Braithwaite, D. W., & Siegler, R. S. (in press). Children learn spurious associations in their math textbooks: Examples from fraction arithmetic. *Journal of Experimental Psychology: Learning, Memory, and Cognition*. doi: 10.1037/xlm0000546 (Anticipated publication date: 2018)

Children Learn Spurious Associations in Their Math Textbooks: Examples from Fraction Arithmetic

David W. Braithwaite, Carnegie Mellon University

Robert S. Siegler, Carnegie Mellon University and Beijing Normal University

Author Note:

David W. Braithwaite, Department of Psychology, Carnegie Mellon University; Robert S. Siegler, Department of Psychology, Carnegie Mellon University and The Siegler Center for Innovative Learning (SCIL), Advanced Technology Center, Beijing Normal University.

Corresponding author: David W. Braithwaite, baixiwei@gmail.com, 1-812-929-1837, address: 5000 Forbes Avenue, Baker Hall, Pittsburgh, PA 15217, or Robert S. Siegler, rs7k@andrew.cmu.edu, 1-412-551-4370, address: 5000 Forbes Avenue, Baker Hall, Pittsburgh, PA 15217.

This work was supported in part by the Institute of Education Sciences, U.S. Department of Education, through Grant R305A150262 to Carnegie Mellon University, and by the Advanced Technology Center and Siegler Center for Innovative Learning, Beijing Normal University. The authors would like to thank Xinlin Zhou for his assistance with data collection, and Tian Jing and Jing Shao for checking the Chinese translations of the materials used in Experiment 3.

Abstract

Fraction arithmetic is among the most important and difficult topics children encounter in elementary and middle school mathematics. Braithwaite, Pyke, and Siegler (2017) hypothesized that difficulties learning fraction arithmetic often reflect reliance on associative knowledge—rather than understanding of mathematical concepts and procedures—to guide choices of solution strategies. They further proposed that this associative knowledge reflects distributional characteristics of the fraction arithmetic problems children encounter. To test these hypotheses, we examined textbooks and middle school children in the US (Experiments 1 and 2) and China (Experiment 3). We asked the children to predict which arithmetic operation would accompany a specified pair of operands, to generate operands to accompany a specified arithmetic operation, and to match operands and operations. In both countries, children's responses indicated that they associated operand pairs having equal denominators with addition and subtraction, and operand pairs having a whole number and a fraction with multiplication and division. The children's associations paralleled the textbook input in both countries, which was consistent with the hypothesis that children learned the associations from the practice problems. Differences in the effects of such associative knowledge on US and Chinese children's fraction arithmetic performance are discussed, as are implications of these differences for educational practice.

Keywords: fraction arithmetic, fractions, associative learning, textbook analysis, spurious correlations

Children Learn Spurious Associations in Their Math Textbooks: Examples from Fraction Arithmetic

Fractions are crucial to numerical development. Individual differences in fractions knowledge in earlier grades predict not only later success in algebra (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014) but also overall math achievement in high school, even after controlling for IQ, reading achievement, whole number arithmetic skill, and familial SES (Siegler et al., 2012). Fractions are also important for occupational success: 68% of American white-collar and blue-collar employees report using fractions, decimals, or percentages in their work (Handel, 2016). Reflecting their importance, fractions are a major focus of US mathematics instruction in third through sixth grade (CCSSI, 2010).

Despite this prolonged instruction, many children in the US fail to master fractions, especially fraction arithmetic (Byrnes & Wasik, 1991; Fuchs et al., 2014; Hecht & Vagi, 2010; Jordan et al., 2013; Lortie-Forgues, Tian, & Siegler, 2015; Newton, Willard, & Teufel, 2014; Siegler, Thompson, & Schneider, 2011). For example, in one recent study that presented all four fraction arithmetic operations with both equal denominator operands (e.g., 3/5+1/5, 3/5×1/5) and unequal denominator operands (e.g., 3/5×1/4, 3/5÷1/4), percent correct was only 46% for sixth graders and 57% for eighth graders (Siegler & Pyke, 2013).

Certain types of problem are especially challenging. On addition and subtraction problems in Siegler and Pyke (2013), unequal denominator problems elicited many more errors than equal denominator ones (45% vs. 20%), but on multiplication problems, unequal denominator problems elicited far fewer errors than equal denominator ones (42% vs. 63%). Errors on unequal denominator addition and subtraction problems often involved using a strategy that would be appropriate for multiplication, that is, performing the arithmetic operation separately on the numerators and denominators, as in 3/5+1/4 = (3+1)/(5+4) = 4/9. Conversely, errors on equal denominator multiplication problems often involved using a strategy that would be appropriate for addition or subtraction, that is, performing the operation on the numerators and maintaining the common denominator, as in 4/5×3/5 = (4×3)/5 = 12/5. Children in other studies of fraction arithmetic have displayed similar patterns of accuracy and specific errors (e.g., Siegler et al., 2011).

To explain the poor overall fraction arithmetic performance and the specific patterns of accuracy and errors, Braithwaite, Pyke, and Siegler (2017) hypothesized that children's strategy choices rely on associative knowledge, rather than correct mathematical rules or conceptual understanding. They further proposed that children acquire this associative knowledge from the statistical distribution of practice problems they encounter. To test these hypotheses, they analyzed all fraction arithmetic problems stated in numerical form (i.e., not as word problems) from the fourth, fifth, and sixth grade volumes of three major US mathematics textbook series—Pearson Education's enVisionMATH (Charles et al., 2012), Houghton Mifflin Harcourt's GO MATH! (Dixon, Adams, Larson, & Leiva, 2012a, 2012b), and McGraw Hill Education's Everyday Mathematics (University of Chicago School Mathematics Project, 2015a, 2015b, 2015c). Results of those analyses are presented in Table 1, which includes similar data to those described in Braithwaite, Pyke, and Siegler (2017), except that the new Table includes problems from third grade textbooks as well as fourth to sixth grade ones, averages the input over the three textbook series, and only includes problems with two fraction operands (problems with a whole number and a fraction were included in the earlier analysis).

========================== Table 1 about here ===========================

The analyses revealed strikingly non-random relations between arithmetic operations and features of operands in the textbooks. As shown in Table 1, 93% of problems with equal denominators involved addition or subtraction (e.g., 3/5+1/5), whereas unequal denominator problems involved multiplication or division almost as often as addition or subtraction. Viewed from another perspective, addition and subtraction problems involved equal and unequal denominator operands with about the same frequency, whereas 90% of multiplication and division problems had unequal denominator operands.

To understand the effects that this unbalanced distribution might have on learning, Braithwaite, Pyke, and Siegler (2017) constructed FARRA, a computational model of fraction arithmetic learning. This model formalized the assumptions that a) children associate operand features with arithmetic operations, b) these associations derive in large part from problems encountered in textbooks, and c) the associations guide children's choices of solution strategies. In particular, distributions of problems in textbooks are hypothesized to lead children (and FARRA) to associate equal denominators with addition and subtraction and to associate unequal denominators at least as much with multiplication as with addition or subtraction.

FARRA displayed all eight phenomena identified in Braithwaite et al.'s review of the literature on children's fraction arithmetic, including the patterns of accuracy and specific errors described above. The similarity of children's performance to that of FARRA suggested that they, like FARRA, might learn associations between operand features and arithmetic operations from textbook input, and rely on these associations to select solution strategies. The textbook problems might well not be the only source of the children's associations; another source might be the requirement of the standard addition and subtraction algorithms that equal denominators always be present at some point in the solution process. However, the distribution of textbook problems does seem likely to be one important source of children's associations between arithmetic operations and operands.

Although consistent with the hypothesis that children learn associations between problem features and operations that are present in textbooks, the evidence is relatively indirect. The present study is an attempt to provide more direct evidence regarding this hypothesis by assessing the associations directly, rather than inferring them from children's arithmetic performance.

To assess whether children learn associations between operand features and operations in textbook problems, we presented three tasks: one that specified the arithmetic operation and required children to generate pairs of operands to accompany it; one that specified the operands, and required children to predict the arithmetic operation that would accompany them; and one that specified two pairs of operands and two operations and required children to match an operation to a pair of operands.

From a mathematical perspective, arithmetic operations are independent of operands: any arithmetic operation can be performed on any pair of operands (with the exception that if the second operand is zero, the operation cannot be division). However, if children learn associations between operations and operands, they should follow the textbook patterns in predicting which arithmetic operation will accompany particular pairs of operands and which types of operands will accompany each arithmetic operation. For example, if presented equal denominator operands (e.g., 4/5 and 3/5), children would be expected to more often predict that the operation is addition or subtraction than that it is multiplication or division.

This theoretical position also implies that children's associations should parallel other patterns that appear in the distribution of textbook problems. Another pattern that Braithwaite et al. found in the textbooks was an association between the arithmetic operation and whether the operands were two fractions or a whole number and a fraction. In the three textbook series that they analyzed, 94% of problems that included both a whole number operand and a fraction or mixed number operand involved multiplication or division (Table 2). In contrast, in the same textbooks, 71% of problems in which both operands were fractions or mixed numbers involved addition or subtraction (Table 2). Thus, we predicted that children would associate operand pairs that included a whole number and a fraction more strongly with multiplication and division than with addition and subtraction.

========================== Table 2 about here ============================

It was far from a foregone conclusion that children would learn such spurious associations between problem features and operations simply because the associations exist in textbooks. Mathematics is a formal system in which frequency of various types of problems and problem features is irrelevant to the rules that should be used. Consistent with the content being learned, mathematics education emphasizes learning the explicit rules that specify the conditions under which each solution strategy should be used and ignoring irrelevant features of problems. To the extent that students learn what their teachers and textbooks are trying to teach them, there is no reason for children to learn relations that are irrelevant to that mathematical content.

On the other hand, implicit learning of statistical patterns in the environment, sometimes termed “statistical learning” (Saffran, Aslin, & Newport, 1996), is a fundamental learning mechanism present throughout the lifespan (Perruchet & Pacton, 2006). For example, it plays an important role in infants’ language development (Pelluchi, Hay, & Saffran, 2009) and in school-age children’s learning of orthographic regularities (Pacton, Perruchet, Fayol, & Cleeremans, 2001; Treiman & Kessler, 2006). In the context of mathematics learning, rather than explicit learning of rules entirely replacing statistical learning, both types of learning mechanism may operate simultaneously.

If this proposal is correct, children may also detect statistical patterns other than operand-operation associations. To test this possibility, we examined another type of statistical information—the frequencies with which specific fractions appear as operands. Some fractions appear often, others rarely, in textbook problems. For example, 9 of the 10 fractions appearing as operands most often in the textbooks analyzed in Tables 1 and 2 were the same in all three series: 1/2, 1/3, 2/3, 1/4, 3/4, 2/5, 1/6, 5/6, and 3/8. Other fractions with single digit numerators and denominators, including 3/6, 1/7, 2/7, 3/7, 6/7, 1/9, 2/9, and 8/9 were not among the 20 most common fractions in any of the series. The most commonly presented fractions tended to have small numerators and denominators, but that is not the only consideration. Unsimplified fractions (e.g., 3/6) and fractions with prime denominators (e.g., 3/7) appeared less often than their numerator and denominator sizes would suggest (e.g., both were rarer than 3/8)[1]. To test whether children learn such frequency information, we calculated correlations between the frequencies with which fractions appeared as operands in textbooks and the frequencies of children generating those fractions as operands.

Experiment 1

Participants in Experiment 1 performed three tasks: the "choose-operation," "generate-operands," and "match-operands-with-operations" tasks (Table 3).

========================== Table 3 about here ==========================

In the generate-operands task, children were shown eight arithmetic problems in which only the operation and two empty boxes were visible. They were asked to write numbers in the boxes that would be likely to appear with that operation: two fractions on half of trials, and a whole number and a fraction on the other half. Based on the textbook input, we expected that children would write two fractions with equal denominators more often when the operation was addition or subtraction than when it was multiplication or division and that children would write a whole number and a fraction more often when the operation was multiplication or division than when it was addition or subtraction.

The generate-operands task also afforded a test of our hypothesis that children are sensitive to the frequencies with which fractions appear as operands. We predicted a positive correlation between the frequency with which children generated specific fractions and the frequency with which fractions appeared as operands in the three US mathematics textbooks.

In the choose-operation task, children were shown 12 pairs of operands with an empty box between them where an arithmetic operation would be. Children were asked to guess which arithmetic operation would most likely appear in the boxes and to choose each operation equally often over the 12 problems. Based on the textbook input, we predicted that when the operands were fractions with equal denominators, children would guess addition or subtraction more often than multiplication or division, and when the operands were a whole number and a fraction, children would guess multiplication or division more often than addition or subtraction. We did not predict any preference among arithmetic operations for trials in which the operands were fractions with unequal denominators, because textbook problems with such operands involved addition and subtraction about as often as multiplication and division (Table 1).

In the match-operands-with-operations task, eight trials were presented, each with two pairs of operands and a box between them. Children were told that one problem involved a certain operation (e.g., addition) and the other a different operation (e.g., multiplication), and were asked to connect the first operation to one pair of operands. We predicted that when choosing between an equal denominator and an unequal denominator operand pair, children would connect the equal denominator pair more often to addition or subtraction than to multiplication or division. We also predicted that when choosing between two fraction operands and a whole number and a fraction, children would connect the pair with the whole number operand more often to multiplication or division than to addition or subtraction.

These predictions are summarized in the leftmost two columns of Table 4.

========================= Table 4 about here ==========================

Method

Participants. Participants were 137 children, 66 sixth graders (mean age = 11.4 years) and 71 eighth graders (mean age = 13.4 years), attending a middle school in Pittsburgh, PA. The percent of children eligible for free or reduced price lunch at this school was 34% (the state median was 54%; Pennsylvania Department of Education, 2016). The experimental sessions were administered by the first author and two female research assistants. The Carnegie Mellon University Institutional Review Board approved this experiment and Experiment 2.

Materials. Two sets of operand pairs were created for the choose-operation and match-operands-with-operations tasks. Each child was randomly assigned to receive one of the two sets. (No operand pairs were created for the generate-operands task, because only arithmetic operations were presented in that task.)

Each set of problems consisted of 12 pairs of operands, four groups of three pairs each (Table S1 in the online supplemental materials). The three pairs of operands in each group had the second operand in common; the first operand was varied to produce one equal denominator fraction-fraction pair, one unequal denominator fraction-fraction pair, and one whole-fraction pair. The first operand was always larger than the second, so that combining the two operands with any of the four arithmetic operations would yield a positive answer. In the whole-fraction pairs, the whole number was always the first operand. All fractions were between 0 and 1.

Procedure. The tasks were presented in a fixed order: generate-operands, choose-operation, match-operands-with-operations.

Generate-operands. On each trial, children were shown an arithmetic operation with an empty box on either side and asked to generate operands with numbers that would be likely to appear with that operation. The first page displayed four problems in which the order of operations was +, −, ×, ÷, and the second page displayed four problems in which the order of operations was ÷, ×, −, +. Children were instructed to insert a pair of fractions for two of the four problems on each page, and a whole number and a fraction for the other two problems on each page. Mixed numbers (e.g., 3 1/4), which children generated on 4.9% of trials, were counted as fractions; classifying these numbers as whole numbers did not change the results of the analyses.

Choose-operation. On each of 12 trials, children were shown an arithmetic problem with the two operands visible and asked to predict which arithmetic operation would appear with each operand pair.

Compared to multiplication and division, addition and subtraction are conceptually more basic and are introduced earlier in the mathematics curriculum. Thus, it seemed possible that without any constraint on their choices, some children might choose addition or subtraction on most or all of the trials, without regard to the features of the operands. To prevent this outcome, children were asked to choose each operation equally often, that is, on 3 of the 12 trials. To make it easier for children to check how often they had chosen each operation, all 12 trials were shown on the same page. Children were randomly assigned to receive the trials in either a fixed random order or the reverse of that order.

Match-operands-to-operations. On each of eight trials, children were presented two operations and two pairs of operands; the task was to choose which pair of operands would be more likely to involve the first operation. On the first two trials, children were told: "In each row below, there are two problems with the operation missing. One of them was an *addition* problem, and the other was a *multiplication* problem. For each row, try to guess which problem was the *addition* problem. In other words, in which problem would you guess the missing sign was a **+** sign? Please circle (a) or (b) in each row." The word "addition" and the "+" sign were replaced by "subtraction" and "−" on the third and fourth trials, by "multiplication" and "×" on the fifth and sixth trials, and by "division" and "÷" on the seventh and eighth trials. In the instructions for the multiplication and division trials, the instruction "the other was a *multiplication* problem" was replaced with "the other was an *addition* problem."

On one trial for each operation, the two response options were the unequal denominator operand pair and the equal denominator operand pair from one of the four groups of number pairs in the stimulus set (e.g., 3/4 ☐ 1/6 and 4/6 ☐ 1/6). On the other trial, the response options were the whole-fraction operand pair and the unequal denominator operand pair from the same stimulus group (e.g., 3 ☐ 1/6 and 3/4 ☐ 1/6). Each group of operand pairs was used for one arithmetic operation. The order in which these two types of trials were presented was determined randomly for each child; for a given child, the order was the same for each arithmetic operation.

Administration of tasks. The experiment was administered in a paper-and-pencil, whole class format. Instructions for each task were read aloud to the class to avoid the possibility of reading difficulties interfering with performance. To allow this procedure, children waited until everyone finished a given task before starting the next one. Children were told to complete each task within 2.5 minutes, but were permitted to finish even if they exceeded this time limit.

Results

We present results from the three tasks in the order in which the tasks were presented. In all cases, we examined effects involving grade (sixth or eighth) on the dependent variable in all *t*-tests and ANOVAs. Most main effects and interactions involving grade were not significant; exceptions are reported below.

Children produced invalid responses on a small percentage of trials (1-2%) on two of the three tasks in Experiment 1 and on fewer than 1% of trials on each of the tasks in Experiments 2 and 3. Data from these invalid trials were excluded from analysis; details regarding the exclusions are provided in the online supplemental materials. On the generate-operands task in Experiment 1, invalid responses were produced on a greater percentage of trials (10%). Reasons for these more frequent invalid responses are described in the next section along with other data from the task.

Generate-operands Task. Children completed an average of 7.16 valid trials (of a possible 8): 3.54 addition/subtraction trials and 3.62 multiplication/division trials. The other 10% of trials were excluded from analysis; the most common reason for exclusion was that children generated whole numbers for both operands (5% of trials). Eight children (five sixth graders and three eighth graders) were excluded from the analyses for predictions 1A and 1B because they did not generate a valid response for any addition/subtraction trial, for any multiplication/division trial, or both.

As predicted (Table 4, prediction 1A), percent fraction-fraction operand pairs in which the fractions generated by children had equal denominators was higher on addition/subtraction trials (46.1%) than on multiplication/division trials (29.0%), $F(1, 97) = 17.75$, $p < .001$, $\eta_p^2 = .05$. (Here and in Experiment 3, this percentage could only be calculated for trials on which children generated fraction-fraction pairs; 16 sixth graders and 14 eighth graders were excluded because they did not generate a fraction-fraction pair on at least one addition/subtraction and one multiplication/division trial. The effect remained when the analysis was performed without excluding these children, using percentage of all responses instead of percentage of fraction-fraction responses as the dependent variable, $F(1, 127) = 22.17$, $p < .001$, $\eta_p^2 = .15$.) Percent fraction-fraction responses with equal denominators was higher among eighth graders (46.1%) than sixth graders (25.5%), $F(1, 97) = 10.26$, $p = .002$, $\eta_p^2 = .07$.

Consistent with another prediction (Table 4, prediction 1B), children tended to generate whole-fraction operand pairs on more than half of multiplication/division trials (54.5%, *SE* = 2.5%), one-sample $t(128) = 1.82$, $p = .070$, $d = 0.16$, and therefore tended to generate whole-fraction operand pairs on fewer than half of addition/subtraction trials (45.1%, *SE* = 2.5%). (The first percentage was compared to chance, rather than to the second percentage, because the two percentages were not independent of each other.)

Finally, as predicted (Table 4, prediction 1C), the frequency with which children generated a fraction was positively correlated with that fraction's frequency as an operand in the textbook problems analyzed in Tables 1 and 2, $r(796) = .778$, $p < .001$. The correlation did not merely reflect fractions with smaller numerators and denominators being more common in both children's responses and textbooks: The partial correlation remained significant, $r(796) = .773$, $p < .001$ after controlling for the inverse of the sum of numerator and denominator. (The inverse of the sum, rather than the sum, was used as a control because it explained a larger portion of the variance in fractions' frequencies in textbooks than the sum did (37.6% vs. 0.2%) and thus constituted a more stringent control for problem size.)

Choose-operation Task. Children completed an average of 11.85 valid trials (of a possible 12): 3.94 equal denominator trials, 3.96 unequal denominator trials, and 3.96 whole-fraction trials. As predicted (Table 4, prediction 2A), when presented with equal denominator operands, children chose addition or subtraction on more than half (68.2%, *SE* = 2.2%) of trials, one-sample $t(136) = 8.24$, $p < .001$, $d = 0.70$. Also as predicted (Table 4, prediction 2B), when the operands were a whole number and a fraction, children chose multiplication or division on more than half of trials (60.6%, *SE* = 2.3%, one-sample $t(135) = 4.65$, $p < .001$, $d = 0.40$. On unequal denominator trials, children also chose multiplication or division on more than half (58.8%, *SE* = 2.1%) of trials, one-sample $t(136) = 4.18$, $p < .001$, $d = 0.36$.

Match-operands-with-operations Task. On this task, children generated an average of 7.82 valid trials (of a possible 8), including 3.94 addition/subtraction trials and 3.88 multiplication/division trials. As predicted (Table 4, prediction 3A), children chose equal denominator operand pairs in preference to unequal denominator operand pairs more often on addition/subtraction trials (73.3%, *SE* = 2.7%) than on multiplication/division trials (43.7%, *SE* = 3.3%), $F(1, 133) = 48.15$, $p < .001$, $\eta_p^2 = .27$. Also as predicted (Table 4, prediction 3B), children showed a marginal tendency to choose whole-fraction operand pairs in preference to fraction-fraction pairs more often on multiplication/division trials (58.2%, *SE* = 3.2%) than on addition/subtraction trials (50.0%, *SE* = 3.0%), $F(1, 132) = 3.05$, $p = .083$, $\eta_p^2 = .022$.

Although no main effect of grade was present in either analysis, grade interacted with arithmetic operation in the latter analysis, $F(1, 132) = 7.10$, $p = .009$, $\eta_p^2 = .051$. Sixth graders chose whole-fraction operand pairs over fraction-fraction operand pairs more often on multiplication/division trials (62.1%, *SE* = 4.6%) than on addition/subtraction trials (40.9%, *SE* = 4.3%), $F(1, 65) = 8.76$, $p = .004$, $\eta_p^2 = .85$. In contrast, eighth graders showed no effect of arithmetic operation (multiplication/division: 54.4%, *SE* = 4.5%; addition/subtraction: 58.8%, *SE* = 3.9%), $F(1, 67) = 0.47$, *ns*. Thus, prediction 3B was consistent with the results for sixth but not eighth graders.

Discussion

The findings of Experiment 1 indicated that US children form spurious associations between arithmetic operations and operand features and that these associations parallel relations in textbook problems. As predicted, children associated equal denominator operands more strongly with addition and subtraction than with multiplication and division. Also as predicted, children associated whole-fraction operand pairs more strongly with multiplication and division than with addition and subtraction, though evidence for this latter association was less consistent. Further as predicted, the frequency with which children generated particular operands was closely related to the frequency with which the operands appeared in textbook problems. These findings indicate that children detect a variety of spurious associations and other distributional features of textbook input, as hypothesized by Braithwaite et al. (2017).

Experiment 2

The choose-operation task is particularly relevant to the assumptions of Braithwaite, Pyke, and Siegler's (2017) FARRA model of children's fraction arithmetic. FARRA uses numeric features of operands as cues for selecting solution strategies. Similarly, in the version of the choose-operation task in Experiment 1, children appear to have used numeric features of operands as cues to predict the most likely arithmetic operation.

However, results of that choose-operation task may have partially reflected the task constraint that children choose each arithmetic operation equally often. For example, children who chose addition or subtraction on more than half of equal denominator trials would necessarily have chosen multiplication or division on more than half of the remaining trials. This fact could explain why children chose multiplication or division more often than chance for whole-fraction operand pairs and unequal-denominator operand pairs.

To test this alternative explanation, we administered a modified version of the choose-operation task in Experiment 2. On this version, children were not required to choose each arithmetic operation equally often, or at all. We predicted that children still would choose addition or subtraction more often than chance for equal denominator operand pairs, and would choose multiplication or division more often than chance for whole-fraction operand pairs (Table 4, predictions 2A and 2B).

Method

Participants. Participants were 168 sixth graders (mean age = 11.2 years), attending the same middle school in Pittsburgh, PA where Experiment 1 was conducted. None of the children had participated in Experiment 1, and the teachers from whose classes the children were drawn were different from the teachers from whose classes the children in Experiment 1 were drawn. Half of the experimental sessions were administered by each of the two female research assistants who administered Experiment 1.

Materials. The same two sets of operand pairs used in the choose-operation task in Experiment 1 were used as stimuli (Table S1 in the online supplemental materials). Each child was randomly assigned to receive one of the two sets.

Procedure. As in Experiment 1, on each of 12 trials, children were shown an operand pair with an empty box between the operands and were asked to choose which arithmetic operation they thought would appear with the operands. Children were randomly chosen to receive the trials in either a fixed random order or the reverse of that order. The only differences between this version of the task and that used in Experiment 1 were that in this version, children were given no instruction regarding whether, or how often, to choose each operation, and rather than circling their chosen arithmetic operation, children wrote the operation directly in the empty box between the two operands.

Results

As predicted (Table 4, prediction 2A), on problems with equal denominator operands, children chose addition or subtraction more often than the chance level of 50% (84.3%, *SE* = 1.9% of choices), one-sample $t(167) = 18.55$, $p < .001$, $d = 1.43$. Also as predicted (Table 4, prediction 2B), when the operands were a whole number and a fraction, children chose multiplication or division more often than chance (83.5%, *SE* = 2.2% of choices), one-sample $t(167) = 15.57$, $p < .001$, $d = 1.20$. On unequal denominator trials, children chose multiplication or division more often than chance (58.6%, *SE* = 2.3% of choices), one sample $t(167) = 3.72$, $p < .001$, $d = 0.29$.

Discussion

Findings from the choose-operation task in Experiment 1 replicated even when children were not required to choose operations equally often. In fact, when presented equal denominator operand pairs, children predicted addition or subtraction more often in Experiment 2 than in Experiment 1 (84.3% vs. 68.2%), and when presented whole-fraction operand pairs, children predicted multiplication or division more often in Experiment 2 than in Experiment 1 (83.5% vs. 60.6%). In Experiment 1, the need for children to attend to the number of times they had chosen each operation may have distracted them from attending to numeric features of the operands, and thereby weakened the effects of those features on children's predictions.

Experiment 3

The results of Experiments 1 and 2 raised the question of how anyone learns fraction arithmetic. Chinese middle school students display much higher accuracy on fraction arithmetic problems than US middle school students (92% vs. 46% correct in Bailey, Zhou, Zhang, et al., 2015; see also Torbeyns, Schneider, Xin, & Siegler, 2015). Does this high accuracy reflect Chinese children not forming spurious associations, either because the associations are not present in Chinese textbooks or because they ignore spurious associations in practice problems? Or do Chinese children form spurious associations but override them in their arithmetic strategy choices? This could occur through strong conceptual knowledge allowing them to understand and choose appropriate procedures for each arithmetic operation, through substantial practice allowing them to recall correct procedures for each operation, or through both processes.

One reason to suspect that strong conceptual knowledge of mathematics prevents learning of spurious associations is that children who understand the conceptual bases of correct solution procedures do not need to rely on mathematically irrelevant associations to select solution procedures. Consistent with this possibility, when presented a mathematics word problem after studying a structurally similar analogue and a superficially similar distractor, mathematical experts were less likely than novices to show negative transfer from the superficial distractor to the test problem (Novick, 1988). Similarly, people with greater understanding of non-mathematical domains, ranging from radiology to baseball, often do not recall irrelevant information that novices and people with moderate amounts of knowledge do recall (e.g., Arkes & Freedman, 1984; Hagen, 1972; Myles-Worsley, Johnston, & Simons, 1988; V. Patel & Groen, 1991; Voss, Vesonder, & Spilich, 1980).

On the other hand, children with stronger mathematical knowledge are often more, not less, likely to encode numerical features of problems. When comparing fractions, mathematically proficient students often solve difficult problems by relying on subtle relations among the numbers involved to choose strategies; less proficient students rarely rely on these subtle but useful features (Fazio, DeWolf, & Siegler, 2016). Similar findings have been obtained with whole number mental arithmetic (Braithwaite, Goldstone, van der Maas, & Landy, 2016). These findings suggest that children with strong mathematical knowledge could learn spurious associations involving numerical features of problems, but override the associations to choose correct fraction arithmetic procedures.

To distinguish among these several possibilities, we examined in Experiment 3 whether spurious associations between fraction arithmetic operations and operand features are present in Chinese textbooks, and if so, whether Chinese children learn the spurious associations. If Chinese textbooks include spurious associations, and children display knowledge of them, those findings, together with previous findings of Chinese children's superior knowledge of fraction arithmetic, would suggest that forming spurious associations between operations and operand features does not necessarily prevent mastery of fraction arithmetic. Such a finding would also suggest that high mathematical proficiency does not prevent acquisition of irrelevant associative knowledge. On the other hand, finding that Chinese textbooks do not include the spurious operation-operand feature associations would suggest that absence of such associations could be one factor contributing to Chinese students' success in learning fraction arithmetic. A third possibility—that Chinese textbooks do include the spurious associations but children do not learn them—would suggest that superior conceptual understanding or more extensive practice prevents them from learning the associations.

To determine whether Chinese textbooks, like US textbooks, exhibit spurious associations between fraction arithmetic operations and operand features, we analyzed the 3rd-6th grade volumes of one textbook series from each of the three major Chinese publishers of primary school mathematics textbooks: Beijing Normal University Press (Beijing Normal University Press, 2014; New Century Primary Mathematics Curriculum Writing Group, 2015), People's Education Press ("Helping You Learn Mathematics Classroom Workbook" Writing Group, 2014; People's Education Press Curriculum Research Center Primary Mathematics Curriculum Research and Development Center, 2014), and Phoenix Education Publishing (Nanjing Oriental Mathematics Education Scientific Research Center & Jiangsu District Primary and Middle School Education Research Center, 2014; Suzhou Education Press Primary School Mathematics Curriculum Writing Group, 2014).

The criteria for inclusion of problems were identical to those employed with the US textbooks. Problems had to have one arithmetic operation and two operands, one of which was a fraction or mixed number. The other operand could be a fraction, mixed number, or whole number. We excluded problems not in numerical form, such as story problems, as well as problems that did not require calculation of an exact answer. Problems were selected according to these criteria by the first author, who is fluent in Chinese. The number of problems meeting all of these criteria for the textbook and workbook was 442 in the Beijing Normal University series, 507 in the People's Education Press series, and 391 in the Phoenix Education Publishing series. It should be noted that many Chinese children solve the problems in more than one workbook, so that the smaller number of problems obtained from the Chinese textbooks does not mean that Chinese children encounter fewer fraction arithmetic problems than US children do.

In all three Chinese textbooks, more than 85% of equal denominator problems involved addition or subtraction (Table 5), and more than 85% of problems with one whole number and one fraction operand involved multiplication or division (Table 6). These patterns were similar to those observed in US textbooks (Tables 1 and 2).

========================== Table 5 about here ===========================

========================== Table 6 about here ===========================

Chinese children performed the same three tasks as US children did in Experiment 1. Because the Chinese textbooks displayed associations very similar to those in US textbooks, we tested the same predictions for the Chinese students as for the US students in Experiment 1.

Method

Participants. Participants were 126 children, 65 sixth graders (mean age = 11.2 years) and 61 eighth graders (mean age = 13.4 years), attending a middle school in Beijing. The first author and two research assistants, one male and one female, administered the experiment. The Beijing Normal University Institutional Review Board approved the experiment.

Materials. The materials used in Experiment 3 were a Chinese language version of the English language materials used in Experiment 1. The materials were translated into Chinese by the first author; all translations were checked by a native Chinese speaker.

Procedure. The procedure in Experiment 3 was the same as in Experiment 1, with the exception that children read the instructions on their own and were allowed to work at their own pace. This was done because the children's teachers indicated that all of the children had sufficiently high reading levels to understand the instructions; the teachers' impression appeared accurate, based on children's low percentage of missing or invalid responses. In the generate-operands task, mixed numbers were again counted as fractions; children generated such numbers on 2.5% of trials, and classifying them as whole numbers did not change the results of the analyses.

Results

As in Experiment 1, we tested for effects and interactions involving grade (sixth or eighth) in all *t*-tests and ANOVAs. None of the main effects or interactions involving grade were significant; they therefore are not described further.

Generate-operands Task. Children completed an average of 7.82 valid trials (of a possible 8): 3.90 addition/subtraction trials and 3.93 multiplication/division trials. As predicted (Table 4, prediction 1A), percent fraction-fraction operand pairs with equal denominators was higher on addition/subtraction trials (42.1%) than on multiplication/division trials (20.6%), $F(1, 99) = 20.92$, $p < .001$, $\eta_p^2 = .07$. The effect remained when the analysis was performed using percentage of all responses instead of percentage of fraction-fraction responses as the dependent variable, $F(1, 123) = 17.00$, $p < .001$, $\eta_p^2 = .12$.

Children generated whole-fraction operand pairs on 46.7% (*SE* = 2.2%) of addition/subtraction trials and on 52.9% (*SE* = 2.3%) of multiplication/division trials. Contrary to our prediction (Table 4, prediction 1B), the percentage of multiplication/division trials on which children generated whole-fraction operand pairs did not differ from chance (i.e., 50%), one sample $t(124) = 1.27$, $p = .205$.

Finally, as predicted (Table 4, prediction 1C), the frequency with which children generated a fraction as an operand was positively correlated with that fraction's frequency in problems drawn from the three Chinese textbooks, Pearson's $r(432) = .927$, $p < .001$. As in Experiment 1, the correlation remained significant after controlling for the inverse of the sum of the numerator and denominator of each fraction, $r(432) = .915$, $p < .001$.

Choose-operation Task. Children completed an average of 11.97 valid trials (out of a possible 12): 3.99 equal denominator trials, 4.00 unequal denominator trials, and 3.98 whole-fraction trials. As predicted (Table 4, prediction 2A), the Chinese children chose addition or subtraction as the operation on more than half of trials with equal denominator operands (72.9% of trials, *SE* = 2.2), one-sample $t(125) = 10.36$, $p < .001$, $d = 0.92$. Also as predicted (Table 4, prediction 2B), the Chinese children chose multiplication or division on more than half (57.4%, *SE* = 2.1%) of trials in which one operand was a whole number and the other was a fraction, one-sample $t(125) = 3.58$, $p < .001$, $d = 0.32$. As in Experiments 1 and 2, these children also chose multiplication or division on more than half (65.5%, *SE* =2.1%) of unequal denominator trials, one sample $t(125) = 7.22$, $p < .001$, $d = 0.64$.

Match-operands-with-operations Task. Children completed an average of 7.94 valid trials per child (of a possible 8): 3.97 addition/subtraction trials and 3.97 multiplication/division trials. As predicted (Table 4, prediction 3A), children chose equal denominator operand pairs in preference to unequal denominator operand pairs more often on addition/subtraction trials (79.8%, *SE* = 3.0%) than on multiplication/division trials (46.0%, *SE* = 3.4%), $F(1, 124) = 52.26$, $p < .001$, $\eta_p^2 = .30$.

Contrary to our prediction (Table 4, prediction 3B), but consistent with the performance of US eighth graders in Experiment 1, Chinese children tended to choose whole-fraction rather than fraction-fraction operand pairs more often on addition/subtraction trials (55.2%, *SE* = 3.6%) than on multiplication/division trials (48.0%, *SE* = 3.9%), $F(1, 122) = 3.07$, $p = .082$, $\eta_p^2 = .025$.

Discussion

Both Chinese textbooks and Chinese children showed spurious associations between fraction arithmetic operations and operand features like those observed in their US counterparts in Experiments 1 and 2. Thus, children in both countries extract associations between arithmetic operations and operand features, even when these associations do not reflect a mathematical rule or principle. Given that Chinese middle school students are highly accurate on fraction arithmetic problems (Bailey et al., 2015; Torbeyns, et al., 2015), the findings indicate that learning spurious associations does not preclude mastery of fraction arithmetic procedures, and that mathematical expertise does not preclude learning spurious associations.

General Discussion

Below, we discuss implications of the present findings for understanding whether children learn spurious associations between operations and operands in the fraction arithmetic problems they encounter, the effects of learning such associations on their fraction arithmetic performance, and how these findings might be used to improve mathematics instruction.

Learning of Spurious Associations in Fraction Arithmetic Input

The present study tested a central hypothesis of Braithwaite, Pyke, and Siegler's (2017) theory of fraction arithmetic learning—that children learn associations between arithmetic operations and features of operands in the practice problems they receive, even when those associations have no mathematical basis. The present findings were largely consistent with that hypothesis and therefore with the theoretical assumption on which it was based.

Braithwaite, Pyke, and Siegler (2017) documented associations between arithmetic operations and operand features in US textbooks and provided indirect evidence that US children detect these associations. The present study yielded much more direct evidence that US children form associations paralleling those in the textbook problems, and extended the analysis to Chinese textbooks and children, demonstrating that learning of mathematically irrelevant associations between fraction arithmetic operations and operand features is not idiosyncratic to

US children. Of particular interest, the greater mathematical expertise of Chinese children did not prevent them from learning such relations.

The choose-operation task was most directly relevant to our model of fraction arithmetic learning, because predicting the arithmetic operation based on numeric features of the operands strongly resembles the model's use of numeric features of operands as cues to which solution strategy to use. Findings from this task were invariably consistent with our predictions, for both US and Chinese children. When presented with two equal denominator fractions, children expected the arithmetic operation to be addition or subtraction; when presented with a whole number and a fraction, children expected the operation to be multiplication or division.

In principle, findings in Experiments 1 and 3 that children expected that whole-fraction problems would involve multiplication/division might have reflected task constraints. That is, if children disproportionately chose addition or subtraction when presented with equal denominator operands, the requirement to choose each operation equally often could have led them to choose multiplication and division more often than chance when presented other types of operands. However, children displayed an even stronger preference for choosing multiplication/division on whole-fraction pairs in Experiment 2, where there was no requirement or even encouragement to choose operations equally often. Thus, the preference for choosing multiplication/division on whole-fraction problems did not depend on requirements to choose operations equally often.

In all three experiments, children chose multiplication or division more often than chance when presented with two unequal denominator fractions. This result was not predicted, because problems with unequal denominator operands involved multiplication or division about half the time in both US textbooks (44.4%) and Chinese textbooks (53.7%). However, among problems with two fraction operands, multiplication and division problems involved unequal denominators far more often than did addition and subtraction problems, in both the US textbooks (89.9% vs. 46.2%, Table 1) and the Chinese textbooks (90.6% vs. 56.6%, Table 5). This large difference may have led children not only to associate multiplication and division with unequal denominators, but also to associate unequal denominators with multiplication and division.

An alternate explanation of our results is that, rather than children's responses reflecting associations learned from textbook problem distributions, both children's responses and the textbook distributions might reflect differences among fraction arithmetic procedures. Adding and subtracting fractions with unequal denominators require converting the fractions to a common denominator and then following the procedure that would be used if the original problem had equal denominators. The fact that equal-denominator addition and subtraction is a component of the procedure for unequal-denominator addition and subtraction could explain why textbooks present large numbers of equal-denominator problems for addition and subtraction; the fact that equal-denominator addition and subtraction is easier could explain why children often paired equal denominators with addition and subtraction in the present experiments.

However, at least three considerations argue against this alternate explanation being the sole source of the present findings (though it could have been one source). First, given that children would not be performing the arithmetic problem regardless of the operation or operands they chose, there was no reason for them to choose the easiest operand-operation pairing. Second, if ease of executing procedures were critical, children should have associated whole-fraction operand pairs with addition, since adding such operands is trivial (e.g., 3+1/6 = 3 1/6). The actual pattern was the opposite; addition was rarely predicted on such problems. Third, the difference in difficulty between equal and unequal denominator addition and subtraction problems is much smaller among Chinese children (e.g., 93% vs. 88% correct in Bailey et al., 2015) than among US children (e.g., 80% vs. 55% correct in Siegler & Pyke, 2013), but Chinese children appeared to associate equal denominators with addition and subtraction at least as strongly as US children.

In the generate-operands task, the operands children generated paralleled textbook problems not only with respect to the association of equal denominators with addition and subtraction, but also with respect to the frequencies of specific fractions. The correlation between frequency of particular fractions in textbooks and children was very strong in both the US (r = .778) and China (r = .927). Thus, besides learning spurious associations between operations and operand features, children also learn the frequency with which specific fractions appear as operands. This finding supports the general perspective that children learn statistical patterns in math practice problems, even when these patterns do not reflect any mathematical principle.

Effects of Learning Spurious Associations on Fraction Arithmetic Performance

Braithwaite et al. (2017) argued that reliance on irrelevant associative knowledge partially explains US children's poor mastery of fraction arithmetic procedures. US children display especially low accuracy on types of problems that are rarely presented in textbooks. Moreover, they often err on such problems by using strategies that would be appropriate for more frequently encountered types of problems (Siegler & Pyke, 2013; Siegler et al., 2011). For example, they often err on equal denominator multiplication problems by using a strategy that would be appropriate for the much more common equal denominator addition or subtraction problems, leading to errors such as $3/5 \times 4/5 = 12/5$. Such errors appear to reflect children associating operand features with arithmetic operations (in this case, associating equal denominator operands with addition and subtraction).

This perspective on potential negative effects of associative knowledge dovetails with previous research on spurious correlation effects in mathematics learning (Ben-Zeev & Star, 2001; Chang, Koedinger, & Lovett, 2003). For example, Ben-Zeev and Star (2001) trained university students to use two algorithms for comparing algebraic fractions. On a subsequent test, students used each algorithm more often for problems similar to the example problems shown for that algorithm during training, although the two algorithms were equally valid for all test problems. Ben-Zeev and Star (2001) dubbed such influences of formally irrelevant problem features on students' strategy choices a "spurious correlation effect."

The present findings suggest that spurious correlation effects also occur in the context of fraction arithmetic. Together with prior data on the high frequency of fraction arithmetic errors that seem to reflect the effects of spurious correlations, the new findings suggest that reliance on such spurious correlations can hinder mastery of fraction arithmetic.

However, the findings of Experiment 3 in the present study also suggest that learning spurious associations between operations and operand features does not inevitably lead to high frequency of incorrect arithmetic strategies. The fraction arithmetic problems in the Chinese textbooks were distributed roughly like those in US textbooks, and children in the two countries formed similar spurious associations, yet Chinese students typically solve fraction arithmetic problems very accurately. For example, in Torbeyns, Schneider, Xin, and Siegler (2015), Chinese sixth and eighth graders scored above 90% correct on the same set of fraction arithmetic problems on which US peers scored below 50% correct. Thus, although Chinese children appear to learn the same spurious associations as US children, Chinese children less often choose incorrect fraction arithmetic strategies based on those associations.

One possible explanation is that Chinese children have better conceptual understanding of fraction arithmetic operations than US children, and this superior understanding allows them to override the influence of the spurious associations on arithmetic strategy choices. Consistent with this hypothesis, although most US sixth and eighth graders incorrectly judge that the product of two positive fractions smaller than one is larger than either operand alone (Siegler & Lortie-Forgues, 2015), most Chinese sixth and eighth graders do not commit this error (Tian & Siegler, in preparation). Further, a far higher percentage of primary school mathematics teachers in China than in the US can explain the rationales for fraction arithmetic procedures (Ma, 1999), making it possible for them to teach the rationales to their students. Chinese children's superior understanding of fraction arithmetic could override their spurious associations between operations and operand features, thereby avoiding detrimental effects of the spurious associations.

Another possibility, not exclusive from the first, is that Chinese students receive more practice with fraction arithmetic, resulting in stronger learning of correct procedures. Primary and high school students spend much far time per week doing mathematics homework in China than in the US (Fuligni & Stevenson, 1995). This finding suggests that the quantity of students' fraction arithmetic practice, like other types of math practice, is greater in China. This greater practice may enable children to overcome effects of spurious associations, due to it creating stronger associations between operations and correct problem-solving procedures.

Instructional Implications

Either or both of these possibilities may account for why Chinese children accurately solve fraction arithmetic problems despite their spurious associations between operations and operand features on such problems. Further, both interpretations suggest possible directions for improving US children's mastery of fraction arithmetic procedures: improve children's conceptual understanding, provide substantially more practice using the procedures, or both.

Less clear, however, is *how* to improve children's conceptual understanding of fraction arithmetic. Conceptual difficulties in this area have proven remarkably resistant to instruction. For example, children's ability to estimate a sum of two fractions without calculating an exact answer has shown almost no improvement despite decades of effort, most recently codified in the Common Core (CCSSI, 2010). When US eighth graders were asked in 1978 to choose the best estimate of 12/13+7/8 from among the options 1, 2, 19, 21, and "I don't know," the correct answer, 2, was chosen by only 24% of children (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). By 2014, percent correct on this problem had only risen to 27% (Lortie-Forgues et al., 2015). Similarly, in 2016, when sixth and eighth graders were asked to estimate the magnitudes of each operand and the sum of the operands in fraction addition problems, half of the estimated sums were smaller than the estimate of one or both operands (Braithwaite, Tian, & Siegler, 2017). This unimpressive performance in no way implies that we should abandon the goal of improving conceptual understanding of fraction arithmetic, but it does illustrate the challenge of doing so.

Giving children substantially more practice with fraction arithmetic would likely be beneficial, but it also is easier said than done. Increasing fraction arithmetic practice without increasing total time spent on mathematics would require decreasing time spent on other areas of mathematics. Increasing total time spent on mathematics would require decreasing time spent on other subjects or increasing total time spent in school. None of these changes would be easy to implement.

An alternative approach to improving US fraction arithmetic instruction would be to present children with more problems of the types to which they currently are rarely exposed, such as equal denominator multiplication problems and addition and subtraction problems involving a whole number and a fraction. This approach could prevent children from forming spurious associations and therefore from relying on such associations when solving problems. In contrast to the approaches described above, this approach would not require far-reaching changes to methods or amount of instruction. Instead, it could be implemented by instructional designers quite easily. The approach would likely be most effective if implemented not only by traditional textbook publishers, but also by designers of alternative resources on which many mathematics teachers rely for practice problems, such as online resources (e.g., Khan Academy, Illustrative Mathematics, education.com).

It remains to be tested whether this approach is effective, and if so, how many additional "underrepresented" practice problems would be needed to achieve the desired effect. A small number of examples can be sufficient to correct misconceptions in mathematics. For example, when placing whole numbers on a number line, second graders' responses are usually distributed logarithmically, but often shift to a linear response pattern after children are shown the correct response on a single trial (Opfer & Siegler, 2007). Similarly, adding a small number of currently underrepresented fraction arithmetic problems to existing curricular materials could prevent children from forming, and relying on, spurious associations. If so, there would be no obvious downside to this approach, as it would not require significant reduction in the number of other problems or increased time in school. On the other hand, if many additional problems are required, the benefits of adding them, relative to the costs of requiring children to solve a greater total number of problems, would need to be assessed. We hope to investigate these issues in future research and hope that others will as well.

Summary

In both the US and China, children learn mathematically irrelevant associations between fraction arithmetic operations and operand features that parallel associations between operations and operand features in mathematics textbooks. The findings were consistent with Braithwaite, Pyke, and Siegler's (2017) hypothesis that children form such associations and thus can use them to choose solution strategies. However, the data from Chinese children demonstrate that forming spurious associations does not predestine children to use them to choose incorrect strategies. How best to avoid the potential drawbacks of forming such spurious associations is an important issue for improving both theory and practice in this crucial area of mathematics learning.

References

Arkes, H., & Freedman, M. (1984). A demonstration of the costs and benefits of expertise in recognition memory. *Memory & Cognition*, *12*(1), 84–89. http://doi.org/10.3758/BF03197000

Bailey, D. H., Zhou, X., Zhang, Y., Cui, J., Fuchs, L. S., Jordan, N. C., … Siegler, R. S. (2015). Development of fraction concepts and procedures in U.S. and Chinese children. *Journal of Experimental Child Psychology*, *129*, 68–83. http://doi.org/10.1016/j.jecp.2014.08.006

Beijing Normal University Press. (2014). *Mathematics (Grades 3, 4, 5, and 6)*. (J. Liu, Q. Kong, & D. Zhang, Eds.). Beijing, China: Beijing Normal University Press.

Ben-Zeev, T., & Star, J. R. (2001). Spurious correlations in mathematical thinking. *Cognition and Instruction*, *19*(3), 253–275. http://doi.org/10.1207/S1532690XCI1903_1

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*(4), 247–253. http://doi.org/10.1016/j.cedpsych.2012.07.001

Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2014). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology*, *118*, 110–118. http://doi.org/10.1016/j.jecp.2013.09.001

Braithwaite, D. W., Goldstone, R. L., van der Maas, H. L. J., & Landy, D. H. (2016). Non-formal mechanisms in mathematical cognitive development: The case of arithmetic. *Cognition*, *149*, 40–55. http://doi.org/10.1016/j.cognition.2016.01.004

Braithwaite, D. W., Pyke, A. A., & Siegler, R. S. (2017). A computational model of fraction arithmetic. *Psychological Review*, *124*(5), 603-625. http://doi.org/10.1037/rev0000072

Braithwaite, D. W. & Siegler, R. S. (in preparation). Rational arithmetic with decimals and fractions.

Braithwaite, D. W., Tian, J., & Siegler, R. S. (2017). Do children understand fraction addition? *Developmental Science*, e12601. http://doi.org/10.1111/desc.12601

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology*, *27*(5), 777–786. http://doi.org/10.1037/0012-1649.27.5.777

Carpenter, T. P. T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. (1980). Results of the second NAEP mathematics assessment: Secondary school. *The Mathematics Teacher*, *73*(5), 329–338.

Chang, N. M., Koedinger, K. R., & Lovett, M. C. (2003). Learning spurious correlations instead of deeper relations. In R. Alterman & D. Kirsch (Eds.), *Proceedings of the 25th Annual Conference of the Cognitive Science Society* (pp. 228–233). Boston, MA: Cognitive Science Society.

Charles, R., Caldwell, J., Cavanagh, M., Chancellor, D., Copley, J., Crown, W., … Van der Walle, J. (2012). *enVisionMATH (Common Core edition)*. Glenview, IL: Pearson Education, Inc.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington, D.C.: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Dixon, J. K., Adams, T. L., Larson, M., & Leiva, M. (2012a). *GO MATH! (Common Core edition)*. Orlando, FL: Houghton Mifflin Harcourt Publishing Company.

Dixon, J. K., Adams, T. L., Larson, M., & Leiva, M. (2012b). *GO MATH! standards practice book (Common Core edition)*. Orlando, FL: Houghton Mifflin Harcourt Publishing Company.

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *42*(1), 1–16.

Fuchs, L. S., Schumacher, R. F., Sterba, S. K., Long, J., Namkung, J., Malone, A., … Changas, P. (2014). Does working memory moderate the effects of fraction intervention? An aptitude–treatment interaction. *Journal of Educational Psychology*, *106*(2), 499–514. http://doi.org/10.1037/a0034341

Fuligni, A. J., & Stevenson, H. W. (1995). Time use and mathematics achievement among American, Chinese, and Japanese high school students. *Child Development*, *66*(3), 830–842. http://doi.org/10.1111/j.1467-8624.1995.tb00908.x

Hagen, J. W. (1972). Strategies for remembering. In S. Farnham-Diggory (Ed.), *Information Processing in Children* (pp. 65–79). Academic Press.

Handel, M. J. (2016). What do people do at work? *Journal for Labour Market Research*, *49*(2), 177–197. http://doi.org/10.1007/s12651-016-0213-1

Hecht, S. A., & Vagi, K. J. (2010). Sources of group and individual differences in emerging fraction skills. *Journal of Educational Psychology*, *102*(4), 843–859. http://doi.org/10.1037/a0019824

"Helping You Learn Mathematics Classroom Workbook" Writing Group. (2014). *Helping you to learn mathematics (Grades 3, 4, 5, and 6)*. (Y. Xu & G. Shen, Eds.). Beijing, China: Popular Science Press.

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology*, *116*(1), 45–58. http://doi.org/10.1016/j.jecp.2013.02.001

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review*, *38*, 201–221. http://doi.org/10.1016/j.dr.2015.07.008

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Taylor & Francis.

Myles-Worsley, M., Johnston, W. A., & Simons, M. A. (1988). The influence of expertise on X-ray image processing. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *14*(3), 553–557. http://doi.org/10.1037/0278-7393.14.3.553

Nanjing Oriental Mathematics Education Scientific Research Center, & Jiangsu District Primary and Middle School Education Research Center. (2014). *Mathematics (Grades 3, 4, 5, and 6)*. (L. Sun & L. Wang, Eds.). Nanjing, China: Phoenix Education Publishing.

New Century Primary Mathematics Curriculum Writing Group. (2015). *Mathematics growing up with you (Grades 3, 4, 5, and 6)*. (X. Chen, Ed.). Beijing, China: Beijing Normal University Publishing Group.

Newton, K. J., Willard, C., & Teufel, C. (2014). An examination of the ways that students with learning disabilities solve fraction computation problems. *The Elementary School Journal*, *39*(3), 258–275. http://doi.org/10.1163/_afco_asc_2291

Novick, L. R. (1988). Analogical transfer, problem similarity, and expertise. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *14*(3), 510–20.

Opfer, J. E., & Siegler, R. S. (2007). Representational change and children's numerical estimation. *Cognitive Psychology*, *55*, 169–195. http://doi.org/10.1016/j.cogpsych.2006.09.002

Pacton, S., Perruchet, P., Fayol, M., & Cleeremans, A. (2001). Implicit learning out of the lab: The case of orthographic regularities. *Journal of Experimental Psychology: General*, *130*(3), 401–426. http://doi.org/10.1037/0096-3445.130.3.401

Patel, V., & Groen, G. (1991). The general and specific nature of medical expertise: A critical look. In K. A. Ericsson & J. Smith (Eds.), *Toward a general theory of expertise: Prospects and limits* (pp. 93–125). Cambridge University Press.

Pelucchi, B., Hay, J. F., & Saffran, J. R. (2009). Statistical Learning in a Natural Language by 8-Month-Old Infants. *Child Development*, *80*(3), 674–685. http://doi.org/10.1111/j.1467-8624.2009.01290.x

Pennsylvania Department of Education. (2016). National school lunch program reports. Retrieved from http://www.education.pa.gov/Teachers - Administrators/Food-Nutrition/Pages/National-School-Lunch-Program-Reports.aspx#tab-1

People's Education Press Curriculum Research Center Primary Mathematics Curriculum Research and Development Center. (2014). *Mathematics (Grades 3, 4, 5, and 6)*. (J. Lu & G. Yang, Eds.). Beijing, China: People's Education Press.

Perruchet, P., & Pacton, S. (2006). Implicit learning and statistical learning: one phenomenon, two approaches. *Trends in Cognitive Sciences*, *10*(5), 233–238. http://doi.org/10.1016/j.tics.2006.03.006

Saffran, J. R., Aslin, R. N., & Newport, E. L. (1996). Statistical learning by 8-month-old infants. *Science*, *274*(5294), 1926–1928.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., … Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–7. http://doi.org/10.1177/0956797612440101

Siegler, R. S., & Lortie-Forgues, H. (2015). Conceptual knowledge of fraction arithmetic. *Journal of Educational Psychology*, *107*(3), 909–918. http://doi.org/10.1037/edu0000025

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, *49*(10), 1994–2004. http://doi.org/10.1037/a0031200

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–96. http://doi.org/10.1016/j.cogpsych.2011.03.001

Suzhou Education Press Primary School Mathematics Curriculum Writing Group. (2014). *Practice and testing primary school mathematics (Grades 3, 4, 5, and 6)*. (W. Zhao & Z. Xu, Eds.). Nanjing, China: Phoenix Education Publishing.

Tian, J. & Siegler, Robert S. (in preparation). Conceptual understanding of decimal and fraction arithmetic among Chinese children.

Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, *37*, 5–13. http://doi.org/10.1016/j.learninstruc.2014.03.002

Treiman, R., & Kessler, B. (2006). Spelling as statistical learning: Using consonantal context to spell vowels. *Journal of Educational Psychology*, *98*(3), 642–652. http://doi.org/10.1037/0022-0663.98.3.642

University of Chicago School Mathematics Project. (2015a). *Everyday Mathematics Assessment Handbook* (4th ed.). Columbus, OH: McGraw-Hill Education.

University of Chicago School Mathematics Project. (2015b). *Everyday Mathematics Student Math Journal Volumes 1 and 2* (4th ed.). Columbus, OH: McGraw-Hill Education.

University of Chicago School Mathematics Project. (2015c). *Everyday Mathematics Student Reference Book* (4th ed.). Columbus, OH: McGraw-Hill Education.

Voss, J. F., Vesonder, G. T., & Spilich, G. J. (1980). Text generation and recall by high-knowledge and low-knowledge individuals. *Journal of Verbal Learning and Verbal Behavior*, *19*(6), 651–667. http://doi.org/10.1016/S0022-5371(80)90343-6

Endnotes

1. These patterns may reflect influences of the Common Core State Standards in mathematics. In the Standards, expectations for fractions in grades 3 and 4 are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100 (CCSSI, 2010). All of these denominators except for 100 appeared more often than any other denominator in all three textbook series. The patterns may also reflect textbook designers favoring fractions that are easy to use in calculations. Arithmetic problems involving fractions with large denominators may be difficult because they are likely to require relatively difficult whole number arithmetic calculations (e.g., calculating 1/18+1/15 requires identifying 90 as the least common denominator of the operands and retrieving the facts 18×5 = 90 and 15×6 = 90, or alternatively, calculating 18×15 = 270). Problems with unsimplified fractions (e.g., 3/6) as operands may be more difficult to solve because children may try to simplify the fractions (e.g., 3/6 = 1/2) before calculating the answer. Problems with prime numbers as denominators (e.g., 3/7) may be difficult to calculate with because adding or subtracting such fractions to other fractions with unequal denominators requires conversion to a large common denominator (e.g., compare 3/7+1/4 = 19/28 to 3/8+1/4 = 5/8).

Table 1. Percent problems classified by arithmetic operation and denominator equality. Data averaged over 3rd-6th grade volumes in three US textbook series (only problems with no whole number operands are included, *N* = 1359 problems).

	Arithmetic Operation			
Operand Denominators	Addition	Subtraction	Multiplication	Division
Equal denominators	18	20	**1**	**2**
Unequal denominators	18	14	17	9

Table 2. Percent problems classified by arithmetic operation and whether operands were both fractions or had one whole and one fraction (mixed numbers classified as fractions.) Data averaged over 3rd-6th grade volumes in the three US textbook series cited above (N = 1972 problems).

	Arithmetic Operation			
Operand Number Type	Addition	Subtraction	Multiplication	Division
Fraction-fraction	25	24	13	7
Whole-fraction	**0**	**2**	17	12

Table 3. Tasks used in Experiments 1 and 3. (A modified version of the choose-operation task was used in Experiment 2.)

Task	Instructions	Example Trial
Generate-operands	"Look at the arithmetic operation, and try to write numbers that you think probably would appear in problems with that operation."	$\square + \square$
Choose-operation	"For each problem, please guess what the arithmetic operation probably was. It could be ×, −, ÷, or +. Draw a circle around the one you guess."	$\frac{4}{6} \square \frac{1}{6}$ $+$ $-$ $\times$ $\div$
Match-operands-with-operations	"Try to guess which problem was the [e.g., addition] problem. In other words, in which problem would you guess the missing sign is a [+] sign?"	(a) $\frac{4}{6} \square \frac{1}{6}$ (b) $\frac{3}{4} \square \frac{1}{6}$

Table 4. Summary of predictions and outcomes in Experiments 1, 2, and 3: "✓" indicates that a prediction was consistent with the results ($p < .05$), "✓(6)" or "✓(8)" that it was consistent with the results for sixth or eighth graders only ($p < .05$), "†"that a marginal effect in the predicted direction was found ($p < .1$), and "✕"that a prediction was not consistent with the results.

Task	Prediction	US Students (Experiment 1)	US Students (Experiment 2)	Chinese Students (Experiment 3)
Generate-Operands	1A. Children will generate two fractions with equal denominators more often for addition and subtraction than for multiplication and division.	✓		✓
	1B. Children will generate one whole number and one fraction more often for multiplication and division than for addition and subtraction.	†		✕
	1C. The frequency with which children generate a fraction will be positively correlated with the fraction's frequency in textbook problems.	✓		✓
Choose-Operation	2A. Children will choose addition or subtraction more than multiplication or division on equal denominator fraction-fraction trials.	✓	✓	✓
	2B. Children will choose multiplication or division more than addition or subtraction on whole-fraction trials.	✓	✓	✓
Match-operands-with-operations	3A. Children will choose equal denominator fraction-fraction operand pairs more often for addition and subtraction than for multiplication and division.	✓		✓
	3B. Children will choose whole-fraction operand pairs more often for multiplication and division than for addition and subtraction.	✓ (6)		✕

Table 5. Percent problems classified by arithmetic operation and denominator equality in 3rd-6th grade volumes of three Chinese textbook series (only problems with no whole number operands are included, $N = 835$ problems).

	Arithmetic Operation			
Operand Denominators	Addition	Subtraction	Multiplication	Division
	Beijing Normal University ($N = 249$)			
Equal denominators	14	7	0	3
Unequal denominators	17	17	28	14
	People's Education Press ($N = 333$)			
Equal denominators	17	13	2	3
Unequal denominators	16	16	21	12
	Phoenix Education Publishing ($N = 253$)			
Equal denominators	12	11	1	2
Unequal denominators	16	15	21	21

Table 6. Percent problems classified by arithmetic operation and whether operands were both fractions or had one whole and one fraction (mixed numbers classified as fractions) in 3rd-6th grade volumes of three Chinese textbook series (*N* = 1340 problems).

	Arithmetic Operation			
Operand Number Type	Addition	Subtraction	Multiplication	Division
	Beijing Normal University (*N* = 442)			
Fraction-fraction	17	14	16	9
Whole-fraction	0	4	24	16
	People's Education Press (*N* = 507)			
Fraction-fraction	22	19	15	9
Whole-fraction	0	4	16	14
	Phoenix Education Publishing (*N* = 391)			
Fraction-fraction	18	17	15	15
Whole-fraction	0	3	14	18