Undergraduate Mathematics Students' Pronumeral Misconceptions

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Despite an emphasis on manipulative algebraic techniques in secondary school algebra, many tertiary mathematics students have mastered these skills without conceptual understanding. A significant number of students with high tertiary entrance ranks enrolled in first semester university mathematics were found to have misconceptions relating to pronumerals. School mathematics teaching at all levels must emphasise that pronumerals represent numbers, not objects, labels or abbreviations. Symbol manipulation must be balanced by problem-solving experiences so that the different roles of pronumerals either as variables, parameters, specific unknown numbers or generalised numbers have meaning for students.

Students' difficulty with algebra has been the subject of a considerable body of research conducted over recent decades. Much of the focus has been on junior secondary mathematics and the early transition to algebra. Important findings of this research have been misconceptions regarding the meaning of letters and understanding of variables in algebra (see, for example, MacGregor & Stacey, 1997). Fundamental misconceptions may persist, unnoticed in students who can successfully implement algorithms and manipulate expressions. For example, earlier research showed that a significant percentage of university nursing students held incomplete or incorrect conceptions related to decimal numbers (see Pierce, Steinle, Stacey, & Widjaja, 2008). The research reported in this paper addresses the question of whether university mathematics students may harbour fundamental misconceptions related to the meaning of letters in algebra.

We conjecture that despite a strong emphasis on manipulative algebraic techniques in secondary school algebra, some students may undertake university mathematics without appropriate conceptual understanding of the symbols they are using. When questions are presented in a non-standard format, it may be evident that these students have failed to understand the concept of a pronumeral standing in the place of a number, still believing, as many beginning algebra students do, that pronumerals represent objects, labels or abbreviations. Or perhaps further misconceptions regarding the role of pronumerals as generalised numbers, as variables, as parameters or as unknown numbers may be exposed when students are given questions in an unusual form. Such misconceptions inhibit the students' ability to develop problem formulation skills, since they do not fully understand the role of pronumeral assignment, for example, which quantity is variable and which is simply unknown. This is compounded by an evident assumption that the domain of all pronumerals is the real numbers, rather than a set that needs to be specified in each instance.

Background

Kieran (2004) describes algebra activities as falling into three groups: *generational*, *transformational* and *global/meta-level*. She notes that much of the early meaning-making

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of algebra occurs in generational activities; that is, forming of expressions and equations. For example, equations containing an unknown, expressions of generality arising from geometric patterns or numerical sequences and expressions of the rules governing numerical relationships. These early activities can be approached via a generalised arithmetic framework or by a functions framework, but "when generalised arithmetic is the underlying framework for generating and interpreting algebraic objects, the unknown takes priority over the variable, and expressions and equations tend to be viewed as representations of numerical processes rather than functional relations." (p. 24). Kieran further asserts that "in much algebra teaching, conceptual understanding of the objects of algebra has tended to be segregated from the development of manipulative skill." (p. 25).

By the upper years of secondary schooling, algebra activities tend to be transformational, with students engaging, for example, in factorising, expanding, substituting, and solving equations. Students' success in these activities depends largely on the acquisition of manipulative skills. Alongside these transformational activities, students may also engage in what Kieran terms global/meta-level activities; that is, activities for which algebra is used as a tool but which are not exclusive to algebra. For example, problem solving, modelling, justifying and proving.

Trigueros and Jacobs (2008) assert that students' confusion over the role of letters in algebra is not surprising due to a lack of precise mathematical definition of the word 'variable', which "has come to be a 'catch all' term to cover a variety of uses of letters in expressions and equations." (p. 3). Sometimes pronumerals may be used as variables related in a function, sometimes as unknown numbers to be found, and sometimes as generalised numbers, but it is common for teachers to use the term 'variable' for each of these cases.

Several resilient misconceptions pervade students' meaning-making associated with the use of pronumerals. There is the well-documented belief that a pronumeral is an abbreviation or label for an object, but even for those students who do understand that a pronumeral stands in place of a number, there is a failure to distinguish between a pronumeral that is standing in place of a generalised number and a pronumeral representing the value of a quantity that can vary. Further misconceptions are that each letter represents a unique number and that two different pronumerals cannot have the same value.

In their early experiences with algebra, many students develop the misconception that letters stand for objects, labels or abbreviations; for example, c stands for a cat and $5c$ stands for five cats (MacGregor & Stacey, 1997). Research has shown this to be a resilient misconception even at tertiary level. Rosnick and Clement (1980) investigated first year engineering students' abilities to translate English word problems into algebraic equations. They noted a common error pattern in very simple word problems which they called the 'reversal error'. They postulated that letters were seen as labels so that "There are six times as many students as professors at this university" was written algebraically by many as $6S = P$. Rosnick and Clement (1980) concluded that "the level of mathematical incompetence of these students is evidence for the shortcomings of an educational system that focuses primarily on manipulative skills" (p. 23). They believe that the answer lies in encouraging students to view an equation as an active operation on a variable that creates an equality, in contrast to "a static statement where the larger coefficient is associated simplistically and incorrectly with the larger variable". (p. 23)

An aim of the pilot research reported in this paper was to explore whether similar misconceptions relating to basic algebra occur in Australian undergraduate mathematics students who have completed at least one mathematics course that included calculus in their final year of schooling.

Method

This paper reports a pilot research study for which an online mathematics quiz (referred to as 'quiz') and a separate online background demographic survey (referred to as 'survey') were made available to first year mathematics and statistics students soon after the start of their first semester at a leading Australian university. Students' attention was drawn to the quiz and survey through emails and verbally by their lecturers. The students accessed both the quiz and survey through the university's Learning Management System. Students' participation in this research study was entirely voluntary and the quiz did not form a formal part of any subject they were studying. The quiz was time limited (35 minutes) and students were given one attempt only. Following the quiz a small number of students whose responses indicated that, despite their success in secondary school mathematics, they may habour persistent misconceptions were invited to take part in a 30 minute 'think-aloud' interview.

The full quiz comprised 16 questions, designed to probe students' understanding of both pronumerals and functions. Findings about students' understanding of functions have been previously discussed (Bardini, Pierce, & Vincent, 2013). The present paper focuses on questions that related to a basic understanding of the use of pronumerals (questions 13, 14, 15 and 16). Question 14 specifically targeted the 'reversal error' described above, while questions 13, 15 and 16 were designed to further investigate findings from Stephens (2005) that indicated students' belief that "when a letter represents a number, usually each letter represents a different number".

Results and Discussion

Of the approximate 2000 students who could have accessed both the quiz and the survey, 427 students answered the quiz and over 600 completed the survey. In both cases, not all of the students answered every question. The responses of the 383 students who attempted the quiz *and* answered the survey were analysed. Only four students agreed to be interviewed. Quiz questions 13 – 16 are presented here together with the response data for 383 students, the corresponding interview questions, and the interview responses of one student whose comments provide us with some insights. Student A had responded incorrectly to three of the four quiz questions 13 – 16. He had completed both an intermediate and advanced level, calculus based subject in his final year at school. At the time of interview he was enrolled in a Bachelor of Science degree and had just completed 'Calculus 1'.

Quiz Question 13

When is the equation $L+M+N=L+P+N$ true?

(a) Always

(b) Never

(c) Sometimes

Table 1 shows the percentages of students who selected each of the given choices, calculated as a percentage of the 383 students and as a percentage of those students who answered the question. 20% of the students did not answer the question and approximately 23% of the students who did believed that the equation was never true.

Table 1
Quiz Question 13 Responses

	% students $n = 383$	% respondents $n = 337$
Always	1.8	2.1
Never	20.4	23.1
Sometimes	65.8	74.8

In the Quiz, Student A's response was 'Never'. Although he responded correctly in the interview, his hesitance suggested uncertainty with his answer.

Interview Question 13

"Is it possible for the following equation $D+E+F=D+G+F$ to be true? Explain."

Student A: ... um ... only if E equals G.
Interviewer: OK.
Student A: … somehow, yeah ...
Interviewer: OK, so it's not always true?
Student A: Mmm.

Quiz Question 14

Let N be the length of the Niger River in kilometres and let R be the length of the Rhine River in kilometres. Which of the following says that the Niger River is 3 times as long as the Rhine?

(a) $N = 3R$
(b) $R = 3N$
(c) $R = 4N$
(d) $N = 4R$

Table 2 shows the percentages of students who selected each of the given choices, calculated as a percentage of the 383 students and as a percentage of those students who answered the question. About 13% of the students did not answer the question. Twenty-nine students (8.7% of those who answered the question) selected 'R=3N'.

Table 2
Quiz Question 14 Responses

	% students $n = 383$	% respondents $n = 333$
$N = 3R$	79.1	91.0
$R = 3N$	7.6	8.7
$R = 4N$	0.3	0.3
$N = 4R$	0.0	0.0

Interview Question 14

Let T be the number of teachers and S be the number of students at Hilltop College. Which of the following equations says that there is one teacher for every eleven students?

(a) $T = 11S$ (b) $S = 11T$

Student A selected option (a).

Student A: I think it would be (a).
Interviewer: Why do you think it would be (a) and not (b)?
Student A: So you want one teacher for every eleven students, so um, yeah, eleven times S, eleven students for one teacher … otherwise you'd assume there'd be more teachers than students.
Interviewer: So the second one would be saying that there are more teachers than students, is this what you're saying?
Student A: Yeah.

As observed by Rosnick and Clement (1980), Clement, Lochhead, and Monk (1981) and MacGregor and Stacey (1997), reversal of the pronumerals in this type of equation was not uncommon. Student A clearly articulated his conceptualisation of the situation: "eleven times S, eleven students for one teacher". Student A's flawed reasoning appears to match the word order matching strategy identified by Rosnick and Clement. Other students who reversed the order may have incorrectly reasoned that since there are more students than teachers, the coefficient 11 should be associated with the 'bigger variable', that is, the number of students. It would appear that many of the students had not internalised the concept that S represents the *number* of students and T represents the *number* of teachers, rather than mere labels. If this concept is understood, substituting values for the number of students, for example, $S = 5$, $S = 10$, would demonstrate the absurdity of the reversed equation.

Quiz Question 15

Some students were asked to find values of x that would make the following equation true: $x + x + x = 12$

Select each student whose answer is correct (Choose as many as apply).

(a) Mary wrote $x = 2$, $x = 5$ and $x = 5$

(b) Millie wrote $x = 9$, $x = 2$ and $x = 1$

(c) Mandy wrote $x = 4$

Table 3 shows the students' responses, calculated as for the previous questions. Forty nine students (about 13%) did not answer the question. Only 76% of students responded correctly, selecting choice c only. Thirty three students thought all three answers were correct, while seven did not select (c).

Student A responded correctly to this question. The students who believed that options a and b were correct have failed to understand the role of x in the equation $x + x + x = 12$, that is while x stands for a variable, within the one equation each separate appearance of x must stand for the *same* number. Had these students been presented with the equation in the form '$3x = 12$' or if they had been asked to simplify the equation first, no doubt they would all have recognised that $x = 4$ was the only correct answer. When students have been used to practising repetitive equation solving in their early algebra experiences, presenting them with a non-routine problem highlights the basic misconceptions which many students manage to carry with them.

Table 3
Quiz Question 15 Responses

	% students $n = 383$	% respondents n = 334
(a), (b) and (c) all correct	8.6	9.9
(a) and (b) only	1.3	1.5
(a) and (c) only	0.3	0.3
(a) only	0.8	0.9
(b) only	0.3	0.3
(c) only	76.0	87.1

Quiz Question 16

Some students were asked to find values of x and y that would make the following equation true: $x + y = 16$.

Select each student whose answer is correct (Choose as many as apply).

(a) John wrote $x = 6$ and $y = 10$

(b) Jack wrote $x = 8$ and $y = 8$

(c) James wrote $x = 9$ and $y = 7$

Table 4 shows the students' responses. About 13% of the students left this question unanswered.

Table 4
Quiz Question 16 Responses

	% students n = 383	% respondents n = 333
(a), (b) and (c) all correct	65.5	75.4
(a) and (b) only	0.3	0.3
(a) and (c) only	18.8	21.6
(a) only	1.0	1.2
(b) only	1.3	0.5
(c) only	0	0

Interview Question 16

Some students were asked to find values of x, y and z that would make the following equation true: $x + y + z = 36$.

Select each student whose answer is correct, choosing as many as apply.

(a) $x = 6, y = 15, z = 15$

(b) $x = 8, y = 8, z = 20$

(c) $x = 7, y = 14, z = 15$

(d) $x = 12, y = 12, z = 12$"

Student A at first stated that (a), (b) and (c) were correct, but when he came to option (d), he suddenly became confused and queried whether x, y and z could have the same value. His confusion then gave rise to uncertainty over options (a) and (b):

Student A: OK, (a) would be correct because you end up with thirty-six.
Interviewer: Mmm.
Student A: Ah, yeah, (b)'s correct.
Interviewer: OK
Student A: … Yeah, (c)'s correct.
Interviewer: Mmm.
Student A: And yeah, (d) would be correct as well … oh … hang on, they're different [letters], can they equal the same thing? … um …
Interviewer: So what is the problem there?
Student A: Ah, because *x* equals twelve, *y* equals twelve, *z* equals twelve … I suppose they can all equal the same number …
Interviewer: Ah, so you're saying definitely (c) because they're all different? So because they're different letters they must be different numbers?
Student A: Yeah, I think that's it, isn't it?
Interviewer: So, only (c) is correct?
Student A: Ah … I think so …
Interviewer: Could you find another possible triplet that would be correct as well? Other values?
Student A: Ah … ah … yeah … four, twelve, twenty.
Interviewer: So as long as the three of them are different, is that what you're looking for?
Student A: I think so. [laughs]

The erroneous belief that each pronumeral stands for a different number is yet another misconception that has remained robust for many students since their earliest algebra experiences, resulting in the belief that two different pronumerals in an algebraic expression cannot have the same value.

Conclusion

Despite the difference in schooling systems and the decades that separate our study from previous studies, for a significant number of first semester undergraduate mathematics students in this study there are similar misconceptions relating to understanding of the use of pronumerals. At upper secondary and tertiary levels, these students will be exposed to increasingly sophisticated problem-solving tasks where they will be expected to assign pronumerals to different aspects of a given problem. So an understanding of what quantities are varying, constant or unknown is critical. They will also be expected to determine a domain for the variables which is appropriate to the context. Students will start to study relationships between variables (functions) and how the variables change in relation to each other (calculus). Hence a deep understanding of different types of pronumerals and how to define them is vital.

The proportion of students with these misconceptions in our study is not large, but nevertheless of great concern, particularly as some of these students are likely to become teachers of secondary school mathematics. It should be kept in mind that these are students who have successfully passed through the school system with sufficiently high scores in their final year of school to be accepted into first year mathematics at a high profile university. It is not unreasonable then to hypothesise that if the study were conducted in first year tertiary mathematics units across the country, the proportion of students with these misconceptions would be much higher.

These findings highlight the importance of diagnostic testing and the use of carefully constructed non-routine questions to uncover students' incorrect or incomplete conceptual understanding of mathematical ideas. Suitable, research informed tests are now readily available 'online' to teachers and their students, see for example Specific Mathematical Assessments that Reveal Thinking (www.smartvic.com).

It seems that important work has still to be done in the classroom to ensure that teachers clearly define the different roles of pronumerals: as unknown numbers, as variables, or as generalised numbers. It is also essential that symbol manipulation is balanced by problem-solving experiences so that these different roles of pronumerals have meaning for students.

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