The Role of Challenging Mathematical Tasks in Creating Opportunities for Student Reasoning

Peter Sullivan
Monash University
<peter.sullivan@monash.edu>

Aylie Davidson
Elsternwick Primary School
<davidson.aylie.a@edumail.vic.gov.au>

The following is a report of an exploration of what mathematical reasoning might look like in classrooms. Focusing on just one lesson in one classroom, data are presented that indicate that upper primary students are willing and able to reason for themselves, especially in classrooms in which the culture for such reasoning has been established. It seems that the opportunities to reason are a product of the tasks that are posed, the structuring of the classroom, and the willingness of the teachers to allow students to engage with the tasks for themselves.

Among the positive contributions to mathematics teaching and learning from the implementation of the Australian Curriculum: Mathematics (AC:M) is the emphasis given to reasoning not only as a process which exemplifies mathematical thinking but also as a strategy for learning the mathematics in the first place. The following report explores ways of incorporating both emphases into everyday mathematics teaching.

There are some assumptions underpinning our approach to incorporating reasoning into everyday mathematics teaching. First, we argue that students are not reasoning if they are merely repeating an argument developed by someone else – the reasoning needs to be their own. Students are more likely to reason if they have developed a strategy, connection or justification for themselves than if they are performing a procedure they have been taught. A second assumption is that such thinking for themselves takes time and that it happens only when students are working on tasks that they do not know how to solve.

The report draws on one aspect of a larger project[i] that recommended that teachers present students with challenging tasks. The project examined what happened when teachers posed challenging tasks to students and reported on ways in which teachers encouraged students to persist. In this report, the focus is on the connections between the posing of challenging tasks and the opportunities for students to reason mathematically.

Theoretical Perspective

While the focus of the data reported below is on student reasoning, our approach relies on teachers posing challenging tasks with which students engage with only limited teacher guidance. It is out of such engagement that the reasoning emerges. These two aspects of our theoretical perspective are elaborated in the following.

Challenge

The focus on challenge is partly based on a perspective of the nature of mathematics. Mathematics is seen as a network of interconnected ideas. To build these networks of ideas it is necessary for students to process different concepts simultaneously, to compare and contrast concepts and to consider their application in different contexts. Connected to this is a perspective on how mathematics is learned. We consider that mathematics learning takes concentration and effort over an extended period of time to build the connections between topics, to understand the coherence of mathematical ideas, and to be able to

2014. In J. Anderson, M. Cavanagh & A. Prescott (Eds.). Curriculum in focus: Research guided practice *(Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia)* pp. 605–612. Sydney: MERGA.

transfer learning to practical contexts and new topics. This perspective is informed by Vygotsky's (1978) notion of the Zone of Proximal Development, Middleton's (1995) descriptions of how motivation is connected to students' learning, the elements of effective teaching described by Anthony and Walshaw (2009) and Sullivan (2011), and the descriptions of inquiry based classroom teaching by Marshall and Horton (2011).

The perspective is also based on the hierarchy of classroom of classroom experiences that Smith and Stein (2011) described as moving from "Memorization" to "Procedures without connections" to "Procedures with connections" to "Doing Mathematics" tasks. We argue that the building of the networks of ideas for themselves can best be done by engaging in experiences which can be described as "Doing mathematics" to which we would add "for themselves".

We take it as given that most students cannot build these networks of ideas, or "do mathematics", without sustained thinking. When confronted by a task that requires them to make decisions on the solution type and solution strategy, the expectation is that the students do not appeal to the teacher for direction but seek to solve the task for themselves especially when the solution is not clear. This requires a willingness to persist.

The theoretical perspective that informs our approach to this willingness to persist is based on the notion of mindsets (Dweck, 2000). Dweck categorised students' orientation to learning in terms of whether they hold either *mastery* goals or *performance* goals. Students with *mastery* goals seek to understand the content, and evaluate their success by whether they feel they can use and transfer their knowledge. They tend to have a resilient response to failure, they remain focused on mastering skills and knowledge even when challenged, they do not see failure as an indictment on themselves, and they believe that effort leads to success. Students with *performance* goals are interested predominantly in whether they can perform assigned tasks correctly, as defined by the endorsement of the teacher. Such students seek success but mainly on tasks with which they are familiar. They avoid or give up quickly on challenging tasks, they derive their perception of ability from their capacity to attract recognition, and they feel threats to self-worth when effort does not lead to recognition. To take up challenges, and therefore to engage with mathematical reasoning, it is beneficial if students have a growth mindset and adopt mastery goals.

The development of growth mindsets and mastery goals takes time and is a product of the classroom culture. In an important meta-analysis of 49 research studies on classroom culture between 1991 and 2011, Rollard (2012) described three significant and relevant findings. First, the meta-analysis found that the middle years of schooling (years 5 to 9) are critical for connecting classroom goal structures and the formation of student attitudes, including an orientation to persist. Second, Rollard (2012) found that classrooms that promote mastery, specifically those that focus on the learning of the content rather than competitive performance, are more likely to foster positive student attitudes to learning. Third, Rollard concluded from the meta analysis that classrooms in which teachers actively support the learning of the students promote high achievement and effort. We interpret this to refer to ways that teachers support students in engaging with the challenge of the task, and in maintaining this challenge as distinct from minimising it.

As an example of establishing this classroom culture, the project suggests to teachers that they use the metaphor of a "zone of confusion" which they invite students to enter for a time, in a sense giving students permission to not know how to proceed with a task.

In other words, creating classrooms that promote a willingness to persist and posing tasks with which students can engage lays the foundation for them to build their own networks of mathematical ideas and creates the opportunities for students to reason.

Reasoning

The second perspective informing the research is identifying ways that engaging in mathematical reasoning can inform learning and doing mathematics. The AC:M (Australian Curriculum and Assessment and Reporting Authority (ACARA, 2011) describes reasoning in the following way:

> Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising.

and

> Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached, when they adapt the known to the unknown, when they transfer learning from one context to another, when they prove that something is true or false and when they compare and contrast related ideas and explain their choices.

This focus was derived from the earlier report by Kilpatrick Swafford, and Findell (2001) which described adaptive reasoning as one of five key aspects of mathematics. In interpreting that statement, Watson and Sullivan (2008) described adaptive reasoning as the capacity for logical thought, reflection, explanation and justification.

This inclusion of reasoning in the AC:M is part of a trend internationally. For example, the NCTM Standards for School Mathematics (2000) includes the following statement:

> Instructional programs from prekindergarten through grade 12 should enable all students to: recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof.

Stacey (2010) argued that reasoning was underemphasised in Australian jurisdictional curriculums. In reporting an analysis of Australian mathematics texts, Stacey reported that some mathematics texts paid some attention to proof and reasoning, but in a way which seemed "… to be to derive a rule in preparation for using it in the exercises, rather than to give explanations that might be used as a thinking tool in subsequent problems" (p. 20).

The real challenge though is finding ways for students to engage in mathematical reasoning for themselves. Helpful advice was offered by Fraivillig (2004) who suggested that teachers should encourage students to solve problems in more than one way, allow students to develop their own approaches, encourage collaboration between students, and use students' explanations as the prompt to explaining the mathematical intent of the task and lesson. Interestingly this advice also serves to exemplify what reasoning might look like in a mathematics classroom.

In elaborating on such advice, Clarke (2013) described three particular and discrete actions for teachers to:

- elicit student mathematical reasoning through both the choice of task and the pedagogies that surround the task;
- encourage reasoning by prompting students to reason, which is done by actions such as celebrating reasoning when it happens, and by probing students to better justifications if they are close; and
- model mathematical reasoning and guide students in its use.

The research reported below is exploring what mathematical reasoning might look like and ways that it can be fostered in everyday mathematics classrooms.

The Research Context

The overall project involved proposing challenging tasks to teachers to match content they were intending to teach, it offered suggestions about ways of encouraging students to persist and it gathered data from teachers and students on their experience. While the project worked with around 35 primary teachers and 20 secondary teachers, each of whom taught and evaluated around 10 lessons, the following is a report of just one lesson in one school on the topic of multiplication. The intent is to provide a finer "grain" of detail, and therefore different insights, than is possible in reports in which data are synthesised across a number of schools and teachers.

The collaboration between the authors is an example of design research which "attempts to support arguments constructed around the results of active innovation and intervention in classrooms" (Kelly, 2003, p. 3). The *intervention* is that the lesson outline was suggested by the first author that was implemented by the second author. The *innovation* refers to both the structure of the lesson and the strategies used to engage students in the task for themselves.

The following is the lesson outline that was proposed to all project teachers as used as the basis of the lesson described below. The Learning task was as follows:

> I did a multiplication question correctly for homework, but my printer ran out of ink. I remember it looked like
>
> 2 __ × 3 __ = __ __ 0
>
> What might be the digits that did not print? (give as many answers as you can)

The lesson documentation also included a second or Consolidating task (see Dooley, 2012, for a rationale for this element) was:

> I did a multiplication question correctly for homework, but my printer ran out of ink. I remember it looked like
>
> 1 __ × 4 __ = __ __ 2
>
> What might be the digits that that did not print? (give as many answers as you can)

The lesson documentation also proposed a rationale for the lesson, a suggested learning intention (Hattie, 2009), enabling and extending prompts (Sullivan, Mousley, & Zevenbergen, 2009), and suggested student solutions.

We consider this task to be challenging, or what Smith and Stein (2011) describe as Doing Mathematics. This is because it allows students opportunities to determine their own approach, to identify and describe patterns and to justify their reasoning about those patterns. The task also has the advantage that it is at the level of the curriculum for these students and there are a number of, but not too many, possible answers. Another advantage is that it is an unusual task for the students in the sense that it was unlikely to have been previously presented in such a way. Students can determine their own methods of solution, record those solutions in their own ways and can communicate their solutions to others.

In advance, we identified three possible approaches to reasoning that we hoped to see on the first task.

> Identifying and using patterns (for example, if $20 \times 30 = 600$ then we can work out $21 \times 30 = 630$ without doing another multiplication);
>
> Systematically exhausting possibilities (such as a 0 in the units digit × *anything* will produce a 0 in the units place, as will 2×5, 4×5 etc.)
>
> Making a generalization (considering the task as $[20 + x][30 + y]$, the units digit is 0 if $x = 0$ or $y = 0$ or xy is a multiple of 10).

We had hypothesized that if any of these were generated by the students, this would be evidence of mathematical reasoning. Our research questions were:

> Does a task such as this one, if supported by the recommended pedagogies, provide students with opportunities to do mathematics and to reason mathematically?
>
> Do students learn the underlying mathematics from the experience of engaging with or observing mathematical reasoning?

Results

The following report is of a single lesson taught by one teacher in a larger project. The intention is to provide insights into the relationship between the type of task, the type of pedagogies, and the responses of the students. The lesson was taught by the second author and observed by the first author. There are three sets of data presented: a videotape record of the lesson that is represented here in an interpretive summary; selected aspects of the written reactions of the teacher to the lesson; and data from the students.

The first set of data was a video record of the lesson that could be interrogated in detail. Basically the video record captured six distinct phases: the posing of the Learning task; students working on that task; students reporting on their explorations; posing of a Consolidating task; students working on that task; and students sharing solutions and strategies for the Consolidating task along with teacher comments. In both phases in which students reported on their explorations there was at least one student who “identified and used patterns”, another who “systematically exhausted possibilities” and at least one who attempted a generalisation indicating that such reasoning is possible. The students worked conscientiously throughout the 65 minute lesson, those who were asked to explain their approaches did so willingly and clearly, and students listened attentively to the explanation of others.

A second set of data is the written reactions of the teacher after the lesson. In reflecting on the lesson, and responding to the prompt “In what ways did the structuring of the lesson in that way give students opportunities to explain their reasoning?” the teacher wrote:

> Breaking down the lesson into a three part cycle, the Launch, Explore and Summary, is essential in providing students with opportunities to explain the reasoning. I refer to it as a cycle as it can occur once, or several times throughout the lesson, when a Consolidating task takes place, or whether you are simply ‘checking-in’ during the Explore phase.
>
> During each phase of the cycle the teacher is required to take purposeful actions in order to set up a reflective and meaningful discussion, which supports students to explain their reasoning in a safe and supportive environment (students need to know that they are accountable, that they will be expected to share and listen to others, but that you’re not setting them up for failure or embarrassment either.)
>
> One big aspect of setting up a reflective summary phase, involves changing teachers’ perceptions on what a ‘share-time’ discussion in maths looks like. It is not simply ‘show and tell.’ We are now moving from a ‘show and tell’ to not only having students explaining their reasoning, but having students listen, re-explain, build on, learn from and challenge other students’ thinking. However, in order to achieve this, teachers must monitor students carefully during each phase of the lesson.
>
> Two major parameters are concerned in supporting students to explain their reasoning. One is the ‘pure mathematical’ aspect, the second, the social aspect. What I mean by each, is that we want students to be able to articulate their thinking by connecting ideas and concepts and using correct mathematical terms and language, however, in order for this to occur, students must have developed an appreciation that they can in fact learn from their peers (not only their teacher.) In this way, we are able to develop a genuine community of learners.

These comments highlight the critical role of giving students time to engage with the task, establishing a classroom culture that allows reasoning to emerge, and specific pedagogies that facilitate the students' engagement with reasoning.

A third set of data is a summary of an analysis of the student learning. An important source of insights into the learning is their responses recorded on their worksheets. As with the video analysis, each of the hypothesised forms of reasoning was observed on the worksheets in this case many times. However, it is not meaningful to summarise student worksheet responses since the students can add to their worksheet while listening to the response of others and so such analysis would not quantify the extent of student reasoning.

One source of useful data was from an item on the pre- and post-test that was posed to all students in the project. The question, termed here *Item One*, was posed as follows:

> I did this multiplication correctly but my printer ran out of ink and one of the digits did not get printed.
>
> 22_ × 5 = 1140
>
> What might be the missing digit? (Student chose from options 2; 4; 6; and 8)

The results for all students in the project and this focus class in particular on both the pre- and post-test are presented in Table 1.

Table 1
Number of Students Correct (and Percentage) for the Overall Project and the Focus Class on Item One

Pre test whole group n = 1226	Post test whole group n = 847	Pre test focus class n = 22	Post test focus class n = 25
550 (45%)	468 (55%)	14 (64%)	18 (72%)

While more students in the focus class proportionally were correct than the overall group, the improvement in both groups is similar indicating that some students learned from the experience. Given the observation of extensive positive learning in the classroom, it is possible that the form of the test (on-line) or the timing resulted in it not being an accurate measure of knowledge.

As a delayed-post-measure, the item was posed again to the focus class on a paper version in test conditions. An additional prompt – "prove your answer" was added to the statement for Item One. Note that this item has only one correct answer. Also on the delayed test, an additional item, termed Item Two, was posed as follows:

> I did this multiplication correctly but my printer ran out of ink and one of the digits did not get printed.
>
> 2_ × _ = _ 6
>
> What might be the missing digits? Give as many answers as you can and explain what you did.

Note that this has a range of possible correct answers.

Three aspects of the responses of each student were coded. Table 2 presents the number of students correct from the focus class on this delayed post test. Noting that only a little over half of the 800 or so students in the overall project answered Item One correctly, that 28 out of 29 students were correct is a strong indication that the students in the class knew how the answer the question. Further, that 19 out of 26 (73%) gave all six possible answers to Item Two indicates that the mathematical point of the various possibilities has been accepted by the class.

Table 2
Responses of Students in the Focus Class to Delayed-Post-Test Items

	Item One	Item Two
No correct answers	1	0
1 correct answer	28	0
2 or more correct answers	n.a.	7
A complete set of answers	n.a.	19

The responses were also inspected and coded on the extent to which the students sought to communicate more than the answers. Table 3 presents the coding of their responses to both items.

Table 3
Coding of Clarity of Explanations to Delayed-Post-Test Items

	Item One	Item Two
Answers only with no attempt to explain	4	6
Answers with an attempt to explain but not clear	7	8
Correct answers and clear explanations given	17	13

Three quarters of students made some attempt to communicate their thinking, with close to half of the responses of students to Item Two being rated as clear. This is evidence that most of the students were willing to attempt to explain their thinking, a key elements of reasoning.

The responses of the students to Item Two were also coded on the type of reasoning that was evident in their response. The codes are presented in Table 4, noting that some responses were coded under more than one category.

Table 4
Types of Reasoning Displayed

	Item Two
Identifying and describing patterns	6
Exhausting possibilities (known facts)	24
Making a generalisation	0
No explanation	6

It is interesting that all of the explanations given were able to be coded in one of the three categories that we had anticipated. That none of the students sought a generalisation is no doubt a weakness in the item, in that there is a limited number of answers. It is also possible that seeing a generalisable solution takes more time than is available in a test context. Nevertheless we argue that systematically exhausting possibilities is a first step toward identifying and proving a generalisation suggesting that these students are engaging in reasoning.

Summary and Conclusion

Recognising the limitations in presenting data from just one lesson, the information presented illustrates that given an appropriate task that allows sustained engagement with the mathematical concept, a supportive classroom culture, and a teacher who structures lessons to prioritise student thinking, it seems that reasoning (in terms of the way it is defined here) is possible.

In terms of the research questions, the students did respond to the content items in ways that suggest they learnt the mathematics. It also seems that the process for learning the mathematics was connected to the opportunities for reasoning created by engaging with the task and either explaining to listening to student reasoning.

In terms of the assumptions, the suggestion that students can and should reason for themselves is affirmed. Likewise, the importance of giving students tasks which are in some ways complex and for which the responses take time is emphasised by these results.

References

ACARA (2011). The Australian Curriculum: Mathematics. Downloaded on July 23 from http://www.australiancurriculum.edu.au/Mathematics/Content-structure

Anthony, G., & Walshaw, M. (2009). *Effective pedagogy in mathematics*. Educational Series 19. Brussels: International Academy of Education; Geneva: International Bureau of Education.

Clarke David (2013). Unpublished comment included as part of a presentation to ACARA, October.

Dooley, T. (2012). Constructing and consolidating mathematical entities in the context of whole class discussion. In J. Dindyal, L. P. Cheng, & S.F. Ng (Eds). *Mathematics education: Expanding horizons* (Proceedings of the 35th conference of the Mathematics Education Group of Australasia, pp. 234-241). Singapore: MERGA.

Dweck, C. S. (2000). *Self-theories: Their role in motivation, personality, and development.* Philadelphia: Psychology Press.

Fraivillig, J. (2004). Strategies for advancing children's mathematical thinking. *Teaching Children Mathematics, 7,* 454-459.

Hattie, J. (2009). *Visible learning: A synthesis of over 800 meta analyses relating to achievement.* New York: Routledge.

Kelly, A. (2003). Research as design. *Educational Researcher, 32*(1), 3-4.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Marshall, J., & Horton, R. (2011). The relationship of teacher-facilitated, inquiry-based instruction to student higher-order thinking. *School Science and Mathematics,* 111.

National Council of Teachers of Mathematics (2000). Executive summary Principles and Standards for School Mathematic. Reston, VA: NCTM.

Rollard, R.G. (2012). Synthesizing the evidence on classroom goal structures in middle and secondary schools: A meta analysis and narrative review. *Review of Educational Research, 82*(4), 396-435.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematical discussions.* Reston VA: National Council of Teacher of Mathematics.

Stacey, K. (2010). Mathematics teaching and learning to reach beyond the basics. Paper presented as part of the ACER conference "Teaching mathematics: Using research-informed strategies". Camberwell, ACER.

Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies* (Australian Education Review, 59). Camberwell, Australia: Australian Council for Educational Research.

Sullivan, P., Mousley, J., & Jorgensen, R. (2009). Tasks and pedagogies that facilitate mathematical problem solving. In B. Kaur (Ed.), *Mathematical problem solving* (pp.17-42). Association of Mathematics Educators: Singapore / USA / UK World Scientific Publishing.

Watson, A, & Sullivan, P. (2008). Teachers learning about tasks and lessons. In D. Tirosh, D. & Wood, T. (Eds.), *Tools and resources in mathematics teacher education* (pp. 109–135). Sense Publishers: Rotterdam.

The Encouraging Persistence Maintaining Challenge project is funded through an Australian Research Council Discovery Project (DP110101027) and is a collaboration between the authors and their institutions. The views expressed are those of the authors. We acknowledge the generous participation of the project schools. We are working with colleagues Jill Cheeseman, Doug Clarke, Jim Middleton, Angela Mornane, Anne Roche and Carly Sawatski