Supporting Students to Reason About the Relative Size of Proper and Improper Fractions

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Fractions are a well-researched area; yet, student learning of fractions remains problematic. We outline a novel path to initial fraction learning and document its promise. Building on Freudenthal's analysis of the fraction concept, we regard *comparing*, rather than *fracturing*, as the primary activity from which students are expected to make sense of fractions. Analysing a classroom design experiment conducted with a class of 14 fourth grade pupils, we identify two successive mathematical practices that emerged in the course of the experiment and indicate how their emergence was supported.

In this paper, we analyse findings from a classroom design experiment aimed at supporting fourth grade students' understanding of fractions as numbers that quantify *relative size* (Thompson & Saldanha, 2003). We focus on the second part of the experiment, in which we were successful in supporting students' reasoning about fractions as numbers that quantify magnitude values that can be smaller than, as big as, or bigger than one. This kind of reasoning is seldom expected from novice fraction learners, as it has been widely documented that conceiving a fraction as a number that accounts for a quantity that is bigger than one (i.e., a whole) can present a major conceptual challenge (Steffe & Olive, 2010).

In the experiment, we tested an instructional approach in which students were never oriented to relate fractions to the equal partition, division, or segmentation of a whole—as it is typically done. Instead, building on Freudenthal's (1983) insights about fraction as comparer, we engaged students in tasks in which the entities that fractions quantify were always separate from the reference unit.

Theoretical Background

Much of the research on fractions adopts a cognitive perspective on learning (Lamon, 2007; Post, Carmer, Behr, Lesh, & Harel, 1993; Steffe & Olive, 2010; Tzur, 1999), where the primary focus is on understanding (and modelling) the learning processes. For instructional design purposes, we found it useful to approach learning from a situated perspective and view it as changes in the forms of students' participation in classroom mathematical practices (Cobb, 2003). We interpret learning as being shaped by means of support, which therefore constitute the explicit focus of our research.

This particular perspective made us aware of another commonality among otherwise diverse studies on fraction learning: the instructional tasks used almost exclusively fall within what Freudenthal (1983) characterises as *fraction as fracturer* situations, where a whole, often a food item, is being cut or split into equal-sized parts. We elaborate elsewhere (Cortina, Visnovska, & Zuniga, 2015) how these types of instructional support result in fraction images that are counterproductive to developing mature understanding of fractions.

Our instructional approach is based on a different type of situations that call for fractions use. In these situations, fractions are used to compare aspects (e.g., lengths) of "objects which are separated from each other or are experienced, imagined, thought as

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such" (Freudenthal, 1983, p. 145). Understanding whether these *fraction as comparer* situations can effectively support student learning is the focus of our research.

Methodological Approach

The classroom design experiment was conducted in a fourth grade classroom in a public school serving low-income students in southern Mexico. The classroom consisted of 14 students, ages 9 and 10. The experiment included 13 instructional sessions, each lasting about 90 minutes. A set of individual pre- and post- interviews was conducted with all the students to document the individual learning. The sessions and interviews were video recorded. In addition, all student work was collected, and a set of field notes was kept.

The design experiment consisted of three phases: planning, classroom experimentation, and retrospective analysis (Gravemeijer & Cobb, 2006). During the planning phase, a hypothetical learning trajectory (HLT) was formulated. In it, we conjectured that it would be possible to support students, early on, to make sense of unit fractions as numbers that account for the relative size of things that are separate from a reference unit; for instance, the length of a rod relative to the length of a unit of measure (see Figure 1).

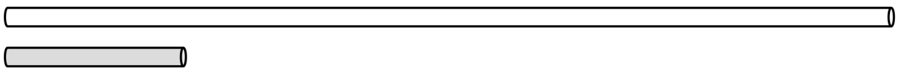

Figure 1. A reference unit and a rod that is $^1/_5$ of its length.

In addition, students would reason about the relative size of unit fractions, primarily, in terms of how many iterations of their size would be necessary to produce the size of one. Hence, a $^1/_5$ rod would have a length such that it would be necessary to iterate it five times to obtain a length as long as the reference unit (see Figure 2).

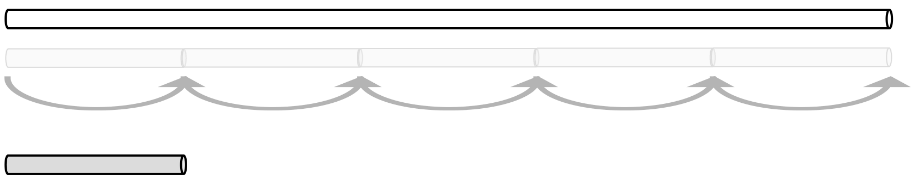

Figure 2. A fifth as a rod of such a size that five iterations of its length are necessary to obtain the length of the reference unit.

The second phase consisted of the actual experimentation in the classroom, and of conducting an ongoing analysis of the student learning. The ongoing analysis served to assess and adjust the HLT in light of ongoing classroom events.

In the final phase of the design experiment, a retrospective analysis of the actual learning trajectory undertaken by the students was conducted, with the benefit of hindsight. We analysed the data using an adaptation of constant comparative method described by Cobb and Whitenack (1996) that involves testing and revising tentative conjectures while working through the data chronologically. As new classroom episodes were analysed, they were compared with conjectured themes and categories, resulting in a set of the theoretical assertions that remained grounded in the data. Given the scope of this paper, we include representative episodes and interactions, where possible, as we build our argument. The viability of the ongoing analysis was revised to account for how the mathematical activity actually evolved in the classroom. This retrospective analysis resulted in reformulation of the HLT, so that the emergence of two identified mathematical practices would be explicitly supported in subsequent iterations of the design.

We now summarise the first mathematical practice (Cortina, Visnovska, & Zuniga, 2014), which involved reasoning about the relative size of unit fractions in ways consistent with what Tzur (2007) called *the inverse order relationship among unit fractions*. Hence, when comparing two unit fractions (e.g., $^1/_7$ vs. $^1/_{10}$) all of the students came to consider the one with the smaller denominator ($^1/_7$) as the one quantifying the bigger size. We then turn to the main focus of this paper—the second mathematical practice—where students could reason about fraction comparisons.

First Mathematical Practice: The More Times it Fits, the Smaller it has to Be

As we have elaborated elsewhere, the first mathematical practice emerged between days 1 and 4 of the design experiment. At the beginning of the instructional intervention, most of the pupils reasoned about the relative size of unit fractions following what Baroody (1991) called *the magnitude comparison rule*. They regarded unit fractions represented by numbers that would come later in the counting sequence as always accounting for larger sizes. Hence, a tenth would represent, for the students, a bigger quantity than a seventh.

Central to the instructional activities with which we helped students make sense of how big numbers can sometimes account for small sizes, was a narrative about how ancient Mayan people measured. The students were presented with a measuring stick (24 cm long) and told that that some archaeologists believed that ancient Mayans used this stick as a tool for measuring lengths. Students were then each given a replica of the stick and were asked to use it to measure the lengths of different things. This activity served to raise a question of how to account for the lengths that the stick did not cover exactly. On day 3, students were presented with the solution that the ancient Mayans could have come up with to systematically and precisely account for such lengths. It involved producing *smalls*: rods of a specific size relative to that of the length of the stick.

Each student then engaged in producing the smalls by cutting plastic straws. For the *small of two*, students were told that its length needed to be such that when used to measure the stick, the measure would have to be exactly two (i.e., a rod $^1/_2$ as long as the stick). Pupils made their small of two, with teacher guidance, by iterating a straw along the stick and adjusting its length. It took about 15 minutes for all the students to produce their small of two. Students were then told that the *small of three* would have a length such that it would fit exactly three times along the stick. Before making it, the teacher briefly discussed with the students if they expected the small of three to be longer or shorter than the small of two. A similar process was followed to produce the smalls of four, five, and six. Then, students were given leeway to produce more smalls, until the session ended. Some made as many as ten.

The activity of producing the smalls (unit fractions), and reasoning about their relative size helped the students develop imagery that was consistent with the inverse order relation (Cortina, et al., 2014). By day 5, pupils made sound comparisons between the sizes of smalls, even if they had not physically seen them. For instance, they regarded a small of 14 as being necessarily bigger than a small of 20. They also seemed to have a clear image of what the size of a small might be, so that when asked about the size of a *small of a hundred*, they responded that it would be very small, and gestured with their hands and fingers to show a tiny length.

Second Mathematical Practice: Reasoning about Fraction Comparisons

The second mathematical practice involved reasoning about fractions as representing lengths that could be either smaller than, as big as, or bigger than the reference unit. These were initially the length of paper-strips that were actually measured by the students, using the smalls. For instance, they could be the length of a paper-strip that was four times as long as the length of the small of three (i.e., $^4/_3$ as long as the stick). Later on, they were presented only as written measures, expressed with conventional fraction notation: $\frac{a}{b}$

The following excerpt from day 11 is representative of students' reasoning at this point of the design experiment. The teacher wrote the fractions $^{99}/_{100}$ and $^5/_5$ on the chalkboard using conventional notation. Several students raise their hands to answer. The teacher pointed at Lourdes.

Lourdes: Five smalls of five is bigger because ninety-nine smalls of one hundred is smaller.

Teacher: And why is that?

Lourdes: Because the bigger the number is it has to be smaller (gesturing with her hands a tiny size) so it fits.

Teacher: But ninety-nine is a lot, no?

Lourdes: Yes, but it needs to be small to fit in the stick.

Carlos: (jumping in) and there is not enough to fill it.

Teacher: Marisol?

Marisol: I think that five smalls of five is bigger because ninety-nine smalls of one hundred is smaller because it is not enough to fill the stick.

Teacher: It is not enough to fill the stick. Carlos?

Carlos: Ninety nine smalls of one hundred is not going to be enough to fill the stick because it is missing one small for it to be one hundred smalls of one hundred, and five of five do fill the stick.

This excerpt depicts several important aspects of students' reasoning in the second mathematical practice. First, it shows how, following what pupils had done in the first mathematical practice, the denominator of a fraction was construed as the length of a rod, relative to the length of the reference unit. Lourdes' comment about the smalls of one hundred being little, illustrates this point. As for the numerator, it was interpreted as a number that accounted for iterations of the length of the smalls, which accumulated into a length. Carlos' comment about 99 smalls of one hundred not being enough to fill the stick is illustrative of this second point.

In Lourdes' responses above, it is possible that she was only taking into consideration the relative size of the smalls involved, and not how many times each small was iterated. This kind of reasoning had emerged several times in the classroom. However, each time it was treated as inadequate or incomplete by the class. In this instance, Carlos decided to jump in and add the important missing facet of the argument. Over time, instances of reasoning about *relative size of smalls only* faded out.

The excerpt also shows how students first came to assess the relative size of a fraction in terms of it representing a length that was enough, or not, to *fill* (cover) the length of the reference unit. It is hence worth highlighting that students were not comparing the fractions relying on numeric facts and patterns (e.g., in the first fraction, the numerator was smaller than the denominator). Instead, they were comparing them quantitatively.

Importantly, the second mathematical practice was not limited to the realm of proper fractions. For instance, in the same session, a few minutes before the conversation above took place, students were asked to compare $^{12}/_{13}$ with $^{6}/_{5}$. All but two of the students chose the latter fraction as the one expressing the bigger length, and their justifications of this choice were mathematically sound. This is how one of the students justified his choice:

Eduardo: Because you need thirteen smalls of thirteen to fill the stick, and with twelve it's not enough. And in the other you need five, but they are six and it even goes further.

This contribution illustrates how, once the second mathematical practice was established, students easily construed both proper and improper fractions as numbers that soundly accounted for the size of a length. By using the comparer approach to fraction instruction from the outset of the design experiment, we had oriented pupils to construe the entities that unit fractions quantify as being separate from the reference unit and, thus, susceptible of being iterated *unrestrictedly*. For the students then, there was no natural boundary (e.g., the length of the unit whole) limiting the extent to which a small could be iterated. The iteration of a small of five ($^{1}/_{5}$) more than five times did not become, at any point of the design experiment, a troublesome issue for any of the students.

Supporting the Emergence of the Second Mathematical Practice

The second mathematical practice we just described emerged from the previous one. The retrospective analysis revealed that two shifts in student reasoning were critical in the emergence of this practice and required supporting: students first needed to come to view the smalls as capable-of-being-iterated measurement units in their own right. The second shift involved students coming to make sense of a new representation introduced by the teacher (see Figure 3) as actually representing the iterations of the smalls.

In the HLT we formulated during the planning phase of the design experiment, we conjectured that the activity of producing the smalls would rather easily lead students to make sense of the equivalence of multiples of unit fractions with *one*. In other words, we conjectured that students would somewhat effortlessly recognise that two iterations of the small of two, $^{2}/_{2}$, would render the same length as three iterations of the small of three, $^{3}/_{3}$, four iterations of the small of four, $^{4}/_{4}$, and so on. During the design experiment, we came to realise that, for the students, making sense of this basic equivalence was not trivial. The following excerpt illustrates how students were thinking about the smalls on day 5.

Teacher: Carlos, how long is the small of three?

Carlos: It has to measure three times that stick… the straw has to measure three times that stick. Until it gives you three.

It is worth noticing that Carlos used the expression *to measure* to describe the act of iterating a straw along the stick. This use of the expression sounds strange in English and in Spanish. Nevertheless, students commonly used it in this way, at this point of the design experiment. Carlos seemed to construe iterating, essentially, as a means to gauge and fix the length of a small. This should come as no surprise, since this is how iterating was used in the activity of producing the smalls. What was initially surprising to us was that even after the accurate smalls were produced, students did not automatically come to see them as units of measure in their own right. Instead, the smalls initially represented to them *only* the result of the construction process.

Aiming to help the students reason about the equivalencies between iterating the length of the smalls a certain number of times, and the length of the stick, we provided the students with a *Measurement Kit* (see Figure 3), which included a stick and four smalls (wooden rods representing 1, $^1/_2$, $^1/_3$, $^1/_4$, and $^1/_6$), and a printed sheet. The sheet had five bars the length of the stick, four of them segmented to match the sizes of the smalls. We conjectured that the sheet would become a useful a resource for reasoning about equivalence and inequalities with *one* (e.g., $1 = {^4/_4}$, $1 > {^5/_6}$) and with other fractions (e.g., ${^5/_6} < {^4/_4}$).

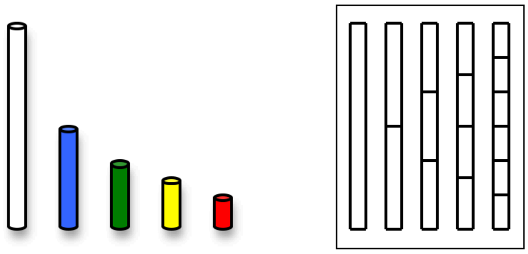

Figure 3. The *Measurement Kit* included a white stick (24 cm), four rods (blue, 12 cm; green, 8 cm; yellow, 6 cm; and red, 4 cm), and the printed sheet. Colours and sizes of rods correspond to plastic straw smalls.

When we first engaged students in activities aimed at supporting them to reason about the relative lengths produced by iterating the smalls, we noticed some unanticipated complications. The first was that in the new type of activities, when the students started to use the rods as a means to measure, they seemed to approach them as if they were independent. They did not reason with the fact that the smalls were produced from the same stick. As a consequence, students would not consider that the specific number of iterations of each small would have to necessarily render the length of the stick. For instance, they would not anticipate that a paper strip that measured two smalls of two would necessarily also have to measure three smalls of three.

The second complication, related to the first one, was that students did not readily regard the printed sheet as a useful resource for determining equivalencies between measures made with smalls. By and large, when it was first introduced, the pupils did not see the sheet as record of the iteration of the smalls, relative to the length of the stick.

It was through engaging students in activities that involved measuring paper strips of different sizes, using different smalls, and by constantly referring them to the sheet, that we eventually succeeded in helping the students recognise the equivalent relation between the iteration of each small and the length of the stick. As we illustrated above, by day 11, most of the students could make correct comparison between the sizes of two fractions, using the equivalence with the stick as a benchmark, even between fractions whose denominators they had not physically produced.

In the finial interviews, it was apparent that all of the students could do correct comparisons between fractions, using the equivalence with the stick as a benchmark. Four of them could do so only when encouraged by the interviewer to reason about the fractions as numbers that accounted for the iterations of smalls, and to reflect whether the outcome of the iteration would produce a length equal to that of the stick. The remaining ten students could do the comparisons rather easily, and could explain their answers in ways similar to Marisol, Carlos, and Eduardo in the excerpts presented above.

In the retrospective analysis we realised that the complications we faced were the result of shortcomings of our original instructional design. On the one hand, we should have provided students with activities that would have allowed them to more directly recognise and reason about the equivalence between iterating the smalls and the length of the stick. On the other hand, we should have introduced the printed sheet in a way that would have allowed students to construe the segmentations on the bars as marks left by the iteration of the smalls more easily. These realisations formed the basis for our revisions of HLT.

Discussion and Conclusions

Student learning documented above is not currently typical in mathematics classrooms. The two mathematical practices that emerged in the classroom with novice fraction learners, within the three weeks over which the design experiment took place, correspond to overcoming the two *developmental hurdles* in fraction learning that Norton and Hackenberg (2010) identified in their review of research in the field. We take the relatively smooth emergence of these practices as an indication of the potential of the tested instructional approach.

The presented analysis of the actual learning trajectory helped us to understand how the emergence of the two classroom mathematical practices was supported in the classroom design experiment, and which forms of student reasoning were crucial to the emergence of these practices. The design research cycle would not be complete without the formulation of the new, revised, HLT that would present a starting point in the next iteration of testing and refinement of instruction. With the hindsight we gained through the analysis, the revisions would include the following:

1. The students did not automatically come to see the reciprocal relation between the size of a small and the size of the stick, as a result of the process by which the small was produced. However, students can be supported to come to see smalls as units of measure in their own right, for instance by engaging in activities, in which they use smalls to construct strips of paper of the pre-determined length, such as $^{3}/_{5}$, $^{5}/_{5}$, or $^{7}/_{5}$.
2. The Measurement Kit sheet did not initially have any history for students and we struggled in supporting them in creating meaning for it and using it effectively. With the hindsight, we would now have students construct this sheet in a series of activities, rather than providing the ready copy to them. We collected some informal indications that this approach is superior.

Our understanding of the shortcomings of our initial design conjectures that led to these revisions constitutes the key theoretical contribution within the type of research we conduct (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). It is reasonable to expect that in the upcoming design iterations, our (and others') improved understanding of how specific means of support shaped forms of student reasoning (including their confusions) will lead to a more effective design. This is the pathway along which we can envision that understanding of fractions as numbers that quantify relative size would become possible for all students.

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