Braithwaite, D. W., & Siegler, R. S. (in press). Developmental changes in whole number bias. *Developmental Science.* (Anticipated publication date: 2017.)

Developmental Changes in the Whole Number Bias

David W. Braithwaite

Carnegie Mellon University

Robert S. Siegler

Carnegie Mellon University and Beijing Normal University

Author Note:

David W. Braithwaite, Department of Psychology, Carnegie Mellon University; Robert S. Siegler, Department of Psychology, Carnegie Mellon University and The Siegler Center for Innovative Learning (SCIL), Beijing Normal University.

This work was supported in part by the Institute of Education Sciences, U.S. Department of Education, through Grants R305A150262, R324C100004:84.324C, Subaward 23149, and R305B100001 to Carnegie Mellon University, in addition to the Teresa Heinz Chair at Carnegie Mellon University and the Siegler Center of Innovative Learning and the Advanced Technology Center, Beijing Normal University.

Correspondence concerning this article should be addressed to David W. Braithwaite, Department of Psychology, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, PA 15213. baixiwei@gmail.com.

Research Highlights

- Whole number bias in children's understanding of fraction magnitudes declines from fourth to eighth grade.
- Whole number bias reflects a mix of different forms of representation, including integrated, componential, and hybrid.
- Decreases from fourth to eighth grade in the whole number bias result from decreasing reliance on componential and hybrid representations and increasing adoption of integrated fraction magnitude representations.
- Substantial individual differences in whole number bias are present from fourth to eighth grade, with a substantial minority of fourth graders relying predominantly on integrated fraction magnitudes and a substantial minority of eighth graders relying predominantly on componential approaches.

Abstract

Many students' knowledge of fractions is adversely affected by whole number bias, the tendency to focus on the separate whole number components (numerator and denominator) of a fraction rather than on the fraction's integrated magnitude (ratio of numerator to denominator). Although whole number bias appears early in the fraction learning process and persists into adulthood even among mathematicians, little is known about its development. The present study demonstrates that with age and experience, whole number bias decreases, and reliance on fractions' integrated magnitudes increases, on both number line estimation and magnitude comparison tasks. In particular, children treated equivalent fractions with larger components (e.g., 16/20) as larger than ones with smaller components (e.g., 4/5), thus indicating a whole number bias, but the amount of bias decreased considerably between fourth and eighth grade. Implications of the findings for children's understanding of fraction equivalence and for theories of numerical development are discussed.

Developmental Changes in the Whole Number Bias

Understanding fractions is crucial for success in mathematics achievement in general, algebra in particular, science, and many occupations (Booth, Newton, & Twiss-Garrity, 2014; McCloskey, 2007; Siegler et al., 2012). Unfortunately, children in the U.S. and many European countries experience great difficulty gaining this understanding (Torbeyns, Schneider, Xin, & Siegler, 2015). The problem often persists into adulthood, for example among community college students (Fazio, DeWolf, & Siegler, 2016; Schneider & Siegler, 2010; Stigler, Givvin, & Thompson, 2010).

A major obstacle to understanding fractions is the whole number bias – the tendency to focus on the whole number components of fractions (numerators and denominators) rather than thinking of a fraction as a single number (Ni & Zhou, 2005). Reflecting whole number bias, children often add or subtract fractions by adding or subtracting both their numerators and denominators (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Siegler, Thompson, & Schneider, 2011). For example, children asked to solve 1/8+1/8 often proposed the answer 2/16 (Mack, 1995).

Whole number bias also interferes with children's understanding the magnitudes of fractions by creating a misperception that fractions with larger whole number components have larger magnitudes. This misperception is reflected in higher error rates and longer response times on fraction magnitude comparison tasks when the larger fraction has smaller components (DeWolf & Vosniadou, 2014; Fazio et al., 2016; Meert, Grégoire, & Noël, 2010a, 2010b). For example, participants are often slower and less accurate at recognizing that 2/5 is larger than 3/9 than at recognizing that 2/5 is smaller than 4/9, despite the distance between the magnitudes of the latter pair of fractions being smaller.

In the remainder of this study, we focus on the effects of whole number bias on understanding of fraction magnitudes, and use the term "whole number bias" to refer specifically to such effects. This focus is justified by the strong relations between fraction magnitude understanding and success in more advanced mathematics (Siegler & Braithwaite, 2016). Fraction magnitude understanding is both correlated with and predictive of later proficiency in fraction arithmetic and overall mathematics achievement, even after controlling for plausible third variables, including academic achievement in other areas such as reading, executive function, and non-symbolic numerical knowledge (Booth et al., 2014; Fazio, Bailey, Thompson, & Siegler, 2014; Siegler & Pyke, 2013; Siegler et al., 2011). Moreover, randomized controlled trial interventions that improve fraction magnitude understanding lead to improvements in fraction arithmetic and varied measures of conceptual understanding of fractions (Fuchs et al., 2013, 2014, in press).

Despite the importance of the whole number bias, little is known about its development. Alibali and Sidney (2015) noted that "the bias is evident, not only in learners who have just been introduced to rational numbers, but also in individuals who have extensive familiarity with rational numbers," including adults. The apparent persistence of whole number bias suggests the following question: Does whole number bias decrease over development?

As we argue below, existing evidence is inconclusive with respect to whether whole number bias decreases at all. Further, even if whole number bias does decrease, neither the timing of this decrease nor the mechanisms underlying it are well understood. The present study addressed these questions by tracking the developmental trajectory of the whole number bias from fourth to eighth grade. To understand the mechanisms underlying developmental changes in whole number bias, we tracked changes over the same time period in the distribution of different types of fraction magnitude representation among different children.

Whole Number Bias

Existing evidence for effects of whole number bias on fraction magnitude understanding comes mainly from fraction magnitude comparison tasks (for examples relating to fraction conversion, see Bright, Behr, Post, & Wachsmuth, 1988; Kerslake, 1986; Ni, 2001). Performance on such tasks improves with age during primary and middle school. For example, between fifth and seventh grade, fraction comparison accuracy improved from 75% to 90% in Meert et al. (2010b), and from 68% to 94% in Gabriel et al. (2013). These improvements could reflect decreasing effects of whole number bias on children's representations of fraction magnitudes.

Alternatively, however, changes in comparison accuracy could reflect changes in strategy use. Despite the name of the task, fraction magnitude comparison can be performed using a variety of strategies that do not involve fraction magnitudes at all. For example, when comparing two fractions, one may simply judge the fraction with the larger numerator to be larger. Although this strategy yields correct answers for some comparisons, using it in all cases would lead to errors consistent with whole number bias, such as the incorrect judgment that 3/9 is larger than 2/5. However, the strategy does not involve fraction magnitudes at all, so its use does not imply biased representations thereof. Similarly, when one denominator is a whole number multiple of the other (e.g., 2/3 and 4/9), people can multiply the fraction with the smaller denominator by N/N and then judge the fraction with the larger numerator as larger. Such a strategy can reflect consideration of magnitudes, but it also can be used mechanically without considering either fraction's magnitude.

These are not just logical possibilities; both children and adults often use such strategies on fraction magnitude comparison problems (Bonato, Fabbri, & Umiltà, 2007; Fazio et al., 2016). For example, in Fazio et al. (2016), individual students at both a highly selective university and a non-selective community college averaged 10-11 distinct strategies in solving a set of 48 fraction magnitude comparison problems. The most common strategies involved component-wise comparison without reference to integrated fraction magnitudes.

To avoid use of strategies specific to the magnitude comparison task that allow comparison without reference to fraction magnitudes, the present study assessed whole number bias using number line estimation, in which participants estimate the magnitudes of fractions by placing them on a number line (Bright et al., 1988; Fazio et al., 2014; Iuculano & Butterworth, 2011; Kerslake, 1986; Meert, Grégoire, Seron, & Noël, 2012; Opfer & Devries, 2008; Resnick et al., 2016; Thompson & Opfer, 2008). Number line estimates are generated for individual fractions in isolation, and thus are not subject to the componential comparison strategies described above. Participants were also presented a fraction magnitude comparison task to provide continuity with previous studies. The two tasks together promised to provide a more accurate depiction of developmental changes in whole number bias than either alone could.

Fraction Magnitude Representations

Another goal of the present study was to understand what type of representations give rise to whole number bias, and how changes in these representations contribute to changes in whole number bias. According to one account, whole number bias results from reliance on componential representations – that is, representations that reflect the sizes of fractions' whole number components and not their integrated magnitudes (Bonato et al., 2007). Consistent with componential representations, in a fraction comparison task, the distance between two fractions' whole number components, but not the distance between their integrated magnitudes, predicted response times (Bonato et al., 2007). An alternate account proposes hybrid representations that reflect influences of both fractions' component sizes and their integrated magnitudes. Supporting this account, several studies have found effects of both componential distance and difference in integrated magnitudes on fraction magnitude comparisons (Meert, Grégoire, & Noël, 2009; Meert et al., 2010a, 2010b; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013).

Evidence for hybrid representations is not conclusive, however, because the above-mentioned studies analyzed data aggregated across participants. This fact leaves open the possibility that some participants relied entirely on componential representations and others relied entirely on integrated magnitudes, creating the appearance of hybrid representations without any individual relying on such representations. In the area of numerical cognition, differences between group and individual data patterns are quite common. For example, Siegler (1989) found that a model of children's subtraction that fit the aggregated data quite well did not fit any individual child. Similarly, Siegler (1987) found that a model of children's addition that fit the aggregated data very well was used on only about one-third of trials.

Thus, it remains an open question whether whole number bias in fraction magnitude representation results from the use of hybrid representations by most or all individuals, use of componential representations by some individuals and integrated magnitude representations by others, or some combination of all three forms of representation. Although we made no specific prediction regarding the absolute frequencies of these models providing the best fit to the data, we expected decreases in the frequencies of componential representations, hybrid representations, or both to accompany any developmental decreases in the whole number bias.

Equivalent Fractions

Equivalent fractions were well suited to examining the above issues. Fractions are equivalent if their components (numerators and denominators) stand in the same ratio (e.g., 4/5 = 16/20). Thus, equivalent fractions have the same magnitude and are interchangeable in tasks relating to magnitude, including magnitude comparison and number line estimation. However, based on prior research regarding whole number bias, we expected that many children would not treat equivalent fractions as having equal magnitudes, but would instead treat fractions with larger components (e.g., 16/20) as larger than equivalent fractions with smaller components (e.g., 4/5). Including equivalent fractions in the present study had the important advantage of permitting manipulation of component size completely independently of fraction magnitude. That is, for a given magnitude, we could include fractions of different component sizes but identical integrated magnitudes (e.g., 4/5 and 16/20).

Equivalent fractions are also an important subject in their own right. Fraction equivalence is at the heart of a crucial difference between fractions and whole numbers: Each whole number has a unique representation using Arabic numerals (e.g., “6”) or spoken language (e.g., “six”), but each fraction can be expressed in infinitely many ways (e.g., 4/5, 8/10, 12/15, 16/20…). Thus, understanding fraction equivalence increases understanding of which properties of whole numbers are not true of numbers in general – a central theme in numerical development (Siegler et al., 2011; Smith, Solomon, & Carey, 2005; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010).

Moreover, understanding fraction equivalence is essential to understanding fraction arithmetic procedures. For example, to understand the standard procedures for adding and subtracting fractions, learners must know that substituting equivalent fractions does not change the magnitude of the operands and therefore does not change the answer (e.g. 4/5 + 1/4 = 16/20 +

5/20 = 21/20). If 4/5 were not equivalent to 16/20, or if 1/4 were not equivalent to 5/20, this substitution and the ensuing answer would have no logical basis.

Previous work has examined children's understanding of fraction equivalence using fraction-model conversion tasks and purely symbolic conversion tasks. In fraction-model conversion tasks, children represent a fraction using a graphical model partitioned according to a different denominator – for example, by marking 5/3 on a number line segmented into 12ths. Children often perform poorly on such tasks (Kamii & Clark, 1995; Ni, 2001), even if they are able correctly to perform purely symbolic conversions such as converting 5/3 into 20/12 (Bright et al., 1988). Thus, children's ability to convert between equivalent fractions symbolically may over-estimate their understanding of equivalence. On the other hand, the use of pre-segmented graphical models in fraction-model conversion task may increase children's use of incorrect counting strategies (Boyer, Levine, & Huttenlocher, 2008), inhibiting performance even among children whose internal representations of equivalent fractions are actually equal. Thus, fraction-model conversion tasks may under-estimate children's understanding of equivalence.

In the present study, we assessed whether children treat equivalent fractions as equal in magnitude comparison and number line estimation tasks when the fractions are presented at different times. These tasks avoids the limitations of the above conversion tasks because they do not explicitly involve conversion between equivalent fractions. Of course, this strength is also a weakness, in that our tasks do not directly assess whether equivalent fractions are recognized as equal. No task is perfect; it was hoped that the measures employed in the present study would offer advantages complementary to those of previous studies (Boyer et al., 2008; Bright et al., 1988; Kamii & Clark, 1995; Ni, 2001).

Experiment 1

Experiment 1 included two tasks designed to test for the predicted effects of whole number bias: fraction number line estimation and fraction magnitude comparison. On each trial of the number line estimation task, children placed a fraction drawn from an equivalent pair onto a 0-1 number line. Fractions with larger components were expected to elicit larger estimated magnitudes than equivalent fractions with smaller components. On each trial of the magnitude comparison task, one of the two fractions from an equivalent pair was compared to a larger comparison fraction, with the particular comparison fraction varying from trial to trial. Children were expected to judge fractions with larger components as larger than the comparison fraction more frequently than equivalent fractions with smaller components. The participants were fourth and fifth graders, to test whether effects of whole number bias on number line estimation and magnitude comparison change over these grades. Children in our sample received considerable fraction instruction during these grades, which increased the likelihood of observing a change in whole number bias over this one year period.

Method

Participants. Participants included 66 children, 33 fourth graders (mean age=9.8 years) and 33 fifth graders (mean age=10.9 years), 30 males and 36 females, all attending an elementary school near Pittsburgh, PA. In the school, 64% of students received free or reduced price lunches. Mathematics achievement at the school was below average for the state in which the study was conducted. On the mathematics portion of the Pennsylvania System of School Assessment (PSSA), the standardized achievement test used in Pennsylvania, 42% of fourth graders and 74% of fifth graders scored below the basic level, compared to 25% and 26% respectively statewide. The student body was 74% African-American, 12% multiracial, 11% Caucasian, and 4% “other.”

Two female Caucasian research assistants administered the experiment, which was conducted near the end of the school year.

Materials. Stimuli were two sets of fractions, each consisting of 13 pairs of equivalent fractions, with no fraction appearing in both sets. Numerators within each set ranged from 1 to 24 and denominators from 2 to 28; each set contained one pair of fractions with magnitudes equal to 0.5 and three pairs of fractions with magnitudes in each quadrant of the range from 0 to 1 (excluding 0.5). Each equivalent pair included one fraction with a single digit denominator (e.g. 4/5) and one with a two-digit denominator (e.g. 16/20); these will be referred to as small component fractions and large component fractions, respectively. The small component fractions were in lowest terms, with one exception[1]. All fractions are listed in the Supporting Information, as are the instructions that children were given.

Procedure. The study was conducted in a whole class format during the children's mathematics class period. Within each grade, 17 children were presented with one of the two fraction sets and 16 children with the other. Children were not told that the tasks involved equivalent fractions, and successive problems never involved fractions from the same equivalent pair. The tasks were presented in a printed packet, which children completed working individually at their own pace. The number line estimation task was always presented first, followed by the magnitude comparison task.

On the number line estimation task, each fraction was presented above a 0-1 line, one item per page. Children were instructed to mark each fraction's position on the number line. The fractions were presented in a fixed random order or its reverse, with half of the children receiving each order. To ensure that all of each child's data for small and large component fractions represented fractions of equal magnitude, equivalent fraction pairs were excluded if either fraction was left blank. This filter resulted in exclusion of 5.9% of trials, including all trials from 2 children, leaving 64 children for analysis. Among those children, 3.0% of trials were excluded.

On the magnitude comparison task, a fraction with components intermediate in size to those in each equivalent pair was chosen as the comparison fraction for both equivalent fractions in the pair. For example, the comparison fraction for the pair “4/5, 16/20” was 10/11. Each of the equivalent fractions was presented once together with the comparison fraction for the pair; children were instructed to circle the larger of the two. Different fractions were used as the comparison fractions for different equivalent pairs.

The equivalent fractions were always smaller than their comparison fractions (by at least 0.05, mean=0.116), so the comparison fraction was always the correct answer. Because the comparison fractions were not marked as such, there was no obvious way for children to know which numbers were considered comparison fractions for our purposes. Equivalent fractions and comparison fractions each appeared equally often on the left or right side of the page. Problems were presented either in a fixed random order or the reverse of that order. Data were filtered in the same way as for number line estimation, resulting in the exclusion of 3.7% of trials, including all trials from 1 child, leaving 65 children for analysis. For those children, the percent of trials excluded was 2.2%.

Results

Number Line Estimation. To assess children’s competence at the number line estimation task, Percent Absolute Error (PAE), the absolute value of the difference between estimate and true value, was calculated on each trial. Mean PAE was 19.3% and did not differ significantly between fourth and fifth graders.

Next, to assess whole number bias, the mean value of each child's estimates was calculated separately for small and large component fractions, and these data were submitted to an ANOVA with component size as a within-subjects variable, and grade and fraction set as between-subjects variables. This analysis yielded a single main effect for component size, $F(1, 60) = 27.37$, $p < .001$, $\eta_g^2 = .130$, and no interactions. Consistent with the whole number bias, on average, children's number line estimates for large component fractions were 0.104 larger than for equivalent small component fractions. Estimates of 67% of children (43 of 64) were larger for large than for small component fractions, $\chi^2(1, N = 64) = 7.56$, $p = .006$.

==================== Figure 1 about here ====================

An items analysis, in which the mean estimate for each fraction was calculated across subjects, was conducted to better understand the effect of component size on number line estimates. These data, shown in Figure 1, were submitted to an ANOVA, treating each equivalent pair as an item and component size (larger or smaller) as a within-items variable. The effect of component size was significant, $F(1, 25) = 52.47$, $p < .001$, $\eta_g^2 = .102$. For 24 of 26 equivalent pairs, mean estimates were larger for the fraction with larger components, $\chi^2(1, N = 26) = 18.6$, $p < .001$.

Magnitude Comparison. Children correctly answered 55.2% of magnitude comparison items, slightly but significantly higher than chance (50.0%), $t(64) = 3.64$, $p < .001$. Accuracy did not differ between fourth and fifth graders, 56.3% versus 54.0%.

An ANOVA with component size as a within-subjects variable, and grade and fraction set as between-subjects variables, revealed that fractions with numerators and denominators larger than those of the comparison fractions were judged to be larger than the comparison fractions more often than equivalent fractions with components smaller than those of the comparison fractions were (62.7% versus 27.0%), $F(1, 61) = 15.6$, $p < .001$, $\eta_g^2 = .189$. Almost 70% of children (45 of 65) more often judged as larger equivalent fractions with components larger than those of the comparison fractions than equivalent fractions with components smaller than those of the comparison fractions, $\chi^2(1, N = 65) = 9.62$, $p = .002$.

==================== Figure 2 about here ====================

Component size and grade also interacted, $F(1, 61) = 5.02$, $p = .029$, $\eta_g^2 = .070$, due to the effect of component size decreasing with grade (Figure 2). Paired *t*-tests indicated that the effect of component size on magnitude comparison judgments was significant in fourth grade, $t(32) = 5.02$, $p < .001$, but not in fifth grade, $t(31) = 1.06$, $p = .300$. The proportion of children who judged large component fractions to be larger more often than small component fractions decreased from 79% of fourth graders to 59% of fifth graders, though this difference was not significant, $\chi^2(1, N = 65) = 2.04$, $p = .154$.

An items analysis, in which each equivalent pair was treated as an item and component size as a factor in a repeated-measures ANOVA, yielded a significant effect of component size, $F(1, 25) = 699.9$, $p < .001$, $\eta_g^2 = .910$. In all 26 equivalent pairs, the fraction with components larger than those of its comparison fraction was judged to be larger more often than the equivalent fraction with components smaller than those of that comparison fraction.

Discussion

Results of Experiment 1 revealed large and consistent effects of component size on children's responses to equivalent fractions, thus providing unambiguous evidence of whole number bias. The results replicated previous findings from magnitude comparison tasks with non-equivalent fractions (e.g. Meert et al., 2010b) and demonstrated for the first time that analogous effects appear in number line estimation. Thus, whole number bias is not merely an artifact of strategy choices on magnitude comparisons; instead, this bias appears to reflect a more general feature of fraction magnitude representations.

The effect of component size on magnitude comparison decreased from fourth to fifth grade, but no decrease was seen with number line estimation. Whole number bias in fraction number line estimation might not decrease with age, or might decrease later or over a larger range of ages than those included in the experiment. To test these possibilities, Experiment 2 included children ranging from fourth to eighth grade.

The magnitude comparison task included only comparisons in which the larger fraction's numerator and denominator were both smaller or both larger than those of the smaller fraction, an approach shared with some other studies of whole number bias (e.g., DeWolf & Vosniadou, 2014). Meert et al. (2010b) obtained results analogous to ours using fraction pairs with one equal component: 10- and 12-year-old children were faster and more accurate when the larger fractions had larger numerators (and equal denominators) than when the larger fractions had smaller denominators (and equal numerators). One other type of comparison is possible, in which the larger fraction has a larger numerator but a smaller denominator (e.g., 3/4 vs. 2/5). We suspect that such comparisons would elicit results intermediate to those found in the two types of comparison included in the present study. However, to our knowledge, no study of fraction magnitude comparison among children has included all three of these types of comparison.

Experiment 2

The first goal of Experiment 2 was to evaluate the presence and degree of whole number bias in fraction number line estimates decrease over the period from fourth to eighth grade. We expected whole number bias to decrease over this period, on the grounds that children's competence at fraction number line estimation increases over the same period (Resnick et al., 2016; Siegler et al., 2011).

A second goal was to determine what changes in the form of fraction magnitudes representations occur over the same period, if any. As reviewed in the Introduction, whole number bias in data aggregated across participants could reflect reliance on hybrid representations, a combination of componential and integrated magnitude representations, or a mix of all three types of representation. Because each of these possibilities would create the appearance of hybrid representations at the group level, we evaluated the fit of different models of fraction magnitude representation at both the group and individual levels, using Bayesian model comparison. Analyzing the distribution of different representations among different individuals also allowed us to investigate whether this distribution changes over developmental time, concurrent with the hypothesized decrease in whole number bias.

A third goal was to exclude a possible alternate interpretation of Experiment 1. Because the smaller component fractions in Experiment 1 were almost all in lowest terms, the findings may have reflected differing understanding of fractions that are or are not in lowest terms, rather than effects of component size as such. Textbooks and teachers appear to usually present fractions in lowest terms, so greater familiarity with fractions in that form provided a plausible explanation for the Experiment 1 results. Experiment 2 included multiple equivalent fractions for each magnitude, allowing us to observe effects of component size among equivalent fractions, none of which was in lowest terms.

Method

Participants. Participants included 137 children; 46 fourth graders (mean age=9.2 years), 49 sixth graders (mean age=11.1 years), and 42 eighth graders (mean age=13.1 years); 62 males and 75 females. All attended an elementary school near Pittsburgh, PA, a different school from that in Experiment 1. In this school, 64% of students received free or reduced lunch. Students scored near state averages on the mathematics portion of the PSSA (the proportions of fourth, sixth, and eighth graders in the school scoring below basic level in were 33%, 21%, and 24% respectively, compared to 25%, 25%, and 38% statewide). The student body was 52% Caucasian, 47% African-American, and 1% “other.” The experiment was administered by two female Caucasian research assistants, and was conducted in the first half of the school year.

Materials. Stimuli were 11 groups of equivalent fractions with 4 fractions in each group, a total of 44 fractions. All 44 were proper fractions with numerators ranging from 1 to 20 and denominators ranging from 2 to 36. Each group of equivalent fractions included one fraction that was in lowest terms; these fractions were 1/5, 2/9, 1/4, 1/3, 3/7, 1/2, 5/9, 2/3, 3/4, 4/5, and 5/6. The remaining fractions were obtained by multiplying the lowest-terms fraction in each group by three different numbers N/N, with N ranging from 2 to 15. Within each group, the lowest-terms fraction and the fraction obtained by multiplying by the smallest N/N, usually 2/2 or 3/3, were classified as small component fractions (e.g., 4/5, 8/10), while the remaining two fractions were classified as large component fractions (e.g., 12/15, 20/25). All fractions are listed in the Supporting Information, as are the instructions that children were given.

Procedure. The number line estimation task was conducted using the same procedure as in Experiment 1, with a few exceptions noted below. To allow time for the larger number of trials on these tasks, the magnitude comparison task was not presented in Experiment 2.

The same set of number line estimation problems was presented to all children in a fixed random order or the reverse of that order, with half of children receiving the items in each order. Trials were excluded if left blank, and if a given child did not provide a response for at least one large and at least one small component fraction within a given group of equivalent fractions, then all trials for fractions within that group were excluded. This filter resulted in exclusion of 0.8% of all trials; no participants were excluded.

Results

Number Line Estimation. Mean PAE on the number line estimation task was 15.4%. Estimation accuracy improved with grade, $F(1, 135) = 5.35$, $p < .022$, $\eta_g^2 = .038$. Mean PAE was 17.9% in fourth grade, 15.5% in sixth grade, and 12.5% in eighth grade; pairwise comparisons between grades did not reach significance.

To assess whole number bias, the mean value of each child's estimates was calculated separately for the half of fractions classified as large and the half classified as small component fractions. These data were submitted to an ANOVA with component size (large versus small) as a within-subjects variable and grade (fourth, sixth, or eighth) as a between-subjects variable. This analysis revealed a main effect for component size, $F(1, 135) = 31.0$, $p < .001$, $\eta_g^2 = .090$. On average, estimates for large component fractions were 0.065 larger than for equivalent small component fractions. In all 11 equivalent fraction groups, the mean estimate across subjects of the two large component fractions was larger than the mean estimate across subjects of the two small component fractions (Figure 3).

==================== Figure 3 about here ====================

Critically, a component size by grade interaction was also present, $F(1, 135) = 11.0$, $p = .001$, $\eta_g^2 = .033$. It reflected the difference in mean estimates between large and small component fractions decreasing with grade (Figure 4). Paired *t*-tests indicated that the effect of component size was significant in fourth grade, $t(45) = 4.66$, $p < .001$, and sixth grade, $t(48) = 3.18$, $p = .003$, but not in eighth grade, $t(41) = 0.93$, $p = .357$. The proportion of children who gave larger estimates for large component fractions decreased from 67% of fourth graders to 59% of sixth graders to 50% of eighth graders, though these differences were not significant, χ^2(2, N = 137) = 2.75, p = .253.

==================== Figure 4 about here ====================

Because the small component fractions in each set always included one fraction in lowest terms, whereas the large component fractions were never in lowest terms, a difference between fractions that are or are not in lowest terms – rather than an effect of component size as such – could have caused the observed differences between small and large component fractions. To test this possibility, the above analysis was repeated with the data for fractions in lowest terms excluded. The effect of component size remained significant, F(1, 135) = 24.7, p < .001, η_g^2 = .047, as did the interaction of size with grade, F(1, 135) = 13.8, p < .001, η_g^2 = .027.

Bayesian Model Comparison. To evaluate alternate accounts of whole number bias and developmental changes therein, number line estimates were submitted to three linear regression models: a componential model, which included numerator and denominator as predictors; an integrated fraction magnitude model, which included fraction magnitude as the only predictor; and a hybrid model, which included all three of these predictors. Bayes Factors were calculated for each model relative to the null (intercept only) model using the method described by Liang, Paulo, Molina, Clyde, and Berger (2008), as implemented in the BayesFactor package for R (Morey & Rouder, 2015). The Bayes Factors were used to calculate the posterior probability of each regression model and the null model via Bayes' Theorem, assuming equal prior probabilities. R^2 and adjusted R^2 were calculated for each model.

These analyses were first applied to the mean estimates for each fraction across all children. As shown in Table 1, the hybrid model had the largest Bayes Factor and posterior probability, and explained significantly more variance than the null model, $F(3, 40) = 900.0$, $p < .001$, the integrated fraction magnitude model, $F(2, 40) = 111.6$, $p < .001$, and the componential model, $F(1, 40) = 716.4$, $p < .001$. The hybrid model's predictions for mean estimates of the large and small component fractions are shown in Figure 3 (dashed lines).

=================== Table 1 about here ====================

The same analyses were then repeated for each child. The posterior probabilities of the models for each child were averaged across children to estimate the distribution of models within the entire sample. Average posterior probability was highest for the integrated fraction magnitude model (54.8%), intermediate for the hybrid (22.2%) and componential (17.2%) models, and lowest for the null model (5.9%). Similar results were obtained when children were classified into groups by selecting the single most probable model for each child: most children were placed into the integrated fraction magnitude group (54.0%), followed by the componential (21.2%), hybrid (16.8%), and null (8.0%) groups. To illustrate the different response patterns characteristic of these four groups, Figure 5 shows mean estimates for large and small component fractions separately for children in each group. In sum, the fit of the hybrid model to the aggregated data was driven by a mix of distinct response patterns among individuals, rather than indicating consistent use of the hybrid model by all, or even most, children.

=================== Figure 5 about here ====================

Two further analyses tested the hypothesis of a shift with age and experience away from emphasis on component sizes and toward emphasis on fraction magnitudes. First, individual children's posterior probabilities for each model were submitted to linear regression with grade as a predictor. These analyses found an increase with grade in the probability of the integrated model, $\beta = 0.063$, $F(1, 135) = 7.94$, $p = .006$, and decreases with grade in the probabilities of the hybrid model, β = -0.032, $F(1, 135) = 4.76$, $p = .031$, and the componential model, β = -0.031, $F(1, 135) = 4.14$, $p = .044$ (Figure 6A). Second, we calculated the proportion of variance uniquely explained by integrated magnitudes (i.e., change in R^2 between componential and hybrid models) and the proportion uniquely explained by components (i.e., change in R^2 between integrated and hybrid models) for each child. Linear regression found an increase with grade in the proportion of variance explained by integrated magnitudes, β = 0.012, $F(1, 135) = 4.53$, $p = .035$, and a decrease with grade in the proportion of variance explained by components, β = -0.037, $F(1, 135) = 9.47$, $p = .002$ (Figure 6B).

==================== Figure 6 about here ====================

Discussion

In Experiment 1, whole number bias in the fraction magnitude comparison task decreased from fourth to fifth grade, and was no longer significant in fifth grade. This finding alone could create the impression that the major developmental changes in whole number bias occur in the relatively short period from fourth to fifth grade. However, the results of Experiment 2 indicate that whole number bias in fraction number line estimation decreases over the more protracted period from fourth to eighth grade. This result highlights the importance of using multiple measures to assess the development of whole number bias. More general implications of the developmental changes identified in Experiment 2 are deferred to the General Discussion.

Children's aggregated data were consistent with a hybrid model of magnitude representation, but only a minority of children exhibited hybrid response patterns. The aggregated data concealed substantial diversity among individuals, much as in other areas of mathematical cognition (Siegler, 1987, 1989). These findings caution against drawing inferences about the form of children's mental representations based on aggregated data alone.

Finally, effects of component size on number line estimates remained significant even when fractions that were in lowest terms were removed from the data. This result allows us to exclude a possible explanation for these effects in Experiment 1 – that they merely reflect a difference between fractions that are or are not in lowest terms – and thus supports our interpretation that the effects reflect influences of whole number bias.

General Discussion

The present findings hold implications regarding the development of fraction magnitude knowledge, fraction equivalence, and numerical knowledge more generally.

Development of Fraction Magnitude Knowledge

The present findings demonstrate that whole number bias decreases with children's age and experience with fractions. Effects of fraction component size decreased, and effects of integrated fraction magnitude increased, between fourth and eighth grade on the number line task in Experiment 2. Even in the brief period between fourth and fifth grade, the impact of component size decreased on the magnitude comparison task in Experiment 1 (the task was not presented in Experiment 2). Thus, although whole number bias continues even in adulthood (Bonato et al., 2007; DeWolf, Grounds, Bassok, & Holyoak, 2014; Fazio et al., 2016; Meert et al., 2009, 2010b; Obersteiner et al., 2013), the size of this bias shows a clear decline over the period during which children study fractions in school (Common Core State Standards Initiative, 2010).

The findings dovetail with those of a previous study involving fraction magnitude comparison (Gabriel et al., 2013). This study used the effect of distance between fractions' integrated magnitudes on difficulty of comparison to assess children's reliance on fractions' integrated magnitudes during the task. Distance effects on response time (though not accuracy) increased from fifth to seventh grade. The present study found analogous developmental trends not only in magnitude comparison but also in number line estimation, a finding that demonstrated that the previous findings were not attributable to use of strategies for magnitude comparison that circumvented directly comparing magnitudes. Instead, both previous and present findings seem to reflect increasing reliance on fraction magnitudes, and concurrently decreasing whole number bias.

The model comparison conducted in Experiment 2 permits a more precise description of these representational changes. According to this analysis, for some children, whole number bias reflects reliance on componential representations of fractions. For other children, whole number bias reflects reliance on hybrid representations, which are influenced by both component size and integrated fraction magnitude, as proposed by Meert and her colleagues (Meert et al., 2009, 2010a, 2010b). Both of these forms of representation become less common with age and experience, whereas integrated representations, which are influenced only by fraction magnitudes, become more common. Even in eighth grade, the estimated probability of using integrated representations was only 66%. Thus, this transition may continue beyond eighth grade.

What could cause the transition from componential and hybrid to integrated fraction magnitude representations? One possible cause is formal classroom instruction about fraction magnitudes. However, such instruction takes place primarily in the third and fourth grades in the U.S. (CCSSI, 2010) whereas the present findings and others (Gabriel et al., 2013; Meert et al., 2010b) suggest that reduction in whole number bias continues well beyond this period.

Another possibility is that understanding of fraction magnitudes improves as a result of experience with related topics encountered after explicit instruction in fraction magnitudes: decimals, percentages, ratios, rates, proportions, and rational number arithmetic. Studying these topics might improve children's understanding of rational number magnitudes in general and that of fraction magnitudes in particular. Illustratively, fraction arithmetic experience indicates that component size is a poor predictor of fraction magnitude. For example, children might initially believe 1/2 to be smaller than 2/6, but exposure to an addition problem such as 2/6+1/6 = 1/2 could lead them to recognize that 1/2 must be larger than 2/6 because the sum of positive numbers must be larger than the addends. Consistent with the possibility that learning fraction arithmetic improves fraction magnitude representations, in a study of U.S. and Chinese sixth graders, effects of national origin (U.S. or China) on fraction magnitude understanding were fully mediated by fraction arithmetic ability (Bailey et al., 2015).

It is uncertain whether all children must pass through the "phase" of componential and hybrid representations. Even among fourth graders in Experiment 2, a substantial minority exhibited integrated magnitude representations. These children may have made the transition away from componential or hybrid representations before the study was conducted, but some may never have possessed such representations at all. It is also uncertain whether the appearance of componential and hybrid representations constitutes a natural development or whether it is contingent on fractions instruction. For example, instruction that leverages children's early understanding of ratios and proportions (Boyer & Levine, 2015; McCrink & Wynn, 2007; Mix, Levine, & Huttenlocher, 1999) might accelerate the transition to integrated magnitude representations, or even allow children to avoid componential and hybrid representations entirely. These are important areas for future research.

Development of Fraction Equivalence

Whole number bias implies that equivalent fractions with different component sizes will not be represented as equal in magnitude. Consistent with this implication, children in the present study estimated that fractions with larger components were larger than equivalent fractions with smaller components. These results complement previous evidence of children's difficulties with fraction equivalence (Bright et al., 1988; Byrnes & Wasik, 1991; Kamii & Clark, 1995; Kerslake, 1986; Ni, 2001). They extend these findings by showing that equivalent fractions are not represented as equal in magnitude even when presented separately, without the need for conversion between them. They also demonstrate that this misperception decreases with age and experience between fourth and eighth grade.

However, it is concerning that the influence of whole number bias on representation of equivalent fractions persisted at least as late as sixth grade. Children are expected to learn procedures that depend on fraction equivalence in fourth and fifth grade (CCSSI, 2010). The typical procedures for adding fractions with unlike denominators, reducing fractions to lowest terms, and canceling during fraction multiplication would make no sense if equivalent fractions were not equivalent. For example, if 3/6 were not equivalent to 1/2, how would it be legitimate to reduce 3/6 to 1/2? Poor understanding of fraction equivalence could impede understanding of related fraction procedures when they are taught, even if understanding of fraction equivalence improves later. Moreover, studies with adults, even expert mathematicians, indicate that the whole number bias never goes away (Alibali & Sidney, 2015; Obersteiner et al., 2013). These observations highlight the importance of developing interventions specifically aimed at improving understanding of fraction equivalence before children are taught procedures that are built on fraction equivalence (e.g., Hunt, 2013).

Knowledge of whole numbers may interfere with understanding of fraction equivalence in an additional way as well. With whole numbers, there is a one-to-one correspondence between symbolic numbers and their magnitudes; stated another way, whole numbers do not have equivalents. To understand fraction equivalence, children must learn that this property is not true of fractions: multiple distinct fractions represent the same magnitude. Children often assume that properties of whole numbers are also true of fractions, and abandon such beliefs slowly if at all (Ni & Zhou, 2005). For example, children experience difficulty understanding that fractions are *dense* – that is, no matter how small the distance between two fractions, there are fractions in between – because this property violates the assumption, based on whole number experience, that every number has a unique successor (Smith et al., 2005; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004). Similarly, understanding of fraction equivalence may develop slowly in large part because it violates the assumption that distinct numbers must represent distinct magnitudes.

Implications for Numerical Development

The present study highlights differences in the developmental trajectories of whole number and fraction magnitude understanding. Both trajectories involve increasing precision in magnitude representations, culminating in formally correct representations in which numbers' represented magnitudes correspond in a linear fashion to their true magnitudes. However, the two trajectories differ with respect to the form and origins of the representations preceding these formally correct representations, and with respect to the speed with these correct representations appear.

The developmental trajectory of whole number magnitude representations has often been described in terms of a logarithmic-to-linear shift. Accurate (linear) representations of whole number magnitudes are preceded by logarithmic representations characterized by exaggerated distances between smaller numbers and compressed distances between larger numbers (Ashcraft & Moore, 2012; Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Berteletti, Lucangeli, &

Zorzi, 2012; Booth & Siegler, 2006; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Geary, Hoard, Nugent, & Byrd-Craven, 2008; Opfer, Thompson, & Kim, 2016; Siegler & Booth, 2004; Siegler & Opfer, 2003). By contrast, the present findings suggest that development of fraction magnitude understanding involves a shift from componential or hybrid towards integrated magnitude representations, which we call a "componential-to-integrated shift."

Previous studies have found no evidence for logarithmic representations of fraction magnitudes among children (Iuculano & Butterworth, 2011; Siegler et al., 2011). Further, logarithmic models of numerical magnitude representation cannot in principle capture the effects of component size observed in the present study, because these models posit that the represented magnitudes of numbers depend only on their true magnitudes, not on the sizes of their components. The same constraint applies to proportion estimation models, which have been proposed as an alternative to logarithmic models in research on whole number development (Barth & Paladino, 2011; Rouder & Geary, 2014). Thus, the componential or hybrid representations that serve as developmental precursors to accurate fraction magnitude representations are fundamentally different from those that serve as developmental precursors of accurate whole number magnitude representations.

The different starting points of the logarithmic-to-linear and componential-to-integrated shifts may reflect differences in the developmental origins of whole number and fraction knowledge. Logarithmic representations of symbolic whole numbers are believed to originate with innate or early-developing representations of non-symbolic quantities, such as dot arrays (Dehaene, 2011; Piazza, 2010). In principle, representations of non-symbolic ratios, such as ratios between dot arrays, could play a similar role in the development of fraction knowledge (Matthews & Chesney, 2015; Matthews, Lewis, & Hubbard, 2016). However, the present findings suggest a different developmental precursor to accurate fraction magnitude representation: knowledge of symbolic whole numbers. Children's knowledge about the whole number components of fractions gives rise to componential and hybrid representations that, for many children, eventually give way to integrated fraction magnitude representations.

The componential-to-integrated shift appears to occur more slowly than the logarithmic-to-linear shift. In a number line estimation task using whole numbers from 0 to 1,000, the proportion of children whose estimates were best fit by a linear model increased from 9% in second grade to 72% in sixth grade – an increase of 63% (Siegler & Opfer, 2003). Opfer and Siegler (2007) observed many children to shift from logarithmic to linear representations in response to feedback after only one trial during a single experimental session! In Experiment 2 of the present study, the posterior probability of the integrated magnitude model increased from 41% in sixth grade to 66% in eighth grade – an increase of only 25% over four years (Figure 6A). Thus, the componential-to-integrated shift occurs more slowly than the logarithmic-to-linear shift and may not occur at all for some children.

In summary, both whole number and fraction development involve progression towards accurate representations of numerical magnitudes characterized by a linear relation between represented and true magnitudes. However, the starting points of the progressions, the speed at which they occur, and the obstacles that learners must overcome all differ for whole numbers and fractions. Development of numerical magnitude representations provides a unifying theme within which to characterize both the similarities and the differences between fraction and whole number development.

References

Alibali, M. W., & Sidney, P. G. (2015). Variability in the natural number bias: Who, when, how, and why. *Learning and Instruction*, *37*, 56–61. http://doi.org/10.1016/j.learninstruc.2015.01.003

Ashcraft, M. H., & Moore, A. M. (2012). Cognitive processes of numerical estimation in children. *Journal of Experimental Child Psychology*, *111*(2), 246–267.

Bailey, D. H., Zhou, X., Zhang, Y., Cui, J., Fuchs, L. S., Jordan, N. C., … Siegler, R. S. (2015). Development of fraction concepts and procedures in U.S. and Chinese children. *Journal of Experimental Child Psychology*, *129*, 68–83.

Barth, H. C., & Paladino, A. M. (2011). The development of numerical estimation: evidence against a representational shift. *Developmental Science*, *14*(1), 125–35. http://doi.org/10.1111/j.1467-7687.2010.00962.x

Berteletti, I., Lucangeli, D., Piazza, M., Dehaene, S., & Zorzi, M. (2010). Numerical estimation in preschoolers. *Developmental Psychology*, *46*(2), 545–551.

Berteletti, I., Lucangeli, D., & Zorzi, M. (2012). Representation of numerical and non-numerical order in children. *Cognition*, *124*(3), 304–313.

Bonato, M., Fabbri, S., & Umiltà, C. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology: Human Perception and Performance*, *33*(6), 1410–1419.

Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2014). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology*, *118*, 110–118. http://doi.org/10.1016/j.jecp.2013.09.001

Booth, J. L., & Siegler, R. S. (2006). Developmental and individual differences in pure numerical estimation. *Developmental Psychology*, *42*(1), 189–201.

Boyer, T. W., & Levine, S. C. (2012). Child proportional scaling: is 1/3=2/6=3/9=4/12? *Journal of Experimental Child Psychology*, *111*(3), 516–33. http://doi.org/10.1016/j.jecp.2011.11.001

Boyer, T. W., & Levine, S. C. (2015). Prompting Children to Reason Proportionally: Processing Discrete Units as Continuous Amounts. *Developmental Psychology*, *51*(5), 615–620.

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology*, *44*(5), 1478–1490.

Bright, G. W., Behr, M. J., Post, T. R., & Wachsmuth, I. (1988). Identifying fractions on number lines. *Journal for Research in Mathematics Education*, *19*(3), 215–232.

Byrnes, J. P., & Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology*, *27*(5), 777–786. http://doi.org/10.1037/0012-1649.27.5.777

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington, D.C.: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Dehaene, S. (2011). *The number sense: How the mind creates mathematics*. Oxford University Press.

DeWolf, M., Grounds, M. A., Bassok, M., & Holyoak, K. J. (2014). Magnitude comparison with different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance*, *40*(1), 71–82. http://doi.org/10.1037/a0032916

Fazio, L. K., Bailey, D. H., Thompson, C. A., & Siegler, R. S. (2014). Relations of different types of numerical magnitude representations to each other and to mathematics achievement.

Journal of Experimental Child Psychology, *123*, 53–72. http://doi.org/10.1016/j.jecp.2014.01.013

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *42*(1), 1–16.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., … Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology*, *105*(3), 683–700. http://doi.org/10.1037/a0032446

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Malone, A., Wang, A., … Changas, P. (in press). Effects of intervention to improve at-risk fourth graders' understanding, calculations, and word problems with fractions. *Elementary School Journal*. http://doi.org/10.1017/CBO9781107415324.004

Fuchs, L. S., Schumacher, R. F., Sterba, S. K., Long, J., Namkung, J., Malone, A., … Changas, P. (2014). Does working memory moderate the effects of fraction intervention? An aptitude–treatment interaction. *Journal of Educational Psychology*, *106*(2), 499–514. http://doi.org/10.1037/a0034341

Gabriel, F. C., Szucs, D., & Content, A. (2013). The development of the mental representations of the magnitude of fractions. *PLoS ONE*, *8*(11), 1–14. http://doi.org/10.1371/journal.pone.0080016

Geary, D. C., Hoard, M. K., Byrd-Craven, J., Nugent, L., & Numtee, C. (2007). Cognitive mechanisms underlying achievement deficits in children with mathematical learning disability. *Child Development*, *78*(4), 1343–1359. http://doi.org/10.1111/j.1467-8624.2007.01069.x

Geary, D. C., Hoard, M. K., Nugent, L., & Byrd-Craven, J. (2008). Development of number line representations in children with mathematical learning disability. *Developmental Neuropsychology*, *33*(3), 277–99. http://doi.org/10.1080/87565640801982361

Hunt, J. H. (2013). Effects of a supplemental intervention focused in equivalency concepts for students with varying abilities. *Remedial and Special Education*, *35*(3), 135–144. http://doi.org/10.1177/0741932513507780

Iuculano, T., & Butterworth, B. (2011). Understanding the real value of fractions and decimals. *Quarterly Journal of Experimental Psychology*, *64*(11), 2088–98. http://doi.org/10.1080/17470218.2011.604785

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior*, *14*(4), 365–378. http://doi.org/10.1016/0732-3123(95)90035-7

Liang, F., Paulo, R., Molina, G., Clyde, M. A., & Berger, J. O. (2008). Mixtures of g priors for Bayesian variable selection. *Journal of the American Statistical Association*, *103*(481), 410–423.

Matthews, P. G., & Chesney, D. L. (2015). Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes. *Cognitive Psychology*, *78*, 28–56. http://doi.org/10.1016/j.cogpsych.2015.01.006

Matthews, P. G., Lewis, M. R., & Hubbard, E. M. (2016). Individual differences in nonsymbolic ratio processing predict symbolic math performance. *Psychological Science*, *27*(2), 191–202. http://doi.org/10.1177/0956797615617799

McCloskey, M. (2007). Quantitative literacy and developmental dyscalculias. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins* *of mathematical learning difficulties and disabilities* (pp. 415–429). Baltimore, MD: Paul H. Brookes Publishing.

Meert, G., Grégoire, J., & Noël, M.-P. (2009). Rational numbers: Componential versus holistic representation of fractions in a magnitude comparison task. *Quarterly Journal of Experimental Psychology*, *62*(8), 1598–616. http://doi.org/10.1080/17470210802511162

Meert, G., Grégoire, J., & Noël, M.-P. (2010a). Comparing 5/7 and 2/9: Adults can do it by accessing the magnitude of the whole fractions. *Acta Psychologica*, *135*(3), 284–92. http://doi.org/10.1016/j.actpsy.2010.07.014

Meert, G., Grégoire, J., & Noël, M.-P. P. (2010b). Comparing the magnitude of two fractions with common components: Which representations are used by 10- and 12-year-olds? *Journal of Experimental Child Psychology*, *107*(3), 244–259. http://doi.org/10.1016/j.jecp.2010.04.008

Meert, G., Grégoire, J., Seron, X., & Noël, M.-P. (2012). The mental representation of the magnitude of symbolic and nonsymbolic ratios in adults. *Quarterly Journal of Experimental Psychology*, *65*(4), 702–24. http://doi.org/10.1080/17470218.2011.632485

Morey, R. D., & Rouder, J. N. (2015). BayesFactor: Computation of Bayes Factors for common designs. R package version 0.9.11-1.

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Educational Psychology*, *26*(3), 400–417. http://doi.org/10.1006/ceps.2000.1072

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52.

Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, *28*, 64–72. http://doi.org/10.1016/j.learninstruc.2013.05.003

Opfer, J. E., & Devries, J. M. (2008). Representational change and magnitude estimation: Why young children can make more accurate salary comparisons than adults. *Cognition*, *108*(3), 843–849. http://doi.org/10.1016/j.cognition.2008.05.003

Opfer, J. E., Thompson, C. A., & Kim, D. (2016). Free versus anchored numerical estimation: A unified approach. *Cognition*, *149*, 11–17.

Piazza, M. (2010). Neurocognitive start-up tools for symbolic number representations. *Trends in Cognitive Sciences*, *14*(12), 542–51. http://doi.org/10.1016/j.tics.2010.09.008

Resnick, I., Jordan, N. C., Hansen, N., Rajan, V., Carrique, J., Siegler, R. S., & Fuchs, L. (2016). Developmental growth trajectories in understanding of fraction magnitude from fourth through sixth grade. *Developmental Psychology*, *52*(5), 746–757.

Rouder, J. N., & Geary, D. C. (2014). Children's cognitive representation of the mathematical number line. *Developmental Science*, *17*(4), 525–36. http://doi.org/10.1111/desc.12166

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology. Human Perception and Performance*, *36*(5), 1227–38. http://doi.org/10.1037/a0018170

Siegler, R. S. (1987). The perils of averaging data over strategies: An example from children's addition. *Journal of Experimental Psychology: General*, *116*(3), 250.

Siegler, R. S. (1989). Hazards of mental chronometry: An example from children's subtraction. *Journal of Educational Psychology*, *81*(4), 497–506.

Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child Development*, *75*(2), 428–44. http://doi.org/10.1111/j.1467-8624.2004.00684.x

Siegler, R. S., & Braithwaite, D. W. (2016). Numerical development. *Annual Review of Psychology*.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., … Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691–7. http://doi.org/10.1177/0956797612440101

Siegler, R. S., & Lortie-Forgues, H. (2015). Conceptual knowledge of fraction arithmetic. *Journal of Educational Psychology*, *107*(3), 909–918. http://doi.org/10.1037/edu0000025

Siegler, R. S., & Opfer, J. E. (2003). The development of numerical estimation: Evidence for multiple representations of numerical quantity. *Psychological Science*, *14*(3), 237–243.

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, *49*(10), 1994–2004. http://doi.org/10.1037/a0031200

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–96. http://doi.org/10.1016/j.cogpsych.2011.03.001

Smith, C. L., Solomon, G. E. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology*, *51*(2), 101–40. http://doi.org/10.1016/j.cogpsych.2005.03.001

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503–518. http://doi.org/10.1016/j.learninstruc.2004.06.015

Stigler, J., Givvin, K., & Thompson, A. (2010). What community college developmental mathematics students understand about mathematics. *MathAMATYC Educator*, *1*(3), 4–16.

Thompson, C. A., & Opfer, J. E. (2008). Costs and benefits of representational change: Effects of context on age and sex differences in symbolic magnitude estimation. *Journal of Experimental Child Psychology*, *101*(1), 20–51. http://doi.org/10.1016/j.jecp.2008.02.003

Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, *37*, 5–13. http://doi.org/10.1016/j.learninstruc.2014.03.002

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, *14*(5), 453–467. http://doi.org/10.1016/j.learninstruc.2004.06.013

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, *28*(2), 181–209.

Endnotes

1. Both sets of fractions were required to include a pair of fractions with magnitudes equal to 0.5. However, only one set could include the fraction 1/2, because no fraction was allowed to appear in both sets. Thus, in the other set, 2/4 was treated as the small-component fraction in the pair equal to 0.5.

Table 1. Summary of linear regression fits. $\beta_{intercept}$, $\beta_{magnitude}$, $\beta_{numerator}$, and $\beta_{denominator}$ denote the coefficients of the corresponding predictors in each model. *BF* stands for Bayes Factor relative to the null model. *P*(M|D) stands for posterior probability of the model given the experimental data. Adj. R^2 stands for adjusted R^2.

Model	$\beta_{intercept}$	$\beta_{magnitude}$	$\beta_{numerator}$	$\beta_{denominator}$	BF	*P*(M|D)	R^2	Adj. R^2
Null	0.492	–	–	–	1.0	~0%	–	–
Integrated	0.182	0.616	–	–	3.6E19	~0%	90.4%	90.2%
Componential	0.436	–	0.032	-0.011	2.1E9	~0%	72.4%	71.0%
Hybrid	0.067	0.702	-0.001	0.004	1.5E33	~100%	98.5%	98.4%

Figure 1. Mean estimates for large and small component fractions plotted against true magnitudes (Experiment 1). The 45° diagonal line indicates the locations of normatively correct estimates.

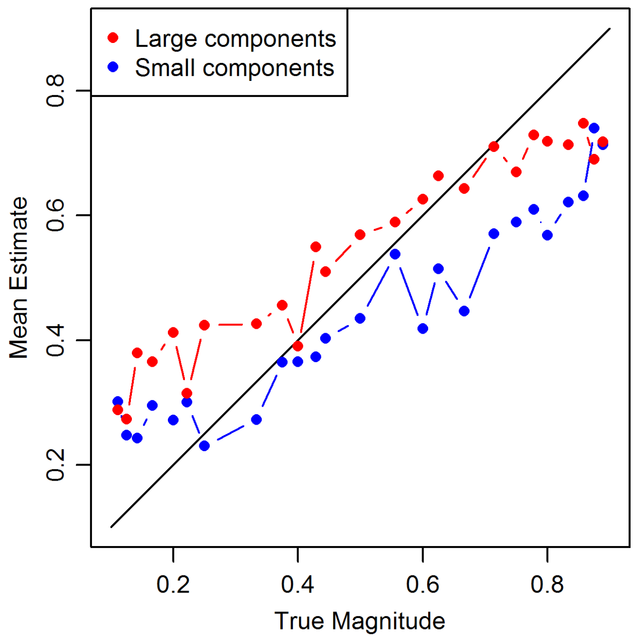

Figure 2. Percent of equivalent fractions judged to be larger than corresponding comparison fractions in Experiment 1. Error bars indicate standard errors.

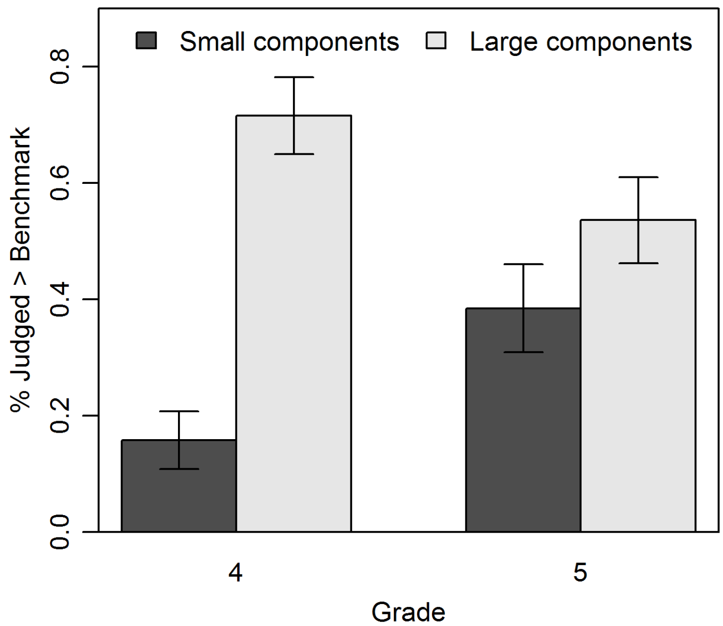

Figure 3. Mean estimates for large and small component fractions plotted against true magnitudes (Experiment 2). Solid lines indicate experimental data; dashed lines indicate predictions of the hybrid model, as described in the text. The 45° diagonal line indicates the locations of normatively correct estimates.

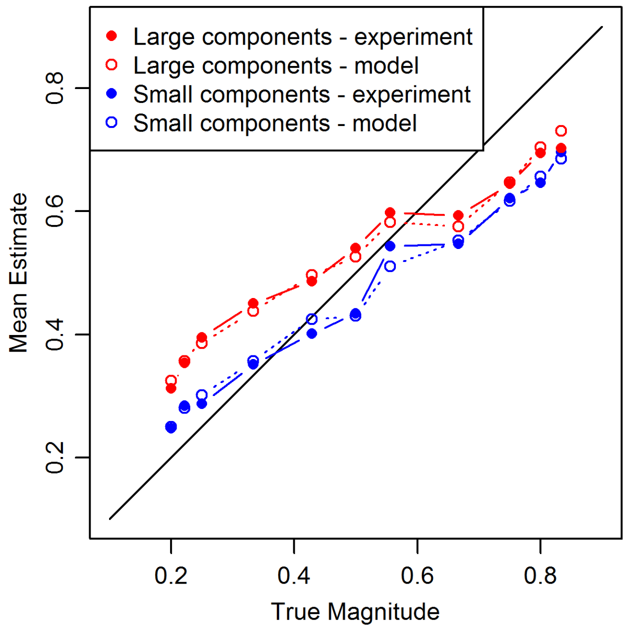

Figure 4. Effect of component size by grade level (Experiment 2), calculated as the difference in mean estimates between large and small component fractions. Error bars indicate standard errors.

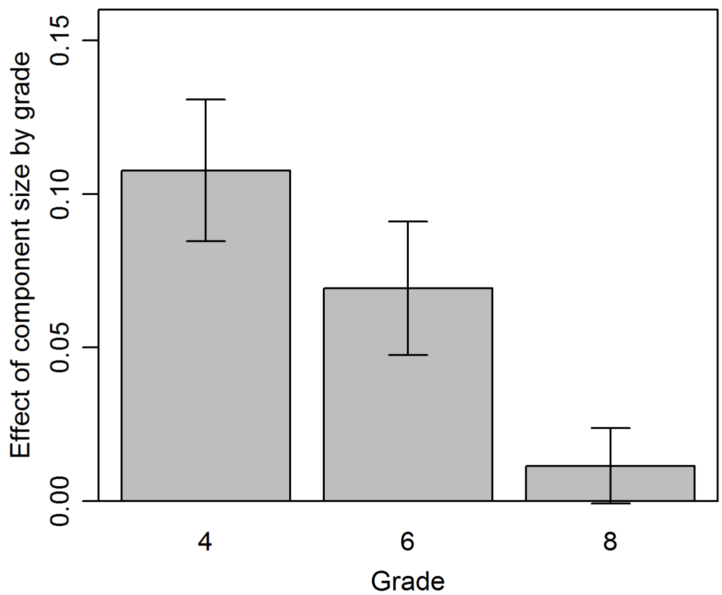

Figure 5. Mean estimates for large and small component fractions plotted against true magnitudes separately for children for whom the model with the highest posterior probability was (A) the null model, (B) the integrated model, (C) the componential model, and (D) the hybrid model (Experiment 2). 45° diagonal lines indicate the locations of normatively correct estimates.

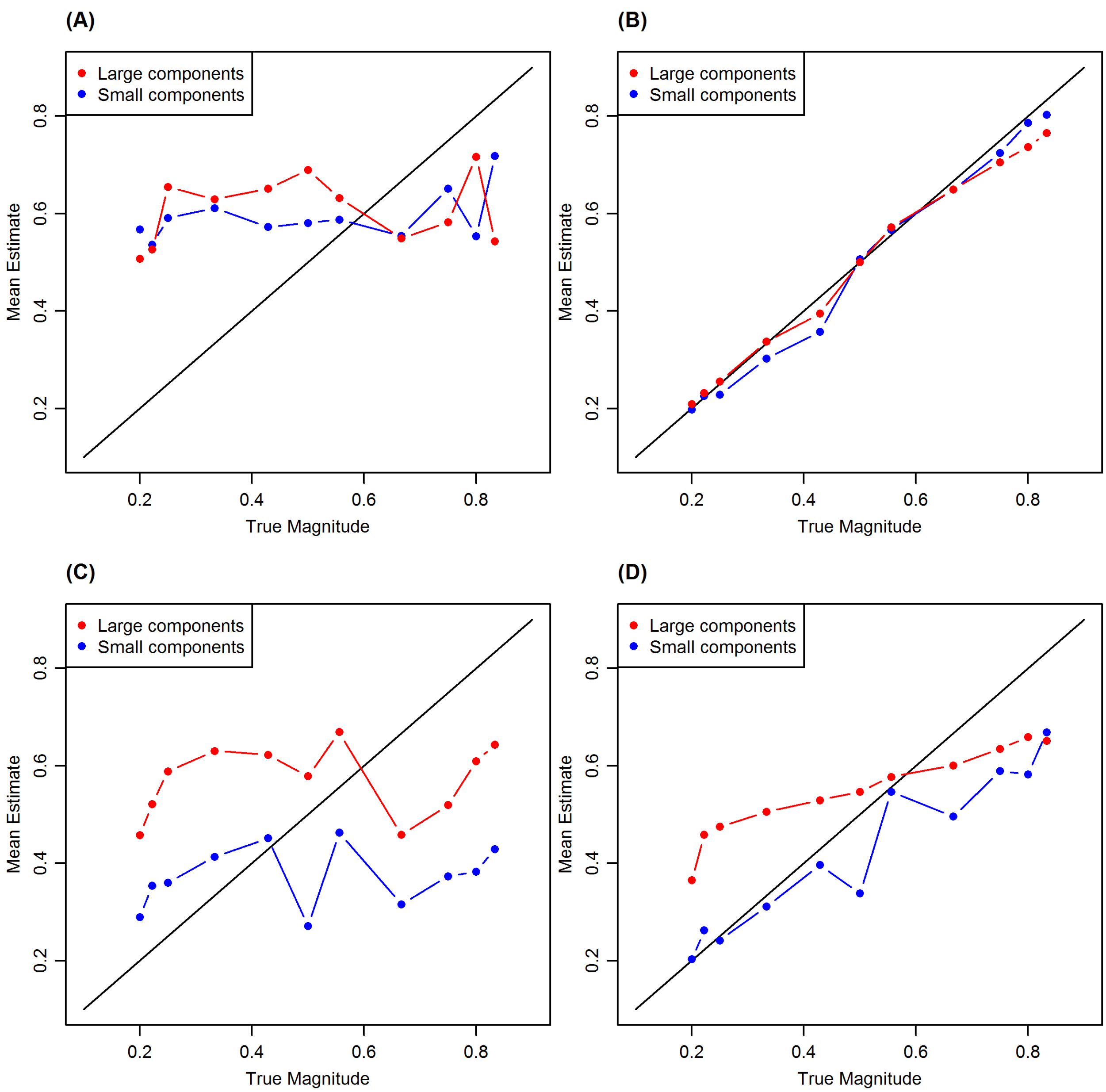

Figure 6. Results of individual participant modeling of number line estimation responses for each grade level (Experiment 2). (A) Posterior probabilities of the integrated fraction magnitude, hybrid, componential, and null models. (B) Proportions of variance uniquely explained by integrated magnitudes and component sizes. In both figures, error bars indicate standard errors.