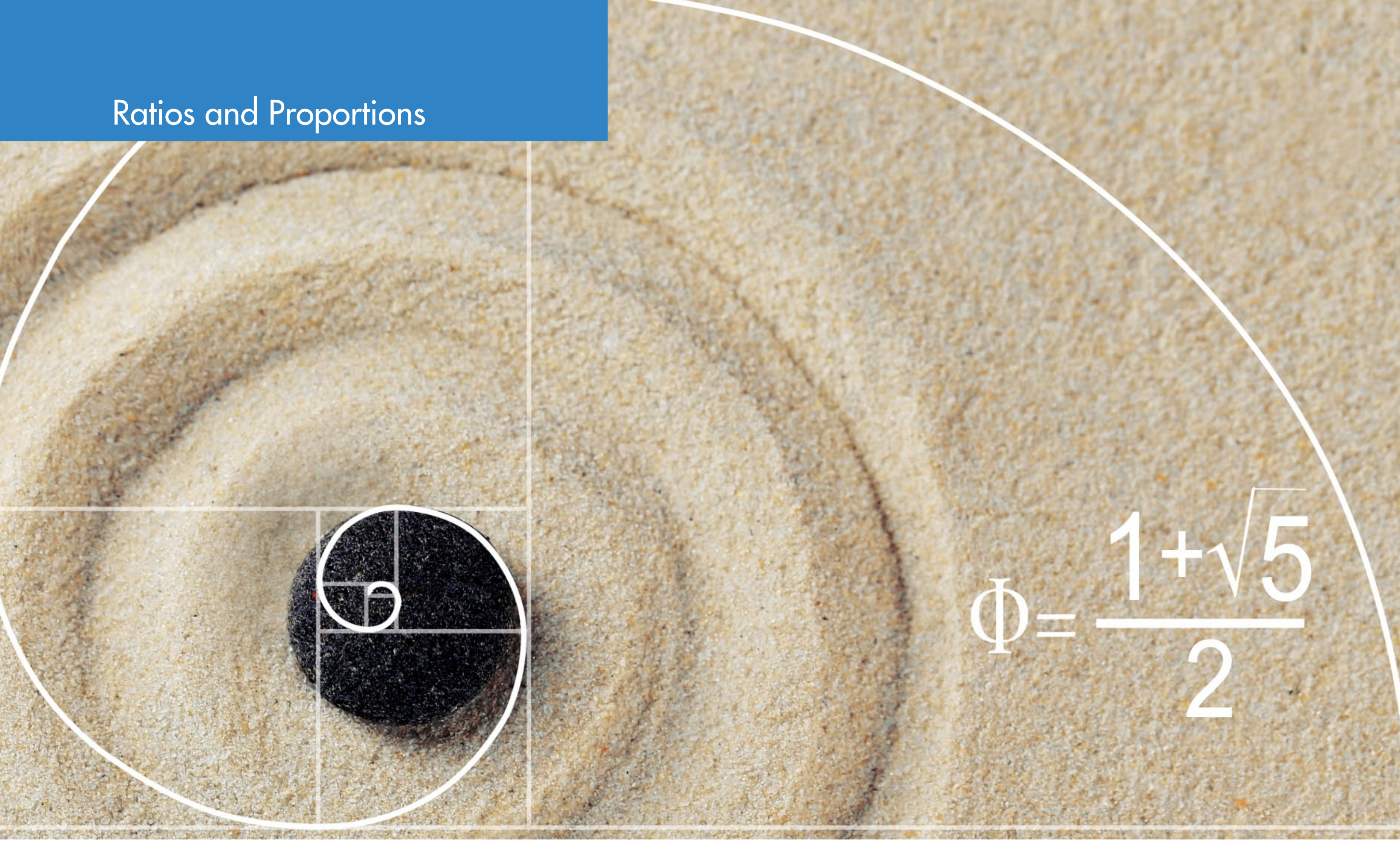

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Helping Students With Mathematics Difficulties Understand Ratios and Proportions

Barbara Dougherty, Diane Pedrotty Bryant, Brian R. Bryant, and Mikyung Shin

Cara, a seventh-grade student with learning disabilities (LD) in mathematics, believes that the ratio 2:3 is equivalent to 4:5 because there is a difference of one between the two numbers in each ratio and there is a difference of two between corresponding numbers in the two ratios (2 + 2 = 4 and 3 + 2 = 5). This misconception affects her ability to find equivalent ratios as well as to work with ratios and proportions in multiple contexts, including determining rates of change with functions and other contextual situations.

Why are ratios and proportions important for students' success in mathematics? What are common misconceptions that challenge students with persistent mathematics difficulties? What lesson components for intensive interventions are necessary to address the instructional needs and misconceptions of students with persistent mathematical difficulties? How can progress-monitoring results inform instruction and address misconceptions? Cara's teacher seeks to find the answers to these questions in order to identify ways to support Cara's understanding of ratios and proportions.

Ratios and proportions are foundational to student understanding across multiple topics in mathematics and science. In mathematics, they are central to developing concepts and skills related to slope, constant rate of change, and similar figures, which are all fundamental to algebraic concepts and skills. Ratios and proportions are used in relationships found in triangles, including trigonometric ones, such as sine, cosine, and tangent, found in later algebraic instruction. In science, they are used when quantities involve density, acceleration, and other comparable derived measures. Even in real-life situations, ratios and proportions are useful when determining amounts to be used in recipes or finding the mileage per gallon of gas. In general, ratios and proportions describe relationships between and among quantities.

The Common Core State Standards in Mathematics (CCSS-M; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) include a cluster of standards associated with developing a deep understanding of ratio and proportional reasoning in Grades 6 and 7. For example, one standard within Ratios and Proportional Relationships is for students to "understand ratio concepts and use ratio reasoning to solve problems," which requires them to:

1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*
2. Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*
3. Use ratio and rate reasoning to solve real-world and mathematical problems (e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations).
   a. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.
   b. Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*
   c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.
   d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities. (CCSSM 6.RPA.1)

The depth and complexity of these standards increase across the two grades and are then used in Grade 8 with other topics, including slope and transformations on the coordinate grid.

The concepts and skills that support students' understanding of ratios and proportions include, but are not limited to, realizing that (a) the relationship between (or among) the quantities in a ratio is multiplicative in nature (not additive), (b) a unit rate can be found even if one of the numbers in the ratio is not a factor (or multiple) of the other number, and (c) equivalent ratios are not necessarily integral multiples of another ratio (e.g., 6:9 is equivalent to 4:6). When these understandings are not well situated within students' knowledge about ratios and proportions, significant difficulties can occur in algebraic contexts. For instance, oftentimes students struggle with ratios and proportional reasoning because of misconceptions that were established in earlier grades due to practitioners' poor instruction. Teachers must understand, recognize, and address these misconceptions so that student learning of ratios and proportional reasoning are not impaired.

Common Misconceptions That Challenge Students With Persistent Mathematics Difficulties

Mathematical misconceptions are faulty and incorrect ideas resulting from students' misunderstanding about a mathematical idea or concept (Allen, 2007). Misconceptions are usually based on applications of inappropriate generalizations or rules, or insufficient teaching (Allen, 2007). Instruction often includes giving students rules that expire, but students hold onto those rules and attempt to apply them even when the situation is inappropriate. For

Figure 1. Example of an Additive Comparison as a Misconception and the Correct Multiplicative Comparison

Additive Comparison

$$\begin{array}{ccc} \text{1 cup of sugar} & & \text{2 cups of flour} \\ \downarrow & & \downarrow \\ \text{1 cup + 2 cups =} & & \text{2 cups + 2 cups =} \\ \text{3 cups of sugar} & & \text{4 cups of flour} \end{array}$$

1:2 ≠ 3:4

Multiplicative Comparison

$$\begin{array}{ccc} \text{1 cup of sugar} & & \text{2 cups of flour} \\ \downarrow & & \downarrow \\ \text{1 cup × 3 cups =} & & \text{2 cups × 3 cups =} \\ \text{3 cups of sugar} & & \text{6 cups of flour} \end{array}$$

1:2 = 3:6

example, students in the elementary grades are often taught to add a zero to the end of a whole number when multiplying by 10. However, this "rule" does not hold when multiplying decimals (e.g., 0.65 × 10 = 6.5 rather than 0.650; Karp, Bush, & Dougherty, 2015). Misconceptions related to ratios and proportional reasoning focus on prerequisite knowledge about multiplication and fractions. So, what are some common misconceptions, such as Cara's misconception, that interfere with the ability to understand ratios and proportional reasoning and to generalize those understandings to algebraic thinking?

Misconception 1: Additive Versus Multiplicative Comparisons

Students such as Cara have not developed an understanding of the relationship between additive comparison and multiplicative comparisons (Fielding-Wells, Dole, & Makar, 2014; Norton, 2005; Van Dooren, DeBock, & Verschaffel, 2010). Students do not understand that *4 more than* (additive comparison) has a different meaning than *4 times* (multiplicative comparison). For example, a recipe calls for 2 cups of flour for every 1 cup of sugar. How many cups of flour are needed if a recipe is increased to 3 cups of sugar? Students can think 1 cup of sugar for 2 cups of flour, 2 cups of sugar for 4 cups of flour, and 3 cups of sugar for 6 cups of flour. Therefore, the amount of flour now needed is 6 cups, which maintains the same relationship as the original one given (1:2). Students who think ratios are additive would have mistakenly thought that 3 cups of sugar is an increase of 2 cups from the original 1 cup. They would then add to the original 2 cups of flour 2 more cups to get 4 cups needed flour (see illustration, Figure 1). For ratios to be in proportion, they must be equivalent and compare the same types of quantities. Equivalent ratios have a multiplicative relationship, so students must understand the concept of multiplicative comparison (Lobato, Ellis, & Charles, 2010). Teachers are advised to first provide explicit instruction with modeling as a corrective measure when students have established a consistent, faulty pattern of responding and then provide multiple opportunities for students to explain their thinking for creating equivalent ratios and use corrective feedback to repair faulty understandings.

Misconception 2: Incorrect Conceptualization of Fractions

When talking about fractions, students and their teachers might use the language *out of*—as in "$\frac{3}{4}$ is 3 parts out of 4 parts"—instead of "3 one fourths." When students read $\frac{3}{4}$ as 3 parts out of 4 parts, the numerator and denominator appear to be two whole numbers, which represents the misconception that the parts are separate quantities. Understanding a fraction, $\frac{a}{b}$, as "*a* one *b*ths" is important for creating a ratio as a multiplicative comparison (Lobato et al., 2010). For example, a quantity representing $\frac{3}{4}$ is 3 times larger then the quantity representing the unit fraction $\frac{1}{4}$. Teachers can model for students the correct language for talking about a fraction, such as "3 one fourths," and provide additional practice for students to use this same language to talk about fraction quantities.

Misconception 3: Lack of Covariational Thinking

When working to find a pattern in a table, students mistakenly look at only the pattern from row to row rather than using *covariational* thinking (i.e., thinking about how two quantities vary together) (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). For example, in Figure 2, in row 3, the difference between the number of triangles and the number of angles is 4. Students who mistakenly do not think about how the two quantities vary together throughout the table might say that in row 4, the answer is 7 (3 + 4 = 7) rather than seeing the pattern of ×3 (e.g., 1 × 3 = 3, 2 × 3 = 6, 3 × 3 = 9).

Teachers should ask students to explain how they determined the missing values and the relationship between the number of triangles and the number of angles. Students should use the language, "There [is/are] _____ for every

Figure 2. Table of Equivalent Ratios to Promote Covariational Thinking

Number of Triangles	Number of Angles
1	3
2	6
3	?
4	?
7	?
?	30

Instructions: Mona started a table that shows the number of triangles and the number of angles. Decide what should go into the blank cells of the table.

______." For instance, students would say, "There is one triangle for every three angles" for the unit rate or ratio of 1:3. Continue to model the covariational aspects, so that students move to thinking about the relationships of the quantities (Dougherty, Bryant, & Bryant, 2016)

These are significant misconceptions that affect students' ability to access the more complex concepts associated with proportional reasoning, such as functions, algebraic equations, and graphing (Lobato et al., 2010). As noted in the CCSS-M, students in Grade 6 should understand ratio concepts and use ratio reasoning and proportional relationships to solve problems (CCSS-M 6.RP; NGA & CCSSO, 2010, p. 42). In Grade 7, students use their understanding of ratios to solve real-world problems involving percentage, interest, tips, and so forth (CCSS-M 7.RP; NGA & CCSSO, 2010, p. 48). In Grade 8, students make connections among proportional reasoning, lines, and linear equations (CCSS-M 8.EE; NGA & CCSSO, 2010, p. 54). Given the importance of understanding ratios and proportional relationships, intensive intervention for students with mathematics disabilities can be structured to promote their understanding of ratios and proportional reasoning and prevent or ameliorate misconceptions.

Responding to the Instructional Needs and Misconceptions of Students

To support students with persistent mathematics difficulties, specific lesson components, which are supported by multiple research findings in mathematics and the Standards for Mathematical Practice (Gersten et al., 2009; NGA & CCSSO, 2010), can be incorporated into instruction. These include explicit, systematic instruction; asking students the right kinds of questions; using multiple representations; and providing student scaffolded instruction.

Explicit, Systematic Instruction

Explicit, systematic instruction (Gersten et al., 2009) is frequently used to describe the type of instruction for students who need intensive interventions to learn mathematical concepts and skills. There is a general structure that is used in these lessons that typically includes modeling, guided practice, and independent practice. According to the National Mathematics Advisory Panel (NMAP; 2008), *explicit instruction* is characterized by problem-solving models, a range of examples, practice with feedback, and students' verbalizations of their thinking processes. Further, the NMAP recommended that students with mathematical difficulties "receive some explicit mathematics instruction regularly. Some of this time should be dedicated to ensuring that these students possess the foundational skills and conceptual knowledge necessary for understanding the mathematics they are learning at their grade level" (p. xxiii). An illustration of this structure is provided in Table 1; however, some of the terminology is altered for teachers to use with older students. To plan for explicit instruction using this structure, teachers can first consider the essential big ideas of mathematics topics and the prominent misconceptions that students may have. Using these two elements, tasks and questions can be developed so that students' attention is focused on the important aspects of the mathematics topic as illustrated in Table 1.

Types of Questions

In many mathematics intervention classes, the types of questions that students are asked focus primarily on low-level information or recall of algorithmic steps (Dougherty & Foegen, 2011). Although factual information and algorithms are important and students should learn both, they also should have opportunities through questioning to think more deeply about mathematical ideas so that they can form connections across older and newer learning (Skemp, 1987).

Students with persistent mathematics difficulties benefit from learning mathematics more deeply through instruction that includes three types of questions—reversibility, flexibility, and generalization—that are important for intensive instruction (Dougherty, Bryant, Bryant, Darrough, & Pfannenstiel; see Table 1 for examples). *Reversibility questions* give students the answer and then they create the question. To include reversibility thinking in instruction, use some of the following guidelines:

- Give students answers to the types of problems being taught and have them identify the questions.
- Ask students how they can show (through manipulatives, pictures, number lines) how to solve the problem.

Flexibility questions ask students to solve a problem in multiple ways or find similarities and differences between and among problems and classes of problems. To include flexibility thinking, use the following guidelines:

- Ask students if they have solved a problem that is similar to

Table 1. Teaching Ratio and Proportional Reasoning and Preventing Misconceptions

<table>
<tr><th>Lesson Component</th><th>Description</th><th>Example</th></tr>
<tr><td>Lesson objectives (CCSS-M, 6.EE.9; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 44)</td><td>Use appropriate content outcomes and associated standards for mathematical practice.</td><td>Students determine the dependent and independent variables in a situation.
Students describe relationships between the dependent and independent variables in multiple ways, including writing an expression or equation.
Students reason abstractly and quantitatively.
Students look for and express regularity in repeated reasoning.</td></tr>
<tr><td>Vocabulary</td><td>Determine the vocabulary that is essential to the lesson.
Use accurate and precise mathematical language.</td><td>Dependent variable
Independent variable</td></tr>
<tr><td>Misconception</td><td>Identify the misconceptions that may be evident in the processes students use to solve problems or in the way in which they think about the task. This may include misconceptions that have developed from previous grades.</td><td>Students confuse the variables. For more complex situations, it may be difficult to determine the relationship between the variables.</td></tr>
<tr><td>Warming up</td><td>Provide short tasks that (a) include a review from previous lessons or (b) act as advance organizers for this or upcoming lessons.</td><td>Example 1: Zach put $25 in his savings account each week. Identify the independent and dependent variables. Use corrective feedback for answers that are not the independent and dependent variables. Review the vocabulary: independent and dependent variables.</td></tr>
<tr><td>Learning to solve (modeling)</td><td>Target instruction on a specific, central concept.
Use explicit instruction to focus students’ attention on specific mathematical structures, ideas, or common misconceptions.
Include small group or pair tasks to engage students as appropriate.
Utilize students’ intuitive or natural approach to build the mathematics.
Use questions that improve critical thinking (reversibility, flexibility, and generalization).
Incorporate multiple representations including natural language, tables, charts, diagrams, and physical materials.</td><td>As we learned in previous lessons, tables can be used to show equivalent ratios. As a review, look at example 2. With a partner, complete the table and identify the independent and dependent variables.
Example 2: Henna made a table to show the amount of money she earns when she works at the craft store.
<table>
<tr><th>Number of hours worked</th><th>Amount of money earned</th></tr>
<tr><td>1</td><td>$5</td></tr>
<tr><td>2</td><td>$10</td></tr>
<tr><td>3</td><td></td></tr>
<tr><td>4</td><td>$20</td></tr>
<tr><td></td><td>$25</td></tr>
<tr><td>6</td><td>$30</td></tr>
<tr><td></td><td>$45</td></tr>
<tr><td>10</td><td></td></tr>
<tr><td></td><td></td></tr>
</table>
</td></tr>
</table>

(continued)

Table 1. (continued)

Lesson Component	Description	Example
		Give students time to complete the example. Ask some students to share their responses. Discuss their responses as appropriate. How would you describe the relationship in the table? Write your description. *(Answers will vary, but focus on students saying that the total amount of money earned is 5 times the number of hours worked.) (Examples of reversibility are shown in rows 6 and 8; answers that vary reflect flexibility.)* As students describe the relationships, focus on those that indicate the multiplicative nature. Let us revisit your table and find another way to describe the relationship. We can write the relationship in the table by using a variable so that it gives us the rule for any number we might use. This is called a generalization. What is this called? (*generalization)* Look at the bottom of the first column. We are going to use a variable, in this case x, to help us write the generalization. The variable x represents the number of hours worked. What does the variable x represent? (*number of hours worked*). We want to write an expression that describes the relationship or the computation that is used on x to give the amount in the second column. You told me that the amount in the second column is 5 times the amount in the first column. We will write 5 times x in the second column. Model for students how to write 5 times x in the second column. Have students do the same. If students are unclear or do not remember how to show 5 times a variable, review the symbols for showing multiplication with a variable. Explain that there are multiple ways to write the multiplication that are equivalent for writing an expression using variables. Examples include $5x$, $5 \bullet x$, or $(5)(x)$. It is not advisable to use the $\times$ because students might confuse the multiplication symbol and the variable. Provide another example of a generalization. What if for every hour worked, the pay was $6.00? How would that change the generalization ($5x$) that we wrote? What would be the generalization now? *(6x)* You can use the generalization to find the amount of pay for any hours worked. If the generalization is $6x$, what is the pay for working 12 hours? *($72)* Continue with similar examples if needed. Conclude by asking the following questions. What generalization did we discuss in this lesson? How would you define an independent variable? How would you define a dependent variable?

(continued)

Table 1. (continued)

<table>
<tr><th>Lesson Component</th><th>Description</th><th>Example</th></tr>
<tr><td>Practicing together (guided practice)</td><td>Allow time for students to work with a partner or small group, independent of the teacher.
Use tasks that promote the use of multiple representations.
Advise students of the time limit for their work and describe what will happen at the end of the time allotted.
Have students publicly share their processes or solution approaches.
Encourage students to show their thinking rather than only sharing an algorithm or step-by-step process.
Consider options for students to critique each other's presentations.</td><td>Work with your partner to complete the following table.
<table>
<tr><th>Number of hours worked</th><th>Amount of money earned</th></tr>
<tr><td>1</td><td>$7.50</td></tr>
<tr><td>2</td><td>$15.00</td></tr>
<tr><td>3</td><td>$22.50</td></tr>
<tr><td>4</td><td>$30</td></tr>
<tr><td>5</td><td></td></tr>
<tr><td>6</td><td></td></tr>
<tr><td></td><td>$75.00</td></tr>
<tr><td></td><td>$150.00</td></tr>
<tr><td>x</td><td></td></tr>
</table></td></tr>
<tr><td>Trying it on your own (independent practice)</td><td>Create problems that will give you an indication of student thinking.
Provide feedback regarding students' responses.
Use reversibility, flexibility, and generalization to create the problems.</td><td>Sample problem:
<table>
<tr><th>Number of math problems</th><th>Number of minutes to complete</th></tr>
<tr><td>1</td><td>2</td></tr>
<tr><td>2</td><td>4</td></tr>
<tr><td>3</td><td>6</td></tr>
<tr><td>4</td><td>8</td></tr>
<tr><td>5</td><td>10</td></tr>
<tr><td>10</td><td>20</td></tr>
<tr><td>20</td><td>40</td></tr>
<tr><td>m</td><td>??</td></tr>
</table>
a. $1m$ because m represents the number of math problems.
b. $2m$ because each math problem takes 2 minutes to solve.
c. $3m$ because it takes 6 minutes to solve 3 problems.
d. $10m$ because it takes 20 minutes to solve 10 problems.</td></tr>
<tr><td>Wrapping it up</td><td>Focus on a single task that culminates the learning experience.
Use tasks that have multiple responses.
Have students share their responses as time allows.</td><td>Write the generalization using a variable that shows:
For every cup of rice, it takes 3 cups of water to cook it.
Discuss student responses as time allows.</td></tr>
</table>

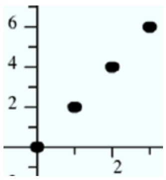
Note. CCSS-M = Common Core State Standards for Mathematics.

Figure 3. Example of Multiple Representations

Verbal: Avah sells tacos for $2 each in her food cart. How much does she make if she sells 3 tacos?

Symbolic: $y = 2x$ where x represents the number of tacos sold, and y represents the amount of money Avah makes.

Graphical:



Tabular:

X	Y
0	0
1	2
2	4
3	6

(or different from) the given problem.

- Ask students how they can use what they know about the problem to solve the new problem.
- Have students solve the problem in a different way using models.

In a related way, *generalization questions* ask students to identify patterns and use those patterns to make conjectures or generalizations. To use generalization questions, apply the following guidelines:

- Have students identify patterns that they notice.
- Ask students to name representations they are familiar with that work for the new problem.
- Have students identify a strategy that they have learned that can be used to solve the problem.

From the introduction of a topic to the final lesson in the instructional sequence, these questions provide opportunities for students to think deeply about significant ideas. If they are used consistently throughout a unit of study, student responses become more and more substantive as they come to understand how to respond to these questions. Each question type is included in each of the lesson components.

Concurrent Use of Multiple Representations

Multiple representations—including physical phenomena or manipulatives, natural language (written and spoken), tables, diagrams, and symbols—can facilitate conceptual understanding by having students represent concepts and talk or write about their representations (National Council of Teachers of Mathematics, 2000). Rather than following a progression of first presenting physical models, followed by diagrams, and finally by symbols, the representations can be presented concurrently to support student learning (Dougherty, 2008). The concurrent presentation helps students make connections across the representations and create "mental residues" of their experiences (Author, 2008)—the images left behind after students have had experiences that build a concept. These images connect

By presenting representations simultaneously, students can see how the symbols represent the action or relationship embodied in the physical models or diagrams.

ideas and can be called upon when students face problems that require them to retrieve ideas from previous learning. These connections can then, in turn, be called upon (prompted) when students face problems that require them to retrieve ideas from previous learning. In addition, by presenting representations simultaneously, students can see how the symbols represent the action or relationship embodied in the physical models or diagrams. Moreno, Ozogul, and Reisslein (2011) noted that using multiple representations fosters problem solving and presenting representations concurrently enhances the likelihood that students can engage in the problems. For example, in ratios and proportions, students can use cubes to model a given ratio, then represent the relationship of associated equivalent ratios in a table or a picture. Figure 3 presents an example of multiple representations including verbal, symbolic, graphical, and tabular representations. The teacher can have students translate between the verbal, symbolic, graphical, and tabular representations to show various relationships of information in problems.

Scaffolds

Scaffolds or instructional supports can help students access the content that may be challenging for them. Teachers can use scaffolds to assist students in attending to and tackling problems that may at first appear too difficult to attempt (Bryant et al., 2014). When concepts and skills are first introduced, supports may include graphic organizers, cognitive strategies, teacher prompts, and "think-alouds"; supports are gradually decreased as students gain proficiency.

Another scaffold teachers can employ is small groups. Table 1 provides examples of using small groups by purposefully situating problems within a small group (either pairs or groups no larger than four students) so that students have a safe environment in which to discuss their ideas. As students share their thinking coupled with teacher facilitation (e.g., questioning, prompting), misconceptions can be identified and, in some cases, self-corrected as students become more proficient with explaining the mathematics to others. Discussions within the small groups also can affirm or correct their thinking and give students confidence in sharing their ideas before the whole class.

Other examples of scaffolds are opportunities to respond and student verbalizations as they learn new concepts and skills. Recommendations in the Standards for Mathematical Practice (NGA & CCSSO, 2010) specifically include the use of multiple opportunities for students to solve problems, reason abstractly and quantitatively, look for and make generalizations, construct and critique arguments, and make use of mathematical structure. A crucial component of opportunities to respond is that students are also given explicit, frequent feedback. Teachers should ensure these recommendations are systematically included across a series of lessons for students to build competence in using and demonstrating these standards.

Informing Instruction Through Progress Monitoring

Teachers often are challenged to find measures that specifically assess important skills and concepts of mathematical elements that are being taught (Foegen & Morrison, 2010). These specific measures are often referred to as *proximal measures*, which are sensitive to student growth; as students learn more, they get higher scores on the assessments and the results can be plotted to demonstrate gains.

As already noted, it is especially important that progress-monitoring measures examine students' misconceptions and ability with regard to reversibility, flexibility, and generalization (Authors, 2015; Krutetskii, 1976). In addition, measures should be designed to include items that assess both procedural knowledge and conceptual understanding. Multiple-choice items can be developed that include response choices indicative of mathematical misconceptions, such as those Cara demonstrates. Students should be able to demonstrate not only what the correct answer to a problem is but also *why* an answer is correct. By examining closely student responses to progress monitoring items, teachers can determine whether the student is benefiting fully from instruction or requires additional teaching.

Figure 4. Example of a Conceptual Multiple-Choice Question

Are the ratios 3:5 and 6:8 equivalent?

a. Yes, they are equivalent because $3 + 2 = 5$ and $6 + 2 = 8$; and $3 + 3 = 6$ and $5 + 3 = 8$.
b. Yes, they are equivalent because 6 is twice 3.
c. No, they are not equivalent because 3 and 5 are odd numbers.
d. No, they are not equivalent because the product of 3 and 8 is not equal to the product of 5 and 6.

Discussions within the small groups also can affirm or correct their thinking and give students confidence in sharing their ideas before the whole class.

As an example, Figure 4 is a trying-it-on-your-own (TIOYO) item that represents conceptual understanding (because there is a *why* component) and offers potential misconceptions as response choices (see Table 1 for another example). Items such as these help teachers understand student thinking.

How students respond to TIOYO items can help inform intensive intervention. When students continue to miss many TIOYO items (e.g., 50% or more), it is important to change or intensify a component of instruction by deploying explicit instruction to correct the identified misconception as the first approach. Other ways to intensify or change a component of instruction include increasing the number of scaffolds presented, regrouping students, administering a supplemental lesson that focuses on similar skills, and so forth. It is unlikely that students who consistently respond poorly to TIOYO items will make satisfactory progress throughout the remaining lessons or do well on high-stakes test items.

Some interventions provide independent practice items at the end of a lesson; others do not. When independent practice items are not available, teachers can create items by examining the skills and concepts that were taught in a lesson. What problems were used as teaching examples or during guided practice? Create four or five similar items for the students to work on at the end of the lesson. If students respond correctly to 75% or more of the items, teachers can be fairly confident that the students benefited from the lesson.

Although the number of progress-monitoring tools for mathematics is increasing, it remains difficult to find measures that assess skills and concepts taught as part of algebra-readiness interventions. However, the National Center for Student Progress Monitoring (http://www.studentprogress.org/) is an excellent resource for gaining information. Measures that are available at easycbm.com also are useful progress-monitoring measures.

Final Thoughts

Teachers who work with students with persistent mathematics difficulties and mathematics disabilities must be aware of misconceptions that can interfere with student learning of ratios and proportional reasoning. In this article, we have provided examples of misconceptions and lesson components for intensive interventions. Teachers can incorporate these components into their own interventions and stay alert to misconceptions that may become evident as students solve problems and explain their solution strategies. Cara's misconception about additive and multiplicative comparisons is a common problem shared by struggling students. Carefully structuring intensive interventions with evidence-based lesson components and using results from progress-monitoring measures show promise in addressing essential needs for learning ratios and proportional reasoning, which are fundamental concepts of algebra.

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Barbara Dougherty, *Research Professor, Learning, Teaching, & Curriculum, University of Missouri, Columbia;* **Diane Pedrotty Bryant,** *Professor, The Meadows Center for Preventing Educational Risk,* **Brian R. Bryant,** *Research Professor, The Meadows Center for Preventing Educational Risk, University of Texas at Austin;* **Mikyung Shin,** *Assistant Professor, Secondary Special Education, Jeonju University, Ewha Womans University, Seoul, South Korea.*

Address correspondence concerning this article to Barbara Dougherty, University of Missouri, 121 Townsend Hall, Columbia, MO 65211 (e-mail: barbdougherty32@icloud.com).

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