A rich assessment task as a window into students' multiplicative reasoning

Ann Downton
Monash University
<ann.downton@monash.edu>

Vince Wright
Australian Catholic University
<vince.wright@acu.edu.au>

This study explored the potential of a rich assessment task to reveal students' multiplicative thinking in respect to a hypothetical learning trajectory. Thirty pairs of students in grades 5 and 6 attempted the task. Twenty-two pairs applied multiplicative structure to find the number of items in arrays. However counting and computational errors resulted in a success rate of less than 50%. The rich task provided valuable data about students' strategic choices and their need to develop computational fluency.

Rich Assessment Tasks

Through assessment educators signal to their students what they value (Clarke, 1997). Contemporary views about what it means to 'do mathematics' value a broadening of student activity from the performance of routine procedures, to include reasoning, flexibility, problem solving, making connections and the development of a productive disposition (Kilpatrick, Swafford, & Findell, 2001). Students' development of mathematical processes requires opportunities to work on cognitively demanding tasks (Clarke Roche, Cheeseman, & van der Schans, 2014), described herein as rich mathematical tasks.

Rich mathematical tasks were originally defined by Ahmed (1987) as possessing several critical characteristics including; intellectual quality, extended engagement, opportunities for collaborative work, multiple entry points and solution strategies, connectedness and affordance for multiple representations. Much is written about the complex interaction between rich tasks and teachers' practice in the development of learning opportunities for students (Stein, Grover, & Henningsen, 1996; Sullivan, Askew, Cheeseman, Clarke, Mornane, Roche, & Walker, 2015). In our work we investigated the usefulness of a rich task for assessment purposes. That is, to establish students' strategic preferences with reference to a learning trajectory for multiplicative thinking.

Progression in Multiplicative Thinking

The construct of a hypothetical learning trajectory (HTL) was first proposed by Simon (1995) who saw a trajectory as a composite of teacher goals, a conjectured growth path in the target mathematical concept, and aligned activities. The meaning of HTL is often narrowed in critique to the conjectured growth path, with arguments against deterministic linear progression (Lesh & Yoon, 2004). However trajectories are usually developed through large scale teaching programmes, in-depth small-scale case studies, or design experiments. In recent work on HLTs, assessment tasks, growth paths and learning opportunities are aligned (see for example, Clements, Sarama, Spitler, Lange, & Wolfe, 2011). In a recent review, Sztajn, Confrey, Holt Wilson, & Edgington (2012) argued strongly for HTLs being at the centre of instructional design and the need for more coordination of research based approaches to their development. For the purposes of this paper we take the narrower view of HLTs as conjectured conceptual growth paths, with a focus on multiplicative thinking.

A considerable body of research evidence points to a degree of consistency in the way students develop multiplicative thinking, though there is divergence in the language used by researchers, the grain size with which they describe progression and views on how thinking develops (Downton, 2013; Wright, 2011). Essentially multiplicative thinking involves an increasingly co-ordinated transfer of count. Composite units of singletons, and the counting of those units, are co-ordinated into a binary operation that, in turn, is reversible (Boulet, 1998; Davydov, 1992; Steffe, 1994). So a trajectory of broad stages involves progression from unitary counting (one by one) to counting of composites (skip counting and repeated addition) to binary operation. Figure 1 summarises this trajectory, citing major contributions from a range of scholars.

Stage Researchers	Count-all	Composite counting Repeated addition			Known product Derived fact	
Kouba, 1989	Direct representation (with physical objects)	Additive	Transitional counting		Recalled number facts	
Anghileri, 1989	Unary — Unitary counting	Rhythmic counting	Number pattern		Binary — Known fact	
Steffe, 1994	Initial number sequence	Tacitly nested number sequence			Explicitly nested number sequence	
Lemaire & Siegler, 1995	Counting set of objects	Repeated addition			Retrieval	Rapid responses
Lefevre, et al., 1996		Repeated addition	Number series		Retrieval	Derived fact
Mulligan & Mitchelmore, 1997	Unitary counting (direct counting)	Repeated addition			Multiplicative calculation	
		Rhythmic counting	Repeated adding, additive doubling	Skip counting	Known multiplication fact	Derived multiplication fact
Sherwin & Fuson, 2005	Count-all	Additive calculation	Count-by		Learned product	Pattern-based, Hybrid

Figure 1. Trajectory for multiplicative thinking (Wright, 2011, p. 37)

Some caution is needed when interpreting the trajectory. The studies mainly used tasks involving single digit factors. A trajectory for multiplicative thinking must also include the relationship between multiplication and division (Thompson, & Saldanha, 2003). Furthermore, Mulligan and Mitchelmore (1997) found that students operated at varying stages for multiplication were dependent on access to manipulatives, problem type and size of the numbers involved. Similarly Sherin and Fuson (2005) argued that students' performance on any given task was also dependent on access to number specific knowledge resources. Given the variety of problem types to which multiplicative thinking can be applied (Greer, 1992) and the complex unit structures involved (Vergnaud, 1994), at best the trajectory might describe students' preferential tendency with an expectation of variability given different tasks.

Work still needs to be done on progression beyond the *Known Product Derived Fact* stage in trajectory above. Jacob and Willis (2003) suggested a further stage labelled

Operating on the Operators at which students treat factors as variables and look for multiplicative relationships. However, students take considerable time to develop a mature sense of when to apply multiplicative relationships appropriately, and frequently confuse additive and multiplicative situations (Van Dooren, De Bock, & Verschaffel, 2010). We investigated students' responses to a rich assessment task with a view to gaining insight into their preferential stages from the trajectory.

Method

The students who provided data for this study came from four Year 5 and 6 classes at a State primary school in North-West Melbourne. There were twenty-eight non-English languages spoken at home by parents and caregivers, indicative of the diversity of ethnic groups in the community. The school was selected for extended in-class support and professional learning of teachers because its leaders responded to an expression of interest. The school had also received funding in the previous year to work on improving mathematics outcomes for students through a cluster model with other schools. The work samples presented in this paper were those willingly provided by the students, in line with the ethic protocols of the study.

The samples come from four similar lessons taught by one of the researchers on a single day. The lesson was based on a humorous adaptation of *The Enormous Turnip*, a traditional fairy tale (Wright, 1996). In the adapted story the old couple create a competition in order to give away the massive turnip. The competition involves finding clever ways to count the number of small turnips in a given patch (Fig. 2). Photocopies of the patch were provided to students so they could record their work in any way they wanted. The students worked collaboratively in pairs and were invited to confer with other pairs once they had established a count themselves. Our interest was in the usefulness of the samples for assessment. In particular to see if the samples reflected a trajectory for multiplicative thinking.
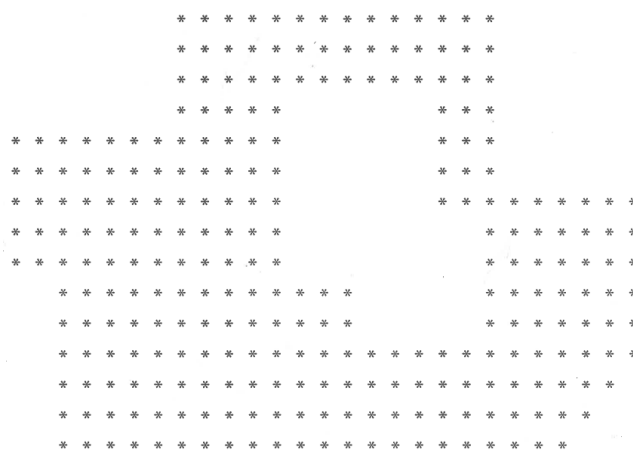
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Figure 2. The turnip patch.

Results

The problem proved to be challenging for many pairs of students. Of the 30 samples collected only 14 (47%) contained the correct total of 273 turnips. The learning trajectory

proposed above proved to be a useful way to sort the samples. Multiplicative structure, in the form of finding the turnips in arrays using products, was the most common type of response (*n*=22, 73%). Interestingly the skip counting or addition of composite approach used by seven students had a higher rate of correct solutions (71%) than more sophisticated multiplicative strategies (41%). Table 1 contains the frequency of trajectory stages and success rates for the task.

Table 1
Trajectory Stages and Correctness Rates of Work Samples

Trajectory Stage	Correct Answers	Incorrect Answers	Total Answers
Unitary Counting	0	1	1
Composite Counting or Repeated Addition	5	2	7
Known Product of Derived Fact	9	13	22

More detailed analysis revealed sub-categories within the stages of the trajectory, particularly in the ways that composite counting and addition was used. Three main types of composite were created by the students, row totals, groups of ten and rows/columns within a bordered array (see Fig. 3). These strategies were additive. However, in one case where tens were used the strategy was almost multiplicative in that the students appeared to know that 27 tens were 270.
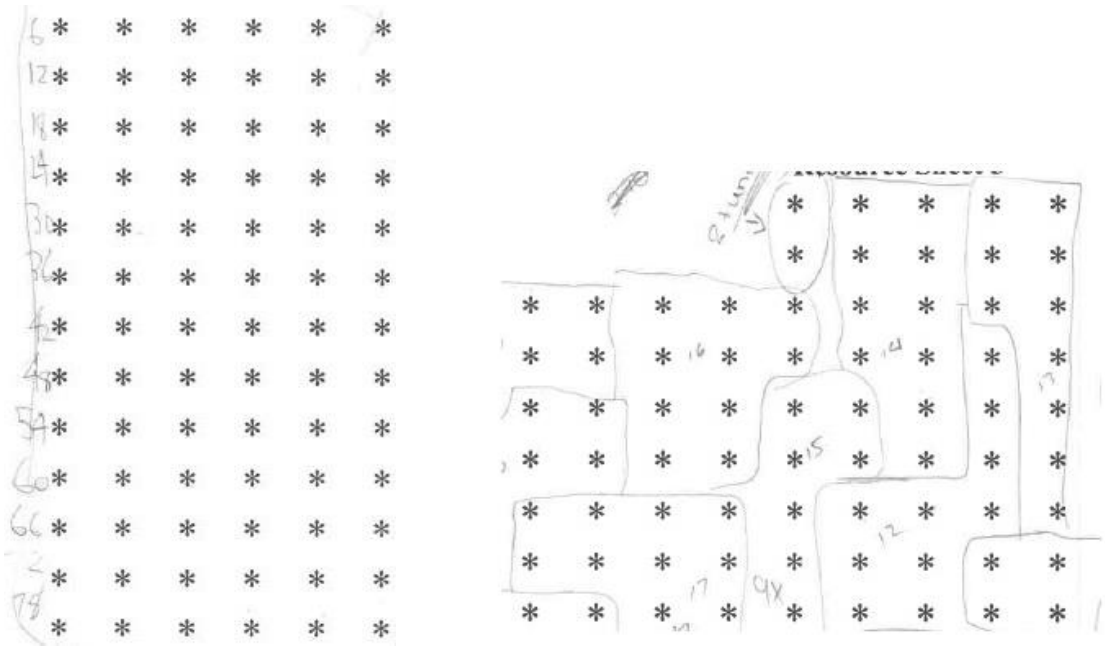

Figure 3. Composite counting strategies.

The students relied heavily on known facts so evidence of deriving was scarce. When they created arrays outside of their fact range the students either used algorithms to find the

product or did not record the calculation. In one notable exception the students used the distributive property to calculate the product of 6 × 13 using 6 x 10 + 6 × 3 (see Fig. 4).
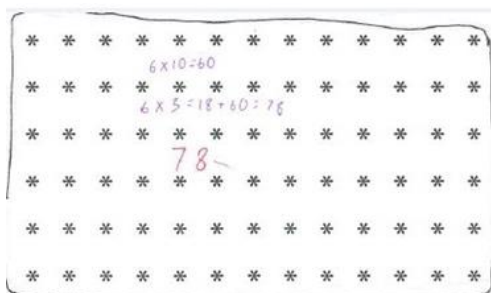

Figure 4: Example of derived multiplication facts.

The 22 samples that exhibited use of array structures were classified in two ways, by correctness of answer and the strategy employed for summing the arrays or sections. The high occurrence of incorrect answers (*n*=13) suggested that there were considerable inaccuracies in counting the number of turnips in arrays or sections (4) and/or calculating the sum of those counts (2) or both types of error (7).

Counting errors included using the incorrect products for an array, double counting or omitting turnips, or miscounting irregular sections. An example of a calculation error is shown in Figure 5. The students correctly divided the turnip patch up into small arrays for which they had known products. In calculating the sum they arranged all the products in vertical order, correctly summed the ones column on the left side but lost track of the tens column when transferring to the right hand column of figures. Their answer of 293 was therefore 20 more than the correct answer, 273.
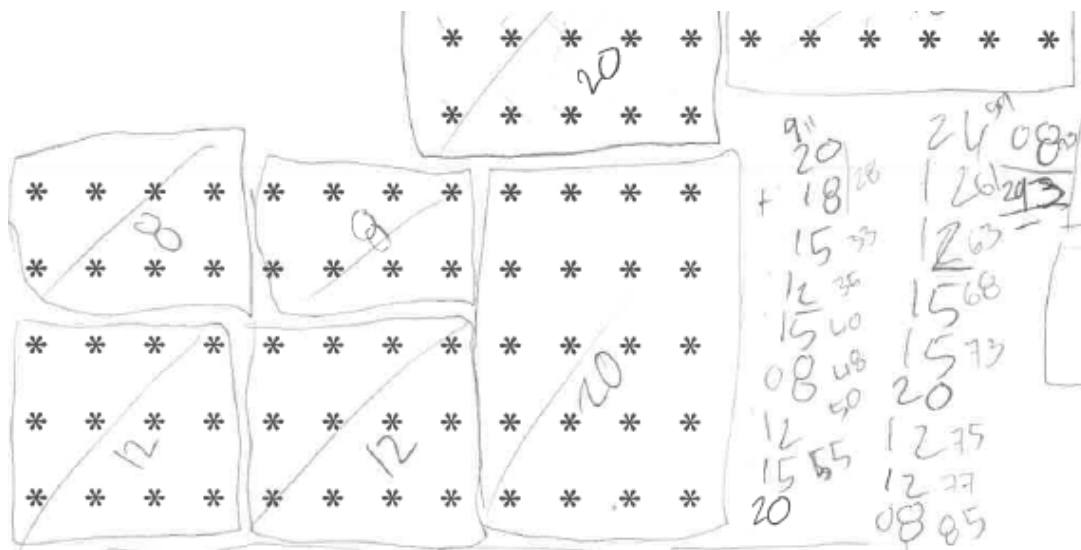

Figure 5. One pair of students correctly found array products but made errors in vertical addition.

Counting or summing errors derailed so many potentially elegant solution strategies. Sorting the array based samples by summing strategies revealed interesting patterns (see Table 2).

Table 2

Strategies for Summing the Products and Counts from Arrays and Sections

	Combining of products	Whole array and subtract negative space	Single vertical algorithm	Cumulative sums	Unclear or not shown
Correct	2	0	1	4	2
Incorrect	1	6	3	2	1

Only three pairs of students combined compatible products to simplify the summing calculation. For example, one pair of students used common factors to combine compatible products (see Fig. 6). In that sample the students combined multiples of three (27 and 12) and of 10 (60 and 20) to make summing easier. The complete negative space strategy involved calculating the whole array as complete, using an algorithm for 15 x 27, followed by subtraction of the missing space. Some students imagined the items in empty arrays or sections while others filled in those spaces with dots or marks. No pairs that used the negative space strategy found the correct total. Six students used cumulative sum strategies in which they added two products, then added a third to the result, then a fourth and so forth. Ironically the success rate of this cumbersome procedure was higher than for a single vertical algorithm.
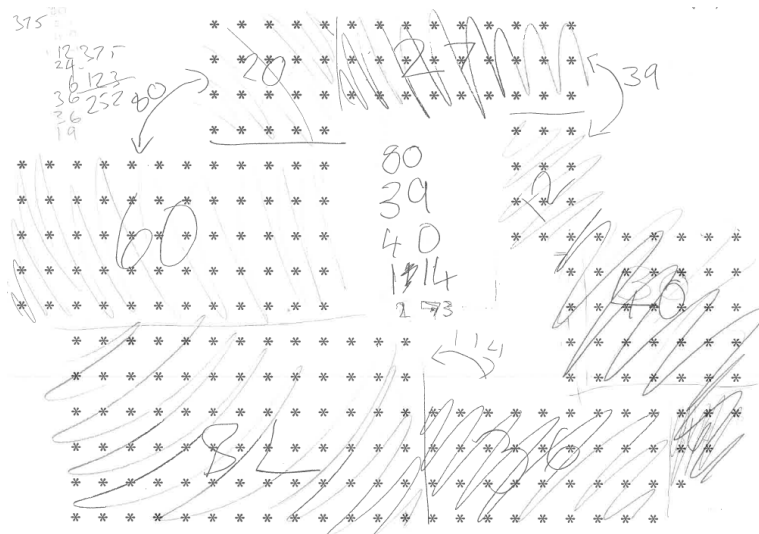

Figure 6. Common factors used to combine compatible products.

These data show that most pairs of students were able to apply multiplicative structure to individual arrays. However, more than half of the pairs were unable to systematically find the total of the arrays and sections. Even successful pairs resorted to cumbersome calculations, rather than the application of mental strategies to combine compatible numbers.

Discussion and Implications

The complexity of this multi-step task was evident in the diversity of strategies students employed. Their recording yielded complex informative data that verified the use of the rich task as an assessment tool. Most samples showed that students operated at the *Known*

product or fact/derived product stage as evidenced by their ability to apply multiplicative structure to arrays. However, eight pairs chose to apply unitary or additive strategies through forming composites such as columns, rows or sets of ten. The task also demanded a systematic approach to finding the sum of arrays and sections. Work samples revealed inaccuracies in students' accounting for all items and calculation of the final total. Few pairs of students chose efficient ways to combine compatible numbers to ease cognitive load and the preference to use written algorithms, often inaccurately, overrode the use of mental strategies. This indicates that most students were emerging multiplicative thinkers with some way to go before they could operate on the operators and solve complex multiplicative problems. Our research suggests that the ability to manage multi-step, combined operations tasks is one possible dimension for extending the learning trajectory for multiplicative thinking. The work samples also provided a means to sub-categorise the counting of composites stage and extend the HLT as it relates to this rich task (See Fig. 7).

Count-all	Composite counting Repeated addition			Known product Derived fact		Extended multiplicative thinking
Count in ones	Group in non-equal composites	Group in equal composites	Partition arrays and skip count	Know multiplication facts	Derive multiplication facts	Multi-step, multi-operations
Count all the items in the space one by one	Count row totals and use addition to find the total	Use tens to make counting easier	Identify equal composites in arrays and skip count	Partition the space into arrays accessible to known facts	Partition the space into efficient partitions and derive totals using properties of multiplication	Partition the space into efficient arrays and combine compatible products to simplify calculation

Figure 7: Subcategorised trajectory of multiplicative thinking.

A significant strength of using the rich assessment task was the opportunity to witness students' strategic choices and note the implications of those choices. Students' strategies for finding the total were all sound, and often creative, but they lacked the computational fluency and flexibility to enact those strategies correctly. Creation of a more detailed and extended HLT may assist teachers in identifying specific student's needs and changes in their strategy preferences over time. This opens up opportunities for further research.

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