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Why Is Learning Fraction and Decimal Arithmetic So Difficult?

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Abstract

Fraction and decimal arithmetic are crucial for later mathematics achievement and for ability to succeed in many professions. Unfortunately, these capabilities pose large difficulties for many children and adults, and students' proficiency in them has shown little sign of improvement over the past three decades. To summarize what is known about fraction and decimal arithmetic and to stimulate greater amounts of research in the area, we devoted this review to analyzing why learning fraction and decimal arithmetic is so difficult. We identify and discuss two types of difficulties: (1) Difficulties that are inherent to fraction and decimal arithmetic, and (2) Culturally contingent sources that could be reduced by improved instruction and prior knowledge of learners. We conclude the review by discussing commonalities among three interventions that have helped children overcome the difficulties of mastering fraction and decimal arithmetic.

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Why Is Learning Fraction and Decimal Arithmetic So Difficult?

In 1978, as part of the National Assessment of Educational Progress (NAEP), more than 20,000 U.S. 8th graders (13- and 14-year-olds) were asked to choose the closest whole number to the sum of 12/13 + 7/8. The response options were 1, 2, 19, 21 and "I don't know". Only 24% of the students chose the correct answer, “2” (Carpenter, Corbitt, Kepner, Linquist, & Reys, 1980). The most common answer was “19”.

This lack of understanding proved not to be limited to fraction arithmetic. The 1983 NAEP asked another large, representative sample of U.S. 8th graders to choose the closest whole number to the decimal arithmetic problem, 3.04 * 5.3. The response options were 1.6, 16, 160, 1600, and “I don’t know”. Only 21% of 8th graders chose the correct answer, 16 (Carpenter, Lindquist, Matthews, & Silver, 1983). The most common answer was “1600”.

In the ensuing years, many efforts have been made to improve mathematics education. Governmental commissions on improving mathematics instruction (e.g., NMAP, 2008), national organizations of mathematics teachers (e.g., NCTM, 2007), practice guides sponsored by the U.S. Department of Education to convey research findings to teachers (e.g., Siegler et al., 2010), widely adopted textbooks (e.g., Everyday Mathematics), and innumerable research articles (e.g., Hiebert & Wearne, 1986) have advocated greater emphasis on conceptual understanding, especially conceptual understanding of fractions. (Here and throughout the review, we use the term *fractions* to refer to rational numbers expressed in a bipartite format (e.g., 3/4). We use the term *decimals* to refer to rational numbers expressed in base-10 notation (e.g., 0.12)).

To examine the effects of these calls for change, we recently presented the above-cited fraction arithmetic question to 48 8th graders taking an algebra course. The students attended a suburban middle school in a fairly affluent area. Understanding of fraction addition seems to have changed little if at all in the 36 years between the two assessments. In 2014, 27% of the 8th graders identified "2" as the best estimate of 12/13 + 7/8. Thus, after more than three decades, numerous rounds of education reforms, hundreds if not thousands of research studies on mathematics teaching and learning, and billions of dollars spent to effect educational change, little improvement was evident in students' understanding of fraction arithmetic.

Such lack of progress is more disappointing than surprising. Many tests and research studies in the intervening years have documented students' weak understanding of fractions (e.g., NAEP, 2004; Stigler, Givvin, & Thompson, 2010). The difficulty is not restricted to the U.S. or to fractions. Understanding of multiplication and division of decimals also is weak in countries that are top performers on international comparisons of mathematical achievement, for example China (e.g., Liu, Ding, Zong, & Zhang, 2014; PISA, 2012).

Given the importance of knowledge of rational numbers for subsequent academic and occupational success, this weak understanding of fraction and decimal arithmetic is a serious problem. Early proficiency with fractions uniquely predicts success in more advanced mathematics. Analyses of large datasets from the U.S. and the U.K. showed that knowledge of fractions (assessed primarily through performance on fraction arithmetic problems) in 5th grade is a unique predictor of general mathematic achievement in 10th grade. This was true after controlling for knowledge of whole number arithmetic, verbal and nonverbal IQ, working memory, family education, race, ethnicity, and family income (Siegler et al., 2012). Other types of data have led to the same conclusion. For example, a nationally representative sample of 1,000 U.S. algebra teachers ranked poor knowledge of "rational numbers and operations involving fractions and decimals" as one of the two greatest obstacles preventing their students from learning algebra (Hoffer, Venkataraman, Hedberg, & Shagle, 2007).

The importance of fraction and decimal computation for academic success is not limited to mathematics courses. Rational number arithmetic is also ubiquitous in biology, physics, chemistry, engineering, economics, sociology, psychology, and many other areas. Knowledge in these areas, in turn, is central to many common jobs in which more advanced mathematics knowledge is not a prerequisite, such as registered nurse and pharmacist (e.g., for dosage calculation). Moreover, fraction and decimal arithmetic is common in daily life, for example in recipes (e.g., if 3/4 of a cup of flour is needed to make a dessert for 4 people, how much flour is needed for 6 people), and measurement (e.g., can a piece of wood 36 inches long be cut into 4 pieces each 8.75 inches long). Fraction and decimal arithmetic are also crucial to understanding basic statistical and probability information reported in media and to understanding home finance information, such as compound interest and the asymmetry of percent changes in stock prices (e.g., the price of a stock that decreases 2% one month and increases 2% the next is always lower than at the outset).

Fraction and decimal arithmetic are also vital for theories of cognitive development in general and numerical development in particular. As with so many other topics, Piaget and his collaborators were probably the first to recognize the importance of understanding of rational number topics, such as ratios and proportions, for a general understanding of cognitive development. Inhelder and Piaget (Inhelder & Piaget, 1958; Piaget & Inhelder, 1975) posited that acquiring understanding of ratios and proportions is crucial to the transition between concrete operations and formal operations that occurs at roughly age 12 or 13 years. Indeed, Inhelder and Piaget's (1958) classic book on development of formal operations placed great emphasis on this type of reasoning, using tasks such as balance scales, shadows projection, and probability to assess the development of understanding of proportionality in preadolescence and adolescence. Understanding fraction and decimal arithmetic requires understanding of the fractions and decimals themselves; indeed, as will be seen, failure to grasp fraction and decimal arithmetic often reflects a more basic lack of understanding of the component fractions and decimals.

Fractions and decimals also have an inherently important role to play in domain specific theories (Carey, 2011), in particular theories of numerical development. Although most existing theories of numerical development have focused entirely or almost entirely on whole numbers (e.g., Geary, 2006; Leslie, Gelman, & Gallistel, 2008; Wynn, 2002), encountering rational numbers provides children the opportunity to distinguish between principles that are true for natural numbers (whole numbers greater than or equal to one) and principles that are true of numbers more generally. For example, within the set of natural numbers, each number has a unique predecessor and successor, but within the set of rational numbers there are always infinite other numbers between any two other numbers. Moreover, every natural number is represented by a unique symbol (e.g., 4), but every rational number can be represented by infinite expressions (e.g., 4/1, 8/2, 4.0, 4.00), and so on. Encountering fractions and decimals also provides children the opportunity to learn that despite the many differences between natural and rational numbers, they share the common feature that both express magnitudes that can be located and ordered on number lines (Siegler, Thompson, & Schneider, 2011).

Similarly, fraction arithmetic provides children the opportunity to learn that the effects of arithmetic operations on magnitudes vary with the numbers to which the operations are applied. For example, multiplying natural numbers never decreases either number's magnitude, but multiplying two fractions or decimals from 0-1 always results in a product less than either multiplicand. Similarly, dividing by a natural number never results in a quotient greater than the number being divided, but dividing by a fraction or decimal from 0-1 always does. Thus, learning fraction and decimal arithmetic provides an opportunity to gain a deeper understanding of arithmetic operations, in particular of multiplication and division. In line with this analysis, Siegler & Lortie-Forgues (2014) suggested that numerical development can be seen as the progressive broadening of the set of numbers whose properties, including their magnitudes and the effects of arithmetic operations on those magnitudes, can be accurately represented.

Consistent with their importance, fractions and decimals recently have been the subjects of an increasing amount of research. In the past five years, studies have examined developmental and individual differences in mental representations of fractions (DeWolf, Grounds, Bassok, & Holyoak, 2014; Fazio, Bailey, Thompson, & Siegler, 2014; Gabriel, Szucs, & Content, 2013; Hecht & Vagi, 2012; Huber, Moeller, & Nuerk, 2014; Jordan, et al., 2013; Meert, Gregoire, & Noel, 2009; Meert, Gregoire, Seron &

Noel, 2012; Schneider & Siegler, 2010; Siegler & Pyke, 2013), developmental and individual differences in mental representation of decimals (Huber, Klein, Willmes, Nuerk, & Moeller, 2014; Kallai & Tzelgov, 2014), predictors of later fraction knowledge (Bailey, Siegler, & Geary, 2014; Jordan et al., 2013; Vukovic et al., 2014), relations between fraction understanding and algebra (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014), and effects of interventions aiming at improving knowledge of fractions (Fuchs et al., 2013, 2014) and decimals (Durkin & Rittle-Johnson, 2012; Isotani et al., 2011; Rittle-Johnson & Schneider, 2014).

Most of this recent research has focused on understanding of individual fractions and decimals (e.g., understanding whether 4/5 is larger than 5/9 or where .833 goes on a 0-1 number line). Fewer studies have investigated fraction arithmetic. However, given the omnipresence of fraction and decimal arithmetic in many occupations and activities, their pivotal role in helping some people gain a deeper understanding of arithmetic operations than that needed to understand arithmetic with whole numbers, and the fact that people can have excellent knowledge of individual fractions or decimals without understanding arithmetic operations with them (Siegler & Lortie-Forgues, in press), development of rational number arithmetic seems worthy of serious attention.

The remainder of this article is organized into four main sections. We first describe the development of knowledge of the four basic arithmetic operations with fractions and decimals, and the instructional environment in which these acquisitions occur. Next, we identify and describe a set of difficulties that are inherent to fraction and decimal arithmetic and that lead to specific types of misunderstandings and errors being widespread. After this, we describe culturally contingent variations in instruction and prior knowledge of learners that influence the likelihood of children overcoming the difficulties and mastering fraction and decimal arithmetic. Finally, we describe several instructional interventions that have been successful in helping students overcome the difficulties in societies where many students fail to do so.

Development of Fraction and Decimal Arithmetic

Understanding development requires knowledge of the environments in which the development occurs. Therefore, we begin our review with a brief description of the educational environment in which students learn rational number arithmetic. The focus here and throughout this article is on acquisition of fraction and decimal arithmetic in the U.S., because more studies are available about how the process occurs in the U.S. than elsewhere. Data from other Western countries and from East Asia are cited when we have been able to find them and they provide relevant information.

Environments in Which Children Learn Fraction Arithmetic

The Common Core State Standards Initiative (CCSSI, 2010) provides useful information for understanding the environment in which children in the U.S. learn rational number arithmetic. The CCSSI has been adopted by more than 80% of U.S. states as official policy regarding which topics should be taught when. Moreover, the CCSSI recommendations have been incorporated on standardized assessments that themselves shape what is taught. Fully 92% of 366 middle school math teachers surveyed by Davis, Choppin, McDuffie, and Drake (2013) reported that new state assessments, which in most cases are designed to reflect the CCSSI goals, will influence their instruction. For these reasons, and because the timing corresponds to coverage in major U.S. textbook series such as *Everyday Math*, we use the CCSSI recommendations as a guide to when children in the U.S. receive instruction in different aspects of rational number arithmetic.

The CCSSI (2010) recommended that fraction arithmetic be a major topic of study in fourth, fifth, and sixth grade (roughly ages 9 to 12). Instruction begins with addition and subtraction of fractions with common denominators, proceeds to instruction in those operations with unequal denominators and to fraction multiplication, and then moves to fraction division. Reviewing the operations and teaching students how they can be applied to problems involving ratios, rates, and proportions are recommended for seventh and eighth grade. To the extent that these CCSSI recommendations are followed, substantial development of fraction arithmetic would be expected from fourth to eighth grade.

Development of Fraction Arithmetic

Even after this period of relatively intense instruction, however, performance is often poor. To illustrate the nature and magnitude of the problem, we will describe in some detail results from a study of the fraction arithmetic knowledge of 120 6th and 8th graders recruited from three public school districts near Pittsburgh, Pennsylvania (Siegler & Pyke, 2013). Sixth graders were studied because they would have received instruction in fraction arithmetic very recently; eighth graders were studied because they would have had experience with more advanced fractions problems (e.g., ratio, rate, and proportion problems), and would have had time to practice and consolidate the earlier instruction in fraction arithmetic. Children from all classrooms reported having been taught all four arithmetic operations with fractions.

Participants were presented 16 fraction arithmetic problems, four for each of the four arithmetic operations. On each operation, half of the problems had equal denominators, and half had unequal denominators. For each arithmetic operation, the four items were generated by combining 3/5 with 4/5, 1/5, 2/3, and 1/4 (e.g., 3/5+ 4/5, 3/5+1/5, 3/5+2/3, and 3/5+1/4). Thus, all numbers in the arithmetic problems were five or less.

The sixth graders in this study correctly answered only 41% of the fraction arithmetic problems, the 8th graders 57%. Performance was highest on fraction addition and subtraction (60% and 68% correct, respectively), followed by fraction multiplication (48%), and fraction division (20%). Equality of denominators had a large impact on accuracy of addition and subtraction: the increased procedural complexity associated with adding two fractions with unequal denominators led to less accurate performance on items with unequal denominators (55% and 62% for addition and subtraction, respectively), than on items with equal denominators (80% and 86%, respectively). Interestingly, even though the standard fraction multiplication procedure is not influenced by whether denominators of the multiplicands are equal, percent correct was lower on problems with equal than unequal denominators (36% vs. 59%). The reason was that when denominators were equal, students often confused the procedure for fraction multiplication with that for fraction addition and subtraction. This led to errors that involved keeping the denominator constant (e.g., 3/5 * 1/5 = 3/5), as with the correct procedure for addition (e.g., 3/5 + 1/5 = 4/5) and subtraction (3/5 – 1/5 = 2/5).

As in whole number arithmetic with younger children, strategy use on the fraction arithmetic problems was strikingly variable. Not only did different children use different strategies, the same child often used different strategies on virtually identical pairs of problems. About 60% of the students used distinct procedures (usually one correct and one incorrect) for at least one arithmetic operation on virtually identical problems (e.g., 3/5 * 1/5 and 3/5 * 4/5). Another type of variability involved errors: Children made the well documented whole number overextension errors (e.g., Ni & Zhou, 2005) that reflect inappropriate generalization from the corresponding whole number arithmetic procedures (e.g., 3/5 + 4/5 = 7/10), but they made at least as many wrong fraction operation errors, in which they inappropriately generalized procedures from other fraction arithmetic operations (e.g., 3/5 * 4/5 = 12/5). This variability was present among both 6^{th} and 8^{th} graders. The findings suggest that the students' problem was not that they did not know the correct procedure, and not that they had a systematic misconception that fraction arithmetic was like whole number arithmetic, but rather that they were confused about which of several procedures was correct. This confusion led to a mix of correct procedures, independent whole number errors, and wrong fraction operation errors.

It is important to note that the pattern of performance of children in the U.S. is not universal. On the same fraction arithmetic problems presented in Siegler and Pyke (2013), Chinese 6^{th} graders scored almost three standard deviations higher than U.S. age peers, and were correct on 90% or more of problems on all four arithmetic operations (Bailey et al., 2015). However, the pattern is representative of results with U.S. children's fraction arithmetic performance (Bailey, Hoard, Nugent, & Geary, 2012; Booth et al., 2014; Byrnes & Wasik, 1991; Hecht, 1998; Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010; Mazzocco & Devlin, 2008; Siegler et al., 2011).

Environments in Which Children Learn Decimal Arithmetic

Decimal arithmetic is introduced slightly later than fraction arithmetic and taught primarily in two rather than three grades. The CCSSI proposes that in fifth grade, the four basic arithmetic operations should be introduced with numbers having one or two digits to the right of the decimal. Multi-digit decimal arithmetic is to be taught in sixth grade. Reviewing decimal arithmetic, like reviewing fraction arithmetic, is suggested for seventh grade, as is translating across decimals, fractions, and percentages.

Despite fractions and decimals both representing rational numbers, the standard arithmetic procedures used with them are quite different. Unlike with fraction arithmetic, the standard procedures used for all decimal arithmetic operations closely resemble those used for whole number arithmetic, with the exception that decimal arithmetic requires correct placement of the decimal point. This exception is important, though, because the rules for placing the decimal point vary with the arithmetic operation and are rather opaque conceptually (to appreciate this, try to explain why 1.23 * 4.56 must generate a product with four decimal places).

Development of Decimal Arithmetic

To convey a sense of the development of decimal arithmetic, we focus on Hiebert and Wearne's (1985) study of 670 fifth to ninth graders' decimal arithmetic performance. For each arithmetic operation, they presented five or six problems that varied in the number of digits to the right of the decimal point of each operand (the numbers in the problem) and in whether there were equal or unequal numbers of digits to the right of the decimal point in the two operands.

As with fraction arithmetic, substantial improvement occurred during this age range, but accuracy never reached very high levels. Between first semester of grade 6 and second semester of grade 9, percent correct improved for addition from 20% to 80%, for subtraction from 21% to 82%, and for multiplication from 30% to 75%.

Patterns of correct answers and errors on specific problems differed in ways that reflect characteristics of the usual computational procedures. Accuracy on addition and subtraction problems was much greater when the operands had equal numbers of digits to the right of the decimal. For instance, sixth graders' percent correct on decimal addition problems was 74% when addends had an equal number of decimal places (e.g., 4.6 + 2.3) but was only 12% when the number of decimal places differed (e.g., 5.3 + 2.42). This difference remained substantial at older grade levels as well. For example, ninth graders generated 90% correct answers on 0.60 - 0.36 but 64% correct answers on 0.86 - 0.3.

In the same study, accuracy of multiplication and division was not influenced by differing numbers of decimal places in the operands. Accuracy did not differ in ninth grade, for example, between 0.4 * 0.2 and 0.05 * 0.4 (67% and 65% correct) or between 0.24 ÷ 0.03 and 0.028 ÷ 0.4 (72% and 70%). On the other hand, performance was much worse (4% vs. 56% correct in 6th grade) on multiplication of two decimals (e.g., 0.4 * 0.2) than on multiplication of a decimal and a whole number (e.g., 8 * 0.6). In the next two sections, we examine the mix of intrinsic and culturally contingent sources of difficulty that lead to these patterns of performance and development.

Inherent Sources of Difficulty in Fraction and Decimal Arithmetic

The previous section documented U.S. children’s weak performance with fraction and decimal arithmetic. In this section, we identify and discuss seven sources of this weak performance that are intrinsic to fraction and decimal arithmetic, intrinsic in the sense that they would be present regardless of the particulars of the educational system and culture of the learners. The difficulties involve 1) fraction and decimal notation, 2) accessibility of fraction and decimal magnitudes, 3) opaqueness of standard fraction and decimal arithmetic procedures, 4) complex relations between rational and whole number arithmetic procedures, 5) complex relations of rational number arithmetic procedures to each other, 6) opposite direction of effects of multiplying and dividing positive fractions and decimals below and above one, and 7) sheer number of distinct components of fraction and decimal arithmetic procedures.

This list of inherent difficulties should not be interpreted as exhaustive. Rather, it specifies some of the factors that contribute to the difficulty that children commonly encounter with fraction and decimal arithmetic. Also, these intrinsic factors may be culturally contingent in the long run, in the sense that people devised the notations and procedures and imaginably could devise different ones that do not pose these difficulties. Finally, intrinsic does not mean insuperable; in countries with superior educational systems and cultures that greatly value math learning, most students overcome the difficulties. With those caveats, we examine the seven intrinsic sources of difficulty.

Fraction and Decimal Notations

One factor that makes fraction and decimal arithmetic inherently more difficult than whole number arithmetic is the notations used to express fractions and decimals.

Fractions. A fraction has three parts, a numerator, a denominator, and a line separating the two numbers. This complex configuration makes fraction notation somewhat difficult to understand. For instance, students, especially in the early stages of learning, often misread fractions as two distinct whole numbers (e.g., 1/2 as 1 and 2), as a familiar arithmetic operation (e.g., 1 + 2) or as a single number (e.g., 12) (Gelman, 1991; Hartnett & Gelman 1998). Even after learning how the notation works, fractions are still effortful to process. Maintaining two fractions in working memory while solving, for example, 336/14 * 234/18 requires considerably more cognitive resources than maintaining the corresponding whole number problem, 24 * 13. The greater memory load of representing fractions reduces the cognitive resources available for thinking about the procedure needed to solve the problem, for monitoring progress while executing the procedure, and for relating the magnitudes of the problem and answer. Consistent with this analysis, individual differences in working memory are correlated with individual differences in fraction arithmetic, even after other relevant variables have been statistically controlled (Fuchs et al., 2013; Hecht & Vagi, 2010; Jordan et al., 2013; Siegler & Pyke, 2013).

Decimals. The notation used to express decimals is more similar to that used with whole numbers, in that both are expressed in a base ten place value system. Nevertheless, the notations also differ in important ways. A longer whole number is always larger than a shorter whole number, but the length of a decimal is unrelated to its magnitude. Adding a zero to the right end of a whole number changes its value (e.g., 3 ≠ 30), but adding a zero to the right side of a decimal does not (e.g., 0.3 = 0.30). Naming conventions also differ (Resnick, 1989). Naming a whole number does not require stating a unit of reference (e.g., people rarely say that 25 means 25 of the 1's units), but naming a decimal does (e.g., 0.25 is twenty-five *hundredth* and not twenty-five *thousandth*). Maintaining in memory the rules that apply to decimals, and not confusing them with the rules used with whole numbers, increases the working memory demands of learning decimal arithmetic.

Accessibility of Magnitudes of Operands and Answers

Whole number arithmetic is influenced by access to the magnitudes of operands and answers. Several paradigms indicate that after second or third grade, whole number magnitudes are accessed automatically, even when accessing them is harmful to task performance (e.g., Berch, Foley, Hill, & Ryan, 1999; Lefevre, Bisanz, & Mrkonjic, 1988; Thibodeau, LeFevre, & Bisanz, 1996). For example, when the task is to respond "yes" if the answer after the equal sign of an addition problem is identical to one of the addends and "no" if it is not, people are slower to respond "no" when the answer is the correct sum (e.g., 4 + 5 = 9) than when it is another non-matching number (e.g., 4 + 5 = 7) (LeFevre, Kulak, & Bisanz, 1991). Similarly, on verification tasks, people respond false more quickly when the magnitudes of incorrect answers are far from correct ones (e.g., 2 + 4 = 12) than when the two are closer (e.g., 2 + 4 = 8) (Ashcraft, 1982).

Fractions. Unlike whole number magnitude, fraction, magnitudes have to be derived from the ratio of two values, which reduces the accuracy, speed, and automaticity of access to the magnitude representations (English & Halford, 1995). Accessing fraction magnitude also requires understanding whole number division, often considered the hardest of the four arithmetic operations (Foley & Cawley, 2003). These additional difficulties have led some authors to suggest that children have to go through a fundamental reorganization of their understanding of numbers before being able to represent fractions (e.g., Carey, 2011; Smith, Solomon, & Carey 2005; Vamvakoussi & Vosniadou 2010). Supporting this point, Smith, Solomon, & Carey (2005) showed that understanding of many concepts related to rational numbers (e.g., the presence of numbers between 0 and 1, the fact that numbers are infinitely divisible) seems to emerge at the same time within an individual. Unfortunately, developing this level of understanding appears lengthy and difficult: only 56% of their older participants (5th and 6th graders) had undergone this reorganization.

Consistent with the fact that fraction magnitude is difficult to access, both 8th graders and community college students correctly identify the larger of two fractions on only about 70% of items, where chance is 50% correct (Schneider & Siegler, 2010; Siegler & Pyke, 2013). Similarly, when the smaller fraction has the larger denominator, fraction magnitude comparisons of both adults and 10- to 12-year-olds are slower than when the smaller fraction has the smaller denominator (Meert, Gregoire, & Noel, 2009; 2010). People with better knowledge of fraction magnitudes (as measured by fraction magnitude comparison or number line estimation) usually perform better on fraction arithmetic, even after relevant variables such as knowledge of whole numbers, working memory, and executive functioning have been statistically controlled (Byrnes & Wasik, 1991; Hecht, 1998; Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010; Jordan, et al., 2013; Siegler & Pyke, 2013; Siegler, Thompson & Schneider, 2011).

Decimals. Representing the magnitudes of decimals without a 0 immediately to the right of the decimal point is as accurate and almost as quick as representing whole number magnitudes (DeWolf et al., 2014). However, representing decimals with one or more "0" immediately to the right of the decimal point is considerably more difficult. When we write a whole number, we do not preface it with "0's" to indicate that no larger place values are involved (e.g., we often write "12" but almost never "0012.") This difference between ways of writing whole numbers and decimals makes representing the magnitudes of decimals with 0's immediately to the right of the decimal point quite difficult. For instance, in Hiebert & Wearne (1986), only 43% of 9th graders correctly identified the largest number among 0.09, 0.385, 0.3 and 0.1814. Similarly, in Putt (1995), only about 50% of U.S. and Australian pre-service teachers correctly ordered from smallest to largest the numbers 0.606, 0.0666, 0.6, 0.66 and 0.060.

The relation between decimal magnitude knowledge and decimal arithmetic has not received as much attention as the comparable relation with fractions. However, the one study that we found that addressed the issue indicated that knowledge of the magnitudes of individual decimals is positively related to the accuracy of decimal arithmetic learning (Rittle-Johnson & Koedinger, 2009).

Opaqueness of Rational Number Arithmetic Procedures

Fractions. The conceptual basis of fraction arithmetic procedures is often far from obvious. Why are equal denominators needed for adding and subtracting but not for multiplying and dividing? Why can the whole number procedure be independently applied to the numerator and denominator in multiplication, but not in addition or subtraction? Why is the denominator inverted and multiplied when dividing fractions? All of these questions have answers, of course, but the answers are not immediately apparent, and they often require understanding algebra, which is generally taught after fractions, so that students lack relevant knowledge at the time when they learn fractions.

Decimals. Decimal arithmetic procedures are in some senses more transparent than fraction arithmetic procedures – they can be justified with reference to the corresponding whole number operation, which they resemble. For example, just as adding 123 + 456 involves adding ones, tens, and hundreds, adding 0.123 + 0.456 involves adding tenths, hundredths, and thousandths. However, some features of decimal arithmetic procedures, particular those related to placement of the decimal point, are unique to decimal arithmetic, and their rationale is often unclear to learners. For example, why is it that adding and subtracting numbers that each have two digits to the right of the decimal point (e.g., 0.44 and 0.22) results in an answer with two digits to the right of the decimal point, that multiplying the same numbers results in an answer with four digits to the right of the decimal point, and that dividing them can, as in the above problem, result in an answer with no digits to the right of the decimal point?

Complex Relations Between Rational and Whole Number Arithmetic Procedures

Fractions. The mapping between whole number and fraction arithmetic procedures is complex. For addition and subtraction, once equal denominators have been generated, numerators are added or subtracted as if they were whole numbers, but the denominator is passed through to the answer without any operation being performed. For multiplication, numerators and denominators of the multiplicands are treated as if they were independent multiplication problems with whole numbers, regardless of whether denominators are equal. For the standard division procedure, the denominator is inverted, and then numerator and denominator are treated as if they were independent whole number multiplication problems.

These complex relations between whole number and fraction procedures probably contribute to the prevalence of independent whole number errors (e.g., adding numerators and denominators separately, as in 2/3 + 2/3 = 4/6). For instance, in Siegler and Pyke (2013), independent whole number errors accounted for 22% of sixth and eighth graders' answers on fraction addition and subtraction problems.

Decimals. The mapping between decimal and whole number arithmetic procedures is also complex. The procedures for adding and subtracting decimals are very similar to the corresponding procedures with whole numbers. However, although aligning the rightmost digits of whole numbers preserves the correspondence of their place values, aligning the rightmost decimals being added or subtracted does not have the same effect when the numbers of digits to the right of the decimal point differ. Instead, the location of the decimal point needs to be aligned to correctly add and subtract.

This difference between whole number and decimal alignment procedures leads to frequent errors when decimal addition and subtraction problems have unequal numbers of digits to the right of the decimal point. For instance, in Hiebert & Wearne (1985), seventh graders' decimal subtraction accuracy was 84% when the operands had an equal number of digits to the right of the decimal point (e.g., 0.60 - 0.36), whereas it was 48% when the operands differed in the number of digits to the right of the decimal (e.g., 0.86 - 0.3)

Complex Relations of Rational Number Arithmetic Procedures to Each Other

Fractions. Complex relations among procedures for different fraction arithmetic operations also contribute to the difficulty of fraction arithmetic. For example, adding and subtracting of fractions with an equal denominator requires leaving the denominator unchanged in the answer, whereas multiplying fractions with an equal denominator requires multiplying the denominators. Inappropriately importing the addition and subtraction procedure into multiplication leads to errors such as 2/3 * 2/3 = 4/3. In Siegler and Pyke (2013), 55% of answers to fraction division problems and 46% of answers to fraction multiplication problems involved inappropriately importing components from other fraction arithmetic procedures.

Decimals. Decimal arithmetic procedures are also confusable with one another. This is particularly evident in procedures for correctly placing the decimal point in the answer for different arithmetic operations. For example, the location of the decimal point in addition and subtraction requiring aligning the decimal points so that numbers with the same place value are being added or subtracted. In contrast, multiplication does not require such alignment, and the location of the decimal point in the answer corresponds to the sum of the decimal places in the multiplicands (e.g., the product of 8.64 * 0.4 will have three numbers to the right of the decimal point). Confusion among decimal arithmetic operations is seen in frequent errors in which the procedure used to place the decimal point in addition and subtraction is imported into multiplication, producing errors such as 0.3 * 0.2 = 0.6. Such errors accounted for 76% of all answers by 6th graders in Hiebert and Wearne (1985).

Direction of Effects of Multiplying and Dividing Proper Fractions and Decimals

Understanding the direction of effects of multiplying and dividing proper fractions and decimals (those between 0 and 1) poses special problems for learners. Multiplying natural numbers always results in an answer greater than either multiplicand, but multiplying two proper fractions or decimals invariably results in answers less than either multiplicand. Similarly, dividing by a natural number never results in an answer greater than the number being divided, but dividing by a proper fraction or decimal always does. Knowing the effects of multiplying and dividing numbers from 0-1 might be made yet more difficult by the fact that adding and subtracting numbers from 0-1 has the same directional effect as adding and subtracting whole numbers, as do all four arithmetic operation with fractions and decimals greater than one. Both middle school students and pre-service teachers show poor understanding of the directional effects of fraction and decimal multiplication and division (Fischbein, Deri, Nello, & Marino, 1985; Siegler & Lortie-Forgues, in press).

Sheer Number of Distinct Procedures

Fractions. Fraction arithmetic requires learning a large number of distinct procedures, probably more than for any other mathematical operation taught in elementary school. It requires skill in all four whole number arithmetic procedures, as well as mastery of procedures for finding equivalent fractions, simplifying fractions, converting fractions to mixed numbers and mixed numbers to fractions, knowing whether to invert the numerator or denominator when dividing fractions, and understanding when equal denominators are maintained in the answer (addition and subtraction) and when the operation in the problem should be applied to the denominator as well as the numerator (multiplication and division).

Decimals. Decimal arithmetic does not require mastery of as many distinct procedures as fraction arithmetic, but it does pose some difficulties beyond those of whole number arithmetic. In particular, the standard procedure for placing the decimal point in answers to decimal addition and subtraction problems is distinct from that used with multiplication problems, and both are distinct from that used with decimal division problems.

Culturally contingent sources of difficulty

Several factors that are not inherent to fraction and decimal arithmetic, but instead are determined by cultural values and characteristics of educational systems, also contribute to difficulties learning fraction and decimal arithmetic. The culturally contingent factors determine the impact of the inherent sources of difficulty. For example, confusability between fraction and whole number arithmetic procedures is an inherent source of difficulty, but high quality instruction and high motivation to learn mathematics leads to this and other sources of difficulty having a less deleterious effect on fraction arithmetic learning in East Asia than in the U.S.

We divided culturally contingent factors into two categories: (1) factors related to instruction and (2) factors related to learners' prior knowledge. In several cases, the only relevant research that we could locate involves fractions, but the same factors might well handicap learning of decimal arithmetic.

These culturally contingent factors, of course, are not unique to rational number arithmetic; rather they extend to mathematics more generally. In an insightful discussion of the issue, Hatano (1990) distinguished between compulsory and optional skills within a culture. Reading was an example of a compulsory skill in both East Asia and the U.S.; regardless of individual abilities and interests, both cultures view it as essential that everyone learn to read well. Music was an example of an optional skill in both cultures; both cultures view musicality as desirable but not necessary, an area where individual abilities and interests can determine proficiency. In contrast, East Asian and U.S. cultures view mathematics differently; in East Asia, mathematics is viewed as compulsory in the same sense that both cultures view reading, whereas in the U.S., mathematics is viewed as optional, in the same sense that both cultures view music. The present review focuses on cultural variables that specifically involve rational number arithmetic, but related cultural variables no doubt influence learning in many other areas as well.

Instructional Factors

Limited understanding of rational number arithmetic operations by teachers. At minimum, a person who understands an arithmetic operation should know the direction of effects that the operation yields. Without understanding the direction of effects of arithmetic operations, people cannot judge an answer's plausibility and judge the plausibility of the procedure that generated it. Indeed, people who mistakenly believe that multiplication always yields a product as great as or greater than either multiplicand might infer that correct procedures are implausible and that incorrect procedures are plausible. For instance, a person who believed that multiplication must yield answers greater than either multiplicand might judge the correct equation "3/5 * 4/5 = 12/25" to be implausible, because 12/25 is less than either multiplicand; the same person might judge the incorrect "3/5 * 4/5 = 12/5" to be plausible, precisely because the answer is larger than both multiplicands.

Fractions. To assess understanding of fraction arithmetic, Siegler and Lortie-Forgues (in press) presented 41 pre-service teachers 16 fraction direction of effects problems of the form, "Is $N_1/M_1 + N_2/M_2 > N_1/M_1$", where N_1/M_1 was the larger operand. For example, one problem was "True or false: 31/56 * 17/42 > 31/56". Two-digit numerators and denominators were used to avoid use of mental arithmetic to obtain the exact answer and use that to answer the question. The problem set included all eight combinations of the four arithmetic operations, with fraction operands either above one or below one. The pre-service teachers attended a high quality school of education in Canada, and their academic performance was above provincial norms.

The main prediction was that the accuracy of such judgments would reflect the mapping between the magnitudes produced by whole number and fraction arithmetic.

Accuracy of judgments of the direction of effects was expected to be well above chance on addition and subtraction, regardless of the fractions involved, and also to be well above chance on multiplication and division with fractions greater than one. This hypothesis was based on the fact that the direction of effects for these six combinations of arithmetic operation and fraction magnitude is the same as with the corresponding whole number operation. However, below chance judgment accuracy was predicted on multiplication and division with fractions less than 1, because these cases yield the opposite direction of effects as multiplying and dividing whole numbers. Thus, despite having performed thousands of fraction multiplication problems with numbers from 0-1, and thus having thousands of opportunities to observe the outcomes of multiplication with such fractions, the adults to whom we posed such problems were expected to perform below chance in predicting the effects of multiplication with fractions below 1. The same was expected for division.

These predictions proved accurate. The pre-service teachers performed well above chance on the six types of problems on which they were expected to be accurate, but well below chance on both multiplication and division of fractions below 1 (Table 1). For example, when asked to judge whether multiplying two fractions below 1 would produce an answer larger or smaller than the larger multiplicand, the pre-service teachers correctly predicted the answer on 33% of trials.

The below-chance accuracy of judgments on the direction of effects task for multiplication and division of fractions below 1 was not due to the task being impossible. Mathematics and science majors at a highly selective university correctly answered 100% of the same multiplication direction of effects problems. The incorrect predictions also were not due to the pre-service teachers lacking knowledge of individual fraction magnitudes or of how to execute fraction arithmetic procedures. The pre-service teachers generated extremely accurate estimates of the magnitudes of individual fractions below 1 (almost as accurate as those of the math and science majors at the highly selective university) and consistently solved fraction multiplication problems, yet they judged the direction of effects of fraction multiplication less accurately than chance. The division results were highly similar.

Converging results have been obtained in studies in which teachers were asked to generate a story or situation that illustrated the meaning of fraction multiplication and division (e.g., Ball, 1990; Depaepe et al., 2015; Li & Kulm, 2008; Lin, Becker, Byun, Yang, & Huyang, 2013; Lo & Luo, 2012; Ma, 1999; Rizvi & Lawson, 2007; Tirosh, 2000). For instance, only 5 of 19 (26%) U.S pre-service teachers were able to generate a story or a situation showing the meaning of the division problem 1 3/4 ÷ 1/2 (Ball, 1990). Similarly, when asked to explain the meaning of 2/3 ÷ 2 or 7/4 ÷ 1/2, the vast majority of U.S. teachers failed to provide an explanation that went beyond stating the "invert and multiply" algorithm; Chinese teachers had little difficulty explaining the same problem (Li & Kulm, 2008; Ma, 1999).

Limited understanding of fraction arithmetic is not limited to U.S. teachers. When Belgian pre-service teachers were asked to identify the appropriate arithmetic operation for representing the word problem "Jens buys 3/4 kg minced meat. He uses 1/3 to make soup balls and the remaining part is used for making bolognaise sauce. How much (sic) kg minced meat does he use for his soup balls," only 19% correctly identified the problem as corresponding to 1/3 * 3/4 (Depaepe et al., 2015).

In contrast, when Chinese teachers were asked to explain the meaning of 1 3/4 ÷ 1/2, 90% generated at least one valid explanation. Taiwanese pre-service teachers were also highly proficient at generating meaningful models of fraction addition, subtraction, and multiplication (Lin et al., 2013). Moreover, when asked to identify whether fraction multiplication or division was the way to solve a story problem, Taiwanese pre-service teachers were correct far more often than U.S. peers (74% versus 34%; Luo, Lo & Leu, 2011). Thus, lack of qualitative understanding of fraction arithmetic by North American and European teachers is culturally contingent.

Decimals. These difficulties in understanding fraction multiplication and division are not limited to fractions. We conducted a small experiment on understanding of multiplication and division of decimals between 0 and 1 with 10 undergraduates from the same university as the one attended by the pre-service teachers in Siegler and Lortie-Forgues (in press). These undergraduates were presented the direction of effects task with decimals, with each common fraction problem from the direction of effects task being translated into its nearest 3-digit decimal equivalent (e.g., “31/56 * 17/42 > 31/56” became “0.554 * 0.405 > 0.554”).

Results with decimals paralleled those with common fractions: high accuracy (88% to 100% correct) on the six problem types in which fraction arithmetic produces the same pattern as natural number arithmetic, and below chance performance on the two problem types that shows the opposite pattern as with natural numbers (38% and 25% correct on multiplication and division of fractions below one). Thus, the inaccurate judgments on the direction of effects task were general to multiplication and division of numbers from 0-1, rather than being limited to numbers written in fraction notation.

Without understanding of rational number arithmetic, teachers cannot communicate the subject in a meaningful way, much less address students' misconceptions and questions adequately (e.g., Tirosh, 2000; Li & Huang, 2008). Consistent with this assumption, a strong positive relation has been observed between teachers' knowledge of fractions and decimals (including arithmetic) and their knowledge of how to teach these subjects to students (Depaepe et al., 2015). At minimum, understanding rational number arithmetic would prevent teachers from passing on misunderstandings to their pupils. For example, it would prevent them from stating that multiplication always yields a product larger than either multiplicand or that division always yields a quotient smaller than the number being divided.

Emphasis of teaching on memorization. For many years, U.S. researchers, organizations of mathematics teachers, and national commissions charged with improving mathematics education have lamented that instruction focuses too much on memorizing procedures and too little on understanding (e.g., Brownell, 1947; NCTM, 1989; NMAP, 2008). Focusing on how to execute procedures, to the exclusion of understanding them, has several negative consequences. Procedures learned without understanding are difficult to remember (Brainerd & Gordon, 1994; Reyna & Brainerd, 1991), especially over long periods of time. Lack of understanding also prevents students from generating procedures if they forget them. Consistent with this perspective, the CCSSI (2010) and other recent attempts to improve U.S math instruction strongly recommend that conceptual understanding of procedures receive considerable emphasis.

It is unclear whether these recommendations have been implemented in classrooms. Even teachers who have been given extensive, well-designed instruction in the conceptual basis of fraction and decimal arithmetic, and in how to teach them, often do not change their teaching (Garet et al., 2011). Unsurprisingly, Garet et al. found that the lack of change in teaching techniques was accompanied by a lack of change in students' learning. Teachers' continuing emphasis on memorization could reflect the greater ease of teaching in familiar ways, absence of incentives to change, not being able to convey knowledge to students that they themselves lack, and desire to avoid being embarrassed by questions about concepts that they cannot answer.

Minimal instruction in fraction division. At least in some U.S. textbooks, fraction division is the subject of far less instruction than other arithmetic operations. Illustrative of this phenomenon, Son and Senk (2010) found that *Everyday Mathematics* (2002), a widely used U.S. textbook series that has a relatively large emphasis on conceptual understanding, contained 250 fraction multiplication problems but only 54 fraction division problems in its fifth and sixth grade textbooks and accompanying workbooks.

To determine whether *Everyday Mathematics* is atypical of U.S. textbooks in its lack of emphasis on fraction division, we examined a very different textbook, *Saxon Math* (Hake & Saxon, 2003), perhaps the most traditional U.S. math textbook series (Slavin & Lake, 2008). Although differing in many other ways, *Saxon Math* was like *Everyday Math* in including far more fraction multiplication than division problems (122 versus 56). This difference might be understandable if fraction division were especially easy, but it appears to be the least mastered fraction arithmetic operation among North American students and teachers (Siegler & Lortie-Forgues, in press; Siegler & Pyke, 2013).

Such de-emphasis of fraction division is culturally contingent. Analysis of a Korean math textbook for Grades 5 and 6, which is when fraction multiplication and division receive the greatest emphasis there, as in the U.S., showed that the Korean textbook included roughly the same number of fraction multiplication problems as *Everyday Math* (239 versus 250), but more than eight times as many fraction division problems (440 versus 54) (Son & Senk, 2010). Viewed from a different perspective, the Korean textbook series included considerably more fraction division than fraction multiplication problems (440 versus 239), whereas the opposite was true of the U.S. textbooks (250 versus 54). The minimal emphasis on fraction division in these and quite possibly other U.S. textbooks almost certainly contributes to U.S. students' poor mastery of fraction division.

To the best of our knowledge, no cross-national comparison of textbook problems have been done for decimal arithmetic. However, U.S. textbooks also give short shrift to decimal division; our examination of 6th grade *Saxon Math* indicated that only 3% (8 of 266) of decimal arithmetic problems involved division.

Textbook explanations of arithmetic operations. Textbook explanations are another culturally contingent influence on learning. In the U.S., whole number multiplication is typically explained in terms of repeated addition (CCSSI, 2010). Multiplying 4 * 3, for instance, is taught as adding four three times (4 + 4 + 4) or adding three four times (3 + 3 + 3 +3). This approach has the advantage of building the concept of multiplication on existing knowledge of addition, but it has at least two disadvantages. First, because adding positive numbers always yields an answer larger than either addend, defining multiplication in terms of addition suggests the incorrect conclusion that the result of multiplying will always be larger than the numbers being multiplied. Second, the repeated-addition interpretation is difficult to apply to arithmetic with fractions and decimals that are not equivalent to whole numbers. For instance, how to interpret 1/3 * 3/4 or 0.33 * 0.75 in terms of repeated addition is far from obvious.

Repeated addition is not the only way to explain multiplication. For example, multiplication can be presented as "N of the M's with whole numbers (e.g., 4 of the 2's) and as "N of the M" with fractions (1/3 of the 3/4). This presentation might help convey the unity of whole number and fraction multiplication and therefore improve understanding of the latter.

A similar point can be made about division. In U.S. textbooks and in the CCSSI (2010) recommendations, division is introduced as fair sharing (dividing objects equally among people). For example, 15 ÷ 3 could be taught as 15 cookies shared equally among 3 friends. Again, this interpretation is straightforward with natural numbers but not with rational numbers, at least not when the divisor is not a whole number (e.g., what does if mean to share 15 cookies among 3/8 of a friend?).

Again, the standard presentation is not the only possible one. At least when a larger number is divided by a smaller one, both whole number and fraction division can be explained as indicating how many times the divisor can go into the dividend (e.g., how many times 8 can go into 32, how many times 1/8 can go into 1/2). It is unknown at present whether these alternative interpretations of fraction arithmetic operations are more effective than the usual ones in U.S. textbook, but they might be.

Limitation of Learners' Knowledge

Deficiencies in prior relevant knowledge also hinder many children's acquisition of fraction and decimal arithmetic. These again are culturally contingent, in that children in some other countries show far fewer deficiencies of prior knowledge.

Limited whole number arithmetic skill.

Fractions. All fraction arithmetic procedures require whole number arithmetic calculations. For example, 3/4 + 1/3 requires five whole number calculations: translating 3/4 into twelfths requires multiplying 3 * 3 and 4 * 3; translating 1/3 into twelfths requires multiplying 1 * 4, and 3 * 4; and obtaining the numerator of the sum requires adding 9 + 4. Even more whole number operations are necessary if the answer must be simplified or if the operands are mixed numbers (e.g., 2 1/4). Any inaccuracy with whole number arithmetic operations can thus produce fraction arithmetic errors.

Whole number computation errors cause a fairly substantial percentage of U.S. students' fraction arithmetic errors. For example, incorrect execution of whole number procedures accounted for 21% of errors on fraction arithmetic problems in Siegler and Pyke (2013). Such whole number computation errors are far less common among East Asian students doing similar problems (Bailey, et al., 2015). More generally, whole number arithmetic accuracy has repeatedly been shown to be related to fraction arithmetic accuracy, and early whole number arithmetic accuracy predicts later fraction arithmetic accuracy, even after controlling for other relevant variables (Bailey et al., 2014; Hecht & Vagi, 2010; Hecht et al., 2003; Jordan et al., 2013; Seethaler, Fuchs, Star, & Bryant, 2011).

Decimals. Decimal arithmetic also requires whole number arithmetic calculations. At least one study indicates that whole number arithmetic accuracy predicts decimal arithmetic accuracy (Seethaler et al., 2011).

Limited knowledge of the magnitudes of individual fractions. Accurate representations of the magnitudes of individual fractions and decimals can support fraction and decimal arithmetic, by allowing learners to evaluate the plausibility of their answers to fraction arithmetic problems and the procedures that produced them (e.g., Hecht, 1998; Hiebert & LeFevre, 1986; Byrnes & Wasik, 1991). Magnitude knowledge also can make arithmetic procedures more meaningful. For instance, a child who knew that 2/7 and 4/14 have the same magnitude would understand why 2/7 can be transformed into 4/14 when adding 2/7 + 5/14 better than a child without such knowledge.

Fractions. Children with less knowledge of the magnitudes of individual fractions also tend to have less knowledge of fraction arithmetic (Byrnes & Wasik, 1991; Hecht, 1998; Hecht et al., 2003; Hecht & Vagi, 2010; Jordan, et al., 2013; Siegler & Pyke, 2013; Siegler et al., 2011; Torbeyns, Schneider, Xin, & Siegler, 2014). The strength of this relation is moderate to high (Pearson *r*'s range from 0.44 to 0.86 in the above-cited experiments). Moreover, the relation continues to be present after controlling for such factors as vocabulary, nonverbal reasoning, attention, working memory and reading fluency (e.g., Jordan et al., 2013). The relation is also present when the arithmetic tasks require estimation rather than calculation (Hecht, 1998), and when the samples are drawn from countries other than the U.S., such as Belgium (Torbeyns et al., 2014; Bailey et al., 2015). Perhaps most important, the relation is causal; interventions that emphasize fraction magnitudes improve fraction arithmetic learning (Fuchs et al., 2013, 2014).

Lack of understanding of fraction magnitudes also may contribute to difficulty in learning algebra. Consistent with this hypothesis, knowledge of fraction magnitudes in middle school predicts later algebra learning (Bailey et al., 2012; Booth & Newton, 2012; Booth et al., 2013).

Limited conceptual understanding of arithmetic operations.

Fractions. Not surprising given their results with pre-service teachers, Siegler and Lortie-Forgues (in press) found that sixth and eighth graders had the same weak qualitative understanding of fraction multiplication and division. For example, the middle school students correctly anticipated the direction of effects of multiplying two fractions below one on only 31% of trials, very similar to the 33% of trials among the pre-service teachers. This relation again was not attributable to lack of knowledge of the fraction multiplication procedure; the children multiplied fractions correctly on 81% of problems. The results with these students also suggested that inaccurate judgments on the direction of effects task could not be attributed to forgetting material taught years earlier. Sixth graders who had been taught fraction division in the same academic year and fraction multiplication one year earlier also performed below chance on the direction of effects task for fraction multiplication and division (30% and 41% correct, respectively).

Decimals. Poor understanding of decimal arithmetic has also been documented. Hiebert and Wearne (1985; 1986) showed that most students' knowledge of decimal arithmetic consists of memorized procedures for which they have little or no understanding. One line of evidence for this conclusion is that students often cannot explain the rationale for the procedures they use. For instance, fewer than 12% of sixth graders could justify why they were aligning the operands' decimal points when adding and subtracting decimals (Hiebert & Wearne, 1986). In the same vein, when students need to estimate solutions to problems, their answers are often unreasonable. For instance, when asked to select the closest answer to 0.92 * 2.156, with the response options 18, 180, 2, 0.00018 and 0.21, the answer “2” was chosen by only 8% of fifth graders, 18% of sixth graders, 33% of seventh graders, and 30% of ninth graders (Hiebert & Wearne, 1986).

Weak understanding of decimal arithmetic is not limited to arithmetic problems presented in their typical format or to U.S. students. When Italian 9th graders (14- to 15-year olds) were presented the problem, "The price of 1 m of a suit fabric is 15,000 lire. What is the price of 0.65 m?" only 40% of the adolescents correctly identified the item as a multiplication problem. By comparison, 98% of the same participants correctly identified the operation on a virtually identical problem where the values were whole numbers (Fischbein et al., 1985).

As in previous cases, children’s ability to surmount the challenges of understanding decimal arithmetic depends on cultural and educational variables. For instance, when 6th graders in Hong Kong and Australia were asked to translate into an arithmetic operation the word problem "0.96 L of orange juice was shared among 8 children. How much orange juice did each child have," 90% of the children from Hong Kong correctly answered 0.96 ÷ 8. By comparison, only 48% of Australian children answered the same problem correctly (Lai & Murray, 2014).

General cognitive abilities.

Fractions. Fraction arithmetic poses a substantial burden on limited processing resources. It requires interpreting the notation of fractions, inhibiting the tendency to treat numerators and denominators like whole numbers, and carrying out a series of steps while maintaining intermediate results and keeping track of the final goal. Sustaining attention during instruction on easily confusable fraction arithmetic procedures also seems likely to place high demands on executive functions. Not surprisingly given this analysis, fraction arithmetic performance is uniquely predicted by individual differences in executive functions (Siegler & Pyke, 2013) as well as by individual differences in working memory and attentive behaviour (Hecht & Vagi, 2010; Jordan et al., 2013; Seethaler, et al., 2011).

Decimals. Although less research is available on the relation between basic cognitive processes and decimal arithmetic, the one relevant study that we found indicated that working memory was uniquely predictive of fraction and decimal arithmetic performance (Seethaler et al., 2011).

Interventions for Improving Learning

The prior sections of this article indicate that many students in the U.S. and other Western countries fail to master fraction and decimal arithmetic; that at least seven inherent sources of difficulty contribute to their weak learning; and that culturally contingent variables, including the instruction learners encounter and their prior relevant knowledge, influence the degree to which they surmount the inherent difficulties. Alongside these somewhat discouraging findings, however, were some more encouraging findings– children in East Asian countries learn rational number arithmetic far more successfully, and Western children who are provided well-grounded interventions also learn well. This final section examines three especially promising interventions that allow greater numbers of children in Western societies to learn rational number arithmetic.

Fractions

One intervention that has been found to produce substantial improvement in fraction addition and subtraction was aimed at fourth graders who were at-risk for mathematics learning difficulties (Fuchs et al., 2013, 2014). The intervention focused on improving knowledge of fraction magnitudes through activities that required representing, comparing, ordering, and locating fractions on number lines. Fraction addition and subtraction were also taught, but received less emphasis than in the control "business as usual" curriculum.

The intervention produced improvement on every outcome measured, including questions about fractions released from recent NAEP tests. The improvement produced by the intervention on fraction arithmetic was large, almost 2.5 standard deviations larger than the gains produced by a standard curriculum that emphasized the part–whole interpretation of fractions and fraction arithmetic.

Decimals

A computerized intervention conducted on typical 6th graders generated improvements in decimal arithmetic (Rittle-Johnson & Koedinger, 2009). In this intervention, students were presented lessons on decimal place values and decimal addition and subtraction. One experimental condition followed the typical procedure in U.S. textbooks of presenting the two types of lessons sequentially (i.e., the lessons on place value were followed by the lessons on arithmetic). The other condition intermixed the two types of lessons. Intermixing the lessons produced larger gains in addition and subtraction of decimals.

Fractions, Decimals, and Percentages

Moss and Case's (1999) intervention produced substantial gains on arithmetic word problems involving fractions, decimals, and percentages. Typical fourth graders were first introduced to percentages, then to decimals, and then to fractions. The rational numbers were always represented as part of a continuous measure, such as a portion of a larger amount of water in a cylinder or a segment of a distance on a number line or board game. Students were taught to identify benchmarks (50%, 25% and 75%) on these objects, and were taught to solve rational number arithmetic problems using the benchmarks. For instance, students were taught to find 75% of a 900-ml bottle by first computing 50% of 900 ml (450 ml), then computing 25% of the 900 ml by computing 50% of the 50% (225 ml), and then adding the two values together (675ml). Students were not taught any formal procedure to carry out fraction arithmetic operations.

The intervention yielded large gain in fraction knowledge. For example, students who received the intervention showed greater gains when solving arithmetic problems of the form "what is 65% of 160" with percentages, decimals, and fractions than did a control group who received a traditional curriculum on rational number arithmetic.

A shared characteristic of these three effective interventions is that all emphasize knowledge of rational number magnitudes. Focusing instruction on fraction and decimal magnitudes allowed children to overcome the limited accessibility of magnitudes of operands and answers that seems to be a major difficulty in learning fraction and decimal arithmetic. Knowledge of fraction and decimal magnitudes is not sufficient to produce understanding of rational number arithmetic. The pre-service teachers in Siegler and Lortie-Forgues (in press) had excellent knowledge of fraction magnitudes between 0 and 1, yet were below chance in predicting the effects of multiplying and dividing fractions in the same range. However, such knowledge of rational number magnitudes does seem necessary for understanding rational number arithmetic. Without understanding the magnitudes of the numbers being combined arithmetically, it is unclear how children could make sense of the effects of arithmetic operations transforming those magnitudes. Together, these interventions demonstrate that by improving standard practices, one can increase the number of students who surmount the challenges inherent to fraction and decimal arithmetic.

Conclusions

Fraction and decimal arithmetic pose large difficulties for many children and adults, despite the prolonged and extensive instruction devoted to these topics. The problem has persisted over many years, despite continuing efforts to ameliorate it. These facts are alarming, considering that rational number arithmetic is crucial for later mathematics achievement and for ability to succeed in many occupations (McCloskey, 2007; Siegler et al., 2012).

Fraction and decimal arithmetic are also theoretically important. Both knowledge of the magnitudes of individual rational numbers and knowledge of the magnitudes produced by rational number arithmetic are important parts of numerical development beyond early childhood. They provide most children's first opportunity to learn that principles that are true of whole numbers and of whole number arithmetic are not necessarily true of numbers and arithmetic in general. Comprehensive theories of numerical development must include descriptions and explanations of the growth of rational number arithmetic, as well as of why knowledge in this area is often so limited.

In hopes of stimulating greater amounts of research on what we believe to be a crucial aspect of numerical development, we devoted this review to analyzing why learning fraction and decimal arithmetic is so difficult. To address this question, we identified seven difficulties that are inherent to fraction arithmetic, decimal arithmetic or both -- their notation, inaccessibility of the magnitudes of operands and answers, opaqueness of procedures, complex relations between rational and whole number arithmetic procedures, complex relations of rational number arithmetic procedures to each other, direction of effects of multiplication and division of numbers from 0-1 being the opposite as with whole numbers, and the large number of distinct procedures involved in rational number arithmetic. These difficulties are inherent to rational number arithmetic-- every learner faces them.

We also considered culturally contingent sources of learning difficulties. These are factors determined by cultural values and characteristics of educational systems. For instance, relative to Western countries, East-Asian countries have highly knowledgeable teachers (Ma, 1999) and place a large emphasis on students solving difficult mathematics problems (Son & Senk, 2010). Moreover, East Asian students come to the task of learning rational number arithmetic with better knowledge of whole number arithmetic (Cai, 1995) and better knowledge of fraction magnitudes (Bailey et al., 2015). These and many other cultural variables influence the likelihood that children will overcome the inherent difficulties of learning fraction and decimal arithmetic.

We divided culturally contingent sources of difficulty into ones involving instruction and ones involving learners' prior knowledge. The instructional factors included limited understanding of arithmetic operations by teachers, emphasis on memorization rather than understanding, minimal instruction in fraction division, and textbook explanations of arithmetic operations that are difficult to apply beyond whole numbers. Sources of difficulties involving learners' prior knowledge included limited knowledge of whole number arithmetic and of the magnitudes of individual fractions, as well as limited general processing abilities. These sources of difficulty, of course, are not independent. Teachers' focus on memorization may reflect reluctance to reveal their own weak conceptual understanding, learners' weak understanding of fraction magnitudes may reflect inadequate prior teaching; and so on. Moreover, more general cultural values, such as whether a society views mathematics learning as compulsory or optional, doubtlessly also influence learning of fraction and decimal arithmetic.

Perhaps the most encouraging conclusion from the review is that interventions aimed at helping children surmount the difficulties inherent to fraction and decimal arithmetic can produce substantial gains in performance and understanding. Interventions that focus on rational number magnitudes appear to be especially effective in helping children learn fraction and decimal arithmetic. Rational number arithmetic thus appears to be a promising area for both theoretical and applied research, one that could promote more encompassing theories of numerical development and also yield important educational applications.

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Table 1.

Accuracy (percent correct) of pre-service teachers on the direction of effect problems

Operations	Fractions below one	Fractions above one
Addition	92	92
Subtraction	89	92
Multiplication	33	79
Division	29	77