Student Understanding of Large Numbers and Powers: The Effect of Incorporating Historical Ideas

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The value of a consideration of large numbers and exponentiation in primary and early secondary school should not be underrated. In Indian history of mathematics, consistent naming of, and working with large numbers, including powers of ten, appears to have provided the impetus for the development of a place value system. While today's students do not have to create a number system, they do need to understand the structure of numeration in order to develop number sense, quantity sense and operations. We believe that this could be done more beneficially through reflection on large numbers and numbers in the exponential form. What is reported here is part of a research study that involves students' understanding of large numbers and powers before and after a teaching intervention incorporating historical ideas. The results indicate that a carefully constructed framework based on an integration of historical and educational perspectives can assist students to construct a richer understanding of the place value structure.

Introduction

In the current era where very large numbers (and also very small numbers) are used to model copious amounts of scientific and social phenomena, understanding the multiplicative and exponential structure of the decimal numeration system is crucial to the development of quantity and number sense, and multi-digit operations. It also provides a sound preparation for algebra and progress to higher mathematics. However, many students continue to experience difficulties with the place value system, even in secondary school. This may be due to the fact that, as observed by Skemp (1989), while outwardly simple, the place value concept contains a number of complex ideas, with some of them at a high level of abstraction. In this paper we report on a study that turned to the history of mathematics for ideas that might improve students' understanding of a general place value system and show the benefits of such an approach.

For some decades now, it has been argued that the history of mathematics can have a positive influence in teaching and learning (Fauvel & van Maanen, 2000). Arguments are that a historical approach can help to improve perceptions of mathematics, and attitudes to it, raising interest by revealing it as part of human culture. It can show up certain difficulties and conceptual obstacles in the development of mathematics that recur in learning, and how these difficulties were overcome. The theoretical framework for the use of historical analysis in contemporary research is sometimes that *ontogenesis* recapitulates *phylogenesis*; that is, the mathematical development in the individual learner repeats the history of mathematics (Radford, 2000; Sfard, 1995). Although a strict historical parallelism is not considered tenable, one possible pedagogical use of history is to incorporate past conceptual developments and the order in which they arose, into a teacher's knowledge so they can be interwoven with the design of classroom activities (Furinghetti & Radford, 2008; Tzanakis & Arcavi, 2000). The goal of this perspective is to enhance students' understanding in mathematics by looking at the results of a process of historical-cultural development. Briefly, our view is that historical awareness may be beneficial in adopting appropriate teaching strategies for the classroom and also providing a toolkit to understand student

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behaviours (Radford, 2000). In this study, in order to develop a didactic framework for a deep understanding of the place value system, a historical-critical methodology (Gallardo, 2001) was adopted. This involved a to and fro movement between history of mathematics texts and related mathematics education writings in order to extract ideas for classroom activities. The historical analysis resulted in the development of an overarching framework that incorporated i) Large numbers including powers of ten, ii) Other ancient numeration systems, iii) Algebraic generalisation and, iv) The compositional structure of the decimal place value system and its generalisation. Teaching sequences for each of these strands were also developed and implemented in the classroom. Due to space constraints, what is reported here is a part of the research study involving students' understanding of large numbers, particularly powers with positive integer exponents.

History of Mathematics

A study of the history of the development of the current numeration system reveals that it originated in India and was then carried to Europe by the Arab mathematicians (Datta & Singh, 2001). It is reasonable to suppose that a study of the historical evolution of this system may hold valuable lessons for today's classroom. This along with the known student difficulties with the place value system provided the impetus for the study. One of the many crucial mathematical developments in India preceding the creation of the present place value system was the consideration of very large numbers, including their representation as powers of ten (Joseph, 2011). What were the types of numbers that were considered in ancient times? A few examples from Indian historical texts are given below:

1. A major milestone in the development of the Hindu-Arabic place value system is a (surprisingly very early) set of number names for powers of ten. In the *Vajasaneyi* (*Sukla Yajurveda*) *Samhita* (17.2) (c. 2000 BC) of the Vedas, the following list of *arbitrary* number names is given in Sanskrit verse: *Eka* (1), *Dasa* (10), *Sata* (10^2), *Sahasra* (10^3), *Ayuta* (10^4), *Niyuta* (10^5), *Prayuta* (10^6), *Arbuda* (10^7), *Nyarbuda* (10^8), *Samudra* (10^9), *Madhya* (10^{10}), *Anta* (10^{11}), *Parardha* (10^{12}) (e.g. Bag & Sarma, 2003; Datta & Singh, 2001). They were aptly called the *dasagunottara samjna* (decuple terms), confirming that there was a definite systematic mode of arrangement in the naming of numbers. The same list of names of powers of ten was then extended to *loka* (10^{19}) (Gupta, 1987).
2. The same list up to *parardha* (one trillion) as in the *Vajanaseyi Samhita* is repeated in the *Pancavimsa Brahmana* (17.14.1.2) with further extensions. The following passage from it will give an idea of the context and the manner in which large numbers were introduced:

   > ...By offering the agnistoma sacrifices, he becomes equal to one who performs a sacrifice of a thousand cows as sacrificial fee. By offering ten of these, he becomes equal to one who performs a sacrifice with ten thousand daksinas (fee). By offering ten of these, he becomes equal to one who sacrifices with a sacrifice of a hundred thousand daksinas... By offering ten of these, he becomes equal to one who sacrifices with a sacrifice of 100 000 million daksinas. By offering ten of these he becomes the cow [one trillion]. (Sen, 1971, p. 141)

3. In the Buddhist work *Lalitavistara* (c. 100 B.C.E), there are examples of series of number names based on the centesimal scale. For example, in a test, the mathematician Arjuna asks how the counting would go beyond *koti* (10^7) on the centesimal scale, and Bodhisattva (Gautama Buddha) replies: Hundred *kotis* are called *ayuta* (10^9), hundred *ayutas* is *niyuta* (10^{11}), hundred *niyutas* is *kankara* (10^{13}),…and so on to *sarvajna* (10^{49}), *vibhutangama* (10^{51}), *tallaksana* (10^{53}). It is to

be noted that there are 23 names from *ayuta* to *tallaksana*. Then follow 8 more such series, starting with 10^{53} and leading to the truly enormous number $10^{53+8\times46} = 10^{421}$! (Menninger, 1969).

4. In the Vedic literature, time is reckoned in terms of *yugas* or time cycles. The four *yugas* are *Satya-yuga*, *Treta yuga*, *Dwapara yuga* and *Kali yuga*. According to Hindu cosmology, the time-span of these four yugas is said to be 1728000, 1296000, 864000, and 432000 years, respectively, in the ratio 4:3:2:1. The total of these four *yugas* was considered as one *yuga-cycle* or *Mahayuga* and was thus 4320000 years (Srinivasiengar, 1967). Moreover, it is believed that 1000 such *yuga-cycles* comprise one day in the life of *Brahma*, which is 4,320,000,000 years and one day and night period is 8.64 billion years which was further extended to 311×10^{12}. As pointed out by Plofker (2009) time in the astronomical works is bound by cosmological concepts. In one *kalpa* which is 4 320 000 000 years, all celestial objects are considered to complete *integer* number of revolutions about the earth.

It is clear, even from the brief examples above, that large numbers were not only known in early Indian mathematical contexts but formed an integral part of the knowledge and practice of the times.

Large Numbers and Pedagogical Perspectives

In the light of the historical examination of the examples of the use of large numbers in Indian history mentioned above, we may ask what lessons can be learned for today's classroom? As can be seen, some truly enormous numbers were dealt with from a very early time period, and although some of these numbers are literally fantastic, it is significant to note the extent to which these were taken. What is also seen is that the multiplicative (exponential) structure was already present from Vedic times, with number names for powers of ten even in the earliest works. There seems to have been an early awareness of the effect of repeated multiplication. Pedagogically, this implies the value of an early and explicit teaching of powers to create a *foundational base* for a sound grasp of the place value system. However, as in the Mayan civilisation, such numbers in Indian history were mostly employed in a practical context such as astronomy (Plofker, 2009) and time measures (for calendar purposes), according to the needs related to social and cultural practices of the different eras. This suggests that today's students, many of whom are also fascinated by large numbers may be more motivated if a meaningful, realistic context is used. This is not difficult, since large numbers are ubiquitous in today's society, and the concept of exponential growth (Confrey, 1994) is the fundamental idea behind compound interest, computer memory, space distances and is also useful for modelling population growth and spread of disease. This common usage of large numbers suggests that there is a need to understand them, and in this regard students may benefit from *naming*, *reading*, *writing* and *computing* with them (Ronau, 1988) in different contexts. Understanding the *naming convention* (Labinowicz, 1985) for reading large numbers is a prerequisite to grasping them and their structure, and may help students to form a link between the place value chart and the place value system. Students may gain an understanding of the current nomenclature (with digits in clusters of 3) from reading and writing numbers involving million (10^6), billion (10^9), trillion (10^{12}) and so on to decillion (10^{33}). However, the problem is that words like *billion* and *trillion*, and the place value system are based on an

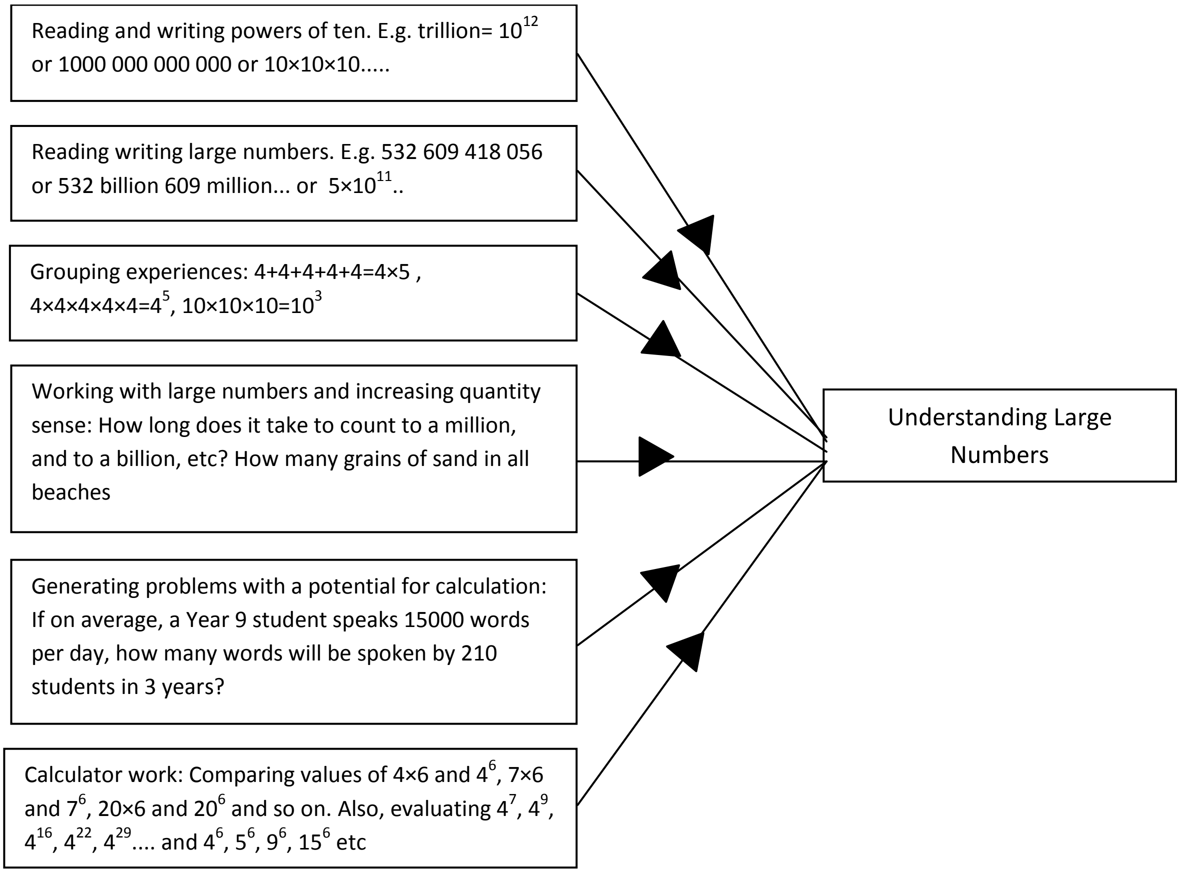

Figure 1. The framework for teaching large numbers and powers.

understanding of *powers* and *exponential growth*, making it difficult to understand one without understanding the other.

This pedagogical *circularity* (Sfard, 1991) or catch-22 situation implies that naming large numbers, and grouping experiences reflecting the embedded nature of repeated multiplication and its shortened notation should go hand in hand. That is, students should be able to move between *multiple representations* (Thomas, 2004, July) making conceptual links between them in order to understand the concept well. Moreover, apart from the nomenclature, it is also important to develop a sense of the size of the numbers and the structure of the powers. This is called *quantity sense* by Wagner and Davis (2010), who define it as a *feel* for amounts and magnitudes. The second historical example given above indicates that groupings of higher order (such as groups of groups of groups) could be employed as a teaching strategy, both to help students understand exponentiation with integer exponents and to develop *quantity sense*. In this study, as a result of viewing the historical analysis within the background of pedagogical research, a framework (see Figure 1) was developed and implemented to teach large numbers and the notion of powers.

The Teaching Method

The second part of the research comprised a case study that involved naming and working with large numbers and then using craft matchsticks to reflect the grouping structure of the place values/powers. The student participants in this research comprising 26 students (13 years old) members of a Year 9 class in a multicultural state secondary school in Auckland, New Zealand were taught by the first-named researcher. The students were representative of a wide range of socio-economic and cultural backgrounds including Korean, Chinese, Indian, European, Filipino, Maori, Pacific Island and New Zealand European ethnicities. Most students were proficient in English, however six students were attending literacy classes and two students were attending ESOL class (English for Speakers of Other Languages). Students were given a pre-test on large numbers at the start of the year and a post-test in the middle of Term 1 after the intervention. The teaching intervention involved naming and writing large numbers both in words and in different numeral forms. Number names such as *million*, *billion*, *trillion* and so on up to *decillion* and their multiple representations were read and written so as to highlight for students the base ten multiplicative structure, and the sub-base of thousand-unit chunks. The names were written out in words and in the expanded form. For example million was written as 10×10×10×10×10 ×10 and as 10^6. Other examples of numbers such as 7 648 309 105 were read and written out as, for example, 7 billion, 648 million, 309 thousand and 105 and also as

$$7 \times 10^9 + 6 \times 10^8 + 4 \times 10^7 + 8 \times 10^6 + 3 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 0 \times 10^1 + 5 \times 10^0.$$

(Initially 10^0 was written as 1). Classroom activity also involved grouping experiences using craft sticks that reflected equal (and unequal) grouping structures (including powers) such as 3+3+3+3 and 3×3×3×3, 10+10+10 and 10×10×10, and linking these to their abbreviated notation. Further, in order to increase quantity sense, the discussion in class centred around the relative sizes of numbers. The teaching unit also included computing with large numbers and problems such as the following were considered: How far does light travel in a year if the speed of light is approximately 300 000 km/sec? How many grains of sand are there in Muriwai beach if it is given that the length of the beach is approximately 50km, width is 50m and the depth is 4m, and if 1 cm^3 contains about 8000 grains of sand? If a million dollars consisting of $1000 bills is stacked to a height of 10 cm, how high is a billion dollars, and a trillion dollars? Following these activities, students were given the post-test on large numbers.

Results

The questions in the large number test involved different ways of writing large numbers, students' ability to compute with large numbers, their ability to produce questions that use large numbers with a potential for calculations, exponentiation and the notion of infinity. While all these areas contribute to understanding student thinking about large numbers, in the ensuing analysis, particular attention was paid to students' answers to questions related to exponentiation, due to its importance in gaining a deep understanding of place value structure.

When the participants' answers to the questionnaires were analysed, from the pre-test to the post-test, every student except two improved their score, leading to a significant improvement in the mean score ($Mean_{pre} = 2.85$, $Mean_{post} = 10.02$, t=10.46 and p<0.0001).

Evidence of the use and understanding of exponentiation was looked for in both the pre-test and the post-test results. In the event, for the first question, which asked 'What is the largest number that you can say in five seconds?' no student answered in terms of powers in the pre-test. On the other hand, in the post-test, although not specifically asked to do so, 11 out of 26 students gave their answer in an exponential form, with (Student 4) S4 writing

99^{99}, S20 50^{150}, and 368^{50} was written by S8. Some students wrote in terms of a much larger base and exponential numbers, for example: 50250640^{100} (S23), 480910^{80} (S24), 1000000^{1000} (S14), who also wrote this number in words. These examples show that although it is unlikely that the students understood the enormous size of the numbers that they had created, they appeared to be aware that writing the number in the exponential form does rapidly increase the value of the number.

Q4 involved writing one trillion in numerals in two different ways, and in the pre-test only one student was able to write it in a power form, but the answer was incorrect. However in the post-test, 17 students gave the correct answer in two different ways including the power form, namely as 1 000 000 000 000, and 10^{12}. Three students further elaborated that $10^{12} = 10×10×10...$ Qs 9,11,12,13 and 14 that were analysed all involve students' understanding of exponentiation as repeated multiplication, and its generalisation, which was a primary focus of this study. The question facilities and the results in the pre- and post-tests are given in Table 1.

Table 1
Question Facilities on Notation for Repeated Multiplication

	Question	% correct in Pre-test (*N*=26)	% correct in Post-test (*N*=26)
Q9	What does 7^{11} mean to you?	23.1 (6)	80.8 (21)
Q11	Write what you understand by 2^{96}	19.2 (5)	84.6 (22)
Q12	What does a^7 mean to you?	26.9 (7)	69.2 (18)
Q13	What is the meaning of 2^y?	3.8 (1)	50 (13)
Q14	Write what you understand by a^y ?	3.8 (1)	50 (13)

Understanding exponentiation as repeated multiplication is a crucial step on the way to number system construction and there was some evidence of numbers written in an expanded form. For Question 9 in the pre-test, 6 out of the 26 students wrote the correct answer, but only 3 of them were able to transfer their understanding of repeated multiplication to the higher powers (Question 11). 50% of the students made the classic error, writing 7×11=77 or a similar answer, while two students did not provide an answer. However, in the post-test, 21 out of the 26 students were successful in giving the correct answer; 7×7×7×...(11 times). Of these 21 successful students, 19 were able to extend their understanding to higher powers. In terms of errors, in the post-test, it is noteworthy that no student made the error of writing, for example, 7×11 for 7^{11} for any of the questions. However, despite explicit teaching during the intervention, one student (S16) did reverse the numbers, writing "11 to the power of 7". Questions 12 to 14 sought to examine whether students could extend the idea of repeated multiplication to the general form. Qs 13 and 14 were especially difficult for students and many did not attempt these in the pre-test. However, as Table 1 shows, students were more successful in the post-test. What is interesting to note is that for some students, writing the expanded form for large numbers was harder than making generalisations. For example, in Q12 seven students were able to give the expanded form a×a×a×a×a×a×a, however, only 5 were able to write that 2^{96} is 2×2×2.....(96 times). As seen in Table 1, questions involving powers such as 2^{96}, 2^y and a^y appear to have been problematic for students in the pre-test, possibly because the *exponents*, rather than the base either involve a large number or a literal symbol. Given that the main focus of the study was on the *grouping structure* and *notation* of *powers*, it is noteworthy that many students experienced some success in the questions on exponentiation. Several students did comment that they found the grouping experiences in class useful in understanding grouping in powers reflecting repeated multiplication.

Discussion and Conclusion

The main purpose of the study was to investigate whether a framework based on an examination of an historical progression of ideas would enhance students' understanding of large numbers, and especially powers of ten. This seems to have been the case. The results appear to substantiate researchers' views (Hewitt, 1998) that, by naming large numbers and working with them, students can construct the naming convention and the base-ten structure of numeration. Further, the evidence of exponentiation in the post-test questions and students' responses, particularly in Q4 tend to confirm Zazkis's (2001) theory that contemplation of large numbers encourages students towards a sense of structure. The teaching module had both an emphasis on operations without actual computation, as proposed by Hewitt (1998), which may help students build abstractions, and also computations with large numbers in support of Wagner and Davis's (2010) view that calculating and connecting to previous experiences may be beneficial to develop a *sense* of *quantity* involved in large numbers.

Another crucial aspect of understanding place value structure is an awareness of powers as repeated multiplication, leading to the grouping structure (Dienes & Golding, 1971) that reflects this (as seen in the second historical example), and the related quantity sense. The results show a degree of success experienced by students pertaining to the exponentiation concept, and this corroborates Zazkis's hypothesis that consideration of large numbers, particularly powers, can help lead learners to structure sense. Students' experiences in grouping, and working not only in base-ten but also in non-decimal bases may also have contributed to their success in responding to questions that involved a generalisation of powers. The test results illustrate that, working with manipulatives, and employing visualisation aspects are important factors in the early stages of concept development. In addition, working with the different *representations* (including concrete materials), as suggested by many researchers (e.g. Nataraj & Thomas, 2009), may have helped some students to generalise to the meaning of larger numbers such as 2^{96}, and to powers with literal symbols.

What surfaced in the study is that, despite the approach used and the varied practice experiences they had, some students still found grouping in powers confusing and difficult. A key aspect in the construction of place value that was highlighted in the research is the importance of *regrouping* as a *whole unit* at each step in the process and keeping a record of the transformation. Another crucial idea is the establishment of links between manipulatives in the grouped form and their representation in the written form, at every stage in the procedure. What also came to the fore in students' work was the presence of obstacles, similar to those experienced historically. For example, students' way of knowing multiplication appears to sometimes stand in the way of their understanding repeated multiplication and this sometimes resulted in the classic error of writing $2\times y$ rather than 2^y.

Notwithstanding the above difficulties, the positive results indicate the usefulness of the framework and the related large number teaching sequence and suggest the value of constructing teaching sequences from a combination of historical ideas and educational research. While some students were still developing the exponentiation concept, for most the early naming of, and working with large numbers including powers of ten and its related groupings, appears to have helped them to understand the naming convention in large numbers and the equal grouping structures involved in powers of ten and other bases.

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