PCK and Average

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This paper considers the responses of 26 teachers to items exploring their pedagogical content knowledge (PCK) about the concept of average. The items explored teachers' knowledge of average, their planning of a unit on average, and their understanding of students as learners in devising remediation for two student responses to a problem. Results indicated a wide range of performance and a wide range of ability in relation to a hierarchical statistical PCK scale. Suggestions are made about developing teachers' PCK.

Historically, understanding of average in its various incarnations, was the first and for many years the most frequently studied statistical concept in mathematics education. From the early 1980s when for example focus was on the weighted mean (e.g., Mevarech, 1983) through the middle 1990s when interest evolved to explore students' developing understanding (e.g., Mokros & Russell, 1995), the interest was in students, not their teachers. Callingham (1997) was one of the first to heed the call of Shaughnessy (1992) and explore teachers' understanding of the concept, finding basic understanding good but the same difficulty with the weighted mean as had been found earlier for students. Finding confusion over the mean, median, and mode by teachers was common into the 21st century (e.g., Begg & Edwards, 1999; Leavy & O'Loughlin, 2006). In a review in 2011, Jacobbe and Carvalho concluded there was little difference between teachers' understanding and that of their students. This is a somewhat disturbing finding in the decade of the 2010s, when the expectations of teachers in terms of their pedagogical content knowledge (PCK) are rising across the entire mathematics.

Since its inception in relation to Shulman's (1987) eight types of knowledge needed by teachers, PCK has often encompassed more than Shulman's original definition. Following the focus on "Mathematical Knowledge for Teaching" by Hill, Rowan, and Ball (2005) and their expansion into six associated domains related to students (Hill, Ball, & Schilling, 2008), reminiscent of Shulman's original groupings, Groth (2007) developed a framework for teaching statistics based on "common" and "specialised" knowledge. Although it is relatively easy to describe the attributes that are desirable in teachers, the question of developing measures to judge their quality is considerably more difficult. For statistics, Callingham and Watson (2011) developed survey items that measured PCK covering (i) content knowledge through giving correct answers to problems previously given to students, (ii) knowledge of students as learners by anticipating students' inappropriate responses to problems and (iii) the pedagogical knowledge on how they would intervene to remediate the student responses they suggested or others selected from student surveys. In this paper, in concentrating specifically on the concept of average, we return to the format of the original profiling instrument of Watson (2001) and employ an item asking teachers to present ideas on planning a unit on average. The purpose is to provide the opportunity to demonstrate content knowledge, general pedagogical and pedagogical content knowledge, curriculum knowledge, knowledge of students, and contextual knowledge.

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Methodology

Sample

The 26 teachers who completed the items analysed here were in the final year of a 3-year longitudinal project providing professional learning in statistics education for teachers and collecting data from them and their students. The teachers were from three Australian states, 11 were female and 15 were male. The years of teaching experience varied from two years to more than 25 years, and they taught in Years 5 to 12, with the main experience being at the middle school level. They had been involved in the project between one and three years.

Instrument

As part of the final survey of teachers in the project they were asked to complete the questions in Figure 1. For Section 1 of the survey they were given about ⅔ of a page, with a box to fill in. Section 5 presented three different problems that had been used in surveys of their own students during the project. Teachers were not asked to solve the problems but to indicate how they would respond to each of two inappropriate answers for each problem. The problem in Figure 1 was the second of the three problems in the section. The space for response (about ⅓ of a page) has been removed.

Section 1: **TEACHING STATISTICS**
You are about to introduce your students to a new unit on average.
Explain how you would prepare for this unit.
How do you think about the different aspects of your teaching of average?
How do these aspects link together?
What aspects do you value most highly?
Please represent your thinking below. You may wish to draw a concept map.

Section 5 (Question 2): **STUDENTS' RESPONSES**
Consider the following problem that students were asked in a survey about chance and data:
The average number of children in 10 families in the neighbourhood is 2.3.
One family with 5 children leaves the neighbourhood. What is the average number of children per family now?
Show your work here.
Consider each of the following answers and explanations given by students in response to the problem. Explain how you would respond to each answer.
5.2a
2.3 x 10 = 23 – 5 = 18 ÷ 10 = 1.8
5.2b
I don't know how many children in each family so how do you work it out?

Figure 1. Teacher survey items on average.

Analysis

Rubrics for assessing the responses were developed based on previous analyses of teachers' responses to PCK tasks. The rubric for Section 1 is given in Table 1. Although reflecting the task as set and increasing attention to students, it also includes aspects of responses observed in earlier interviews of teachers. The highest levels, codes 3 and 4, mirror the expectations of Watson, Callingham and Nathan (2009) to employ content specific strategies and to construct a shift to more general aspects of the situation, respectively. Table 2 displays the rubric for the two parts of Section 5.2, again with the higher level responses reflecting increasing student involvement with development of the solution to the problem.

Table 1
Rubric for Section 1, Unit Plan

Code	Description
0	No response.
1	Minimal response – isolated idea/s, no sequencing.
2	Listing of sequential topics that are relevant to the unit with little linking among them. Generally teacher focused instructions.
3	Presentation of range of related topics with explicit links among them.
4	Presentation of linked structure including a student focus of activities and appreciation of extension ideas.

Table 2
Rubric for two questions in Section 5.2

Code	Description	5.2a 2.3 x 10 = 23 – 5 = 18 ÷ 10 = 1.8	5.2b I don't know how many children … so how do you work it out?
0	Not engaging the mathematics	No response.	Unsure/ out of context
1	Single isolated question or suggested approach	"Get student to explain thinking"	"Discuss average and how to work out"
2	Extended comment related to formula (content only)	Comment on number of families or equation	Extended explanation of formulas involved
3	Suggestions beyond the formula	Questioning related to either: number of families or equation structure	Suggestions that go beyond the formula to model the problem
4	Sequencing task for student	Questions leading student to complete the task	Not observed

The coded responses for all PCK items in the survey were analysed using Rasch measurement (Bond & Fox, 2007) with Winsteps 3.75.0 software (Linacre, 2012). This approach provided (i) a validation of the underlying PCK construct and (ii) a way to identify the relative difficulty of each of the average items, including the planning task (Section 1), and to compare the items with an initial scale of PCK for teaching statistics previously identified based on data from a 2007 survey (Callingham & Watson, 2011), which identified four hierarchical levels of PCK. These average items were not included in the initial survey, so there was interest in seeing whether they conformed to the scale.

Results

Table 3 contains the frequencies for each level of response for the three questions on average. Figure 2 illustrates a minimal approach to the task to describe a plan for a unit of work. Half of the teachers were able to present the relevant topics related to a unit on average (Code 2) but had more difficulty linking them effectively, focusing on their students, or extending to other related aspects.

Table 3
Frequency of response levels for questions on average

Code	0	1	2	3	4
Section 1, Unit Plan					
n (%)	1 (4%)	3 (12%)	13 (50%)	4 (15%)	5 (19%)
Section 5.2a					
n (%)	1 (4%)	3 (12%)	9 (35%)	10 (38%)	3 (12%)
Section 5.2b					
n (%)	3 (12%)	9 (35%)	8 (31%)	6 (23%)	0 (0%)

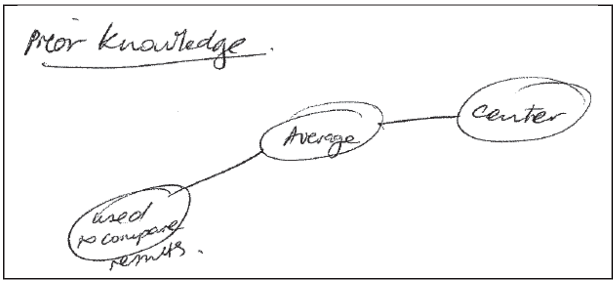

Figure 2. Code 1 response to Section 1, Unit Plan.

Figure 3 shows a concept map that was given a level 3 code for the unit plan. Although complex, it failed to receive a level 4 code because it did not include any student focus. In contrast, one of the code 4 responses came from a Year 5 teacher who focused on her students, finding out what the students knew, working in pairs and keeping records for future assessment, planning exploration activities with "buddies," relating to the real world examples (listed for the year level), and questioning if finding the average is always the best way of describing a group.

Figure 3. Code 3 response to Section 1, Unit Plan.

For the question asking for a response to an incorrect answer in the form of a "run-on" equation that did not represent equality (5.2a), most teachers showed recognition of one of the errors, usually the number of families left, but only about half suggested engaging the student to some extent in devising an appropriate solution. One of the code 1 responses was, “Explain what you have done and give reasons for your method.” A code 2 response that focused only on content, displaying an appropriate explanation but no reference to the student, is shown in Figure 4. Code 3 responses included questioning of the student, such as: “Well done, you’ve realised that there were originally 23 kids and now there’s only 18. How many families do those 18 kids belong to?” A code 4 response included information similar to that in code 3, but also included some aspects of teaching, including a focus on a series of questions for students.

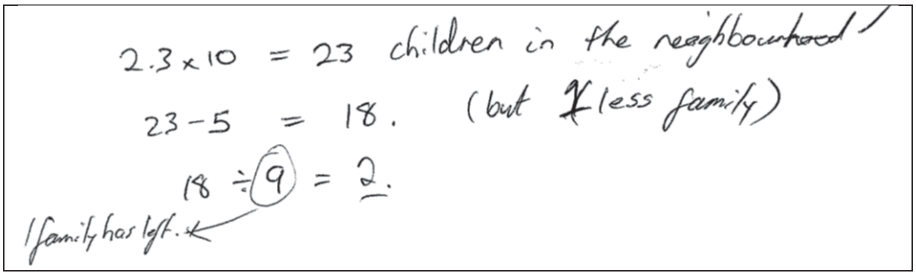

Figure 4. Code 2 response to 5.2a.

For the answer where the student did not know where to start (5.2b), the majority of responses (66%) pointed to the generally accepted way to approach the problem, either very briefly or in more detail (Codes 1 and 2), and about a quarter of responses made suggestions beyond the formula or context of the problem (Code 3). One code 0 response said “not sure” and another suggested that the student was considering social issues of families leaving the district rather than the mathematics. Code 1 responses were brief suggestions, such as “go back to the basics.” Code 2 explanations went further, often posing questions for the student as the problem was laid out: “How many children must there be? … If the large family moves out, how many children left? … families left?” At code 3, responses included suggestions such as modelling 10 families with children in the class or using examples from goal kicking in sports. There were no code 4 responses.

The Rasch analysis indicated that the complete set of PCK items formed a single, unidimensional construct with good fit to the model and high reliability (Infit MSQ=0.99; zSTD Infit=0.0; α=0.83). The item map produced by the software is shown in Figure 5. The average items are highlighted. The number following the item name shows the code. The distribution of teachers’ abilities is shown on the left hand side. Of particular interest is the fact that the second student response (“I don’t know…”) was considerably more difficult for teachers than the one that was mathematical in nature. A code 1 response to item 5.2b—a single suggested approach—was at the same level of difficulty as a code 2 response to item 5.2a—pertinent comments on the mathematical ideas. It seemed that without the “clues” provided by the incorrect mathematical solution, teachers found it difficult to suggest a sound intervention. By comparing the relative difficulty of the items with others on the scale, an approximate mapping of the different codes of the average items could be made to the four levels of the scale of PCK for teaching statistics described by Callingham and Watson (2011).

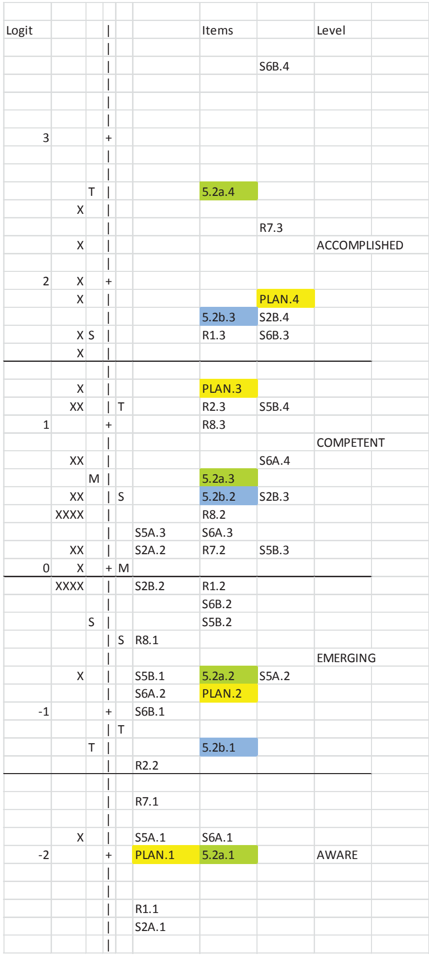

Figure 5. Item map for PCK highlighting levels of average items.

The lowest level of the planning task (PLAN.1) and 5.2a.1 map onto the Aware level, in which teachers suggest only a single response and are not able to address students' understanding. The next group (PLAN.2, 5.2a.2, 5.2b.1) are in the Emerging level, where teachers display some level of understanding of the statistics but only generic classroom intervention, such as "asking questions". At the third Competent level (5.2b.2, 5.2a.3, PLAN.3), teachers recognise students' likely difficulties, and are beginning to suggest statistically appropriate interventions, supported by fairly traditional classroom activities. The top level, Accomplished includes the highest observed code levels of all average questions (PLAN.4, 5.2b.3, 5.2a.4). At this level teachers demonstrate a deep connected knowledge of both the statistical content and of ways to develop student understanding.

Discussion

The results of this study raise several issues for researchers in the area of PCK. Although the literature questions to some extent teachers' content knowledge of average (Jacobbe & Carvalho, 2011), the fact that average has been in the mathematics curriculum longer than any other statistical topic (e.g., Pendlebury, 1896; Smith, 1866) suggests that teachers should be well aware of its significance. The teachers in this study displayed no misunderstanding of average but there were no s pecific questions asking them for definitions or distinctions between mean and median or asking them to solve weighted mean problems. Most teachers displayed an appreciation of the deficiency shown in the student solution in 5.2a. What is more concerning is the teacher-centred perspective taken in many of the responses that covered the content but did not take into account the students as learners. This was evident not only in some of the unit plans but also in the responses to the difficulties seen in the student answers to the "2.3 children" problem (e.g., Figure 4).

As suggested by the authors when they included knowledge of students as learners in their description of PCK (Callingham & Watson, 2011), the challenge is to develop the pedagogy to introduce the content in a way that the students whose responses are presented in 5.2a and 5.2b w ill understand and become successful. The problem for mathematics educators is how to help pre-service and in-service teachers develop remedies for these and other contingencies. In an attempt to accelerate an appreciation of beginning discussion of concepts at the students' level of understanding rather than with the "correct" mathematical solution, Chernoff and Zazkis (2011) conducted an interesting study with pre-service secondary mathematics teachers. First pre-service teachers were presented with a probability problem based on sample spaces that they had encountered previously. This was followed by an inappropriate school student response and a request to intervene. Even after considerable discussion the pre-service teachers had difficulty seeing the limited understanding "through the eyes of a learner" (p. 24). The pre-service teachers were then given a much more difficult problem based on sample spaces where they had no previous experience. There was much debate and no consensus on the approach to a solution. The pre-service teachers were then told "you are incorrect … do it like this" in an approach similar to that taken by the pre-service teachers themselves for the previous problem. When the pre-service teachers rejected the approach because they did not understand, a pedagogically appropriate approach was followed starting with the pre-service teachers' own solutions. This modelling helped the pre-service teachers realise the importance of starting remediation at the student's level.

In this study, question 5.2b, "I don't know …", provided the greatest opportunity for teachers to start at the student's level and the qualitative responses showed the most variation of the three tasks. As noted, one teacher admitted being unsure of what to do. At the other extreme a teacher offered multiple suggestions.

> Depends on the age and ability of student. If they have some algebra skills … If the student is younger or the approach above doesn't "click" I'd use a picture approach – 10 houses …

On the one hand the assessment of one teacher was

> This student has little/no concept of averages and how they work and would need a structured unit of work/hands-on activities to grasp the concepts. Even then doing the reverse operations might be beyond this student's understanding.

On the other hand with a more positive approach, several teachers referred back to "how we calculated average in the beginning," perhaps hoping to capitalise on why the student felt he/she needed to know how many children were in each family.

Having considered the outcomes of this study, involving the most well-known concept in the school statistics curriculum, the authors are led to imagine using the problem and student responses in Section 5.2 of the survey as a basis for discussion and group work with pre-service or in-service mathematics teachers. Part 5.2a may give them a sense of comfort in dealing with the student's response but surely part 5.2b will engender considerable debate. Various approaches could be trialled and critiqued by the group.

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