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Developmental Growth Trajectories in Understanding of Fraction Magnitude From Fourth Through Sixth Grade

Ilyse Resnick, Nancy C. Jordan, Nicole Hansen, Vinaya Rajan, and Jessica Rodrigues
University of Delaware

Robert S. Siegler
Carnegie Mellon University

Lynn S. Fuchs
Vanderbilt University

Development of fraction number line estimation was assessed longitudinally over 5 time points between 4th and 6th grades. Although students showed positive linear growth overall, latent class growth analyses revealed 3 distinct growth trajectory classes: Students who were highly accurate from the start and became even more accurate (n = 154); students who started inaccurate but showed steep growth (n = 121); and students who started inaccurate and showed minimal growth (n = 197). Younger and minimal growth students typically estimated both proper and improper fractions as being less than 1, failing to base estimates on the relation between the numerator and denominator. Class membership was highly predictive of performance on a statewide-standardized mathematics test as well as on a general fraction knowledge measure at the end of 6th grade, even after controlling for mathematic-specific abilities, domain-general cognitive abilities, and demographic variables. Multiplication fluency, classroom attention, and whole number line estimation acuity at the start of the study predicted class membership. The findings reveal that fraction magnitude understanding is central to mathematical development.

Keywords: fractions, numerical magnitude representation, numerical development, number line, mathematics achievement

Reasoning about fractions is vital to our daily lives as well as for learning more advanced mathematics and science (National Mathematics Advisory Panel [NMAP], 2008; Siegler et al., 2012). However, fractions are difficult for many children (e.g., Bailey, Hoard, Nugent, & Geary, 2012; Ni & Zhou, 2005) and even many adults (Schneider & Siegler, 2010; Vosniadou, Vamvakoussi, & Skopeliti, 2008). For example, when asked to estimate the sum of 12/13 + 7/8 from the response options 1, 2, 19, and 21, 55% of 13-year-olds, 36% of 17-year-olds (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981), and 15% of college students at a major university (Lewis & Hubbard, 2015) estimated the sum to be either 19 or 21. These errors suggest that many students do not understand the relation between the numerator and the denominator; in other words, they do not have a good sense of fraction magnitudes.

In the present study, we chart the developmental course of students' fraction magnitude understanding, a main source of difficulty in fraction learning (Siegler, Fazio, Bailey, & Zhou, 2013). In particular, we examine growth in fraction number line estimation acuity between fourth and sixth grades, a critical period for fraction learning (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2006). Examining the development of fraction learning during this time period is particularly important, as students who leave sixth grade with weak fraction knowledge experience cascading mathematics difficulties through the remainder of middle school and in high school (Siegler et al., 2012; Siegler & Pyke, 2013).

Role of Fractions in Numerical Development

Although fractions play an integral role in mathematical learning, many theories of numerical development either fail to include fractions in their frameworks (Siegler et al., 2013; Siegler, Thompson, & Schneider, 2011) or view the development of numerical knowledge as a segmented process, in which whole number knowledge is acquired naturally and fraction knowledge is later acquired with great difficulty (e.g., Geary, 2004, 2006; Gelman & Williams, 1998). Knowledge of whole number principles (e.g., one-to-one correspondence) is even viewed as interfering with fraction learning (e.g., Gelman & Williams, 1998; Wynn, 1995). Siegler and colleagues (Siegler & Lortie-Forgues, 2014; Siegler et al., 2011) argue, however, that the development of all real numbers

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Ilyse Resnick, Nancy C. Jordan, Nicole Hansen, Vinaya Rajan, and Jessica Rodrigues, School of Education, University of Delaware; Robert S. Siegler, Department of Psychology, Carnegie Mellon University; Lynn S. Fuchs, Department of Special Education, Vanderbilt University.

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Correspondence concerning this article should be addressed to Nancy C. Jordan, School of Education, University of Delaware, 206E Willard Hall, Newark, DE 19716. E-mail:njordan@udel.edu

(e.g., integers and fractions) can be unified as one continuous process.

According to the "integrated theory of numerical development," number learning involves understanding that all real numbers have magnitudes that can be ordered and located on a number line (Siegler & Lortie-Forgues, 2014). Although the time course of different developments within the process overlap, young children typically begin by learning nonsymbolic magnitude representations (e.g., which set of dots has more), link these nonsymbolic understandings to symbolic representations of small whole numbers, gradually acquire accurate representations of larger whole numbers, and eventually come to see that an infinite set of rational numbers can be represented.

Fractions are a particularly important part of this process, as they require a reorganization and deeper understanding of numerical knowledge; children must see that properties of whole numbers do not always apply to fractions (Siegler & Lortie-Forgues, 2014; Siegler et al., 2011). Although whole number and fraction operations share the same conceptual structures (Alibali & Sidney, 2015), those operations can have different outcomes; for example, multiplication of whole numbers never leads to an answer smaller than either operand and division of a whole number by a whole number never leads to an answer larger than the number being divided, but multiplication and division with fractions less than one always produce such outcomes. However, one property that unites both fractions and whole numbers is that both types of numbers have magnitudes that can be represented on a number line (Case & Okamoto, 1996; Siegler & Lortie-Forgues, 2014; Siegler et al., 2011).

Indeed, there is growing evidence to suggest that understanding magnitude is key to mathematics learning. The ability to represent whole number magnitudes, for example, predicts fraction learning (e.g., Hansen et al., 2015; Jordan et al., 2013; Vukovic et al., 2014). Further, the representation of magnitudes for whole numbers (e.g., Booth & Siegler, 2006, 2008; Halberda, Mazzocco, & Feigenson, 2008; Holloway & Ansari, 2008; Jordan et al., 2013; Sasanguie, Göbel, Moll, Smets, & Reynvoet, 2013) and fractions (Bailey et al., 2012; Siegler & Pyke, 2013; Siegler et al., 2011, 2012) both predict overall mathematics knowledge.

Development of Fraction Magnitude Understanding

Little is known about the developmental trajectory of fraction magnitude understanding. There is some evidence that even prior to receiving formal instruction, children already possess rudimentary fraction knowledge (Mix, Levine, & Huttenlocher, 1999); 4-year-olds can complete fraction calculations using spatial patterns (Mix et al., 1999), 5-year-olds can systematically divide a whole into equal shares (Hunting & Sharpley, 1988), and 6-year-olds can perceptually identify which of two boxes has a relatively greater proportion of objects (Spinillo & Bryant, 1991).

During formal instruction using fraction notation, which typically occurs in U.S. schools from fourth through sixth grades, children relate fractions to their knowledge of whole numbers. Unfortunately, this association often results in the overgeneralization of whole number principles, which in turn interferes with fraction understanding (Gelman, 1991; Lamon, 1999; Ni & Zhou, 2005; Vosniadou et al., 2008). For example, when adding fractions, a common error is to treat the fractional components as if they are whole numbers, adding numerators together as well as denominators. Connecting fraction symbols with the magnitudes they represent, which is required in tasks such as placing fractions on a number line, poses lasting challenges (Siegler et al., 2012).

Presumably, fraction number line acuity is sensitive to change during the critical period between fourth and sixth grades, when the bulk of fraction instruction occurs in U.S. schools (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Although McMullen, Laakkonen, Hannula-Sormunen, and Lehtinen (2014) report that elementary school students' development of fraction magnitude understanding (i.e., students' ability to circle the larger of two fractions) is relatively slow, the researchers only examined growth over a 1-year period. It is possible that examination of fraction magnitude understanding over a longer time frame is required to capture more evidence of growth. It also seems likely that students follow different growth trajectories in fraction magnitude understanding; some students' knowledge may stay relatively flat, whereas others' knowledge may exhibit steeper growth in response to fraction instruction. Identifying individual differences in developmental growth trajectories from fourth to sixth grade, the primary period of fractions instruction, can help uncover potential problems in magnitude understanding before they become entrenched.

It is also important to examine whether development of fraction magnitude estimation differs according to type of fraction, that is, proper, improper and mixed numbers. A proper fraction's numerator is smaller than the denominator (e.g., 3/4); thus, its magnitude is always less than one. An improper fraction has a numerator equal to or greater than the denominator (e.g., 4/3); therefore, its magnitude is always one or greater. A mixed number is a combination of a whole number and a proper fraction (e.g., 1 3/4). If numerical development is the process of learning the magnitudes of increasing ranges and types of numbers (Siegler & Lortie-Forgues, 2014), it seems likely children do not learn to reason about all types of fractions at the same time.

Different types of fractions may elicit different magnitude estimation strategies (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Schneider & Siegler, 2010). Early fraction instruction usually emphasizes proper fractions, leading many children to view all fractions as numbers between 0 and 1 (Vosniadou et al., 2008). Although a simple strategy of defining fractions as "numbers between 0 and 1" can result in reasonable estimates of proper fractions, the same strategy works poorly for reasoning about improper fractions. For example, a student who incorrectly views any number in the format of a/b as being "between 0 and 1" will exhibit greater error when estimating improper fractions (e.g., 11/4) compared to proper fractions (e.g., 4/11). Thus, accurately representing the magnitudes of improper fractions requires deeper understanding of the relation between the numerator and denominator than do proper fractions.

Relationship Between Fraction Magnitude Understanding and Mathematics Achievement

Fraction magnitude understanding provides an underlying structure for learning a range of fraction concepts (Hecht, 1998; Siegler et al., 2011) and procedures (Hallett, Nunes, & Bryant, 2010; Hecht & Vagi, 2010; NMAP, 2008; Siegler et al., 2011; Vamvakoussi & Vosniadou, 2010). The predictive capability of fraction magnitude estimation extends beyond fraction knowledge to include algebra knowledge (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014; Brown & Quinn, 2007) and overall mathematics achievement (Bailey et al., 2012; Siegler & Pyke, 2013; Siegler et al., 2011, 2012), even though these types of tests typically assess fraction magnitude understanding minimally if at all (Siegler et al., 2011). Moreover, relations among fraction magnitude understanding, fraction arithmetic, and prealgebra/algebra knowledge strengthen between sixth and eighth grades (e.g., Bailey et al., 2012; Siegler & Pyke, 2013). Additionally, fraction and whole number symbolic magnitudes are more predictive than nonsymbolic magnitude understanding (Fazio, Bailey, Thompson, & Siegler, 2014). It has been recently argued that the relationship between number line acuity and mathematics achievement is mutually supportive, with each skill influencing the other in a bidirectional fashion (Friso-van den Bos et al., 2015). That is, number line acuity influences mathematics achievement, which in turn influences number line acuity.

Overview of the Present Study

The present study assessed development of fraction number line estimation acuity longitudinally, from fourth through sixth grades, over five time points. Although we expected students on average to grow linearly, we also predicted that empirically distinct trajectory classes would be revealed by latent class growth analyses. That is, although we expected fraction number line estimation to be difficult for most students at the beginning of fourth grade, we predicted subgroups of students could be identified who vary in their subsequent growth rates. The predictive capability of observed class membership was assessed for general fraction knowledge and for overall mathematics achievement on a high-stakes state test at the end of sixth grade.

Additionally, we assessed factors at the start of the study that we hypothesized would predict growth class membership. Domain-specific, domain-general, and demographic factors all predict overall mathematics achievement (Geary, 2004) and fraction knowledge in particular (Hecht & Vagi, 2010). In the present study, we examined two domain-specific precursors—whole number line estimation (third grade) and multiplication fluency (fourth grade). According to the integrated theory of numerical development, children learn an increasingly wide range and type of numbers (Siegler & Lortie-Forgues, 2014); thus, understanding whole number magnitudes should provide the initial structure for learning fractions. Indeed, whole number arithmetic skill in first grade predicts fraction arithmetic skill in middle school, even after controlling for a range of demographic factors and general cognitive variables (Bailey, Siegler, & Geary, 2014). Calculation fluency also is a key component of mathematical competence (NMAP, 2008), with multiplication skill being especially important to fraction learning (Hansen et al., 2015). For example, it is hard to see that 4/12 and 6/18 are equivalent without knowing that each denominator is three times the numerator. Our model also included a measure of classroom attention, a domain-general ability consistently linked to mathematics proficiency (e.g., Fuchs et al., 2005, 2006) and fraction skills (Hecht & Vagi, 2010). We assessed reading fluency as a divergent measure and included demographic information about students' age, gender, and family income in our models.

No previous studies have examined the development of fraction magnitude estimation over this critical 3-year period of fraction instruction. If understanding of different types of fractions develops at different rates, this would suggest that children are continually expanding their repertoire of numerical magnitudes, which would provide support for the integrated theory of numerical development. By examining estimation patterns, we can identify common strategies and misconceptions in fraction magnitude understanding. The findings might also provide a learning progression structure for educators regarding how and when students learn about different types of fractions and provide information about variability among students in these acquisitions. Characterizing distinct growth trajectories of students for specific fraction types and notations can also help identify students in need of intervention and inform the development of targeted teaching materials.

Method

Participants

Students were drawn from nine elementary schools within two adjacent school districts serving families of diverse socioeconomic backgrounds. Letters describing the study, along with consent forms, were sent to families of all children in third grade. A total of 517 students returned consent forms to participate in this study, of whom 36 opted out before the first assessment. Students were followed longitudinally through third, fourth, fifth, and sixth grades. The sample was replenished in fourth grade ($n = 27$ new children) and fifth grade ($n = 28$ new children), resulting in a total sample of 536 students. Attrition rates were as follows: by the end of third grade, 23 students were no longer participating in the study, an additional 68 by the end of fourth grade, an additional 48 by the end of fifth grade, and an additional 39 by the end of sixth grade. Reasons for attrition included students moving to another school district out of the study (67%), no information being provided on students' elementary to middle school transition (23%), and students withdrawing from the study (10%).

The sample was 47% male, 51.9% White, 40.0% Black, 5.7% Asian/Pacific Island, and 2.5% American Indian/Alaskan Native; 17.7% of children identified their ethnicity as Hispanic. The majority of students (60.9%) participated in a school free/reduced lunch program and thus were classified as low income. Children's mean age at the start of the study was 105.9 months ($SD = 5.35$). The sample contained 10.6% English learners (ELs), and a separate 10.6% of students who were receiving special education services. Starting in fourth grade, all students were taught with curricula that followed the content and sequence of Common Core State Standards in mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Measures

Fraction number line estimation. Fraction magnitude understanding was assessed using a fraction number line estimation task (Siegler et al., 2011). Students estimated the location of fractions on 0–1 and 0–2 number lines. Each number line was 17.5 cm long and presented in the middle of the screen on a laptop computer using DirectRT v2012 (Empirisoft, New York, NY). Fractions were presented one at a time beneath the middle of the number line. For each item, the cursor was set at 0; students used the arrow keys to slide the cursor along the number line, and then pressed a different key to indicate their response. After making their estimate, a new blank number line and a new fraction were presented and the cursor was reset to 0. Students had no time constraints, although most responded within a few seconds per trial.

For the 0–1 number line task, students began by observing the assessor demonstrate where 1/8 would be located, and then completed a practice trial estimating the fraction 1/4 without feedback. The students then estimated the locations of 1/5, 13/14, 2/13, 3/7, 5/8, 1/3, 1/2, 1/19, and 5/6 in that order. The same procedure was used on the 0–2 number line task, with the assessor modeling where fraction 1/8 and 1 1/8 would be located. For the 0–2 number line, the students estimated the location of the fractions and mixed numbers 1/3, 7/4, 12/13, 1 11/12, 3/2, 5/6, 5/5, 1/2, 7/6, 1 2/4, 1, 3/8, 1 5/8, 2/3, 1 1/5, 7/9, 1/19, 1 5/6, and 4/3 in that order. Thus, students estimated the locations of 28 fractions and mixed numbers.

Participants' percent absolute error (PAE) was calculated by dividing the absolute difference between the estimated and actual magnitudes by the numerical range of the number line (1 or 2), and then multiplying by 100 for each estimate. For example, if a child was asked to locate 7/4 on a 0–2 line and marked the location corresponding to 5/4 the PAE would be 25% [$|(1.75 - 1.25)|/2 \times 100$]. Each student was assigned a single score by taking his or her mean PAE. Internal reliability on the 0–1 and 0–2 number line task was greater than .87 at all time-points in the study.

Whole number line estimation. Whole number estimation was assessed using a whole number line estimation task (Siegler & Opfer, 2003). A 25-cm line with 0 at the left end and 1,000 at the right was used. Students first indicated the locations of 0 and 1,000 on the number line and then estimated the location of 150, with feedback regarding its correct location. No feedback was given during the test trials. Task items in the order of presentation were: 56, 606, 179, 122, 34, 78, 150, 938, 100, 163, 754, 5, 725, 18, 246, 722, 818, 738, 366, 2, 486, and 147. Paper and pencil were used for problem presentation and responses. The score on the task was the mean PAE, which was calculated using the same procedure as the fraction number line estimates. Internal reliability between all whole number estimates was .89 for the present sample.

Multiplication fluency. The Multiplication Fluency subtest of the Wechsler Individual Achievement Test (WIAT; The Psychological Corporation, 1992) was used. The multiplication fluency subtest has a reliability coefficient of .90 in fourth grade. Students were given 1 minute to solve as many of 40 multiplication problems as they could. The WIAT is a paper-and-pencil-based task. All items had multiplicands between zero and 10 inclusive. Students earned one point for each correct response.

Reading fluency. Reading fluency was assessed using the Sight Word Efficiency subtest of the Test of Word Reading Efficiency (TOWRE; Torgesen, Wagner, & Rashotte, 1999). The TOWRE has a reliability coefficient exceeding .90 in fourth grade. Students were given 45 s to read aloud as many words as they could from a list of 104 written words. Students earned one point for every correctly read word.

Attention. Classroom attentiveness was assessed using the Inattentive Behavior subscale of the SWAN Rating Scale (Swanson et al., 2006). This instrument's nine items are based on the criteria for attention-deficit/hyperactivity disorder for inattention of the *Diagnostic and Statistical Manual of Mental Disorders* (4th ed.; American Psychiatric Association, 1994). In its present use, teachers rated children's attention during their fourth-grade mathematics classes on a scale of 1 (*below average*) to 7 (*above average*) for each item. Thus, raw scores could range between 9 (a score of 1 on each item) and 63 (a score of 7 on each item). The Inattentive Behavior subscale had high internal consistency for this sample ($\alpha = .97$).

General fraction knowledge. General fraction knowledge was assessed using a combined measure of concepts and procedures. Fraction concepts were evaluated using three items that required students to shade sections of a polygon or set of polygons corresponding to a given fraction (Hecht, Close, & Santisi, 2003), and 25 released fraction concepts items from the National Assessment of Educational Progress (NAEPs; U.S. Department of Education, 2007, 2009). Thus it included 28 items. Fraction procedures were assessed using 26 fraction computation items adapted from Hecht (1998). Students earned one point for each correct answer. Test items had an internal reliability of .92 in the spring of sixth grade.

Mathematics achievement. The DCAS is a statewide, standardized test that was given to all students in the participating school districts in sixth grade (American Institutes for Research, 2012). It is a computer-based multiple-choice test that automatically adjusts item difficulty based on student performance. Internal consistency for the mathematics section of the DCAS for sixth grade exceeds .88, and is aligned with the Common Core State Standards, providing strong construct validity. Students are classified with scaled scores of 1 (*well below standards*), 2 (*below standards*), 3 (*meets standards*), or 4 (*advanced*).

Procedure

These data come from a larger longitudinal study on students' mathematical development. Each student was assessed on fraction number line estimation at five separate time points: winter and spring of fourth grade; fall and spring of fifth grade; and winter of sixth grade. Whole number line estimation was assessed in the winter of third grade and multiplication fluency (WIAT), reading fluency (TOWRE), and attentive behavior (SWAN) were assessed in the winter of fourth grade. The fraction knowledge and mathematics state test (DCAS) was administered the spring of sixth grade.

The DCAS was administered by the school district following published guidelines. Trained assessors from our research team administered the remaining tasks and read aloud each set of instructions. All measures were administered individually in a quiet setting, with the exception of the multiplication fact fluency and fraction knowledge tests, which were given in a whole group classroom setting.

Age of entry into third grade served as the age variable. Binary variables of gender and income status, respectively, were coded as 1 (female; participated in school's free/reduced price lunch program) or 0 (male; did not participate in the school's free/reduced price lunch program).

Results

Little's (1988) Missing Completely at Random (MCAR) test results for all measures used in the present study were significant (χ^2 = 347.977; *df* = 273; *p* = .001), indicating data were not missing completely at random. Students who were missing more data (typically due to moving outside participating schools) were approximately 70% low income, which is significantly higher than the sample as a whole (χ^2 = 6.328; *df* = 1; *p* = .012). Mobility patterns similar to these are common for low-income students (Jordan, Kaplan, Ramineni, & Locuniak, 2009; Smith, Fien, & Paine, 2008). No significant differences in the demographics of the students were observed between students who took one or two (out of five) of the fraction number line estimation assessments (*n* = 130) and those students who took three, four or five of the fraction number line estimation assessments (*n* = 406). Additionally, once students who did not complete any fraction number line estimation assessments were removed from the analysis, the remaining data was missing completely at random (χ^2 = 252.88; *df* = 223; *p* = .083).

Table 1 shows the means and standard deviations on all measures. For ease of interpretation, grade-based percentile scores are presented for standardized measures. For most analyses, raw scores were used; the one exception was the multinomial regression analyses, in which all nonbinary predictor variables were standardized for ease of interpretation. Correlations among all variables are shown in Table 2.

Growth curve and latent class growth analyses both utilize maximum likelihood estimation. In practical terms, maximum likelihood estimation allows researchers to use all data that was collected, but does not impute missing data. Therefore, any participant who took the fraction number line estimation task at least once during the study period was eligible to be included in these analyses. List-wise deletion was used for subjects (*n* = 64) who were missing fraction number line estimation scores at all time points (see analysis above regarding attrition). By using analyses that can handle missing data, the majority of the sample could be conserved, leading to less loss of statistical power. Most children (*n* = 406) took the fraction number line estimation task at least three times.

The results of the baseline growth curve model are shown in Table 3. The fraction number line estimation score at the end of sixth grade was significantly different from zero, and as can be seen from the slope coefficients, the linear decrease in fraction number line estimation PAE was statistically significant as well. This suggests that students become more accurate at estimating the location of fractions on a number line from fourth to sixth grade.

A goal of this study was to examine underlying latent classes of growth in fraction number line estimation. Again, the intercept was set at sixth grade. Latent class growth analysis uncovered three distinct growth classes for fraction estimation (see Figure 1). Students in Class 1 (*n* = 154) are characterized by a high level of accuracy beginning in fourth grade that continued through sixth grade (starts accurate; ends accurate). Students in Class 3 (*n* = 197) exhibited a low level of accuracy from fourth through sixth grade (starts inaccurate; ends inaccurate). Students in Class 2 (*n* = 121) were inaccurate in fourth grade, similar to those in Class 3, but their estimation accuracy in sixth grade was comparable to those in Class 1 (starts inaccurate; ends accurate). Table 4 shows the intercept and slope for each class.

Table 1
Means and Standard Deviations of Measures

Measures	*M* (*SD*)	*n*
Predictor measures		
Whole number line estimation (PAE)	10.94 (6.72)	463
Multiplication fluency (WIAT Multiplication; percentile)	61.02 (27.20)	423
Attention (SWAN; 63)	38.08 (12.76)	415
Reading fluency (TOWRE; percentile)	60.45 (22.37)	421
Fraction number line estimation outcome measures		
Winter 4th grade		
0–1 (PAE)	25.79 (11.20)	421
0–2 (PAE)	23.86 (8.13)	421
0–1 and 0–2 combined (PAE)	24.45 (8.51)	421
Spring 4th grade		
0–1 (PAE)	20.33 (11.38)	418
0–2 (PAE)	18.64 (9.00)	418
0–1 and 0–2 combined (PAE)	19.16 (9.33)	418
Fall 5th grade		
0–1 (PAE)	19.87 (11.97)	409
0–2 (PAE)	18.51 (9.72)	409
0–1 and 0–2 combined (PAE)	19.16 (10.46)	409
Spring 5th grade		
0–1 (PAE)	16.65 (12.16)	401
0–2 (PAE)	15.33 (9.92)	401
0–1 and 0–2 combined (PAE)	15.75 (10.26)	401
Winter 6th grade		
0–1 (PAE)	13.42 (11.08)	362
0–2 (PAE)	12.74 (9.30)	362
0–1 and 0–2 combined (PAE)	12.96 (9.52)	362
Fraction knowledge (54)	32.65 (9.36)	356
Mathematics achievement (DCAS; 1–5 scale)	2.78 (1.02)	342

Note. All scores are raw scores unless indicated otherwise. Number in parentheses indicates total number of items. Number line estimation is coded as percent absolute error; therefore, higher scores indicate poorer performance. PAE = participants' percent absolute error; WIAT = Wechsler Individual Achievement Test; SWAN = Strengths and Weaknesses of ADHD symptoms and Normal Behavior Rating Scale; TOWRE = Test of Word Reading Efficiency; DCAS = Delaware Comprehensive Assessment System.

Despite the large gains achieved by children in Class 2, all latent class groups remained significantly different from each other on fraction magnitude estimation accuracy at the end of sixth grade, $F(2, 359) = 559.69$, $p < .001$. Sixth graders in Class 1 (starts accurate, ends accurate) outperformed those in Class 2 (starts inaccurate; ends accurate; mean difference = −4.55, *SE* = .64, $t(141.02) = -10.96$) and Class 3 (starts inaccurate, ends inaccurate; mean difference = −18.6, *SE* = .58, $t(170.28) = -32.31$). Sixth graders in Class 2, in turn, outperformed peers in Class 3 (mean difference = −14.06, *SE* = .67, $t(233.00) = -21.14$).

A multinomial logistic regression was conducted to examine the odds of Class assignment (1, 2, or 3) based on initial characteristics and skills. See Table 5 for a summary of odds ratios. Class 2 was set as the comparison class, because it was of particular interest to distinguish between students in Class 2 and Class 3 in order to identify factors that led some students to show more improvement than others. Results from the multinomial regression analysis indicated that the model provided a statistically significant improvement over the constant-only model, (χ^2 = 200.43, *df* = 14,

Table 2
Correlations Among All Variables

Variables	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13
1. Income	—												
2. Female	−.036	—											
3. Age	.140**	−.032	—										
4. Whole NLE	.171**	.195**	.159**	—									
5. Multiplication fluency	−.212**	−.054	−.220**	−.420**	—								
6. Attention	−.167**	.198**	−.123*	−.309**	.438**	—							
7. Reading fluency	−.205**	.049	−.252**	−.265**	.455**	.354**	—						
8. Fraction NLE (Winter 4th grade)	.261**	.030	.187**	.464**	−.422**	−.428**	−.260**	—					
9. Fraction NLE (Spring 4th grade)	.258**	.123*	.239**	.469**	−.426**	−.395**	−.279**	.714**	—				
10. Fraction NLE (Fall 5th grade)	.264**	.122*	.206**	.451**	−.471**	−.387**	−.304**	.674**	.861**	—			
11. Fraction NLE (Spring gth Grade)	.197**	.159**	.255**	.500**	−.462**	−.423**	−.294**	.592**	.792**	.808**	—		
12. Fraction NLE (Winter 6th grade)	.228**	.079	.256**	.502**	−.473**	−.423**	−.263**	.575**	.699**	.697**	.833**	—	
13. Fraction knowledge (Spring 6th grade)	−.198**	.050	−.245**	−.477**	.583**	.557**	.333**	−.608**	−.643**	−.634**	−.638**	−.649**	—

Note. NLE = number line estimation. Number line estimation is coded as percent absolute error; therefore, higher scores indicate poorer performance.
* $p < .05$. ** $p < .01$.

$p = .001$). Nonsignificant Pearson and deviance statistics to assess goodness-of-fit showed good model fit with the data ($p > .05$).

The Wald statistic is used to evaluate the unique contribution of each coefficient to the model. Here, the Wald test identified four predictors: age, multiplication fluency, attention, and whole number line estimation. Students who were one standard deviation older than average at the start of third grade were 1.4 times more likely to fall into Class 3 (starts inaccurate; ends inaccurate) than Class 2 (starts inaccurate; ends accurate). Better-than-average multiplication fact fluency, attention, and whole number line estimation emerged as protective factors. Students with multiplication fluency scores one standard deviation above average were approximately 35% less likely to be in Class 3 than Class 2. Students with attention ratings one standard deviation above average were 1.6 times more likely to be in Class 1 (starts accurate; more ends accurate) than Class 2. Finally, students with whole number line estimation scores one standard deviation lower (i.e., more accurate) than average were 70% more likely to be in Class 1 than Class 2. Students with whole number line estimation scores one standard deviation higher (i.e., less accurate) than average were approximately twice as likely to be in Class 3 compared to Class 2.

Table 3
Latent Growth Curve Model for Fraction Number Line Estimation (0–1 and 0–2)

	Estimate	*SE*
Intercept	12.891***	.478
Slope	−.376***	.018
Var (intercept)	86.857***	6.880
Var (slope)	.056***	.010

Note. Var () represents the variance of parameters in parentheses.
*** $p < .001$.

Estimates for fractions on the 0–2 number line were assessed by comparing the mean estimate to actual magnitudes for each whole number, proper fraction, improper fraction and mixed number (from smallest to largest) for each latent class (1, 2, and 3) and grade (spring of fourth and fifth, and winter of sixth). Estimates on the 0–1 number line are not analyzed here because, due to the nature of the task, there were no improper fractions or mixed numbers. Overall, proper fractions and mixed numbers were easier than improper fractions for most children (see Figure 2). While fourth graders in class 1 estimated accurately overall, they were more accurate on proper fractions and mixed numbers than on improper fractions; they tended to estimate the location of improper fractions between the location of one half and one on the line. By fifth grade, after fraction instruction, children in Class 1 were extremely accurate. Peers in Classes 2 and 3, on the other hand, had difficulty with estimates of all three types of fractions in fourth grade; they estimated proper and improper fractions equidistant between the locations of 0 and 1, and they estimated the locations of mixed numbers near 1 on the number line. In fifth and sixth grade, the gap in estimation accuracy between children in Class 2 and Class 3 widened. Children in Class 3 showed little change across the grades, but the estimation accuracy of children in Class 2 increased and approached that of children in Class 1.

Fraction number line estimation class membership was associated with general fraction knowledge at the end of sixth grade, $F(2, 398) = 154.46$, $p < .001$, even after controlling for age, gender, income, attentive behavior, whole number line estimation, multiplication fluency, and reading fluency, $F(2, 274) = 50.94$, $p < .000$. Students in Class 1 (starts accurate; ends more accurate) scored higher on the assessment of general fraction knowledge compared to those in Class 2 (starts inaccurate; ends accurate; $t(239.81) = 9.35$, $p < .001$, $d = 1.21$) and Class 3 (starts inaccurate; ends inaccurate; $t(286.18) = 16.59$, $p < .001$, $d = 1.96$).

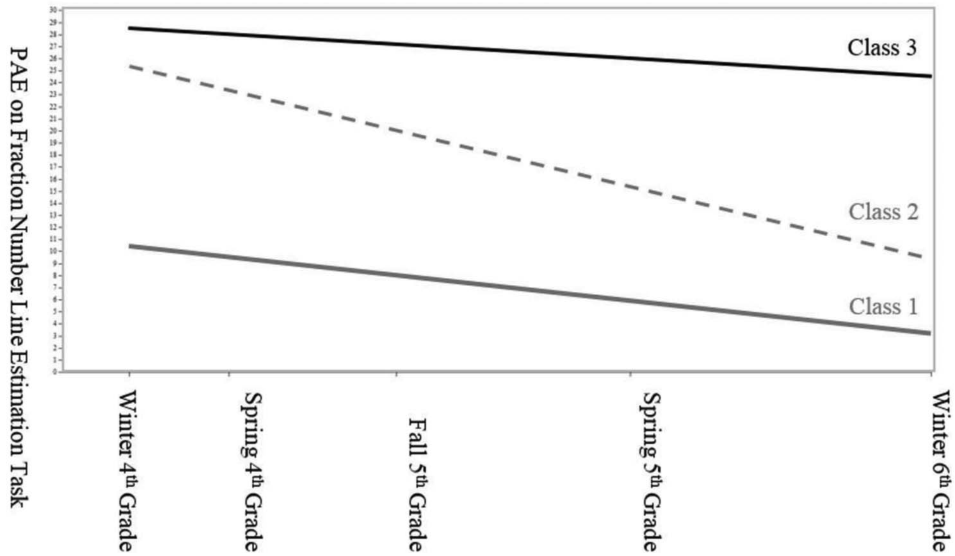

Figure 1. Growth trajectories in fraction number line estimation. *Note.* PAE = mean percent absolute error.

Children in Class 2, in turn, showed greater fraction knowledge than those in Class 3, $t(260.64) = 8.16$, $p < .001$, $d = 1.01$.

Finally, fraction number line estimation class membership was strongly associated with the sixth grade DCAS mathematics proficiency group levels ($\chi^2 = 201.78$, $p < .001$, Cramer's $V = .543$; see Table 5). This relationship held when we controlled for age, gender, income, attentive behavior, whole number line estimation, multiplication fluency, and reading fluency using an ordinal logistic regression ($\chi^2 = 236.38$, $p < .001$). Over half (66.4%) of students in Class 1 scored as "advanced" on the DCAS. Over half (63.4%) of students in Class 2 received a rating of "meets standards" on the DCAS. Students in Class 3 were evenly distributed across "well below standards" (35.3%), "below standards" (31.7%), and "meets standards" (31.7%).

Discussion

Although numerical magnitude representations are central to mathematical learning, no previous research had examined longitudinal growth in fraction number line estimation during the developmental period when fractions are emphasized in most U.S. schools. Over the course of the present study, students gradually increased their estimation accuracy, as measured by the percent absolute error (PAE) on 0–1 and 0–2 number lines. The findings show that fraction number line estimation acuity changes greatly in late elementary and early middle school. The data extend downward previous cross-sectional research, which showed that eighth graders are more accurate than sixth graders in fraction number line estimation (Siegler et al., 2011).

The group data, however, do not tell the complete story. We uncovered three empirically distinct growth trajectory classes in fraction magnitude understandings: students who started with accurate performance and became even more accurate (Class 1); students who started with inaccurate performance but exhibited relatively steep growth (Class 2); and students who started with inaccurate performance and showed little growth (Class 3). Strikingly, about 42% of students left sixth grade with minimal understanding of fraction magnitudes. Another 26% of students seemed to benefit a great deal from fractions instruction in late elementary and early middle school but still were not as accurate as roughly one third of the students who started the study with considerable strength in estimating fraction magnitudes.

Importantly, growth trajectory class predicted performance level on a high-stakes general mathematics test at the end of sixth grade, even after controlling for other factors that may contribute to overall mathematics achievement. Although 95% of Class 1 and 83% of Class 2 students met or exceeded state standards in mathematics, only 33% of Class 3 students did. Moreover, students in Classes 1 and 2 had a distinct advantage over those in Class 3 on a sixth-grade test of fraction knowledge that included curriculum-based concepts and procedures. Identification of these growth trajectory classes between fourth and sixth grade helps explain the large gap between low-achieving and high-achieving students in fraction knowledge and overall mathematics achievement in eighth grade (Siegler & Pyke, 2013).

Analysis by fraction type showed that estimation skill with improper fractions is less accurate and develops later than skill with proper fractions and mixed numbers. This finding supports the integrated theory of numerical development, which posits that numerical development involves gradually widening the range and type of number understood as magnitudes that can be accurately

Table 4
Intercept and Slope of Each Latent Class

	Intercept	Slope
Class 1 (starts accurate, ends accurate)	3.019***	−.310***
Class 2 (starts inaccurate, ends accurate)	8.385***	−.714***
Class 3 (starts inaccurate, ends inaccurate)	23.662***	−.211***

*** $p < .001$.

Table 5
Odds of Class Assignment Compared to Class 2 (Starts Inaccurate; Ends Accurate)

Class 3 (starts inaccurate; ends inaccurate)		Class 1 (starts accurate; ends accurate)	
Predictor	Odds ratio	Predictor	Odds ratio
Low income	1.420	Low income	.607
Female	1.135	Female	.614
Age	1.412*	Age	.920
Whole number line estimation	2.034***	Whole number line estimation	.352***
Multiplication fluency	.647*	Multiplication fluency	1.342
Attention	.825	Attention	1.565**
Reading fluency	1.296	Reading fluency	1.362

Note. All nonbinary predictor variables standardized to improve interpretation. $N = 382$.
* $p < .05$. ** $p < .01$. *** $p < .001$.

located on a number line (Siegler & Lortie-Forgues, 2014). Previous studies have documented that whole number magnitude extends from relatively small whole numbers to increasingly larger whole numbers (Booth & Siegler, 2006; Siegler & Opfer, 2003). Siegler and Lortie-Forgues (2014) suggest that children first learn fractions between 0 and 1, and then learn fractions greater than 1 (Siegler & Lortie-Forgues, 2014). For example, while sixth and eighth graders do not differ in estimation accuracy with fractions on a 0–1 number line, eighth graders are more accurate on a 0–5 number line (Siegler et al., 2011). However, Siegler and colleagues did not differentiate between estimates of improper fractions and mixed numbers. Our data suggest that less accurate estimation of improper fractions may be the driving force behind the observed difference between estimation accuracy with rational numbers above and below 1.

While the integrated theory of numerical development suggests whole number development involves understanding increasingly larger magnitudes (Siegler & Lortie-Forgues, 2014), our data suggest that fraction learning involves understanding the properties of different types of fractions as well as their magnitudes. Magnitude estimation of improper fractions requires students to attend to the relationship between the numerator and the denominator, including the order of the numerals (i.e., a/b vs. b/a). Fourth graders and low-growth students in fifth and sixth grades consistently estimated both proper fractions and improper fractions as falling between 0 and 1, and estimated mixed numbers as just more than 1. This observed estimation pattern suggests that younger students and older ones with mathematics difficulties may simply define fractions as "really small" or "less than 1." Such an approach would never be successful with estimating improper fraction magnitude on a number line.

This interpretation supports Vosniadou et al.'s (2008) assertion that the emphasis on proper fractions in early fraction instruction leads children to view all fractions (a/b) as numbers between 0 and 1. This issue raises the question of whether students should be taught fractions on number lines greater than 1 earlier in development. Number lines from 0–2, for example, encourage reasoning about the multiplicative relation among proper fractions, improper fractions, and mixed numbers, a key goal of the CCSS in mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

That fourth graders and low-growth students estimated both proper and improper fractions between 0 and 1 suggests these students are not engaging effectively in more strategic processes, most likely because of their limited understanding of the relation between the numerator and denominator and weaknesses in multiplicative reasoning. Siegler et al. (2011) found that sixth and eighth graders reported a variety of adaptive strategies on a fraction number line task. For example, one strategy was to divide the number line in half or in the case of lines larger than 1, into whole numbers. Other strategies include transforming fractions by rounding (e.g., 6/10 is a little more than 1/2) simplifying (7/4 is 1 3/4

Table 6
Fraction Number Line Estimation Class Membership in Relation to DCAS Scores

		DCAS proficiency level groups assessed Spring 2014				
		1 (well below standards)	2 (below standards)	3 (meets standard)	4 (advanced)	Total
Fraction number line estimation class	Class 1 (Starts accurate; ends more accurate)	2	4	31	73	110
	Class 2 (Starts inaccurate; ends accurate)	3	13	59	18	93
	Class 3 (Starts inaccurate; ends inaccurate)	49	44	44	2	139
Total		54	61	134	93	342

Note. Only children who were assigned to a latent growth trajectory class and had state test data available in sixth grade ($N = 342$) were included in this analysis. DCAS = Delaware Comprehensive Assessment System

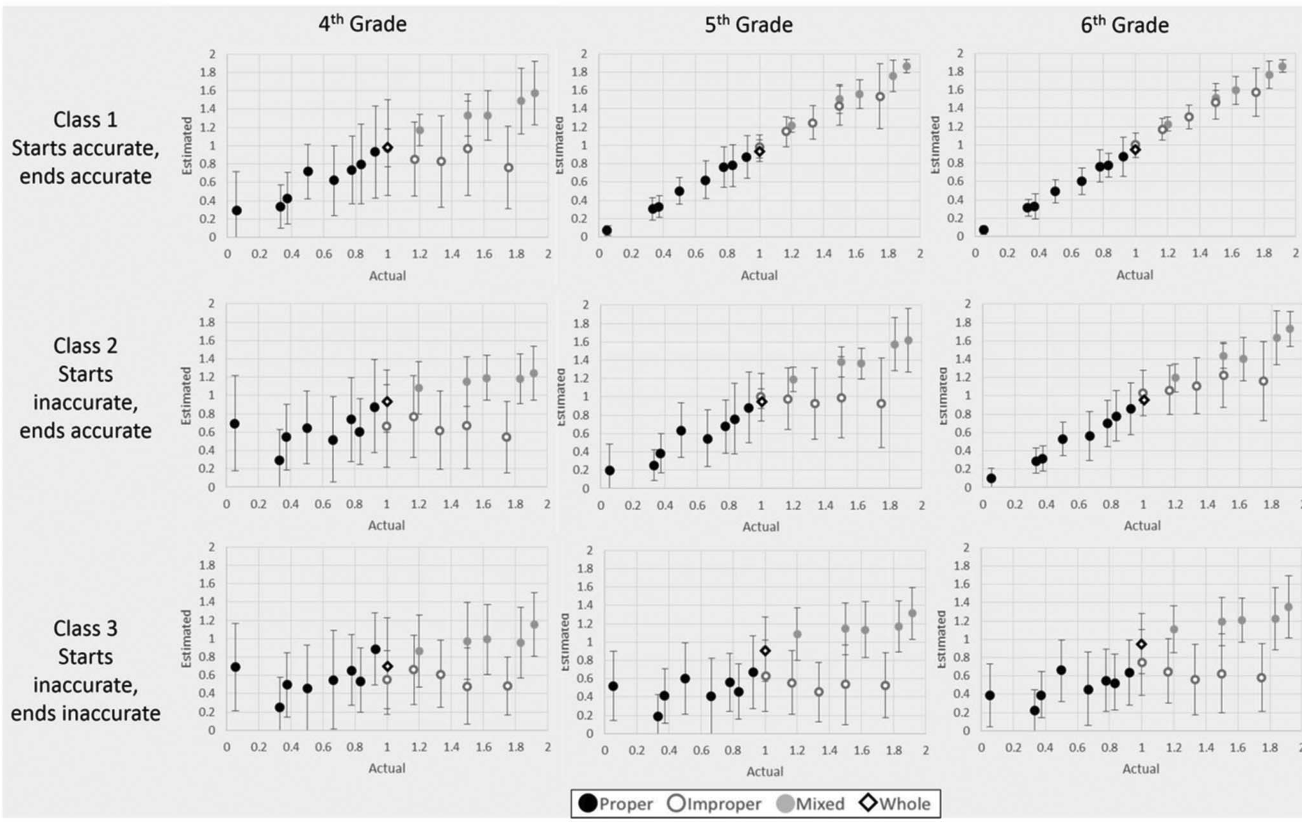

Figure 2. Scatterplot of estimated versus actual magnitudes for each class by grade on the 0–2 fraction number line estimation task. *Note.* Error bars represent standard deviation. Fractions and mixed numbers from smallest to largest were: 1/19, 1/3, 3/8, 1/2, 2/3, 7/9, 5/6, 12/13, [5/5, 1]*, 7/6, 1 1/5, 4/3, [3/2, 1 2/4]*, 7/4, 1 5/8, 1 5/6, 1 11/12. *Brackets indicate fraction equivalence.

which is close to 2), and translating the fraction into a different form (e.g., 3/5 is about 60%). While the high-achieving fourth graders in our sample might be engaging in conscious strategic processes, the majority of students are not, as seen by their inaccurate estimates. Low-growth children, in particular, do not learn how to make use of these strategies over time.

We also investigated the extent to which a range of processes and skills predicted fraction magnitude latent class membership. Attentive behavior, accurate whole number line estimation, multiplication fluency, and age (favoring younger students) emerged as unique predictors of fraction growth. Reading fluency did not predict fraction growth trajectory class, suggesting that general fluency is not uniquely important for fraction development. However, it should be noted that attentive behavior, multiplication fluency, whole number line estimation, and age all were moderately correlated with reading fluency. Thus, to some extent, these skills may support academic achievement more broadly.

Students were nearly 1.6 times more likely to be in Class 1 (starts accurate; ends more accurate) versus Class 2 (starts inaccurate; ends accurate) if their teachers rated them as having above-average attention. Attentive behavior allows students to stay on task and to acquire relevant knowledge and skills in their mathematics classrooms. Attentive behavior also likely facilitates strategy application on number line tasks; in order to understand the magnitude of a given fraction, students need to attend to the numerator and denominator simultaneously and to inhibit ineffective whole number strategies (Bonato et al., 2007; Meert, Grégoire, & Noël, 2009). Namkung and Fuchs (2016), however, reported no relation between classroom attention and fraction number line estimation skill in low-achieving fourth graders (below the 35th percentile on a mathematics achievement test), although they did not examine growth; the correlation between attention and fraction number line estimation may be depressed somewhat in a sample of achievers in the bottom third of the continuum. In our sample, which included a range of achievement levels, the correlation between attention and fraction number line estimation was moderately strong. Our findings also suggest that adaptive strategy application on the fraction number line estimation task (as indicated by increased accuracy), which requires attentional control and inhibitory processes, does not develop until later for most children, if at all, especially those in the low-growth group.

Students who estimated whole numbers accurately in third grade were 70% more likely to be in Class 1 than in Class 2. Likewise, students with inaccurate whole number line estimation skills were approximately twice as likely to be in the low-growth Class 3 than in the steep-growth Class 2. It is not surprising that students who enter fourth grade with strong whole number estimation skills have an advantage when they encounter fractions (Bailey et al., 2014;

Vukovic et al., 2014). Siegler and Lortie-Forgues (2014) assert that the number line unifies numeral development for all real numbers. Students may use similar strategies on both tasks (e.g., dividing the line into parts), although whole number line estimation skill appears to be a more automatic and better learned skill (Siegler & Opfer, 2003).

Students with higher than average multiplication fact fluency at the start of the study were approximately 35% less likely to fall into low-growth Class 3 than steeper-growth Class 2. Fast and accurate multiplication skill facilitates reasoning about fractions (Hecht et al., 2003; Seethaler, Fuchs, Star, & Bryant, 2011). For example, students must quickly see multiplicative relations between equivalent fractions (1/4 is the same as 2/8) as well as between improper fractions and mixed numbers (6/4 is the same as 6 × 1/4, or 1 1/2).

Finally, older students were about one and a half times more likely to fall in low-growth Class 3 compared to the steeper-growth Class 2. This negative relation is likely a result of students starting school later than usual or being held back because of learning or developmental delays.

Conclusions and Future Directions

As early as fourth grade, there are large individual differences in students' fraction magnitude understanding, as revealed by performance on a fraction number line estimation task. Most interesting are the students tracked in this study who started out with low fraction magnitude understanding but showed relatively steep growth between fourth and sixth grade. Do these students learn in the same way as students who initially have a strong sense of fraction magnitude but just at a slower pace? Indeed, steep-growth students' estimation patterns in fifth and sixth grade reflect the high-performing students' estimation patterns in fourth and fifth grade, respectively. Children in the steep-growth group seem to benefit from the mathematics curriculum in school, although future research is needed to align the emergence of specific fraction competencies with instructional approaches.

Being able to reason about whole number magnitude appears to support fraction learning. Class 1 (start accurate, ends accurate) is the most accurate whole number line estimation acuity, Class 2 (starts inaccurate, ends accurate) has the next highest accuracy, and Class 3 (starts inaccurate, ends inaccurate) has the lowest accuracy. Having a strong representation of whole numbers may support learning fraction magnitudes because: having a precise mental number line for whole number magnitudes may provide structure for learners to "fill in the empty spaces" in between whole numbers, whole numbers and fractions may both require the same encoding relative to other numbers (e.g., 75 on a 0–100 number line and 3/4 on a 0–1 number line, both represent 75% of either scale), and/or whole numbers and fractions may be linked through intermediary decimals (Bailey et al., 2014). Additionally, Class 1 and Class 2 both have stronger multiplication fluency compared with Class 3, suggesting the importance of developing this skill in all learners. Multiplication fluency may be important in understanding fraction magnitude, because the numerator and denominator in a given fraction can be expressed as a multiplicative relationship: identifying that 4/8 and 5/10 are equivalent fractions, for example, without knowing that each denominator is two times larger than the numerator.

Children with early fraction estimation acuity are likely to excel in mathematics throughout late elementary and early middle school. Students who cannot accurately place fractions on a number line, especially on the 0–2 line, by about fifth grade will probably continue to struggle in mathematics, at least without instructional intervention. Fraction number line estimation acuity reflects deepening knowledge of the relation between the numerator and denominator as well as knowledge of fraction equivalence. These understandings are enhanced by whole number sense, computational fluency, and strong attention.

Fraction sense in struggling learners appears to be malleable, and interventions centered on a number line can sharply boost mathematics performance (e.g., Fuchs et al., 2013; Saxe et al., 2007). The present research suggests that targeting improper fractions and their relation to proper fractions and mixed numbers on number lines greater than 1 would be especially beneficial. Improving understanding of multiplicative relations and their connection to fraction magnitudes also is likely to increase learning. In addition, students should be encouraged to focus their attention on both end points of the number line and to develop strategies for representing different fractions relative to those endpoints (e.g., the closer the numerator is to the denominator, the closer the fraction is to 1, so on a 0–1 line, numerators that are close to denominators indicates locations near the right end; when the numerator is larger than the denominator, the number is always greater than 1; different fractions can have the same place on the line if the relation between numerator and denominator is equivalent, etc.). Interventions using such approaches may be particularly beneficial early in fraction instruction so that fraction difficulties do not continue to cascade.

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