Pedagogical Content Knowledge in Mathematical Modelling Instruction

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This paper posits that teachers' pedagogical content knowledge in mathematical modelling instruction can be demonstrated in the crafting of action plans and expected teaching and learning moves via their lesson images (Schoenfeld, 1998). It can also be developed when teachers shape appropriate teaching moves in response to students' learning actions. Such adaptive development of teachers' pedagogical content knowledge may in turn be supported by their knowledge of the mathematical modelling process and Ang's (to appear) proposed framework for planning mathematical modelling instruction.

Introduction

The enactment of mathematical modelling instruction within the context of our Singapore mathematics curriculum structure requires a major paradigm shift in teachers, who must adopt new approaches and new roles to such mathematical modelling instruction, as well as the acquisition of new knowledge. To implement such innovations in mathematical modelling instruction successfully will therefore require teachers to have the requisite knowledge in mathematical modelling instruction that provides a basis for classroom practices. To achieve such changes, teachers will need to learn for teaching mathematical modelling (Sherin, 2002; Shulman, 1986b). Such "learning for teaching" approach requires a more adaptive style of instruction in which teachers understand the connections of mathematical ideas in these tasks, and adjust their pedagogical strategies to augment the implementation of these tasks (Chazan & Ball, 1999; Lampert, 2001, Sherin, 2002).

Even though secondary mathematics teachers have access to materials and resources in the form of booklets containing collections of modelling and application activities (for example, Ang, 2009; Galbraith and Carr, 1987), the lack of knowledge in mathematical modelling instruction may inhibit novice teachers of mathematical modelling from structuring and engaging students with meaningful and effective mathematical modelling learning experiences (Shulman, 1986a; Ball, 2000).

Recently, a new framework for mathematical modelling instruction (Ang, to appear) had been proposed to guide and facilitate novice teachers of mathematical modelling in translating their modelling ideas into a sequence of modelling lessons. It seems likely that Ang's framework for mathematical modelling instruction was derived based on his pedagogical content knowledge (Shulman, 1986a) or craft knowledge in teaching mathematical modelling. Such craft knowledge comprises the accumulated and integrated set of knowledge and beliefs developed over a period of time while teaching and practising mathematical modelling.

The discussion to follow is based on the premise that the challenges faced by novice teachers of mathematical modelling in our secondary school classrooms may be due to a lack of teacher knowledge in structuring and implementing mathematical modelling learning experiences rather than to a lack of mathematical modelling instructional resources or ideas. Although much has been written about aspects of mathematics teachers' content and pedagogical content knowledge in specific mathematics topics, there has been a scarcity

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of studies on the kinds of teacher knowledge required for mathematical modelling instruction.

In the subsequent sections, we present some background considerations which are necessary as a basis for what is to be discussed. Specifically, we will attempt to discuss how Ang’s (to appear) proposed framework and knowledge of mathematical modelling process can be used to support the adaptive development of teachers’ pedagogical content knowledge in the continuum of mathematical modelling instructional contexts.

Mathematical Modelling Process

We begin by clarifying the mathematical modelling process and how its components relate with one another. We posit that such considerations of the modelling process can influence the cognitive and metacognitive aspects of structuring mathematical modelling instructions using the proposed framework.

Essentially, the mathematical modelling process is characterized by the iterative negotiation of learning between the real and mathematical world. A typical mathematical modelling learning task traces the following trajectory.

The mathematical modelling process begins with a mathematical modelling problem motivated by practical concerns oriented in the real world. The real world problem is then formulated into a mathematical problem. The process of formulating the real world problem into a mathematical problem entails four more sub-processes which are not claimed to be very original but can be useful for understanding the model formulation process:

1. The modelling purpose is first clarified in terms of identifying the predictive, explanatory or prescriptive functions of the model. Its corresponding modelling approach would then be employed. Some of the common modelling approaches include empirical modelling, simulation modelling and deterministic modelling (Ang, 2009).
2. Other subject discipline knowledge may then be used to identify the relevant elements, relations and structures that characterize the real world situation to be modelled.
3. The necessary assumptions and conditions are then made to consider an idealized real world situation so that its mathematical representation is made more tractable.
4. The mathematical problem is then formulated by being mindful of past mathematical associations or any theoretical connections from other subject discipline that can be used to translate the relevant elements, relations and structures in the idealized real world situation into its corresponding mathematical objects, relationships and structures.

The mathematical problem is analyzed and solved using known mathematical methods. The mathematical solution is then being interpreted into some plausible real world solution. Validation of the mathematical model necessitates checking the accuracy of the mathematical solution, or relating it with observed data, or comparing it with other models and established theory for a measure of the degree of representation of the idealized situation. The reasonableness in the mathematical model will then determine if the process needs to be iterated by re-examining how the mathematical problem has been formulated.

Forms of Teachers’ Knowledge

Teachers' knowledge for mathematical modelling instruction can determine to a large extent how teachers perceive and respond to curriculum innovation efforts to infuse mathematical modelling learning experiences in the secondary mathematics syllabus. To think about knowledge of mathematical modelling instruction requires going beyond knowing the mathematical modelling process. It may require understanding the complex interplay among aspects of other forms of teacher knowledge in the mathematical modelling teaching and learning environment.

Shulman (1986b) argued that behavioural research on teaching actions can lead to overly simple prescriptions of teacher effectiveness. Shulman’s (1986b) discussion of the categories and forms of teacher knowledge has provided a more comprehensive consideration of the factors influencing teaching performance. Of particular note is his introduction of the broader domain of pedagogical content knowledge that highlighted that teacher knowledge depends on more than subject matter knowledge. Shulman (1986a) represented pedagogical content knowledge as the blend of content and pedagogy for the understanding of how particular aspects of the subject matter are organized, adapted and represented to teach for understanding by considering students’ conceptions, abilities and interests.

Following the work of Shulman (1986b), other researchers had examined the nature of pedagogical content knowledge in more detail (Chazan & Ball, 1999; Ball, 2000; Lampert, 2001, Sherin, 2002). Ball (2000) had deconstructed mathematical content knowledge and mathematical pedagogical content knowledge into its key components. Like Shulman (1986b), Ball (2000) argued that both content and pedagogical content knowledge are critical for effective teaching. However, such models of teachers’ knowledge generally do not explain how this knowledge is used in teaching and learning situations.

Schoenfeld (1998) had developed a model of teaching in context. The model described the mechanisms in which teachers’ beliefs, goals, and knowledge interact in specific teaching situations. This serves as a framework that fosters meaningful discussion about factors that may influence teachers’ decisions and actions during the teaching process. The concept of lesson image can be particularly useful here. Lesson image is organized in chunks of action plans that may take the form of routine, script, mini-lecture and simple talk. Teachers’ content and pedagogical content knowledge are in turn organized and accessed to frame these action plans.

Schoenfeld (1998) noted that the model worked well as an analytic tool to understand the act of teaching as it unfolded in particular contexts. These include lesson contexts where teachers had clear and un-conflicting learning goals and their lesson images and agendas were not co-constructed with students during the lesson. In this regard, we argue that the mathematical modelling lessons structured by Ang’s (to appear) proposed framework may be of the types described by Schoenfeld. This point will be elaborated later when we consider the proposed framework in more details. Therefore, it is possible that Schoenfeld’s model of teaching in context can contribute in part towards theoretically grounded understandings of the teaching process and development of teachers’ knowledge in mathematical modelling lessons structured by Ang’s (to appear) proposed framework.

Ang’s proposed framework for mathematical modelling instruction is basically a set of decision procedures aimed at scaffolding novice teachers of mathematical modelling to translate their ideas into a series of modelling learning tasks (see Table 1).

Some researchers have pointed out that cognitively demanding modelling tasks need to be appropriate with respect to the kinds of students’ prior knowledge in order for them to complete the tasks successfully (Galbraith, 2006; Stillman et al., 2007). Three levels of learning experiences in mathematical modelling were defined to guide teachers in pitching their modelling tasks at a level where students have the necessary skills and competencies to complete them successfully (Ang, to appear).

Specifically, level 1 tasks focus on students acquiring mathematical modelling skills that they may relate to in future modelling tasks. Level 2 tasks are essentially about guiding students in applying models known to the students to new modelling situation. This is done with the purpose of developing in students the necessary modelling competencies for meaningful engagement in the modelling process. Level 3 tasks focus on facilitating students in undertaking a mathematical modelling task where they should be ready to build models deem new to them or to modify known models.

Table 1

Framework for Planning/Designing Mathematical Modelling Learning Experience (from Ang, to appear, pg. 4)

Framework Component	Explanation
1. WHICH Level of Learning Experience?	Decide which level (Level 1, 2 or 3) of mathematical modelling learning experience that we wish to focus on.
2. WHAT is the Skill/Competency?	List all the specific skills and competencies (mathematical or modelling) that we target in this learning experience;
	State the problem to be solved, if applicable.
3. WHERE is the Mathematics?	Write down the mathematical concepts or formulae or equations that will be needed in this learning experience.
4. HOW to Solve the problem/model?	Prepare and provide plausible solutions to the problem identified in this learning experience.
5. WHY is this experience a success?	List factors or outcomes that can explain why this experience is considered successful and look out for them during the activity.

We note that the first two questions in the proposed framework can guide teachers in specifying appropriate and clear learning goals for the modelling tasks. Doyle (1988) argued that it is important for teachers to be cognizant of the extent to which students are explicitly expected to demonstrate understanding of the mathematics underlying the activities in which they are engaged. The third question in the proposed framework makes provision for that by asking teachers to draw explicit connections between the mathematical ideas and the modelling tasks being planned.

The fourth question encourages teachers to work out and be familiar with the solution space of the modelling task. This can help teachers in facilitating students' learning during the task implementation (Blum & Borromeo Ferri, 2009). Importantly, this may help teachers determine if the modelling task is really a good fit for the learning goals and hence the need to iterate the planning process back to the first two questions. The fifth question encourages teachers to monitor the progress of the modelling task. Such self-monitoring can increase teachers' sense of competence and control and in turn, their motivation to carry out such modelling tasks.

Structuring Mathematical Modelling Lessons

Arising from the use of this framework and integrating knowledge of mathematical modelling process and mathematical content knowledge, one could construct lesson images to structure mathematical modelling lessons. In this section, we show how two lesson images can be constructed to structure mathematical modelling lessons based on two level 2 modelling tasks.

Task A To explain how it is possible for a detective to estimate a perpetrator's height based on the size of the footprint in a crime scene investigation situation. Teacher A1 intends to structure a level 2 task for students to apply known direct proportional relationship model to explain the novel situation, thereby developing in students the mathematical modelling competencies of model formulation and validation.

Lesson Image A1

Since the purpose of the modelling task is explanatory in nature, empirical modelling approach will be employed. After discussing Task A situation, Teacher A1 will lead students to realize that they will have to investigate if a relationship exists between a person's height and his foot length. Teacher A1 will question students on what they need to know before they can estimate a person's height based on his foot length.

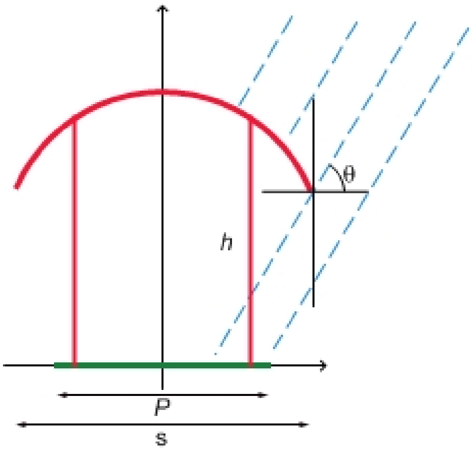
Teacher A1 then question students on the types of data that needs to be collected in order to investigate the existence of the relationship between height and foot length. Teacher A1 will discuss with students factors that may affect one's height, such as genes, race, gender, nutrition, maturation etc. The discussion of such factors can then lead students to see the need for making necessary assumptions with regards to the type of data to be collected. For example, students can make the assumption that they are only examining the data from a particular type of population. At this point, teacher A1 will then provide students with the required data set from internet sources.

After plotting the data points on their graphing tools, teacher A1 will prompt students to reason their choice of linear function that will fit the observed trend of data points. Where there are students who do not get the expected direct proportional relationship for height and foot length due to some solution error, teacher A1 will ask students to check their model with some Science theory that states a taller person will need a longer foot span to maintain balance. Students will then be asked to use their model to explain how they can verify their own height based on their measured foot length. Students will also be asked to explain any deviations based on the implications of the assumptions made earlier.

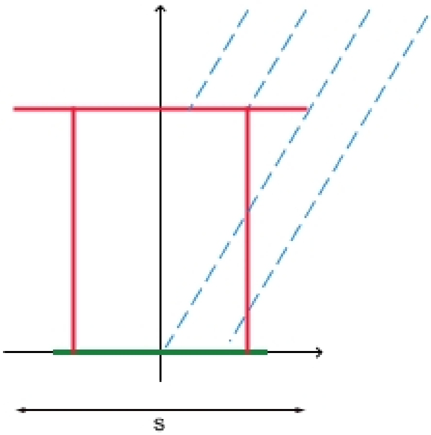
Task B To determine the "best" design for the cover of a link way. Since the purpose of the modelling task is prescriptive in nature, deterministic modelling approach will be employed. Teacher B1 intends to structure a level 2 task for students to apply known quadratic function models and trigonometric ratios of acute angles to solve the design problem.

Lesson Image B1

Teacher B1 will lead students to see the need to quantify functionality of the cover in terms of its "cover effectiveness" from rain. Teacher B1 will get students to take pictures of their school's link way when it is raining. Teacher B1 will then get students to note and abstract the important features of the sheltered link way in the raining situation from their pictures and generate a similar diagram as in Figure 1. These important features will translate into relevant factors that affect the "cover effectiveness" in terms of the amount of pavement that gets wet. These factors include width of pavement (indicated as "P"), span of cover (indicated as "s"), height of support (indicated as "h") and angle of strike of rain from the horizontal (indicated as "θ"). Teacher B1 will also lead students to see the need to assume these factors to be constant in order to compare the extent of cover provided by the two different designs.



(a) parabolic top



(b) flat top

Figure 1: Diagram that shows two possible cross-sectional shapes of the cover for the walkway.

In order to facilitate students to determine the dimensions of the parabolic cross-sectional shape of the cover that will provide maximum cover from the rain, teacher B1 also worked out a plausible mathematical solution as follow:

Referring to Figure 1(a),

Let cross-sectional curve of shelter be $f(x) = ax^2 + b$
Assuming some span of cover, s, width of pavement p such that $s > p$, height of support h, and angle of strike of rain, θ, then,
Cross-sectional length of pavement and some constant length of $\frac{s}{2} - \frac{p}{2}$ that will get wet from rain =

$$f\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2} - \theta\right)$$

$$= \left(\frac{as^2}{4} + b\right)\tan\left(\frac{\pi}{2} - \theta\right), \text{where } a\left(\frac{p^2}{4}\right) + b = \mathrm{h}$$

$$= \left(\frac{as^2}{4} + \mathrm{h} - \frac{ap^2}{4}\right)\tan\left(\frac{\pi}{2} - \theta\right)$$

Therefore, maximum cover occurs when

$$\left(\frac{as^2}{4} + \mathrm{h} - \frac{ap^2}{4}\right) = 0.$$

That is, $a = \frac{4\mathrm{h}}{s^2 - p^2}$ and $b = \frac{-\mathrm{h}s^2}{p^2 - s^2}$.

We note that teacher A1 has integrated his knowledge of mathematical modelling process to a large extent in using the proposed framework to structure and craft his lesson image. This lesson image therefore holds the potential for students to meaningfully experience the mathematical modelling process. This adaptive style of teaching may also develop new pedagogical content knowledge in teachers implementing the lessons as they shape appropriate teaching moves in response to their interpretations of novel student learning actions (Chazan & Ball, 1999; Sherin, 2002).

On the other hand, the lack of deep understanding of the mathematical modelling process may limit teachers in using the proposed framework to structure the lesson in a way that provides students only with procedural understanding of the mathematical modelling process. Teachers are also likely to evoke associated familiar or fixed teaching and learning moves in response to unexpected students' learning actions, hence hindering the development of their pedagogical content knowledge.

In addition to integrating his knowledge of mathematical modelling process in structuring the lesson, teacher B1 has also rigorously applied the framework to structure his lesson. By not working out plausible mathematical solution to the design problem as teacher B1 has done, teachers may encounter conflicting learning goals and hence a coherent sequence of modelling process may not be experienced by the students. For example, the modelling problem may be defined as designing a more functional sheltered walkway, but the latter part of the lesson may have been planned for students to be engaged in subjective ranking of various images of shelters based on some criteria which may not be quantifiable.

The proposed framework is in fact currently being tested in a trial with several secondary schools in Singapore. Teacher participants have been introduced to the proposed framework and given some training in using it to develop modelling lessons. Data, in the form of artifacts such as teachers' notes and documentation, students' sample work and videos of lesson implementations, are currently being collected. Further analysis of the match between teachers' lesson images and the actual lesson implementations will be carried out to examine these teachers' development of pedagogical content knowledge in mathematical modelling instruction. Results of this trial are not available as yet but will be presented in future reports.

Nonetheless, through this discussion, we have elucidated the importance of structuring mathematical modelling lessons via teachers' lesson images, augmented by Ang's proposed framework and knowledge of the mathematical modelling process. Such lesson images can form an important starting point to the adaptive development of teachers' pedagogical content knowledge in the continuum of mathematical modelling instructional contexts.

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