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Title:

Arithmetic and Cognitive Contributions to Algebra

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Abstract Body

Background / Context:
Algebra is a prerequisite for access to STEM careers and occupational success (NMAP, 2008a), yet algebra is difficult for studens through high school (US DOE, 2008). Growth in children's conceptual and procedural arithmetical knowledge is reciprocal, although conceptual knowledge has more impact on procedural knowledge than the reverse (Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001). However, we do not have a complete picture of whether and how algebra procedural skill and conceptual knowledge are distinct versus related. We expect each to have different patterns of relations with arithmetical and cognitive resources. Arithmetic predictors of algebra include arithmetical concepts (Alibali et al., 2007; NMAP, 2008b), arithmetical computational fluency (Tolar et al., 2009; 2013), and fractions and proportional reasoning (Brown & Quinn, 2007; NMAP, 2008a). Cognitive predictors include number sense, language, working memory (Tolar et al., 2009; Lee et al., 2004), and spatial visualization (Tolar et al., 2009). Arithmetic skills are likely strongly related to algebra, and cognitive processes influence algebra directly or indirectly through arithmetic. Relations between cognition and arithmetic are more established relative to what is known about how either influence algebra.

Purpose / Objective / Research Question / Focus of Study:
The broad purpose is to identify skills important for algebra with the ultimate goal of improving algebra preparedness. This presentation focuses on 3 inter-related components: (a) predictors of procedural and conceptual algebra performance (School Algebra); (b) predictors of algebraic precursor skills such as arithmetic concepts, fractions, and proportional reasoning (Skills); and (c) experimental manipulation of procedural and conceptual algebraic skill (Experimental).

Setting:
These studies take place in diverse intermediate/middle schools in Texas and Tennessee.

Population / Participants / Subjects:
There were 3 research components/groups of students: School Algebra, Skills, and Experimental.

There were 114 students in the School Algebra component, across 5 schools in Grades 7 or 8 at one site. They were taking Algebra I as a course in their middle school, and therefore represent skilled learners. Students were assessed on cognitive and arithmetic skills, as well as algebra (procedural and conceptual), in Fall of their Algebra I year. All students were instructed in English, and 57 (50%) were female; the majority (51%) were Caucasian, with a mix of other ethnicities. 107 students were also assessed in Spring of their Algebra I year.

Participants in the Skill component were 162 students from three schools in Grade 6. These students completed measures of cognitive and arithmetic skills in Spring, 2012. The average age was 12.39 (SD=0.53), and all students were instructed in English, though 17% were classified as second-language-learners. Forty-seven percent were female. The majority (77%) were Hispanic, with 20% African American, and 3% Caucasian.

101 students with no exposure to formal Algebra I participated in the Experiment component in Grade 6. They were from the same schools as in the Skills component, 94 of whom completed the experiment (described below).

Intervention / Program / Practice:
For the School Algebra and Skills components, there was no intervention. The Experiment

component addressed two questions: (1) Are procedural and conceptual knowledge separable? and (2) Does sequencing of procedural and conceptual instruction affect gains in knowledge? There were 20 hours of scripted instruction over 4 days, comprising 16 total sessions. Content focused on Relations, Functions, Linear Functions, and Linear versus Non-linear functions. Each session comprised (a) a didactic lesson delivered with worked examples; (b) individual work with scaffolded assistance offered by research staff; and (c) paired practice. Content for individual and paired work was derived from the didactic lesson. Two versions of each lesson were developed. The Procedural version focused on step-by-step computational procedures for determining solutions to algebraic problems. The Conceptual version focused on mathematical terms and concepts including definitions and explanations. The three experimental conditions ordered procedural (P) and conceptual (C) instruction differently. In the P→C condition, students received pretest, then all P lessons over the first two days, then post-test I, followed by the all C lessons over the other two days, and finally post-test II. The C→P condition received the opposite. The PC condition received both P and C lessons *within* each session.

Research Design:

For the School Algebra component, the design was both descriptive and longitudinal predictive. For the Skills component, the design was descriptive. For the Experiment component, the design was a quasi-experimental pre-post design. Each of three teachers taught each condition at each school. Students were randomized to condition when there were multiple conditions for a timeslot, but sometimes there was only one condition available for a timeslot that fit a student's schedule. No student, parent, or researcher selected a timeslot by experimental condition.

Data Collection and Analysis:

Five cognitive measures were used. For Quantity Comparison (Cirino, 2011; School Algebra, Skills), students select which of two Arabic numerals is larger as "quickly but as accurately" as possible. For Number Line Estimation (Booth & Siegler, 2006; Siegler & Booth, 2004; School Algebra, Skills), students identify where on a 0-1000 number line a given Arabic numeral should appear. The WJ-III Visual-Auditory Learning subtest (Woodcock et al., 2001; School Algebra, Skills) asks students to associate a rebus to a word, and "read" sentences using the symbols, and requires memory, symbolic association, and syntactic manipulation. Automated Symmetry Span (Conway et al., 2005; Unsworth et al., 2005) requires students to maintain a series of spatial locations in memory while making judgments about the symmetry of figures. Finally, the Mental Rotations Test (MRT, Peters et al., 1995; Vandenberg & Kuse, 1978) assess spatial visualization by asking students to identify which 2 of the 4 are rotated versions of a target figure and is correlated with algebra performance in college students (Tolar et al., 2009).

There were 5 Arithmetic measures. Arithmetical Concepts (Tolar, 2012; School Algebra, Skills, Experiment) has items of associative, commutative, distributive, and identity properties, and of equality and zero. Subtraction and Multiplication (Ekstrom et al., 1976; School Algebra, Skills, Experiment) has alternating rows of 2-digit problems in these areas, completed in 2 min. The Fraction Competency Test (Brown & Quinn, 2006; School Algebra, Skills, Experiment) has 6 categories of skill, and performance and is correlated r =.58 with an Algebra final exam (Brown & Quinn, 2007). The Diagnostic Assessment of Proportional Reasoning (Misailidou & Williams, 2003; Skills, Experiment) has 13 items calibrated with 303 students (10-14 years) using item response theory techniques. Finally, WRAT-4 Math Computation subtest (Wilkinson & Robertson, 2008; Experiment) is a broad based measure of computational skill.

For Algebra, the Algebra Procedural Skill and Conceptual Knowledge Experimental Tests (Tolar, 2012; School Algebra, Experiment) were utilized. Per the name, both procedural and conceptual items are scored. In 50 minutes, students respond to questions via multiple choice (School Algebra) or through constructed responses (Experiment). Items were developed to be consistent with state standards and to align with the experimental lessons.

Analytic procedures utilized included regression for the School Algebra and Skills components, and ANCOVA-based procedures for the Experiment component.

Findings / Results:

School Algebra Results. 99 students had all data across cognitive predictors. In predicting beginning-of-year algebra performance, the combination of predictors was not significant, though vocabulary was a unique predictor. 112 students had all data across arithmetic predictors. The combination of predictors was significant, ($p<.001$, Adj. $R^2=.23$). In this model, arithmetic concepts ($p=.006$, squared semi-partial r or unique variance=.06), fractions ($p=.003$, $r=.06$), and addition/subtraction correction ($p=.014$, $r=.03$) were unique predictors. When only significant predictors were included across cognitive and arithmetic domains, the overall model was significant, ($p<.001$, Adj. $R^2=.22$, although here, only arithmetic concepts ($p=.010$, $r=.05$) and fractions ($p=.011$, $r=.05$) uniquely predicted beginning of year algebra.

We repeated analyses for end-of-year algebra performance (covarying for pretest). The combination of cognitive predictors was significant, ($p<.001$, Adj. $R^2=.32$), with pretest ($p<.001$, $r=.18$) and vocabulary ($p<.012$, $r=.05$) unique predictors. Arithmetic predictors were also significant, ($p<.001$, Adj. $R^2=.40$). In this model, pretest ($p<.001$, $r=.07$), arithmetic concepts ($p=.021$, $r=.03$), fractions ($p<.001$, $r=.08$), and single digit addition ($p=.025$, $r=.03$) were unique predictors. Finally, when only significant predictors were included across cognitive and arithmetic domains, the overall model was significant ($p<.001$, Adj. $R^2=.36$), although here, only pretest ($p<.001$, $r=.08$) and fractions ($p=.008$, $r=.04$) uniquely predicted end-of-year algebra.

Skills Results. Algebra was not an outcome, as these students in Grade 6 had not yet taken Algebra I. Instead, cognitive predictors were used to predict arithmetic performance, including school administered achievement measures of math. In predicting single digit addition, the overall model was significant, ($p<.001$, Adj. $R^2=.29$). Number line deviation ($p=.006$, $r=.05$) and symbolic comparison response time ($p<.001$, $r=.10$) were the only unique predictors. For single digit subtraction, the overall model was significant, ($p<.001$, Adj. $R^2=.18$). Vocabulary ($p=.013$, $r=.04$) was the only unique predictor. In predicting single digit multiplication, the overall model was significant, ($p<.001$, Adj. $R^2=.19$). Number line deviation ($p=.039$, $r=.03$), number line response time ($p=.034$, $r=.03$), symbolic comparison response time ($p=.001$, $r=.06$), and symmetry span ($p=.012$, $r=.04$) were unique predictors. In predicting addition/subtraction correction, the overall model was significant, ($p<.001$, Adj. $R^2=.37$). Number line deviation ($p=.006$, $r=.04$), number line response time ($p=.014$, $r=.03$), symbolic comparison response time ($p<.001$, $r=.14$), and symmetry span ($p=.014$, $r=.03$) were unique predictors. In predicting subtraction/multiplication, the overall model was significant, ($p<.001$, Adj. $R^2=.28$). Symbolic comparison response time ($p<.001$, $r=.15$) was the only significant predictor.

In predicting arithmetic concepts, the overall model was not significant. In predicting fractions, the overall model was significant, ($p=.003$, Adj. $R^2=.12$), although this model had no unique predictors. In predicting proportional reasoning, the overall model was significant, ($p<.001$, Adj. $R^2=.28$).Vocabulary ($p=.002$, $r=.06$), number line deviation ($p=.023$, $r=.03$), and symbolic comparison response time ($p=.038$, $r=.02$) were significant predictors. Finally, In predicting the school-administered benchmark math test, the overall model was significant, ($p<.001$, Adj. $R^2=.31$). Vocabulary ($p=.023$, $r=.03$), number line deviation ($p<.001$, $r=.06$), and Weber fraction ($p=.009$, $r=.04$) were unique predictors.

Experiment Results. Table 1 shows WRAT-4 Arithmetic and algebra pretest scores by condition. ANOVA compared the experimental groups on WRAT-4 and algebra pretest scores. There was a group effect for conceptual pretest scores, $F(2,91) = 3.24$, $p<.05$, with the P→C group higher than the C→P group. WRAT-4 and procedural pretest scores were not significant (both $p>.05$). Figure 1 shows gains in procedural and conceptual knowledge by experimental condition. ANCOVA compared the three experimental groups on gains after the first two days of instruction (posttest 1). There was a significant group effect on conceptual posttest 1 scores controlling for pretest scores, $F(2,90)=20.42$, $p<.05$. As expected, C→P students who had received conceptual instruction on all lessons scored significantly higher than P→C students who had not received any conceptual instruction. PC students who had received conceptual instruction on half the lessons also scored significantly higher than P→C students, but unexpectedly they also scored significantly higher than C→P students. There was also a significant group effect on procedural posttest 1 scores, $F(2,90)=11.85$, $p<.05$. Surprisingly, both PC students who had received procedural instruction on only half the lessons and C→P students who had received no procedural instruction scored significantly higher than the P→C students who had received procedural instruction on all the lessons. The PC and C→P students did not differ. ANCOVA also compared the experimental groups on gains after all four days of instruction (posttest 2), when all groups had procedural and conceptual lessons, albeit in different orders. There were no significant group effects on either procedural posttest 2 gains, $F(2,90)=1.56$, $p>.05$ or conceptual posttest 2 gains, $F(2,90)=0.49$, $p>.05$.

Conclusions:

This group of studies sought to identify cognitive concomitants of arithmetic, as well as cognitive and arithmetic predictors of algebra, and to examine the manipulation of procedural and conceptual instruction.

Cognitive predictors of arithmetic in Grade 6 include language (vocabulary), line estimation, magnitude estimation (symbolic and nonsymbolic), and working memory. The dominant predictors were clearly line estimation and symbolic magnitude estimation. Both involve identification and mapping of symbolic information, quickly, which is consistent with the nature of written arithmetic, as well as the timed nature of many of these measures.

For the Algebra study, vocabulary was a significant cognitive contributor to algebra skills, but remained so even when arithmetic predictors were included. Arithmetic concepts and fractions were dominant arithmetic predictors. Even when pretest performance was included, the same pattern of predictive variables remained relevant.

For the Experimental study, results suggest that procedural and conceptual knowledge are partially separable. Students may acquire procedural knowledge without conceptual knowledge, but may not necessarily acquire conceptual knowledge without also acquiring some procedural knowledge. However, the evidence also suggests that the order in which students receive procedural and conceptual instruction (i.e., all procedural followed by all conceptual, all conceptual followed by all procedural, or procedural and conceptual together) does not affect the overall procedural or conceptual learning.

Appendices

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Appendix A. References

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Appendix B. Tables and Figures
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Table 1. Mean (SD) Scores by Experimental Condition

Measure	P→C (*n* = 28)	C→P (*n* = 33)	PC (*n* = 33)
WRAT-4 SS	104.1 (11.8)	105.4 (12.2)	108.6 (11)
Algebra Score (%)			
Procedural	19.9 (16.8)	19.7 (12.1)	21.7 (11.6)
Conceptual	20.0 (14.8)	11.9 (11.2)	17.4 (12.6)

For P→C and C→P conditions, students received all lessons of one type (procedural, P, or conceptual, C) the first two days then all of the lessons again of the second type during the next two days. In the PC condition, the students received procedural immediately followed by conceptual type instruction for each lesson. They were instructed on half the lessons the first two days and the other half of the lessons the next two days.

Figure 1. Algebra Procedural and Conceptual Gains by Experimental Condition