Quantitative Relationships Involving Additive Differences: Numerical Resilience

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This case study describes the ways in which problems involving additive differences with unknown starting quantities, constrain the problem solver in articulating the inherent quantitative relationship. It gives empirical evidence to show how numerical reasoning takes over as a Grade 6 student instantiates the quantitative relation by resorting to guess-and-check trials. Although our study focuses on a single case study and a set of limited tasks, analysis of the data brings forth the necessity to give more explicit curricular attention to additive differences.

The solution to a mathematical problem requires the identification of the variables in the problem so that the relationships among them can be articulated. This is often facilitated when one can reason quantitatively (Thompson, 1993), i.e. make sense of the relationship(s) among quantities rather than working with particular values of the quantities. Going a step further, the solution to the mathematical problem becomes more accessible when the problem solver can reason algebraically, where variables can be defined symbolically to establish signified relationships. However, in the absence of algebraic tools or quantitative reasoning skills, one is more inclined to reason numerically by successively incrementing or decrementing particular guesses until the relationship in the problem is satisfied (Alsawaie, 2008; Nathan & Koedinger, 2000). The articulation of relationships among quantities becomes even more demanding when unknown quantities are involved in the problem.

As students transition from primary school to high school, they are called upon to extend their prior knowledge and skills in operating on numbers to a form of generalised arithmetic that is constitutively referred to as algebra. In this generalisation process, some researchers use the term pre-algebra to indicate situations where students have to articulate relationships between sets of numbers or measures rather than particular numbers or measures (Carraher, Schliemann, Brizuela and Earnest, 2006). This includes working with unknown quantities, beyond the instantiation of particular values. By instantiation of particular values, we mean assigning values to the quantities, instead of working with the relationships among the quantities in the problem.

This paper focuses on problems involving additive differences with unknown starting quantities. In problems involving additive differences, the difference between two quantities is specified rather than the individual values of the two quantities. For example, in Task 1 (see Table 1), the additive difference between the number of rolls made on Saturday and Sunday is given, that is "15 more rolls", rather than the quantities on Saturday and Sunday. The problems in this study originate primarily from the Singapore Primary School Leaving Examination (national examination) and the Australian National Assessment Program – Literacy and Numeracy. We would like to highlight that in Singapore, students are introduced to the model method (see Ng & Lee, 2009) to deal with the types of problems chosen in this paper. Essentially, the model method allows students (without symbolic algebra background) to solve problems involving unknown quantities.

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By representing quantities in terms of boxes, it aims at prompting students to establish quantitative relationships.

Research has not given sufficient attention to problems involving additive differences. Hence, the aim of this study was to trace the extent to which a six grader (who has no formal knowledge of algebra and the model method) could reason quantitatively in a specific set of situations involving additive differences between quantities. The following two questions guided the study: (1) How does a student without formal algebraic knowledge and skills handle problem situations involving additive differences? (2) What features of complex additively structured problems constrain the articulation of the inherent quantitative relationships?

Theoretical Framework: Quantitative Reasoning

We analysed the data using the quantitative reasoning framework (Thompson, 1990, 1994, 1995) on the basis of the nature of the tasks. The sixth grade participant in this study had not yet been taught formal algebra at school and we wanted to probe the extent to which she could articulate the relationships among the given quantities in the problem situations. Quantitative reasoning involves analysing the quantities and relationships among quantities in a situation, creating new quantities, and making inferences with quantities. We explain the three main constructs in Thompson's theory, namely quantity, quantitative structure and quantitative operation in relation to additive differences as they constitute the central focus of this study. We use Task 2 (see Table 1) to explicate our interpretation of the constructs. The two explicit quantities are Machine A and Machine B, and the measures of these two quantities are not known. The unit of measurement of these two quantities is the number of pages printed per minute or per three minutes, depending on how the problem solver decides to conceptualise the problem. The quantitative difference (Thompson, 1993) between Machine A and Machine B is the amount by which Quantity A exceeds Quantity B. In other words, the difference between two quantities is equally a quantity. Thus, the triplet 'Quantity A', 'Quantity B' and 'difference between Quantity A and Quantity B' constitutes a quantitative structure in that the difference between Quantity A and Quantity B can only have meaning with reference to Quantity A and Quantity B. The mental operation that allows us to conceptualise the difference between Quantity A and Quantity B as a new quantity is referred to as a quantitative operation. Thompson (1993) further prompts us to differentiate between a quantitative difference and a numerical difference. A numerical difference merely refers to the result of subtraction. The reader is referred to Ellis (2007) for an elaboration of the constructs used in Thompson's theory. Another construct that was useful in interpreting the data was *multiple identifications* (Thompson, 1995). It refers to the interpretation of the same quantity in relation to different referents. We show the application of this concept in the discussion section for Task 7.

Method

An in-depth understanding of the ways in which quantitative relationships are articulated necessarily requires a case study approach. This case study involved two task-based 35-minute interviews. To get rich data, two interviewers were engaged in synchronously following the respondent's answers as she solved the problem. The second interviewer focused the camera on what the student was writing on a moment-by-moment basis, asking supplementary and clarifying questions on the answers given to the questions posed by the first interviewer. Knowing that most of the problems chosen in this study were relatively demanding, we purposively chose a participant (identified by the pseudonym Pam) that had a record of above average achievement in mathematics at school. Pam is presently in Grade 6 and she attends advanced mathematics lessons in a high school every Wednesdays. Before asking Pam to work out the seven tasks in this study, we requested her to solve some preliminary questions to make a first assessment of her ability to make sense of additive differences. Her swift answers to the preliminary questions showed that she could clearly interpret comparative terms such as ‘more than’ and ‘times as many’. Pam also affirmed that she had not come across the types of problems we posed to her in this study. At school she uses a calculator and we allowed her to use it in the interviews. The first interview involved Tasks 1 to 4 while the second interview dealt with Tasks 5 to 7 (Table 1). In the following section, we present the data on a task-by-task basis.

Table 1
Set of Tasks

1. A baker made a total of 175 rolls on the weekend. She made 15 more rolls on Saturday than on Sunday. How many rolls were made on Sunday? (ACARA, 2009)
2. Every minute Machine A prints 12 pages more than Machine B. Machine A and Machine B together print a total of 528 pages in 3 minutes. At this rate, how many pages does Machine B print in 1 minute? (SEAB, 2010)
3. Siti started saving some money on Monday. On each day from Tuesday to Friday, she saved 20 cents more than the amount saved the day before. She saved a total of $6 from Monday to Friday. How much money did she save on Monday? (SEAB, 2010)
4. Lili spent 4 days making paper dolls for her friends. Each day she managed to make 2 paper dolls more than the day before. She made a total of 24 paper dolls. How many paper dolls did she make on the last day? (modified from SEAB, 2013)
5. Gilbert and Hazel have some postcards. After Gilbert gives 18 postcards to Hazel, he has 20 postcards more than her. How many more postcards than Hazel does Gilbert have at first? (SEAB, 2013)
6. In a school hall, chairs are arranged in rows such that there were exactly 9 chairs in each row. For a concert, Mr Ong brought 6 more chairs into the school hall and rearranged all the chairs. There are now exactly 7 chairs in each row and 12 more rows than before. How many chairs are in the school hall for the concert? (SEAB, 2013)
7. Hassan had 5 tins of marbles. At first, each of the tins contained the same number of marbles. He took 18 marbles from each tin. After that, the total number of marbles left in the 5 tins was equal to the total number of marbles in 2 of the tins at first. What was the number of marbles in each tin at first? (SEAB, 2013)

Data and Analysis

Pam’s Strategy to Solve Tasks 1 and 2: Working with Quantitative Differences

In Task 1, Pam readily established the relationship between the two quantities, i.e. rolls on Saturday and Sunday. She recognized that she had to remove the additive difference (15 rolls) to make the quantities on Saturday and Sunday equal as can be inferred from her statement: “If the baker makes 15 more on Saturday, so I minus 15 from 175. So I could divide. A weekend has two days. So 160 divided by 2 is 80.” She further explained the meaning of the intermediate quantity 160: “If she makes the exact same number both on Saturday and Sunday.” She interpreted the additive difference between the two quantities as a quantitative difference, as one entity.

The second problem further revealed Pam’s flexibility to articulate the relationship between two quantities when an additive difference is involved. Her strategy was to remove the additive difference between the two quantities so that each of them has the same value: “First, I got 12 times three because it's three minutes and every minute machine A prints 12 pages more than machine B. So 12 times 3 is 36. And then 528 pages is how many pages they print in three minutes. So, I have done 528 minus 36 which is how much more machine A prints than machine B in 3 minutes.” She cleverly observed that she had to remove three times the difference, i.e. (3 x 12). When we asked her what does the resulting amount, i.e. 528 – 36 = 492 represented, she immediately divided 492 by three to obtain 164 and in turn divided 164 by 2. Her justification “because that's for both machines in one minute without the 12 which is more”, showed that she could hold the quantitative relationship between the two quantities even when the additive differences were involved in multiplicity.

Pam’s Strategy to Solve Task 3: Systematic Guess and Check

Compared to Tasks 1 and 2, where she could readily work with the relation between the two quantities, in Task 3 she worked with instances of the quantitative relation by plugging in different values to verify if the relation held. She started by assigning zero cents on Monday (see Figure 1(a) left) and incremented successively by 20 cents for each day up to 80 cents on Friday. She then summed the amounts on her calculator to verify whether they add up to $6. Observing that this was not the case, she increased the starting amount on Monday to 20 cents and performed a similar procedure to end with 100 cents on Friday. Once again, she observed that the amounts did not sum to $6. Her next strategy was to increase the 100 cents on Friday to 120 cents and work backwards, i.e. by assigning 100, 80, 60, 40 to Thursday back to Monday. She continued the procedure until she obtained the sum of $6. She represented her final solution in Figure 1(a) (right).

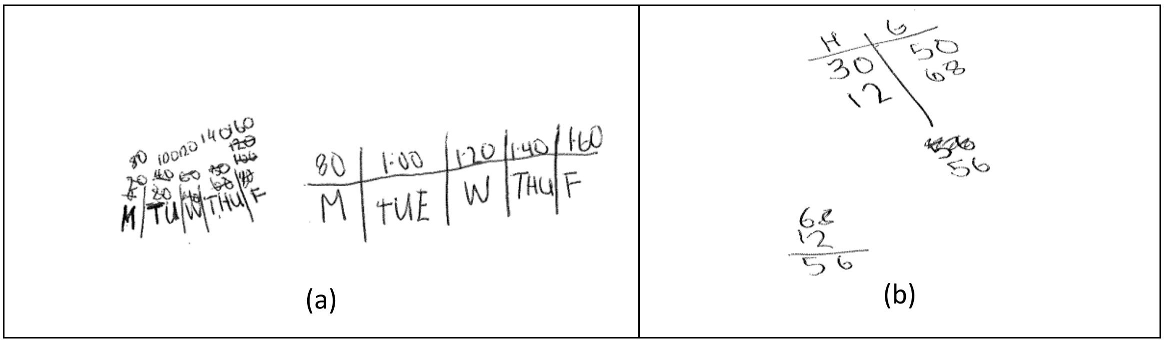

Figure 1. (a) Solution to Task 3, (b) Solution to Task 5

Pam’s Strategy to Solve Task 4: Removing Constant Additive Differences rather than Cumulative Differences

In Task 4, Pam’s strategy was to remove the constant additive difference between the successive days which, in her thinking, would make the quantities equal for the four days. She assumed that if Lili had made the same amount everyday, she would have made 18 (i.e. 24 – 6) paper dolls, where 6 represented 3 (days) × 2 (dolls). This inference prompted her to divide 18 by 4. The lack of divisibility led to a cognitive conflict: "That does not work". In the absence of any solution path, she started to plug-in numbers in the relations given in the problem. What was the missing element in Pam's thinking? We argue that she could not conceptualize the cumulatively increasing sum in the problem rather she focused only on constant additive difference. In other words, the constant difference between the number of dolls on two consecutive days was 2 but cumulatively the number of dolls was equally increasing by 2 on each day.

Pam's Strategy to Solve Tasks 5: Working with Instances of the Quantitative Relation

Pam was initially uncertain and she asked us to clarify the problem statement in Task 5. It appears that this uncertainty occurred because neither of the quantities had any specific values. She had to instantiate values to be able to conceptualise the relationship between the quantities. She mentioned that she randomly started with a guess: "I randomly started with 12 for Hazel. And then. If Gilbert gives her 18, she has 30 and then Gilbert has 20 more postcards than her, which is 50. So if I take away 18 from 30 from Hazel and add it to Gilbert, I got 12 for Hazel and 68 for Gilbert and then I took away 12 from 68." (see Figure 1(b)). In this problem, the measures of the two quantities (Gilbert and Hazel) were not given. Rather a difference relationship between the two quantities was given.

Pam's Attempt to Solve Task 6: Using a Diagrammatic Approach

She started the problem by making a diagrammatic arrangement of 9 chairs horizontally and 12 chairs vertically (see Figure 2(a) right). As she made her way to interpret the problem, she wrote 'after' and 'before' on the worksheet to denote the change in arrangement when more chairs were brought in. She attempted to 'move' the chairs from the 'before' situation to the 'after' situation (see Figure 2(a) left) to be able to conceptualise the problem visually. For instance, she crossed out 4 chairs in the 'before' column and added 8 chairs in the after column (apparently the 8 remaining chairs). She could not make much progress even though we gave her about 10 minutes.

Pam's Strategy to Solve Tasks 7: Further Numerical Resilience

Her determination to satisfy the relationships between the quantities using her strategic numerical trials was equally apparent in Task 7. From the interview excerpts and her inscriptions, we deduced that Pam was articulating the relationship $2x = 5(x - 18)$ in Task 7, although she did not signify this relationship algebraically. For instance, in one of her systematic trials, she assumed that initially there were 25 marbles (see Figure 2(b)) and multiplied that amount by 2 to get 50 marbles. Then she calculated the number of marbles remaining per tin by subtracting 18 from 25 and multiplied the resulting 7 marbles by 5 to get 35. She observed that the number of marbles in two of the tins (50) was greater than the number she obtained (35), notifying her action by writing $50 > 35$. This observation prompted her to further increase the initial trial. We hypothesize that she imagined a placeholder for the unknown quantity to input different values until the relationship held. It is more likely that she did not hold the relationship as concisely as in its algebraic form but worked with it on a part-by-part basis. The numbers that she skilfully substituted in her 'mental equation' served to instantiate the quantitative relations and gave her a foothold to think about the situation.

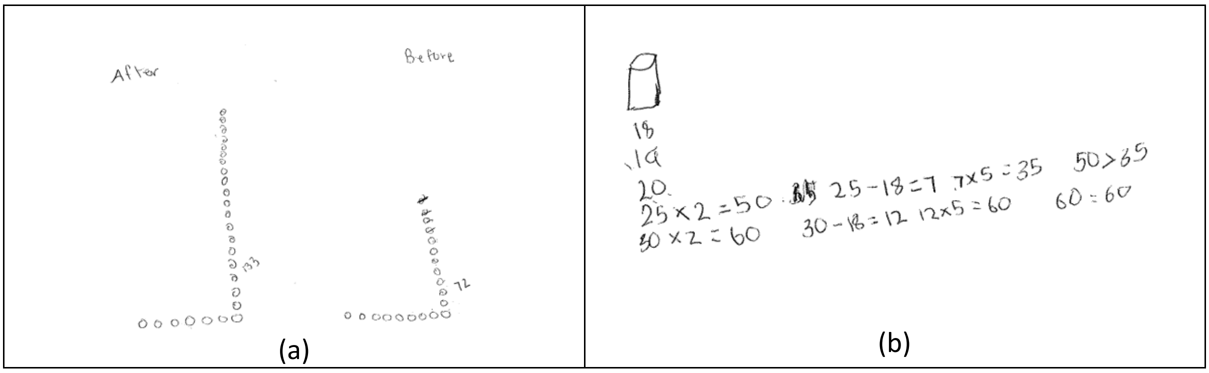

Figure 2. (a) Solution to Task 6, (b) Solution to Task 7

Discussion and Conclusion

In this study, we mapped the extent to which the problem solver could articulate the relationship among the quantities in the additively structured problems and the constraints that she encountered. We now attend to the two research questions which guided our study.

How does a student without formal algebraic knowledge and skills handle problem situations involving additive differences?

The tasks specifically selected in this study involve working with the difference between quantities, qualified by the comparative term 'more'. From Thompson's perspective, the conceptual use of the comparative term 'more' involves a quantitative structure, consisting of the referent quantity, the compared quantity and the difference between the two. In Tasks 1 and 2, Pam showed much flexibility to handle the quantitative difference between the two quantities in each of the respective situations. Her difficulties to coordinate multiple differences became evident in Tasks 3 and 4. In Task 3, she had to resort to numerical reasoning by plugging in a trial starting value as she could not articulate the quantitative structure between the referent quantity, compared quantity and difference, given that the starting referent quantity was unknown. She neither had algebraic tools nor was she aware of the model method to deal with unknown quantities. Such numerical fall back has been observed in previous research by Nathan and Koedinger (2000). In Task 4, she could not keep track of the cumulative increasing sum and this led her to fall back on the systematic guess and check strategy. In Tasks 5 and 7, failure to articulate the relations quantitatively once again led her to fall back on assigning trial values to the quantities, working with the numerical difference rather than the more demanding quantitative difference between the two quantities. Task 6 was the most demanding situation for Pam as she could not coordinate the exchange between the two quantities. The data reinforces previous findings (e.g., Alsawaie, 2008) that students tend to use their knowledge of numbers to cope with situations that they may not have encountered previously. Such numerical resilience compensates for the unavailability of quantitative reasoning or algebraic reasoning skills.

What features of complex additively structured problems constrain the articulation of the inherent quantitative relationships?

As argued by Thompson (1993), the complexity of the problem situation itself may impose much cognitive load such that it is demanding to reason quantitatively. The tasks in this study involve relatively complex relations. We comment on three features of the tasks that made them cognitively demanding from a quantitative reasoning perspective. We focus our discussions on Tasks 3 to 7, where observable constraints were displayed. The first aspect that is evident is the involvement of multiple constant differences, with unknown starting quantities in Tasks 3 and 4. Since the starting quantities in these problems are unknown and only the additive difference and sum are available, the problem solver is faced with the challenge of how to start the problem. A student equipped with algebraic tools can assign a placeholder (x) and work forward with the differences. Another ancillary aspect of Tasks 3 and 4 is that it requires the problem solver to keep track of the cumulative sum of the quantities.

Task 5 displays a second feature that makes quantitative reasoning demanding. Neither of the two quantities has a specified measure or value in this problem. What is available is a relatively complex additive relationship between the two quantities. The problem solver, without algebraic knowledge, is constrained in setting the relationship between the two quantities.

Task 6 can be interpreted as consisting of two scenarios. Initially, the number of chairs (i.e. 9) is known and the number of rows is unknown. After the addition of 6 chairs, the number of rows is the unknown (expressed in terms of an additive difference, 12 more rows) and the number of chairs is known (i.e. 7). We hypothesise that three inferences are important to be able to articulate the relationship quantitatively. First, in each row, in the 'after' situation, there are two chairs less than the 'before' situation. Conceptualising the difference between 'the number of chairs per row' in the 'before' and 'after' situation as one thing, as a quantity is the critical piece that opens the solution path and allows the problem solver to think about the relation quantitatively. Secondly, the 12 extra rows amount to 84 chairs. However, these 84 chairs equally include the additional 6 chairs that were brought in. The difference 84 – 6, i.e., 78 represents the difference in number of chairs in the 'before' and 'after' situation. Thirdly, realising that if there is a difference of 'two chairs per row' and there is a total of 78 chairs, then there are 78 ÷ 2, i.e. 39 rows.

We use the construct of *multiple identifications* from Thompson (1995) to explain how problem 7 could have been alternatively conceptualized. The number of marbles removed ($5 \times 18 = 90$) can be interpreted with respect to two different referents: (i) with respect to the initial five cans stated in the problem, (ii) with respect to the three cans (that can be complementarily deduced). The number of marbles removed ($5 \times 18 = 90$) is equivalent to the number of marbles in three of the cans at the beginning since the remaining ones fill only two of them. In other words, the challenging inference in this problem is the coordination between the quantity removed (5×18 marbles) and its equivalence to the number of cans with the original number of marbles.

This study enhances our understanding of the ways in which the articulation of quantitative relations can be demanding in additive situations. Although the study is limited to a single case study, our analysis of the interactions of the student with the problems is informative in terms of how a problem solver may be constrained to hold quantitative relations. It brings further light to previous work carried out by Thompson (1993) in relation to additive differences in terms of the conceptual challenges they may generate. The participant in the study had no difficulties in understanding the language used in the problems given, as she explained clearly what the requirements of each problem were, when prompted to do so. However, a limitation of the study is the narrow range of additive difference problems that were used to understand her reasoning.

Reflecting on the constraints that Pam encountered and the analysis of the problems from a quantitative perspective, prompt us to conjecture that instruction is necessary for students to develop fluency in articulating the relatively complex quantitative relations involving additive differences. Cognisant that it may not be intuitively appealing to identify a given quantity in relation to different referents in a problem situation, Thompson (1993) suggested that teachers ask students to identify what the various numbers given in problems represent via *instructional conversations*. In such conversations, students are also encouraged to explain their reasoning and pose questions about quantities and relationships about quantities. Importantly, such discussions provide opportunities for students to develop a disposition to think about "what one does with quantities' values in specific situations and with patterns of known information" (Thompson, 2011, pp. 42-43). Complementarily, teachers should provide more instructional attention to quantitative reasoning as a form of mathematical thinking. In addition to the above, we suggest that more explicit curricular attention be given to problems involving unknown starting quantities. A broader range of additive difference problems (including multiplicative comparisons) offers possibilities for further research.

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