Educating Boris: An Examination of Pedagogical Content Knowledge for Mathematics Teacher Educators

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This paper presents a framework for examining the Pedagogical Content Knowledge (PCK) required of mathematics teacher *educators* as they endeavour to build the PCK for teaching mathematics that is required of the pre-service teachers with whom they work. The framework builds on existing research into PCK, and provides a series of filters through which to examine the complexity of the work of teacher education. The usefulness of the framework is trialled by using it to study the PCK used by the first author in working with pre-service teachers to build understanding about ways of interacting with students.

Since Shulman (1986) introduced the idea of *pedagogical content knowledge* (PCK) it has been a key area of research. The vast majority of this work has focussed on teachers of school students in school subject discipline areas, especially including mathematics. The purpose of the present paper is to contribute to broadening the field to consider the PCK required for helping teachers acquire PCK for teaching school mathematics; an area that has received relatively little attention until recently (Zazkis & Zazkis, 2011). The knowledge that mathematics teacher educators (MTEs) require to teach prospective mathematics teachers has been described as encompassing the knowledge needed by school mathematics teachers, in the same way that school teachers need to know the mathematics that they teach (Zaslavsky & Leikin, 2004). It can, therefore, be thought of as a kind of meta-knowledge that may be analysable in ways analogous to those that have been used to examine PCK for teaching school mathematics (Beswick & Chapman, 2012). In this paper we posit such a framework for the PCK required by MTEs, and test the framework by examining the work of the first author in teaching her pre-service secondary mathematics teachers (PSTs) in an online environment.

Background and Framework for Mathematics Teacher Education PCK

There is an extensive and growing literature that examines PCK for teaching school mathematics (hereafter SMPCK) and associated constructs (see, e.g., Ball, Thames, & Phelps, 2008; Chick, Baker, Pham, & Cheng, 2006; Davis & Simmt, 2006; Hill, Ball, & Schilling, 2008; Kraus, Brunner, Kunter, Baumert, Blum, Neubrand, & Jordan, 2008; Rowland, Huckstep, & Thwaites, 2005). In all cases the focus is on what needs to be known (or drawn upon) to teach school mathematics. In some cases knowledge is treated as a static entity, categorised into more specific knowledge types such as knowledge of students' conceptions (e.g., Ball and colleagues; Chick et al.; Kraus et al.); in other cases knowledge is treated more dynamically, including the capacity to know how to act in contingent moments (e.g., Davis & Simmt; Rowland et al.). Each approach has advantages and there is overlap, but they are, nevertheless, not all examining the same constructs.

The framework of Chick et al. (2006; presented more completely in Chick, 2007) identified a range of component knowledge areas, associated with teaching school mathematics, that captured "the particular form[s] of content knowledge that embod[y] the aspects of content most germane to its teachability" (Shulman, 1986, p. 9) . The order of presentation in the framework was not intended to be hierarchical; rather, the areas were

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presented within one of three places on a continuum indicating the blend of pedagogy and content knowledge. Overlap and interdependence among the categories was also assumed. The framework offered a set of filters for examining PCK for teaching school mathematics (i.e., SMPCK), thereby allowing the identification of key features of teachers' work.

The authors' recent projects with various colleagues (see the Acknowledgements, and Chick, 2011), led to a desire to examine more closely the PCK required of teacher educators—"us"—as they ("we") work in their ("our") domain. We wanted to consider the PCK of MTEs teaching PSTs. A major component of the "mathematics teacher education" that we teach is the SMPCK of the initial frameworks, so our study is concerned about the PCK for teaching SMPCK. To facilitate our study we took the Chick (2007) framework and attempted to adapt it to this new context, with PSTs taking the place of students, and SMPCK taking the place of mathematics as the teaching domain. This framework is shown in Table 1, along with examples to illuminate each knowledge category.

Table 1
Pedagogical Content Knowledge for teaching School Mathematics PCK Framework

PCK Category	*Evident when the maths educator …*	*Example*
Clearly PCK		
Teaching Strategies	Discusses or uses general or specific strategies or approaches for teaching an SMPCK concept or skill	Gives PSTs two contrasting examples of student responses and requests a "mark" for each, in order to address issues associated with assessment
PST Thinking	Discusses or addresses PST's ways of thinking about an SMPCK concept	Knows PSTs may struggle to recognise that misconceptions may be based on generalising sound ideas inappropriately
PST Thinking - Misconceptions	Discusses or addresses PST misconceptions about a SMPCK concept	Knows that PSTs may believe that identifying the learning outcomes of a lesson is sufficient for lesson planning
Cognitive Demand of SMPCK Task	Identifies aspects of the task that affect its complexity	Knows PSTs may struggle to simulate responses of children with different decimal misconceptions because of the different causes for the symptoms
Representations of SMPCK Concepts	Describes or demonstrates ways to model or illustrate an SMPCK concept	Contrasts different representations (e.g., MAB and LAB) and what they offer for mathematics teaching
Explanations	Explains an SMPCK topic, concept or procedure	Explains what "cognitive demand" means, and why this is important for teachers to know
Knowledge of SMPCK Examples	Uses an example that highlights an SMPCK concept	Uses an example of student work to show a misconception and find evidence of how it arose
Knowledge of SMPCK Resources	Discusses/uses resources that support SMPCK teaching	Uses research papers about an aspect of SMPCK as a stimulus for class discussion with PSTs
SMPCK Curriculum Knowledge	Discusses how SMPCK topics fit into the teacher education curriculum	Explains that knowing "mathematical structure and connections" helps with lesson sequencing
Purpose of SMPC	Discusses reasons for SMPCK content being included in the teacher	Explains how knowledge of misconceptions aids the design and

Knowledge	education curriculum or how it might be used	interpretation of multiple choice test distractors
(SMPCK) Content Knowledge used in a Pedagogical Context		
Profound Understanding of Fundamental SMPCK	Exhibits deep and thorough conceptual understanding of identified aspects of SMPCK (cf. PUFM (Ma, 1999))	Recognises that a teaching activity is "good", and identifies and explains the underlying SMPCK principles that allow students to build understanding
Deconstructing SMPCK Content to Key Components	Identifies SMPCK components within a concept that are fundamental for understanding and applying that concept	Identifies critical aspects of content knowledge, cognitive demand, misconceptions, and representations that affect the design of an assessment task
SMPCK Structure and Connections	Makes connections among SMPCK concepts and topics, including interdependence of concepts	Recognises that the original PCK framework has overlapping components (e.g., PUFM can help evaluate the cognitive demand of a task)
Procedural Knowledge	Displays skills for solving SMPCK problems (conceptual understanding need not be evident)	Uses the strategy of "reverse questions" as a potential way for constructing an open ended task
Methods of Solution	Demonstrates a method for solving an SMPCK problem	If a PST asks how to help a student, the teacher educator suggests possible models or explanations that might help
Pedagogical Knowledge used in a (SMPCK) Content Context		
Goals for Learning	Describes a goal for PSTs' learning	Plans to build PSTs' capacity to develop appropriate questioning techniques for running whole class discussions
Getting and Maintaining PST Focus	Discusses or uses strategies for engaging PSTs	Shows student work sample and asks PSTs to find and explain the error
Classroom Techniques	Discusses or uses generic classroom practices	Uses "think-pair-share" as a way for PSTs to come up with a counterexample for a child's erroneous reasoning
Assessment Strategies	Discusses or uses assessment tasks purposefully to evaluate PSTs' knowledge of SMPCK	Uses a "lesson-plan assignment" to examine PSTs' understanding of planning for student learning

Although we could construct examples of each category from our experiences, and believed that the categories "covered" the extent of the knowledge we felt we used, we were interested to see if the framework was useful for analysing the real moment-by-moment application of knowledge in the work of mathematics teacher education.

Context for the Study

In 2012 the first author taught a secondary mathematics curriculum unit to a small cohort of PSTs, of whom 11 were studying online. This was her first time teaching the unit, and her first major experience of online teaching. The topic for week 7 (of nine) was questioning, and Helen wanted to provide the PSTs with the opportunity to work with and question a student. During the week's recorded PowerPoint presentation she introduced the PSTs to Boris (see Figure 1), a puppet who would be playing the part of a "hypothetical school student". She explained that Boris had been given the "Three Hungry Men" problem (Watson, 1988), which asks solvers to determine how many apples were originally in a bag if three men successively each eat one-third of the apples remaining in the bag and discover eight are left at the end. She then presented the first of Boris' solutions to the problem (see Figure 2). Following the presentation PSTs were asked to contribute to an online discussion "as if you were Boris's teacher talking to Boris". PSTs were told that Boris would respond, and that, since his response might require further teacher interventions, they should return to the thread and, if necessary, continue working with Boris to help him with the problem. The PSTs were informed that Boris's responses were being generated by Helen, and that she might also contribute to the discussions directly in her role as lecturer.

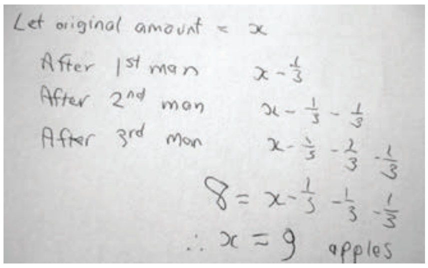

Figure 1. Boris (created by Ian Edwards). *Figure 2.* Boris' first solution to the Three Hungry Men problem.

The data for this study comprise the posts made to the discussion threads, together with Helen's post-hoc recollections of her thought processes in developing the activity and responding to PSTs both as "Boris" and as the lecturer. The data were examined to identify when Helen, as the MTE, appeared to use some form of knowledge for decision-making that was intended to develop SMPCK in the PSTs. This teaching situation—the recorded PowerPoint presentation and the subsequent online discussions—comprise the case for this study, and the purpose of the research is to determine evidence for the existence of (at least some of) the different types of PCK for MTEs posited in Table 1.

Results

We present data from the discussion board posts that provide evidence for a number of different PCK types. There were usually parallel threads on the discussion board, as each PST engaged with Boris individually, but at times the PSTs also contributed to each others' interactions. The data are presented thematically but also respect the chronology within a given thread, which means that the order of the presented PCK types does not necessarily follow Table 1. Space constraints limit our consideration to a few illustrative key themes. In the excerpts text in [] was added as an authorial comment for this paper.

Teaching Strategies

Helen wanted to devise a way for PSTs working in an online environment to engage in an actual questioning sequence intended to diagnose difficulties and help a student. The puppet, Boris, was presented as a pseudo-student, with whom PSTs could interact. It is likely that Boris's Muppet-like appearance increased the PSTs' attention to the task (see also *Getting and maintaining PST focus* in the framework), and may have been less intimidating than an actual student. Helen believed that PSTs would develop their SMPCK by engaging in conversation with "someone" who responded like a real student (although this study does not test this). The use of Boris also allowed her to "control" the types of student understanding to be made evident, as a function of the PSTs' input, which otherwise would have been difficult to achieve with a real student. She thus introduced the PSTs to Boris, and set up a discussion board thread in which PSTs could "talk" to Boris about the problem solution that he had "produced." Boris could then "reply", with further conversation as required to resolve Boris's difficulties. The online discussion environment allowed Helen to interact with individual PSTs, both as Boris and in her role as lecturer. This teaching strategy—allowing PSTs to engage with a "student"—had SMPCK as its focus, as will become evident in the excerpts below.

There were at least two constraints on the activity that may have impacted on its effectiveness. The asynchronous nature of the discussion board environment meant that the conversations with Boris took place over days or weeks, rather than the few minutes that might be typical of a face-to-face classroom. Second, Helen should have ensured Boris had his own login in order to further separate him as student from Helen as lecturer, although she gave clear indications about who was speaking and in what role.

PST Thinking - Misconceptions

Many of the PSTs launched straight into "telling" Boris what he needed to do. For some this may have been because they had not yet understood that Boris really would respond and interact with them, but Helen also knew that the PSTs would struggle to fight this tendency even if they had this understanding. For example, one PST's initial post in response to Boris's solution in Figure 2 included a substantial set of suggestions.

PST1: If you want to explain how Boris work it he is write [sic] in saying let x = original apples. To remove 1/3 is to remove 1/3x as it is a third of the original. It then gets complicated doing it this way as you then remove another 1/3 of (x-1/3x). So after the second man has eaten you have (x-1/3x) - 1/3(x-1/3x). The next step is another 1/3 of this. In other words I would get him to work from the end of the question back to the start

Helen: Hold back! I know you've now thought through things. What's the first thing you would say to Boris though, to give him a chance to figure this out for himself? Boris is here. He's waiting your teacherly input! He'll probably make a studently reply, but that's his job … and that's what students do!

PST1: Oh yeah. Ok Boris, you've started well by knowing that 1/3 needs to be removed each step, but 1/3 of what? Hopefully you agree that it is 1/3 of the peaches [apples] which were? Yes X. You have now worked out the first step! Can you now tell me how we work out the next step after the 2nd man has eaten?

Helen: You could even stop after the first question, and see what Boris does.

The tendency of teachers to "tell" is well-known; Helen's PCK was sufficient to know this, and part of the purpose of the activity was to address it. She also attempted to avoid telling the PST *how* to "avoid telling".

Profound Understanding of Fundamental SMPCK

Immediately after Helen's response in the previous exchange, Boris started to participate in the conversation with PST1.

Boris: Ahhhh. Of. It's not 1/3, it's 1/3 OF the apples ... which is 1/3xX. Is that X/3 or 1/3X???

PST1: That's right Boris, it is 1/3xX. You are spot on in saying that that is also X/3 because (1/3)x(x/1) = x/3 (show cross multiplication). You can use either method Boris, just do what you think is the least complicated for you!

Helen: In fact, Boris' 1/3X option was supposed to be interpreted as 1/(3X) but I couldn't write it that way […], which really does highlight the ambiguity of certain ways of writing things and order of operations.

Here Helen attempted to introduce a typical school student misconception for the PST, associated with expressing 1/3 of *x* algebraically. This reflected her knowledge of SMPCK. Unfortunately the limitations of the textual online interaction appeared to reduce the impact of the issue, or the PST failed to identify the full significance of Boris's query.

Boris: So, I have X apples to start off with, and the first man eats X/3 of the apples, so then there'll be X - X/3 apples. Oooooooookaaayyyyyyyy … so then the second man comes along … and he eats 1/3 … no, wait, he eats 1/3 of the apples. Is that 1/3xX again?

Helen, through Boris, has introduced a different potential student misconception. Students may keep using "*x*" any time they need an unknown to be acted upon, even if that unknown is the result of modifying an already defined unknown (see Kieran, 1992, for a discussion of algebra misconceptions). The PST notices Boris's correct start but perhaps not the new misconception and, because of what the PST wrote next, Helen chose not to pursue it because the response raised another area of SMPCK that could be addressed.

PST1: That's right Boris, you do have X - X/3 apples remaining. Can you simplify that for me mate? Remember he is eating only one third of what you have remaining so it won't be a third of X, but a third of what you have just told me, (2X)/3. So it will be 1/3((2X)/3). It is going to start getting complicated so be careful when you multiply out this step!

Boris: Hang on, where did the 2X/3 come from?

Helen had not anticipated the sudden introduction of $2x/3$ in the response from the PST, and had to deal with this contingency (cf. Rowland et al., 2008). She saw the potential of having the PST address a different aspect of student thinking, namely to help the student see that some students may not be able to get to $2x/3$ from $x - x/3$ as readily as a teacher.

PST1: Ha, remember you simplified X-X/3 into (2X)/3. I was jumping ahead a bit and assumed he could do that when I asked.

Helen: Sometimes jumps for students are bigger than we think, and in this case it's possibly tied up in matching notation to concept. There are (at least) two ways of getting from X-X/3 to (2X)/3: (i) by algebraic manipulation or (ii) by thinking of taking off 1/3 of something and being left with 2/3 of the something. Route (i) takes a bit of effort for students still developing their algebra skills; route (ii) requires students to turn the idea into the notation, which is not always a straightforward step either when fluency is embryonic.

As can be seen, as the thread develops Helen has had to draw on her own knowledge of SMPCK—specifically, her knowledge of typical student conceptions—in order to develop the PST's awareness of this aspect of SMPCK as the scenario unfolds.

Cognitive Demands of an SMPCK Task, PST Thinking, and Goals for Learning

The following excerpt in a separate discussion thread with a different PST started again from Boris's solution in Figure 2.

PST2: Boris if x is the original amount and you take one third away from it 2 thirds may be left. 8 is the 2 thirds left so the one third eaten must be 4. I would get out some counters and place the 8 into 2 groups and look at the missing group being 4.

Boris: So, x-1/3 = 2/3? And 2/3 = 8? Is that how I write it? Where does the x go in what you said?

Helen: Boris isn't being deliberately obnoxious here. He wants to use x to represent the number he's trying to find. He thinks it should be possible to use algebra and manipulate the equations to come up with the answer.

Boris's response was intended to provoke some conversation about the use of equals signs and appropriate algebraic notation. Helen was aware that PST2 did not have a strong mathematical background, thus demonstrating her knowledge of PST thinking, which is why she added an extra comment as "Helen" to direct the PSTs' attention to the use of algebra. PST3 then joined the conversation, but missed Helen's point and Boris's concerns. At this stage, and as the conversation progressed as seen in what follows, Helen became aware that she might have to adjust the cognitive demands of this activity in the thread because of the PSTs' backgrounds.

PST3: PST2's idea of using counters has helped Boris to realise 2/3 is left and 8 is the 2/3 left.

Helen: Boris knows there is 2/3 left (he probably did at the beginning too, although he was thinking more about removing 1/3) and he knows this is 8. The question is what has happened/what needs to happen to Boris's x, and how do you write "8 is the 2/3 left" (which is what Boris was asking with "2/3 = 8")

PST3: Boris's x should only be present in the first part after the first man ate 1/3 apples as 2/3 is used as a starting point for the other two men. ???

Helen: If we're going to use x—and it very definitely is possible to do so (and Boris could get there with the way he is thinking, but he is missing a vital aspect) — then I suspect that x is going to have to hang around. It is obviously there right at the beginning, since it is the number of apples at the start. [Helen presents an argument for how the number of apples at each step depends on how many apples there were at the start, concluding:] how many are left at the end WILL DEPEND ON how many were there at the start ... which was x. So I'm expecting Boris to have an expression that has x's in it, that represents all the eating that has taken place, and can be set equal to 8 because that's what's left after all the eating has taken place.

Immediately after, Helen explicitly stated the goals for the activity, the knowledge required from the PST, and whether or not this knowledge was readily available.

Helen: [Helen thinks out loud for the student about her own teaching;] As a lecturer I am now grappling with a couple of things: I want you to learn how to deal with Boris, but I think there is also some development of your own algebra skills that needs to take place. I am going to continue for a little while to see if we can do both […].

Helen: Without worrying about how to help Boris (for just a tick), can you think about what expression Boris should/could write down after the first man has eaten. There are at least two expressions (the ones I am thinking of are equivalent) that you could legitimately write.

Helen's awareness of PST3's algebra skills came from a separate thread that PST3 had started which was running in parallel and taking a slightly different approach, but which still required the issue of notation to be addressed. This indicates that SMPCK itself may also need to be utilised directly as part of the MTE's work. The conversation among PST3, Boris, and Helen continued in a separate lengthy thread, and involved a complex interplay of addressing SMPCK and mathematical knowledge issues.

Conclusion

Although space constraints have limited the presentation of data, there is evidence here for the existence of at least some of the categories in the framework in Table 1. In particular, the categories do seem to be useful for identifying aspects of knowledge that can be brought to bear in critical moments of mathematics teacher education.

In adapting the framework we were aware that although a simple substitution of PSTs for school students and SMPCK for school mathematics appeared to make sense, there was a danger that this might hide subtle differences. For example, in what ways is an SMPCK resource similar to and different from a PCK resource? Identifying examples was, therefore, a crucial part of checking that the adapted constructs were meaningful. Questions also remain to be explored about the extent to and ways in which MTEs' PCK incorporates SMPCK. The case presented demonstrates the dual focus of MTEs' work that requires them to move between SMPCK and PCK with flexibility that appears to demand a simultaneous focus on both. Finally, just as SMPCK is not the entirety of knowledge required by school mathematics teachers, PCK is not all that MTEs need to know.

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