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MATHEMATICAL PRACTICES IN A LEARNING ENVIRONMENT DESIGNED BY REALISTIC MATHEMATICS EDUCATION: TEACHING EXPERIMENT ABOUT CONE AND PYRAMID

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Abstract:
The purpose of the study is to identify the classroom mathematical practices developed within a learning environment designed by Realistic Mathematics Education for teaching cone and pyramid to preservice teachers. A teaching experiment including five-week instructional sequence by a hypothetical learning trajectory about the solids of cone and pyramid was conducted to five preservice middle school mathematics teachers. Their learning was examined in this teaching experiment performed based on case study as a qualitative research design. The social learning environment in the classroom was investigated by three-phase methodology of Rasmussen and Stephan (2008) developed based on Toulmin's model of argumentation. According to the findings, four mathematical practices emerged in the current study.

Keywords: cone, mathematical practices, pyramid, realistic mathematics education

1. Introduction

Geometry with two and three-dimensional shapes have important place in real life and education. It is one of important learning areas of mathematics and it takes essential place in middle school mathematics (Gürbüz & Durmuş, 2009) since it is a science representing the process of reaching the solution by thinking and imaging about the figures (Hızarcı, 2004). It provides a natural environment in which the learners develop their skills of thinking and proving (National Council of Teachers of Mathematics [NCTM], 2000). It has critical importance in mathematics programs with the role of real life contexts and mathematical concepts since understanding geometry based on rich perspective enhances the understanding of other learning areas in mathematics. For example, the students knowing how to draw the graphs using coordinate system can think analytically about slope, perpendicular and parallel lines so that they can provide different ways to make computations by the formulas of length, area and volume (Van de Walle, Karp, & Bay-Williams, 2014). Hence, it is vital to focus on geometry teaching.

Geometry teaching starts when the children see, know and understand the world around them and then it continues by the development of attaining geometric thinking skills (Ubuz, 1999). Geometry has more abstract concepts than other learning areas of mathematics. Especially, the students need more complex thinking skills in the concept of geometric solids (Yıldız, 2009). Geometric solids and the objects formed by them take place in our lives (Baykul, 2014). In the nature, there are several objects and places composed of geometric solids. Especially, many minerals are in the shape of geometric solids (Yemen-Karpuzcu & Işıksal-Bostan, 2013). Understanding geometric solids encourages the understanding of the concept of space and spatial thinking. Therefore, the students learn the concept of geometric solids by examining the properties of them, computing the surface area and volume of them in the schools (Baykul, 2014).

Many research show that the teachers and especially preservice teachers do not have sufficient knowledge about the concepts of prism, pyramid and cone (Altaylı, Konyalıoğlu, Hızarcı & Kaplan, 2014). Moreover, Gökkurt, et al. (2015), the preservice teachers have difficulty in defining the cone and computing the surface area and volume of it. Also, it has been observed that preservice teachers have problems and misconceptions about geometric solids especially about the cone and pyramids (Alkış-Küçükaydın & Gökbulut, 2013; Bozkurt & Koch, 2012; Gökkurt & Soylu, 2016; Koç & Bozkurt, 2011; Linchevski, Vinner, & Karsenty, 1992). Therefore, the present study was conducted with the aim of supporting preservice middle school mathematics teachers with sufficient knowledge and skills about the geometric solids of cone and pyramid. This teaching experiment was conducted in order to test the effectiveness of Realistic Mathematics Education (RME) by the designed instructional sequence and hypothetical learning trajectory (HLT). In other words, based on the properties of RME, a hypothetical learning trajectory including tasks, tools and imagery was prepared. Then, five-week instructional sequence was conducted to preservice teachers by this HLT. The geometrical tasks and instructional sequence were designed and organized for the preservice middle school mathematics teachers based on their mathematical backgrounds. This current study focuses on their learning by the classroom mathematical practices emerged through the study. More specifically, this paper seeks the answer of the following research question: What are the classroom mathematical practices that are developed within realistic mathematics education learning environment with the aim of teaching cone and pyramid to preservice middle school mathematics teachers (PMSMT).

2. Theoretical Framework

2.1 Mathematical Practices

In the literature, learning and teaching have been examined from different perspectives. In the present study, they were investigated based on communities from sociological points of views representing the process of emergence of classroom practices (Ball & Bass, 2000; Cobb & Bauersfeld, 1995). Based on this view, mathematical learning is performed and made in the social context of the classroom (Cobb & Bauersfeld, 1995; Cobb, Stephan, McClain, & Gravemeijer, 2001). Through this perspective, Cobb et al. (2001) have produced the term of mathematical practice focusing on learning through individual and social processes, neither occurring without the other and nor dominating to each other. Also, classroom mathematical practices are defined as "*it is feasible to view a conjectured learning trajectory as consisting of an envisioned sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices*" (Cobb et al., 2001, p. 125). In this definition, mathematical practices are stated as taken-as-shared ways of reasoning and discussing mathematically. Hence, they are formed through talking about the solutions of mathematical problems including symbolization and notations in the classrooms (Cobb, et. al., 1997).

In the process of emergence of mathematical practices, the individuals develop their own mathematical reasoning by participating in the activities, discussing and analysing others' interpretations so that they encourage mathematical practices by developing their reasoning (Cobb et al., 1997). In this respect, mathematical practices emerge in social learning environment and the main focus points are on both individual learning and collective learning. Also, Cobb and Yackel (1996) add that "*students actively contribute to the evolution of classroom mathematical practices as they reorganize their individual mathematical activities, and conversely, that these reorganizations are enabled and constrained by the students' participation in the mathematical practices*" (p. 180). The mathematical practices are formed through the operations of understanding, reasoning, expressing and convincing others in a learning community in a mathematics classroom by the taken-as-shared way for specific mathematical concepts in the engagement of particular mathematical tasks and ideas (Cobb et al., 2011; Stephan, Bowers, & Cobb, 2003). In other words, mathematical practices are content-specific, happening differently based on social and socio-mathematical norms (Stephan, Bowers & Cobb, 2003).

2.2 Realistic Mathematics Education

The focus point of Realistic Mathematics Education (RME) is the idea that mathematics results from human activities (Freudenthal, 1973) and is not "*as a closed system, but rather* *as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics*" (Freudenthal, 1968, p. 7). Hence, the learners are encouraged by the mathematical tasks and recovering the mathematical ideas by organizing didactically rich realistic contexts (Gravemeijer, 1994).

In this respect, "*one sees, organizes, and interprets the world through and with mathematical models. Like language, these models often begin simply as representations of situations, or problems, by learners*" (Fosnot & Dolk, 2005, p. 189) in order to mathematize. Hence, the main goal of the RME is to illustrate mathematics education to the learners by enhancing their reinvention of mathematics. Based on this goal of the RME, the instructional goal of this teaching experiment was identified to deepen PMSMT's understanding of geometry concept of the cone and pyramid. An environment and instructional design was designed and provided for PMSMT based on RME necessitating deeper understanding. Hence, the theory of RME focuses on construction of mathematics rather than reproduction of it (Streefland, 1991).

3. Methodology

3.1 Teaching experiment

The teaching experiment methodology provides researchers opportunities to examine students' progress through mathematical communications and connect teaching with research and theory and practice. Hence, this methodology facilitates the examination of learners' making sense of mathematics and the process of researchers' making sense of students' thinking (Steffe & Thompson, 2000). In this respect, teaching experiments are useful to focus on the progress of the learners acquired over extended time periods (ibid. 2000). This progress can be investigated by reaching the data directly and combining the practice and theory since "*students' mathematics is indicated by what they say and do as they engage in mathematical activity, and a basic goal of the researchers in a teaching experiment is to construct models of students' mathematics*" (ibid. 2000, p. 269). This methodology examined the students' progress focusing on the teaching episodes including some elements which are (a) a sequence of teaching episodes, (b) a teacher, (c) one or more students, (d) an observer to witness the teaching episodes, and (e) a method to record what happens in the teaching episodes (Steffe & Thompson, 2000). By these elements, teaching experiments proposes a repeated process including designing instructional sequence, testing it in a classroom, analysing the learning and the instructional sequence and making revisions on the instructional sequence by retrospective analysis. This iterative process encourages the development of instructional theory (Gravemeijer, 1998; Gravemeijer et al., 2003). In the study, PMSMT's learning provided by designed A Hypothetical Learning Trajectory (HLT) was focused on in this iterative process.

3.2 Instructional Design

HLT was designed at the beginning of the study. The researcher prepared six activities based on Realistic Mathematics Education (RME). This HLT was composed of four phases. The first phase included the mathematical tasks about the formation of the cone and pyramid. These tasks were designed based on the definition, examples and non-examples of the geometric shapes. The previous research show that it is important to define the mathematical concepts in order to understand effectively (de Villiers, Govender, & Patterson, 2009; Leiken & Zazkis, 2010; Tsamir, Tirosh, Levenson, Barkai & Tabach, 2014) Also, Özyürek (1984) and Senemoğlu (1997) state that the examples and non-examples encourage the student understanding of the attributes and critical properties of the concept by producing generalization about it. Furthermore, the Egypt pyramids and tent are the common real life examples and used in the textbooks of the mathematics curriculum (Tahan, 2013). Therefore, these mathematical activities were designed in this way. The second phase of the HLT focused on the properties and main elements the cone and pyramids.

By using the manipulatives in the geometric solids set, the participants examined the properties and elements of them by the task designed based on the research of Gökkurt (2014). In the third phase of the HLT, the surface of open cone and pyramid was taught since Alkış-Küçükaydın and Gökbulut (2013) and Gökkurt (2014) emphasized the importance of the surface of open geometric solids to understand them. The last phase of the HLT was about the area and volume of the cone and pyramid. The mathematical tasks were designed based on the previous research of Gökkurt (2014) and the middle school mathematics curriculum (Ministry of National Education [MoNE], 2009). By these HLT, five-week instructional sequence was conducted to PMSMT. They engaged in these activities through four weeks. In the fifth week, an activity sheet including seven problems about all of the learning goals in the HLT was conducted to the participants.

Table 1: Hypothetical Learning Trajectory for the Cone and Pyramid

Phase	Learning Goals	Concepts	Supporting tasks	Tools/ Imagery	Possible Discourse
1	Reasoning on drawing cone and pyramid	Definition of the cone and pyramid Examples and non-examples of cone and pyramid Drawing of the cone and	Egypt Pyramid Form a tent	Isometric paper Paper, ruler, scissors, band	Definitions of cone and pyramid Examples and non-examples of cone and pyramid Types of cone and pyramid Various

		pyramid			appearances of cone and pyramid
2	Reasoning on the properties and the main elements of the cone and pyramid	Properties and main elements of cone and pyramid	My properties and elements	Manipulatives	Main elements of cone and pyramid Properties of cone and pyramid
3	Reasoning on the surface of open cone and pyramid	Other geometric shapes related to the open cone and pyramid	Can you close me?	Colorful figures	The possibility of the closure of the shapes
4	Reasoning on area and volume of cone and pyramid	Surface area Base area Lateral area Volume	Cover me Fill me	Manipulative Colorful paper, ruler Sand or liquid Beaker	The formulas of surface area, base area, lateral area and volume

3.3 Participants and Data Collection

The participants of the current study was composed of five preservice middle school mathematics teachers enrolled in the program of elementary mathematics education at a university in the northern part of Turkey. They were junior students at the program of elementary mathematics education and selected by criterion sampling strategy. They were selected based on the criterion that the participants were expected to take the undergraduate courses of *Geometry* and *Analytic Geometry* in previous semesters.

In the current study, the teaching experiment was conducted to five preservice middle school mathematics teachers. This teaching experiment was made with the purpose of examination of preservice middle school mathematics teachers' collective learning and reasoning about the geometric solids of cone and pyramid (Steffe & Thompson, 2000). Based on the nature of this design, the teaching experiment composed of a sequence of teaching episodes lasting five weeks were implemented by an instructor with five PMSMT in the classroom. An independent observer having the PhD degree in the program of mathematics education participated in the teaching experiment, observed the instructional sequence and helped the data collection process including videotape, audiotape, and field notes and data analysis (Steffe & Thompson, 2000). The collective learning supported by the instructional sequence and hypothetical learning trajectory was examined by identifying the mathematical practices emerged in the study. Hence, analysis of the instructional sequence and collective learning period represents the mathematical practices and the learning of the PMSMT effectively.

The data collection period lasted five weeks in which the instructional sequence took place based on the hypothetical learning trajectory for the cone and pyramid by Realistic Mathematics Education. This HLT was conducted to another group of PMSMT and tested. Then, the revised form of the HLT was conducted to the participants of the current study. Data were collected through different sources including transcriptions of the video recordings of whole class discussions and audio recordings of peer group works, and worksheets of the tasks that the participants engaged in. In the instructional sequence, the participants studied on the worksheets designed based on RME with their peers. Their discussions in the small groups were recorded by the audio recordings. After they completed peer group works, they discussed about the mathematical tasks related to these worksheets under the guidance of the instructor who was the researcher of the present study. At the end of the period of discussion in each week, the worksheets were collected. By examining these data, the classroom mathematical practices were identified. The identification of classroom mathematical practices was related to the data collection from different data sources since mathematical practices emerged in a process including social and socio-mathematical norms (Cobb et al., 2003).

3.4 Data Analysis

The data analysis process was composed of two methods in order to analyze the qualitative data obtained through different sources. These data were investigated by using constant comparative data analysis method as an analysis method of grounded theory. This data analysis method provides an inductive process including identifying and forming relationship between different incidents, incidents and categories and categories so that categories can be grounded in the data (Glaser & Strauss, 1967). In this respect, the elements of the Toulmin's argumentation model and raw data were compared constantly in the current study by the way explained by Creswell (2009). The data collected in a particular time period at a week were compared by the data itself, the data gathered at the same week and the data done across different weeks. Hence, the patterns in the data were determined effectively and mathematical practices could emerge in the study. The second method for the analysis of qualitative data illustrating the classroom mathematical practices as taken-as-shared mathematical ideas was three-phase methodology of Rasmussen and Stephan (2008) based on Toulmin's argumentation model (1969). This method helps the analysis of the learning in the way of taken-as-shared by documenting collective learning activities forming classroom mathematical practices (Stephan & Cobb, 2003). This methodology includes three phases based on Toulmin's (1969) model of argumentation. The first phase is composed of the actions of transcribing the whole class discussions. Through the second phase, argumentation logs and taken-as-shared mathematical ideas emerge. The identification of taken-as-shared mathematical ideas is performed by two criteria. The backings and/or warrants of the argumentation do not appear in the whole class discussion anymore and the mathematical idea produced in an argument is used in future arguments in order to justify by taking the roles of the data, warrant, or backing

(Rasmussen & Stephan, 2008). In the last phase of the model, taken-as-shared mathematical ideas were collected and named by common titles related to the mathematical tasks. The Toulmin's (1969) model of argumentation as the basis of this model is composed of three elements, which are the data, claim (conclusion), and the warrant used. By this model, an individual produces a claim focusing on the data representing the attained knowledge or given information. Then, the validity of the claim is provided by the warrant in order to emphasize the connection of the data and the claim and the way of reaching conclusion by the data. In some variants of this model, there is another element as backing increasing the validity of warrant by providing further evidence (Stephan et al., 2003).

4. Results

4.1 Mathematical Practice 1: The definition of the cone and pyramid

When the participants were asked to draw a cone, it was observed that they drew circular and right cones. Therefore, the instructor started the discussion period in order to determine whether the participants' knowledge about the cone was limited by circular right cone and to develop their limited knowledge. In this respect, the instructor guided the discussion through the drawing and the definition of the cone as follows:

Instructor: Is there any different representation for the cone?
C: We can draw oblique cone.

It was observed that the participants tended to draw and explain prototype cone which was circular cone since it was the cone most commonly used to teach cones in the textbooks. Then, the instructor guided the discussion about the definition of cone as follows:

Instructor: Can you define the cone?
A: The cone is a triangular geometric object having a circular base.
B: The cone is a geometric object formed in a way that all of the points in its circular base is connected by the line segments at a point.

The instructor realized that the participants' knowledge about the cone was limited to circular cone because they stated that the shape of the base of cone was circle. Then, the instructor directed the discussion in order to help them realize their missing knowledge as follows:

Instructor: The term of circular cone is stated in the books. If the base of the cone were in the shape of circle, why would this term be explained?
C: So, the shape of the base does not have to be only circle but also to be ellipse. Moreover, the base can be any shape as in the figure.

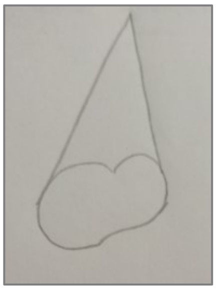

Figure 1.1

At this episode of the discussion, they realized that the shape of the base could be geometric shapes. They understand that this shape was not only circle. Then, the instructor guided the discussion about other critical properties of the cone to define it.

A: The base can be any line segment.
B: It cannot be. For example, this figure is a cone.

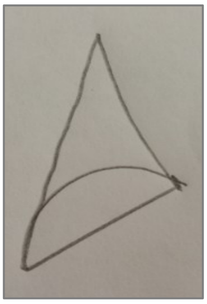

Figure 1.2

A: There exist line segments on the base.
C: …the base of the cone can be any geometric shape.
Instructor: So, what can you say about the base?
C: It is a geometric shape.
Instructor: Does this shape represent a cone?

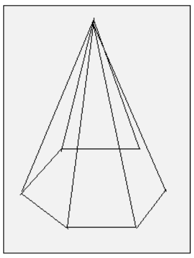

Figure 1.3

A: No, it does not. The geometric shape representing the base has to be a closed one.
Instructor: Right. Then, does this shape illustrate a cone?

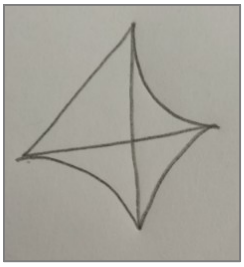

Figure 1.4

B: Yes. It is closed and there exists an apex.
Instructor: The line segments beginning from the points of the edge of the base concur at a point as apex.
B: These lines are linear.
Instructor: So, what are the critical properties of the cone?
A: The base is in the shape of a closed geometric shape. There exist apex and the line segments connecting the points of the base with the apex.

At this episode of the discussion, the participants accurately explained the critical properties of the cone. Then, they were asked to define the cone by using its critical properties and the connection of these properties as follows:

Instructor: What is the definition of the cone?
B: The cone is a geometric solid whose base is in the shape of a closed curve and connected to the apex with line segments. For example, this solid is a cone.

Figure 1.5

Instructor: Then, how does the surface representing the line segments connecting the edges of the base with apex take shape?
C: This surface is formed by the line segments connecting the base to the peak point. Let's think that these lines are moved and this surface is formed by moving these line segments.
Instructor: This surface is conic surface. Let C be the closed curve and S as a point on a different plane. Conic surface is surface formed by a line passing through S and moving

based on C. Can you produce another definition of the cone by using the term of conic surface?
A: Cone is a geometric solid which is limited by the apex and plane intersecting the conic surface.

In light of the discussion including the participants' explanations and ideas, the critical properties of cone were realized and then they formed the appropriate and sufficient definition of cone. They explained the base, conic surface and the intersection of conic surface and plane. Through this discussion, they accurately produced the definition of cone.

A and B provided an incomplete claim about the definition of cone since their knowledge about the cone is limited to circular cone. Then, through the discussion guided by the instructor, A provided explanations about the base and the conic surface of the cone. With the help of instructor and discussion, A provided warrant by explaining the critical properties of the cone and the way of connection of these properties and the formation of the cone by using them. At the end of the discussion, B and A provided the claim of the argumentation by stating the definitions of the cone.

After this discussion, the instructor guided the discussion about the pyramid as follows:

Instructor: Right. What about the definition of the pyramid?
D: The pyramid is a geometric object whose base is in the shape of square and formed by connecting the points on the edge of the base by the line segments to the peak point.

When the definition of the pyramid made by D was examined, it was observed that this definition was not sufficient and necessary. He stated that the base was in the shape of square but the base could be different geometric shapes such as heptagon, triangle. Hence, the critical property of square base in this definition was unnecessary. Unnecessary part of this definition was discussed and revised in the discussion as follows:

C: … the base of a cone can be in the shape of triangle or rectangle. It does not have to be square.
A: Yes, the shape of the base can be all types of polygons.
Instructor: What are the critical properties of the pyramid as a solid object?
A: The shape of the base is polygon and there is apex as peak point.

When the explanation of A was examined, it was observed that the critical properties were necessary but not sufficient since the critical property of triangular lateral faces was missing. In order to help the participants realize this missing property, the instructor drew Figure 1.5 and continued the discussion as follows:

Instructor: What do you think about his object? It has the properties that A says. Is it a pyramid?

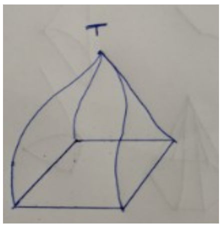

Figure 1.6

A: Yes, the line segment from the apex intersects the base perpendicularly.
C: Not right. This line segment does not need to be perpendicular. For example, this is also a pyramid.

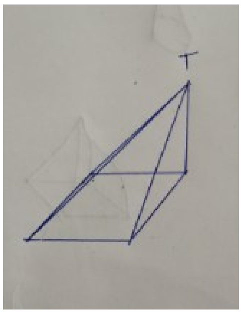

Figure 1.7

C drew a pyramid and made explanation accurately in order to help A realize her mistake. Then, the discussion was directed in order to find and understand the missing property of the pyramid. Hence, the instructor asked them to state different properties and then A provided correct explanation as follows:

A: The lateral faces are triangular. The apex as peak point is connected with the vertices of the base by the line segments so that triangular faces are formed.
Instructor: So, what is the definition of the pyramid?
A: Pyramid is the geometric solid having triangular lateral faces and formed by connecting the vertices of a polygon and a point outside this base with line segments.

At this episode of the discussion, the participants understood the critical properties of pyramid and made its definition. Through the discussion about the definition of the pyramid, A provided the data by explaining the base, apex and lateral faces for the argument. Then, with the help of the mathematical ideas of other and under the guidance of the instructor, A stated warrant by talking about the connection of the critical properties of the cone and the formation of it by them. At the end of the argumentation, A stated the claim as the sufficient and accurate definition of the pyramid.

In the following tasks, the participants were asked to identify the typical and non-typical examples of cone and pyramid and make explanation for their identifications by reasoning. Through the discussion about this activity sheet, the mathematical ideas about the definition of cone and pyramid and relationship between them were used as data and warrant and they became self-evident and also taken-as-shared. In the discussion period, the participants talked about the geometric solids represented in the activity sheet, and made identifications about the examples and non-examples of the cone and pyramid by reasoning and explaining. They usually provided claims that the particular figure was the cone and pyramid. They examined the critical properties and definitions of them and provided the data and warrant by using them. It was observed that they provided the explanation about the identification of the cone and pyramid, they benefited from statements by the visualization and definitions of them. For example, D explained that "*it is pentagonal pyramid since its base is in the shape of a polygon and it has apex, triangular lateral faces formed by combining the points of the edge of the base and peak point with line segments. It is pyramid and also a cone*". Moreover, they benefited from the mathematical idea about the relationship between cone and pyramid to identity the examples and non-examples of cone and pyramid. For example, A stated "*… the fourth one is a cone but not pyramid since the shape of its base is not a polygon*". After providing the correct definition of the pyramid, the instructor directed the discussion in order to understand the relationship between pyramid and cone.

4.2 Mathematical Practice 2: Reasoning on the main elements of cone and pyramid

In the argumentation about the main elements of circular cone, A provided the claim by explaining the main elements of it. For this claim, C explained the data by using the definition of a cone. Also, C and A provided warrant of the argumentation by stating the connection way of these elements to construct a cone. The mathematical task in the second week was about the main elements of cone and pyramid. They examined the main elements through discussion taking place as follows:

D: … peak point and base.

C: Cone is a geometric solid whose base is in the shape of closed curve and connected to the apex with line segments. Hence, by using them, we cannot form a cone. They are not enough. We need the line segments connecting the apex with the points of the edge of base. These line segments are main lines.

Instructor: What else?

A: … the altitude because the cone can be constructed differently based on the length of the altitudes. Also, if the cone is circular cone, radius is also main element of a cone.
Instructor: Right. The, what are the main elements of a circular cone?
A: … apex, base, lateral face, main line, radius and altitude.
Instructor: Then, what are the main elements of a pyramid?
C: Pyramid is a special kind of cone so its main elements are apex, base, lateral faces, main line and altitude. The base is not circle so the radius is not main element of a pyramid.
A: The shape of the base is polygon. There are edges and the intersection points of the edges with the base are the vertices.
C: The vertices are not explained for a cone.
A: The apex is also a vertex but we name this vertex as apex.

Through the discussion, the participants understood the main elements of pyramid as apex, base, lateral faces, edges, vertices and the altitude. Through the discussion, they used the knowledge about the definitions of cone and pyramid and the relationship between them. Also, this represented another instance that these mathematical ideas were used and observed in the discussion.

In the following activity sheets, there was a mathematical task about identifying the truth of the statements including the properties of cone and pyramid and the knowledge about these geometric solids learned through instructional sequence. They focused on diagnostic branched tree including true and false expressions about cone and pyramid. Through this mathematical task, they used the mathematical idea about the main elements of cone and pyramid. Through the argumentation about this mathematical task, the participants explained the claims about the truth and incorrectness of the statements taking place in the diagnostic branched tree. Then, the definitions and main elements of cone and pyramid were used as data and warrants of the argumentation. The following discussion period represents an example for this period.

Instructor: Let's think about the diagnostic branched tree.
D: For example, the pyramid is a right pyramid when the base and the line segment connecting base and peak point are perpendicular. The measure of the angle formed by base and this line segment is not 90^0 in oblique pyramids.

This discussion period represented another instance in which the mathematical idea about the main elements of cone and pyramid was used so that it became self-evident and taken-as-shared.

4.3 Mathematical Practice 3: Reasoning on the surface of open cone and pyramid

The mathematical task at the third week of the instructional sequence was about the examination of surface of opening form of cone and pyramid. They investigated the geometric shapes forming these geometric solids and the ways of closing these shapes to form them.

Instructor: What do you think about the opening form of a cone?
C: A cone is a solid object limited by a plane intersecting a part of it and it's all main line segments. Conic face infinitely extends and it is intersected by a plane as it's base so that a cone is formed.

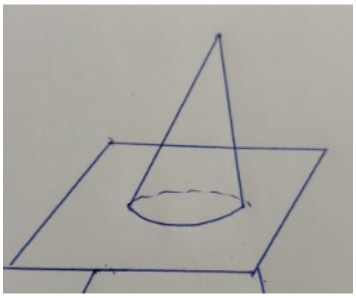

Figure 2.2

Instructor: Right. Cone is a closed empty object having a base and surrounded by a limited surface. The shape of B is not closed. (Figure 2.2)
C: The opening form of a cone is this shape.

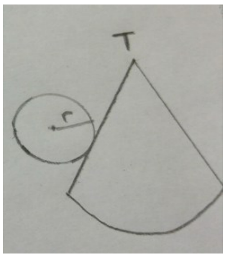

Figure 2.3

A: The place of the base is not correct. We cannot form a cone by this shape through closing it. The correct shape of it is this.

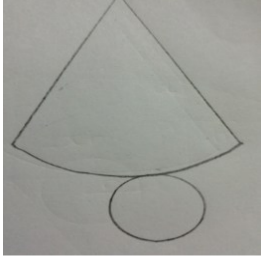

Figure 2.4

Instructor: Right. Does the shape of the base have to be circle?
C: it is circular base for circular cylinder.
Instructor: What do you think about other opening forms of cones?
C: … like this one.

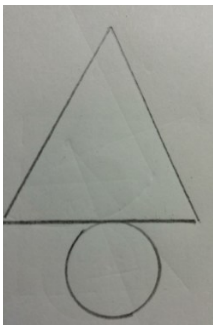

Figure 2.5

At this episode of the discussion, the participants examined and understood the surface of the opening form of cone. Through forming its opening form, they used the knowledge about the definition of cone, its critical properties and main elements. They focused on and discussed about their ideas related to about the elements, their connection and possible places based on its definition. By talking about its main elements and the places of them for particularly base, they used the knowledge about two-dimensional geometric shapes such as polygons. Moreover, the placements of the shapes composed of the cone were considered through closing them to form a cone. They imagined the ways and processes of closing these shapes.

In the argumentation about the opening form of the cone, C and the instructor provided data and warrant by explaining the definition and construction of cone and conic surface. Then, the participants discussed about their ideas considering the opening form of cone and they formed the correct explanation. At the end of the argumentation, the instructor provided accurate claim, data and warrant by summarizing the participants' ideas explained through the discussion. The instructor stated the opening form of cone based on main elements and critical attributes of the cone and construction of it.

Instructor: Right. What about a pyramid.
A: When the shape of the base is polygon, there exist triangles on the opening form of a cone. For example, the shape of lateral faces of a square pyramid is triangle and there are triangular regions.

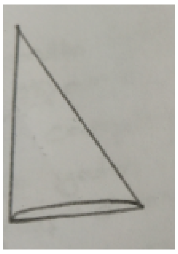

Figure 2.8

D: The number of triangular regions is related to the number of edges of the base. We can accept all version of the opening forms of all types of pyramids are also opening forms of cones since pyramid is a special kind of cone. Let's think about pentagonal pyramid.

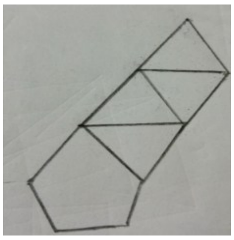

Figure 2.9

C: … the number of lateral faces is five in pentagonal pyramid. Let's think about its closed form. A net for a pentagonal pyramid must have one pentagon and five triangles. For example, there are three triangular regions in triangular pyramid.

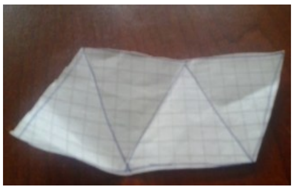

Figure 2.10

When the drawings of the participants about the opening forms of cone and pyramid, B provided accurate prototype drawings while A and C made non-prototypical representations. In the discussion about the opening form of the cone, B ignored the base of the cone. C did not think about the placement of the geometric shapes in this form and closing the shapes to form a cone. He drew these shapes randomly ignoring the possible intersection points of these shapes to form a cone through closing them. Through the discussion, the participants understood that it was important to consider closing the shapes illustrating the opening form of cone and pyramid and necessity and role of the base of them.

In the argumentation related to opening form of a pyramid, C provided the claim based on pentagonal pyramid. He stated that there were a pentagon and five triangular regions connected at particular possible places. For the data of this argumentation, A used the knowledge of main elements and definition from the placement and connection of bases and triangular regions.

In the following weeks, the instructor asked the participants to identify the examples and non-examples of cone and pyramid and explained the reasons of these identifications. Through the discussion, the participants made these identifications correctly by using the knowledge about the definition of cone and pyramid and relationship between them. This discussion happened in the classroom as follows:

A: … the first object is just a cone. There is peak point and the shape of the base is closed curve for a cone. However, the shape of the base of pyramid is polygon. Hence, this object is not pyramid.
B: The shape is also a circular cone.
D: The shape is closed to form a cone so it is cone.

When the discussions of the participants were examined, they provided explanation by reasoning based on the properties of the solid objects and visualization. A benefited from reasoning by using the critical properties of cone and the main elements of it such as base as closed curve and peak point. In other words, she used the knowledge about the definition and main elements of the cone. Moreover, she emphasized the base as polygon for the pyramids. However, B and D reasoned based on the visualization of the solid objects. A provided the claim by explaining that the figure was about the opening form of the cone. Then, she used the knowledge about opening form of cone and its definition as data and warrant of the discussion.

4.4 Mathematical Practice 4: Surface area and volume of cone and pyramid

At the beginning of the discussion, the instructor asked the participants the volume and surface area and area of lateral faces of prototype and non-prototype cones and pyramids.

Instructor: Let's talk about square pyramid.
D: The area of base is a^2 since the length of the edge of the square is a. Also, it is composed of four triangles and a square. The surface and lateral area is the sum of the areas of all of them.

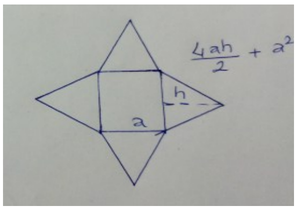

Figure 3.1

At this episode of the discussion, he used the opening form of the solid objects.

C: … the area of lateral faces is different from surface area. While computing surface area, the areas of all of the geometric shapes forming the geometric solid are used. However, the area of the base is not used to find lateral area.

At this episode of the argumentation, C provided the claim about the differences between surface area and lateral area. D provided data by the knowledge about area of base and the main elements of pyramid and warrant by the knowledge of opening form of the pyramid.

Instructor: Right. What about the volume?
B: The volume can be computed by multiplying the measures of the area of the base and the altitude. It is, $a^2.h$.
C: B. Let's think about the volume of the parts of the pyramid, it decreases through moving from base to the peak point.
B: Ok. What is the difference?
C: For example, compare the volume of a square prism and square pyramid whose base's edges' lengths are equal and their altitudes are also equal in length.
B: Ok. Let's fulfil this prism with this pyramid. Three pyramids fulfil this prism so the volume of the pyramid is $(a^2.h)/3$.

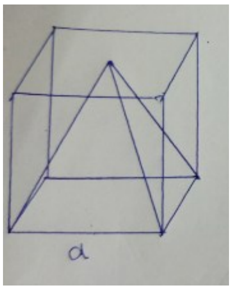

Figure 3.2

At this episode of the argumentation, B claimed that the volume of the pyramid could be computed by the formula of $(a^2.h)/3$. For this claim, B provided data benefiting from the formula by multiplying the area of base and the length of the altitude. This explanation had missing parts so C provided accurate data by providing missing knowledge using the difference between square prism and square pyramid. Then, C explained warrant benefiting from the drawing of these solids. Then, the instructor guided the discussion about lateral area, surface area and volume of cone.

A: We can draw the opening form of a cone. There are a segment and a circle. For example, in a right cone, the lateral area is $(\pi.a^2)/4$.

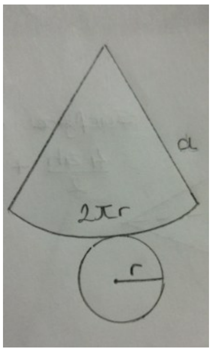

Figure 3.3

C: … the measure of the angle of a segment in a right cone may not be 90^0. It may be acute angle or obtuse angle. The critical property of right cone is not related to the measure of the angle of the segment but it is about the altitude of the cone.
A: Right. The measure of the angle of the segment is not needed to be right.
C: So, the length of the arc of the segment is equal to the circumcenter of the circle. Then, the lateral area can be computed by the lengths of the radius of the segment and arc.

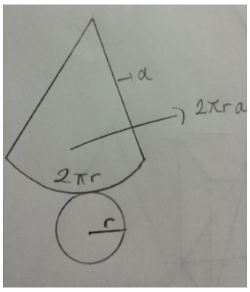

Figure 3.4

Instructor: Why do we multiply these lengths to compute lateral area?
B: I think so because we compute the area.
Instructor: Ok. Let's draw the circle of the segment and the circumcenter of this circle is equal to 2.π.a.
B: 2πra is not correct since lateral area of a cone is smaller than this value found by this formula.
Instructor: Right. Let the angle measure of the segment be arbitrary. We can compute the circumcenter of the segment can be computed in two different ways.

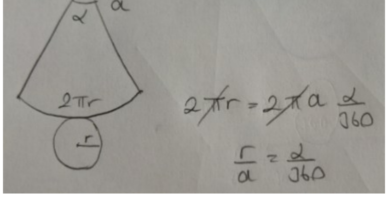

Figure 3.5

C: So, the surface area can be computed by (π.r.a)+(π.r^2).
Instructor: Right. What about the volume?
B: (π.r^2.h)/3 as in the pyramid. The volume of the parts of the cone decreases moving from the base to the peak point. The volume of a cone is the third of the volume of a cylinder whose lengths of the radius and altitude are equal to the ones of the cone.

The knowledge about lateral area, surface area and volume of cone and pyramid was used in the problems asking these values for different solid objects in different problem situations. The discussion about the cone took place in the similar way to the discussion related to the pyramid. The participants produced the claim illustrating the formula of surface area, lateral area and volume of the cone. For this claim, they used the knowledge of definition, main elements and opening from of the cone for data. Then, they produced warrant benefiting from the opening form of the cone and cylinder. Also, they used the knowledge about the area of circle and segment.

The discussion taking place about the scenarios about the lateral area, surface area and volume of cone and pyramid, represented another instance that this mathematical idea was used. The participants used the knowledge about the mathematical idea about surface area, lateral area and volume of cone and pyramid as data and warrant. For the solutions of these problems, they provided solutions benefiting from these ideas.

5. Discussion and Conclusions

In the present study, the concepts of cone and pyramid were taught to the preservice middle school mathematics teachers focusing on their definitions, opening and closed forms, main elements, surface area and volume through argumentation. This teaching experiment provided that the participants understood and attained conceptual knowledge about cone and pyramid. The participants learned the definitions of them correctly. Also, they understood the concept of opening and closed forms of them by examining different forms and representations of them. The researcher supported the interaction and discussion in the classroom focusing on their justifications, proofs and counter explanations so that they could learn cone and pyramid effectively. Hence, the present study aimed to have information and obtain view about how to develop classroom mathematical practices through instructional sequence based on Realistic Mathematics Education about cone and pyramid.

Four mathematical practices were formed: (1) the definition of the cone and pyramid, (2) reasoning on the main elements of cone and pyramid, (3) reasoning on the surface of open cone and pyramid, and (4) surface area and volume of cone and pyramid. These mathematical practices emerged through challenging mathematical ideas and then benefiting from them in different context or problem solutions with unchallenged usage. It was observed that the emergence of these mathematical practices was enabled by the learning environment designed by RME. This environment allowed the participants to produce their ideas, examining and discussing about them in order to produce the correct mathematical idea. Therefore, the study shows that, RME may enhance emergence of effective mathematical practices and developing conceptual understanding in addition to the other advantages explained in the literature. To conclude, the current study represents the happenings in a collective learning of a classroom community using RME about the geometry concept of cone and pyramid. The mathematical ideas accepted as important and formed in the classroom are grouped under four mathematical practices. It is hoped that these mathematical practices are useful to design learning environments for preservice middle school mathematics teachers about cone and pyramid. Moreover, similar research can be designed and conducted about the other geometric solids by using mathematical discussions and RME. Moreover, it is important to emphasize that although the learning focuses on both social interactions and individual reasoning in teaching experiments, the present study represents only the social aspect of learning environment. A similar approach is represented in different studies in the literature (Rasmussen & Stephan, 2002; Uygun, 2016).

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