning group is framed and influenced by both the beliefs of the teachers and their knowledge about mathematics.

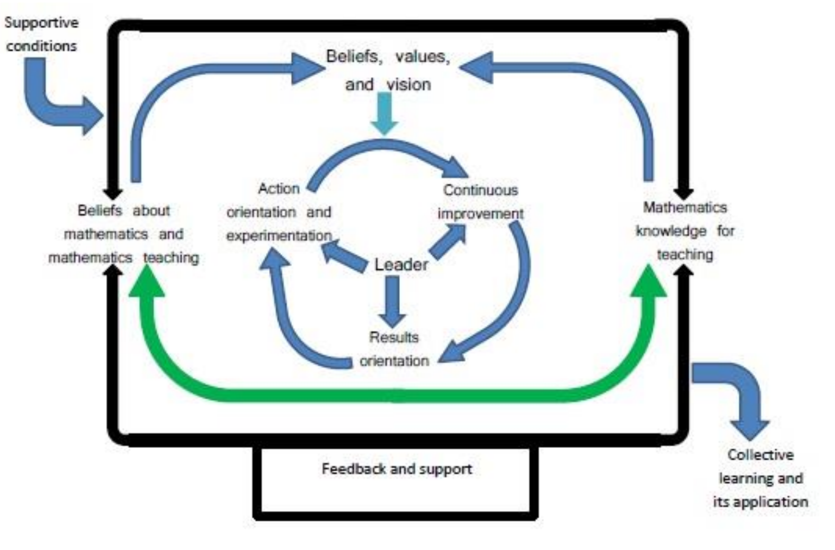

Figure 1. A model of professional learning groups.

Beliefs about mathematics and mathematics teaching need to be considered and challenged through the professional learning group. As shown by Wesley, it was not until a specific experience challenged his beliefs that he began to make changes. The knowledge the teachers have about mathematics teaching is also going to frame the professional learning group. As shown by Blaine, it can also influence the participation of teachers in the group. Conversations addressing mathematics knowledge need to be part of the discussions to support the specialised knowledge growth of all teachers.

Both the beliefs of the teachers and their knowledge of mathematics are going to influence the beliefs, values, and vision of the professional learning group. My model differs from previous research in that the beliefs, values, and vision were not shared by all the members of the group. Although each teacher held the goal that they wanted to improve the mathematics learning of their students, the teachers differed in how they felt this should be accomplished. This in turn influenced the group activities and conversations.

The center of the model consists of a need for continuous improvement, action orientation and experimentation, and a results orientation. The conversations of the professional learning group revolved around this cycle of improving practice and focusing on student results. It was this action research focus that ensured that the group continued to function to make improvements to their teaching to better support students. In the end, this continual testing of the new topics discussed ensured the teachers tried new things to move forward instead of just creating new tests to maintain the current state of teaching as was observed in one professional learning group by Kajander and Mason (2007).

The most critical piece of the professional learning group model is the leader at the center of my model. The leader became an important component of the professional learning group to ensure the conversations kept moving forward. This leader could be a person, a piece of literature, or a workshop that guides the group and keeps the conversations on track toward a common goal (and is not necessarily the person who organizes the meetings). To give you an example of the role of the leader in the professional learning group, I will share with you an interaction between Wesley and the 'leaders', two elementary school teachers. Wesley presented the example question in Figure 2 to the group for factoring binomials.

$$\frac{x}{3}+\frac{1}{4}=\frac{2x}{6}+\frac{1}{2}$$

$$\frac{x}{3}^{\times 12}+\frac{1}{4}^{\times 12}=\frac{2x}{6}^{\times 12}+\frac{1}{2}^{\times 12}$$

$$\frac{x}{\cancel{3}}^{\times \cancel{12}^{4}}+\frac{1}{\cancel{4}}^{\times \cancel{12}^{3}}=\frac{2x}{\cancel{6}}^{\times \cancel{12}^{2}}+\frac{1}{\cancel{2}}^{\times \cancel{12}^{6}}$$

$$4x+3=4x+6$$

Figure 2. Example presented by Wesley for factoring binomials.
Note: This is exactly what was presented by Wesley, but would not work as an example.

What Wesley wanted students to do was to immediately recognize that all of the denominators are factors of 12 and multiply each fraction by 12 as a first step (instead of in pieces as he had seen previously). His goal was to have students solve as shown in the remainder of Figure 2. He strongly felt that if the elementary teachers would teach this to their students then he would be able to do more with his students in grade 10. The 'leaders' pushed the discussion away from this by noting "*I don't just want them to be procedurally fluent, I want them to have some conceptual understanding*" and that elementary students are not ready to complete this procedure because they are still just trying to understand algebra. The 'leaders' served the role of keeping the conversations on productive tracks that both pushed and questioned the thinking of the members of the group.

The entire model, as well as professional learning groups, rests on the need for feedback and support for the teachers involved. To the teachers, this was an essential piece of the professional development and one of the main reasons they appreciated the group meetings. For some of the teachers, it was critical to their development because they were the only intermediate mathematics teachers in their schools, so without the professional learning groups, they would not have opportunities to share resources or interact with other teachers in their division.

In the end, the outcome of a professional learning group should be collective learning that is applied in the classrooms of the teachers involved. All of the teachers interviewed noted there was something they learned as a result of the meetings and applied to their classrooms.

DISCUSSION

Professional learning groups are a complex phenomenon with many important characteristics that are necessary to be considered in order for it to be successful in supporting changes in mathematics teaching. For instance in the example given above, although the leaders pushed the discussion toward strategies that would be effective to support elementary students understanding algebra, the conversation did not change Wesley's mind. He was adamant that this would be an effective strategy for grade 7 and 8 students and was determined to bring it up again. The elementary teachers continuously attempted to push the conversations away from discussions about teaching better rules to younger students and towards ways that they could incorporate models and problem solving to give students a deeper understanding of the concepts. The role of the leader was important to the development of many of the teachers because Wesley was so vocal in the conversations about simply using rules and procedures to teach.

In the end, the push that Wesley needed to start thinking about mathematics differently was the meeting with David Stocker at the university. Although he met with failure on his attempt to use the strategies, he was determined to try again with "*better students*". As such, it is possible that it would not need to be a member of the group that could encourage a push toward changing beliefs about teaching mathematics. The leader plays a critical role in the professional learning group discussions by helping to steer conversations toward the goals of the group to improve mathematics teaching practices by supporting effective pedagogy.

Finally, the beliefs and knowledge a teacher has about teaching mathematics play a role in how the conversations will unfold in a professional learning group. To be effective, mathematics knowledge for teaching must be a part of the discussions in order to better support teachers.

In order to ensure that professional learning groups in mathematics are discussing reform-based practices, then some teachers in the group need to believe that these are effective practices, *or* there needs to be a catalyst to cause changes to the beliefs. Professional learning groups do have the potential to be used for professional development for mathematics teachers when the knowledge of the teachers is supported, the beliefs are considered and possibly challenged, and a 'leader' is included to help maintain the action research focus of the group.

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MATHEMATICS PROBLEMS AND THINKING MATHEMATICALLY IN UNDERGRADUATE EDUCATION

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INTRODUCTION

This PhD thesis is about learning to think mathematically. Undergraduate mathematics education is the increasing focus of international discussion because of its influence in other academic disciplines and industrial fields. But what are the goals of an undergraduate mathematics education? Most mathematics in school emphasizes the formal system of mathematics, and there are many reasons for this. But mathematics exists simultaneously as a *formal system* and as a *mental activity* (Tall & Vinner, 1981). Many people—mathematicians and otherwise—see the emphasis on formal mathematical training as a tremendous disservice to individuals and to our greater society. I will make a case that mathematical thinking is a priority in contemporary learning theories, and that it is of value to employers who seek creative and analytic problem solvers.

Arguably, both a student's mathematical knowledge and her conception of mathematics are shaped by the problems with which she struggles, the content, the presentation, and the social context in which the mathematics is being learned. And a good problem may stay with a person long after the course has finished. Yet how can we emphasize the mental activity of thinking mathematically in undergraduate mathematics education? In my thesis I give a detailed account of mathematical thinking and propose that thinking mathematically can be learned through carefully designed and administered *ill-structured* problems. I describe different design variables of mathematics problems and discuss how these might be related to processes of mathematical thinking that are valued. In particular, I highlight the creative activity necessary for the re-formulation of ill-structured problems. Finally I discuss extending these ideas to inform problem design and pedagogical design that promotes mathematical thinking.

PHILOSOPHY

The discipline of mathematics includes objects, relationships, procedures, and proofs, but it is also formed of intelligent navigation though this system and involves a continual interaction between the individual and the community. In the professional world of mathematics the communication of formal mathematical ideas is primarily realized through writing, especially in published papers, which are then read by other mathematicians. If these written mathematical ideas are validated and accepted by the community, they become part of the collective body of mathematical knowledge. The communicated artifacts of mathematics can be referred to as the *formal system* of mathematics. Consider on the other hand the thinking involved in this knowledge system. New mathematics is created by individuals: drawing from the pool of existing mathematical knowledge, they intelligently form, re-form, and rigorously defend these ideas. The activity of creating and validating formal ideas can be referred to as the *mental activity* of mathematics. Figure 1 presents a model uniting these distinct yet simultaneous mathematical activities. Knowledge is created as ideas progress through a cycle of reading, creating, writing, and validating, all of which take place within a social community. Written and communicated mathematical material comprises the formal system of mathematics, while the acts of creating and validating mathematical material are the mental activities of mathematics.

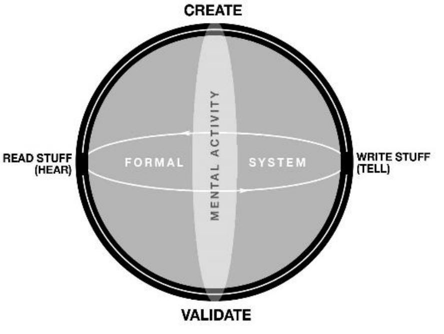


Figure 1. Mathematical knowledge creation.

A shift in mathematics education away from *the product of mathematical thought* toward the *process of mathematical thinking* was strongly advocated by mathematician-psychologist Richard Skemp (1971). This enculturative approach to education describes a learning environment in which activities of the student resemble expert mathematicians' activities, but at the level of the learner. Here the community is made up of teachers, tutors, and peers; the communication of ideas can take on a multitude of forms. Furthermore, 'new mathematical knowledge' can be conceptualized from the perspective of the individual as well as within a social community. An example of this comes from a class that I recently taught in which a student discovered the concurrency of medians in a triangle. Although this is a centuries-old piece of mathematical knowledge, the student herself created this knowledge. This mathematical thinking is 'doing mathematics' as much as the formal proof of concurrency in triangles. The feeling of discovery and ownership is perhaps the most rewarding experience in mathematics, and I am not the only one to strongly advocate for this as a central goal in mathematics education (Lockhart, 2002/2008).

The foundations of a constructivist philosophy of mathematics education are academic discourse and the creative cycle—*validation* within the accepted framework and the *creation* of new mathematics as presented in Figure 1. A fundamental aim of mathematics education from this perspective is mathematical thinking. According to this philosophy, carefully crafted mathematics problems, taught within an enculturative environment, are at the heart of a rich and balanced mathematics education.

OVERVIEW OF THE STUDY

The research in this thesis is an investigation of relationships between mathematics problems and mathematical thinking. To be clear, I do not talk about instructional practices at any length. Instead, I present an analysis of a learning context that emphasizes mathematical thinking can be seen. This research has both a theoretical part and an experimental part. In the theoretical component I present a characterization of the mental processes that make up the activity of thinking mathematically, and discuss these with respect to the characteristics of mathematics problems. In the experimental component I began by collecting demographic data and affective information through an online survey. I then observed pairs of undergraduate students while they worked on carefully crafted mathematics problems. This study was designed to answer the following questions.

1. What is a useful theoretical framework for identifying processes of mathematical thinking that have been highlighted in the literature?
2. What is the relationship between mathematical thinking and problem design?
3. Is there a relationship between a student's affective variables and her/his mathematical thinking?
4. What are the pedagogical implications of this research? Can educators readily use this information in designing problems to teach mathematical thinking?

By conducting the research on pairs of students I was able to capture a great deal of information on thinking processes, more than would have been evident from individuals working alone, because in pairs the students repeatedly discussed and explained their thinking with one another. The application of several mixed-methodological analyses drew out connections between specific problem design variables and particular processes of mathematical thinking, as well as between thinking and affective student variables.

THINKING MATHEMATICALLY

Mathematical thinking resonates more with the mental activity of mathematics than with the formal system, but the two are not so easily separated: the formal system, which includes points and curves, space, change, roots, magnitude, algorithm, categorization, and representation, is both the product and the medium of mathematical thinking. To think mathematically is to ask questions about relationships between these ideas, and to reason through answers to these questions, thereby creating new mathematics, as well as to represent non-mathematical states and processes in this language of mathematics.

I created a cohesive categorical structure of mathematical thinking from a survey of themes of mathematical thinking. Three categories emerge, distinguished by cognitive function. From Hadamard's (1945) reflection on Poincaré's philosophy of mathematical creation, I borrow the term *discovery* to describe subconscious acts in which thoughts arise and intelligent ideas are created. The term *structuring*, from Selden and Selden (2005), describes conscious acts of identifying and arranging mathematical ideas in workable forms. *Justification* is the third category, describing cognitive and metacognitive acts of reflecting on work that has already been carried out. An overview of these categories is shown in Figure 2.

Discovery	Structuring	Justification
• Intuition • Intelligent ideas • Newly nascent subconscious ideas • Extensions • Abstractions	• Symbolic representation • Visual representation • Defining, classifying, characterizing • Constraints and assumptions	• (Local) assessment of product • (Local) assessment of process • (Global) Verification • Formalization • Metacognition
Acts of creation.	Acts of arranging.	Acts of reflection.

Figure 2. Three categories of thinking mathematically.

Naturally, thinking happens not along a single trajectory but in many directions at once, and often more than one idea is developing at any given time, and to think mathematically is to simultaneously engage in multiple types of thinking. This research is directed at drawing out relationships between the categories of mathematical thinking identified here and design elements of mathematics problems.

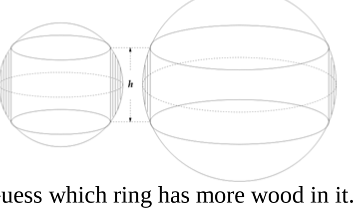
PROBLEM DESIGN

Problems are the heart of all activity in undergraduate mathematics education. Most problems used in mathematics classes are well-structured tasks (DeHaan, 2005)—textbook exercises designed for students to practice techniques that are new to them. An undergraduate mathematics education built largely on these types of problems gives students an entirely incorrect view of what mathematics is (Lockhart, 2002/2008). On the other hand, an *enculturative* education, which supports the practice of learning through more authentic mathematical experiences, integrates the formal system and the mental activity of mathematics and engenders a more cohesive conception of mathematics. "Instead of gaining a deep understanding of mathematical knowledge and the nature of mathematics, students in conventional classroom environments tend to learn inappropriate and counterproductive conceptualizations of the nature of mathematics" (Roh, 2003, p. 1). Practical exercises are essential to deepening an understanding of any discipline, but do not represent the full spectrum of mathematical activity. Constructivist pedagogies propose a different sort of mathematical task: the investigation and resolution of messy, ill-structured, often interdisciplinary problems.

Kilpatrick (1978) presented a thorough analysis of mathematical task design variables, which was then refined by Goldin and McClintock (1979). I have chosen to focus this research on the task design variables *content* and *structure*, which according to Webb (1979) are the "main essence of mathematics problems" (p. 77). Content variables are the key words and vocabulary used to describe a problem, as well as the given information and the goal information, and the equipment available for use. This includes the *kernel* of the problem—the core mathematical idea at its heart (Webb, 1979). In essence, the kernel of a problem can be held constant while the problem statement is moulded to fit a situation. Let us consider an example of an analysis of these variables by contrasting two mathematical problems, below (see Figure 3).

A. Consider two spheres of different diameters. Each sphere has a cylindrical core removed along a pole such that the height of the cylindrical void within each sphere is equal. Which spherical object do you think has more volume, and why?

B. Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both napkin rings have the same height *h*, as shown in the figure.



i. Guess which ring has more wood in it.

ii. Check your guess: use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius *r* through the center of a sphere of radius *R* and express the answers in terms of *h*.

(Adapted from Stewart, 2006, p. 437)

Figure 3. Problem1: A and B.

Contrasting attributes of these two problems can be seen through an analysis of the content and structure of Problem A: no strategy or solution procedure; wording is vague and requires interpretation; no picture is provided (visual representation is significant); though the goal information is clear, the formality of solution unclear. Content affects a student's understanding of a problem (Kulm, 1979) and emerges most prominently in the reading and analysis of a problem. Structure variables describe the representation of a mathematical problem. Problem A on the left is an ill-structured problem while Problem B, in contrast, is well-structured.

Jonassen (2000) suggests a link between ill-structured problems and mathematical thinking activities such as providing a problem with structure when there is none, metacognition and argumentation, and disambiguating important from irrelevant information. The re-formulation of a problem is the act of providing the problem with the structure necessary to go forward; an ill-structured problem is posed in such way that it requires re-formulation activities. These can include realizing the true form of objects and processes, precisely identifying the underlying structure or question (discarding 'fluff'), re-stating the problem in a way that sets the stage for problem solving, and imposing constraints so that the problem has meaning and a solution might be reached. These are fundamentally the same thinking processes that are involved in problem-posing. Thus, the act of re-formulating a problem seems to present a way for students to learn and practice the art of problem posing, but on a smaller scale. This activity might be described as scaffolding mathematical thinking.

Three mathematics problems were selected/designed for this study. The first is Problem 1A presented above. The other two are:

> **Problem 2:** For the 2011 Dodge Ram 2500 the consumer may choose from two tire sizes, one wider than the other. The tire specifications are as follows.
>
> Option A: 245/70R17
> Option B: 265/70R17
>
> When referring to the purely geometrical data, to take a common example, 195/55R16 would mean that the nominal width of the tire is approximately 195 mm at the widest point, the height of the side-wall of the tire is 55% of the width (107 mm in this example) and that the tire fits 16-inch-diameter (410 mm) wheels.
>
> With this information, which tire do you estimate would wear out first?
>
> **Problem 3:** In how many ways can you change one half-dollar?

All three problems used in this study are ill-structured problems, in part because no procedure is provided with which to solve them. Further distinctions can be seen in the given information and the goal information. The intention was that these problems would be unfamiliar to the participants at the time of the study, but that neither the concepts nor possible solution procedures would be entirely unfamiliar.

In this thesis I propose that re-formulation can bring about Discovery via *insight*, Structuring via *constraining*, and Justification via *metacognitive control*. Furthermore, the process of re-formulating an ill-structured problem can also elicit retrospective questions about relationships posed in the problem, which can lead to Discovery-type realizations about altering relationships, abstracting, and the creation of new, related problems. An example of this opportunity comes from an analysis of Problem 3 in which the word *change* is used but there is no mention of the denominations of the coins to be used. Consider the quarter, dime, nickel, and penny and the fact that Canada has now eliminated the penny. The Structuring activity directed at the recognition of assumptions might bring about new creative insight, and acts of Discovery. In this research I observed and analyzed the mathematical thinking that arose while students engaged with these three problems. In this summary I provide only details from Problem 1 of the research.

STUDENT WORK AND RESULTS

Students were paired according to their metacognition, as measured in the initial survey of affective variables. This study reveals that students with well-developed metacognition engage in more frequent, more varied, and more advanced mathematical thinking than their less metacognitive peers. Compare the work of the weakly metacognitive Jesse and Kez to the other two pairs who are identified as highly metacognitive. Note the coding notation for Discovery (D), Structuring (S) and Justification (J).

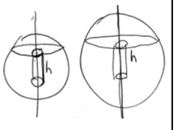
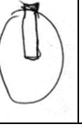
Ang and Bre		
A:	Yeah. I'm going to draw pictures… [*Both draw and read. A's picture is a bird's-eye view, while B's is a side view.*] 	**S**
A:	So my guess is that… they're asking us to calculate the volume within the sphere minus the volume of this cylinder. That's within the sphere. 	**Re-formulation**
B:	So the part that's missing only defines the height though, right? It doesn't say anything about the diameter.	**S/J**
B:	So you don't know if this one goes all the way over or if it meet, or with this one [?].	**D/S**
A:	And we don't know if goes right to the edge, either [*indicates the top pole of the sphere*].	**D/S**
B:	So then it would also matter… if it was… the top [*drawing*] or the middle, right?	**S**
B:	Because you'd be missing that little bit [*indicates the part of the sphere that lies above the cylinder*] wouldn't you? Maybe that's not significant. 	**S**
B:	We agree that the pole has to go right through, right?	

Figure 4. Ang and Bre.

Jesse and Kez		
J:	So, the spheres… And, like, whatever different diameters [*drawing*].	**S**
J:	So each of them are going to have the same… amount. So it's going to look like [*drawing*]. Like that? 	**S**
K:	Uh huh. Yep.	
J:	Oh, it should be smaller. So it's like the same here? [*Indicates height of intended cylinder in the second sphere.*] [*Drawing*] Sorry, I don't draw so good.	
J:	Something like that?	**S/J**
K:	And the heights are equal?	
J:	Yeah.	**S/J**
K:	So [*re-reading*] which spherical object do you think has more?	
	[*Silence; assessing*]	
K:	How come it isn't that? [*Indicates the larger sphere.*]…because the diameter is bigger.	**Conjecture**
J:	Is there like a trick to it? Okay, [*re-reading*] Consider two spheres of different diameters. Okay. Each sphere has a cylindrical core removed along a pole… So that just means straight, I'm guessing.	**J**

Figure 5. Jesse and Kez.

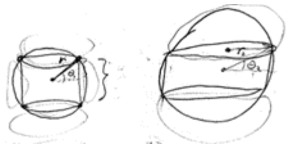
Pat and Quinn		
P:	Well, the height of both of them is equal. So it would be like this is one, and then if you have a bigger one, then it would have to be taken like farther out so that you would have, uh… 	**D/S**
Q:	But, uh, does it have to like touch the surface of the sphere, or can it be contained within the sphere? It doesn't specify…what the diameter of this…circular face of the cylinder is. Or whether it's circular at all.	**S/J**
P:	I think that along an axis that [*laughing*] I just think that along an axis then that means like it has to be like touching [*indicates on picture where shells of sphere and cylinder meet*].	**J**
Q:	But the axis is arbitrarily chosen. Well, it's a sphere so it's, it's going to be spherically symmetric, so... I, I just don't see why it implies that the cylinder has to touch the sphere necessarily…the outer surface, right, of the sphere, necessarily. Are we allowed to ask questions?	**S/J**
P:	I mean, I think it's kind of like… like you could be anything if you, if you could do like, you know, uh, it has to be that.	**D/J**
Q:	Yeah.	**Re-formulation**
P:	You know otherwise there's almost like no question.	

Figure 6. Pat and Quinn.

Conjecturing is rooted in Discovery. In this study we see many students posing conjectures based on a combination of intuition and rational argument; this action is immediately followed by a conscious and directed search for more information and formal justification. Presenting a solution to an ill-structured problem requires Structuring and Justification, in part because students' perceptions of a final, formalized solution are wildly varied. The results of this research demonstrate that ambiguous wording can occasion the conscious act of constraining while the absence of visual representation in a problem statement forces the development of mental and physical imagery. Both acts of Structuring—constraining and visualizing—also involve Justification in the early part of a problem, which is then referred back to in the later implementation and verification of the solution.

Many activities that arise through ill-structured problems lead to the development of new mathematics. Learning how to re-formulate a problem may be an introductory lesson in learning how to create new mathematics problems. Guessing, conjecturing, heuristic activities such as constraining and considering special cases, analyzing perceived coincidence, and later, reflective observations can all lead to the development of new mathematics problems. As we see in this study, some ideas arise but are undeveloped while others seem just under the surface. Discovery and Structuring are concentrated during the initial work on an ill-structured problem.

This framework provides educators and education researchers, as well as students, a means for discussing mathematical thinking. It helps teachers and students to better understand the processes of thinking mathematically, builds a relatively simple educational tool for designing problems, and helps teachers and students begin to make changes to their teaching and learning to reflect these aims and highlight these processes. Future directions of this research include gaining a better understanding of the initial re-formulation of a problem, and the analysis of a single problem kernel in multiple representations.

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UNDERSTANDING LEARNING IN MATHEMATICS THROUGH THE METAPHOR OF AUTHORING

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INTRODUCTION

Mathematics education reforms have emphasized students' personal development of mathematical ideas (NCTM, 2000; WNCP, 2008). Absent from these reforms and from experiences in many high school mathematics classrooms is explicit discourse about the processes of learning—both identifying the strategies students use to learn (e.g., homework, study groups) and how to adapt strategies. While study 'how-to' books abound (e.g., Ooten & Moore, 2010), the publications are didactic in their approach and divorced from individual students' intentions and processes for learning mathematics.

Within mathematics education, research into metacognition has supported improvement in students' mathematical thinking (e.g., Hamilton, Lesh, Lester, & Yoon, 2007; Schoenfeld, 1987) and has identified successful students as those who understand their personal learning processes (Dahl, 2004; Smith, 1999). Emphasizing a shift from knowing to learning, in the reported study I explored students' development toward becoming capable mathematical learners by addressing the question: *How can we understand students' learning as they actively engage in developing their processes of learning math?* Results demonstrated that students began to see themselves as capable learners as they developed personal learning processes that supported mathematical understanding.

MODE OF INQUIRY

I framed the study methodologically using constructivist grounded theory [CGT] (Charmaz, 2014), which returns to the symbolic interactionist root of grounded theory while looking through a constructivist lens as an interpretive process for inquiring into dynamic phenomena. Within this postmodern orientation, theory is constructed by a researcher on a provisional basis and contingent to the context. There is "emphasis on examining processes, making the study of action central" (p. 16), recognizing that shifts in people's actions and experiences signify growth and changes within the people and their interactions. CGT supports a focus on students' development of mathematical understanding as a processual phenomenon by interpreting empirical data of students' experiences, rather than applying extant theories. The use of CGT also responds to the growing importance in theorizing to make progress within the field of mathematics education (Hiebert, 1998).

PERSPECTIVE

Within a symbolic interactionist approach, I adopted Blumer's (1954) notion of *sensitizing concepts* to "merely suggest direction along which to look" (p. 7). The researcher's sensitivities—what the researcher attends to because of her/his experiences of conducting research, scholarship in the field, and interests—are explored and employed as a starting place in data analysis. My sensitizing concepts developed out of two related research projects (Mason & McFeetors, 2007; McFeetors, 2006), and include: *intentions*, *voice*, *identity*, and *relationships with sources of knowledge*.

Intentions are internal constructs that give meaning to actions. These thoughts and desires arise from attention to previous experiences and to the consequences of actions, often through reflection. When students are intentional, they are acting with the intentions they have formed and hold, to move toward a particular end-in-view. Intentions point to what students want to do or achieve, an aim, and a notion of how they might go about doing, a process (see Searle, 1983).

Voice points toward having confidence to say things and to do so, making sense of experience through conversation and being implicated in shaping oneself. Voice is a dynamic concept, where a student's voice is continually being refined through experience and through the voicing of experience and growth of self (see Baxter Magolda, 1992; Confrey, 1998).

Identity is an understanding or sense of self. (Re)forming identity is shaping a way of being in the world and understanding that way of being, and is continually undertaken through experiences and relating with others. Shaping an identity is the ongoing negotiation of a student's relationship with mathematics, learning, schooling, others—identity is malleable and complex (see Britzman, 1994; Sfard & Prusak, 2005).

Relationships with sources of knowledge (e.g., teachers, textbooks) point to students' epistemological stances informing their mathematical learning. Relationships characterized by dependence, independence or interdependence are illustrated through where mathematical authority is ascribed and in ways of learning (see Belenky, Clinchy, Goldberger, & Tarule, 1997; Chickering & Reisser, 1993).

RESEARCH CONTEXT AND METHODS

The study occurred in a suburban school in a Western Canadian city. Thirteen grade 12 students participated in the study. In addition to their *Pure Mathematics* course, they were enrolled in a course called *Mathematics Learning Skills*. In *Learning Skills*, students received support for their mathematical learning by working individually and cooperatively on homework and requested help from the teacher. Within the course, I assisted the teacher in providing mathematical support and coaching students to improve their approaches to learning mathematics, simultaneously constructing data.

Data construction occurred over four months, where the class met four times per week. I wrote field notes after observing each class. Students wrote bi-weekly interactive journal writings (Mason & McFeetors, 2002) where they described how they were learning mathematics. I replied in order to interact with their ideas and model thinking about learning. Students took part in one of three small groups to develop a learning process as a group (transition from notes to homework, develop big ideas from homework, and create summary sheets). Small groups met three to five times. Students also participated in two informal interviews, midway and at the end, as a retrospective look at their progress in shaping their learning strategies. While the interactions were intended as multiple sources of data, they also afforded students opportunities to develop learning processes to support their mathematical understanding.

Guided by my sensitizing concepts, I analyzed the data with line-by-line coding and the constant comparative method (Glaser & Strauss, 1967). I initially used *in vivo* codes (e.g., "*making quick notes*", "*connecting*") to mark students' development and use of learning processes across all forms of data. Codes were refined and abstracted through several passes through the data, using phrases like "*explain to self*" and "*seek help*". The interpretive level of coding occasioned awareness of processes and the conversational context within which students were developing as mathematical learners.

Coding facilitated intimacy with the data, but did not support the elevation of codes to categories. Dey (2010) explains, "coding does not exhaust the analytic process" (p. 167) and that "often categorization does not proceed through the invocation of rules at all but through comparison with recalled or prototypical exemplars" (p. 169). Proceeding with both codes and prototypical exemplars, I constructed three frameworks that explicated related categories concerning students' learning. Identifying the interrelationships among the categories in the frameworks and the frameworks themselves, I theorized that students' mathematical learning and learning about their learning processes could be understood through a metaphor of *authoring*.

RESULTS: THREE ANALYTIC FRAMEWORKS

Each of the following three frameworks represents different interpretive moments with the data, demonstrating the complexity of and interconnectedness of the students' experiences of learning to learn mathematics. The individual frameworks each foreground different aspects of the students' growth in learning mathematics, and each draws together related categories of analysis. The first framework presented takes a wide view in making sense of the opportunities for conversations about learning. The second framework zooms in to portray how students' developed learning processes within the conversational spaces. The third framework looks within the processes to examine how students came to understand mathematical ideas in their learning.

FOSTERING LEARNING-BASED CONVERSATIONS IN MATHEMATICS

Students participated in learning-based conversations, where inquiring into ways of learning mathematics were foregrounded against the backdrop of content. As well, students were improving their approaches to learning mathematics and growing as mathematical learners through the conversations. The four features of opportunities for learning-based conversations include: *preparation*, *presence*, *mode*, and *pace*. The features represent qualities of providing opportunities for students to talk about and improve their learning strategies. *The Framework for Fostering Learning-based Conversations in Mathematics* focuses on occasioning learning-based conversations, rather than of the qualities of conversations themselves.

The *preparation* feature points to the varying degrees of advanced planning that took place in providing opportunities for the students to attend to their learning. This feature has a temporal dimension, from spontaneous to deliberate interactions. In offering help in class, I recorded a field note where Teresa "*asked me if it was like a question in her notes…I encouraged her that she had used a great strategy*" for getting unstuck—an example of being alert to a spontaneous moment to shift the focus to learning. When I deliberately showed students a list of learning strategies they used during an interview, Grace exclaimed, "*That's a lot!…I thought I only had two or three ways to learn math.*" The deliberate planning for learning-based conversations allowed for exploration of the processes and meaning of learning for the students.

The *presence* feature refers to the composition of members in the conversation, ranging from internal dialogue to including teachers and peers. While self-talk often focused on mathematical thinking, Danielle described, "*sitting on the bus, and I was thinking ... how would I be able to separate my ideas and stuff*" as a catalyst for a new learning process. In small group sessions, students suggested and considered different ways of learning mathematics content. Students valued different perspectives on learning processes, not looking for experts to inform them but rather a responsiveness to turning round ideas in conversation with fellow inquirers.

The *mode* highlights the dialogic nature of learning-based conversations through various forms, including spoken interactions, textual artifacts, and a hybridity of these two modes. Spoken discourse occurred mainly through one-on-one interactions and small groups. It required moving students from noticing content to identifying how learning occurred. Interactive journal writing, as a textual artifact, gave students time to pause and consider their approaches to learning. It was a safe space to share emergent thoughts about learning and as I wrote back I could scaffold by drawing learning into view. Upon returning a journal, Kylee exclaimed, "*This idea for the cue card is great! I'm going to try it tomorrow*"—a hybrid of student, me, and text.

The *pace* feature indicates a shift in classroom rhythm that allowed for a suspension of time from content to explore issues of learning. When Shane explained, "*sometimes I just think about how I learn*," it was within a relaxed feeling of learning-based conversations contrasted with the rapidity of mathematics content. Opening up brief moments mattered to students. For instance, Ashley identified, "t*he [small] groups that we're doing, it's mostly concentrated there*," for conversations where learning processes were developed that helped her succeed in mathematics. The different intensity fostered students' choice to engage in learning-based conversations where they inquired into ways of learning mathematics.

DEVELOPING PROCESSES OF LEARNING MATHEMATICS

Learning strategies (e.g., study, copy notes, do homework) were ways students were told to learn mathematics and were labels that did not make clear how to enact the strategies. As students interrogated learning strategies, they engaged in developing learning processes. Learning processes are ways students make sense of mathematical content and are developed by students in response the individuality of the learner. Understanding students' engagement resulted in the *Framework for Developing Processes of Learning Mathematics*. The five facets of the framework portray how students developed personal processes of learning mathematics. Being enacted simultaneously and in an interrelated fashion, *becoming aware*, *incorporating suggestions*, *verbalizing possibilities*, *developing intentions*, and *seeing themselves as learners* were the ways in which students shaped how they were learning mathematics.

The students were *becoming aware* of the learning strategies, the limitations of those strategies, and the personal nature of their mathematical learning. Kylee, who had previously created a cue card system for learning biology, noticed limitations for mathematics in writing, "*I realize how much of my time I waste making Q-cards (sic) before my test when I could instead be studying them*," and found an opening to refine an existing process of learning. Shane's insight, "*I would focus on learning how these numbers work and now I guess how the numbers work is a concept in itself, but I never thought of it that way*," signals a new awareness that his mathematical learning could fit his identity as a conceptual learner. Growth in students' awareness was situated in a space where their voices could be heard and valued, even in its tentative state.

As the students shaped ways of learning, they were *incorporating suggestions* from their peers and from me. Kylee described my interactions with her as, "*you weren't telling me to do something or getting mad because I did that on a math test. You were just encouraging*," as I offered alternatives and worked from students' current capabilities in learning mathematics. Drawing on both peers' and my support, Elise exemplifies incorporating suggestions by actively modifying suggestions. She modified Danielle's approach of using sticky notes on summary sheets as, "*I kind of like the way she does it, but I think it works better for me the way I do it*." While students listened to the ideas of others, they recognized that they were expert sources of knowledge about how they learned.

When students deliberated on ways of learning and put their ideas into words, they were *verbalizing possibilities* for learning processes. Danielle demonstrated that sometimes these were internal conversations, where "*I was just sitting on the bus, and I was thinking…how would I be able to separate my ideas*." Later on, Danielle wrote in a journal, "*With the method I developed, my ideas are organized and laid out in a way that really helps me understand*." While handing in the journal she explained aloud how she would use different colours of sticky notes to represent different kinds of content. Her oral utterance is an example of her first attempt to articulate for herself emerging details for her learning process. As we worked together in a small group on this way of studying, Danielle not only refined her explanations for the use of sticky notes as an organizational technique, but used them to demonstrate how she connected mathematical ideas across a unit of content. Verbalizing possibilities demonstrated that students could be sources of knowledge for how they learn and perceived their voices as being valued in the learning context.

Students were *developing intentions* for particular ways of learning mathematics as they moved away from an unquestioning use of learning strategies. Grace articulated a shift from submitting to external expectations toward intending to understand. Her growing intention of putting mathematical ideas in her own words "*so it's easier for me to understand*" impacted all her learning processes. Her study group's talk also shifted from "*how you got the answers*" to "*we discuss why you're doing it*" demonstrating the intention she had for her collaboration with peers evolved from comparing answers to mathematical discussion. As students became aware that learning processes could be adapted to fit who they were as learners, the intention was not only how the process would support their mathematical learning, but they also became intentional about improving their own learning of mathematics.

Within each of the preceding forms of engagement, students were *seeing themselves as learners* as they shaped learning processes that they saw as personal. That learning is personal in nature was highlighted by many of the students, as they explained that they needed to uniquely shape learning processes for themselves. Laurel points to growth in seeing herself as capable of learning mathematics when she explained, "*I was just fed up, and I was like, 'I can't learn math.' …I really wanted to kind of show that I could do math…it was nice to figure out that I can do it*." In figuring out how she learned, Laurel's perception of who she was as a learner shifted to a learning-based orientation in the world. This allowed Laurel to do more than go through the motions of completing homework—she was actively learning from homework. The (re)forming of each student's identity meant that they were in the process of growing personally and also contributed to a stronger sense of their agency in shaping mathematical learning processes.

STUDENT-GENERATED MECHANISMS FOR UNDERSTANDING MATHEMATICS

The students acknowledged that a memorization-based approach had left them struggling, and they subsequently engaged in an inquiry approach out-of-class that enabled them to understand mathematical ideas. The *Framework for Student-generated Mechanisms for Understanding Mathematics* represents the students' processes for understanding mathematical content through four categories: *breaking down*, *putting together*, *connecting*, and *writing down*. The students named the processes as integral to their sense-making of mathematics. Through this generation and an understanding of mathematics, the students saw themselves as sources of knowing, able to shape mathematical ideas through verbalizing their understanding and ways of coming to understand.

Students developed the mechanism of *breaking down* when making meaning of a symbolic system recorded for worked solutions in their homework or notes. Elise noticed that her learning process of identifying key ideas "*broke down what it actually meant. So, I understand what to do when I have a question*," as a way to understand each component part in a single lesson. The students saw the teacher's presentation of a lesson as a whole in which the parts were largely inaccessible. Breaking down allowed students to understand of each of the mathematical elements within a lesson for themselves. The students needed the opportunity to see and make sense of all the component parts of a lesson, ascertaining these for themselves.

The action that often followed breaking down was to reassemble ideas into a coherent whole: *putting together*. Chelsea emphasized that in order to understand mathematics topics, "*You have to put it all together*." Nadia explained, "If *I have the main idea and then everything that could possibly be underneath that and I continue doing that for all the different information, they you can see that I really have everything put together*." The collecting of fragments of mathematical ideas reconstructed a whole that had been presented in class in a new way. Putting together existed on the scale of a singular lesson. The whole of the mathematical ideas in a lesson were now of the student's making and having, rather than the teacher's providing.

Relationship-building also occurred across multiple lessons, which students named as *connecting*. Rather than understanding the intricacies of a specific procedure as with *putting together*, students identified relationships between previously learned ideas and new content in connecting. Danielle explained the process of connecting when making unit summary sheets: "*I don't really connect them to, till the end, then I get it more…It's mostly in my head. I just make those connections by myself*." Ashley demonstrated connecting directly on her summary sheets as she would "*highlight or draw after in arrows how these ideas connect to the different parts of the chapter*." In connecting, students were generating relationships for themselves instead of taking on the teacher's structuring of content.

Writing down signified an intention of understanding mathematical ideas, in contrast to scribing notes in class explained by Shane as being "*like a drone—copied down the notes*." Danielle described, "*I just wrote everything down. Make sure I understand them*." Danielle would only commit to paper that which she understood. Often writing down was seen as a culminating activity; however, Kylee would refine her provisionary understanding multiple times throughout a unit: "I*'ll write down…to make sure I understand—it's all definitely quite an active process.*" The students would commit to words only those ideas they had rendered sensible.

DISCUSSION: THEORIZING WITH THE METAPHOR OF AUTHORING

I have often wrestled with the metaphor of *constructing*. While it provides a way to view students as actively shaping their mathematical understanding, the metaphor points to inanimate edifices, which could serve to dehumanize the act of learning. By foregrounding content, the metaphor has the effect of minimizing the student as a person being shaped by experiences of learning—students' (re)forming of identity through learning mathematics.

As I listened to the students and sought to develop an abstract interpretation of their experiences in learning to learn mathematics, an alternative metaphor seemed to better represent their growth. Baxter Magolda (2001) viewed post-secondary students' maturation as "*self-authorship—the capacity to internally define* [...] *beliefs, identity, and relationships*" (p. xvi). In mathematics education, the authoring of self is seen as a component part of identity formation (Graven, 2012). I propose that authoring can be explanatory of an ontological orientation toward mathematical learning more broadly.

Authoring is a generative activity of making meaning of experiences and interactions that shapes self and the world. Engaging in the act of authoring implicates the author in self-making as he/she expresses understanding with a sense of authority through his/her voice to an audience. Authoring, then, can be used to understand learning in its complexity as it relates to epistemological and ontological growth through experience.

Throughout the project, students were making meaning of their learning in a dialect between emerging awareness and (re)forming intentions through actively generating effective learning process for mathematics. Elise stated she was "*figur[ing] out that there are ways that I can be creative…I can learn it in a way that works for me*." Students, as authors, invested themselves into their learning processes as they tentatively verbalized possibilities for a process as they deliberated on suggestions offered by others. Danielle commented for one learning process that it was "*the method I developed*." She was developing internal authority as she became aware of process that fit who she was as a learner. The students saw the learning processes they had authored as their own because they had imbued the personal processes with their own intentions for mathematical learning.

The students were generative as they were putting together component parts of a mathematical topic and connecting across topics, transforming mathematical content previously given by their teachers. Through this authoring, students viewed the mathematical ideas as meaningful. Grace commented, "*You can tell how I'm improving…I actually understand what I'm doing*." Students' emerging authority was exhibited in the mechanism of writing down, where Kylee provides an example of a sense of authority, "*writing it down is definitely learning it in my own ways instead of just how it is in the workbook*." She was voicing the mathematical content she had authored as her own.

In addition to *authoring* interpreting the cases of learning to learn mathematics and learning mathematics, it also attends to the learner as the one authoring himself or herself into the world. Through experiences of learning, the learner is in the process of being and becoming. In the final interview, Laurel explained in three related moments, "*it was nice to figure out that I can—do it*"; "*I've figured out what works best for me*" in her process of learning math; and, that high school was "*about learning to learn*." In these, Laurel exemplifies on behalf of her peers that students voiced self-authoring as individuals capable of doing mathematics, learning mathematics, and learning to learn mathematics. As the students (re)formed their identity in self-authoring, the students' growth as mathematical doers and thinkers indicated their *becoming*.

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HOW ELEMENTARY STUDENTS LEARN TO MATHEMATICALLY ANALYZE WORD PROBLEMS: THE CASE OF ADDITION AND SUBTRACTION

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ABSTRACT

Mathematical problem solving, and more specifically the ability to mathematically analyze and model a situation, is an extremely complex phenomenon. The lack of a nuanced understanding of the reasoning involved prevents teachers from effectively meeting students' needs.

While the developmental approach (Davydov, 2008) was implemented to teach problem solving to Grade 2 elementary school students, I used the grounded theory methodology to study how the students solved additive problems. One of the research questions was "What kind of mathematizing do students use to solve additive word problems?"

In this paper, I present some theoretical models stemming from my PhD thesis (Polotskaia, 2015), which can be used to interpret students' reasoning while solving word problems.

RÉSUMÉ

La résolution de problèmes mathématiques, et plus particulièrement la capacité d'analyser et de modéliser mathématiquement une situation, est un phénomène extrêmement complexe. Le manque de compréhension nuancée de ce phénomène empêche les enseignants de répondre de façon efficace aux besoins de leurs élèves.

Une approche développementale (Davydov, 2008) a été utilisée pour enseigner la résolution des problèmes aux élèves de la deuxième année du primaire. Au cours d'une année, en appliquant la méthodologie de la théorisation ancrée, j'ai étudié le raisonnement qu'employaient ces élèves pour résoudre des problèmes additifs. Une des questions de la recherche était « Quels sont les moyens de mathématisation que les élèves utilisent pour résoudre des problèmes écrits? ».

Dans la présente communication, je discute quelques modèles théoriques issus de ma thèse doctorale (Polotskaia, 2015), qui peuvent servir dans l'interprétation du raisonnement des élèves en résolution de problèmes écrits.

> Complexity "is a hallmark of educational settings".
> (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 9)

INTRODUCTION AND PROBLEM

Mathematical problem solving, and more specifically the ability to mathematically analyze and model a situation, is one of the most important aspects of teaching and learning mathematics in school (Lesh, Doerr, Carmona, & Hjalmarson, 2003). There is a core of research (Carpenter, Fennema, & Franke, 1996; Fagnant, 2005; Julo, 2002; Ng & Lee, 2009; Novotná, 1999; Stacey & MacGregor, 1999) that examines various ways of supporting students in problem-solving activities, including representing and modeling. Arithmetic problems with simple additive structures that can be solved using one addition or subtraction operation are not usually considered to require any mathematical modeling. Usually students use various drawings of objects or manipulations with small objects to support their reasoning and calculation to solve such problems. It is well described in the research (e.g., Nesher, Greeno, & Riley, 1982) that in the first stages of development, students understand a word problem as an instruction to manipulate and can find the answer as a result of the direct execution of this instruction. For example, "*more marbles*" results in the addition operation.

> Peter had 8 marbles. Mara gave him 5 more marbles. How many marbles does Peter have now?
>
> Solution: 8 + 5 = 13

At the final stage, students are able to see the problem as a whole and their reasoning becomes flexible. "[The student] is able to read the word 'more' in the text, and yet perform a subtraction operation" (Nesher et al., 1982). For example, "*more marbles*" yields the subtraction operation.

> Peter had 8 marbles. Mara gave him some more marbles. Now, Peter has 14 marbles all together. How many marbles did Mara give Peter?
>
> Solution: 14 – 8 = 6

However, our understanding of the development of students' reasoning from direct manipulation to more powerful abstract strategies is still not clear. Research has yet to produce a satisfactory explanation for whether this developmental path is the most efficient and how we can better support students in such development. For example, there is no empirical evidence in research of how and why learning to mathematically model word problems with simple structures may affect the development of students' mathematical reasoning (Lingefjärd, 2011). The lack of nuanced understanding of ways of reasoning that students may use to analyze and model a problem prevents teachers from efficiently meeting students' needs.

In my research, I identified holistic flexible reasoning within the context of additive word problems as a learning target and studied how students reason while modeling simple additive word problems.

TWO PARADIGMS IN WORD PROBLEM-SOLVING DEVELOPMENT

Within the domain of mathematics education, we can distinguish two paradigms. Within the *Operational Paradigm*, knowledge about word problem solving develops from the initial understanding of arithmetic operations (as proposed by Nesher et al., 1982).

Within the Operational Paradigm, the initial knowledge of operations as processes (adding or removing) serves as a basis for students to interpret a word problem. Thus, the interpretation is often sequential and the problem is understood as a sequence of events. (See Figure 1.)

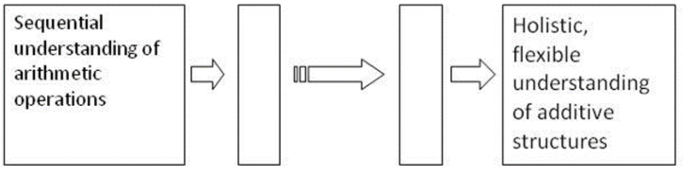


Figure 1. The Operational Paradigm.

Within the *Relational Paradigm* (as proposed by Davydov (1982)), the same holistic and flexible understanding should be developed from a holistic understanding of the additive relationship studied with physical objects such as ropes, liquids and surfaces. Davydov (1982) describes the concept of additive relationship as "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229). Thus, in order to understand a word problem, students need to identify the additive relationship involved (the whole and its two parts).

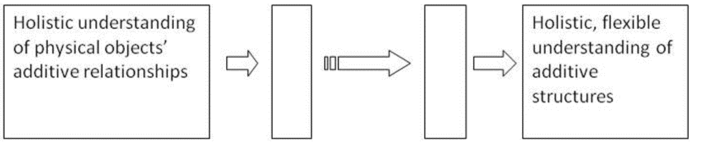


Figure 2. The Relational Paradigm.

In my study, I adopted the *Equilibrated Development Approach* (EDA), proposing to develop a holistic flexible understanding of additive word problems using both elements: operations (addition and subtraction) and additive relationships. Within EDA, prior to solving problems, students develop some knowledge about addition and subtraction as operations and learn to identify the parts and whole of physical objects with regard to their length (additive relationship). To solve a problem, students were asked to analyze the text to identify the additive relationship involved, model this relationship and manipulate the model in order to suggest the arithmetic operation. (See Figure 3.)

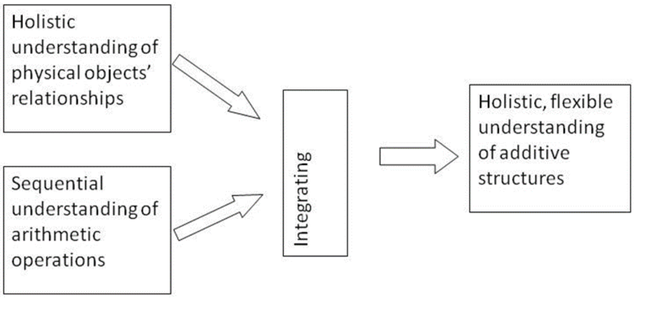


Figure 3. The Equilibrated Development Approach.

To model additive relationships, we used *Arrange-All diagrams*, a method similar to one described by Davydov and some others (Davydov, 1982; Fan & Zhu, 2007). An example of an Arrange-All diagram is as follows:

> Peter had 8 marbles. Mara gave him some more marbles. Now, Peter has 14 marbles all together. How many marbles did Mara give Peter?

> **Solution:** One needs to start with the whole and hide the known part: 14 – 8 = ?

I studied students who participated in a larger research project, headed by Dr. Savard, and were exposed to the EDA training since the beginning of their Grade 2 elementary school year. One of my research questions was "*What ways of mathematizing do students use while solving additive word problems?*"

METHODOLOGY

To understand and describe students' reasoning within the Relational Paradigm, I applied grounded theory methodology. Researchers (Bruce, 2007; Goldin, 2000; Zazkis, 1999) suggest grounded theory methodology as an approach to developing new models and theoretical insights based on observations. This methodology mainly includes qualitative methods, such as individual interviews with students. These methods allow researchers (Krutetskii, 1976; Piaget, 1974; Vygotsky, 1997; Vygotsky & Luria, 1930/1993) to create a new vision of their research subjects and contribute to the theory's development.

Bruce (2007) highlights that grounded theory research is not fully an inductive process. The existing theoretical knowledge can be used to initiate data collection, which then becomes driven by emergent data. Following this suggestion, I used two simplified theoretical models to start interviewing students and interpret their production and reasoning in problem solving.

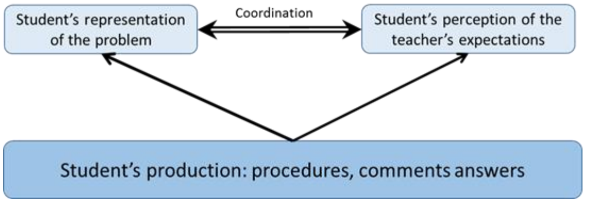


Figure 4. M1: Simplified model of student's production interpretation according to DeBlois (2011).

According to the first model (see Figure 4), the student's production in problem solving is affected by her mental representation of the problem and perception of the teacher's expectations.

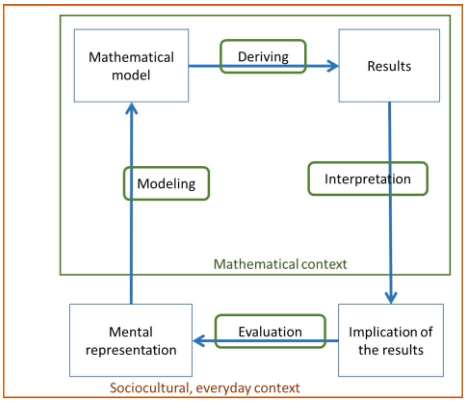


Figure 5. M2: Simplified model of problem-solving process according to Savard (2008).

According to the second model (see Figure 5), in order to solve a word problem, the student should interpret (model) the problem mathematically, moving from the sociocultural context to the mathematical context and back.

I started with these two models, then used the collected data to identify various characteristics of students' ways of mathematizing and to modify the models to make them coherent with the data.

CONTEXT AND PARTICIPANTS

Two Grade 2 classes in the same school in the French-speaking community agreed to participate. Two experienced teachers were participating in the professional development program created within the scope of the larger research project. I selected six students from each class: two relatively high performers in mathematics, two average performers, and two relatively low performers (total of 12 students). To select my participants, I considered their results in a written problem-solving test and the judgement of their teachers.

DATA COLLECTION

In this paper, I discuss only the first part of the study: the analysis of individual observations of students solving word problems. I video recorded 91 observations over 4 interview sessions in November, January, March, and May of the same school year. In each interview, I asked the student to solve specially designed word problems involving addition and subtraction operations.

THEORETICAL RESULTS

The two chosen theoretical models provided a good start in interpreting the students' reasoning. One of the popular ways of mathematizing used by the students, mimicking the problem, can be well explained by these models. In the following example, the student uses the 'teacher-approved' method (can be explained by M1) of drawing small circles to mimic the events described in the problem.

> There were 34 logs in the pack that dad bought to make a campfire. The fire burned for 48 minutes. Some logs were already burned. There were 27 logs left in the pack. How many logs were burned?
>
> **Student's explanation:** *I will draw 34 logs and then remove till there are 27. So, I will see how many were burned.*

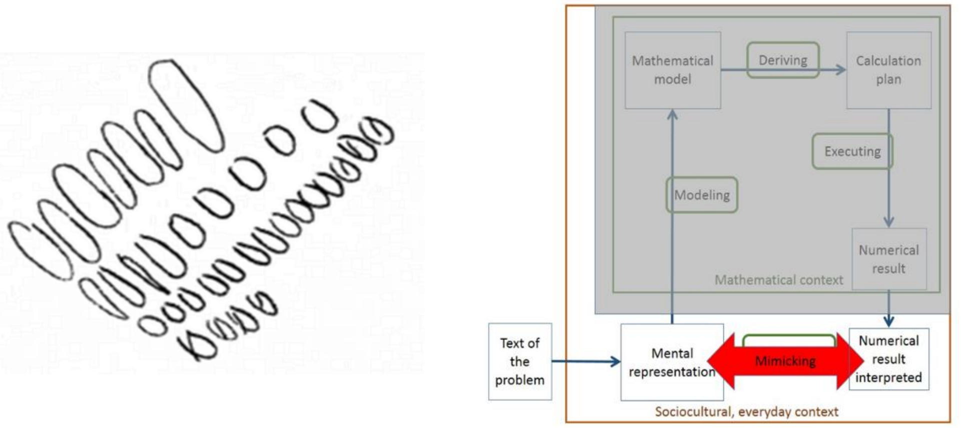


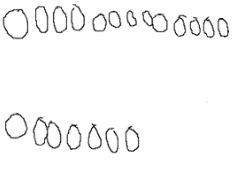
Figure 6. Student solution analysis with M2.

As the M2 model reveals, the student never entered into the mathematical context (see Figure 6). She directly played the event (burning logs) and visualized the result as the final state of the event.

Many characteristics of a problem's text can directly affect students' reasoning (Voyer, 2011). In my study, many students strongly associated the verb *burned*, used in the problem above, with a well-known semantic meaning (remove). This led them to an incorrect interpretation of the text and the intended mathematical task. From my analysis, it followed that the text of a word problem and its special characteristics should be included at the basis of each model.

Further analysis revealed that students' previous knowledge together with their perception of the teacher's expectations could directly affect their mental representation of the problem. For example, analyzing the following problem, one student pointed with her finger to the two numbers, put on the paper the corresponding two groups of small circles and counted them all together. She explained that, "*numbers are important*." For this student, the text of the problem was a source of numerical information and not so much a meaningful description of a situation.

> I took 13 tokens. I hid 7 of the tokens in my left hand and the rest in my right hand. How many tokens are in my right hand?



Furthermore, students' production can depend on the form of communication (media) chosen by the teacher. In the case of the following problem, the verbal explanations given by a student, the diagram constructed and the mathematical operation were not coherent with each other.

> Tomas made a bunch of snowballs for his snowball fight with his friend Greg. After 15 minutes of fighting, Tomas had thrown 37 snowballs. He now has 25 snowballs left. How many snowballs did Tomas make?

Student's first explanation: *After, he threw 37, there were already 37, then he removed, because he threw them, he removed them. ...There are 25 remaining.*

The student's diagram, which seems to be coherent with the text and their first explanation, was constructed mechanically by putting the two known numbers into the part-part-whole diagram template (see Figure 7).

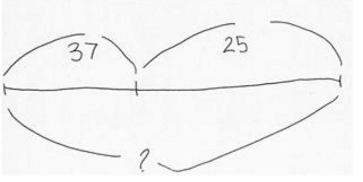


Figure 7. Student's part-part-whole diagram.

Student's explanation of the operation: *There are already 37. ... Then 25, we removed them. 37 – 25. Minus because he removed. Because when he throws, it makes minus.*

In this case, none of three communications is meaningfully linked to another.

Summarizing the analysis of all data, I present here the two modified models:

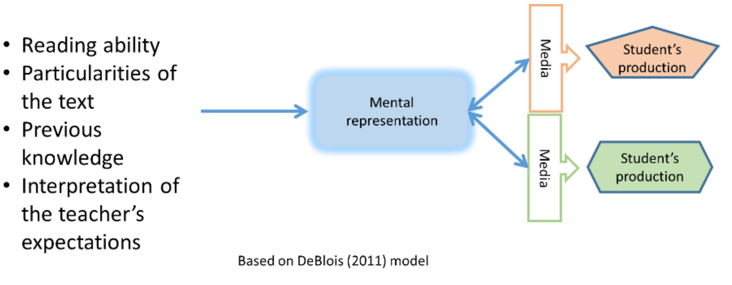


Figure 8. M1.1: Interpreting student's production in word problem solving.

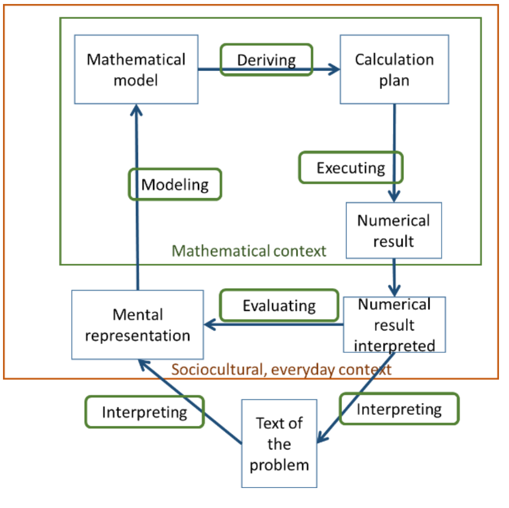


Figure 9. M2.1: Model of solving simple word problems via mathematical modeling (solving cycle).

PRACTICAL CONCLUSIONS

The limits of this communication do not allow for more detailed discussion. The following describes some conclusions from my PhD project.

The Relational Paradigm appears to be a valuable tool for research and practice in mathematics education. The use of this paradigm is very important in studying how students learn to mathematically model problems and situations. If teaching conforms to the Relational Paradigm, the development of holistic relational reasoning for solving word problems can be initiated from the very beginning of mathematical instruction in school. This type of reasoning can grow in parallel with other types of reasoning.

The use of Arrange-All diagrams as a modeling method has a strong effect on students' learning. The diagram representation method itself as well as the multiple forms of communication requested from the learners proposes an additional challenge in coordinating reasoning and communication. The difficulty students have coordinating different media (M1.1) seems to be caused by the absence of appropriate knowledge and ability and might thus be considered a learning target (Vygotsky, 1984). From my observations, it follows that the practice of this specific coordination of reasoning is the main engine of the development of explicit holistic analysis and modeling in students. The meta-cognitive process of solving a word problem organized as a problem-solving cycle (M2.1) can provide effective and efficient support for such development.

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LE CONCEPT DE VARIABILITÉ CHEZ DES ENSEIGNANTS DE MATHÉMATIQUES AU SECONDAIRE

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RÉSUMÉ

Cette recherche visait à explorer les connaissances didactiques des enseignants reliées au concept de variabilité. Douze enseignants de mathématiques du secondaire ont d'abord été testés sur leurs connaissances relatives au concept de variabilité. Les sujets ont ensuite été interrogés afin de voir s'ils étaient en mesure de reconnaître dans l'action, à partir de simulations d'enseignement, des conceptions inadéquates liées à l'étude du concept en question et de proposer des interventions lorsque confrontés à des réponses et raisonnements d'élèves. L'analyse des réponses des enseignants a permis d'explorer leurs connaissances didactiques relatives au concept de variabilité et de porter un regard sur l'enseignement de ce concept, notamment en documentant des pistes d'interventions possibles pour l'enseignant. De plus, les résultats montrent que les raisonnements précédemment observés chez des élèves et étudiants universitaires ont également été observés chez les enseignants de mathématiques du secondaire.

INTRODUCTION

La place tenue par la statistique dans la société actuelle fait en sorte qu'il devient important de s'interroger sur l'enseignement de cette discipline. Or, son enseignement qui est inclus de façon générale, du moins au niveau pré-universitaire, dans les cours de mathématiques pose un défi majeur dû au fait que le raisonnement statistique comporte des spécificités qui le distinguent considérablement des raisonnements associés aux autres domaines des mathématiques scolaires. Le degré d'incertitude présent en statistique contraste avec les conclusions déterministes auxquelles les mathématiques scolaires nous ont souvent habitués. Je me suis arrêté au concept de variabilité qui est un concept clé pour le développement de la pensée statistique dans la mesure où la statistique est définie comme étant la science de la variabilité, la variabilité des phénomènes naturels et sociaux du monde qui nous entoure (Wozniak, 2005).

Prendre conscience de la variabilité d'un phénomène, c'est constater que les résultats sont sujets à variation, c'est concevoir que les résultats sont imprévisibles, c'est considérer les fluctuations d'échantillonnage, c'est faire le deuil de la certitude et s'engager dans le monde de l'incertain. Renonçant à des connaissances assurées, on peut alors à l'aide de méthodes statistiques accéder à une maîtrise relative de l'incertitude pour estimer, prévoir et prendre des décisions avec risque consenti. Voilà l'enjeu véritable du raisonnement statistique (Vergne, 2004).

Alors si on veut promouvoir le développement de la pensée statistique chez les élèves, l'enseignement du concept de variabilité est fondamental. Il convient donc de se demander quelles sont les connaissances des enseignants à ce sujet. J'ai donc entrepris de répondre, de manière exploratoire, à la question suivante: *quelles sont les connaissances didactiques des enseignants en rapport avec le concept de variabilité compte tenu de leurs connaissances disciplinaires* ?

LES CONNAISSANCES DISCIPLINAIRES ET LES CONNAISSANCES DIDACTIQUES

Inspiré des travaux de Shulman (1986, 1988), je distingue deux types de connaissances: disciplinaire et didactique, ce que Shulman (1986) décrivait en termes de « *subject matter content knowledge* » et « *pedagogical content knowledge* » (p. 9). Shulman (1988) définit les connaissances disciplinaires, « *subject matter content knowledge* », comme étant la compréhension appropriée des objets d'étude par un spécialiste du contenu dans le domaine. Quant au « *pedagogical content knowledge* », j'y réfère dans ce texte comme étant des connaissances didactiques puisqu'elles sont spécifiques au contenu, mais relatives de l'enseignement et de l'apprentissage de la discipline. Les connaissances didactiques font référence à la façon d'aborder et de représenter un sujet afin que celui-ci soit compréhensible aux yeux des autres. Avec ce type de connaissances, on va au-delà de la connaissance du contenu pour se concentrer sur une autre dimension soit la connaissance du contenu à des fins d'enseignement. Les connaissances didactiques incluent une compréhension de la part de l'enseignant de ce qui rend l'apprentissage d'un contenu spécifique facile ou non pour les élèves en se référant notamment à leurs conceptions, à la connaissance de leurs erreurs les plus fréquentes ainsi qu'aux interventions possibles pour aider les élèves à se rendre compte de leurs erreurs et à surmonter les obstacles dont elles témoignent. Mais comment évaluer les connaissances didactiques?

Selon Vergnaud (2002), il existe deux formes de connaissances : les connaissances prédicatives et les connaissances opératoires. La forme prédicative représente la connaissance mise en mots, telle qu'on la trouve notamment dans les manuels. Les connaissances prédicatives permettent d'énoncer les propriétés et les relations des objets de pensée. La forme opératoire de la connaissance, quant à elle, est celle qui est mise en œuvre en situation. Les connaissances opératoires sont donc mobilisées dans l'action, elles permettent d'agir en situation. Selon Vergnaud (2006), la plus grande partie de nos connaissances se situent dans leur forme opératoire, souvent de manière implicite, voire inconsciente, puisqu'il existe un décalage parfois impressionnant entre ce qu'une personne peut faire en situation, et ce qu'elle est capable d'en dire. Par exemple, dans le cas qui nous concerne, il peut être difficile pour une personne de formuler *a priori* des conceptions inadéquates liées à l'étude du concept de variabilité, mais cela ne signifie pas pour autant que cette personne n'a pas de connaissances à ce sujet; si elle sait les reconnaître dans l'action, alors nous pouvons considérer qu'elle a des connaissances en acte (Vergnaud, 2002). Les connaissances didactiques des enseignants, telles que définies, relatives au concept de variabilité n'étant pas documentées, c'est donc dans l'action que nous y aurons vraiment accès.

MÉTHODOLOGIE

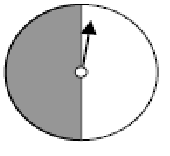
La recherche décrite dans ce présent article a été menée auprès de douze enseignants de mathématiques du secondaire au Québec (Vermette, 2013). L'expérimentation s'est déroulée en deux temps. D'abord, les enseignants devaient répondre à un questionnaire comportant six questions faisant intervenir le concept de variabilité. Cette phase de l'expérimentation a permis de porter un regard sur les connaissances disciplinaires des enseignants sur le concept de variabilité. Ensuite, suivait une entrevue dont l'objectif était d'explorer dans l'action les connaissances didactiques des enseignants sur le concept en question à l'aide de simulations d'enseignement. Pour chaque item du questionnaire auquel les enseignants avaient déjà répondu, une mise en situation faisant intervenir une tâche didactique était proposée. Dans l'exercice de ses fonctions, un enseignant est appelé à réaliser différentes tâches, certaines qu'il est possible de qualifier de didactique, car elles font appel à ses connaissances didactiques. Dans le cadre de notre étude, les tâches didactiques ciblées étaient l'analyse de problèmes, l'analyse de réponses et de raisonnements d'élèves et l'intervention.

Ci-dessous, une mise en situation est présentée à titre illustratif qui confronte les enseignants à une réponse et un raisonnement d'élèves et, par le fait même, les amène à analyser cette réponse et ce raisonnement afin de proposer une intervention. Elle est précédée de l'item du questionnaire sur la variabilité auquel elle se rapporte.

MISE EN SITUATION

La question disciplinaire[1]

Un enseignant fournit une roue identique à celle ci-dessous à chacun de ses élèves leur demandant de faire 5 séries de 50 tours et de comptabiliser, pour chacune des séries, le nombre de fois où l'aiguille s'arrête dans la partie ombragée. Vous avez préalablement réalisé l'expérimentation. Écrivez une liste de nombres qui décrit le nombre de parties ombragées que vous avez obtenues ? Pourquoi avoir choisi ces nombres ?



La question didactique

Un élève trouvant l'expérimentation trop longue décide de tourner la roue à 5 reprises et de multiplier le résultat obtenu par 10. Il effectue la même démarche pour chacune des séries. Que pensez-vous de sa démarche ? Comment interviendriez-vous auprès de cet élève ?

Cette situation met de l'avant le concept de variabilité dans un contexte d'échantillonnage à caractère probabiliste. La tâche proposée à l'enseignant ici découle du raisonnement fautif d'un élève qui ne tient pas compte de la taille de l'échantillon, comme si la grandeur de l'échantillon n'avait pas d'influence sur sa variabilité des résultats. Par ce raisonnement, l'élève sous-entend que les résultats obtenus seraient les mêmes pour chacune des dix répétitions. Selon l'étude de Reading et Shaughnessy (2004), certains élèves en fin de secondaire utilisent un raisonnement proportionnel afin de lier les proportions des échantillons aux proportions de la population. Dans ce cas-ci, un élève qui obtiendrait 4 parties ombragées sur 5 tours de roues déduirait qu'il en obtiendrait alors 40 sur 50 tours de roues. Toutefois, les caractéristiques d'un échantillon aléatoire se rapprochent des caractéristiques statistiques de la population plus la taille de l'échantillon augmente. Par conséquent, la variabilité d'un

[1] Adaptée de Watson, Kelly, Callingham et Shaughnessy (2003).

échantillon de taille 5 est plus grande que pour un échantillon de taille 50. Aussi, en tournant la roue à 5 reprises, il devient impossible d'obtenir la valeur correspondant à la probabilité théorique soit 50% de parties ombragées. La démarche préconisée par l'élève ne permet pas non plus de s'en approcher, car au mieux il est possible d'obtenir deux parties ombragées pour trois parties non-ombragées ou vice versa.

La tâche qui précède a permis de voir si les enseignants interrogés étaient en mesure de reconnaître dans l'action, à partir d'une simulation d'enseignement, une conception inadéquate liée à l'étude du concept de variabilité et a permis de prendre connaissance par le fait même des interventions qu'ils auraient faites et ce, afin de porter un regard sur leurs connaissances didactiques liées au concept en question.

RÉSULTATS

Cette recherche a permis d'identifier deux types d'intervention. Le premier consiste en une explication référant explicitement aux enjeux mathématiques (la taille de l'échantillon par exemple) dans le but d'aider les élèves à clarifier le concept et à répondre par le fait même à la question. Dans ce cas, rien n'est proposé qui pourrait engager explicitement les élèves dans un processus de réflexion, de recherche et de validation de leurs connaissances. Le second type d'intervention consiste en une confrontation. Celle-ci présente des conditions permettant aux élèves de mettre en doute leur raisonnement erroné et la réponse obtenue. Ce type d'intervention a le potentiel de créer un conflit cognitif chez les élèves, les obligeant à s'interroger sur leurs conceptions et à les rectifier.

La plupart des enseignants qui n'ont pas été en mesure de proposer d'interventions n'avaient pas répondu correctement à la question disciplinaire. Il était difficile pour certains enseignants de refuser les raisonnements erronés des élèves puisque eux-mêmes avaient répondu de la même façon. La recherche a ainsi permis de constater que les conceptions déjà observées chez des élèves et chez des étudiants universitaires se retrouvent également chez des enseignants du secondaire.

Afin d'illustrer ces propos, ci-dessous les résultats obtenus à la mise en situation précédemment présentée.

MISE EN SITUATION

<u>La question disciplinaire</u>

À cette question, dix sujets ont produit une liste comportant une variété de nombres pour représenter le nombre de parties ombragées obtenues d'un échantillon à l'autre. De ce nombre, neuf ont justifié les écarts proposés en fonction de la valeur attendue soit 25 dans ce cas-ci tandis que l'autre a affirmé que c'était le fruit du hasard.

Les deux sujets qui n'ont pas proposé de fluctuations d'échantillonnage ont donné une liste de nombres concentrés sur la valeur attendue correspondant à la probabilité théorique. Reading et Shaughnessy (2004) avaient d'ailleurs observé chez des élèves du secondaire, que l'étude des probabilités dans le cadre des mathématiques scolaires influence la prise en compte du concept de variabilité. Il m'apparaît raisonnable de croire que cette influence est en partie dû aux manuels où l'on y retrouve une part très faible de questions menant à l'analyse de fluctuations d'échantillonnage et à l'interprétation de l'incertitude au profit d'exercices de nature plus déterministe qui mettent l'accent sur le calcul de probabilités théoriques (Vermette, 2013).

La question didactique

Pour la question didactique, huit sujets ont vu l'enjeu de la taille de l'échantillon et ont proposé une intervention. De ces huit enseignants, six avaient donné une liste variée pour représenter le nombre de parties ombragées obtenues d'un échantillon à l'autre à la question disciplinaire. Les deux autres ont donné une liste basée uniquement sur la valeur attendue correspondant à la probabilité théorique.

Intervention de type explication

Un enseignant a expliqué les effets de la taille de l'échantillon sur les fluctuations d'échantillonnage. En acte, il a manifesté des connaissances relatives aux enjeux conceptuels traduits en explication à l'élève.

Intervention de type confrontation

Différents types de confrontation se sont manifestés.

- **Confrontation à une expérimentation** : Les deux enseignants qui n'ont pas considéré de fluctuations d'échantillonnage dans leur réponse à la question disciplinaire ont tout de même pris conscience que le raisonnement de l'élève risque de donner un faux portrait de la réalité. Ils ont donc proposé à l'élève d'effectuer des séries de tailles différentes et de comparer les résultats obtenus afin qu'il puisse voir les différences entre les séries. En acte, ils ont manifesté des connaissances relatives au rôle de l'expérimentation pour explorer l'influence de la grandeur de l'échantillon sur sa variabilité des résultats.
- **Confrontation à un cas extrême** : Trois enseignants ont confronté l'élève à un résultat extrême : à 0 partie ombragée obtenue lorsque l'on fait tourner la roue 5 fois serait associé 0 partie ombragée lorsque l'on fait tourner la roue 50 fois.
- **Confrontation à un cas particulier** : Un enseignant a confronté l'élève en exagérant son raisonnement. Par exemple, en lui demandant d'appliquer la même démarche, mais en faisant tourner la roue trois fois au lieu de 5 fois, le pourcentage du nombre de parties ombragées obtenues s'éloignera de la valeur correspondant à la probabilité théorique soit 50% puisqu'il ne pourra obtenir qu'une partie ombragée pour deux parties non-ombragées ou vice versa.
- **Confrontation à un cas analogue dans un autre contexte** : Un enseignant a suggéré de situer l'expérimentation dans un autre contexte empêchant ainsi l'atteinte de l'un des événements lorsque l'on transfère par raisonnement proportionnel (raisonnement utilisé par l'élève) les résultats obtenus pour un petit échantillon à ceux d'un grand échantillon.

  « *Si je lance un dé 5 fois, je n'obtiendrai pas les 6 résultats possibles si l'on fait référence à un dé à 6 faces. Ainsi, si je multiplie le résultat obtenu par 10, cela signifie que je ne pourrais qu'obtenir 5 résultats dans mon expérimentation, ce qui n'est pas impossible, mais peu probable; c'est-à-dire que l'on obtiendrait au moins 1 fois chaque résultat sur 50 lancers.* »

En acte, ces enseignants ont manifesté des connaissances relatives au rôle du contre-exemple pour confronter l'élève ainsi qu'aux variables sur lesquelles jouer pour construire ce contre-exemple : le résultat possible d'une expérimentation (cas extrême), le nombre d'expérimentations (cas particulier) et le contexte.

DISCUSSION ET CONCLUSION

La présente recherche met en exergue l'importance d'étudier les connaissances disciplinaires et didactiques des enseignants relatives au concept de variabilité, un concept qui est au cœur de la pensée statistique. L'exploration dans l'action des connaissances didactiques des enseignants sur le concept en question s'est fait à l'aide de simulations d'enseignement. Les situations didactiques proposées confrontaient les enseignants à des réponses et raisonnements d'élèves qui faisaient ressortir des conceptions relatives à la variabilité. Nous pouvons penser que les connaissances des conceptions relatives à un concept particulier permettent aux enseignants non seulement de mieux planifier leur enseignement, mais aussi de mieux organiser et gérer l'activité des élèves dans la classe de façon à ce que ceux-ci rencontrent les éléments d'un savoir mathématique visé. Les connaissances des conceptions relatives à un concept sont donc des connaissances didactiques puisqu'elles sont spécifiques au contenu, mais relatives de l'enseignement et de l'apprentissage de la discipline.

L'ensemble des résultats de cette étude montre que les connaissances didactiques des enseignants sont dépendantes de leurs connaissances disciplinaires. En général, ceux qui ont pu repérer les erreurs des élèves (qui ont manifesté des connaissances didactiques relatives à l'analyse des raisonnements des élèves) sont ceux qui avaient résolu adéquatement le problème mathématique. Toutefois, non seulement faut-il connaitre les conceptions liées au concept de variabilité, il est aussi nécessaire de pouvoir intervenir afin de contrer celles qui sont erronées ou inadéquates.

Les questions didactiques exploitées dans le cadre de cette recherche permettent de porter un regard sur l'enseignement du concept de variabilité et de mieux connaitre les connaissances didactiques en action des sujets, notamment en indiquant et en documentant des pistes d'interventions possibles, basée soit sur l'explication, l'expérimentation ou la confrontation, pour l'enseignant qui fait face à des raisonnements et à des réponses d'élèves. Certaines des interventions se sont révélées plus créatives, tout en mettant de l'avant de bonnes conditions pour permettre aux élèves de se rendre compte de leurs erreurs.

Des résultats émergent des considérations pour la formation de futurs enseignants. Bien entendu, les interventions mises de l'avant dans ces résultats constituent une amorce de pistes d'interventions qui pourraient être exploitées en classe et servir par le fait même à la formation de futurs enseignants. Mais plus important encore est le constat que certains pouvaient réagir sur le champ aux réponses et raisonnements d'élèves en proposant des interventions pertinentes, alors que d'autres non. Ce contexte soulève de nombreuses préoccupations et pointe vers des besoins de formation importants. Il est nécessaire de mettre en place une façon d'aller évaluer les connaissances didactiques des enseignants et d'expliciter leurs connaissances en acte afin que celles-ci deviennent conscientes de sorte qu'ils puissent les mobiliser au moment opportun dans leurs pratiques de classe. La simulation est une piste à envisager, car elle permet de faire réagir les sujets spontanément face à une situation comme cela se produit en classe et est plus simple à mettre en place que l'observation en classe pour laquelle il est difficile de cerner l'étude d'un concept qui s'étend sur plusieurs années. Il va de soi, que la recherche sur l'apprentissage des élèves est nécessaire puisqu'elle doit servir de base à la création de situations didactiques.

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Ad Hoc Sessions

Ad hoc Session

LESSON AND LEARNING STUDY: COLLABORATIVE LEARNING STRUCTURES FOR PRESERVICE TEACHER EDUCATION

Diana Royea
University of British Columbia

Inspired by a session about collaborative learning groups as a means of in-service teacher professional development, this *ad hoc* discussed the benefits and challenges of using Lesson Study and Learning Study with preservice teachers. Lesson Study, a form of teacher-led action research and professional development, involves the collaboration of teachers engaging in cycles of lesson design and implementation focused on teacher-generated problems (Lewis, 2002; Stigler & Hiebert, 1999). Inspired by Lesson Study and design experiments, Learning Study employs the same cyclic lesson planning, implementation, and modification with the addition of pre- and post-evaluations and an explicit learning theory (Marton & Runesson, 2015).

A number of potential benefits and challenges of these approaches as means of supporting preservice teachers' development of mathematical knowledge for teaching emerged. While Lesson Study reportedly helps in-service teachers improve their mathematical content knowledge, knowledge of students, and pedagogical knowledge, several factors that impede the effectiveness of Lesson Study have been identified (Chokshi & Fernandez, 2004; Fernandez, 2005). Time constraints and insufficient content and pedagogical knowledge were highlighted as factors likely to be exacerbated for preservice teachers engaging in Lesson Study.

Using Variation Theory (Marton & Booth, 1997) to guide lesson design, analysis, and implementation benefits the Lesson Study framework (Holmqvist & Mattisson, 2008). For preservice teachers, however, learning about and how to use Variation Theory may present additional obstacles to overcome. The potential of Learning Study for in-service teachers' learning has been widely researched yet very little attention has been paid to potential benefits of enculturating preservice teachers in Learning Study as part of their initial teacher education. This *ad hoc* suggests this as a future area of research in mathematics teacher education.

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Mathematics Gallery

Gallérie Mathématique

HEAR MY VOICE: PROSPECTIVE TEACHERS' STORIES

Ann Kajander and Jennifer Holm
Lakehead University

The goal of this presentation was to showcase a methodology to illuminate qualitative data in a novel way. Preservice elementary teachers were interviewed at the beginning and end of their curriculum and instruction course in junior/intermediate mathematics. However rather than using transcription with a traditional qualitative analysis methodology, actual participant voices from the audio files of the interviews were used to create audio montages, in order to showcase major themes in the data without losing the emotional messages inherent in their voices.

A common theme in the research literature relates to the profound mathematical insecurity felt by many preservice elementary teachers (Holm & Kajander, 2012), yet we have long felt that traditional qualitative methodologies, even case study methodologies, fail to expose the depth of the emotional impact of these beliefs on the individuals involved. In other aspects of our on-going work related to teacher growth, it has been observed that even the most initially reticent participants can make extraordinary gains, when they experience learning in a rich environment (Holm & Kajander, 2012). The excitement shared by participants when describing such journeys is often somewhat masked when only written transcripts of their words are examined.

Here we sought to authentically illuminate the profound transformations which may be experienced in even the short time-frame of a one-year course. The audio clips were chosen by the first author, and then assembled into audio files with the help of an audio engineer. Each audio file was drawn from responses to a particular interview question or major theme, and ordered by time and theme, as elementary teacher candidates experienced aspects of the one-year mathematics curriculum and instruction course.

Participants overwhelmingly described traditional experiences in their own school learning of mathematics and a general feeling of uncertainty and dislike of mathematics at the start of the mathematics methods course. When asked, for example, how they might teach students about the area of a rectangle, a number of participants initially responded that they did not know. This response had an increased impact, we felt, when heard repeatedly being given by different participants, as expressed in a defeated voice. One participant then added that perhaps it was "*one of those letters times letters things*". From such unlikely starting points, we continue to be amazed at the profound development felt by many participants. In particular, the participant voices also illuminated the increased confidence and deep satisfaction with their learning that many shared in the post-interviews. "*It's so worth it*", said one, and another added "*I love it, I love the new methods*". We feel the audio presentation adds a new dimension to understanding teacher emotions, and should be explored further as a research methodology.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

· Reading and writing in the mathematics classroom
· The NCTM "Standards" and Canadian reality
· Explanatory models of children's mathematics
· Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

· Fractal geometry in the curriculum
· Socio-cultural aspects of mathematics
· Technology and understanding mathematics
· Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

· Research in undergraduate teaching and learning of mathematics
· New ideas in assessment
· Computers in the classroom: mathematical and social implications
· Gender and mathematics
· Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

· Theories of mathematics education
· Pre-service mathematics teachers as purposeful learners: issues of enculturation
· Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

· Autonomy and authority in the design and conduct of learning activity
· Expanding the conversation: trying to talk about what our theories don't talk about
· Factors affecting the transition from high school to university mathematics
· Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

· Teacher education: challenges, opportunities and innovations
· Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
· What is dynamic algebra?
· The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

· Awareness and expression of generality in teaching mathematics
· Communicating mathematics
· The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
- Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe / Proof in mathematics and in schools
- The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 *Brock University, St. Catharines, Ontario*

- MOOCs and online mathematics teaching and learning
- Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
- Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
- What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
- Mathematics curriculum re-conceptualisation

2014 *University of Alberta, Edmonton, Alberta*

- Mathematical habits of mind / Modes de pensée mathématiques
- Formative assessment in mathematics: Developing understandings, sharing practice, and confronting dilemmas
- Texter mathematique / Texting mathematics
- Complex dynamical systems
- Role-playing and script-writing in mathematics education practice and research

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l’activité mathématique en classe
	N. BOULEAU	La personnalité d’Evariste Galois: le contexte psychologique d’un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES	Old and new mathematical ideas from Africa: Challenges for reflection
	M. WALSHAW	Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries
	W. HIGGINSON	Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM
2013	R. LEIKIN	On the relationships between mathematical creativity, excellence and giftedness
	B. RALPH	Are we teaching Roman numerals in a digital age?
	E. MULLER	Through a CMESG looking glass
2014	D. HEWITT	The economic use of time and effort in the teaching and learning of mathematics
	N. NIGAM	Mathematics in industry, mathematics in the classroom: Analogy and metaphor

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242

NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.