# Competence with Fractions in Fifth or Sixth Grade as a Unique Predictor of Algebraic Thinking?

Catherine Pearn
*The University of Melbourne*
<cpearn@unimelb.edu.au>

Max Stephens
*The University of Melbourne*
<m.stephens@unimelb.edu.au>

Researchers have argued that there are strong links between primary school students' competence with fraction concepts and operations and their algebraic readiness. This study involving 162 Years 5/6 students in three primary schools examined the strength of that relationship using a test based on familiar fraction tasks and a test of algebraic thinking utilising number relations and equivalence. A strong relationship was found between the two, and with some fraction tasks embodying high potential to anticipate algebraic thinking.

## Introduction

Many researchers argue that a deep understanding of fractions is important for a successful transition to algebra. The National Mathematics Advisory Panel (NMAP, 2008) stated that the conceptual understanding of fractions and fluency in using procedures to solve fractions problems are central goals of students' mathematical development and are the critical foundations for algebra learning. Teaching, especially in the primary and middle years, needs to be informed by a clear awareness of what these links are before introducing students to formal algebraic notation.

Students from three Victorian schools were assessed using two paper and pencil tests. The paper and pencil assessment included the Fraction Screening Test (Pearn & Stephens, 2014) and an Algebraic Thinking Questionnaire based on Mason, Stephens, and Watson (2009) and Stephens and Ribeiro (2012). This paper aims to examine the strength of primary school students' competence with fraction concepts and operations and their algebraic readiness using a test based on familiar fraction tasks and a test of algebraic thinking utilising number relationships and operations.

## Previous Research

Siegler and colleagues (2012) used longitudinal data, from both the United States and United Kingdom, to show that, when other factors were controlled, competence with fractions and division in fifth or sixth grade is a *uniquely accurate predictor* of students' attainment in algebra and overall mathematics performance five or six years later. They controlled for factors such as whole number arithmetic, intelligence, working memory, and family background. We need to extend these important findings to highlight for teachers the specific areas of fractional knowledge that impact directly on algebraic thinking.

Researchers such as Kieren (1980) and Lamon (1999) believe that much of the basis for algebraic thought rests on a clear understanding of rational number concepts and the ability to manipulate common fractions. According to Wu (2001) the ability to efficiently manipulate fractions is "vital to a dynamic understanding of algebra" (p. 17). But these authors have not focussed on the specific aspects of fractional knowledge and competence needed for success in algebra.

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Particularly relevant to this paper are the following studies. Lee and Hackenburg (2013) showed that fractional knowledge appeared to be closely related to establishing algebra knowledge in the domains of writing and solving linear equations; highlighting the importance of multiplicative operations to transform a known fraction to the whole. This capacity will later be fundamental to the solution of algebraic equations. Empson, Levi and Carpenter (2011) also point to understanding fractions as relational quantities; that is a conceptual understanding which connects any given fraction to its corresponding unit fraction which in turn is related to the whole.

This study extends the research of Empson et al. (2011) by using reverse fraction tasks to investigate students' capacity to establish an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole. In addition, we investigate how students track successive transformations of the given fraction and the quantities represented.

## This study

In this paper we discuss the results for 162 Year 5 and 6 students from three Victorian schools who were assessed at the end of the 2015 school year using two paper and pencil tests. Two were Melbourne metropolitan schools and the third was a large country school. These students had not been taught formal algebraic notation. Table 1 shows the number of students from each of the three schools at each Year Level.

Table 1
*Number of students by school and year level (N =162)*

| Year Level | School A | School B | School C | Total (%) |
|---|---|---|---|---|
| Year 5 | 18 | 0 | 26 | 44 (27%) |
| Year 6 | 27 | 40 | 51 | 118 (73%) |

### *The Tasks*

The Fraction Screening Test was developed from an earlier version of 20 items used in many schools with students from Years 5 – 8 (Pearn & Stephens, 2014). All of the item types could be found in commonly used text books for these year levels. The Fraction Screening Test is divided into three parts. Part A includes 12 routine fraction tasks such as equivalent fractions, ordering fractions and recognising simple representations. Part A also includes a simple reverse fraction task where students are asked to determine how many lollies are in the whole group if the four lollies shown are one-half of the whole group.

Part B includes five number line tasks based on Pearn and Stephens (2007). Part C includes items that require students to order a group of fractions from largest to smallest; match four fractions to their equivalent decimals; and two tasks which ask students to circle the one that does not belong e.g. in ¼, 25%, 0.4 0.25. Three items require students to use reverse thinking with less common fractions (see Figure 1). Following the research of Lee and Hackenburg (2013), these three fraction tasks specifically require students to relate a given fraction to an equivalent number of objects, and when transforming the fraction to make a whole to carry out corresponding operations on the number of objects.

Each reverse fraction question, shown in Figure 1, was marked out of three. One mark was given for a correct response with no explanation or if there was some evidence of correct diagram an initial representation which the student did not take further (starting point). Two marks were given for a correct answer with limited explanation and three marks were given for a correct answer with adequate explanation. Zero was given when the question was not attempted or an incorrect response was given.

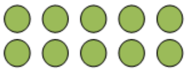
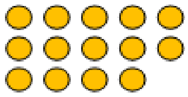
| Reverse Fraction Task 1 | Reverse Fraction Task 2 | Reverse Fraction Task 3 |
|---|---|---|
| This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.<br><br>a. How many counters did I start with?<br>b. Explain how you decided that your answer is correct. | Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs.<br>How many CDs does Kay have? _____<br>*Show all your working.* | This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.<br><br>a. How many counters did I start with?<br>b. Explain how you decided that your answer is correct. |

Figure 1: The three reverse thinking fraction questions

The Algebraic Thinking Questionnaire has two components, one focussed on multiplication ATQ (M), which is the focus of this paper, and the other on division. Four key ideas drawn from the literature identify specific features of the transition from arithmetic or calculation-based thinking to thinking about number sentences as mathematical expressions. These are:

1. Keeping a number sentence in uncalculated form and so being able to view it as a group of numbers in relation to each other according to the operations involved (Britt & Irwin, 2011; Jacobs, Franke, Carpenter, Levi , & Battey, 2007);
2. Utilising the idea of equivalence to solve missing number sentences (Kaput, Carraher, & Blanton, 2008);
3. Exploring variation, compensation and equivalence, identifying numbers that stay the same and numbers that vary in equivalent expressions (Britt & Irwin, 2011);
4. Identifying rules that that underlie relationships in equivalent expressions and being able to express these relationships in the form of a generalisation (Mason, Stephens & Watson, 2009; Stephens & Ribeiro, 2012).

ATQ(M) uses whole numbers and fractions (see Table 2) and commences with two tasks, asking students to find a missing number. This may be found either by calculating the known side and then ascertaining the value of the missing number, or alternatively by using multiplicative relationships between different terms in these equivalent expressions. Tasks 3 and 4 involving fractions and are not intended to be solved easily by calculation.

Table 2

*Algebraic Thinking Questionnaire (Multiplication): Question 1*

| | | | |
|---|---|---|---|
| Task 1 | $36 \times 25 = 9 \times \square$ | Task 2 | $48 \times 2.5 = \square \times 10$ |
| Task 3 | $\frac{2}{3} \times \square = 18$ | Task 4 | $\frac{2}{5} \times \frac{\square}{\square} = 1$ |

Students were asked in each task to: "Write a number in the box to make a true statement and explain your working briefly".

Question 2 (Table 3) focused on equivalence relationships with two unknown numbers represented by Box A and Box B, and by symbolic representations of two unknowns. The tasks included both whole numbers and fractions. Task 1 of Question 2 (not included in Table 3 below) presented students with the same number sentence as used in Task 2 and asked them to write specific numbers in Box A and Box B giving two correct instances. In Tasks 2 – 5, students were required to explain the relationships between the respective unknown numbers or the given symbolic representations (*c* and *d*; or *a* and *b*).

Table 3

*Algebraic Thinking Questionnaire (Multiplication): Question 2*

| Question 2, Tasks 2 - 5 | |
|---|---|
| T2. When you make a correct sentence what is the relationship between the numbers in Box A and Box B? | 5 × ☐ (Box A) = 10 × ☐ (Box B) |
| T3. If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly. | |
| T4. What can you say about *c* and *d* in this mathematical sentence? | $c \times 2 = d \times 14$ |
| T5. What can you say about *a* and *b* in this mathematical sentence? | $a \times \frac{3}{4} = b \times 1\frac{1}{2}$ |

## *Scoring ATQ (M)*

All four tasks for Question 1 (Table 2) were scored on a 0, 1, 2, 3 scale. A score of 0 was given for no attempt. A score of 1 was given for an attempt which failed to give the correct answer. A score of 2 was given for a correct answer with no explanation or working shown, or for a correct answer that had been obtained by calculation or trial and error. A score was 3 was only given for a correct answer that was accompanied by an explanation involving equivalence relationships.

Question 2 consisted of five items, four items worth three marks, and one item (Task 1) worth two marks. The total possible score for ATQ (M) was 26 marks. In Task 1, students were given a score of 1 for each correct instance of the number sentence. Subsequent questions were scored on a scale of 0 to 3. In Task 2, a full score of three was given if students were able to describe the mathematical relationship between Box A and Box B which made the respective number sentence true; and the same for Task 3, if they could explain why the number sentence could still be true if any number was used in Box A. Finally, for the two sentences involving *c* and *d* or *a* and *b*, students were given a score of 1 if they gave a specific pair of values for *c* and *d* or *a* and *b* which made the sentence true; they received a score of 2 if they were able to give a partial explanation of the relationship between *c* and *d* (or *a* and *b*); and a score of 3 if they were able to describe correctly the relationship between *c* and *d* or *a* and *b* which made the sentence true. For example, a fully correct response to Question 2 Task 5 for multiplication might take the form "*a* is always double the value of *b*", or $a = 2b$, or "*b* is half of *a*".

# Results

The results of Algebraic Thinking Questionnaire (Multiplication) enabled the cohort to be divided into four approximately equal sized groups, based on the closest whole number score. As is clear from Table 4, the four groups have unequal ranges on ATQ (M) with three-quarters of the cohort scoring less than half the possible score. While there were many low scores, there were, however, 19 students who scored 18 or more on ATQ (M).

Table 4
*Performance by groups*

| Groups N= 162 | ATQ (M) Range | Reverse fraction tasks: Mean Score | Fraction Screening Test: Mean Score | |
|---|---|---|---|---|
| | | | Reverse fraction tasks included | Reverse fraction tasks excluded |
| G1 n = 47 | $0 \le r \le 2$ | 3.4 | 35.9 | 32.5 |
| G2 n = 48 | $3 \le r \le 6$ | 3.8 | 42.9 | 39.1 |
| G3 n = 29 | $7 \le r \le 11$ | 6.9 | 50.4 | 43.5 |
| G4 n = 38 | $12 \le r \le 26$ | 7.5 | 53.9 | 46.4 |

The mean score for the Fraction Screening Test was 44.8 out of a possible 60 marks. The mean score for the three reverse thinking tasks was 5.1 out of a possible 9 marks while the mean score for the Fraction Screening Test excluding the three reverse fraction questions was 39.7 out of a possible 51 marks.

The differences between the mean scores of the bottom half and the top half is quite striking in respect of the three reverse fraction tasks. Between the first two groups there is only a small increase in the mean score on these three questions. The big jump is between the second and third group, from 3.8 to 6.9; and from 6.9 to 7.5 out of a possible 9, from the third to the fourth group.

In the first group there were, however, four students who scored full marks, completely answering the three reverse fraction questions. All four students used a mix of additive and multiplicative methods. Three students in the second group scored full marks on the reverse fraction questions. Two students used a mix of additive and multiplicative methods. However, in the third and fourth groups there were 14 (out of 29) and 25 (out of 38) students respectively who answered all three questions correctly. Almost all of these students employed multiplicative methods. In the third and fourth group only two students scored 0 on the reverse fraction questions.

### *Relationship between ATQ (M) and the Fraction Screening Test (FST)*

The fourth and fifth columns of Table 4 examine the performance of the four groups based on ATQ (M) and students' corresponding performances on the FST. There is again a clear progression between the scores of each group on the ATQ (M) and students' overall scores on the FST. Once more, when comparing the first and fourth groups, there is a large difference of 18 points in mean scores on the FST. Similar differences are observed between the lower and upper halves. Even when the scores of three reverse thinking fraction questions are removed from the total score of the FST, the fifth column of the table reflects the same large differences between mean scores for the first and fourth groups and also for the lower and upper half of the cohort; showing that increasing fractional competence is clearly associated with improved performance on the ATQ (M).

While ATQ (D) was not used in the above analysis, both components of the ATQ were both correlated (0.47) with students' performances on the three reverse fraction questions in Figure 3. Students who scored highly ATQ (M) also scored highly on ATQ (D). Higher performances on the Fraction Screening Test overall were consistently matched by comparably higher performances on the ATQ (M) with both tests having an overall correlation coefficient of 0.59. Students with higher scores on the reverse thinking fraction questions were far more likely to be in the upper two groups of performance on the ATQ (M). From these results we are entitled to say that fractional competence is a predictor of algebraic thinking, but not a unique predictor.

## *Specific student responses to the three Reverse Fraction Tasks*

Some students used purely visual methods especially Reverse Fraction Task 1 arguing that an extra row was required. Visual methods were rarely successful for the third task except where some students identified seven pairs of two counters. Very few students were able to create a visual representation for Reverse Fraction Task 2. The following examples are taken from students who either relied entirely on multiplicative thinking or used a mix of multiplicative and additive thinking to solve these three questions. The following sample responses are drawn from students whose scores put them in the top group on ATQ (M).

*Two responses to Reverse Fraction Task 1:* After making the connection between the number of objects in the group and the fractional part, these students determined the number of objects in the unit fraction by dividing by the numerator of the given fraction. Student JL (Figure 2) uses equivalence between the 10 counters and "two parts out of three". This reasoning anticipates exactly what will later be needed to solve the related algebraic equation $\frac{2}{3}x = 10$. Student JL moves from "two parts" to "one part" to "three parts" without needing to state the fractional value represented by each "part". As each "part" is transformed so is the number of counters represented by each "part", until Student JL correctly concludes: "You started with 15 counters".

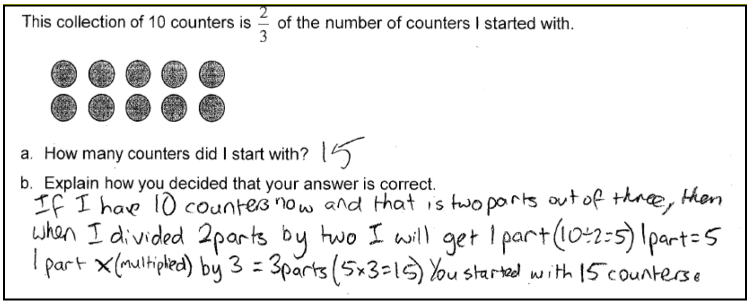


Figure 2: Student JL response

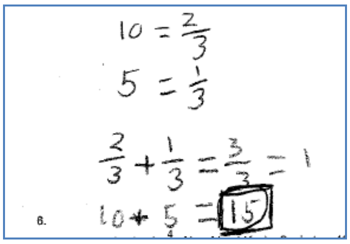


Figure 3: Student JB response

In Figure 3 Student JB again starts with the statement that ten (counters) equals two-thirds; correctly finds the unit fraction but uses an additive method to find one whole, matching all three fractions to the number of objects represented. We call this a mixed method as distinct from the fully multiplicative method used by Student JL. The equivalent expression $10 = \frac{2}{3}$ is also treated like $\frac{2}{3}x = 10$.

*Two responses to Reverse Fraction Task 2:* These responses comprise one that is fully multiplicative and one that involves a mix of multiplicative and additive methods. Student DC (Figure 4) uses no verbal elaborations. The fraction four-sevenths is not stated explicitly but is implied in the division by four. In the second line Student DC states the equivalence between the number of objects and the fraction one-seventh while the third line shows the transformation needed to go from one-seventh to a whole without needing to refer to the fraction. This method in fact anticipates how one would solve $\frac{4}{7}x = 12$

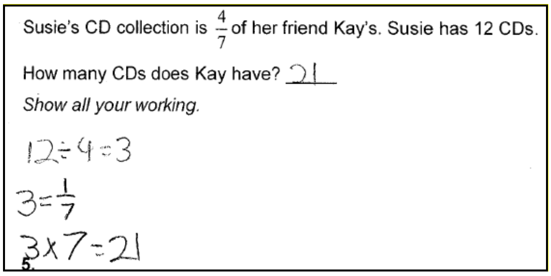



Figure 4: Student DC response

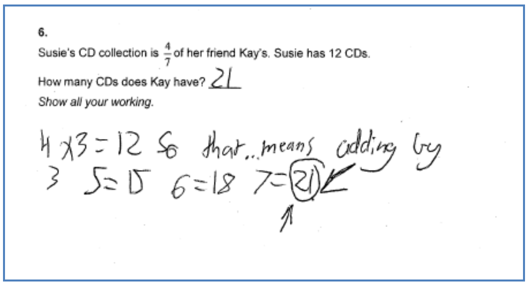



Figure 5: Student RH response

In the next example (Figure 5), Student RH uses a different pathway to find the unit fraction and to state its equivalence to 3 CDs. This student then returns to the initial relationship (four-sevenths being equivalent to 12 CDs) and successively finds the next three sevenths to reach a whole. At each stage Student RH relates the increasing value of the fraction to the quantities it represents.

*One response to Reverse Fraction Task 3:* Student RY shown above in Figure 6 uses a fully multiplicative approach, with no circling, dividing by the numerator of the given fraction to find one-sixth, and then multiplies this by six to find the whole. Student RY treats the initial relationship the same as one would the corresponding algebraic expression, $\frac{7}{6}x = 12$. Some other students found the whole by a mixed method using subtraction.

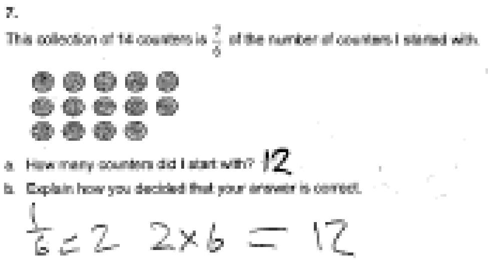



Figure 6: Student RY

## Conclusion

Since these students in Years 5 and 6 have not yet met equations written in algebraic form, it is not surprising that their underlying reasoning was expressed sometimes in forms that were constructed and sometimes using mathematical shorthand that some might regard as unconventional. Students who used mixed methods demonstrate conceptual understanding by moving from the given fraction to the unit fraction. Scaling up the unit fraction and its related quantities to find a whole can then be achieved additively only by relating successive fractions to the quantities they represent. Students who use fully multiplicative thinking know how to transform a given fraction to obtain the corresponding unit fraction. Having obtained the unit fraction, these students successfully scale up the unit fraction and its related quantities multiplicatively to find the whole. These multiplicative methods, which most clearly mirror the thinking needed to solve the corresponding algebraic equations, can be seen as uniquely accurate predictors of algebraic thinking.

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