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This study of *Preservice Secondary Mathematics Teachers’ Beliefs about Teaching Geometric Transformations (GTs) using Geometer’s Sketchpad (GSP)* aimed to explore the beliefs hold by preservice secondary mathematics teachers about teaching geometric transformations with Geometer's Sketchpad. In this study, I applied three methodological iterations. The first iteration was a pilot study of Preservice Secondary Mathematics Teachers' Beliefs about Teaching Mathematics with Technology. The second iteration was a study of Beliefs about Teaching Geometric Transformations Using Geometer's Sketchpad: A Reflexive Abstraction. The third and final iteration was the study of Preservice Secondary Mathematics Teachers' Beliefs about Teaching Geometric Transformations Using Geometer's Sketchpad. This study used five assumptions of radical constructivist grounded theory (RCGT) - symbiotic relation between the researcher and the participants, the participants and the researcher's voice, research as a cognitive function, research as an adaptive function, and praxis as quality criteria - synthesized from radical constructivist epistemology and grounded theory methodology. Five task-based interviews in the problematic contexts of teaching GTs by using GSP were administered with each of the two participants. The first analytical and interpretive approach generated six major categories associated with beliefs about action, affect, attitude, cognition, environment, and object of teaching GTs with GSP. The constructivist re-interpretation approach entered into epistemic scaffolding with holistic findings of the participants' beliefs in terms of reflective and reflexive beliefs that were associated with their anticipated practices of using GSP for teaching GTs. Some implications of the study have been discussed.

Keywords: Preservice teacher beliefs, radical constructivist grounded theory (RCGT)

PRESERVICE SECONDARY MATHEMATICS TEACHERS' BELIEFS ABOUT TEACHING GEOMETRIC TRANSFORMATIONS USING GEOMETER'S SKETCHPAD

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DEDICATION

I would like to dedicate this work to my family members. With a special dedication going to my parents Ghanashyam Belbase and Nama Devi Belbase for their encouragement, support and upbringing with love, care, and freedom. As well, I would like to dedicate this work to my daju (elder brother) Birendra Belbase, bhauju (sister-in-law) Kanti Belbase, sisters Sulochana Marasini and Rupa K.C. for their encouragement, love, and support. Finally, and most importantly, I dedicate this work to my wife Basanta Belbase and daughter Anjila Belbase for everything they have given me and shared with me throughout this journey.

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ABSTRACT

This study of *Preservice Secondary Mathematics Teachers' Beliefs about Teaching Geometric Transformations (GTs) using Geometer's Sketchpad (GSP)* aimed to explore the beliefs hold by preservice secondary mathematics teachers about teaching geometric transformations with Geometer's Sketchpad. In this study, I applied three methodological iterations. The first iteration was a pilot study of Preservice Secondary Mathematics Teachers' Beliefs about Teaching Mathematics with Technology. The second iteration was a study of Beliefs about Teaching Geometric Transformations Using Geometer's Sketchpad: A Reflexive Abstraction. The third and final iteration was the study of Preservice Secondary Mathematics Teachers' Beliefs about Teaching Geometric Transformations Using Geometer's Sketchpad. This study used five assumptions of radical constructivist grounded theory (RCGT) - symbiotic relation between the researcher and the participants, the participants and the researcher's voice, research as a cognitive function, research as an adaptive function, and praxis as quality criteria - synthesized from radical constructivist epistemology and grounded theory methodology. Five task-based interviews in the problematic contexts of teaching GTs by using GSP were administered with each of the two participants. The first analytical and interpretive approach generated six major categories associated with beliefs about action, affect, attitude, cognition, environment, and object of teaching GTs with GSP. The constructivist re-interpretation approach entered into epistemic scaffolding with holistic findings of the participants' beliefs in terms of reflective and reflexive beliefs that were associated with their anticipated practices of using GSP for teaching GTs. Some implications of the study have been discussed.

CHAPTER 1: INTRODUCTION

In this chapter, I established the rationale for the study of preservice secondary mathematics teachers' beliefs about teaching mathematics with technology in general and teaching geometric transformations (GTs) with Geometer's Sketchpad (GSP) in particular. I introduced the context and issues of teacher beliefs and it highlighted the rationale for the questions by situating the importance of the research questions. I also outlined some limitations and delimitations of the study.

Context and Issues for the Study

Several studies of beliefs have been carried out in a variety of fields, for example, mathematics (e.g., Nguyen, 2012), artificial intelligence (e.g., Boukhris, Benferhat, & Elouedi, 2012; Huber & Schmidt-Petri, 2009), sociology (e.g., Boudon, 2001), psychology (e.g., Hofer & Pintrich, 2009), and education (e.g., Fang, 1996; Pepin & Roesken-Winter, 2015). In education, belief structures of teachers have been the object of much research (e.g., Chang, 2008; Elen, Stahl, Bromme, & Clarebout, 2011; Thompson, 1984, 1992). In mathematics education, study of preservice and inservice teacher beliefs has been accepted as an important area to consider for reforming mathematics teaching and learning (Pepin & Roesken-Winter, 2015). In this context, both preservice and inservice mathematics teachers' beliefs seem to be critical in the teaching and learning of mathematics. They seem to be critical because the beliefs they hold may impact their perception and practice of teaching (Thompson, 1984 & 1992), knowledge of content and method of knowing (Elen, Stahl, Bromme, & Clarebout, 2011), and method of teaching and learning (Chan, 2008; Sing & Khine, 2008; Voss, Kleickmann, Kunter, & Hachfeld, 2013). They may also influence their professional development (Mohamed, 2006; Richards, Gallo, & Renandya, 2001) and curriculum practice (Burkhardt, Fraser, & Ridgway, 1990; Koehler & Grouws, 1992). Therefore, teacher beliefs are important aspects of teacher education in general and mathematics teacher education in particular (Leatham, 2002).

The general context of studying preservice mathematics teachers' beliefs can be linked to the goal of engendering belief structures that support quality mathematics instruction (Pepin & Roesken-Winter, 2015). The process of stimulating positive beliefs may facilitate change of preservice teachers' beliefs as well as mechanisms to change beliefs through improving their pedagogical and mathematical proficiency (Li & Moschkovich, 2013). It also may help us in exploring criteria for richer beliefs. It may even go further to ethics of changing beliefs, emotional aspects of beliefs, measurement of beliefs, and beliefs in the classroom contexts (Raths & McAninch, 2003). Some issues related to teacher beliefs can be emphasized with the following questions:

1. How do beliefs influence practices of teaching mathematics?
2. Can we change beliefs of other persons (e.g., mathematics teachers)?
3. What are the mechanisms for change of mathematics teachers' beliefs?
4. What beliefs do we want to change or promote in relation to teaching mathematics?
5. How does ethics play a role in changing teacher beliefs?
6. What are the areas of concern for changing teacher beliefs about teaching mathematics?
7. How does the interrelation of cognition and emotion affect one's beliefs?
8. How can we measure or assess teacher beliefs about teaching mathematics?
9. Why do school experiences of teachers matter in relation to their beliefs?

The issues related to these questions are highlighted in Chapter 2 in detail.

The issues of teacher beliefs are related to individual efforts of teacher educators, institutional goals and programs of teacher education, and epistemological paradigms and practices of teacher education researchers (Eichler & Erens, 2015). These issues may also relate to personal emotions and cognitions of the preservice or inservice teachers, and autonomy of prospective teachers to teach and learn during their practicum (de Klerk, Palmer, & van Wyk, 2012). These issues reflect the present context of preservice or inservice secondary mathematics teachers' beliefs about teaching and learning mathematics in general and teaching GTs with GSP in particular. These issues related to teacher beliefs can be viewed broadly from three lenses - relational, institutional and praxis that I have discussed in the next sub-section while framing the statement of the problem for this study.

Statement of the Problem

The contextual issues from the literature explained in chapter two can be analyzed through three lenses in the area of preservice secondary mathematics teachers' beliefs about teaching GTs with GSP. These lenses are - relational, institutional, and praxis-oriented (Table 1). The relational lens focuses on the interrelation of teacher beliefs and practices (OECD, 2009). The institutional lens is related to teacher education programs and practices for changing teacher beliefs in a positive direction (Tatto, 1998). The praxis lens observes teacher education as a process or product in terms of teacher beliefs (Fives & Gill, 2015; Hartley & Whitehead, 2006). These three lenses might be helpful to analyze the contextual issues synthesized from the literature. They were helpful to synthesize the statement of the problem for the study. These issues are explained in further depth in the literature review (Chapter 2).

Table 1. Lenses in the Study of Preservice/Inservice Teacher Beliefs

Lenses	Viewpoint	Literature
Relational	Interrelation of beliefs and practices	Lampert & Ball (1998), Kolb (1984), Walker, Brownlee, Exley, Woods, & Whiteford (2011)
Institutional	Teacher beliefs as focus of teacher education programs	Beswick (2007, 2012), Chai, Wong, & Teo (2011), Drageset (2010), Goodson (2012), Grant (1984), Kuntze (2012), Nespor (1987), Peterson, Fennema, Carpenter, & Loef (1989), Quinn (1998 a & b), Richard (2010), Richardson (1996), Schmidt & Kennedy (1990), Tillema (1995), Witherspoon & Shelton (1991), Yilmaz & Şahin (2011), Zakaria & Musiran (2010)
Praxis	Teacher education as product or process	Grundy (1987), Streibel (1991)

The relational, institutional, and praxis problems indicated a need for research to explore preservice or inservice mathematics teachers' beliefs about teaching GTs with GSP. These problems indicated that there was a gap in the literature of teacher beliefs about teaching GTs with GSP. This situation raised two questions in my mind – Why should this research focus on beliefs about teaching GTs? And, why is it important to study preservice mathematics teachers' beliefs about use of GSP for teaching GTs? I would like to discuss these questions under separate sub-sections.

Why Geometric Transformations?

Geometric transformations have been a key aspect of mathematics curricula in middle and high schools (NCTM, 2000). The Common Core State Standards for Mathematics (CCSSM) recommends teaching and learning transformation geometry at the high school level. It highlights "Experiment with Transformations in the Plane" with five specific standards.

> CCSS.MATH.CONTENT.HSG.CO.A.1
> Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
> CCSS.MATH.CONTENT.HSG.CO.A.2
> Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other

> points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
> CCSS.MATH.CONTENT.HSG.CO.A.3
> Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
> CCSS.MATH.CONTENT.HSG.CO.A.4
> Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
> CCSS.MATH.CONTENT.HSG.CO.A.5
> Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)

Likewise, the Principles and Standards for School Mathematics (NCTM, 2000) also emphasized teaching and learning of geometric transformations as a key aspect in geometry of middle and high schools. However, there is limited research (e.g., Harkness, 2005; Hoong & Khoh, 2003) in the area of teaching and learning geometric transformations in general and teachers' beliefs about teaching GTs in particular. That means there is a gap in research literature of preservice mathematics teachers' beliefs about teaching GTs. This study will add into the literature in this field.

Why Geometer's Sketchpad?

Use of technology for teaching and learning mathematics in general and GTs in particular has been emphasized in literature (e.g., Hollebrands, 2007; Hoong, 2003; Hoong & Khoh, 2003; Hooper & Rieber, 1995; Hoyles & Lagrange, 2010). There are many studies of using GSP or other technologies in teaching geometry (e.g., Hannafin, Burruss, & Little, 2001; Jackiw, 2001; Leatham, 2002; Nordin et al., 2010), but there are very few studies (e.g., Stols & Kriek, 2011) in the area of teacher beliefs about using GSP in teaching GTs. This shows that there is lack of ample literature on studies of preservice or inservice teacher beliefs about teaching GTs with

GSP. Hence, this study is expected to contribute in the literature of teacher beliefs about teaching GTs with GSP.

The synthesis of eight contextual issues from the literature, three lenses to view the different perspectives, and literature gap in the study of teacher beliefs about teaching GTs using GSP inspired me in identifying the key issues and formulating the purpose and research questions for the current study.

Purpose and Research Questions

The purpose of this study is to explore preservice secondary mathematics teachers' expressed beliefs about teaching geometric transformations (GTs) using Geometer's Sketchpad (GSP). Particularly, it aims to find out different categories and characters of teacher beliefs in the context of the problem solving situations and anticipations of teaching GTs with GSP. This study explored the following research questions:

(1) What beliefs do preservice secondary mathematics teachers hold about teaching geometric transformations with Geometer's Sketchpad?

(2) What beliefs do preservice secondary mathematics teachers hold about their future practices of teaching geometric transformations with Geometer's Sketchpad?

The preservice secondary mathematics teachers' beliefs were explored through discussions, interactions, and reflections on teaching and learning GTs with GSP during five task-based interviews. The first research question focused on the participant's categorical beliefs about teaching GTs with GSP and the second research question focused on their holistic beliefs about teaching GTs with GSP. This study explored preservice teachers' beliefs in relation to their thinking, reasoning, and problem solving within GTs and anticipation of teaching GTs with GSP.

I drew inferences of such beliefs from their constructions, descriptions, explanations, reasons, comments, and reflections on different GTs by using GSP.

More specifically, the research question (1) intended to explore the different dimensions of the participants' beliefs. This research question anticipated to address some aspects of relational problems stated in the statement of the problem. The nature of preservice secondary mathematics teacher's beliefs about teaching GTs with GSP had been nested to their anticipated practices in their future teaching. The research question (2) intended to characterize their holistic beliefs about teaching GTs with GSP in terms of belief objects either as external or internal to the believer. These research questions sought to understand their beliefs in relation to relational, institutional and praxis issues related to preservice secondary mathematics teachers' beliefs about teaching GTs with GSP. These issues served as the basis for exploring the participants' beliefs about teaching GTs with GSP with outcomes that might help mathematics teacher educators to consider forming and changing preservice teacher beliefs at the core of the mathematics teacher education programs. Therefore, the study had some potential significance in mathematics teacher education.

Significance of the Study

This study has potential of epistemological, methodological, and pedagogical significance. The study explored the beliefs of preservice secondary mathematics teachers in a variety of situations within teaching GTs with GSP. The participants acted upon the problems posed to them and expressed their beliefs during the five task-based interactive interviews. The descriptions, analyses, and interpretations of the interview data were helpful to understand the categorical and holistic beliefs of the preservice mathematics teachers about teaching GTs with

GSP. Five assumptions synthesized from the literature were used as guiding principles for the study. These assumptions are related to symbiotic relation, voice, cognitive and affective function of research, and praxis dimensions (these assumptions have been discussed in chapter 3 in detail). In this sense, the study has some epistemological significance because it used task-based problematic situations of GTs within the dynamic environment of GSP to explore the participants' beliefs in those contexts and how those beliefs can be interpreted in terms of their anticipated practice of teaching GTs with GSP. This study has methodological and pedagogical significance in understanding preservice mathematics teachers' beliefs about knowledge and anticipated practices in teaching GTs with GSP. The study intended to explore experiences of participants in a variety of mathematical situations related to teaching and learning of GTs with GSP. This study also focused on participants' experiences while participating in mathematical activities and interactions of GTs within the dynamic geometry environment of GSP in terms of layers of categorical and holistic beliefs.

This study is a qualitative interpretive research that focussed on making sense of the preservice secondary mathematics teachers' categorical and holistic beliefs about teaching GTs using GSP. While doing this study, there were certain constraints in terms of research input (e.g., data), process (analytical and interpretive approach), and product (the outcome). I have discussed these constraints in terms of limitations inherent in the process and delimitations I set for the operationalization of this research.

Limitations/Delimitations

This study was grounded in the data of preservice secondary mathematics teachers' expressed beliefs in the task-based interactive interview contexts of teaching GTs with GSP. The pre- and post- interview surveys supplemented the interviews. That means the sources of data were mainly limited to the interviews. Due to time and resources constraints, I delimited the sources of data to the five task-based interactive interviews with each participant.

Grounded theory study, in general, keeps the number of participants open. The interview process continues until the data gets saturated. The idea of theoretical sampling demands further interviews with additional samples until all the core categories or major themes get saturated. However, the number of participants in this study was limited to two for the purpose of in-depth interviews. I interviewed each participant five times for the purpose of data saturation. Therefore, the study on the two cases may have a limited scope of generalizability.

Some grounded theorists (e.g., Glaser, 1978) preferred that the researcher not do analysis and interpretation of the data with preconceived theoretical lens. For them, any preconceived theories might affect the categories and concepts that should come out of the data. However, this study used radical constructivist grounded theory (RCGT) for the data construction, analysis, and interpretation of the interviews (Belbase, 2013, 2015, Charmaz, 2006). As soon as I framed the study, research questions, interviews, and analysis of data, I began interpreting the contexts, problems, and a way of exploring preservice teachers' beliefs. I could not remain neutral to the study in terms of my personal perspectives, theories, and worldviews; and it was not practically viable to maintain neutrality in terms of objectivity. That means, this study had been shaped and affected my ontology (my view about the nature of knowledge), epistemology (my view about method of knowing), axiology (my personal values about research as a process), phenomenology (my being in the world as a student, teacher, and researcher), and methodology (the theory-method assemblage). Therefore, my bias and orientation to a particular theory, epistemology, and philosophy might have influenced the input (data is constructed, but not collected), process (data is analyzed from both reductionist and constructivist approach), and outcome (the report is my interpretive account).

Chapter Conclusion

In this chapter, I introduced the eight different contextual issues from the literature and three lenses to view these problems – relational, institutional, and praxis, which highlighted the need to study preservice mathematics teachers' beliefs about teaching GTs with GSP. The purpose of this study was to explore preservice secondary mathematics teachers' beliefs about teaching GTs with GSP in terms of categorical and holistic beliefs. The two research questions focused on the relational, institutional, and praxis problems of preservice secondary mathematics teachers' beliefs about teaching GTs with GSP. The study was significant in terms of potential epistemological, methodological, and pedagogical implications of teacher beliefs and nature of their beliefs in teaching and learning of geometry in general and GTs in particular. I outlined some limitations and delimitations of the study in terms of scope of input, process, and outcome of the study and influence of researcher's personal biases.

In the next chapter, I discussed meaning of belief with respect to cognitive, affective, and pedagogical dimensions followed by eight contextual issues. I further synthesized three lenses to view these issues. Finally, I synthesized teacher beliefs in terms of traditional, constructivist, and integral beliefs about teaching mathematics with technology in general and teaching GTs with GSP in particular followed by a brief introduction of reflective and reflexive beliefs.

CHAPTER 2: NAVIGATING THROUGH LITERATURE

After setting the basics for the study in terms of knowing the context and problems, conceptualizing research questions, anticipating the significance, and delimiting the boundary, I explored literature on teacher beliefs in this chapter. In this chapter, I presented meaning and dimensions of beliefs, contextual issues, three lenses to view these issues, teacher beliefs about mathematics, pedagogy of mathematics, and technology integration in mathematics education. I further reconceptualized teacher beliefs about mathematics, pedagogy, and technology in terms of traditional, constructivist, and integral beliefs and how they are interrelated to each other and how these views make sense in relation to teaching GTs with GSP. Finally, I discussed teacher beliefs in terms of reflective and reflexive beliefs.

Conceptualization of these beliefs was an ongoing process throughout the research, not just before data collection, analysis, and interpretation. Synthesis of teacher beliefs in terms of traditional, constructivist, and integral was pre-analytic (before carrying out analysis and interpretation of the data). Further synthesis of beliefs as reflective and reflexive beliefs was a post-analytic (after carrying out analysis and interpretation of data). Hence, pre-analytical review of literature contributed and informed the analytic and interpretive process to construct categories of beliefs through theoretical sensitivity. The re-interpretation of beliefs from a holistic constructivist approach informed the post-analytical review of literature for re-synthesis of beliefs in terms of reflective and reflexive beliefs.

Now I would like to introduce the meaning and dimensions of beliefs followed by different contextual issues, three lenses, and further synthesis of teacher beliefs.

Meaning of Belief

There is diversity in how psychologists, educationists, and philosophers conceptualize beliefs (Leder, Pehkonen, & Törner, 2002). There are also a variety of views about beliefs in the literature of mathematics education (Furinghetti & Pehkonen, 2002). Goldin (2002) defines beliefs as "multiply-encoded, internal cognitive/affective configurations, to which the holder attributes the truth value of some kind" (p. 59). For Schoenfeld (1985) "belief systems are one's mathematical world view" (p. 44). He further clarifies the notion of beliefs "as an individual's understanding and feelings that shape the ways that he or she conceptualizes and engages in mathematical behavior" (Schoenfeld, 1992, p. 358). According to Lester, Garofalo, and Kroll (1989), "beliefs constitute the individual's subjective knowledge about self, mathematics, problem solving, and the topics deal within problem statements" (p. 77). Hart (1989) states, "belief is a certain type of judgment about a set of objects" (p. 44). According to cognitive activation (COACTIV) theory, "beliefs are psychologically held understandings and assumptions about phenomena or objects of the world that are felt to be true, have both implicit and explicit aspects, and influence people's interactions with the world" (Voss, Kleickmann, Kunter, & Hachfeld, 2013, p. 249). These definitions of beliefs clearly outline two views - static and dynamic views of beliefs. The static views of beliefs are limited to certain objects or phenomena whereas the dynamic views are related to adaptive processes to look at the world in relation to one's identity and others.

Goldin, Rösken and Törner (2009) elaborate four aspects to define beliefs- ontological, enumerative, normative, and affective aspects. The ontological aspect is related to the existence of a belief object. According to Goldin et al. (2009), these objects can be "personal, social, or epistemological in nature" (p. 3). The enumerative aspect is associated with belief states. The normative aspect of beliefs seems to associate with one's belief contents in the form of a fuzzy set with differing weights. Affective aspect appears to associate with feelings, perceptions, values, and attitudes.

From the above discussion, we may ascertain that a belief is a certain mental state in which one makes a judgment of something and he or she expresses a degree of confidence toward it. This view means that belief is associated with the processing of functional interaction between mental state and brain state, (Belbase, 2013 b) of our thinking and reasoning about an object (both tangible and intangible) or phenomenon (of the world) or noumenon (things in mind). This processing may generate a certain mental state expressing one's degree of confidence toward the object, phenomenon, or noumenon considering it as true. The thinking and reasoning part seem to associate with cognition, the degree of confidence is related to affective bond, and the truth-value may relate to strength of belief. Belief may influence one's action and pedagogy in mathematics education. Hence, there are three dimensions of beliefs - affective dimension, cognitive dimension, and pedagogical dimension. I have discussed these dimensions in the following sub-sections.

Affective Dimension of Beliefs

Beliefs and affect are interrelated within a dynamic system (Pepin & Roesken-Winter, 2015). Affect is associated with emotions that may influence one's beliefs. The emotional factors of beliefs are perceptions, feelings, appreciations, motivations, values, and attitudes (Furinghetti & Pehkonen, 2002). At the basic level of affect, a teacher may have some degree of awareness about what to teach, how to teach, and what technology to use while teaching mathematics. He or she may demonstrate certain preferences over others, initiate certain actions over others, and may justify why he or she indicated a certain sequence of actions. He or she then seems to exhibit certain attitudes or behavior toward mathematical contents, pedagogy, and the use of technology in teaching and learning. His or her intentions toward actions through the selections and judgments may confer his or her highest endowment to his or her beliefs (McLeod, 1988). It seems that an affect (feeling and emotion) interact with cognition (mind and brain processes) to shape the beliefs of an individual or a group.

Cognitive Dimension of Beliefs

A teacher's beliefs about teaching and learning of mathematics may have intricate connections to cognition. The cognitive dimension of beliefs seems to associate with knowledge and conceptions. Some researchers (e.g., Thompson, 1992) consider beliefs as part of knowledge and conceptions. Therefore, one's beliefs are associated with what he or she can recall, describe, and identify as knowledge of mathematics, pedagogy of mathematics, and technology for mathematics. A teacher may believe that the mathematical rules and theories can be analyzed, synthesized, prioritized, and categorized by an individual or group, or one can assimilate and adapt them to solve problems. He or she may believe that the mathematical knowledge in the forms of content, pedagogy, and technology come through an authority, or such knowledge can be constructed or created by an individual or group.

Cognition and beliefs interact, influence and inform each other through affect (Eichler & Erens, 2015). The cognitive dimensions may play a significant role in shaping one's beliefs about mathematical content, teaching and learning processes, and the integration of technology in teaching and learning mathematics. Cognition provides a content that one believes about something (Spezio & Adolphs, 2010). Affective and cognitive states of a person may interact and influence with the third dimension – pedagogical dimension of his or her beliefs.

Pedagogical Dimension of Beliefs

The pedagogical dimension of beliefs is relational (Hult, 1979), private (Churchill, 2006), and praxis-oriented (Grundy, 1987). The relational aspect of the pedagogical dimension seems to focus on how a teacher relates himself or herself to the content or subject matter, the process of teaching and learning, students, and the environment. Another aspect of this dimension is a private theory (Garcia, 2009) of beliefs. A teacher has his or her individual preferences of what to do with the teaching and learning of mathematics with technology and how to do it. These private theories may come from his or her experiences, observations, learning from the literature, and general inferences (Churchill, 2006). The praxis aspect may be related to the fundamental human interests and ethics of education (Habermas, 1972). A teacher may believe and value technical (empirical analytic ways of knowing the world), practical (historical hermeneutics ways of knowing the world), or the emancipatory (critical methods of knowing the world) aspects of knowledge and knowing (Grundy, 1987; Streibel, 1991).

I think these three dimensions of teacher beliefs are associated with contextual issues of forming and changing beliefs, quality and ethics of beliefs, systemic relation of beliefs with emotion and cognition, measurement of beliefs, and school experience as source of beliefs. I discussed these issues in the next sub-section.

Contextual Issues for Studying Teacher Beliefs

I synthesized eight areas of contextual issues from the literature on teacher beliefs. These issues are related to change of teacher beliefs, mechanisms for change of teacher beliefs, quality of belief system, ethics of change of teacher beliefs, areas of concern for changing teachers' beliefs, interrelation of emotion and cognition in shaping beliefs, measurement of beliefs, and influence of school experience on teacher beliefs. These issues were introduced in the Chaper 1 with some questions. Table 2 summerizes the context and issues of studying teacher beliefs. I discussed each of these issues in detail under the separate sub-sections.

Table 2. Contextual Issues for Studying Teacher Beliefs

Context	Area of Issue	Literature
1	Change of teacher beliefs	Fenstermacher (1979), Green (1971), Peterson, Fennema, Carpenter, & Loaf (1989), Richardson (2003), Schoenfeld (2010), Stipek, Givvin, Salmon, & MacGyvers (2001), Thompson (1992)
2	Mechanism for change of beliefs	Alexander & Sinatra (2007), Bendixen, Schraw, & Dunkle (1998), Cano & Cardelle-Elawar (2005), Chandler et al. (1990), Kardash & Howell (2000), Lundeberg & Levin (2003), Part (2009), Qian & Alvermann (2000), Schommer et al. (1992), Sinatra (2005), Tatto & Coupland (2003)
3	Quality of belief system	Schommer-Aikins (2011), Stahl (2011)
4	Ethics of change of beliefs	Raths (2001)
5	Areas of concern for change of beliefs	Ashton & Gregoire (2003), Goldin, Rösken, & Törner (2009), Prawat (2003)
6	Interrelation of emotion and cognition	Mercer (2010)
7	Measurement of beliefs	Adamson (2004), Bellemare, Kröger, & van Soest (2005), Hyland, Shevlin, Adamson, & Boduszek (2013), Luft & Roehrig (2007), Stipek, Givvin, Salmon, & MacGyvers (2001)
8	Influence of school experience	Raths & McAninch (2003)

Change of Teacher Beliefs

One of the goals of the teacher education program in general and mathematics education in particular is associated with forming and changing of beliefs of preservice and inservice teachers. This goal is to form and change teacher beliefs through mathematical content and pedagogical knowledge including other social, affective and cognitive facets (Schoenfeld, 2010). Therefore, mathematics teacher education seems to aim to form and change their beliefs despite challenges in ongoing practices of teacher education (Richardson, 2003). Even in traditional teacher education programs the teacher educators seem to have a sense of necessity to change preservice teachers' beliefs. Mathematics education researchers (e.g., Peterson, Fennema, Carpenter, & Loaf, 1989; Stipek, Givvin, Salmon, & MacGyvers, 2001; Thompson, 1992) emphasized forming and changing teacher beliefs about mathematical content and mathematical pedagogical knowledge for a change in teaching and learning practices. In this context, the role of mathematics teacher education is and should be to form and influence preservice teachers' beliefs and practices in a positive way (Fenstermacher, 1979; Green, 1971). Hence, one of the goals of mathematics teacher education is and should be the transformation of beliefs about teaching and learning (Fenstermacher, 1979). Then, a question comes in my mind: How to change preservice and inservice teachers' beliefs? This question points to the methodological issue of how to form or change their beliefs and mechanism for change of beliefs. I discussed this issue in the next sub-section.

Mechanism for Change of Beliefs

The process of forming positive beliefs in preservice secondary mathematics teachers about teaching and learning of mathematics is related to mechanisms for forming and changing their beliefs. These mechanisms are related to epistemic, cognitive, and affective factors associated with their beliefs system. Literature shows that an academic culture plays a crucial part in the development of epistemic beliefs. There could be different models for the mechanism of forming and changing beliefs. A model of domain learning can be helpful in understanding the complexities of knowledge evolution and construction leading to epistemic change (Alexander & Sinatra, 2007; Sinatra, 2005). The process of forming or changing beliefs is related to personal epistemology with reasoning (Chandler et al., 1990), and making strategies (Schommer et al., 1992). This process may further relate to conceptual understanding and change (Qian & Alvermann, 2000), cognitive ability (Kardash & Howell, 2000), moral reasoning (Bendixen, Schraw, & Dunkle, 1998), and academic performance (Cano & Cardelle-Elawar, 2005) to provide us a mechanism for forming and changing beliefs.

Many teacher education programs in general and mathematics teacher education in particular focus on teacher beliefs as part of their interventions (Part, 2009). The results of The Program for International Student Assessment (PISA) and Trends in International Mathematics and Science Study (TIMSS) have some influence on a study of teacher beliefs and change of beliefs. These results have motivated some teacher educators to develop new programs for teachers that aimed to form or change beliefs (Part, 2009). These efforts indicate interest of mathematics teacher educators to include beliefs and change in beliefs as a part of the teacher education program. Then the question comes in my mind: What kind of beliefs or beliefs system are we going to promote through teacher education? This question points to an effort to develop such programs to form or change beliefs may lead to the next issue, and that is quality of beliefs intended and formed or changed.

Quality of Belief System

While beliefs can be a central part of teaching, learning, and research in teacher education programs, the next issue, to me, is related to the quality of beliefs formed or changed either as better or worse. In mathematics education, especially for mathematics teachers, a belief system that adapts to the cognitive flexibility with the "ability to make appropriate epistemological judgments" (Stahl, 2011, p. 38) can be considered as having better beliefs compared to "cognitive rigidity, cognitive indecisiveness, and unresponsiveness" (Schommer-Aikins, 2011, p.

74). That means forming or changing beliefs may result into a positive direction with epistemological pluralism, inclusiveness, and value-laden academic atmosphere. It may go in the reverse course too if the leadership in the teacher education program is cognitively rigid (does not allow flexibility), epistemically unresponsive (does not respond to the change of context), and systematically very inorganic (does not promote dynamism). Hence, the programmatic effort to form or change beliefs is influenced by ideological standpoints and decisions taken by the teacher educators as leaders. When teacher beliefs need to be changed in a positive form, it raises a question of ethics: What is ethics of change of other's beliefs through a program? Since the implicit ideology of a person may influence the formation and transformations of beliefs and quality of such beliefs, it raises the issue related to ethics associated with change of other person's beliefs.

Ethics of Change of Beliefs

Another issue relates to whether a change of preservice or inservice teachers' beliefs is within the ethics of mathematics teacher education or not. This issue again takes me back to Raths's (2001) question, "What are the ethics involved in making a concerted effort to change the beliefs of another person?" (n. p.). Raths (2001) characterizes such change of beliefs of other persons, especially when motivated by ideology or politics, like 'brainwashing'. The ethical question is related to which beliefs to form or change and to what effects.

Mathematics teacher education may focus on beliefs related to - nature of mathematics, ways of knowing mathematics, teaching and learning mathematics, technology integration in mathematics, and cognitive and affective aspects of mathematics education. It may further associate beliefs with treatment to children as learners, creating an environment to learn with equal opportunity, differentiation of instruction based on children's ability, and diversity of students in the classroom. From an educational point of view, change of preservice or inservice mathematics teachers' beliefs in relation to these areas of concern should not be a problem. Change of teacher beliefs for better education, environment, and opportunity should be within ethics of good education. The only problem is to make decisions focusing particular types of beliefs to form or change (Raths, 2001). These decisions may be associated with curriculum, teaching and learning, assessment, technology integration, and educational environment. In this context, change of teacher beliefs is not against general ethics of education, however it raises another question: What are the areas of concern for change of teacher beliefs? The change might have a long-term impact not only on the person holding these beliefs, but also the community and society where they live, work, and influence others (as teachers, parents, and citizens). Then there might be some focal areas of beliefs from psychological, philosophical, and even political viewpoints to center the formation or change of beliefs around. These viewpoints of teacher educators and other education leaders may constrain or widen the opportunity for teachers to transform their beliefs for reform oriented mathematics education.

Areas of Concern for Change of Beliefs

The other issues associated with teacher beliefs and changes in their beliefs are philosophical. It seems that the philosophical issues relate to the realist-nominalist debate in teacher education that influences preservice teachers' beliefs. Some teacher educators (e.g., Prawat, 2003) consider this as a false dichotomy that does not help improve one's practice of teaching. Prawat (2003) suggests 'realist constructivism' may overcome the problems from this dichotomy. Other teacher educators (e.g., Ashton & Gregoire, 2003) discussed emotion as an important part of forming or changing teacher beliefs. They indicated to the severe limitation of educational and psychological literatures on the issues of teacher thinking and practice. They emphasized the relationship of emotions to beliefs that have not been adequately paid attention to in teacher education. Ashton and Gregoire (2003) proposed a model for preservice teachers' beliefs formation and change of beliefs focusing on positive emotion and cognition.

Goldin, Rösken, and Törner (2009) emphasized the role of teacher beliefs in teaching and the subsequent influence on students' learning of mathematics in relation to their problem solving and sense making. Some affective structures associated with deeply rooted beliefs are- "don't disrespect me; check this out; stay out of trouble; it's not fair" (Goldin et al., 2009, p. 12). These affective structures related to one's identity, avoidance of problems, and sense of equity takes us to think about interrelation of emotion and cognition. Then the question is: How do emotion and cognition interact to form or modify one's beliefs? This question points to formation or change of beliefs is associated with one's emotions and cognition. Hence, interaction of emotion and cognition could be an issue in relation to the study of teacher beliefs.

Interrelation of Emotion and Cognition

Another issue is related to emotional and cognitive aspects of preservice or inservice mathematics teachers' beliefs. Literature indicates that beliefs and emotion are closely associated, but not the same.

"An emotional belief is one where an emotion constitutes and strengthens a belief and which makes possible a generalization about an actor that involves certainty beyond the evidence. The experience of emotion is not a mere product of cognition or a reaction to a belief. It is not an afterthought. Feelings influence what one wants, what one believes, and what one does." (Mercer, 2010, p. 2)

Mercer further clarifies that emotion and cognition meet at a point, which is a belief. That means emotion and cognition of a person are balanced by and acted upon by his or her beliefs. Both emotion and feeling are very delicate constructs in psychology that may undergo a change at any time. An emotion may influence one's experiences, and feeling may come within the experience as part of consciousness toward the experience. Then, the relation can be feelings as a momentary reaction to the experiences within an emotion. Both feeling and emotion act on beliefs in an uncertain way whereas they act on knowledge relatively in a more certain way (Mercer, 2010). However, beliefs may also act on emotion and feelings in a reverse order.

The way one reacts to certain situations may provide connections to what he or she believes. How to change beliefs through a change in one's emotion? Perhaps, a positive experience can produce positive emotions leading to forming and changing beliefs in a positive way. Emotions, feelings, and beliefs are qualitative mental and psychological states. Knowing the extent of these states might help in forming and changing beliefs. However, the issue is how to measure beliefs and extent of change of beliefs. Then a question related this issue is: What tools or techniques do afford the best approximation or description of one's belief state?

Measurement of Beliefs

The next issue is related to assessing beliefs. Some researchers devised tools to assess beliefs - attitudes and beliefs scale surveys (Hyland, Shevlin, Adamson, & Boduszek, 2013; Stipek, Givvin, Salmon, & MacGyvers, 2001) and semi-structured interviews (Adamson, 2004). Other researchers developed unstructured interviews (Luft & Roehrig, 2007) and probability scale of beliefs (Bellemare, Kröger, & van Soest, 2005), to name a few. The measurements with these tools are based on research by participant self-reports and the subjective quantification of their unique belief states. The issue is how sufficiently precise these tools can measure or unveil teacher beliefs. There are limitations of these self-reports. One has to depend on such reports exploring teacher beliefs. Even in clinical interviews, the respondents' reports are recorded as their belief statements or expressed belief states, but not true measures of beliefs. Task-based interviews may provide a context in which the respondents may express their beliefs in a context, to some extent.

A researcher can use the available tools to assess teacher beliefs and use the results to plan for teacher education, interventions in the professional development, and create an environment for positive beliefs. However, the concern may not be restricted to a programmatic choice of teacher education. It can go beyond the school of education. It maybe related to the schools where preservice teachers graduated from and where they spent most of their life as students. That means schools as a place to develop beliefs at an early stage are important not both at the time when the preservice teachers were students, but also when they go to teach as prospective teachers (during their practicum). This raises the next question in my mind: How do school experience influence teacher beliefs?

Influence of School Experience

The last issue in this discussion is related to the influence the schools have in changing or retaining preservice or inservice teacher beliefs (Raths & McAninch, 2003). The professional development and the teaching experiences go side-by-side. The preservice and inservice teachers can implement their ideas in relation to the content, pedagogy, and technology in their teaching. Their actions in classrooms may speak about their beliefs to some extent if they have an autonomous working environment (Raths & McAninch, 2003). Schools where preservice teachers go for their practicum seem to have a huge influence on retaining those beliefs they formed or changed during their study at colleges of education in general and mathematics education in particular (Lampert & Ball, 1998; Op’t Eynde, De Corte, & Verschaffel, 2002). Hence, many teacher educators and researchers seem to accept that classroom experience as a student and as a teacher has a great influence in shaping teacher beliefs.

These eight contextual issues discussed above seem to have interrelation to each other. Therefore, addressing one issue might influence other issues too. For addressing these issues, I tried to view them from three lenses in general – relational, institutional, and praxis lens to understand the underlying problems, challenges, and their roots.

Three Lenses to View the Issues

Relational Lens

The interrelation of preservice secondary mathematics teachers’ beliefs and their practices seems to be a subject of many studies in mathematics education research. Researchers (e.g., Walker, Brownlee, Exley, Woods, & Whiteford, 2011) claim that there is a strong relation between beliefs about knowing, knowledge of preservice mathematics teachers’ learning outcomes, and their teaching practices. It seems to me that a change in practice requires a change in their beliefs. Despite efforts to change preservice teachers’ beliefs and practices, it appears that there is not much impact of such efforts in teaching and learning of mathematics in the long run. This issue seems critical to mathematics teacher education programs in colleges of education. Most of the teachers teach mathematics the way they learned it in their high school and college classes (Lampert & Ball, 1998). Lampert and Ball mentioned that teacher educators are complaining about their preservice teachers not teaching the way they were trained to teach.

That means either they did not have a value of experience in the methods classes for their teaching or they could not use those methods and approaches they learned in the mathematics education program. This view may contradict with the idea of experiential learning theory that defines learning as “the process whereby knowledge is created through the transformation of experience” (Kolb, 1984, p. 41).

In a recent publication, Pepin and Roesken-Winter (2015) discuss teacher and student beliefs in relation to dynamic affect system and how beliefs and affect within this system interact each other and inform each other. This seminal work highlights the issues of student-teacher participation in affective system in forming and shaping beliefs, methodological issue in studying belief and affective system in mathematics education, epistemology and theory of beliefs, teachers’ efficacy beliefs in mathematics education, domain specific beliefs, influence of motivation on belief system, and complexity of student and teacher beliefs. Their work emphasizes that it is necessary to make sense of teacher and student beliefs in mathematics education from systems perspective.

It appears to me that there is relational problem between having beliefs and translating beliefs in the classrooms. This problem raises a question: What is institutional role of mathematics education program to impart positive beliefs in preservice teachers and translate their beliefs into actions? This problem relates to the institutional problem either in the teacher education or the school system. One way to address this issue is to change the institutional roles of teacher education programs. Hence, one needs to look at this issue from institutional lens.

Institutional Lens

There is an abundance of research and theories on teacher beliefs. These studies before 1990 (e.g., Grant, 1984; Nespor, 1987; Peterson, Fennema, Carpenter, & Loef, 1989; Rokeach, 1968; Stonewater & Oprea, 1988; Thompson, 1984) shed more light on the status of teacher beliefs than on the processes of forming or changing teacher beliefs. Various studies from 1990 to 2000 (e.g., Battista, 1994; Brosnan, Edward, and Erickson, 1996; Brown & Baird, 1993; Jones, 1991; Kagan, 1992; Perry, Howard, & Tracey, 1999; Quinn, 1998 a & b; Richardson, 1996; Schmidt & Kennedy, 1990; Tillema, 1995; Witherspoon & Shelton, 1991) focused on the nature and measurement of teacher beliefs. These studies could not explicate how teacher beliefs are formed or changed with experience and how these changes impacted teaching practices. The problem appears to be related to a lack of adequate studies on evolving or developing teacher beliefs through teacher education programs and courses (Skott, 2015), poor interdisciplinary coordination (mathematics and mathematics education), and inadequate university and school collaboration for developing preservice teachers.

Blömeke, Hsieh, Kaiser, and Schmidt (2014) reported in Teacher Education and Development Study – Learning to Teach Mathematics (TEDS-M) different issues related to teacher knowledge, beliefs, and teacher education in the different parts of the world. The authors in this seminal work highlighted methodological, cultural and historical, developmental, social, and economical challenges at different parts of the world. Various studies within TEDS-M program focused on the integration of teacher beliefs with mathematical content knowledge (MCK), mathematical pedagogical content knowledge (MPCK), pedagogical content knowledge

(PCK), general pedagogical knowledge (GPK), mathematical knowledge for teaching (MKT), continuous professional development (CPD), and opportunities to learn (OTL).

It seems that mathematics teacher education program has a bigger role in the process of changing and forming positive beliefs about teaching and learning mathematics. This role is materialized through teacher educators who might consider the teacher education as a process or product. Therefore, these problems can be viewed in relation to the praxis lens.

Praxis Lens

Mathematics teacher education can be viewed from two perspectives - either as a product or as a practice of process (Grundy, 1987). If it is a product, then it is an a priori construct with a program of study that lays out the direction even before the actual action. If it is a practice of process, then it is an unfolding process based on students' interest, teacher beliefs, and the general educational environment. The first view seems static, and the second view seems dynamic (Streibel, 1991). There is a greater chance of ignoring preservice teacher beliefs as part of teacher education program in the first view. A static view of the program as a product aligns with Habermas's technical interest that focuses on traditional instruction and empirical-analytic ways of knowing about the world. There is no room for personal theory and beliefs in relation to learning and teaching. The dynamic view may embrace practical and emancipatory aspects of education. The practical aspect of teacher education seems to embrace historical and hermeneutical perspectives of knowledge and knowing (Pepin & Roesken-Winter, 2015). How one makes sense of the world may depend on what he or she believes, conceptualizes, and values. The emancipatory aspect focuses on beliefs and awareness toward existing social, political, or cultural contexts. It embraces a positive change of beliefs and practices (Streibel, 1991).

The current practice of mathematics teacher education seems inclined toward the technical aspect more and the practical and emancipatory aspects less. The measurements of teacher beliefs in current research practices seem to be heavily driven by technical aspects. The measurements of beliefs and change of beliefs through belief-scales may not truly measure one's beliefs. There should be a depth of exploration of one's beliefs through his or her reflections, actions, and anticipations. The relational, institutional, and praxis problems indicate a need of such research that focuses on preservice or inservice teacher development through forming or changing their beliefs and anticipated practices.

I think the cognitive, affective, and pedagogical dimensions of teacher beliefs add to the contexts and problems for this study. These dimensions can be the basis for studying one's beliefs about teaching and learning GTs with GSP. To possess an open mind to the overall view of teacher beliefs, I conceptualized teacher beliefs within domains of mathematics, teaching mathematics, learning mathematics, and technology integration in mathematics education. There are various ways to characterize teacher beliefs. Different authors have adapted to different categories of beliefs. The most common approach of characterizing the teacher beliefs can be done with their axiology (what they value), ontology (what knowledge exists) epistemology (how they come to know), methodology (what approaches do they use), and pedagogy (how they play their role as a teacher). These philosophical paradigms indicate that teacher beliefs about subject matter, pedagogy, and technology highlight three belief profiles – traditional, constructivist, and integral beliefs.

Beliefs about Mathematics

Many researchers discussed preservice and inservice teachers' beliefs in general and their beliefs about mathematics as a discipline in particular. Some researchers (e.g., Aguirre, 2009) reported teachers' beliefs about content domains such as geometry, algebra, and statistics claiming that their beliefs may influence the relative degree of abstractness of the subject matter. The greater the abstractness of the domain the more negative their belief is. Many teachers seem to not see algebra as a valuable subject due to less applicability in daily life. Others (e.g., Dionne, 1984) claimed teachers' beliefs align with the traditional, the formalist, and the constructivist views.

The traditional view considers mathematics as an objective knowledge independent of human cognition. This view comes from the Platonism as a philosophy of mathematics. It seems that it is an absolutist and positivistic view. The formalist view considers mathematics as a rigorous and formal body of knowledge with axioms, proofs, and logical structures (Eichler & Erens, 2015; Ernest, 1991). Constructivists consider mathematics as a body of knowledge constructed by an individual and society (Ernest, 1991; Prawat, 1992). Mathematical objects are constructions of human endeavors. They seem to accept that mathematical knowledge does not exist out of human cognition and experience (Ernest, 1991; von Glasersfeld, 1989).

Therefore, by using these views, I reconceptualized the teacher beliefs about mathematics into three levels - the traditional (highly objective and structured), constructivist (relatively subjective and unstructured), and integral level (highly subjective and inclusive) (Table 3). The third level of conceptualization was more abstract than the first two from the available literature because of obscurity in beliefs. I discussed each of these levels under separate sub-sections.

Table 3. Beliefs about Mathematics

Beliefs about Mathematics	Characteristics	Literature
Traditional	Mathematics is objective, independent, and external to human cognition.	Dionne (1984), Furinghetti & Morselli, (2011), Handal (2003), Shahvarani & Savizi (2007), Oystein (2011), Törner (1998)
Constructivist	Mathematics is practical, man-made, subjective, contextual, and science of every person.	Ernest (1989 a), Shahvarani & Savizi (2007), Thompson (1992), Törner (1998), White-Fredette (2010), Zakaria & Musiran (2010)
Integral	Mathematics is a product of social, historical, and cultural practices.	Dede & Uysal (2012), Ernest & Möller (2012), Furinghetti & Morselli (2009), Leatham (2002), Nkhwalume (2013)

Traditional Beliefs about Mathematics

Many mathematics education researchers and scholars outlined mathematics teachers' traditional belief about mathematics (Dionne, 1984; Furinghetti & Morselli, 2009 & 2011; Handal, 2003; Törner, 1998). These beliefs consider mathematics as objective, independent, and external to human cognition (Ernest, 1991). These beliefs align with the Platonist view in the philosophy of mathematics (Oystein, 2011). The traditional believers consider that mathematics is abstract, and it is independent of the knower, like other objects in the nature. The traditional view appears to be a positivist and a realist stance about mathematics (Tracey, Perry & Howard, 1998). Within this belief system, mathematics is considered as exact science that is formulated by deduction (Felbrich, Kaiser, & Schmotz, 2014).

The traditional belief system has a negative effect on the innovative curriculum practice and research-based teaching and learning (Handal, 2003). The teachers with traditional beliefs about mathematics view it as universal, consisting of certain rules and facts, and as the science of elites (Shahvarani & Savizi, 2007). These beliefs are associated with negative emotional dispositions and general lack of confidence in mathematical reasoning. These beliefs associate with mathematical rules, formulas, and theories to be recalled; mathematics as a dry subject that does not have room for subjectivity; and that the formal mathematics in school does not make any sense in one's everyday life (Martino & Zan, 2011). These themes carry the negative emotional dispositions about mathematics and lack of connection in day-to-day life. The traditional beliefs about mathematics also align with instrumentalist views that consider mathematics as a collection of facts, rules, and skills (Eichler & Erens, 2015). Preservice or inservice teachers with such beliefs emphasize external justification of mathematical knowledge from the world out there (i.e., external authorities) (Ernest, 1991). They consider mathematics as part of empirical science that can be objectively verified, and there is no role for one's subjectivity (Ernest, 1991). Teachers with traditional beliefs about mathematics may consider mathematical proofs as a product (Furinghetti & Morselli, 2009), not a mental process. Preservice or inservice mathematics teachers may develop such a belief systems in schools and colleges when they were students (Skott, 2015).

Constructivist Beliefs about Mathematics

Constructivists believe that mathematics is practical, man-made, subjective, contextual, and a science of every person (Shahvarani & Savizi, 2007). Ernest (1989 a) named this kind of belief as a problem solving view that considers mathematics as "dynamic, continually expanding, a field of human creation, an invention, and a cultural product" (p. 249). Mathematics is a body of knowledge resulting from the individual and social construction of mathematical rules, facts, axioms, concepts, ideas, theories, and abstractions (Zakaria & Musiran, 2010). Within this belief system, mathematics is considered as "problem solving process, discovery of structure and regularities" (Felbrich, Kaiser, & Schmotz, 2014, p. 211).

Mathematical objects are abstractions of mathematical relations created by mathematicians (Dionne, 1984) and practitioners of mathematics, and they describe processes in relation to mathematical objects. This view seems close to the process view that considers mathematics as a process of solving problems, thinking, and reasoning (Törner, 1998). Mathematics is a mental activity with mathematical conjectures, structures of proofs, refutations, contradictions, and the mathematical arguments formed within a social and cultural background (Thompson, 1992). Hence, constructivists believe that knowledge of mathematics is "fallible, corrigible, tentative, and evolving" (Hersh, 1979, p. 43 as cited in Ernest, 1991, p. 19).

Integral Beliefs about Mathematics

Some beliefs about mathematics may range in between traditional and constructivist and sometimes beyond both. Integral believers seem to consider mathematics as a tool for reasoning and developing other sciences, it is an abstract language, and it is an ideal form of conceptualizing the world. There is no other mathematical truth except the one we (human beings) construct it as a part of culture from a humanist perspective (Ernest, 1991). These beliefs seem to be partly traditional and partly constructivist (Dede & Uysal, 2012). Others follow the system view of mathematics that is broader than the toolbox view. This view considers mathematics as the logical study of mathematical axioms, theories, and abstract relationships (Törner, 1998). Within this belief system, mathematics is considered as "a science which is relevant for society and life" (Felbrich, Kaiser, & Schmotz, 2014, p. 211).

Some teachers describe their beliefs about mathematics very negatively. For example, "one can learn mathematics only at school, mathematics is difficult, mathematics is abstract, and it has no connection with everyday life…" (Perkkilä, 2003, p. 2). Often these negative beliefs connote traditional beliefs, but not all the traditional beliefs about mathematics are negative. Hersh (1979) assumes that mathematics is product of social-cultural-historical creation of human actions and conscious efforts. That means mathematics is an interface between physical and mental actions and models.

Beliefs about Mathematics Teaching

Different researchers discussed teacher beliefs about mathematics teaching with contradictory views. Some researchers (e.g., Kuhs & Ball, 1986) classified teachers' beliefs about mathematics teaching in terms of learner-focused, content-focused with emphasis on conceptual understanding, content-focused emphasizing performance, and emphasis on the collective whole with classroom-focused. The learner-focused belief seems to emphasize teaching of mathematics by engaging children in the construction of their mathematical knowledge. This kind of belief aligns with constructivism. The emphasis on content and student performance emphasizes teaching of mathematics with a focus on mastery of rules, procedures, and skills. These teachers embrace traditional beliefs of teaching mathematics as a transmission of mathematical knowledge from teachers to the students. The second (content-focused with emphasis on conceptual understanding) and fourth (emphasis on the collective whole with classroom-focused) categories seem to be more integral perspective. These teachers teach mathematics for understanding with a focus on the whole classroom (Richard, 2008).

Van Zoest, Jones, and Thornton (1994) proposed a different framework to assess preservice mathematics teachers' beliefs about teaching. This framework emphasized three components for assessment of teaching beliefs: learner-focused interaction, conceptual understanding, and performance. This frame is similar to a frame suggested by Kuhs and Ball

(1986), but it lacks one component from this, classroom-focused teaching. It does not include other affective (emotional) components associated with teacher beliefs. The framework is a mechanistic one that only focuses contents, concepts, and interactions. It does not have the components of thinking, constructing, and transforming mathematics education through teaching. Teachers' beliefs about teaching mathematics can be viewed from their axiology, epistemology, methodology, and pedagogy. These criteria of teacher beliefs about teaching mathematics synthesized three major belief profiles from the literature and theories of teaching beliefs. Table 4 highlights these beliefs in terms of traditional, constructivist, and integral belief profiles.

Table 4. Beliefs about Mathematics Teaching

Beliefs about Mathematics Teaching	Characteristics of Beliefs	Literature
Traditional	The teacher assumes the authority; he or she dictates mathematical rules, formulas, and problems emphasizing speed, accuracy, and memorization.	Dede & Uysal (2012), Kuhs & Ball (1986), Perkkilä (2003), Van Zoest, Jones, & Thornton (1994), White-Fredette (2010)
Constructivist	Teaching mathematics is playing a multidimensional and contextual role as a facilitator. The teacher may encourage students to act as mathematicians to argue on the mathematical facts, rules, relations, and theories to construct mathematical reasoning and knowledge.	Anderson (1996), Beswick (2007 & 2012), Day (1996), Dede & Uysal (2012), Perkkilä (2003), Zakaria & Musiran (2010)
Integral	The teacher helps students to teach and learn themselves by searching new ways to calling on them, hearing the students' voices about the problems and their questions, leading the students towards construction of mathematics.	Giroux (1992), Leatham (2002), Perry, Howard, & Tracey (1999), Silver (2003)

Traditional Belief about Mathematics Teaching

The traditional belief about mathematics teaching assumes teaching as a transmission of mathematical knowledge from a teacher to students (Ernest, 1991). This perspective is an instrumentalist view that focuses on the teaching of facts, procedures, and skills (Dede & Uysal, 2012). Teaching mathematics in the traditional view refers to teacher-centered teaching with lectures, drills-and-practices, and demonstrations. The teacher assumes the authority as a mathematician, and he or she dictates mathematical rules, formulas, and problems emphasizing speed, accuracy, and memorization. He or she emphasizes mathematical contents focusing performance and accurate outcomes. He or she focuses classroom activities that are heavily content driven with an emphasis on accurate performance (Kuhs & Ball, 1986).

From a traditional viewpoint, a teacher's role is an instructor, and the students' role is instructee, and to be passive recipients of knowledge (Ernest, 1989 a). A teacher with this belief emphasizes correct performance by using appropriate skills in solving mathematics problems. He or she emphasizes procedures and rules rather than processes. This kind of belief system results in reluctance to reform-oriented curriculum and practice (Perkkilä, 2003). Such a belief system is associated with prior experience of mathematics that is poor, inflexible, and authoritative. These teachers focus on rote learning of rules and formulas with one correct solution to the problem. They also seem to believe that textbooks are the main resource for teaching mathematics (Perkkilä, 2003). For them, the children's ability to produce right answers is more important than their thinking and reasoning. Some preservice mathematics teachers appear to believe in telling students what to do or demonstrating to them how to do mathematics as the major part of teaching mathematics (Van Zoest, Jones, & Thornton, 1994).

Constructivist Beliefs about Mathematics Teaching

The constructivist teachers assume that the child is at the center of the teaching and learning process. This process associates with the child-centered teaching focusing on reasoning, creative thinking, and problem solving. According to these views, teaching mathematics involves students in "constructing their own meaning as they confront with learning experiences which build on and challenge existing knowledge" (Anderson, 1996, p. 31, as cited in Dede & Uysal, 2012). The teachers with such beliefs may apply either content-focused with an emphasis on understanding or learner-focused with an emphasis on construction of mathematical knowledge (Kuhs & Ball, 1986). The teacher's role is believed to be a facilitator (Ernest, 1989 a) and students' role is active co-constructors of mathematical knowledge (Zakaria & Musiran, 2010).

Some preservice teachers believe in cooperative and collaborative activities (Perkkilä, 2003). Teaching mathematics through cooperative activities can help students learn from each other (Ernest, 1991). These activities emphasize teaching for understanding, teaching of problem solving, using manipulatives for concept teaching, and helping children produce their solutions. For some teachers, mathematics teaching is a creative function that makes students do mathematics and construct their mathematical ideas. Some preservice mathematics teachers seem to be slow in adopting and implementing the constructivist view of teaching. They advocate problem solving phases for constructivist teaching: (1) stating the problem and clarifying the variables, (2) exploring the different possible solutions, (3) phase of relief from the dead end, and (4) presenting one's solutions and interpreting the solutions. These phases indicate teachers' constructivist beliefs in teaching mathematics with a clear statement of problems, alternate solutions, avoiding the dead end, and presenting the solutions (Van Zoest, Jones, & Thornton, 1994).

Integral Beliefs about Mathematics Teaching

In some cases, it may not be suitable or possible to identify whether certain beliefs about mathematics teaching are either traditional or constructivist. It is not necessary that one's beliefs about teaching mathematics should be within these paradigms. One's belief is a very complicated mental construct. In other cases, one´s own belief about teaching mathematics is difficult to characterize due to the complexity associated with it.

For example, some beliefs about teaching include elements related to how teachers plan and use instructional actions in a problematic circumstance for learners, where teachers should identify students' errors and misunderstandings in problem solving environments. Teachers should negotiate social norms and values that develop in supportive learning environments so that students can learn and construct their mathematical knowledge (Perry, Howard, & Tracey, 1999). These characterizations of teacher beliefs do not explicate whether these beliefs are strictly traditional or constructivist; rather they move toward both directions.

It seems that teaching mathematics can move toward an integral approach beyond the traditional and constructivist dichotomy as a border-crossing (Giroux, 1992; Silver, 2003) through decentralization of disciplinary knowledge and interdisciplinary movement of mathematics education research and classroom practices. The idea of border-crossing is a vision of an interdisciplinary and intradisciplinary integrated approach. The border-crossing goes beyond traditional and constructivist teaching of mathematics toward a critical and postmodern perspective that focuses on 'deconstruction of current mathematics teaching' (Nkhwalume, 2013). The deconstruction of the formal curriculum and formal models of teaching extends to mathematics teaching in the context of local and global mathematical processes.

The postmodern view of mathematics teaching can go even further with self-reflexivity (Cain, 2011) about the content and pedagogy through retrospection, prospection, and idiosyncratic construction of meanings (Belbase, 2013 a). This view appears to be more of a decentered form of mathematics teaching with scope of authority for knowledge construction placing in students and teachers together. The decentering process of teaching mathematics lends opportunities for the teachers to plan for a students' active engagement in the construction and reconstruction of mathematics (Goss, Powers, & Hauk, 2013). Then the act of teaching of mathematics may not have clear-cut steps and boundaries within which one can define or enact (Smitherman, 2006) and hence it is provisional, contextual, and subject to adjustment within the classroom environment.

Beliefs about Mathematics Learning

There are different views about mathematics learning. Some scholars (e.g., Fisher, 1992) view mathematics learning in terms of mathematics knowing. Mathematics learning as part of knowledge of mathematics relates to mathematical cognition. The process of knowing may focus on a variety of learning and degrees of learning through reception, assimilation, adoption, adaptation, construction, evaluation, and extension. Skemp (1971, 1978) proposed views about mathematics learning as relational and instrumental. The teachers who believe instrumental learning focus on traditional methods in the classroom with drill-and-practice, memorization, and repetition of problem solving (Ernest, 1991). The instrumental learning of mathematics emphasizes the use of symbols and formulas without acquiring a greater adaptation to the process (Idris, 2006).

The teachers who believe in relational learning focus on contextual learning of mathematics with self-discovery or guided discovery of mathematical concepts by the students (Kim & Albert, 2015). The learners of mathematics may construct a relational schema (web or pattern of thoughts), and they can use this schema to relate concepts from one area to another area and prior concepts to the current concept (Skott, 2015). The earlier schema can form new schemas from such adaptations and transformations (Idris, 2006).

Some preservice teachers may believe that mathematics learning can be an active process through involvement of students and others may consider that mathematics learning takes place through students following teacher direction and instruction (Wang and Hsieh (2014). The former view is known as active learning view in which students are expected to participate in the construction of their strategies to solve mathematical problems. The latter view is a passive learning which assumes that "students learn mathematics through following explanations, rules, and procedures transmitted by the teachers" (Wang and Hsieh, 2014, p. 265). Table 5 summarizes the beliefs about learning mathematics in terms of traditional, constructivist, and integral belief profiles.

Table 5. Beliefs about Learning Mathematics

Beliefs about Learning Mathematics	Characteristics of Beliefs	Literature
Traditional	Learning is memorizing mathematical rules, facts, and formulas. Mathematical rules, facts, and theories are transmitted from the outside world or authority (i.e., a teacher) into the empty minds of students.	Dengate & Lerman (1995), Ernest (1995), Perkkilä (2003), Dunn, M. L., 2002; Schwier & Misanchuk (1993), Zakaria & Musiran (2010)
Constructivist	Mathematics learning is an individual and social process of conceiving meaning of mathematical concepts, procedures, and theories.	Dengate & Lerman (1995), Ernest (1995), Furinghetti & Morselli (2009), Lo & Anderson (2010), Steffe & Kieren (1994)
Integral	Mathematics learning involves self-reflectivity and reflexivity in problem solving, integrates ideas from different disciplines and also different areas within the same discipline.	Leatham (2002), Nagata (2004), Steffe & Gale (1995)

Traditional Beliefs about Mathematics Learning

Traditional belief about mathematics learning seems to align with exogenic philosophy and theories of learning. This view assumes that the authority (e.g., teacher) may transmit mathematical rules, facts, and theories into the empty minds of students (Dengate & Lerman, 1995). Teachers with this belief consider learning as memorizing mathematical rules, facts, and formulas (Ernest, 1991). The metaphor of mind is a tabula rasa and the metaphor of the world as the outer absolute world (Ernest, 1995). It is not clear whether the outer world is the absolute Newtonian world (that focuses on determinism and materialism), or it is a socially constructed world (that values human agency and creativity). The teachers (both preservice and inservice) with such beliefs seem to consider learning mathematics as passively receiving mathematical knowledge from the external authorities, such as a teacher, without becoming skeptical of such knowledge processes. A large number of teachers, still today, seem to embrace this kind of belief. For example, Zakaria and Musiran (2010) claimed that the majority of preservice teachers in their study believed that the learning of mathematics involves the memorization of rules, formulas, and procedures, and their beliefs aligned with this exogenic view. Perkkilä (2003) reported beliefs of some of the preservice mathematics teachers toward learning mathematics as memorizing or rote learning to produce accurate answers. Ernest (1989 a) argued that the models of mastering skills and reception of knowledge guided the traditional beliefs about mathematics learning. These models appear to focus on passive reception, submissive learning, and compliant learning.

Constructivist Beliefs about Mathematics Learning

Constructivist beliefs about mathematics learning seem to align with the endogenic philosophy and theories of learning. That means it is more inclusive in the sense of adapting to the learning from different sources across the discipline, culture and day-to-day life (Dengate & Lerman, 1995; Steffe & Kieren, 1994). The metaphor of the mind is an active constructor of mathematical knowledge and the metaphor of the world as the inner cognitive and the experiential world (Ernest, 1995). Some studies revealed that many preservice and inservice mathematics teachers embrace this kind of belief about mathematics learning. For example, Lo and Anderson (2010) reported that the majority of the preservice teachers, in their study, believed that mathematics learning could be enhanced by creating a challenging but supportive environment that build upon students' prior experiences. For them, mathematics learning is an individual and social process of conceiving meaning of mathematical concepts, procedures, and theories. They considered that a student constructs meaning of mathematical concepts on his or her own through self-study, self-reflection, and creative thinking and reasoning. He or she may also work collaboratively with peers and learn from each other (Brodie, 2010). Learning mathematics is an inductive process in which students work with specific cases, examples, and problems and then shift their goal from specific to the general understanding. Ernest (1989 a) discussed models of constructivist mathematics learning as the active construction of mathematics by children, exploration of mathematical ideas by the children, and autonomy in learning mathematics by the children at their pace. Some authors claim that mathematics learning is a smooth process from simple to complex construction of ideas projecting from known to unknown with new meanings (Furinghetti & Morselli, 2009 & 2011).

Integral Beliefs about Mathematics Learning

Preservice and inservice mathematics teachers' beliefs about mathematics learning are not clearly explicated either as traditional or constructivist in some cases. Some of these beliefs are- everyone could learn mathematics, learning mathematics does not need to be a natural talent, and the teacher should create a learning environment (Leatham, 2002). It maybe, in some cases, difficult to recognize whether a belief about learning is exogenic or endogenic.

According to a postmodern view of learning, the learner actively engages in the reflexive process while solving mathematical problems. The reflexive process involves students' retrospective, prospective, and idiosyncratic reasoning and thinking while solving mathematical problems and learning from it (Belbase, 2013 a; Nagata, 2004). Self-reflexivity in problem solving integrates ideas from different disciplines and also different areas within the same discipline. It is an integral process to accommodate a wide variety of resources available to an individual in terms of experience, feeling, thinking, reasoning, and being mindful while dealing with the problem at hand. A mathematics teacher with such beliefs seems to encourage students to learn their limitations and increases their potential of being self-learners and problem solvers (Steffe & Gale, 1995).

Belief about Technology Integration

Technology integration in mathematics teaching and learning has been a critical issue in recent time (Adamides & Nicolaou, 2004). Many researchers and scholars of mathematics education (e.g., Ertmer, 2006; Lampert & Ball, 1998) agree that teaching and learning of mathematics have been affected by teachers' beliefs about content, pedagogy, and technology. The technological aspect may play a crucial role in making teaching and learning of mathematics interesting to the students with a deeper sense of mathematical problem solving and reasoning (Ertmer, 2006). The technological environment may support in the visualization of mathematical relations, operations, and phenomena otherwise not possible to see (Foley & Ojeda, 2007). It also helps students and teachers to collaborate in teaching and learning process using a variety of constructive tools. Hence, technology plays a significant role in modern classroom practices providing a creative, constructive, and flexible learning environment to the students (Garry, 1997).

Technology has added a new dimension to thinking, reasoning, and constructing mathematical rules, formulas, proofs, and theories. In this context, many researchers and scholars (e.g., Benninson & Goos, 2010; Chai, Wong, & Teo, 2011; Chrysosotomou & Mousoulides, 2009; Ertmer, 2006; Hoong, 2003; Leatham, 2002 & 2007; Lee & Hollebrands, 2008; Lin, 2008; Niess, 2005; Nordin et al., 2010; Quinn, 1998 b; Teo, Chai, Hung, & Lee, 2008; Wachira, Keengwe, & Onchwari, 2008) discussed issues and problems of technology integration in mathematics teaching and learning. The majority of these studies analyzed beliefs of preservice or inservice teachers toward technology integration in education in terms of either as positive or negative impressions. Other views about technology integration in mathematics education include – technology support algebraic thinking and reasoning, technology for interpretation of data, and technology for contextualization of mathematics (Polly, 2015). Table 6 summarizes the teacher beliefs about technology integration in mathematics education in terms of traditional, constructivist, and integral belief profiles.

Table 6. Beliefs about Technology Integration in Mathematics Education

Beliefs	Characteristics of Beliefs	Literature
Traditional	The use of technological tools in teaching and learning is very formal and rule bound with formalization of mathematical procedures to find correct answers, follow mechanical step-by-step directions. Technology is a tool to solve mathematical problems.	Baker, Gearhart, & Herman (1994), Hopper & Rieber (1995), Teo et al. (2008)
Constructivist	Teachers focus on technology integration in mathematics teaching and learning to help students construct mathematical ideas (e.g., conjectures, proofs, and explanations). Technology is a tool to generate new mathematics and interpretations.	Cox & Cox (2009), Hooper & Rieber (1995), Leatham (2002), Mistretta (2005), Olive et al. (2010), Teo et al. (2008)
Integral	Technology integration in mathematics education makes teaching and learning interactive, dynamic, explorative, and active. Technology is a part of new culture embedded in mathematics.	Karagiorgi (2005), Karen-Kolb & Fishman (2006), Newhouse (2001)

Traditional Beliefs about Technology Integration

A mathematics teacher who believes in teacher-led instruction with drill-and-practice is less likely to integrate technology in teaching and learning mathematics (Teo et al., 2008), especially for understanding. That means some teachers who prefer direct teaching with lectures and problem solving with repeated practice may value the use of technology less, except for use on some other than difficult computations. For them, technology integration may not be a good thing, especially for performance-based teaching and learning. These teachers may undermine the importance of technology in mathematics education.

Hopper and Rieber (1995) introduced a model of technology integration in education with five stages: formalization, utilization, integration, reorientation, and evolution. The formalization and utilization stage seems to be aligned with traditional beliefs. Teachers having traditional beliefs may introduce technology in teaching and learning mathematics with limited use in demonstrating mathematical ideas. The use of technological tools in teaching and learning is formal and rule-bound. From this viewpoint, technology can have uses in the formalization of mathematical procedures to find correct answers. The integration, reorientation, and evolution stages seem to be aligned with constructivism and integralism. Hopper and Rieber (1995) claimed that teachers who believe in the traditional curriculum, teaching, and learning do not often reach the level of evolution. Often they do not even reach the level above integration. They may use technology in teaching and learning mathematics for accuracy and speed.

Constructivist Beliefs about Technology Integration

Technology integration in mathematics teaching and learning may provide a new dimension for teachers and students to think, reason, solve problems, and extend their vision beyond the classroom contexts (Cennamo, Ross, & Ertmer, 2010). The creative use of technology can help students and teachers to project infinite possibilities toward problem solving and reasoning (Wilson & Cooney, 2002). Technology seems to transcend the mathematics beyond current thinking and reasoning towards new horizons. Technology appears to change mathematics, mathematics teaching, learning, and ways of thinking. Hence, some teachers consider the role of a teacher in technology integration in mathematics education as a change agent (Teo et al., 2008). Teachers who believe constructivist teaching and learning of mathematics may also hold constructivist beliefs about technology integration in mathematics education. They may use technological tools to extend the classroom context to the real world. Such teachers seem to focus on technology integration in mathematics teaching and learning to help students construct mathematical ideas (e.g., conjectures, proofs, and explanations). Constructivist teachers believe about technology integration in teaching mathematics allows teachers and students to exploit the technological environment for meaningful teaching and learning, empowering, and providing access of mathematics to all students.

Integral Beliefs about Technology Integration

Many researchers discussed teacher beliefs about technology integration in terms of positive or negative, rather than traditional or constructivist (Karen-Kolb & Fishman, 2006). Teachers' who have positive beliefs about technology try to integrate technological tools in their teaching. They seem to think that the technology can help them teach better compared to teaching without it. Nevertheless, there are teachers with negative beliefs about technology integration in mathematics education. They consider that technology does not help them in effective teaching. The use of technology in a classroom can be a disordered task with poor organization and lack of motivation (Karagiorgi, 2005; Newhouse, 2001).

One can synthesize the integral (holistic) beliefs about technology integration in mathematics education as a natural part of mathematics class, constant access to technology, frequent use of technology, and the need for knowledge of technology use. Some categorical beliefs are- technology enhances mathematical procedures and concepts, technological tools help students to project ideas accurately, and one should not let the tool do everything. Other categories are- it replaces tidy hand written works, the use of it makes math joyful, it can help in career preparation, and it can provide students opportunities to learn in different ways (Butler, Jackiw, & Laborde, 2010; Forgasz, Vale, & Ursini, 2010; Hoyles & Lagrange, 2010). These beliefs seem to neither strongly signify traditional nor constructivist, rather they appear to be more inclusive and hence integral. Integration of technology provides a better access of mathematics to the students with equity and inclusiveness (Cennamo, Ross, & Ertmer, 2010).

Hence, teacher beliefs about mathematics, teaching and learning mathematics, and technology integration in mathematics have been discussed from the viewpoint of three belief profiles – traditional (instrumental), constructivist (relational), and integral (volitional). There could be another way of analyzing and characterizing beliefs in terms of belief objects. One's beliefs could be characterized as reflective and reflexive beliefs in terms of beliefs associated with external objects and phenomena or one's internal ability, attitude, and action.

After doing analysis and interpretation of the interview data in a holistic way, I carried out the next round of review of literature but in a different way. This time, I wanted to know more about reflective and reflexive beliefs from the literature and theory of beliefs.

Reflective and Reflexive Beliefs

Mathematics teaching is both a reflective and reflexive process (Lowery, 2003; Smitherman, 2006). One's beliefs about actions could be termed as reflective beliefs. This belief is associated with how one looks back on his or her practices, and how he or she makes sense of them in terms of "learning from experiences" (Wilson, Shulman, & Richert, 1987 as cited in Lowery, 2003, p. 23). Teachers' lived experiences can help them in forming their personal theories about practice, and that may shape their beliefs about teaching mathematics (Shulman, 1987). Whereas, reflexive belief is associated with one's active and conscious roles and actions of teaching mathematics through recursive and thought generating dialogues (Smitherman, 2006). Reflexive belief is not only associated with thinking of oneself in relation to practice and action, but it is also about belief toward one's own ability and confidence gained through such dialogue and experiences. Hence, reflexive belief is relational thinking within the community of practice (Jawarski, 2006).

Teachers' reflective beliefs about teaching and learning mathematics may influence their observation of how they make sense of their teaching and how they can learn better from their teaching process (Leikin & Zazkis, 2010). Their reflexive beliefs may affect their ability and action to be a critical self-learner with self-awareness and consciousness to the subject matter, context, and relationship with students (Whittock, 1997). Reflective beliefs seem to be dependent on the external source such as a phenomenon, action, or object that has a reliable representation (Sperber, 1997). Whereas reflexive beliefs do not have such external source and hence they are non-representational. Table 7 summarizes the major characteristics of reflective and reflexive beliefs about teaching mathematics with technology. Each of them has been discussed under separate sub-sections.

Table 7: Reflective and Reflexive Beliefs

Belief Type	Belief Object	Characteristics	Literatures
Reflective Belief	Action, phenomenon	Reflective belief is related to an action or phenomenon. This kind of belief is related to the external world in which one gains experiences. It is explicit.	Adams, Turns, & Atman (2003); Adler (1993); Barrett & Lanman (2008); Bucciarelli (1984); Duffy (2009); Edwards (1999); Engel (2000); Loughran (2002); Lowery (2003); Schön (1983); Shulman (1987); Sperber (1997); Thompson & Thompson (2008); Wilson, Shulman, & Richert (1987).
Reflexive Belief	Self-other relation, own biography and identity	Reflexive belief is related to one's awareness and consciousness to the self-other relations. This belief is about the internal world. It is implicit and self-referential.	Barry, Britten, Barbar, Bradley, & Stevenson (1999); Braude (1995); Campbell (2010); Jaworski (2006); Koch & Harrington (1998); Österholm (2011); Smitherman (2006); Van der Hart, Nijenhuis, & Steele (2006).

Reflective Beliefs

It seems that reflective beliefs tend to focus on one's awareness toward actions, objects, and phenomena. "Reflective beliefs are those beliefs we consciously hold; truths we explicitly endorse" (Barrett & Lanman, 2008, p. 111). The reflective beliefs may be rooted in reflective thinking and reflective practice. Reflective practice in teaching and learning has been widely discussed in the research literature of various disciplines- for example, nursing (e.g., Duffy, 2009; Jarvis, 1992; Hargreaves, 2004), engineering (e.g., Adams, Turns, & Atman, 2003; Bucciarelli, 1984), and sports management (e.g., Edwards, 1999). Other fields that apply reflective beliefs are - medical education (e.g., Koepke, 2009) and teacher education (e.g., Adler, 1993; Fletcher, 1997; Harford & Mac Ruairc, 2008). Reflective beliefs can be associated with these reflective thinking and practices. Schön (1983) introduced reflection-in-practice, reflection-on-practice, or reflection-for-practice that may generate a state of mind with confidence toward an action or practice. One's beliefs that arise from such reflective actions are reflective beliefs.

Reflective beliefs have been historically discussed in the literature in various forms and mostly in terms of reflective practice. In modern education, John Dewey (1933) is credited with beginning reflective practice in education. He discussed human thought processes as reflective phenomena. He implied to “a state of perplexity, hesitation, doubt, and act of search or investigation directed toward bringing to light further facts associated with certain belief” (p. 9). It seems that Dewey’s version of reflective practice is helpful to develop some key aspects of reflective beliefs in terms of the course of action of the past, present, or the future. Then one may have pre-action beliefs, during action beliefs, and post-action beliefs.

Further, theory of reflective beliefs can be associated with Schön’s (1983) ‘the reflective practitioner: how professionals think in action’. Schön’s (1983) work outlined reflection-in-action and reflection-on-action as part of reflective beliefs people develop in terms of professions they are involved. In this sense, reflective beliefs could be anticipatory (future-oriented), retrospective (past-oriented), and contemporary (current) (Loughran, 2002). Thompson and Thompson (2008) extended Schön’s (1983) idea of reflection-on-action and reflection-in-action by adding one more domain as reflection-for-action. They discussed ways to promote critical reflective practice by focusing whether one is reflecting-on-action or reflecting-in-action or reflecting-for-action. One may acquire reflective beliefs in a “reflective mode” even through “partial understanding of their true conditions” (Engel, 2000, p. 24). Reflective beliefs are, therefore, associated truth-values ascribed to one’s actions or any external phenomena. The degree of the truth-values might relate to strength of one’s beliefs. These beliefs are not isolated from one’s self-awareness, consciousness, and deeply rooted values. The relationship of self-awareness, consciousness, and values to one's actions and thoughts generate the next level of beliefs known as reflexive beliefs.

Reflexive Beliefs

Van der Hart, Nijenhuis, and Steele (2006) discussed reflexive beliefs in terms of how traumatized persons believe about themselves in relation to others. These beliefs seem to associate with "feelings, prejudice, suggestion, and restricted views of ourselves and others" (Van der Hart et al., 2006, p. 181). Such beliefs even lead the believer to "fixed cognitive schemas, that is, maladaptive core beliefs about self, others, and the world" (p. 181). Reflexive beliefs are related to one's awareness of self and others, especially "beliefs about the referents of one's states" (Braude, 1995, p. 72). That means reflexive beliefs are self-referential to person's awareness, consciousness, and values. Campbell (2010) proposed the theory of consciousness that, "…an entity is conscious if and only if it has reflexive beliefs" (p. 9). Hence, reflexive beliefs are about self and other relation and core beliefs in terms of one's consciousness. This kind of belief even relates to one's being among others that reciprocates roles and responsibilities as a teacher and student (Smitherman, 2006).

Van der Hart, Nijenhuis, and Steele (2006) discussed different action tendencies as part of reflexive beliefs. The lower action tendencies serve short-term goals for living with basic reflexes, presymbolic and basic symbolic actions. The intermediate action tendencies include reflective and reflexive actions. The higher-level action tendencies include prolonged reflective, experimental, and progressive actions. These tendencies are associated with reflexive beliefs that may act at different levels of consciousness and awareness of the person. According to Jovchelovitch (1996), reflexive beliefs of teachers are associated with "who they are, how they understand themselves and others, where they locate themselves and others, and which are the cognitive and affective resources that are available to them in a given historical time" (p. 125). Braude (1995) relates reflexive beliefs to indexicality of one's own mental state. This indexicality refers to his or her epistemological state that has a relational property in terms of experience and cognitive state of the mind (Braude, 1995).

Jansen (2008) introduced the idea of reflexive beliefs in terms of discourse in mathematics classrooms. She interrelated students' psychological factors to their participation and opportunities to participate at individual level and classroom as a community. Their reflexive beliefs in terms of 'who they are' influenced their engagement in the classroom discourse. Davis and Harré (1999, p. 37) proposed the idea of "reflexive positioning" in which an individual "positions oneself" based on his or her personal self-referential beliefs. Despite these examples, there is still a lack of literature in mathematics education that explicitly focuses on reflexive beliefs of inservice or preservice mathematics teachers.

Chapter Conclusion

This chapter discussed the meaning of belief as one's internal cognitive and affective structures with some truth values, worldviews, subjective knowledge, judgment, psychological state of understandings and assumptions, and one's degree of confidence toward events and objects. The affective dimensions of beliefs are associated with perceptual emotional factors, the cognitive dimensions relate to one's knowledge and conceptions, and the pedagogical dimensions are relational, private, and praxis-oriented. Mathematics teachers' beliefs about mathematics, teaching mathematics, learning mathematics, and technology integration in mathematics education have been discussed in terms of three belief profiles - traditional, constructivist, and integral beliefs. The traditional beliefs are aligned with behaviorist and positivist epistemology, the constructivist beliefs are aligned with poststructuralist and relativist epistemology, and integral beliefs are aligned with critical, feminist, and postmodernist views. Finally, reflective and reflexive beliefs have been introduced in terms of either an action or phenomenon-oriented beliefs or beliefs associated with self-other relation and one's sub-conscious and conscious states of mind.

The next chapter deals with the method of inquiry that highlights grounded theory approach, five assumptions for the study, two iterative processes, and the final iteration as a the current research design.

CHAPTER 3: METHOD OF INQUIRY

In this chapter, I presented my approach to this study focusing on grounded theory in general and radical constructivist grounded theory in particular followed by five theoretical assumptions for the study. Then I discussed iterative approach of pilot study and analysis of self-reflexive interview. Finally, I explained the process of this study including data construction (collection), analysis, and interpretation followed by setting the quality criteria of the research.

Grounded Theory

Grounded theory in qualitative research methodology has a backdrop in Glaser and Strauss's (1967) publication of 'The Discovery of Grounded Theory: Strategies for Qualitative Research' that focused on the discovery of a theory grounded in data. Afterwards, many publications on grounded theory (e.g., Bryant & Charmaz, 2007; Charmaz, 2000, 2005, 2006; Clarke, 2005; Corbin & Strauss, 2008; Gibson, 2007; Glaser, 1978, 1992, 2005, 2007; Mruck & Mey, 2007; Strauss, 1987; Strauss & Corbin, 1990, 1998) show a diversity of interpretations of this approach in qualitative research in terms of researchers' ontology, epistemology, methodology, axiology, and methods. Some use it as more realist and positivist approach (e.g., Glaser & Strauss, 1967; Glaser, 1978, 1992, 2005, 2007; Urquhart, 2013) and others apply it in qualitative research as more interpretive (Corbin & Strauss, 2008; Strauss & Corbin, 1990, 1998), constructive (Bryant & Charmaz, 2007; Charmaz, 2006), and postmodern design (Clarke, 2005).

Grounded theory approach to qualitative research focuses on coding process and paradigms to generate meaningful categories from the data (Charmaz, 2006; Clarke, 2005;

Strauss & Corbin, 1990). In grounded theory methodology, coding is the process of giving name to a conceptual unit of qualitative data (Corbin & Strauss, 2008). According to Glaser and Strauss (1967), the categories are conceptual units of the grounded theory. The process of categorizing is organizing the most open and axial codes to construct major codes as categories or sub-categories. While doing the process of coding and categorizing, the researcher uses his creativity, knowledge of theory, and awareness to the phenomena to generate most appropriate meaning units as codes. The researcher's sensitivity to the phenomena is guided by his or her personal theories, knowledge, and epistemology. Hence, theoretical sensitivity is the awareness of the researcher toward the subtle meaning of concepts and categories derived from qualitative data (Strauss & Corbin, 1990).

During the process of analysis of data with codes and categories, the researcher reads the data transcripts time and again to get a better sense of it. He or she keeps notes of thoughts, hunches, and meanings of the data and the process. Hence, memoing (or memo writing) is anything that a researcher records in his or her personal reflection about the research process, especially codes, categories, and any other related things or ideas about the study. Theoretical memoing is a part of regular memoing but with greater focus on the theoretical codes and categories in terms of dimensions of the theory (Charmaz, 2006; Glaser, 1978).

The analysis of the data and the construction of new data may go side-by-side in a grounded theory methodology. That means the data collected at the beginning is analyzed and interpreted immediately. The initial codes and categories serve as foundations for further questions and it guides the researcher to collect more focused and relevant data. The concepts and codes guide the new data construction that needs further grounding. The new data source and process of construction of the new data is determined by the researcher based on the concepts or categories to be saturated with additional data. This process is called theoretical sampling. Hence, the researcher continues theoretical sampling and analyzing of new data as a parallel process.

The aim of parallel process of theoretical sampling and analyzing data is to orient the research process to key categories. While constructing categories (axial or selective), the researcher compares each code to other codes, categories, and contexts in the data. The analysis of data by comparing attributes, meanings, incidents, and contexts is the process of constant comparison (Glaser & Strauss, 1967; Strauss & Corbin, 1990, 1998). Construction of a grounded theory (either as substantive or formal) involves the organization of the core categories that form different dimensions of the phenomenon under study. This interrelation of the themes or core categories around a central phenomenon under the study forms either a substantive theory or formal theory depending on the level of abstraction and the ability to generalize the phenomenon (Birks & Mills, 2010).

Any study is influenced by my personal ontology and epistemology that guided the methodology and method. I assumed nominalist (non-realist) ontology for this study. For me, a person, in his or her mind, constructs the reality of the beliefs as mental states. There is no ontological reality of belief outside one's mental schema. My epistemology is guided by radical constructivism. According to this epistemology, "Knowledge is not passively received either through the senses or by way of communication, but it is actively built up by the cognizing subject; and the function of cognition is adaptive and serves the subject's organization of the experiential world, not the discovery of an ontological reality (von Glasersfeld, 1996, p. 2)". I assumed that construction of any knowledge is an individual process at first, and then it may go to a shared construction. The function of constructing knowledge is an active process that is advanced by the adaptation to new experiences. Hence, for me, the act of this research is a process of constructing knowledge of preservice secondary mathematics teachers' beliefs about teaching GTs with GSP, and it is governed by the principles of radical constructivism.

While formulating these assumptions, I realized that anything I construct as knowledge from this study is "partial, provisional, and perspectival" (Mauthner & Doucet, 2003, p. 416) and such knowledge is situated within the time of my interaction with the participants and the data and in the intellectual space within which the interaction, analysis, and interpretation took place.

I viewed the entire grounded theory process of coding, categorizing, memoing, theoretical sampling, and continuous analysis and interpretation of data with constant comparison method from the epistemological viewpoint of radical constructivism. For this, I conceptualized five assumptions for guiding the entire research process within radical constructivist grounded theory approach (RCGT). These assumptions integrated the principles of radical constructivism with grounded theory approach to guide the process of the study. I believe in knowing the whole from the part and knowing the part from the whole. These five assumptions together formed a theoretical frame for this study to know the beliefs of the participants in a deeper sense – both part to whole and vice-versa. These assumptions also outlined my role as a researcher and the participants' role in the research process. Hence, it is modification of grounded theory into my personalized approach. I discussed each assumption in the next sub-section with essential theories from the literature and how the assumption make sense in the current study.

Theoretical Assumptions for the Study

I conceptualized five theoretical assumptions of RCGT from the literature of radical constructivism (e.g., von Glasersfeld, 1978, 1984, 1996, 2000, 2002) and grounded theory (e.g., Charmaz, 2006; Clarke, 2005; Strauss & Corbin, 1990) to guide the methodology of the study including data construction, description, analysis, and interpretation. Table 8 summarizes the major characteristics of these assumptions with a list of related literature.

Table 8. Five Assumptions for Guiding the Study

Assumptions	Characteristics	Literature
1. Symbiotic Relation	The researcher and participants have a symbiotic relationship based on mutualism. Both the researcher and participants are benefited from the research in terms of cognitive or pedagogical gain from the research process.	Bryant & Charmaz (2007), Charmaz (2006), Clarke (2005), Corbin & Strauss (2008), Dickson-Swift, James, Kippen, & Liamputtong (2006), Goldin (2000), Maher (1998), Steffe & Thompson (2000), Von Glasersfeld (2000 & 1995)
2. Voice of Researcher and Participants	The research values researcher and participants' voice through their reflective and reflexive practices. The participants have their voice through their direct quotes (protocols) and researcher has his or her voice through interpretive accounts.	Bergkamp (2010), Charmaz (2006), Denzin & Lincoln (2000), Guba & Lincoln (1985), Hertz (1997), Mills, Bonner, & Francis (2006), Orb, Eisenhauer, & Wynaden (2000), Warfield (2013)
3. Research as Cognitive Function	The processes of data construction, coding, categorizing, theoretical sampling, theoretical memoing, theoretical sensitivity, and constant comparison are active cognitive processes. Both the participants and the researcher are active cognizing subjects.	Bailyn (1977), Charmaz (2006), Clarke (2005), De Gialdino (2009), Grolemund & Wickham (2012), Morse (1994), von Glasersfeld (1995)
4. Research as Adaptive Function	The construction or invention of a grounded theory in a dynamic environment is a self-adaptive process, and the grounded theory undergoes reorganization with changing data, context, and interpretation. The researcher and participants' views, conceptions, and experiences co-evolve and adapt to the new information, experiences, and contexts.	Charmaz (2006), Glaser & Strauss (1967), Layder (1998), Lichtenstein (2000), Rodwell (1998), Strauss & Corbin (1990 & 1998), Welsh (2009)
5. Fit and Viability (Praxis) of the theory	It is related to the question of 'To what extent the theory constructed or invented from the data resonates with the context?' And 'Can the theory constructed or invented from the data explain a similar phenomenon?'	Charmaz (2006), Confrey (2002), Elliott & Higgins (2012), Glaser & Strauss (1967), Guba & Lincoln (1989 & 2001), Rodwell (1998), Strauss & Corbin (1990 & 1998), Von Glasersfeld (2002)

These assumptions integrate essential features of grounded theory methodology in qualitative research within interpretive and radical constructivist epistemology. These assumptions focus on mutualism as a symbiosis between the researcher and participants, balancing voice of research participants and the researcher, notion of research as a cognitive and adaptive function, and praxis as criteria to examine theory constructed or hypothesized from the data. Hence, these assumptions synthesized from the literature of grounded theory and radical constructivism helped me to conceptualize the research input, process, and outcomes. Each assumption has been discussed under a separate sub-section.

Assumption 1: Symbiotic Relation

I assumed that the participants and researcher have a symbiotic relationship while co-constructing knowledge from the research. Both radical constructivists (e.g., Steffe & Thompson, 2000; von Glasersfeld, 1995, 1996 & 2000) and grounded theorists (e.g., Charmaz, 2006; Clarke, 2005; Corbin & Strauss, 2008) accept that the researcher and participants have a symbiotic relationship. The researcher and the participants share different epistemic roles during the research process. The project and research process guides the mutual relation between the researcher and participants. To me, the researcher is a seeker of data from the participants' actions, experiences, and reflections. The participants are collaborators and implicit benefactors of the research that has a transformative agenda. The researcher may construct data through teaching experiments (Steffe, 2002; Steffe & Thompson, 2000), clinical interviews, or task-based interviews (Goldin, 2000; Maher, 1998) in which the participants contribute to the study through their participation, construction of narratives, and reflections on their experiences. At the same time, they may learn something new from the research process. When the participants go through the task-based interactive interview sessions, they may learn new ideas of mathematical problem solving. The researcher and participants may co-create such a bond through which they negotiate the roles and responsibilities and aim to solve a research problem.

The symbiotic relationship between the participants and the researcher can take on different forms depending on the nature of the project. When a research process is an organic activity, there could be three kinds of relationship between the participants and the researcher - mutualism (benefitted both participants and researcher), commensalism (researcher may be benefitted without harming the participants), and parasitism (researcher may be benefitted at the cost of participants).

In this project, the researcher-participants relation was a mutualism. The participants went through a series of problem-based task situations. These situations provided them with new insights or experiences in terms of teaching GTs with GSP. At the same time, I had their immediate reflections, points of view, and artifacts as data for the study. In the ongoing study of the beliefs held by preservice secondary mathematics teachers about teaching GTs with GSP, the different task situations were used as contexts for interaction. Here, the participants and I discussed the problem of constructing images of a polygon under reflection, rotation, translation, and composite transformations using GSP, where these situations were linked to the exploration of beliefs and anticipated practices of the participants in relation to the subject matter presented and discussed. While doing this, I thought that the participants might become more aware of the teaching GTs, using GSP while teaching GTs, and other pedagogical issues related to teaching GTs with GSP. In this study, I not only asked the participants, but also reflected back to my beliefs while constructing codes and categories and making sense of them. Hence, all the participants, including I, had a cognitive gain, that was they learned from this study. Therefore, the research process functioned as a model of cognitive symbiosis of mutualism between the participants and I (the researcher). This mutualism helped me to resonate with the voice of the participants and my voice discussed in the next sub-section.

Assumption 2: Voice of the Researcher and the Participants

I assumed that a qualitative research might carry participants' voice in different forms (e.g., vignette, protocols, narratives, life stories, to name a few). These voices are direct expression of the participants in different genres (both verbal and non-verbal). At the same time the researcher expresses his or her voice through interpretation and re-interpretation of participants' voice. Hence, the research carries the voice of the participants as the first person perspective (direct voice) and the researcher as the second and the third person perspectives (indirect voice) in the form of reflexivity (Hertz, 1997). Here, reflexivity of the researcher may relate to his or her sense, awareness, and consciousness to the issues through deep abstraction of meanings.

There can be different approaches of representing researcher and participant voice in the research. According to Hertz (1997), there are three dimensions of voice- author, participants, and self in the research. It is a matter of concern that who's voice should we focus in the research because voice may influence "empirical problem, methodology, and theoretical tradition" (Hertz, 1997, p. xii). Many grounded theorists (e.g., Bergkamp, 2010; Mills, Bonner, & Francis, 2006; Warfield, 2013) used researcher and participants' voice as part of their study. Classical grounded theory states the researcher's voice should be neutral and objective, whereas the constructivist grounded theory assumes that the voice of the researcher is reflexive (Charmaz, 2006; Warfield, 2013) that reveals the nuances of both the participants and the researcher voice.

The research process for this study was different from the classical grounded theory in that it focused on participants and my voice through reflective and reflexive practices, whereas the latter only focuses on the conceptualization of the experiences of the participant(s). The layers of data analysis and interpretation included both the participants' and my voice together. These voices were reflected in the layered protocols and interpretive accounts. The research brought out the voice of the participants as the first person perspectives and my voice as both the second and the third person perspectives in the forms of reflective and reflexive interpretive accounts (Hertz, 1997). This process allowed me to decenter my voice keeping the participants' voice upfront (Pierre, 2009) in the forms of narrative protocols. The initial data were coded and categorized to find out major concepts about the beliefs of preservice mathematics teachers in relation to the teaching GTs with GSP. Their voice that reflected their beliefs were integrated together in the narrative protocols as their first person perspectives.

Presenting voice of the participants empowered them in the world of practice (Orb, Eisenhauer, & Wynaden, 2000). Here, I was aware of establishing the voice of participants in my writing by being true to their expressions through protocols without losing my own voice during interpretation and re-interpretation while maintaining the authenticity of their voice (Denzin & Lincoln, 2000; Guba & Lincoln, 1985). The authenticity of voice related to keeping participants' as well as researcher's voice as close to the actual narrative as possible in the writing of research report. I represented my voice in the research in relation to the participant's beliefs about teaching GTs with GSP through the second and third-order interpretations. While doing this, I had to reconstruct their voice through my interpretive accounts. I was aware of the fact that their voice might not "speak on their own" (Mauthner & Doucet, 2003, p. 418) rather I made deliberate choice in construction of protocols to re-present their voice. In the process of re-presenting their voice, I was presenting 'my voice of their voice'. This process of constructing voice through narrative protocols and interpretive accounts was not a linear one, but it went through back and forth analysis and interpretation that constituted this study as a cognitive function.

Assumption 3: Research as a Cognitive Function

Next, I assumed that the processes of data constructing, coding, categorizing, theoretical sampling, theoretical memoing with theoretical sensitivity and constant comparison are active cognitive processes. The meanings of the participants' beliefs through codes and categories or themes are possible through the cognitive structures in the research process. Both the participants and the researcher are active cognizing subjects. A researcher constructs data through interviews and observations. While doing this, he or she decides what to ask, observe, and record and perform in situ data construction (Friedhoff, Zu Verl, Pietsch, Meyer, Vomprass, & Liebig, 2013). The choice of questions in the interviews, the choice of time and space and participants, and ways to record data makes this process flexible, and goal oriented. The researcher does the coding and categorizing of the initial data. While doing this, he or she maintains a theoretical memo in order to keep track of the process, his or her thinking, and personal hunches (Charmaz, 2006). The construction of categories and codes is guided by the research context, purpose, and his or her theoretical sensitivity. He or she does theoretical sampling and constant comparison of concepts or categories with additional data. All of these functions are related to active cognitive processes (Bailyn, 1977).

In the study of preservice secondary mathematics teachers' beliefs about teaching GTs with GSP, both participants and I engaged in interactions during the task-based interviews. The interactive, reflective, and constructive moments in the interview sessions provided with them a pathway to look at their beliefs in conjunction with knowledge and comprehension of teaching GTs using GSP. Bloom's (1964) taxonomical hierarchy provided a pathway along which the adaptive function of analysis and interpretation took place in the study. The participants and I discussed ways to use GSP in teaching GTs. We analyzed the situations of using GSP. Then I carried out the further task of synthesis of codes and categories to generate concepts to formulate a theory of participant's beliefs about teaching GTs with GSP. While doing this, we went through a series of cognitive interactions. These cognitive interactions helped us in "organization of the experiential world" (von Glasersfeld, 1990, p. 19) in forming and shaping our knowledge and beliefs. Hence, the entire research process was a cognitive function.

The research process as cognitive function was a dynamic adaptation to the new situations, experiences, and reflections through construction of new codes, categories, and meanings. Hence, the cognitive process of the research was not static and linear, but a complex dynamic phenomenon that went through adaptation to new conditions, codes, concepts, and categories. That means the idea of research as a cognitive function led me to another assumption of the research as an adaptive function.

Assumption 4: Research as an Adaptive Function

Next, I assumed that any endeavor to construct a new knowledge is an adaptation to a new experience, context, and challenges. The construction of codes and categories in the grounded theory approach is not one time activity, but it goes through series of new codes, categories, and meanings. The researcher does not stop the analysis and interpretation as one slot function, but learns from the earlier analysis and interpretation and makes adjustments to new codes, categories, and meanings with additional data. Hence, the construction or invention of grounded theory is an adaptive process (Lichtenstein, 2000). That means the grounded theory undergoes reorganization with changing data, context, and interpretation.

The construction or invention of a substantive (or formal) theory is neither a linear nor a final one. The theory is constructed or invented from the data analysis and interpretation. While doing this, the theory goes through different stages of evolution, as an embryo developing within an ovary with time and new inputs in terms of nutrients. The nutrients for the theory are new concepts, categories, and dimensions that come out of the qualitative data. The premature theory with the beginning data or a small amount of data grows and develops to maturity with additional clarity in its dimensions.

In the study of preservice secondary mathematics teachers' beliefs about teaching GTs with GSP, I constructed data in three phases. In the first phase, I constructed initial data through task-based interviews with the two participants. The data were analyzed with open, axial, and selective coding. I constructed three major conceptual categories that included beliefs about ability, action, and attributes. Then in the second phase, these three concepts were taken as a basis for further interviews. Then two more interviews were conducted with two participants in relation to their beliefs about teaching GTs with GSP focusing on the key aspects of ability, action, and attributes. The analysis and interpretation of additional data produced six major categories of beliefs related to action, affect, attitude, cognition, environment, and object. Finally, the last interviews were conducted with a focus on the major categories of beliefs already built up. These series of interviews and analyses helped me in forming a structure of participants' beliefs in terms of action, affect, attitude, cognition, environment, and object with data saturation. In this sense, the study was associated with qualitative, constructive, and adaptive grounded theory (Layder, 1998; Welsh, 2009).

The construction of six major categories from the categorical analysis of data constituted the core belief structure of the participants. The belief structure with action, affect, attitude, cognition, environment, and object may have qualities to represent the participants' beliefs about teaching GTs with GSP. However, these qualities should be examined in relation to their fitness to the context and viability to represent beliefs of the preservice mathematics teachers in a similar context. That means the major categories can be examined with praxis criteria of fit and viability. This process can be linked with the Denzin and Lincoln's (2005) three crises in qualitative research – crisis of representation, legitimation, and praxis.

Assumption 5: Fit and Viability of Theory (Praxis)

Finally, I assumed that the analysis and interpretation of data provides major categories that can be examined with praxis criteria. The question remains whether the categorical findings from the data resonate with the context or not, and whether they provide a viable interpretation of the phenomenon or not. These issues can be viewed from the criteria of fit and viability. A researcher has to examine the theory constructed or invented from the categorical findings of the study (Charmaz, 2006; Glaser & Strauss, 1967; Strauss & Corbin, 1990 & 1998). One can observe the quality of the categorical findings or theory of the research from the praxis dimension. This dimension is made up of dimension of fit and dimension of viability.

Praxis dimension of fit relates to whether the theory constructed or invented from the data seems suitable in the context. It is related to the question of "To what extent the theory constructed or invented from the data resonates with the research context?" The praxis of viability relates to whether the theory grounded on data carries a possible explanation of the phenomenon under study, and whether it can be transferred to a similar but another phenomenon. That means it is related to the question of "Can the theory constructed or invented from the data explain a similar phenomenon?" Praxis dimensions focus on the "transformative possibilities of the research process and product…." (Rodwell, 1998, p. 79). This dimension may help us in judging the quality of study and the structure or theory out of it in terms of "give and take between the researcher and the researched, between the data and theory" (Rodwell, 1998, p. 79). Hence, it seems that the greater the qualitative fit and viability of the theory the more authentic is the research process and product.

In the study of the beliefs of preservice secondary mathematics teachers about teaching GTs with GSP, I constructed six belief categories and two constructs of their holistic beliefs. In this context, I attempted to see 'to what extent these constructs fit within their belief system and whether they provide a viable explanation of their beliefs'. For this reason, stepwise interviews and analyses were carried out by following the participants up in the subsequent interviews with new questions and sending to them part of transcripts and concepts generated from the analyses for member checking. I also sought help from a fellow graduate student to carry out peer debriefing, reviewing, and auditing the codes, categories, and concepts that came up from the data (Rodwell, 1998). The peer reviewer and I worked together to make sure that the inquiry process was going to accomplish the goal. The peer reviewer was a key "affective and intellectual dimensions of the inquiry…" (Rodwell, 1998, p. 194).

These five assumptions of RCGT guided the methodology in the study of Preservice Secondary Mathematics Teachers' Beliefs about Teaching Geometric Transformations Using Geometer's Sketchpad by clearly outlining the roles and responsibilities of the participants and I (this researcher) and identifying our position in the research process. These assumptions shaped the approach to this study through three different iterations. I would like to discuss them under separate sub-sections within approach to this study.

Iterative Approach to the Study

I conceptualized and modified this study at three different iterations until it reached the present state. The first iteration was a piloting study of three preservice secondary mathematics teachers' beliefs about teaching mathematics with technology, the second iteration was self-reflexive analysis of my beliefs about teaching GTs with GSP, and the last iteration was the current study of two preservice secondary mathematics teachers' beliefs about teaching GTs with GSP. The first and second iterations have been introduced briefly followed by the approach to the current study.

First Iteration: A Pilot Study

I conducted a pilot study of Preservice Mathematics Teacher's Beliefs about Teaching Mathematics with Technology[1] in the fall of 2011 and the spring of 2012. The main purpose of the pilot study was to explore the beliefs of preservice secondary mathematics teachers in

[1] Belbase, S. (2015). A preservice secondary mathematics teacher's beliefs about teaching mathematics with technology. *International Journal of Research in Education and Science, 1*(1), 31-44.

relation to teaching mathematics with technology and how these beliefs unfolded with experiences within task situations and teaching experiences. Another purpose of the pilot study was to set the stage for this dissertation research by narrowing down the scope of the study and increasing the depth.

During their methods of mathematics teaching and practice teaching in schools, five semi-structured task-based interviews were conducted with three participants. These interviews used problematic contexts of teaching multiplication of fractions, functions and limits, geometric transformations, and statistical data to construct qualitative data about their technological-pedagogical beliefs (Goldin, 2000). An additional unstructured interview was conducted with the same participants after their practice teaching. To inform the research methods, the interview data from one selected participant was analyzed and interpreted in two layers. The first-order analysis produced their narrative portrayals of beliefs about teaching mathematics with technology. The second-order analysis and interpretation constructed seven major themes of the participant beliefs - the subject's beliefs about teaching materials, teaching strategy, bridging activities, technological tools, mathematical concepts and meanings, activities and transformation, and issues and challenges.

This study created a foundation for the further study on beliefs about teaching GTs with GSP through self-interview in the second iteration.

Second Iteration: A Self-Interview Analysis

I conducted the second iteration of research with a method that included self-interview, analysis and interpretation of the interview, and the scope of the study on Preservice Secondary Mathematics Teachers' Beliefs about Teaching Geometric Transformations with Geometer's

Sketchpad[2]. This process was helpful in shifting the focus from a broader area of beliefs about mathematics teaching with technology to a more specific area, that was on specific content (i.e., geometric transformation) and specific technology (i.e., GSP). The narrowing down of the scope of the study helped me in focusing on specifics of content, process, and technology integration.

In this iterative process, "I (the researcher) changed my role from a researcher to a participant in an imagined interview. The interior other (David) interviewed me as a researcher. I conducted a single interview session lasting for about three hours" (Belbase, 2013 a, p. 15). I performed coding, recoding, and interpreting the data by using a grounded theory method (Charmaz, 2006). Here, some major categories from the data were derived as - "beliefs about the advancement of pedagogy, beliefs about pedagogical environment, beliefs about the role of student and teacher, beliefs about self as a future teacher, beliefs about teaching-learning activities, and beliefs about transitions in teaching learning" (Belbase, 2013 a, p. 15).

These two iterations discussed above served as integral parts of ongoing development of methodology for the study of Preservice Secondary Mathematics Teachers' Beliefs about Teaching GTs with GSP. These iterative processes helped me in conceptualizing the research problem, scope, and method from a very broad (multiple contents and technology) to a narrow (one content and technology) area of study. These iterative processes helped me in formulating the final research design as the third iteration. This iteration constitutes the process of this study starting from getting the approval from the Institutional Review Board (IRB) to complete and analysis and interpretation of the data.

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[2] Belbase, S. (2013). Beliefs about teaching geometric transformations with Geometer's Sketchpad: A reflexive abstraction. *Journal of Education and Research, 3*(2), 15-38. DOI: http://dx.doi.org/10.3126/jer.v3i2.8396

Process of the Current Study

The first and second iterations discussed earlier helped me in developing a clearer purpose and research questions for the study. These processes were helpful in reframing the research questions, interviews, and data analyses and interpretations. These processes guided the research approach to the final study of two preservice secondary mathematics teachers' beliefs about teaching GTs with GSP. The final iteration constituted getting approval from the Institutional Review Board (IRB), recruitment of participants, construction of interview guideline, administration of interviews, writing theoretical memos, analysis and interpretation, and setting up quality criteria for the study. I would like to discuss each of them under separate sub-sections.

Approval from the IRB

This study involved human subjects as the participants to construct data for analysis and interpretation. When there was involvement of human subjects in the study, the first step in materializing the research was to get approval from the Institutional Review Board (IRB) to get ethical clearance for the study. I updated my Collaborative Institutional Training Initiative (CITI) training for human subject through online refresher training. I submitted a proposal and interview protocol to the IRB for review, feedback, and approval. The proposal outlined the key aspects of this research focusing on how it was going to protect the research participants as human subjects in this study mentioning potential risks and benefits to the participants. This proposal informed the research process in brief together with rights and responsibilities of the researcher and the participants. With the expedited review process, the IRB approved my proposal on November 18, 2013 (Appendix – B) to begin the study.

Recruitment of the Participants

Initially, I planned for recruiting four research participants for this study. I contacted four potential research participants with the help of my major advisor and invited them for the interviews through email communication. Three of them responded positively and one did not respond to my request through email. After participating in one initial interview, another participant withdrew from the process. Then I recruited two preservice secondary mathematics teachers from a pool of students taking second methods of teaching secondary mathematics course in the fall of 2013 in a University in the Mid-West of the U.S. The selection of two participants depended on access, availability of their time for interviews and their interest to volunteer in the study. A task-based semi-structured interview protocol was developed and used with the two participants as a major data source for this study.

Construction of Interview Guideline

I constructed a semi-structured interview guideline for the interviews (Appendix-A). Different interview guidelines were used in the pilot study and self-interviews. However, they helped me in formulating the interview guideline for the current study. The interview guideline constituted of task situations of reflection, translation, rotation, and composite transformations followed by list of potential interview questions from each of these domains of GTs. The interview questions in the interview guideline presented in Appendix-A were modified from the earlier interviews by including additional elements for cognitive, affective, and pedagogical dimensions of beliefs.

After analyzing the first two interviews with each participant, I reframed the questions based on new codes and categories constructed from the data. I reformulated multiple questions and prompts for each major category and sub-category. After conducting two more interviews (interviews 3 and 4), I modified the interview guideline based on the new codes and categories from the data. The final interview focused more on reconfirming their beliefs they already expressed in the earlier interviews about the major categories of action, affect, attitude, cognition, environment, and object of teaching GTs with GSP.

Administration of Research Interviews

I administered a pre and post-interview surveys with the participants to get initial and final idea about theirs beliefs. The surveys consisted of five-point Likert-scale items from focal areas of beliefs about teaching GTs with GSP and were administered both using a pre and post-interview survey for the individual participants. These survey result was complementary to the interviews and hence I did not report it separately.

The first two interviews took place during the end of fall of 2013 term when the participants were taking the methods of teaching mathematics course and rest of the three interviews in the spring of 2014 term when they were doing their student teaching internship. The first interview episode was designed for teaching reflection with GSP. The second interview episode was designed for teaching translation with GSP. The third interview episode was designed for teaching rotation with GSP. The fourth interview episode was designed for teaching of the composite transformations with GSP. The last interview focused on confirming their beliefs expressed in earlier interviews. Each interview episode was designed with task situations, discussions, and reflections. Each interview episode took 37-86 minutes to complete. All the interview episodes were recorded using a digital video recorder. I conducted the first two interviews myself and recorded them for transcribing and analyzing. For the rest of the three interviews, I was assisted by a graduate student of mathematics education to conduct the interviews. The interview guidelines were used to conduct these interviews and allowed to change my role to that of researcher-observer in the rest of the three interview sessions. Table 9 provides information about the different interviews, focus of these interviews, and time of interviews and analysis.

Table 9. *Administration of Interviews*

Interview No.	Interview with Cathy		Interview with Jack		Focus of Interviews	Data Analysis
	Date of Interview	Time Duration	Date of Interview	Time Duration		
1	Dec. 5, 2013	00:43:25	Dec. 9, 2013	01:24:02	Teaching reflection with GSP	Dec. 5, 2013 – Feb. 28, 2014
2	Dec. 6, 2013	01:13:30	Dec.16, 2013	01:26:34	Teaching translation with GSP	
3	Mar. 3, 2014	00:37:47	Mar. 5, 2014	00:42:38	Teaching rotation with GSP	Mar. 3, 2014 – Mar. 30, 2014
4	Mar. 7, 2014	00:49:11	Mar. 6, 2014	00:46:03	Teaching composite transformation with GSP	Mar. 7, 2014 – Mar. 30, 2014
5	Apr. 2, 2014	00:59:43	Apr. 3, 2014	01:12:45	Teaching GTs with GSP (overall)	Apr. 2, 2014 – May 15, 2014

The video data were transferred to an external hard drive for storage by labeling them with the date and pseudonym of participants, and each copy was stored for each interview record in a laptop computer for transcribing. I transcribed the data gleaned from the interviews for further analysis and interpretation, where the transcripts were labeled with dates, pseudonyms of the participants, and areas of interviews.

Writing Theoretical Memos

I wrote a reflective memo after each interview to support the analysis and interpretation of the data. These memos included major points that were discussed during the interview sessions, the observations of the interview process, and non-verbal expressions of the participants. These memos helped me in constructing themes and categories during the coding process. It helped him in constant comparison of the codes, categories, and themes. This process helped me in keeping track of major theoretical constructs, ideas, concepts, and categories during data generation, analysis, and interpretation. This process also enabled me to reflect on the key ideas, events, and processes at each stage of data generation, analysis, and interpretation. The initial codes from the data were connected together to a broader concept or category leading to the construction of final categories of beliefs. The writing of memos also helped me in forming second and third-order interpretive accounts during the interpretation of the data.

Analysis and Interpretation

The analyses and interpretations of the interview data were carried out in two phases. In the first phase, a classificatory analysis and interpretation was done using the principles of grounded theory (Charmaz, 2005; Strauss & Corbin, 1990, 1998). This kind of analysis was based on grounded theory approach to find concepts and categories from the pieces of data. In the second phase, the data were analyzed and interpreted using a holistic approach (Hall, 2008). This analysis was based on constructivist approach to find meanings of data as a whole. The whole analysis and interpretive approach was guided by the five assumptions of RCGT.

Classificatory analysis and interpretation

I transcribed the interview data verbatim for each interview episode, and the transcribed texts were used for the analysis and interpretation. I then carried out analyses of data with open coding, axial coding, and selective coding to construct codes, categories, and themes (Corbin & Strauss, 2008; Strauss & Corbin, 1990, 1998). At first, I read the interview transcripts line-by-line and marked important concepts in the words, phrases, sentences or paragraphs with a code. Some of these codes came from the language of the interviewees and others were constructed based on general meaning of those units (words, phrases, sentences, or paragraphs). Then these open codes were organized into a matrix with three columns.

The first column was filled with these open codes that were identified and that could be grouped under a theme. They were then coded axially together with a common name in the second column. These common names served as a pivot (axis) around which the open codes were grouped together. Once these the pivotal or axial codes were identified they were then re-organized in the matrix, in the second column. I then considered the pivotal or axial codes as a basis to identify key categories or themes from the data, and compared each axial code with other axial codes to see the differences and similarities. I re-organized similar axial codes in a group to form a key or selective code as a category. The third layer of codes, that is selective codes were organized in the third column to serve as major categories of beliefs. These categories served as the basis to form a structure (or a categorical units) of beliefs of preservice secondary mathematics teachers about teaching GTs with GSP. For this process, the codes were used from the first and second iterations as a coding paradigm for initial analysis of the interview transcripts. This coding paradigm served as a guide to help me in identifying the key concepts, codes, and categories of preservice secondary mathematics teachers' beliefs about teaching GTs with GSP. However, the codes and categories were not limited to the coding paradigm.

The first two interviews were conducted and then the data were analyzed for initial codes and categories (Charmaz, 2006). The initial analyses identified preliminary categories by constant comparison of codes with codes, codes with categories, codes with events, categories with categories, categories with events, and events with events. The interview guideline was revised with new questions based on initial codes and categories. Then two more interviews were conducted with more focus on specific categories and sub-categories. The final interviews were conducted after modification of the earlier interview protocol. The final interview focused on the major categories and sub-categories derived earlier to have a better understanding of their beliefs associated with these categories. The theoretical sensitivity was maintained as I remained open and reflective to the elements of theoretical importance in the data (Charmaz, 2006; Corbin & Strauss, 2008). The codes and categories for the pilot study and self-interview data served as a basis to maintain this theoretical sensitivity of preservice secondary mathematics teachers' beliefs about teaching GTs with GSP. For this process, I constantly tried to be open-minded in keeping all kinds of possibilities to emerge from the data. Prior conceptions and beliefs helped me in identifying and constructing key concepts and categories of the data.

This study included five interviews with each participant; I expected the theoretical saturation from the moment the last interview was over. Theoretical saturation was a process of gaining good fit between major concepts or categories and data. At this stage, additional data generation did not produce any new significant properties of categories. Participants were sent emails when any concepts or categories needed further clarification. This way a theoretical saturation of the data was achieved. Finally, I integrated key categories organizing them into a theoretical construct of beliefs about teaching GTs with GSP.

The classificatory analysis of data produced six key categories and twenty-one sub-categories of preservice secondary mathematics teachers' beliefs about teaching GTs with GSP. However, these categorical beliefs from the reductive approach did not reflect the nature of beliefs in a holistic sense. The radical constructivist approach of qualitative study considers observing the data both piece-wise and as a whole. The next phase of analysis and interpretation of data was carried out in the form of holistic analysis and interpretation to gain a complete picture of their belief structures.

Holistic analysis and interpretation

I assumed the data from the interviews as a whole for further analysis and interpretation without breaking the data into segments of codes and categories (Halls, 2008; Lyons & Coyle, 2007). The whole data was reviewed in order to understand the nature of the participants' beliefs in terms of when they are formed and how they might influence each other. I read and re-read the interview transcripts and the narratives from the earlier analysis in order to understand the stories of the participants in relation to their beliefs and interrelation of those beliefs (Lyons & Coyle, 2007). I observed the whole data from the viewpoint of interrelation between the believers, belief objects, and nature of beliefs. The next phase of the study after analysis and interpretation of data was writing the research report.

Reporting the Analytical and Interpretive Findings

In constructing the next chapter, I produced and reported the findings in two stages. In the first stage, he outlined the primary findings in terms of major categories and related sub-categories. I discussed these categories in three layers in the forms of reflective abstractions of the data. The first layer included belief narratives of participants as the first-order description and analysis in the forms of protocols associated with each sub-category. The second layer included researcher's reconceptualization of their beliefs about teaching GTs with GSP as the second-order interpretation. The third layer included an overall synthesis of beliefs as the third-order interpretation of their beliefs in relation to literature and relevant theory of teacher beliefs.

In the second stage, I looked at the entire data from a holistic view rather than pieces of codes and categories (Lyons & Coyle, 2007). He observed the data in terms of belief and associated objects either as external or internal to the believer. In this context, I used the framework of temporal dimensions of beliefs to characterize the participants' beliefs about teaching GTs with GSP. He sampled belief statements within these domains and discussed them in relation to the literature and theory of teacher beliefs. The structure of participants' beliefs were analyzed and interpreted in terms of layered accounts of beliefs using dialogic and interpretive approach. There were three layers of analyses and interpretations of the interview data as discussed below.

First-order description and analysis

I constructed narrative protocols to present the participants' voice by using pieces of interview data. I then organized the belief narratives or protocol pieces in the order of categories. These narratives constituted a first-order description and analysis of the data. The belief narratives included the description of beliefs in relation to the sub-categories in first person perspectives of the research participants. I constructed the narrative protocols from the interview transcript in the form of question-answer to make the participant voice relevant to the questions and contexts. These narratives reflect, to some extent, our 'speech act' in the interaction within the task situations.

Second-order interpretation

I reconstructed my narrative account of their beliefs elaborating and interpreting the major belief concepts within the protocols of sub-categories. I used the major belief concepts to construct the interpretive accounts of the participants' beliefs about teaching GTs with GSP. The protocols served as the foundation for constructing second-order analysis and interpretation.

Third-order interpretation

I then reconstructed an overall synthesis of the participants' beliefs from different theoretical points of view from the literature and my interpretive accounts of their beliefs. Therefore, it formed an abstract interpretation of the participants' beliefs integrating ideas from the first and second-order interpretation and extant theories of beliefs from the literature.

I tried to maintain the quality of the research with trustworthy and verisimilitude of the data, analysis and interpretation, and in reporting this study. I adopted the guidelines of quality criteria of the study as discussed in the next sub-section.

Quality Criteria

The question of quality issues in qualitative research is critical. Bergman and Coxon (2006) rightly stated, "quality concerns play a central role throughout all steps of the research process in qualitative methods, from the inception of research questions and data collection to the analysis and interpretation of research findings" (p. 1). There might be different ways to address the quality of study- for example, trustworthiness. In this study, I dealt with the issues of the triple crises (Denzin & Lincoln, 2005). Every qualitative research seems to be threatened by triple crises: crises of representation, legitimation, and praxis (Denzin & Lincoln, 2005).

The crisis of representation relates to the level of attention on the research questions. One aspect of this research is to create pedagogical "thoughtfulness and wakefulness" (Van Manen, 1990) among the participants regarding the issues arising in their beliefs about teaching GTs with GSP. I then adapted to Denzin and Lincoln's (2005) notion of the democratic relationship between the participants and I to resolve this issue. The multiple interviews in relation to beliefs about teaching GTs with GSP addressed the issue of representation. These interviews in the task-based situations helped me in representing their expressed beliefs in a subtle and subjective way. I tried to stay focused on their beliefs through my reflective and reflexive abstractions of their beliefs. The layers of account of their beliefs in the forms of first-order description and analysis (protocols), second-order interpretation, and third-order interpretation helped me in addressing this issue. I asked one of my fellow graduate students to help him (as a non-native speaker of English) by facilitating in the interviews and auditing some transcripts and codes. With her confirmations and clarifications, I was able to represent accurately the wordings, expressions, and meanings of each participant's voice.

Both the crises of legitimation and the crisis of praxis were deeply connected with the crisis of representation in this study. According to Denzin (1997), a form of legitimation was political in nature due to its relationship to power and ideology. I maintained the legitimacy of the study through a rigorous interview practice with depth of focus on their beliefs in the task-based interview context and richness of the data about their beliefs. The richness of data was achieved through five interviews with each participant.

The notion of verisimilitude was another criterion for resolving the crisis of legitimacy. According to Denzin (1997), verisimilitude was to construct a vivid textual representation of the phenomenon as if it resonated close to the real. In this research, verisimilitude dealt with such relation between the vivid expressions of beliefs through the text and textualities of the phenomenon related to task situations and their reflections that rendered contextual meaning through protocols and layered interpretations. "Meanings are not permanently embedded by an author in the text at the moment of creation. They are woven from symbolic capacity of a piece of writing and the social context of its reception" (Van Maanen, 1988, p. 25). Therefore, this study embraced Van Manen's (1990) notion of praxis as "thoughtful action: action full of thought and thought full of action" (p. 128).

I addressed the crisis of praxis in the form of 'fit' and 'viability' of the belief structures constructed from the data that resonated within the context of teaching GTs with GSP. The final interview protocols were sent to the participants for their final review (member check). The participants confirmed that protocols truly represented their voice, views, and beliefs toward teaching GTs with GSP. This helped me in legitimizing the process and bringing their voice into the study as true as possible.

Other criteria- transferability, dependability, and confirmability were related to originality, usefulness, and resonance of data as suggested by Charmaz (2005). The originality of data was maintained by considering new insights, concepts, and theoretical significance, and challenge to the existing ideas and practices (Charmaz, 2005, 2006). I addressed the issue of usefulness by considering the applicability of concepts in mathematics education, generic processes, extension of ideas to different contexts, and contribution to wider society (Charmaz, 2005). Likewise, the data addressed the issue of resonance through the fullness of explanation of experience, revelation of biases, links between larger social and personal interests, and a deep sense of lives and worlds (Charmaz, 2005) in relation to participants' beliefs about teaching GTs with GSP. These quality criteria of the study were supplemented with some ethical issues and how I addressed them for the quality of the study to be ethically balanced. I discussed some principles of ethical consideration in the next sub-section.

Ethical Considerations

I considered the principle of non-maleficence, principle of informed consent, principle of participant voice, principle of free wish, principle of integrity, and principle of protection of participants' identity to maintain the ethical balance in the research process. I would like to introduce each of these principles in brief.

Principle of Non-Maleficence

This study observed the principle of non-maleficence and thus any harm to the participants was avoided (Cohen, Manion, & Morrison, 2000) by conducting the interviews in a safe environment.

Principle of Informed Consent

This study was abided by the principle of informed consent. For this, I asked the participants to read the informed consent form and then sign on it as a proof of consent to participate in the study.

Principle of Participant Voice

Voice of each participant was emphasized with full control of what they wanted to share about their beliefs toward teaching GTs with GSP. I was aware of my relation with the participants during the interviews, and helped them to express their beliefs with free-will having no prejudice or reservation.

Principle of Freedom

The participants had the freedom to withdraw their participation anytime from the study. The consent form clearly stated that their participation was voluntary and they could withdraw their participation any moment they wished and there was no consequence for doing that.

Principle of Integrity

The research process was carried out by maintaining integrity of data in accordance with the complexities of interrelation of phenomena and the persons involved (explicitly and implicitly) in the study (Glen, 2000).

Principle of Safety

The participants' identity was protected in the study by using their pseudonym in this report and I will continue using the same pseudonyms in the future publications or disseminations of this study.

Chapter Conclusion

This chapter discussed the theoretical assumptions of Radical Constructivist Grounded Theory (RCGT) as a guideline for this study. Five assumptions of RCGT were synthesized from the literature of radical constructivism and grounded theory to guide the entire research process. These assumptions were related to – symbiotic relation of researcher and participants, voice of researcher and participants, research as a cognitive function, research as an adaptive function, and praxis criteria of the beliefs constructs from the study. The study problems and questions were formulated with the three iterative processes. The first iteration of the pilot study provided a broad picture of the research on preservice secondary mathematics teachers' beliefs about teaching mathematics with technology. The second iteration in the self-reflexive interview analysis helped me in understanding my personal beliefs about teaching GTs with GSP. These two iterations helped me in framing the research questions, interviews, and the analysis and interpretive pathways. The third and final iteration encompassed this study in terms of task-based interviews, layered process of analyzing and interpreting the data, and writing this report. The quality criteria of the study were addressed by considering the triples threats and methods of resolving them. Finally, some ethical considerations were discussed in relation to how to maintain ethics of good research in relation to protecting the rights of the research participants and their identity throughout the research process and after the research.

The next chapter deals with the findings and discussions in two parts. The first part highlights the categorical findings and discussions with six major categories and twenty-one sub-categories of participants' beliefs about teaching GTs with GSP. The second part resynthesizes preservice secondary mathematics teachers' belief structures from a holistic approach in terms of pre-, in-, and post reflective and reflexive beliefs.

CHAPTER 4: RESULTS AND DISCUSSION

In this chapter, I presented the results of data analysis, interpretation, and subsequent results with discussion. At first, I introduced the research participants, and the major task situations presented and discussed on teaching geometric transformations (GTs) with Geometer's Sketchpad (GSP) during the five interview sessions with the two participants. Then I discussed the results of the study in terms of six categories of action, affect, attitude, cognition, environment, and object that emerged from the categorical analysis of the data. The holistic analysis of the data resulted into two major belief characteristics as reflective and reflexive beliefs. Each of these categories has sub-categories, which were discussed in detail in three analytical and interpretive layers. In the first layer, the protocols that related to the sub-categories were presented. This layer presented the participants' voice in their words as first-order analysis and interpretation of their beliefs. In the second layer, I summarized key aspects of their beliefs in terms of second-order analysis and interpretation. In the third layer, I discussed their beliefs associated with the sub-categories in relation to the literature and theories of teacher beliefs.

Participant Description

I interviewed two participants who were the undergraduate seniors taking mathematics methods class in the fall of 2013 and teaching in schools as part of their practicum in the spring of 2014. I would like to introduce them in brief.

Participant 1: Cathy

Cathy (name changed) was a senior undergraduate Secondary Mathematics Education major at a University in the Rocky Mountain region of the United States. She was from a city in the Northern Rocky Mountain region. She moved from a Mid-North state when she was in Kindergarten. She was an English Language Learner (ELL) in her early school days until the second grade. She loved mathematics since early childhood. She has a sister and two brothers, and she is the third child of her parents. She was an athletics major at the beginning of the college and later she shifted her major to mathematics education. She wanted to be a mathematics teacher after her graduation. She likes to teach at the upper secondary level (grades 10^{th} through 12^{th}). She likes the outdoor activities like - camping, hiking, skiing, kayaking, and fishing.

Participant 2: Jack

Jack (name changed) is an undergraduate senior with a major in Secondary Mathematics Education in a University in the Rocky Mountain Region. He grew up in a small town in the Northern Rocky Mountains and moved to the university town for college. He went to college for two years and majored in Social Studies for Secondary Education. He wanted to teach multiple subjects in the social sciences. Later, he took a full-time job and left college (after two years). The only mathematics class he took at his early college level was Mathematics Problem Solving for college students. This course was compulsory to get the two-year degree. After working full-time for ten years as a local cell-phone retailer, he returned to college as a non-traditional full-time student. He changed his major and became a Secondary Mathematics Education. He wants to be a high school mathematics teacher after graduation. He is very outgoing person. He likes many different kinds of sports including golf, water skiing, and baseball.

Now I would like to discuss the activities of GTs with GSP that I used during the task-based interactive interviews with these participants.

GT Activities with GSP

I used brief activities of constructions of GTs followed by discussions on the uses of GSP in teaching and learning of GTs at the beginning of each interview session. The major activities were – construction of the image of a polygon under reflection (interview 1), translation (interview 2), rotation (interview 3), composite transformations (interview 4), and backward thinking and problem solving (interview 5). Here, I would like to present their sample works and discuss them briefly under separate sub-headings. For the purposes of this research, GT refers to plane geometric transformation of reflection, rotation, translation, and composite of them.

Reflection

Both Cathy and Jack worked individually on constructions of a triangle in GSP and reflected on a line of reflection. Different properties of reflection transformation were discussed, conjectured, and proved. Also, the possible uses of them in the classroom in the teaching and learning of GTs were discussed. The following diagram (Fig. 1) is a sample of their works during the task-based interview episode 1.

Cathy was asked to construct a polygon and reflect it about a line segment. She constructed the triangle ABC at first (Fig. 1). Then she constructed the segment ED as part of the mirror line. She then selected the segment as a mirror in GSP. She selected the triangle ABC and reflected it through the segment ED. This reflection produced the image triangle A'B'C'. She then constructed a table of areas of the two triangles by dragging a vertex of triangle ABC to change the areas. She then plotted the two sets of areas to see the linear relationship. She also measured the corresponding sides of the triangles. The constrution work was followed by interactive interview session in relation to participants' beliefs about teaching reflection transformation with GSP.

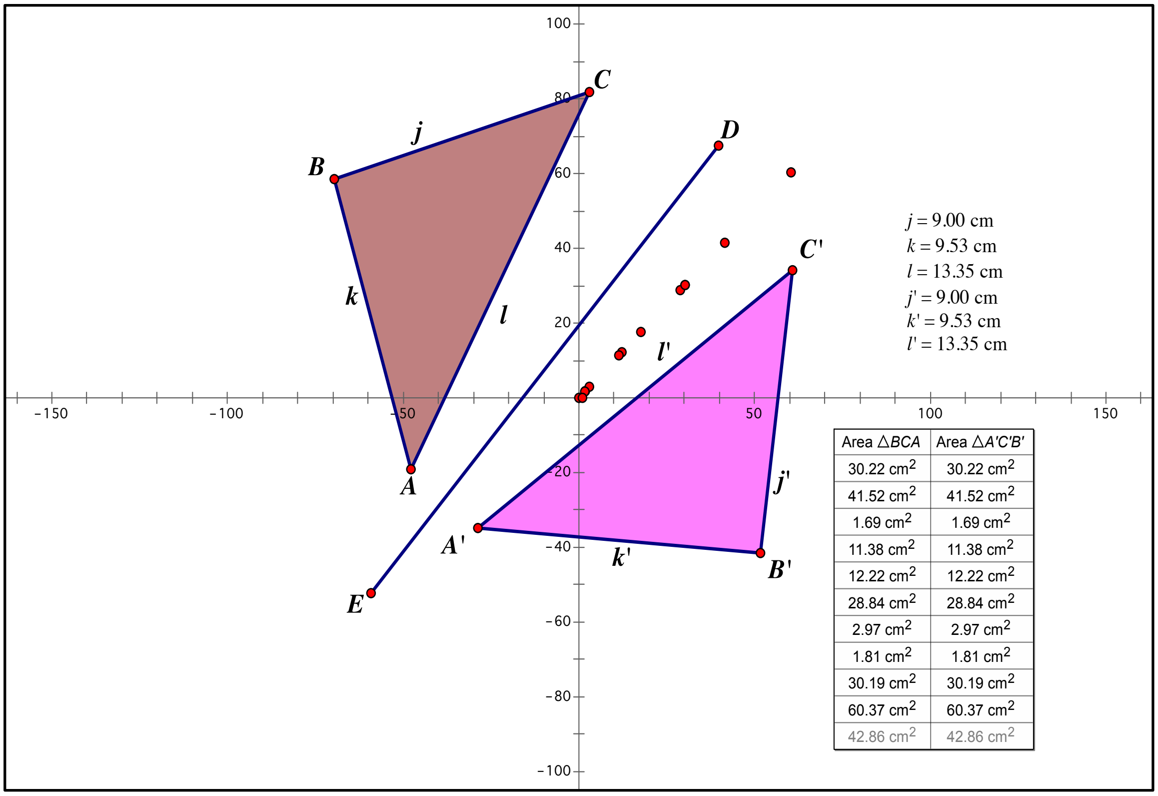
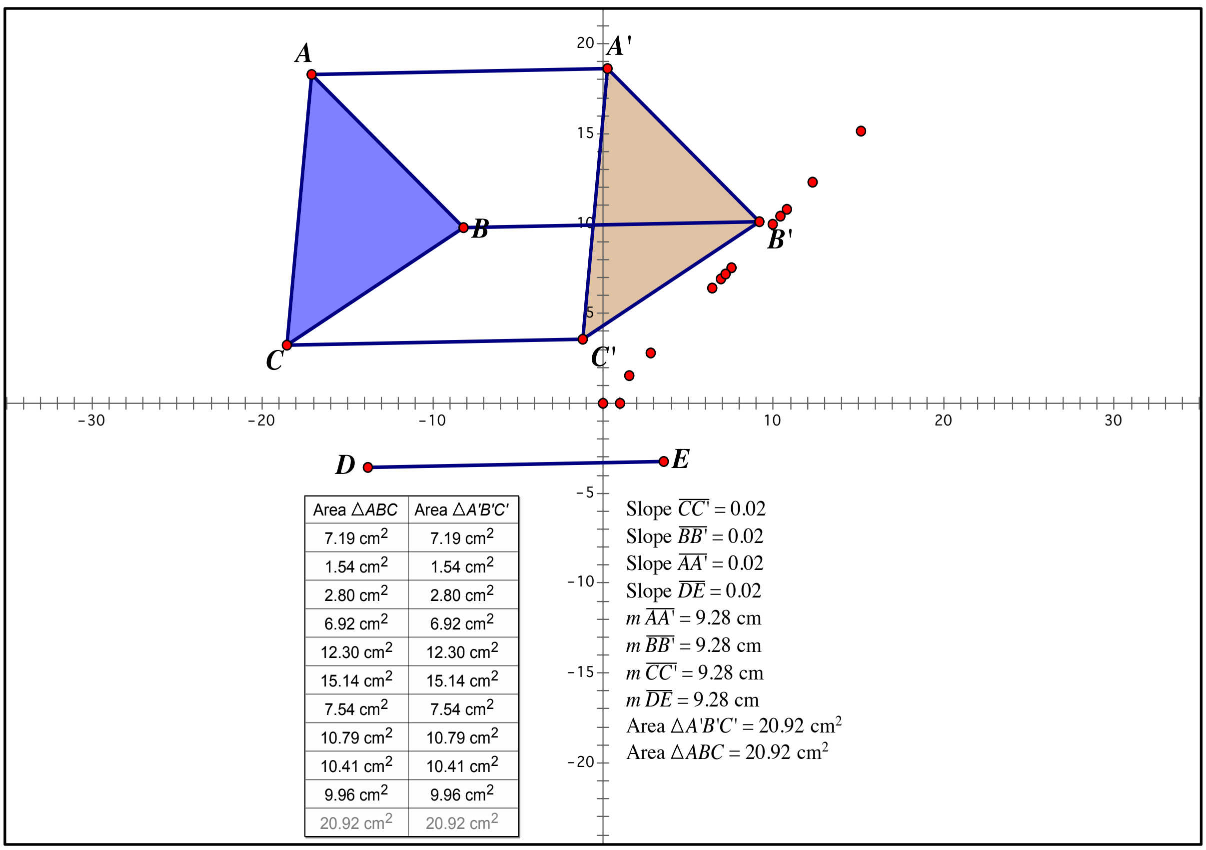


Figure 1. Cathy's construction of an image of a triangle under reflection about a line

Translation

Both Cathy and Jack each worked on construction of a triangle and translated it along a vector (directed line segment). Then we discussed different properties of translation transformation including conjecturing and proving them. We also discussed the possible uses of them in the classroom teaching and learning of GTs. The following diagram (Fig. 2) is one of the sample works they did during the task-based interview episode 2.

Jack was asked to construct a polygon (e.g., a triangle) and translate it along a line segment. He quickly constructed the triangle ABC at first (Fig. 2) and then he constructed the segment DE as a vector for translation (both in distance and direction). He selected the segment as a vector in GSP. He selected the triangle ABC and translated it along the segment DE. This translation produced the image triangle A'B'C'. He constructed a table of areas of the two triangles by dragging a vertex to change the areas. He fixed the number of (x, y) coordinates to ten in the table. He plotted a line-graph of the areas to see the linear relationship. He also measured the lengths and slopes of the segments joining the object and the image vertices that were equal to the length and slope of the segment DE (vector for the translation), respectively. The constrution work was followed by interactive interview session in relation to participants' beliefs about teaching translation transformation with GSP.



Area $\triangle ABC$	Area $\triangle A'B'C'$
7.19 cm^2	7.19 cm^2
1.54 cm^2	1.54 cm^2
2.80 cm^2	2.80 cm^2
6.92 cm^2	6.92 cm^2
12.30 cm^2	12.30 cm^2
15.14 cm^2	15.14 cm^2
7.54 cm^2	7.54 cm^2
10.79 cm^2	10.79 cm^2
10.41 cm^2	10.41 cm^2
9.96 cm^2	9.96 cm^2
20.92 cm^2	20.92 cm^2

Figure 2. Jack's construction of an image of a triangle under translation along a vector

Rotation

In their individual interviews, Cathy and Jack individually worked on construction of a polygon in GSP and rotated it around a point in an angle. Then we discussed different properties of rotation transformation including conjecturing and proving them. We extended our discussion to possible uses of them in the classroom teaching and learning of GTs. The following diagram (Fig. 3) is one of the sample works they did during the task-based interview episode 3.

I

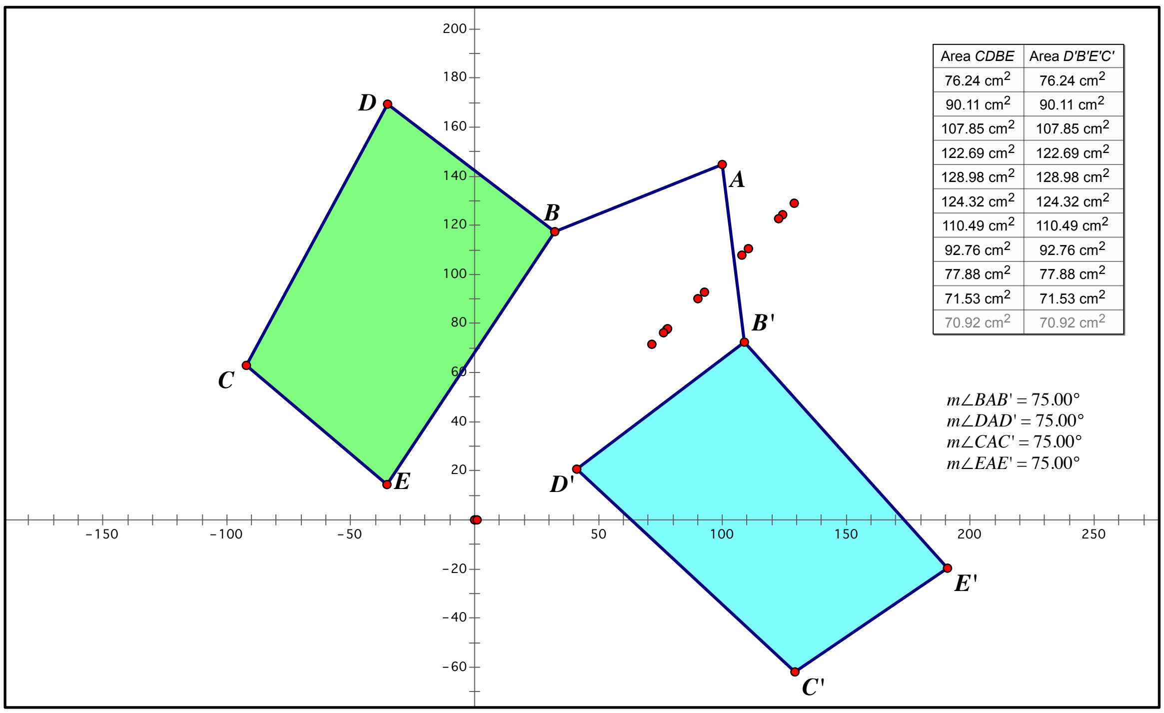


Figure 3. Cathy's construction of an image of a quadrilateral under rotation about a point through an angle

In this interview episode, the interviewer asked Cathy to construct a polygon (e.g., a quadrilateral) and rotate it about a point. She constructed the quadrilateral BDCE at first (Fig. 3). She then constructed a point A as the center of rotation. She then selected the point A as the center for rotation in GSP. She selected the quadrilateral BDCE and rotated it about the point A through an angle of 75 degrees. This rotation produced the image quadrilateral B'D'C'E'. She constructed a table of areas of the two quadrilaterals by dragging a vertex to change the areas. She plotted a line-graph of two sets of areas to see their linear relationship. She also measured the angle of rotation for each vertex, and that was seventy-five degrees. The constrution work was followed by interactive interview session in relation to participants' beliefs about teaching rotation transformation with GSP.

Composite Transformations

Cathy and Jack each worked on construction of a triangle in GSP and reflected, rotated, and translated it. Then we discussed different properties of composite transformation that included conjecturing and proving them. We further discussed the possible uses of them in the classroom teaching and learning of GTs. The following diagram (Fig. 4) is one of the examples they did during the task-based interview episode 4.

Jack was asked to construct a polygon (e.g., a triangle CDE) and reflect, translate, and rotate it. He constructed the triangle CDE at first. He then constructed the segment HI as a mirror for reflection. He selected the segment HI as a mirror in GSP. He selected the triangle CDE and reflected it through the segment HI. This reflection produced the image triangle C'D'E'. He then constructed a segment AB. He selected the segment AB as a vector of translation. He selected the image triangle C'D'E' and translated it along the vector AB and produced another image C''D''E''. Instead of constructing a separate point, he selected the vertex C'' as the center of rotation. He fixed the angle of rotation at 75 degrees. He rotated the triangle C''D''E'' about the point C'' through an angle of 75 degrees, and he got a new triangle C'''D'''E''' as an image under the rotation. He constructed a table of areas of the four triangles by dragging a vertex to change the areas. He fixed the number of (x, y) coordinates ten in the table. He plotted the two sets of areas to see the linear relationship. He also measured slope of the line plot of areas and he found it to be one and the equation as y = x. The constrution work was followed by interactive interview session in relation to participants' beliefs about teaching composite transformations with GSP.

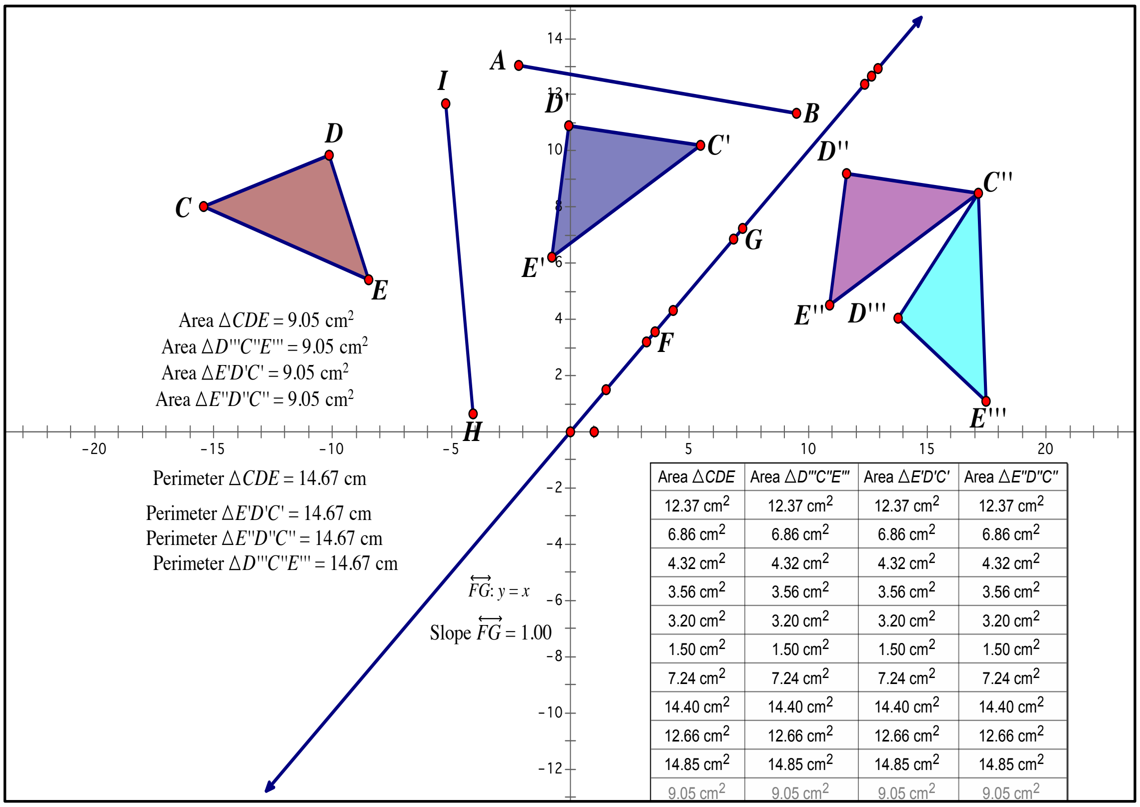


Figure 4. Jack's constructions of images of a triangle under composite transformations

Reverse Thinking

In the final stage of data collection, Cathy and Jack each worked on a ready-made construction of a triangle and its image under a rotation in GSP where the center of rotation was hidden. They worked through it to find the center of rotation. Then we discussed different properties of the center of rotation. The following diagram (Fig. 5) is a sample of their works during the task-based interview episode 5. The discussion was not limited to this construction. The discussion further focused on the overall processes of GTs using GSP.

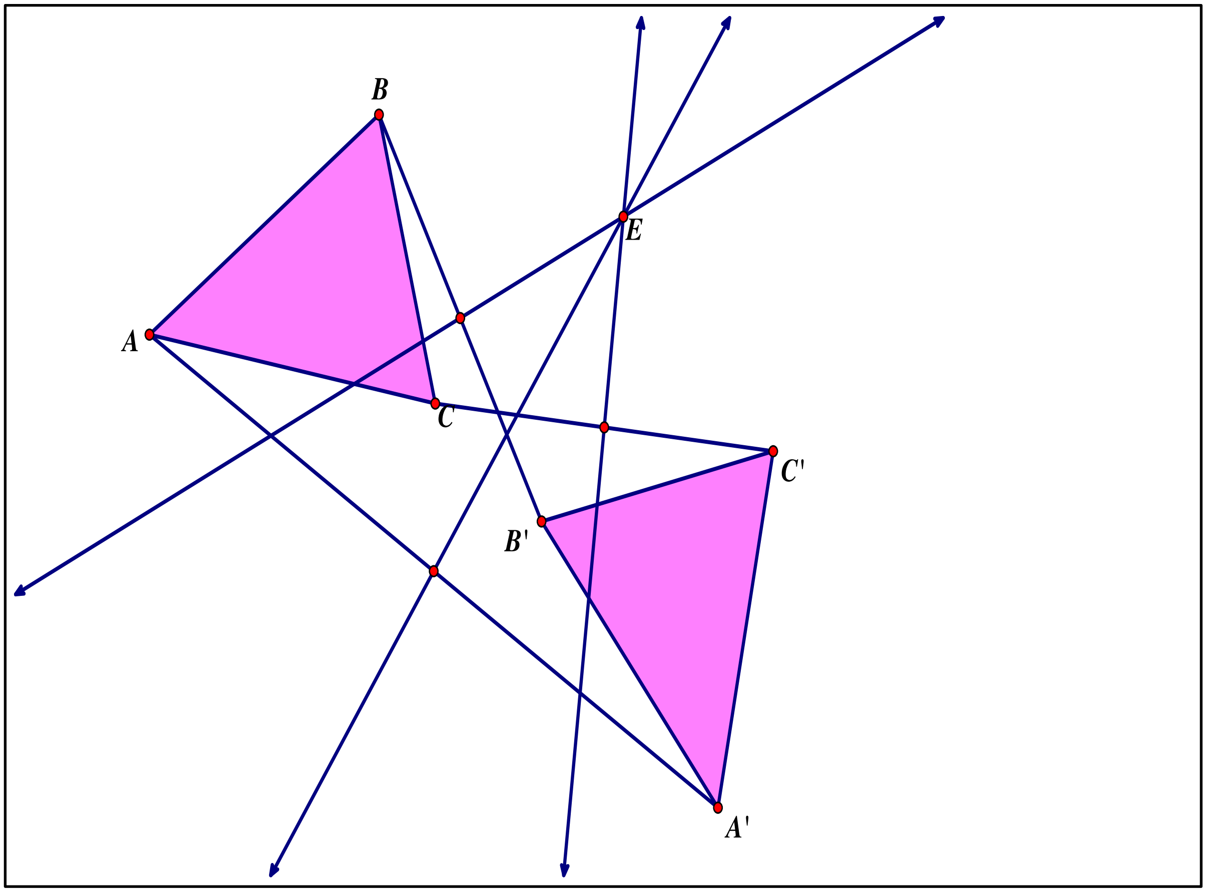


Figure 5. Cathy's construction of the center of rotation from given object and image triangles

In this interview episode, Cathy was asked to construct the center of rotation for the given triangles ABC and A'B'C' as object and image, respectively, under a rotation through an angle. She got puzzled for a while. She expressed that the center of rotation would be somewhere in between the two triangles. When she had a hard time figuring out the center, the interviewer asked her to think backward. What relation would the center have with respect to the object and image vertices? She then joined the object and image vertices. She was able to find the center of rotation as the point where the perpendicular bisectors of AA', BB', and CC' would intersect each other. The constrution work was followed by interactive interview session in relation to participants' beliefs about teaching of reverse thinking with any GTs with GSP.

Conceptualizing Beliefs

I conceptualized beliefs of two preservice secondary mathematics teachers through a series of analytical, interpretive, and generative processes. In the first phase of data analysis, three broad categories of beliefs about ability, action, and attributes emerged from the data as initial categories of participants' beliefs in relation to teaching GTs with GSP. The sub-categories of beliefs about ability were beliefs related to affect, cognition, and pedagogy. The sub-categories of beliefs about actions were beliefs related to constructive action, developmental action, and explorative action. The sub-categories of beliefs about attributes were related to attributes of constructions, attributes of objects, and attributes of pedagogy. I reframed the interview questions for the third and fourth interviews based on these categories and sub-categories that emerged from the first phase of analysis and interpretation.

The second phase of analysis and interpretation emerged into a different set of codes and categories grounded on the codes and categories emerged in the first phase. These codes and categories were constructed from the analysis and interpretation of data from the third and fourth interviews. Six major categories emerged in this phase of analysis and interpretation of the data. These categories were: beliefs related to action, affect, attitude, cognition, environment, and object. I restructured the questions for the fifth interview focusing more on the six major categories and sub-categories emerged in the second phase. However, the analyses and interpretations of data from the final (fifth) interview helped me in the reconstruction of some of the sub-categories based on the new findings from the data. All these major codes and categories from the second and third phase of analyses and interpretations were identified as primary findings of the study.

In the next and final level, I reconsidered the data for re-interpretation from a nested relationship of one-self with others in terms directions of beliefs toward outer objects and phenomena or toward participants' 'selves'. I observed the data holistically beyond reducing them into codes and categories. He observed the data from the viewpoints of belief objects either as external or internal (as directions of beliefs) in terms of reflective beliefs and reflexive beliefs. While observing the entire data and thinking through the nature of beliefs in terms of temporal dimensions of the participants' beliefs, I identified pre-, in-, and post- reflective and reflexive beliefs in their belief narratives. I then sampled some of these beliefs from the interview transcripts and their belief narratives for further analyses and interpretations in a holistic sense. The sampling of already constructed belief narrative was carried out purposefully to reinforce the meaning of belief structures in terms of pre-, in-, and post- reflective and reflexive beliefs.

Now, I would like to discuss the participants' beliefs in terms of six categorical findings and their associated sub-categories followed by the layered interpretations.

Categorical Findings and Discussion

Now I would like to discuss the inferred beliefs of the two participants in relation to teaching GTs with GSP. The results and discussion follow with a composite protocol representing their narrative-voice in the conversational form. The presented protocols include the indicators of their beliefs through unedited segments of narratives to represent their voice and meanings as true as possible. A brief discussion of their belief narrative is followed by second-order interpretation and connections to the literature as the third-order interpretation. Here, participants have been coded as C for Cathy, J for Jack and R for me (this researcher) in the dialogical narrative of participant beliefs. The numbers in the square braces [ ] at the end of belief narratives in the protocols represent the number of interviews that the excerpts of narratives were taken from.

Figure 6 summarizes all the major categorical belief categories and sub-categories that emerged from the data. It shows the interrelation of beliefs associated with an action, affect, attitude, cognition, environment, and object and their sub-categories.

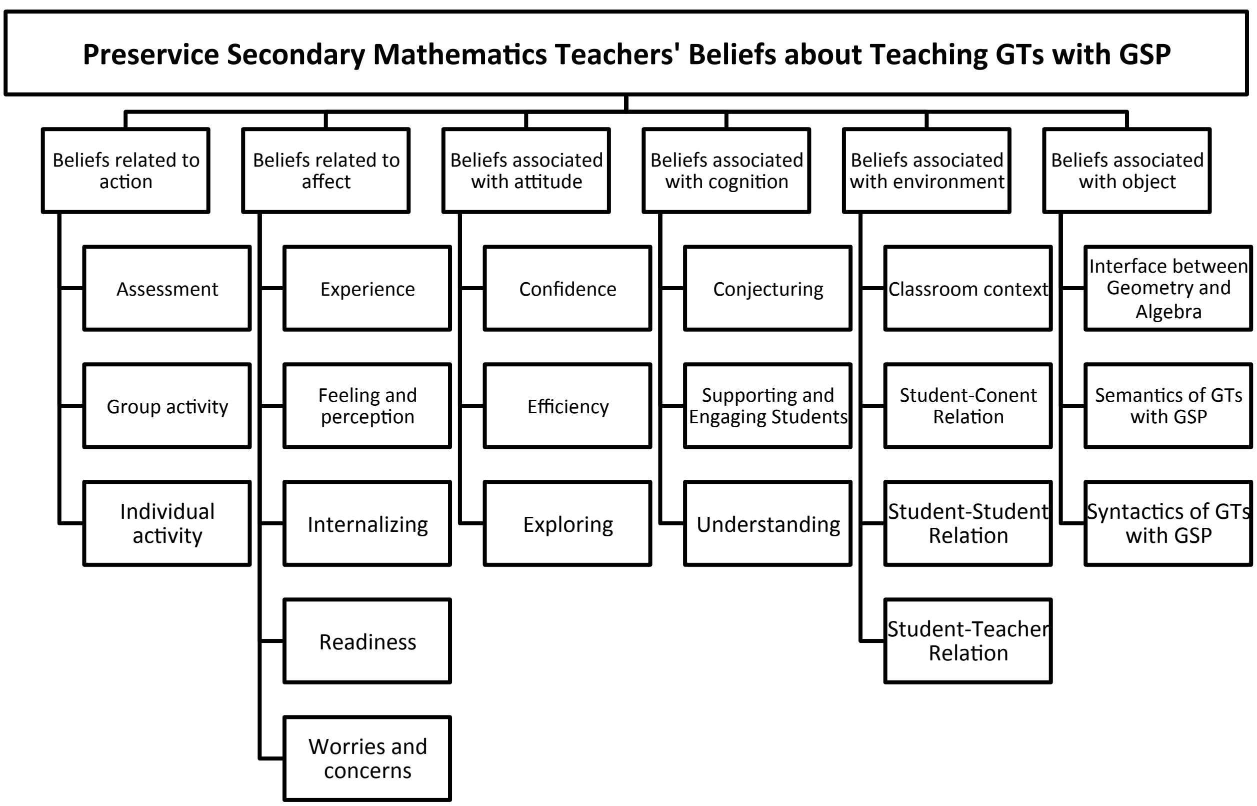


Figure 6. *Preservice secondary mathematics teachers' categorical beliefs about teaching GTs with GSP*

The beliefs associated with actions have sub-categories – assessment of student learning: looking at their creation; group activity with GSP: collaboration with the jigsaw; and individual activity with GSP: saying things out loud versus construction. The beliefs associated with affect have the sub-categories – experience with GSP: suck it up and do it; feeling and perception about the use of GSP: a fine line between wasting time and being productive; internalizing the use of GSP: is it loving the tool or feeling confident?; readiness for using GSP: getting more comfortable; and worries and concerns about using GSP: knowledge, attention, and access. The beliefs associated with attitude have sub-categories – developing confidence with GSP: temporal, visual, and relational; efficiency of using GSP for teaching GTs: enhancing, exploring, and understanding; and exploring GTs with GSP: surface stuff versus depth and learning curve (Fig. 6).

The beliefs associated with cognition have the sub-categories – conjecturing about GTs using GSP: conservation of properties and vocalizing what is done; supporting and engaging students in learning GTs: describing, holding attention, and laying out steps; and understanding GTs with GSP: skipping or quickening steps, visualizing, and solidifying. The beliefs associated with the environment have the sub-categories- classroom context: recognition and contradiction; student-content relationship: making another point of connection and real life application; student-student relationship: glorious conversation, hindrance, and helping out; and student-teacher relationship: being independent. The beliefs associated with an object (e.g., GSP) have the sub-categories – interface between geometry and algebra: connecting hands-on and minds-on; semantics of GTs with GSP: constructions are not freehand and do not give a free thing; and syntactics of GTs with GSP: visualization debunks meaning and power (Fig. 6).

These six categories of participants' beliefs about teaching GTs with GSP are not mutually exclusive. They share many commonalities and hence they seem to be bearing similar meaning in many instances with a very subtle difference. Also, these categories are not exhaustive. There could be other categories arising from the same data. However, these categories were the dominant ones within the limit of data, time, and space. These major categories and their associated sub-categories have been discussed in separate sub-sections.

Beliefs Associated with an Action

The participant's beliefs about actions associated with teaching GTs with GSP emerged into three sub-categories- assessment of student learning: looking at their creation; group activity with GSP: collaboration with the jigsaw; and individual activity with GSP: saying things out loud versus construction (Fig. 7).

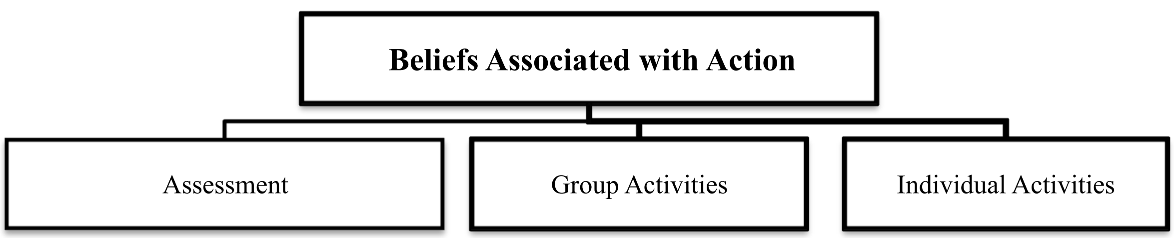


Figure 7. Beliefs associated with an action

Assessment of student learning: Looking at their creation

The participants expressed their beliefs about them as teachers assessing student's learning while teaching GTs with GSP. Here, beliefs about assment of student learning includes both formative and summative assessment. However, the participants' beliefs are more about formative assessments of student learning than the summative assessment. The following protocol 1 is an example of how they expressed their beliefs within this sub-category.

Protocol-1

R*: Do you think that you can assess students' learning of GTs by using GSP?*

C: If I get the document, it depends on what it would be. If I want the procedural then no, but if I want conceptual understanding, then yes. If I get the document (students' constructions) not just the printed thing, in manipulating their construction. [5]

R: How would you decide that students have learned something when you teach rotation using GSP? How will you know you are convinced with their learning?

C: Um, great question. Haven't really thought about assessment of this sort of. (Pause) We can have them get a new activity completely. They will need to recreate a picture or something. You have a picture here, but we want them to draw a mountain. I want them to rotate. Everybody finds it. Send them for rotate about the lake or something. Um, that way I can see they can rotate it and that they know what rotation means, and they are not just following the procedure. [3]

J: You can do a formal assessment, like written or you can do if they are familiar with GSP, you can have them create it. That's gonna take a lot more time. You have to get them familiar with it and might even take it longer. Like, if you actually familiarize them with some of the programs, maybe the kids are super smart and now maybe they catch on with that. [3]

R: What will convince you that they have learned?

C: Um, you can print out the thing (their construction), but I mean to know that it was truly rotated. In the beautiful world, you know we have drop boxes to put in something. Or, I can just walk up to them and ask, 'Show me what you have done?' We can do that. I would ask them like 'where did you rotate it?', tell me what point they rotate about and then say I want to rotate the top of the mountain to the left.[3]

J: A formal assessment like, even if you like oral assessment they are gonna explain it to me. Some kids can explain to me what the rotation is doing. You can do it if it is just a rotation and it is on a paper, you can have two shapes then you can talk about angles of rotation. If you have GSP, they can measure it. If you have paper and pencil, they have to use a protractor to measure it. But, if you use GSP they can quickly measure the angles and talk about how each one goes, which is really cool. That's why I like it. [3]

Second-order interpretation

Cathy expressed here that the student works on GTs with GSP could be documented and used for assessment of their learning. Here, the documentation means not only written explanation about their learning, but also the real tasks saved on the computers when they do different GTs with GSP. She seems to consider that 'having students done new activities' may have some uses as part of an assessment of their learning. At the same time, students' construction of an object and image under a rotation (or any GT) provides some clues about their learning. She seems to believe that asking students, “Show me what you have done?” helps in the assessment of their learning of GTs with GSP based on their responses to the question.

In the same vein, Jack also reports written assessment as a part of judging students’ learning. For him, their tasks on GTs with GSP show how familiar they are with the tools and contents and how they make the connection between them. Further, he suggests that the teacher may use prompts to know their level of understanding of rotation (or a GT) and use of GSP. He says, “A formal assessment like, even if you like oral assessment they are gonna explain it to me” expressing his view on the assessment. Hence, he also seems to believe that the students’ explanations and ability to manipulate the processes in GTs can be used for assessment of their learning.

Hence, for both Cathy and Jack, students’ constructions of GT activities with GSP can be helpful in assessing thir learning in the context. “Show me how you have done?” could be a possible prompt for assessing student learning of GTs with GSP in the classroom. The participants’ narrative about assessment of students’ learning has some relevance to the literature and theories of beliefs, and that has been discussed in the next sub-section.

Third-order interpretation

Some key points derived from the above discussion are: assessment of the procedure versus concept, construction and measurement, and demonstration and explanation. Other key ideas from the literature are – authentic assessment, self-assessment, alternative assessment, process assessment, and content assessment. It seems that the participants could not explicate these categorical approaches to assessment. One of the easiest and most appropriate informal assessments could be ‘walk-around’ approach. Cathy and Jack seem to believe this approach

(See protocol 13 and 17). Cole (1999) introduced the idea of ‘walking around’ as a form of informal assessment. She states, “Walking around is the most immediate, efficient assessment method that we have available to us” (p. 225). Walking around allows the teacher to look at the entire classroom setup, group organization, individual attention, and build up a relationship with students for an informal assessment. He or she can do two kinds of assessments –observation or conference (Cole, 1999). In the first kind, the teacher can do a short observation or focused observation of individuals or a group. The teacher can use on-the-spot observation chart to possess a record of what the students are doing. The second assessment needs face-to-face meetings between the teacher and students. The teacher can use students’ work as an initial step to begin the conversation. He or she can use prompts to know more about what the students are doing and thinking (Cole, 1999). Cathy and Jack’s view about using students’ consturction for assessment of their learning has some relevance to the observation as suggested by Cole (1999).

Bloom’s (1964) taxonomy could provide a basis to develop assessment for different levels of understanding of GTs with GSP. It may include simple procedures with facts, principles, and definitions at the lower level and integrated solution, evaluation, and creation at the highest level. Cathy and Jack’s view of assessment in terms of procedure versus concept connects lower level actions of construction and measurement to a higher-level demonstration and explanation. Hence, there are some points of relevant connection between Cathy and Jack’s beliefs about assessment of students’ learning of GTs with GSP with the help of their tasks and prompting during the classroom activities with the literature.

Now I would like to discuss the next sub-category related the participants’ beliefs about group activity in the classroom with the use of GSP in the sub-section.

Group activity with GSP: Collaboration with the jigsaw

The participants expressed their beliefs about the use of group activities while teaching GTs with GSP. The following protocol 2 is an example of how they expressed their beliefs within this sub-category.

Protocol-2

R: How would you organize a group activity (while teaching GTs with GSP)?

C: Well, it's hard. I think it's hard because you want both students have hands-on. So, maybe if you are gonna have a partner out of a computer you need to draw this and you (other one) need to describe. You are gonna describe and you are gonna draw this. Halfway through you need to switch. I think it's hard because you need to involve with it to have a conversation. So, it would be a lot better if they all were able to do it and have the conversation. So with GSP partner work is not ideal cause one person is not gonna do anything. [3]

J: You can have them get together and design something that involves rotation (or other GT). Like, if you have a whole block, you can really come up with something cool, creative. Like, have them come up with something they could realize. They can even talk about video games and how the things are moving. But, having them talking about just design something or like that, make sure that it's something that they can physically draw in GSP. [3]

R: Do you think a jigsaw method works with GTs with GSP?

J: It could. I don't know how much you can use GSP with it. But, you can totally do like grouping them for every different transformation and having them go back and teaching in their original group. It should be fun. I mean if they learn about all these different ones from each other and they have combined them that would be fun. It's time consuming, but fun. [4]

J: If I have the technology, like the access, yes. If you have something like laptop cart and so they can move around. Jigsaw requires a lot of moving around. Also, if they are doing different things on computers and switching computers that's not gonna be useful. If you have the access, then yes. [5]

C: Yea. I think so. Like if these guys do reflections, those guys do whatever they decide and how they are gonna teach each other. You can use GSP that way. [5]

Second-order interpretation

Cathy says that group activity could be a difficult one in teaching and learning of GTs with GSP. For her, it could be difficult work to engage students in the group-work and to have them learn effectively. For some students, it could be very nice because they might learn from their peers. For others, it may not be productive because they may not participate equally in learning and teaching with their peers. She also expresses her doubt that group activity helps all students in learning. She points out that one student in a pair possibly constructs and other can describe and vice-versa. She says, “It would be a lot better if they all were able to do it and have the conversation” expressing her views about group-work. However, she claims that sometimes only one may do both. She suggests that switching their roles could be important for their learning. Nonetheless, she accepts that having them engaged in conversation can be a challenge. She asserts that partner work or group-work may not be a good idea all the time. She expresses that sometimes it may work and other times it may not.

Cathy accepts that GSP can have an application for the jigsaw method of learning. It is a method of learning in groups. Students are divided into separate groups of 4-5 students. Each group members divide their role and responsibility equally to teach each-other different concepts. All the groups have the members with those roles and responsibilities. The members having similar roles and responsibilities from each group come together and discuss their ideas on a concept or topic. They learn and teach each other and try to become expert or proficient in that concept or topic. Once they complete the discussion and gain the confidence in what they discussed, the members return back to their respective groups. The group members teach each-other the concepts or topics they have learned to their peers in the group turn-by-turn. Despite her skepticism about the effectiveness of peer works, Cathy appears to believe in collaborative works that one group of students can do one type of GT and others do another GT with GSP, and they teach each other different GTs.

On the other hand, Jack considers that group-work is important for learning and teaching of GTs with GSP. He suggests that the teacher can engage the groups in the class in designing something about GTs with GSP. While doing this, he claims that students can do 'something creative and cool' that they can enjoy and learn from it. He asserts that the group members can do some constructions in a group and study the properties of GTs with GSP. For him, the jigsaw approach can be a part of group activity. He states, "…grouping them for every different transformation and having them go back and teaching in their original group" expressing his view about group-work in a jigsaw. He appears to believe that the learning in a group is a fun activity, especially when group members learn from and teach each other through the jigsaw.

Both Cathy and Jack consider that group-work, like jigsaw, can be used for student engagement and group learning while teaching GTs with GSP despite some sketicism toward equal participation of group members in this method. Their views with some key points can be connected to the literature and other beliefs about student assessment in the next level of interpretation.

Third-order interpretation

Some key points from this discussion are- working with a partner, have a conversation with a partner, creating designs in a group, grouping students for every different transformation, and teaching and learning from each other. Some researchers and authors (e.g., Hodges & Conner, 2011; Santucci, 2011) discussed group activity in terms of making student-lead presentations, self-gauzing of mathematics knowledge, peer reviewing of each-other's works, and stratifying students' works. "Grouping the investigations….allows teachers with the time to choose the parts most relevant to their curriculum" (Siegel, Dickinson, Hooper, & Daniels, 2008, p. 491). Muller (2010) considers group-work as a way to facilitate a shy student to work in a group and present in the group. This way the teacher can engage the student, who is shy in the class to share his or her individual work, in a group activity to help him or her to present among others. One of the major benefits of group-work in the class is that "students take ownership of both their individual and their group learning. They take pride in their work and collaborate to ensure that their product is excellent" (Bossé & Adu-Gyamfi, 2011, p. 297). Cathy and Jack's view about engaging students in group works in teaching GTs with GSP through partner conversation and jigsaw have connection to cooperative learning, to some extent.

There are theories of group-work for classroom, for example, cooperative learning and the jigsaw method. Group-work in the classroom helps in social cohesion and cooperation among students for learning (Andrini, 1991; Belbase, 2007; Johnson & Johnson, 1990; Obara, 2010). From the literature, one can outline some characteristics of cooperative learning – promotion of interaction among the group members, group members taking responsibility and accountability for their learning and action, development of group skills, development of personal and social identity, development of group process through mutual care and trust, and achievement of common and individual goals (Belbase, 2007; Johnson & Johnson, 1990). Both Cathy and Jack expressed their beliefs toward group-work through the jigsaw method. They seem to believe that the students could be divided into clusters to work with the different transformations and then the cluster members come together to teach each other different transformations. In this group-work, students may have a better access to mathematics (GTs) and technology (GSP) (Cleaves, 2008).

Individual activity with GSP: Saying things out loud versus construction

The participants expressed their beliefs about using individual activity in the classroom while teaching GTs with GSP. Here, individual activity has been considered as what students can do in the class without getting involved with groups. This can be constructions, proofs, and explanations. The following protocol 3 is an example that shows their expressed beliefs about the use of individual activities in the classroom while teaching GTs with GSP.

Protocol-3

R: What kind of individual activity could students in a class work with?

C: Um, so I think the individual activity would be the discussion. I think that in geometry it is incredibly important to discuss their ideas. And, because that is informal proof and so saying those things out loud solidifies. In geometry, we do it a lot. I think that discussion is incredibly helpful for them. I can go up to them, and they may ask me if they are doing this right. Then being able to say words with their actions is really helpful. [3]

J: You could have them design something and spin it, if they are able to do that, some sort of real world thing. I really like the Ferris wheel. That's really cool. I would also do a tilted wheel which is about eight different rotations, that's really fun. It just depends on how much time you have and how you answer this. If they understand and really do stuff, that's really cool. [3]

Second-order interpretation

Cathy seems to believe that the students' discussion on their ideas as a part of individual activity in the class. She expresses, "I think that in geometry it is incredibly important to discuss their ideas." For her, the discussion is a part of 'informal proof' in which students can say things out loud that 'solidifies' their learning. Another point she makes is 'going up to the students and asking them if they are doing right' as part of individual activity. She claims that when a student can verbalize (say in words) what he or she has done, and this could be helpful in his or her learning as a part of personal action. That means, for Cathy, a student should be able to express his or her thoughts, ideas, and works to the teacher (and others) as a distinct personal activity.

On the contrary, Jack considers students' designs (constructions) as part of individual activity. He says, "You could have them design something and spin it." For him, this design could be related to 'real world thing.' According to him, one such thing could be a 'Ferris wheel or a tilted wheel.' He expresses that designing a 'tilted wheel' with eight different rotations could be very interesting. He contends that how to engage students in such individual activities depends on time factor, students' understanding, and doing different kinds of stuff (on their own). He does not focus much on the discussion side as individual activity, but he is more toward the construction side. He appears to believe that the students' engagement in creative constructions (e.g., Ferris wheel or tilted wheel) can be an activity focused on individual tasks related to a GT with GSP.

Cathy seems to believe students' verbal expression as part of individual action whereas, Jack seems to believe their constructions and designs of different transformations as a part of individual task in teaching GTs with GSP. Their different points of view about individual activity could be linked with the liturature and theory of teacher beliefs about teaching mathematics with technology in the next level of interpretation.

Third-order interpretation

Some key points from this discussion are – discussion of informal proof, explaining one's work, and design something as individual activities. There can be different activities with the use of GSP for engaging students in individual activities such as – open-ended learning environment may support individual students based on their pace of learning and use of tutorials to engage individual students for learning about GSP (Wyatt, Lawrence, & Foletta, 1997). Dickinson (1994) and Manouchehri et al. (1998) have mentioned the importance of individual learning of mathematics with technology (e.g., GSP). A teacher could engage students in constructing, conjecturing, and verifying about GTs using GSP that may develop their aptitudes (Dickinson, 1994). Students can learn GTs with GSP with free exploration (learning of tools in relation to GTs), semi-structured activities (discover some GTs with the tasks provided), fully guided (or structured) activities (follow step-by-step constructions, explorations, conjectures, and verifications), and then back to independent explorations (explore GTs and their properties beyond what was taught) (Manouchehri et al. 1998).

Cathy and Jack seem to have these beliefs implicitly because their expressions do not show these different layers of individual activities, and that is obvious because they do not have experience of using GSP in their actual teaching (Wyatt et al., 1997). However, their view about verbalization and construction of GT processes with GSP have a strong connection to theory of constructivist learning and teaching of mathematics (e.g., Depaepe, De Corte, & Verschaffel, 2015; Skott, 2015).

Beliefs Associated with Affect

The participants' affective beliefs are their mental states of happiness or sadness, emotions, and subsequent states of feeling anxiety or comfort. The beliefs associated with affect have the sub-categories – experience with GSP: suck it up and do it; feeling and perception about the use of GSP: a fine line between wasting time and being productive; internalizing the use of GSP: is it loving the tool or feeling confident?; readiness for using GSP: getting more comfortable; and worries and concerns about using GSP: knowledge, attention, and access (Fig. 8). I discussed each of these sub-categories separately.

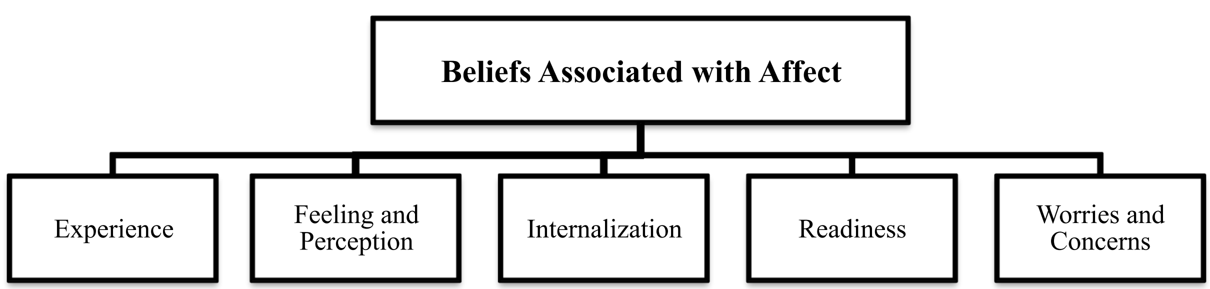


Figure 8. Beliefs associated with affect

Experience with GSP: Suck it up and do it

The participants shared with me their beliefs in relation to their experiences of learning GTs and using GSP in the context of teaching and learning GTs. I considers that beliefs and affect are different psychological or mental states and they might influence each other and inform each other (Philipp, 2007). The following protocol 4 shows their affective experience and related beliefs about teaching GTs with GSP that reflect their beliefs associated with affect.

Protocol-4

R: Tell us something about your experience of learning GSP.

C: Wow! Kind of let me guess. Here is your homework. Here are the proofs. Everything needs to be constructed in GSP. Get on. I think it was like aakhr...it was so hard. I guess we didn't explore tutorial because I learned better without it. That had been helpful except those tutorials that said like 'you can go here and like do a construction of a square and like create a tool'. While you have to create, but then you don't have to go back to create a square. And so, like I had to struggle through all and kinda do all my own. People would have helped me obviously if I had asked. I remember pretty much everything I have done and I am not gonna forget it because I did it. [4]

I was able to like suck it up into it. But, I don't think that a lot of my students would be up to 'suck it up and do it'. But, if they are doing this, and I do not have to teach them every single step the way they had already struggled through it, in a word if they went through it then I would be more comfortable. I guess I would not be able to do this every single time. [4]

J: It consists of about two class periods in the college. [4]

R: How do you describe them as unfolding?

J: We had technology presentations in methods class where each person, maybe, talk something with GSP, and that was about it. We were given step-by-step instructions. Build this and here is how you do it. You can add a little bit to it. That was very procedural and huge instructions to go to it. Very little actual hands-on. [4]

R: Do you think that any conceptual experience unfolded through that?

J: Not really. I mean it was like 'here is what you are doing. This is what it should look like, if it doesn't then you messed up'. [4]

Second-order interpretation

Cathy had an initial experience of using of GSP in her Foundations of Geometry class. This class covered various topics in geometry, including transformations. She explains, "I guess we didn't explore tutorial because I learned better without it." They (the students) did not use the tutorials to learn GSP in the class, and there were no separate sessions to learn the basic tools in GSP. She did not even use the tutorials while learning about GSP tools on her own. Although she thought some of her classmates hated the use of GSP, she contemplated about her self-struggle with the tools in it. She continued with her self-practice by making mistakes and persisting through the difficulties that were inherent in the process. She expresses, “I had to struggle with all and kinda do all my own.” She liked ‘sucking it up’ in GSP for her geometry problem solving. That means she accepted the challenges and difficulties in the process of learning to use GSP in her geometry course. She gained more confidence as she learned by self-practice. She did not even ask for help. She, later on, had some experience of using GSP in her mathematics methods class before her field experience. Therefore, she seems to believe that her learning of GSP through ‘self-struggle’ led to developing her skills and abilities to use it for herself and for teaching GTs with a greater confidence (in the future).

On the other hand, Jack had gone through limited experience of using GSP in his Mathematics Education Program. He had a limited experience of learning GSP in his mathematics methods class that consisted of only two class sessions. He explains that the class had technology presentations. Each student in the class had the opportunity to talk about something with GSP in the presentations. These presentations focused more on the procedural learning (with step-by-step instructions) of using GSP and they did not necessarily emphasize how to use GSP for teaching GTs. He did not have the opportunity to mess up or struggle with the tools in GSP. He says, “We were given step-by-step instructions.” However, he found the use of GSP in teaching and learning GTs as a fun activity. His experience with GSP was like, “Here is what you are doing and this is what it should look like; if it doesn't then you messed up.” He does not have ‘conceptual experience’ in using GSP for teaching and hence, he appears to believe that his experience of learning and practicing with GSP is not adequate for teaching GTs.

It seems that their experiences of working with GSP in Geometry or Methods classes provided them a sense of confidence or lack of confidence in using the tool for teaching. Their confidence or lack of confidence shaped their beliefs about the future use of GSP in teaching GTs. Their convictions about how confident they are in using the tool has a great significance in the liturature and theory of teacher beliefs (Kuntze & Dreher, 2015) that has been discussed in the third-order interpretation.

Third-order interpretation

It seems that both Cathy and Jack went through the same course, but with different instructors while they were in the Foundations of Geometry. They also went through the same course with the same instructor when they were in the mathematics methods course for secondary level teachers. Cathy used GSP in her entire geometry course. She learned to use GSP to solve geometry problems and writing proofs. However, she appears to struggle with this process while learning about GSP. This self-learning process seemed challenging to her, but rewarding for her learning experience. She became “active learner through self-discovery enabled by the software (the tools in GSP)” (Sarracco, 2005, p. 6). She was a self-directed learner of GSP in the context of geometry class (Merriam, 2002). She seemed to assume her responsibility for learning by herself (Knowles, 1975; Tough, 1971) although she could ask the instructor(s) for help.

Whereas, Jack did not have to use GSP while he was in the geometry course. His only experience of learning and using GSP was limited to the mathematics methods course, which was further limited to only experiencing GSP in two class sessions. He learned to use GSP with step-by-step instruction and practice in those classes (Cantürk-Günhan & Özen, 2010). The way he learned to use GSP influenced his intention to use the tool in the future classroom teaching (again with step-by-step instruction) (Also, see protocols 5 & 13). The step-by-step learning of GSP is a common practice among most preservice mathematics teachers (Cantürk-Günhan & Özen, 2010). Jack appears to follow this model (i.e., step-by-step) whereas Cathy seems to follow open-ended explorations (i.e., self-directed approach) for learning and using GSP (Key Curriculum Press, 2009). There was a huge difference between Cathy and Jack in terms of their experiences with learning and using GSP for teaching GTs. These differences were noted in other aspects of their beliefs about using GSP for teaching and learning GTs. Hence, their experiences were a critical factor in shaping their beliefs about teaching GTs with GSP (Ng, Nicholas, & Williams, 2010). Their educational experience at the college courses might have shaped their beliefs about teaching GTs with GSP, and these beliefs might have influenced their intentions of using GSP in future teaching (Martin, Prosser, Trigwell, Ramsden, & Benjamin, 2000). Also, their experiences with GSP led them to perceived usefulness of GSP for teaching GTs in a different way (Cowen, 2009; Davis, 1989).

Many first generation GSP learners and users (e.g., Boehm, 1997; Brumbaugh, 1997; Cuoco & Goldenberg, 1997; Morrow, 1997) learned the tool for teaching GTs and other geometry contents through self-learning and motivation to use it in the class. Boehm (1997) experienced tremendous shift in her practice of teaching geometry in general and GTs in particular by making "visibly uncomfortable" (p. 74) to "comfortable", facilitating "which are not readily explained using static drawing" (p. 74), and revitalizing teaching and learning of geometry. Brumbaugh (1997) experienced how his middle school students constructed, explored, argued, and negotiated the meaning of area of a triangle within dynamic geometry environment of GSP. He then states, "…we as teachers can begin to explore the world of mathematics in new ways" (p. 70) using GSP. It may not be an appropriate way to compare and contrast how Cathy and Jack's experiences unfolded to their beliefs about teaching GTs with GSP to other researchers who are already experts in using the tool for teaching and learning. However, the theoretical constructs from the literature serve as a foundation to understand their beliefs about teaching GTs with GSP in a broader sense and reconceptualize them in the context of what Cathy and Jack experienced and how those experiences speak to their unfolding beliefs.

Feeling and perception about the use of GSP: A fine line between wasting time and being productive

The participants shared with me their beliefs related to feeling and perception in relation to using GSP in the context of teaching and learning GTs. Feeling and perceptions seem to be strong determinants of one's beliefs and subsequent practices in the classroom (Philipp, 2007). The following protocol 5 shows the participants' feeling and perception about the use of GSP during the teaching and learning of GTs. These feelings and perceptions were visible in another parts of our interactions during the interviews.

Protocol-5

R: Do you have positive/negative feelings about using GSP?

C: Yes. I don't start teaching with it. Cause I want most of the students to work on it. But, I think it is great for exploration. Once we know what rotation is I don't want them to draw a rotation every single time. I want them to quickly go to it (GSP) and then explore with it. I think it can be used as a great practice tool. [5]

J: I love GSP. I don't have to use it very much, but it's really fun. And, the visual stuff you can put on the screen for the kids. I mean it is so much more than just do it on paper and pencil. Paper and pencil, *especially for rotation, it is really hard to visualize it. I know with GSP you can do animations. You can show them it is very cool. I think it is a great tool. I really like it. [3]*

J: I like it a lot. It is useful. You can visualize. They can see what's happening. Negative would be just like time and resources at school. Junior High does not have license for it. [5]

R: How about your feeling of using the tool in your future teaching?

J: I am half way there, and I am not there. The more I play with it the more I like it. My biggest fear would just be if you wasted too much time during teaching the program. We don't have a week to teach just GSP. We have got to be able to, time management is the key. If I can do it, I would definitely do it. You have a fine line between wasting time and being productive. [4]

R: Do you think that you have perceived the different roles of GSP (e.g., teaching and learning, exploring, conceptual, procedural, visual, and dynamic)?

C: Yes. Not so much procedural. I guess more of the exploring, conceptual, visual, and dynamic. [5]

R: How do you feel about the role of GSP in teaching?

C: I would like it to be more of learning, but more towards practice. I don't want them to start learning with it. But, once they are learning, then they can bring it into their learning. Teaching, yea, I could use it as a teaching tool, but I want more areas to connect with it, so I can bring it as practice. I am gonna try something else with teaching. [4]

J: I think it's a kind of equal. I mean it's gonna enhance your teaching. If you know how to use it, it can give you things that you can show to the kids. It's gonna enhance their learning if they can do it themselves. If it's a struggle, it's not. I mean it's gonna go other way. I mean you are gonna get frustrated. Kids can't make it happen. You have to have hands-on time with them and one-on-one time with them. It's kinda hard to agree at that in large classes. [4]

Second-order interpretation

Cathy expresses that GSP is a great exploration tool. She explains that it can have uses in exploring different properties of GTs. She expresses, "I want them to quickly go to it (GSP) and then explore with it." She points out that because of direct tools available in GSP for constructions it skips many steps. Hence, for her, it is a very efficient tool in terms of time and effort that students put into it. Cathy seems to have a relatively longer experience of learning and using GSP (See protocol 4). Her experience of using GSP might have made her feel more comfortable in using the tool for herself and future teaching of GTs. She indicates that GSP is more a learning tool than a teaching tool. She does not like to use it in her teaching of GTs. Cathy reveals her belief that GSP is fast and efficient for conceptual learning once procedures are done. She does not accept GSP as a teaching and learning tool (See protocol 10). Hence, she appears to believe that it is more of a practice tool or a tool for exploration.

Jack, on the other hand, considers that GSP is a fun tool. He indicates that it (GSP) is a better way to present things in class while teaching GTs than just using paper and pencil. For him, teaching GTs by using the tools available in GSP is a 'cool thing.' That means for him using GSP in teaching GTs is acceptable, and it is even better than without using the tool. He considers GSP both as a teaching tool and also a learning tool, 'kinda equal-equal.' However, he expresses his belief that it is a teaching tool. He says, "I mean it's gonna enhance your teaching." He also claims that it is a learning tool. He expresses, "It's gonna enhance their learning if they can do it themselves." However, he claims 'there is a thin line between being productive or just wasting time' while using GSP in a classroom for teaching GTs. He asserts that students should be able to use the tools comfortably in order to develop concepts. He admits that GSP is everywhere - as a teaching tool, learning tool, exploring tool, motivating tool, and constructing tool. Nonetheless, he appears to believe that he is not yet prepared to use the tool in his classroom (he means future classroom). He says that he is only halfway into GSP and does not seem prepared to use it in the class without adequate practice by himself. Hence, it appears that his beliefs have been influenced by his limited experience of using GSP.

It seems that the participants' feeling and perceptions about using GSP for teaching GTs have influenced their beliefs about how they are going to use the tool in their future classrooms. Their beliefs about using GSP for teaching GTs have been greatly shaped by their past and present feelings and perceptions about nature of the tool eihter as procedural, conceptual, teaching tool, learning tool, and exploring tool. These affective aspects of their beliefs about using GSP for teaching GTs have some relevance to literature and theory of teacher beliefs (Philipp, 2007; Zambak, 2015) that has been discussed in the next level of interpretation.

Third-order interpretation

Both Cathy and Jack are prospective teachers of mathematics. Their feelings and perceptions about use of GSP for teaching GTs seem different in terms of purposeful use, appropriation, and cautionary thinking. Cathy feels that GSP is a practice tool that comes after some level of teaching and learning of GTs. This belief seems to align well with what Leatham (2007) describes as "post-mastery beliefs" (p. 186). This aligns with the purposeful use of the tool with "robust drawings" in terms of construction that preserves essential features (Saenz-Ludlow & Athanasopoulou, 2012, p. 1037) and "operative reasons" (Leatham, 2007, p. 201) such as a foundational understanding, visualizing, and spatial reasoning (Meng, 2009). Jack's perception about the use of GSP is more balanced in terms of using the tool for both teaching and learning and this view aligns with "exploratory beliefs" (Hanzsek-Brill, 1997, p.176; Leatham, 2007, p. 185). Although Jack expresses a lack of adequate experience in learning and using the tools in GSP, he seems to have a perception of GSP as a useful tool for enhancing teaching of GTs. He also seems to believe that GSP will promote student learning of GTs when it is used independently (Wilson, 2011).

Moreover, Cathy has less sense of attachment to GSP for teaching GTs compared to Jack despite her longer experience of learning and using it. She does not want to use it in the classroom until students have some level of understanding of the procedures and concepts of GTs (with other hands-on activities). Whereas, Jack wants to use it for both teaching and learning processes side-by-side. In this way, they emphasized the appropriations of the tool differently. Cathy seems not to act on GSP as a tool to be used in any activity associated with GTs, but she is appropriating its use based on where the students might be in terms of understanding properties of GTs, both conceptually and procedurally. Jack feels that GSP is more useful than the paper and pencil method of teaching GTs. He feels that he can use it for any possible activities in the classroom while teaching and learning GTs despite his perception of inadequate preparation to use the tool (Laffey & Espinosa, 2003).

Cathy and Jack have differences in their beliefs in terms of being cautious toward the use of GSP for teaching (or learning) of GTs. Cathy's preference toward the use of GSP in a classroom after learning procedure and concept shows her thoughtfulness to use the tool. She does not say that she cannot use it for teaching, but she emphasizes the use of the tool meaningfully to make connections with other activities. Jack, on the other hand, feels that there is a fine line between the use of GSP being productive or it is a waste of time for teaching GTs depending on 'where the students are' in terms of their knowledge and how the tool is used in the classroom. This issue is related to positive or negative dispositions (Smith, Moyer, & Schugar, 2011) toward using GSP in teaching GTs in terms of how they experience the tool through their course works and how they think about using the tool in their future teaching (Meagher, Ögün-Koca, & Edwards, 2011).

Use of GSP for teaching GTs or any geometry lesson has generated different positive feelings and perceptions about the tool. Hofstadter (1997) reflects his feelings about the use of GSP in exploring geometry, "Somehow, this program precisely filled an inner need, a craving, that I had to be able to see my beloved special points doing their intricate, complex dances inside and outside the triangle as it changed" (p. 13). Likewise, Morrow (1997) expresses that, "While visualization in itself is a powerful problem solving tool, the capacity for students to make instantaneous and precise variations to their visual representations adds new dynamic dimension whose implications are only beginning to be understood " (p. 47). That means GSP as a tool or artifact of teaching and learning GTs could have different uses depending on what the teachers believe, as for Cathy it is a practice tool and for Jack it is a tool for both teaching and learning.

Internalizing the use of GSP: Is it loving the tool or feeling confident?

The participants shared with I their beliefs in relation to internalization of using GSP in the context of teaching and learning GTs. Here, the concept of internalization has been considered as participants' conscious or sub-conscious orientation and intention of using GSP as an artifact or a tool in teaching GTs. This process is related to embodied thought, action, cognition, and knowledge of using the tool (GSP) for achieving the goal of teaching GTs. The following protocol 6 shows an example how they expressed their beliefs in this regard.

Protocol-6

R: Do you think that you have internalized teaching GTs with GSP?

C: I think it's not more than like a Mira. I know I have used it. I know how to use it. But, I think the kids show me every time what to produce. Come back and like 'oh oh this is what you did'. But, like I can do this. I mean I feel a lot more confident than the other stuff that I am exploring. [4]

J: Oh, absolutely. I love the Sketchpad. And, I think that kids will like it. That's the important thing that they want to play on computers. When I rented iPads in the class (during practicum), they were so excited. So they can get hands-on technology instead of constantly calculate what x is. Yea, I think so. I would definitely use it if I have the resources to do it effectively. I won't do it half way. [4]

J: Yea, with practice I can do it. I have such a limited thing (experience). In methods class, we used it a little bit. [3]

J: Short of. Um, I need more practice. But,… yea. [5]

Second-order interpretation

With her long experience of using GSP, Cathy explains reflection transformation as a Mira. She says that she has used it and internalized the use of GSP as a tool. The internalization of using GSP is also related to what the students produce. If students are able to show her something that they could produce she feels like ‘she is in it’. She expresses an ownership to the practice with GSP due to her struggle with it and a self-learning practice (See protocol 4). Although she has not used the tool for her actual classroom teaching experience, she says that it is useful for students to practice and develop conceptual understanding of GTs. She seems conscious to the use of GSP for practicing and extending students’ understanding once they have the procedures done. Hence, she seems to believe that she has internalized the use of GSP for her own learning and practice, but not yet for teaching. Therefore, she explains, “I feel a lot more confident than the other stuff that I am exploring.” Her feeling of confidence shows her consciousness toward the effective use of the tool. However, the internalization might not be yet complete due to lack of her actual classroom uses of the tool for teaching GTs.

Jack, on the other hand, shows his strong like toward the tools in GSP, and he appears to think that kids will be excited to work within GSP environment. He claims, “That's the important thing that they want to play on computers.” He considers that using GSP in teaching GTs is not only helpful for a student to visualize what is going on, but it is also a fun activity. He suggests that the use of GSP in the teaching of GTs depends on ‘where the students are’ (conceptually) and ‘what they want to get out of it’ (in terms of meaning). That means he takes students’ interest as part of his decision about how to use it in the classroom. He anticipates future use of GSP in his teaching of GTs. However, he foresees that he needs more practice on it. Hence, he appears to believe that his internalization of using GSP for teaching and learning mathematics (i.e. GTs) is affected by limited experience and practice in addition to students’ ease in using the technology. Hence, it seems that the internalization process is not complete for him and he needs more exposure to the tool and its use in the classroom teaching of GTs. The process of internalization of using GSP for teaching GTs by Cathy and Jack is nested to their knowledge of the tool, commitment to use the tool in classrooms, and their value system in using GSP as a tool or artifact. These aspects can be discussed in relation to the literature and theory of teacher beliefs in the next level of interpretation.

Third-order interpretation

Internalization of any technological tool in general and GSP, in particular, is critical for the use of the tool "in order to capitalize the power of technology" (Kor & Lim, 2009, p. 70). That means the preservice teachers, as users of GSP for their learning and teaching GTs (or other contents) in their future classes, have to internalize the tool first. Some key points from the above discussion are- students' work as part of internalization, sense of confidence, students' interest in technology (e.g., computer), and experience as part of internalization. It shows that the process of internalization involves four important elements- knowledge, value, commitment, and use (Chen & Wang, 2006). It is a process of making explicit knowledge tacit by means of value, commitment, and use (Nonaka, 1991). The reverse process of making tacit into explicit is externalization. The important point here is internalization of knowledge of GSP at first hand for teaching GTs. It seems that without internalization there is no externalization. Both Cathy and Jack seem to understand the value of using GSP for teaching GTs to some extent. However, the level of commitment for the use of the tool for teaching seems not the same.

Cathy seems to think that she has internalized the tool (i.e., GSP) because she says she knows how to use it, and she feels confident about using it. She appears to value this tool as a means to practice GTs, not much for initial teaching and learning (See protocol-10). That means she is not yet strongly committed toward the use of GSP for teaching and learning. Jack seems to value the use of GSP for the learning and teaching GTs. He says that he likes the tool, and he anticipates that his students will like it. He also says that he would use it (in his future teaching) if he has the resources. He seems to be committed more strongly for using the tool compared to Cathy. Her knowledge of GSP with self-practice has helped Cathy to internalize the tool. However, she is not committed toward converting the explicit knowledge of GSP for teaching GTs into tacit (Nonaka, 1991). Jack seems to have a stronger commitment and value toward the tool because he wants to use it for any activity related to teaching GTs, and he positively thinks that students will like it, despite the fact that he has limited experience of GSP. Hence, both Cathy and Jack have a sense of internalization of GSP for teaching GTs to some extent, but there is a lack a strong commitment (for Cathy) and lack of practice (in actual teaching of GTs) because they have not used it in their actual teaching.

The discussion on the internalization of using GSP for teaching GTs might connect to Tee and Lee's (2011) model of socialization to internalization. They designed a pathway for internalization of technology in teaching and learning varying subjects, not just mathematics. The process of internalization begins from socialization. It is a process through which students share their learning experiences with peers or other group members in an informal environment. The next phase is externalization. An individual or group member in a class constructs the meanings of a mathematical process through externalization. They negotiate the meaning of the mathematical process. Here, the students may negotiate the meaning of various processes in GTs while using GSP. The third stage is a combination. In this stage, the students may synthesize their concepts and meanings in the forms of proofs, logics, and constructions (Galindo, 1998).

The final stage is internalization. In this phase students (and also the teacher) can apply reflective and reflexive thinking to have a deeper understanding of the geometry behind any GTs with the help of GSP. They transform the explicit geometry of GTs into a tacit understanding with GSP (Tee & Karney, 2010). The preservice teachers in the present study (i.e., both Cathy and Jack) not necessarily explicated these steps within internalization, however, their views about teaching GTs with GSP has reflected some of these elements implicitly (Tee & Lee, 2011).

Readiness for using GSP: Getting more comfortable

The participants shared with me their beliefs in relation to their readiness for using GSP in the context of teaching and learning GTs. The readiness is associated with their knowledge and willingness of using the tool for teaching GTs. The following protocol 7 is an example.

Protocol-7

R: Do you think you are mentally ready to teach concepts of GTs using GSP?

J: I need to refresh with GSP. I have spent two classes on GSP, very limited actual uses of GSP. [3]

Yes or no, probably more toward no. I just need to get more comfortable with it. The concepts are gonna the same. It's just getting to figure out the things so that you don't fumble around in front of the students. [5]

R: So, with practice?

J: Yea, with practice I can do it. I have such a limited thing (experience). In methods class, we used it a little bit. Junior High does not have it (He is teaching there for practicum.). So, I haven't just messed up with it. [3]

R: Are you just mentally ready?

C: Um, yes. I can teach some of these in two weeks. I think the activity would be more than I like to trick them. [3]

R: Do you think you are ready to use GTs with GSP?

C: Yes, I could, but I would not use it to teach. [5]

R: Why not?

C: Cause I think it is more a conceptual, not a procedural (tool). I think you need to understand the procedure before. And then GSP can do a lot of cool things that you have already understood. How this is all working? Then you can really, why it's all working, you can figure it out like what can we do with it. [5]

Second-order interpretation

Cathy expressed her thoughts about being ready to teach GTs with GSP with a greater confidence than Jack. She says, "I can teach some of these in two weeks. I think the activity would be more than I like to trick them." She thinks that her activities would be more engaging the students by using GSP for learning GTs with practice. However, she feels that GSP is not for teaching. This tool could be used for learning. She considers this tool as a practice tool once the students know the basics of GTs. A teacher has to use prompts based on what the students are doing with it (GSP). She expresses, "How is this all working?" This kind of prompt could be a place to help them (students) know their mistakes. She then focuses on what students already know as a part of the journey to carry on further explorations of GTs. "And then GSP can do a lot of cool things that you have already understood" She claims. She shows her readiness to use GSP in her future classrooms after doing some paper and pencil works to introduce the ideas of different GTs (See protocol 8, 14, 17 & 21). Hence, she seems to believe that she is mentally ready to use the tool for classroom activities.

Jack, on the contrary, says that he needs more practice on GSP in order to use the tool in teaching and learning of GTs. He expresses, "I need to refresh with GSP." He has a sense of very limited experience and practice with GSP. He seems to think that he needs to refresh the practice with GSP to be able to use in his future teaching. He also links his readiness to the access that the school has to GSP. He appears to have a low sense of readiness toward using GSP for teaching GTs. He says, "I haven't just messed up with it." That means he has not struggled through the process of learning and using GSP for teaching GTs. Therefore, he seems to believe that he is not yet mentally ready to use GSP for teaching GTs without adequate practice by himself. However, he expresses his commitment to use the tool in teaching GTs in his future classrooms. The participants' beliefs about their sense of readiness to use GSP for teaching GTs can be connected to the literature and theory teacher beliefs in the next level of interpretation.

Third-order interpretation

There can be different elements of readiness that may influence the future use of GSP in teaching mathematics (or GTs) by these preservice teachers. Some of these elements are- attitude toward GSP for teaching GTs, perceived usefulness of the tool, perceived ease of the tool, and self-efficacy toward the tool (Kumar, Rose, & D'Silva, 2008). So far their attitude is concerned, it is related to how these preservice teachers feel about the use of GSP for teaching GTs. The actual usage of the tool for their future teaching of GTs depends on their current feeling and perception toward the tool. Cathy seems to consider that it is not suitable to use the tool at the beginning of teaching GTs whereas, Jack feels that it could be used either for teaching or learning of GTs (See protocol-2). Positive attitude toward the use of GSP in teaching GTs may result in a higher chance of actual usage of the tool (Ropp, 1999; Shih, 2004). Past researchers (e.g., Hu et al., 2003; Kumar et al., 2008) have shown a positive relation between the teachers' perceived usefulness of technology with the actual usage of the tools. Cathy feels that she has not figured out the procedural side of GSP for teaching GTs, and that might have limited her perceived usefulness of the tool for teaching (See protocol 5). On the other hand, Jack seems to feel that GSP could be a useful tool for his future teaching depending on the availability of the resource (See protocol 5) and his further practice on it.

Past studies (e.g., Davis, Bagozzi, & Warshaw, 1992; Luarn & Lin, 2004; Venkatesh & Morris, 2000) highlighted the perceived ease of technology in relation to actual usage of the tools in teaching mathematics. These studies discussed correlation between teachers' beliefs about easiness to use technology and their actual uses in the teaching of mathematics. The results were mixed. In this study, Cathy seems more proficient in constructing various GTs and discussing their properties than Jack. Her proficiency seems higher than that of Jack because she has used it for a long time in her courses (both in geometry and mathematics methods courses), but Jack, on the other hand, has experience of learning and using the tool for the constructions of GTs only in two class sessions. Therefore, Jack seems not confident enough in doing some of the constructions of GTs with GSP. Hence, there was a difference in their perceived and practical ease of using GSP for their future teaching. Despite the fact that they were different in terms of their experiences and actual practices with the use of GSP, both of them had higher self-efficacy toward using the tool. Cathy expresses that she would use the tool for more exploration and practice in the class while teaching GTs. Jack considers that he would like to use the tool for conjecturing, proving conjectures, and exploring different properties of GTs in the classroom. (See protocols 2, 9, 10, & 11).

Some researchers (e.g., Badri, Mohaidat, & Rashedi, 2013) used four dimensions to examine the readiness of teachers to use technology for teaching (any subject). These dimensions are – optimism, innovativeness, insecurity, and discomfort. They found a high correlation ($r = 0.638$) between innovativeness and optimism. Similarly, the correlation between discomfort and insecurity ($r = 0.595$) was also moderate high. That means the teachers who feel that they can use technology in a creative way are also innovative in using them. Whereas, the teachers who do not feel comfortable in using technological tools feel insecurity in using it. Hence, according to Badri et al. (2013), "insecurity and discomfort are inhibitors of technology readiness" (p. 7).

These quantitative results might resonate with qualitative expressions of Cathy and Jack. Jack's conviction that he lacks adequate experience in using GSP for teaching GTs may create a state of discomfort. This state of discomfort may lead him to avoid using the tool in his future classroom teaching despite his commitment to it.

Worries and concerns about using GSP: Knowledge, attention, and access

The participants shared with me their beliefs in relation to worries and concerns of using GSP in teaching and learning of GTs. Here, worries and concerns are associated with participants' mental states of fears and weaknesses in using GSP for teaching GTs. The following protocol 8 is an example that reveals their worries and concerns in relation to teaching GTs with GSP.

Protocol-8

R: Do you think you have some worries about using GSP?

C: No, I don't think so. I like the Sketchpad. [5]

J: Um, it's time-consuming. Basically, building that background knowledge of the students. If they already know it, then it would be lot easier. But, taking that time actually to get them focused on. [5]

R: Do you have some concerns about using GSP?

C: Yes. Well, they will not understand that's the center of rotation. I did not get back to it. I am their teacher, and I did not know that. Yep, that would be my worry. I want to know that (the center) before this happens. [5]

J: Not really. I like it. I like that you have to construct and see to understand what's going on. But, I also like that there are short cuts that they cannot understand it. [5]

R: Do you have worries about teaching rotations with GSP or the worries about the products they would have?

C: Yea. I would. I guess I would have worries about paper and pencil on this stuff. [3]

R: More, so on GSP?

C: Yea. I think GSP is great, I think it is very good. But, I would have worries about a student looking on another student's screen. They are not experiencing GTs, so they are just trying to learn the technology instead of learning what we want them to learn. [3]

R: Ok. If they did know the technology, would you still have any concerns?

C: Um, I would have a lot less. I would have some concerns that they are just not paying attention. Probably, I would not give these directions orally. I would give them both written and orally. And, since they don't like reading directions, maybe skipping step or missing something and then it doesn't work. Now I am going to help one person but there are other people that, all that people need help, but I can help only one (at a time). [3]

C: If you are in GSP and off-task, I mean this wouldn't be learning what they are supposed to be learning, but still they are learning. [1]

R: What worries you while teaching rotations or any other transformations?

J: Um, the knowledge base like the solid knowledge they have to have and be able to use the program and the time that takes. Another thing, how expensive it is and computer access. You have to be in a computer lab for all the kids to do it. They can watch you, but it's a lot more fun and better if they can get hands-on with it (GSP). [3]

R: Any other worries?

J: Keeping them on task actually doing the program. You get kids onto the computer, and they do whatever they want. [3]

They are doing something constructive or they are off-task, depends on what they are doing. If they are doing their assignment and exploring the program, you want that. [1]

Just keeping them on task. You can't always see what they are doing. Um, another concern is- if you only use GSP you might take away some of the physical learning, but this is gonna supplement that, that's gonna be in my mind a little better. [1]

Second-order interpretation

Cathy seems to think that she has few worries in using GSP once the students work on the procedures with paper and pencil. For her, the worries would be using paper and pencil activity in conjunction with GSP. She expresses, "They are not experiencing GTs, so they are just trying to learn the technology instead of learning what we want them to learn." She expresses her thought that students would focus on GSP instead of content. She seems to be less concerned about using GSP in teaching and learning and more concerned about using it for practice and exploration by the students to have a conversation about what they have done (See protocol-2). Students' attention to their tasks on GTs rather than looking at other's computer screen would be another area of concern. She also reveals her thinking that it is difficult to help all the students in a class at a time when they need individual help. When she is helping a student and others also need her help at the same time, then it may cause a worry to her. She reveals her view that she cannot go to all the students if they need help on different matters. She could provide them written and oral instructions for step-by-step procedures, but still, as she worries, they would not follow the same and may get stuck somewhere. Hence, she appears to believe that her ability to help all students individually would be a matter of worries and concerns.

On the other hand, Jack expresses his thought that he is not well prepared to use GSP in teaching and learning of GTs. He shows a lack of a broader understanding of the tool. His limited experience with this tool makes him worried about using it successfully in the class (See protocols 1, 2, & 3). He considers a strong knowledge base in geometry and GSP as necessary for teaching and learning GTs. He expresses his worries, "building that background knowledge of the students." For this, he thinks, there should be an adequate number of computers in the school and classrooms. Limited access to computers in general and GSP in particular makes him worried. Even when students have access to computers and then GSP, keeping them on task is another area of his concern. When there are many students in the computer lab, then he worries about his ability to see what they are actually doing. He even thinks that students might not pay attention to constructions, explorations, and problem solving in GTs with GSP. They may use GSP for other fun activities or even go off-task (e.g., to play on the Internet). He even thinks that excessive use of GSP might take away some physical manipulations about GTs. Hence, he appears to believe that he has worries and concerns related to his ability to use GSP in the class, students' access to the tool, and students being off-task during practice sessions in the computer lab.

Hence, Cathy and Jack's beliefs associated with worries and concerns are related to focus on the content or technology, application of GSP for exploration or practice, reaching out to all the students for help, knowledge-base for integration of content and technology, access to the tool, and student orientation to the task situations. These issues provide a basis to extend the interpretations of their beliefs related to concerns and worries to the relevant theories of beliefs in the literature, and they are interpreted in the next level.

Third-order interpretation

There are some critical issues that can be raised from this discussion in relation to worries and concerns of using GSP for teaching GTs. First, the teachers themselves have to learn the technology beforehand when they want to use GSP in the classroom teaching. As Cathy mentioned, "I am their teacher, and I did not know that (the center of rotation). That would be my worry." These worries and concerns are much more related to content-related integration of technology rather than generic integration (Wachira & Keengwe, 2011). The teacher educators are now concerned about teaching with technology and student teachers' ability to integrate content and technology (Niess, 2005). The second point is learning technology instead of learning content through technology (i.e., GSP). The intent of using GSP for teaching and learning GTs is not to focus on technology, but to focus on content learning. Cathy's worries and concerns about learning content with technology seem important, and it should be a major focus for technology integration in mathematics education (NCTM, 2011).

The third point is student attention toward using GSP for learning GTs. Cathy points out that her concern would be students not paying attention to her instruction and not following the step-by-step procedures. This issue raises two questions - either the students in such a situation are not interested in the content and technology, or they are not able to follow the instruction. Cathy's worry toward her future class points to the possible challenges for a teacher in teaching GTs with GSP. Her beliefs associated with such worries or concerns may produce some level of pre-teaching anxieties and they may help her prepare for coping with such challenges. The fourth point to note here is the access of students to the technology. Jack's view that students may have limited access to GSP and computers in the school while learning GTs comes from his current experience of teaching in a school. His current school, where he is doing student teaching, does not have GSP, and he is not able to use it in the class (Niess, 2005). Hence, these two preservice teachers' beliefs associated with their worries and concerns are related to the four points – teacher preparation, teaching content with GSP, student attention, and access to GSP.

Beliefs Associated with Attitude

The beliefs associated with attitude have sub-categories – developing confidence with GSP: temporal, visual, and relational; efficiency of using GSP for teaching GTs: enhancing, exploring, and understanding; and exploring GTs with GSP: surface stuff versus depth and a learning curve. Figure 9 shows the interrelation of attitudinal beliefs that may affect one's ability to use the tool for teaching and learning GTs with GSP.

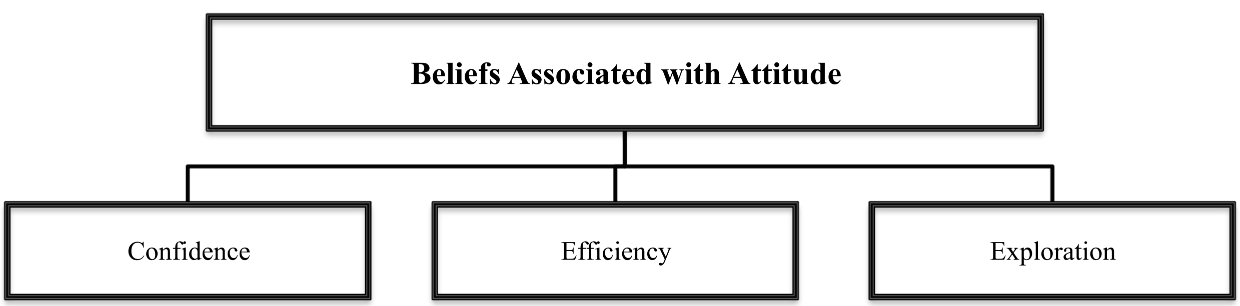


Figure 9. Beliefs associated with attitude

Developing confidence with GSP: Temporal, visual, and relational aspects

The participants shared with me their beliefs in relation to their confidence in using GSP in the context of teaching and learning GTs. The following protocol 9 is an example where they expressed their confidence in using GSP for teaching GTs.

Protocol-9

R: Do you think you have/have not developed confidence in using GSP?

C: Yes, I do have confidence in using GSP. [5]

J: Yea. I am getting there. I would be more confident when I have time to do it. I just haven't spent enough time on it. I would like to spend more time on it. [5]

R: What has brought up that confidence?

C: Foundations of Geometry class, that wonderful class. [5]

R: In your experience how does GSP help in developing confidence in learning transformations?

J: Let's say kids are struggling, even with they say graphing, but they are really good in computer. Think how confident they are gonna be if they can do this, and they can see it. They are actually gonna be able to construct it. You know, maybe a kid is struggling with it. All of a sudden that kid sees it in the computer what's happening. Maybe that just turns him around and builds his huge confidence of what he is seeing. [3]

C: Yes, but only if the teacher is able to make that because I could see it as an older teacher does not build their confidence to be saying 'yes' or 'let me check'. So if you as a teacher are confident enough then you can look at that and be able to prompt them if it is wrong. Why did you do that? ...bla bla bla... So, if you are confident enough, you can definitely build your students' confidence. If you are not confident, then they just gonna be the same. [3]

Second-order interpretation

Cathy expresses her confidence in using GSP for classroom activities while teaching GTs. She re-iterates that she learned about the use of GSP in the Foundations of Geometry course she took a year ago. She says, "…that wonderful class" meaning that for her the Foundations of Geometry was a milestone in learning about GSP. The use of GSP in problem solving and writing proofs in geometry seems to develop her confidence level high. She expresses her thought that she is confident in doing different explorations and constructions about GTs with GSP. Seeing the process of GTs in the dynamic environment of GSP developed her confidence and she seems to think that it can also help students develop their confidence with using the tool when they can see what is going on with reflection, rotation, translation, and their composites. She expresses, “If you as a teacher are confident enough then you can look at that and be able to prompt them if it is wrong.” She considers that students’ confidence in using the tool and solving various problems related to GTs with GSP depends on teachers’ confidence in doing them. She seems to believe that the teacher confidence increases the student confidence about the tool and processes while working on GTs.

On the other hand, Jack shows limitation in his confidence with using GSP for teaching GTs. He says, “I would be more confident when I have time to do it.” He considers that even when some students are struggling with content of GTs and their properties, they can be good in computers. Their level of confidence is influenced by what they see. He claims, “Maybe that just turns him around and builds his huge confidence with what he is seeing.” Even a student who is struggling in learning GTs may have good computer skills, and he or she can use GSP to learn more about GTs.

When a student sees the process of reflection, rotation, or translation in the computer, it just changes his or her ability to grasp the ideas in GTs. For Jack, when a student can see it (GT process) in the computer (with GSP) that ultimately adds to his or her confidence. Hence, Jack appears to believe that one’s confidence level depends on the opportunity to use the tool, ability to see what is going on in the processes with a GT, and how long he or she has gone through struggles during the learning. Hence, Cathy and Jack’s beliefs about developing confidence with GSP in teaching GTs are associated with temporal, visual and relational factors that provide a theoretical basis to associate thsese beliefs to the literure of teacher beliefs in the third-order interpretation.

Third-order interpretation

One can see three important points from this protocol in relation to Cathy and Jack's sense of confidence in using GSP to teach GTs. First, it is about students' ability to construct. Jack reveals his thinking that his confidence with using GSP may be reflected in his students' ability in doing construction about GTs with the tool. The second point is about what his students can see when they construct GTs with the use of GSP. If the students can see what is going on within a GT that may change their whole understanding of GTs by using GSP. The third point comes from Cathy's view about prompting on what students are doing and developing their sense of confidence. That means a teacher's confidence in using GSP for teaching GTs can be viewed indirectly from the students' ability to use the tool and follow the prompts and construct something about GTs with the tool.

Students' construction of an idea within any GTs by using GSP lends two possibilities. The first one is the construction as a mechanism to build on what he or she already knows. Students construct new ideas on their old ideas when they actually construct any polygon or geometrical figure and do a transformation (Ernest, 1986). The construction seems to be a cognitive function. The second is that construction as a transformation of students' thinking and reasoning is metacognitive function.

When students construct any geometrical object and then reflect, rotate, or translate it under a given condition, then they transfer their constructions in terms of cognitive construction through building up new ideas (von Glasersfeld, 1989). At the same time, they gain higher level of thinking and reasoning through meta-cognitive construction when they can see what is happening with what they are constructing with GSP. That means what they can see opens the door into the metacognitive world. A teacher's prompt takes them to a deeper level of understanding into the meta-cognitive world. Hence, all the three points are inter-connected in building confidence of students' learning of a GT with GSP.

Many researchers and authors (e.g., Eu, 2013; Keller, 2010) have highlighted the issue of student learning associated with their confidence in using technology. Lack of confidence in the use of GSP creates dilemma, and it leads to avoidance of using the tool for teaching and learning of GTs in the classroom (Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2015). Cathy seems confident in using the tool for teaching and learning of GTs, but she does not see it important at the beginning. She prefers prompting over using the tool to enhance students confidence in the concepts of GTs whereas, Jack despite his limited experience with GSP feels that students' creation and visualization of GT processes enhance their confidence. It is teacher's personal strategic choice of using an instructional approach where the literature remains open to all kinds of possibilities.

Efficiency of using GSP for teaching GTs: Enhancing, exploring, understanding, and being able to access

The participants shared with me their beliefs in relation to efficiency of using GSP in teaching and learning GTs. Here, efficiency is associated with the comfort and competence of using the tool and doing things in a shorter time with the ease of teaching and learning. The following protocol 10 shows an example of preservice teachers have the sense of efficiency in using GSP for teaching GTs.

Protocol-10

R: Do you think that GSP is an appropriate tool for teaching transformations?

C: Not to teach, but to enhance teaching. [5]

R: GSP is a tool for teaching GTs or learning GTs?

C: Kinda neither. I would say it is for exploring a lot like. I would teach with it, and I would not expect kids to learn from it. But, I would want them to explore and deepen their understanding. [5]

R: Do you think/don't think that GSP is useful in teaching GTs?

C: I think it could be. I have to honestly get more comfortable with the idea of teaching with it, because I have to figure out the procedural side. But, like again you can only rotate. Ok, you have to say rotate. Maybe if we can do this- 'How you find the center of rotation?'. This is how I am gonna do it. I don't know. If I have figured out the procedural side, I would feel more comfortable in teaching. [5]

R: Does the use of GSP make teaching transformations easy?

C: Yea. I don't think you have to, like if you were a poor teacher in my opinion, teach it. You can go into rotations now. Go to GSP, click rotate. Figure out what you need to do. So, I think, yes it would make really easy because you don't have to do anything. With that even if you use it use it correctly. You make sure that they know what a rotation is, not that just you go and click that button. [5]

J: I would like to think so. It depends on your access (of GSP) in the classroom. If you have an access toward showing the students actually see what's happening. If you have access, that's nice. [4]

R: Does it make it faster?

C: Yes, so much faster cause you don't have to hand draw. [4]

J: I don't think it's faster. If they already have the knowledge base, and you are just going and saying we are doing this, then yes. If you have to build on to their knowledge and how to use the program first, I think it's gonna go slower. That takes a lot of time. [4]

R: Does GSP make it more interesting?

C: To an average student that hates math and is not really into it, I think GSP makes it a lot interesting. Personally I like math, so I am gonna find interest on pencil and paper cause I am constructing it still while students don't. Early they find it tedious. They don't want to do that. They are not getting into a point fast enough. So they are getting lost. For them, it is a lot more interesting. [4]

J: Oh, it definitely makes lessons more interesting. Like as I said I think in Sketchpad you can import pictures. You can lay stuff over on top of them and show them real life applications. They can build the Ferris wheel. They can build their own stuff and see it. It's not like a picture and trying to spin it. That's just a picture. With this (GSP) they can actually build it. [4]

Second-order interpretation

Cathy seems to consider that GSP is not a teaching tool for her to use at first. However, she accepts that it enhances teaching. She considers GSP neither as a teaching nor as a learning tool. She could use it for teaching, but she does not like doing that at first. She expresses, “I would want them to explore and deepen their understanding.” She wants her students do constructions and other procedures with paper and pencil. She expresses her thought that students could be engaged in exploring other properties without going into details of constructions by using shortcuts that GSP offers. For her, GSP does not do the procedure, but it can show the concepts in a clearer way. She reveals her view that it is an easy tool for a poor teacher who is not well prepared to deal with the content. For such teachers, GSP does everything very quickly and easily. He or she does not have to go through the details of constructions and procedures. For Cathy, it lends a straight forward approach to enhance teaching GTs. Cathy expresses her thinking that GSP makes learning of GTs interesting to those students who either hate math or find it tedious. Hence, she appears to believe that GSP is not a teaching and learning tool, but it is an exploring and practicing tool for students. She says, “I have to, honestly, get more comfortable with the idea of teaching with it.”

For Jack, teaching GTs with GSP is easy if it is accessible to all the students in the class. He thinks that GTs can be taught very well by using GSP. He accepts that a greater accessibility of GSP in the classroom makes teaching of GTs easier, faster, and more meaningful. He says, “If you have an access toward showing the students actually see what's happening.” However, he considers students’ foundational knowledge of geometry and GSP as the factors of efficiency. If students are not ready with such foundational knowledge of geometry or the technology (i.e., GSP) then they may take a longer time while working with GSP. It can be interesting for some students who know the basics, but it may be frustrating for others due to a lack of foundations. Jack further expresses, “You can lay stuff over on top of them and show them real life applications." However, a teacher can make lessons on GTs interesting by using GSP and importing pictures in it, for example. The teacher can use the pictures for different GT functions, not just the geometric patterns or polygons. He seems to believe that the students can build on the things they already know by using GSP, provided that they have access to this tool.

Hence, the sense of efficiency is different for Cathy and Jack in using GSP for teaching GTs. Cathy's focus is on developing conceptual understanding of GTs before exploration of different properties with GSP. The short-cuts of GSP may not help students to understand the details of GT processes. For her, other hands-on activities with paper and pencil are more important than using GSP for conceptual teaching of GTs. Whereas, for Jack foundational knowledge of the content and technology and accessibility of the tools in the classroom make teaching and learning of GTs more efficient. These views lend a connection to the literature of teacher beliefs in relation to the use of technology for efficient teaching of mathematics.

Third-order interpretation

Some researchers and authors (e.g., Tajuddin, Tarmizi, Konting, & Ali, 2009) discussed instructional efficiency of technological tools in teaching and learning mathematics. They focused on availability (or accessibility), portability, manipulability, applicability, performability, and precision as part of efficiency of technological tool for teaching mathematics. I think that the elements associated with efficiency discussed above are associated with their attitude toward the appropriateness of GSP in teaching GTs, usefulness of the tools in GSP for teaching GTs, easiness to use the tools in GSP while teaching GTs, temporal factors related to GSP for teaching GTs, and their level of interest in the future use of GSP for teaching GTs. Cathy expresses her thought that GSP is not an appropriate tool to introduce reflection. She prefers other hands-on materials and activities to introduce GTs in her future classes. Jack, on the other hand, wants to use GSP or other materials to introduce GTs. Although both of them think that GSP is very useful in learning GTs, Cathy does not see it as useful for teaching as it is for practicing or exploring. Jack sees this tool useful for both teaching and learning.

Cathy finds it easy for students to practice on GTs and Jack finds it easy for demonstration in the class and visualization of GTs. Cathy reveals her thinking that GSP saves time by skipping details of constructions behind a GT. She considers that it provides shortcuts to a GT. Jack expresses his thought that it may take a longer time to teach GTs with GSP if he has to teach about the tool every single time. Cathy is interested to use GSP for exploring after teaching procedures and concepts of GTs. Jack is interested to use GSP for teaching concepts once procedures are done with paper and pencil. It seems that both agree on the view that students can learn geometric transformations with a greater efficiency by constructing, conjecturing, and exploring properties of GTs with GSP (Almeqdadi, 2000; Kor & Lim, 2009; Niess, 2005). The difference is only about purposeful uses and timing of using the tool for teaching and learning GTs. Hence, both Cathy and Jack believe that GSP provides a greater efficiency for either teaching or learing or practice of exploring GTs.

Exploring GTs with GSP: Surface stuff versus depth and learning curve

The participants shared with me their beliefs in relation to exploring different properties of GTs using GSP in the context of teaching and learning. Here, the meaning of exploring is associated with investigation of properties using conjectures and proofs (both formal and informal). I identified some key points of the participants' beliefs about exploring GTs with the use of GSP. These points are discussed in the second and the third-order interpretations of their beliefs. The following protocol 11 is an example to show their beliefs about the use of GSP for exploring properties of GTs. The properties of GTs discussed are associated with the geometric properties.

Protocol-11

R: Do you think you can/cannot explore properties of GTs with GSP?

C: Yes, I can and we should explore that. [5]

R: Why should you?

C: Because I want to explore, it is not just what's subject, you are not gonna want surface stuff but want to dig into the topic. [5]

R: Does this GSP help in exploring properties of any transformations or even composite transformations?

C: Yes, it does. Um, you can do, there is on reflections, it's like a Mira, stand up like this, you don't have to draw it every time. You have to set it next to it and look through it. That's the one. There you can. You kind have to look through and draw it if you gonna explore. So, that makes that part have them faster, I guess, and get the result of their conjectures a lot faster. And, if they are exploring area they don't have to go and find the height and the base to find the area (of a triangle). They can just directly go to the area because that's the conjecture, and that's what they need. [4]

J3: Um, you could talk about how the shapes are congruent. You can talk about angle of rotation. You can talk about distance between them (points), but I don't know if that's what we gonna do with rotation. [3]

I know my high school classes were of direct instructions. Kids can be bored. In a group-work, they can explore some real life stuff on their own too. It's basically keep them not just listening to me rather see them each other as groups that is more effective sometimes. [1]

You can explore like generalized formulas with them which allows a greater algebraic expression. GSP has a lot of learning curve. [1]

Second-order interpretation

Cathy seems to think that she has the ability to explore the properties of GTs by using GSP. She also iterates that a teacher has to do it in order to dig deeper into the topic. She expresses, "…you are not gonna want surface stuff but want to dig into the topic." She accepts that GSP is helpful to explore different properties of not just a single transformation, but it is helpful to explore the properties of composite transformations. For her, use of GSP makes the process of GTs faster, and this supports in proving conjectures. Also, GSP being a short-cut tool, and hence Cathy seems to believe that students do not have to go through details of steps while measuring areas, for example. She claims, "They can just directly go to the area because that's the conjecture, and that's what they need." If the conjecture is about showing that a GT preserves area, then they can directly go to measure the areas of the object and the image and compare them. Therefore, Cathy appears to believe that exploring GTs with GSP is helpful to develop conceptual understanding. However, this view contradicts with her earlier view that she would not use GSP in the classroom to teach the concepts.

In the same line, Jack reveals his thinking that GSP can help him explore the shapes (geometric structures) if they are congruent. For him, GSP helps in exploring angles of rotations, distance between points (from the object to the image), and create proofs. He sees limitations of direct instruction in terms of student motivation, and he prefers group-work for exploring real life stuffs related to GTs by using GSP. He expresses, "I know my high school classes were of direct instruction. Kids can be bored." He even thinks that students can explore the algebraic properties of reflection (or other transformations) relating to the coordinates of the object and the image points. Further, he accepts that students can generalize the algebraic expressions of a GT process. In this way, he seems to believe that GSP has a lot of 'learning curve'.

Here, Cathy and Jack seem to agree that GSP is helpful in exploring the properties of GTs. Cathy accepts that it lends into a deeper understanding the topic (properties). It also provides a shortcut to explore the properties. Jack focuses on different properties, for example, angle of rotation and distance. The process of exploring GTs with GSP is also related to ownership. When students explore the properties themselves using the tool, they own it. These beliefs may have a connection to the literature and theory of teacher beliefs and hence can be further re-interpreted in the next level.

Third-order interpretation

The major points from the above discussion are- a sense of necessity to promote exploration with GSP, deeper understanding of GTs with GSP, direct exploration without details of constructions, scope of explorations with GSP, and learning curve with GSP. Many researchers and authors (e.g., De Villiers, 1998; Dywer & Pfiefer, 1999; Keyton, 1997; Manouchehri, Enderson, & Pugnucco, 1998; Olive, 1992, 1993, 1998; Shaffer, 1995; Tyler, 1992) highlighted the use of GSP for exploring various geometric properties during teaching and learning of mathematics. The two preservice teachers in this study seem to have a sense of necessity to explore different properties of GTs with GSP. The only point of difference is about when to begin exploring the properties of GTs. Cathy considers that it could be after students know procedures and concepts. Jack considers the use of GSP could be at any time during teaching and learning of GTs. The purpose of using GSP in teaching GTs is not just to solve problems, but also "to explore the potential of GSP" (Nordin, Zakaria, Mohamed, & Embi, 2010, p. 114). Both Cathy and Jack seem to believe that GSP has a great potential to explore different properties of GTs using its dynamic features. Cathy considers that it helps students to dig deeper into a topic within GTs, and it could be linked with Van Hiele's idea of geometrical thinking (Abdullah & Zakaria, 2013). Van Hiele's five levels – visualization, analysis, informal deduction, formal deduction, and rigor could be well integrated with teaching and learning GTs with GSP (Usiskin, 1982; Van Hiele, 1985 & 1986).

While students are working on GTs with GSP, they can conjecture the relationships of sides and angles of object and image polygons (or any shapes), and they can measure them and prove them (Lange, 2002). Both Cathy and Jack seem to believe that GSP can help students in doing these proofs (See protocols 9 & 10) (Santos & Machado, n. d.). The next point to be noted here is the real-world application of technology, not just solving geometry problems with GSP. Real-world application can be "practice of drawing, develop visualization skills, compare real-life objects, communicate with geometric terminology, and interact with the software" (Marinas & Furner, 2006, p. 159).

Another point is related to generalization of formulas. In geometric transformations, we (this researcher and the participants) discussed some algebraic manipulations using coordinates and matrices. Jack expresses that use of GSP can be helpful in generalization of algebraic formulas for reflection, rotation, and translation. However, they did not reach that far except the manipulation of algebra of reflection on coordinate axes and lines such as Y = X and Y = -X. The last important point to note in this discussion is 'developing a learning curve.' Jack expresses his thought that GSP offers so much of a learning curve while teaching GTs. It seems that the notion of the 'learning curve' is very powerful in terms of using GSP for teaching and learning of GTs. The learning curve may follow Van Hiele's (1985) levels- visualization, analysis, informal deduction, formal deduction, and rigor or it may go along another path of learning GTs with GSP in a non-linear fashion. The cluster of geometrical thinking may not follow Van Hiele's (1985) model (Bennett, 1994; Clements et al., 1999; Mayberry, 1983; Burger & Shaughnessy, 1986). However, an exploration of GTs may turn into a foundation for constructing proof about congruence or similarity (Giamati, 1995). Hence, Cathy and Jack's beliefs about exploring GTs with GSP have potential to extend the pedagogical affordance of the tool. For Cathy, GSP provides shortcut processes for studying properties of GTs, and for Jack it has a lot of learning curve.

Beliefs Associated with Cognition

The beliefs associated with cognition have the sub-categories – conjecturing about GTs using GSP: conservation of properties and articulating what is done; supporting and engaging students in learning GTs: describing, holding attention, and laying out steps; and understanding GTs with GSP: skipping or quickening steps, visualizing, and solidifying (Fig. 10). These cognitive beliefs are related to cognitive functions, for examples, conjecturing, reasoning, supporting, engaging, and understanding. I discussed each of them under separate sub-headings.

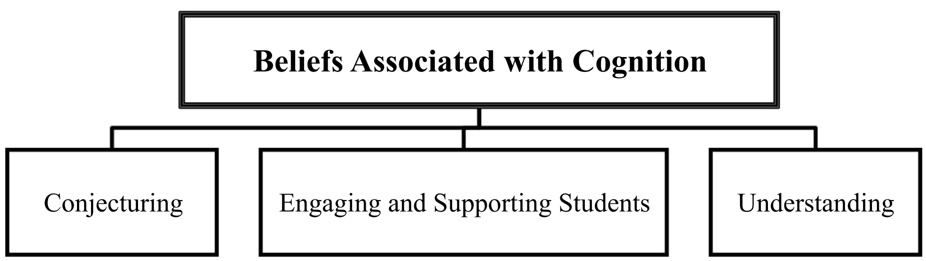


Figure 10. Beliefs associated with cognition

Conjecturing about GTs using GSP: Conservation and articulation of properties

The participants shared with me their beliefs in relation to making conjectures for teaching and learning GTs with GSP. Conjecturing is a part of teaching and learning mathematics in which students and the teacher may guess a relationship, property, or a solution to a problem. It is like a hypothesis in a research. The difference is the scope. A conjecture has a narrow scope within a concept or a problem whereas a hypothesis has a broader scope and stronger impact. The following protocol 12 is an example that shows what they believe about conjecturing. The conjecturing in this section are related to their view of making conjectures from the constructions they did in the figures 2 (reflection), 3 (translation), 4 (rotation), and 5 (composite transformations) at the beginning of this chapter.

Protocol-12

R: What conjecture we or students can make from this (construction of object and image under a rotation)?

C: As in like angles stay preserved, side lengths are preserved. Orientation is not preserved but, perimeters preserved, areas preserved. I think that would be more difficult for them, I guess. I think them explaining, easily explain and come up with, I guess not a proof, but a kind of proof about side lengths. They can come up with that and make it very, almost exact, they can see it, know it why is that, but then you get to the area, I think even the orientation changing, the area would look different for them. So they would have difficulty in that in perimeter even with explaining that all side lengths are the same. They might have more difficulty than saying getting from all side lengths are the same, what does that mean about the perimeter. [3]

J: It gonna be pretty much the exploration or that kind of thing. I would choose angles, how they are related to each point and each object. Even you could talk about with angles and sides and how they are the same. [3]

R: So what conjecture could we or the students make about this composite transformation?

C: Like the area is congruent throughout these shapes, or perimeter or side lengths, like you can reflect down into. [4]

C: They can explain to me like what they are doing. So, like they decided that they are gonna explore the area. So, they can say based on their construction when they reflected over this line, they may translate it, and they rotate it. Um, that it is a construction. So, they don't move independently. And, like I guess explain what they did. They decided that they were gonna measure the areas. And areas depend on base and height cause this is a triangle, and it depends on one half base times height. Instead of measuring the bases and creating heights they decided to just measure the areas because they knew that what the area is meant. The triangles are all the same. [4]

J: Probably it just maintains all those properties. When you get into the translations students like to list the properties each one maintains different things, reverses orientations. So, that they could talk about that. [4]

I would choose angles, how they are related to each point and each object. [3]

It would be interesting to see if they can actually put it into words what they are seeing. It would be interesting to hear their own language what they say. That would be a kind cool to hear. But, I don't know I mean you would see how they understand congruence and similarity, equal angles and linearity. You can see if they really understand those kinds of things that making by them. Write something or vocalize what you are actually doing. One thing I think just kind of fun to make them do like a presentation. If you have the opportunity to put that on your screen up there and have them try to tell in class to other. Sometimes they put into words what you can't, like another student understands them from reflecting, like to do jigsaw such a fun. [4]

Second-order interpretation

Cathy considers that GSP can serve as a tool for making and proving conjectures related to different properties associated with GTs. She seems to think that it is possible to make and test the conjectures about side lengths, angles, orientations, areas, and perimeters. She accepts that these conjectures are helpful in understanding the properties of different GTs. However, she does not think that students are able to prove these conjectures in a formal way (See protocol 3). She expresses, "…they can say based on their construction when they reflected on this line, they may translate it, and they rotate it." They can verbally explain their task, and they can explore by measurement that area under any GTs (within reflection, rotation, and translation) is preserved. For this process, they do not have to go through the measurement of bases and heights of triangles to find their areas. They can use the built-in tool for area measurement. She explains, "Instead of measuring the bases and creating heights they decide to just measure the areas because they know that what the area is meant." She accepts that they can give verbal explanations of the conjectures and discuss them with measurements, but that is not a formal proof, she contends. Hence, she seems to believe that making and proving conjectures about GTs is possible by using GSP; however, the students (high-schoolers) are not able to do it in a formal way. She claims, "I think them explaining, easily explain and come up with, I guess not a proof, but a kind of proof about side lengths (informally)."

Jack also seems to think that it is possible to use GSP for making conjectures about different geometrical properties associated with GTs and proving them. He expresses his thought that students can use GSP to explore different geometric properties about GTs. He says, "Even you could talk about with angles and sides and how they are the same." He expresses his thought of these properties, for example, congruence, similarity, lengths, and angles, to be preserved under a GT. However, he is not sure about how the students can prove these conjectures by using the tools in GSP. He claims, “It would be interesting to see if they can actually put it into words what they are seeing. It would be interesting to hear their language what they say.” He expresses his thought that it would be good to hear the students’ explanations of proofs about conjectures. Even he considers making presentations by students as an approach to sharing their verbal explanations of proofs to others. Sometimes their words could be more interesting than the teacher. Hence, he seems to believe that making conjecture is an important part of teaching and learning of GTs with GSP within the limitations of students’ ability to produce formal proofs, not just a verbal explanation.

Both Cathy and Jack accept that the use of GSP can be helpful in making and proving conjectures about properties of GTs. Cathy seems to focus on properties of lengths, perimeters, areas, and orientations. However, she thinks that it might be difficult for students to prove them. Then she focuses on students’ explanation as part of their informal proofs of conjectures. Whereas, for Jack, students would relate points, angles, sides and each object on the image and pre-image. He is also interested in seeing if students can put their proofs into words what they are seeing in the processes of GTs by using GSP. These views provide some foundations for theoretical interpretation of their beliefs in a broader sense in the third-order interpretation.

Third-order interpretation

Some of the key ideas emerged from this discussion are- conjectures about preserving different structures (sides, angles, areas, and perimeters), process in the conjecturing (construction, measurement, and articulation of the process), and sharing information (making presentation in the class or doing a jigsaw). Hoyles and Jones (1998) state that, "In developing dynamic geometry contexts the following factors are critical - encouraging the learners to make conjectures focusing on the relationships between geometrical objects and providing the means for the learners to explain their actions and their results" (p. 124). This view is clearly reflected in Cathy and Jack's reflection about making conjectures related to different properties that are preserved under a GT. Crane, Stevens, and Wiggins (1996) discussed different kinds of conjectures that can be explored with GSP. Some of these conjectures are related to angles (vertical and exterior), lines (lines in a pair, parallel lines, and tangents to circles), and polygons (triangles, quadrilaterals, trapezoids, rhombus, parallelograms, and rectangles). These conjectures could be a part of the discussion even in any GT process with GSP. They (students) may discuss different kinds of processes involved in the conjectures in terms of construction of geometrical structures, and measurement of angles, lengths, and areas.

According to Giamati (1995), students make conjectures, and then they vocalize them through peer sharing or group sharing. She mentions, "….students can use it to test a wide variety of conjectures once they have mastered this construction tool" (p. 456). Students can make a variety of conjectures, and all of them may not lead to a reasonable proof or verification. In such a case, students may learn why the conjecture was not reasonable (Giamati, 1995). Giamati (1995) further admits that her students gained "deeper understanding of the problem using their scripts to explore it and make conjectures than they would have if the results had merely been explained to them" (pp. 457-458). Tamblyn (2009) proposes important pedagogical strategies in relation to students making conjectures and proving or verifying them. He proposes that students' conjecturing that lead to problem solving and class discussion may help in examining their conjectures. In these activities, he finds GSP as a tool for exploring and investigating different relationships. Hence, Cathy and Jack's view of making conjectures, explaining them, and sharing them in the classroom have a connection to the literature as important aspects of teaching GTs with the use of GSP.

Engaging and supporting students in learning GTs: Describing, holding attention, and laying out steps

The participants expressed their beliefs about engaging and supporting students while teaching GTs with GSP to me. The following protocol 13 is an example.

Protocol-13

R: What do you think would engage them (students) while learning GTs with GSP?

C: Um, honestly I think the assessment at the end, like the exploration, is good enough. Like for exploring I don't think if we can take it out too long, but I think the assessment would be right after it. I think them being able to draw a picture and describing what that picture is. More like being able to make their own picture instead of saying draw a triangle and rotate the triangle. But, like doing that and then maybe using only rotation or using the transformations we have already used. Make your picture cool. I don't know how that instruction would be, but then can be- reflect it and rotate it. [3]

J: I can just think about what's gonna interest them, what's gonna hold their attention. If you are just talking about rotation (or any GT), you are just using formal terms, I think you gonna lose their attention in ten minutes. One of the cool things about GSP is that's where video games come from. Kids love video games. You can talk about, pull them, and see what kinds of games are like. Even you can find a YouTube clip to show them and how they pick out a rotation. You can show them that's where it is coming from. [3]

R: How will you help your students to test and explore that conjecture (about area of object and image polygons under a transformation)?

C: Um, so, start them out with a polygon or a line. Maybe a triangle, cause triangle is easier to see and find the area of. So, you can take the measure of each side, find the length of each and compare them. They can measure all of them. [3]

C: I don't think that they understand proof. Or I think they would think that construction is the proof. I think they could verbalize may be a proof. I don't know if they can make any formal but may be verbalize, yea. [5]

J: You can design an activity in which they are doing these kinds of stuffs. I would give them step-by-step instructions so they get it. It saves time. Because one thing about Sketchpad is it's time consuming if you are having them reteach the program. If you put step-by-step instruction in front of them and you can walk around and help them with it if they need. I think that would be better and it saves time. We have a little time in the classroom for that stuff. [4]

Second-order interpretation

Cathy considers formal assessment as a tool to engage students in learning. She claims, "Like for exploring I don't think if we can take it out too long, but I think the assessment would be right after it." She appears to think that the assessment should be done right after the teaching and learning activities so that students are engaged in the process of learning. Most often a teacher can move around and see what students are doing. A teacher can use students' construction of any GT as a means to assess their learning and engage them with it. For Cathy, a teacher's prompts like 'rotate it and reflect it' and students' responses to the prompts can be helpful tools for continuous assessment of learning in the class and motivating and engaging them in the learning. She says, "…you can take the measure of each side, find the length of each and compare them." She focuses on engaging and helping students in measuring and comparing side lengths of object and image polygons. Cathy seems to think that helping students in making conjectures and proving them are important aspects of teaching and learning GTs with GSP. She expresses her doubt that students are able to prove a conjecture in a formal way although they may talk and verbalize the proof. She expresses, "I don't think that they understand proof. Or I think they would think that construction is the proof."

In a similar vein, Jack considers that a formal assessment of learning can be done by engaging students in hands-on constructions with GSP. He considers that such constructions related to GTs with GSP allow students to see what is going on, and they can explain it to the class. While doing this, in his view, student interest and attention should be the central aspect of teaching and learning. Otherwise, the teacher may lose the attention of the students within a few minutes. He claims, "I can just think about what's gonna interest them, what's gonna hold their attention." Hence, he accepts that the assessment is a part of engaging students in making connection with other relevant things (in life). He explains, "Even you can find a YouTube clip to show them and how they pick out a rotation. You can show them that's where it is coming from." That means a teacher can pull out relevant examples from other media files to demonstrate animation features and can link them with GSP and GT process. He seems to think that step-by-step instruction could help students to follow thoroughly and can reach a level where they can begin their work independently. He expresses, "If you put step-by-step instruction in front of them and you can walk around and help them with it if they need."

Cathy and Jack expressed their beliefs in relation to student assessment as a part of engaging and supporting them in learning. They seem to accept that construction should not be left without description and students' explanation of their construction will help them make sense of what they have done. Also, student interest and attention are key factors in engaging them in meaningful way, and finally, visualization of the GT process motivates them in making connection between mathematics and the real world. These views can be interpreted further with the literature on teacher beliefs in the third-order interpretation.

Third-order interpretation

From the above discussion, one can pick up some key points related to what engages students while teaching and learning of GTs with GSP. Some of these points are- assessment, construction followed by a description, interest and attention, and visualization of how video games are built. These ideas offer different ways to engage students in learning. Tamblyn (2009) prefers hands-on activities for engaging students instead of just presenting information through constructions, graphs, and tables. Hollebrands (2007) states that students can be engaged in different mathematical activities where they not only construct image and pre-image (object) under a GT, but they also interpret the results and compare (assess) the results when their construction changes by dragging.

These activities of engaging students through constructions and explanations may lead to the mathematical and logical abstractions at three levels – “empirical, pseudo-empirical, or reflective” (Hollebrands, 2007, p. 165). The experience of moving an object and observing corresponding change in the image may relate to empirical abstraction. The experience of change in the area or perimeter or orientation while making change in the side lengths (by dragging a vertex) could be a pseudo-empirical abstraction. Here, the student may observe direct change in length (empirical) and can abstract indirect change in the surface area (pseudo-empirical). At the depth of this process, the students may observe the changes in the side lengths, angles, areas, and perimeters together with his or her own action and the process that lead to the changes. The abstraction seems to be reflective abstraction that takes place at a profounder level than just empirical or pseudo-empirical abstraction (Hollebrands, 2007). That means students can be engaged with the actions and operations of GTs with GSP at different levels. The last elements as noted earlier ‘interest and attention’ and ‘visualization of how video games are built’ may lead toward the reflective abstraction.

Some other points discussed above are- lack of understanding a proof, measurement for exploring properties, step-by-step instruction, and walk around and help students. Jiang and McClintock (1997) view that university teacher educators need to create an environment for preservice teachers to “ develop new learning styles, performing investigations and experimentation, using the results of investigations, formulating thoughtful conjectures, verify the conjectures, discuss the possible extensions, generalizations, and applications" (p. 129). They seem to think that the use of GSP can help in achieving these objectives in mathematics teacher education. They also accept that the use of GSP can help students in "geometric intuitions" (p. 134). Hence, teachers can help students in the process of "investigate-conjecture-verify" (Jiang & McClintock, 1997, p. 135).

So far there is a concern about proof and Cathy's view that students don't understand the proof, it is noteworthy to make a connection to Bucher and Edwards (2011). Bucher and Edwards (2011) state that, "Although we discuss the importance of proof frequently in our geometry courses, curiously we ignore our own advice when teaching students about geometric transformations" (p. 716). They also claim that textbooks also don't focus much on formal justification or proof about geometric transformations. That means the proof has not been a central part in high school geometry (in the US). This resonates with Cathy's voice, "I don't think that they understand proof." This issue points to the current practice of "algorithmic approach to transformational geometry- one that omits rigorous proof- misses the opportunity to connect the rigid motions to a host of other geometric topics" (Bucher & Edwards, 2011, pp. 716-717).

The measurement of sides, angles, areas, and perimeters may lend an easy way for teachers to teach GTs with GSP, and the same view is reflected in Jack's expression. However, the students' understanding of the properties may still be incomplete due to a lack of logical/abstract proof. The step-by-step process may provide a clear pathway for the students to follow, do constructions, and observe properties. This kind of practice seems to be linear, and teacher directed. Nonetheless, this approach provides students clear direction and guidelines to begin from a certain state and end at a certain point (through measurement) without loss of time and without being confused. Cole (1999) states, "As a teacher, you are always assessing your students. You walk around as they work on the task or projects, observing groups, conversing with students, spot-teaching concepts and skills, and checking for understanding" (p. 224). Cole (1999) further says, "Teachers walk around, observing and talking with students, for many reasons, making sure that the students are doing what they are supposed to is just one of these reasons" (p. 225). Cole connects walk around activity with making observations, doing on the spot assessment, making notes, and balancing equity in the classroom.

Engaging and supporting students are central to the effective pedagogical practice that may enhance student learning. Cathy and Jack seem to believe that students can be engaged in exploring and assessing GT processes with GSP. Hence, the issues pointed earlier in relation to Cathy and Jack's beliefs may have a connection to the literature in an implicit way, but it needs more work to make it explicit.

Understanding GTs with GSP: Skipping or quickening steps, visualizing, and solidifying

The participants expressed their beliefs about teaching GTs with GSP and reflected their views in terms of confidence, a sense of efficiency, and the process in exploring different properties of GTs. The following protocol 14 is an example.

Protocol-14

R: Do you think that GSP is helpful in the procedural understanding of rotation? How?

C: Yes, I think so. I do think that paper and pencil is very important. So I would like to, after this, go and maybe give them paper and pencil type situation and say now rotate this shape over this and then make sure that they really, actually, know what they did, but I think they have to mark the point where they gonna rotate. They have to graph the figure and rotate the figure. I think that procedurally it (GSP) skips steps, but not really skipping steps. It is just quickening the steps. [3]

J: That it gets cool. It's because I can see exactly what's happening. Especially, if you can show how to do animation where they can see it's spinning. It is procedural. You rotated through the seventy- five degrees where's my point gone and I think that's kinda cool. [3]

R: So, what about conceptual understanding of rotation (by using GSP)?

C: I think that's why I would go back to the paper and pencil to see that they actually get the conceptual. It won't take you that long to do that in paper and pencil. With paper and pencil you have to absolutely know how to do it. So, while GSP would give you conceptual understanding with the assessment part. I would show them paper and pencil activity and solidify; yes you did know conceptually what you were doing. [3]

J: I think it helps with the concept of the angles, especially when you draw something like this, where it is seen, which angle which, and what's happening. I think that helps with the concept of a rotation. Most of them know what a rotation is just by the definition of the word. But, can they visualize it? Being able to draw the lines and measure these angles up here I think that really helps. [3]

Second-order interpretation

Cathy expresses her thought that GSP skips steps of details of procedures. That's why she prefers paper and pencil activity first to develop procedural understanding of any GT. She claims, "I do think that paper and pencil is very important." That means although she values using GSP in the classroom practice of GTs, but she prefers paper and pencil activities at first to give the concepts. The dynamic feature in GSP can show what happens when object is reflected, rotated or translated. However, the detail of the GT process may not visible in the shortcut tools. For this, one has to use GSP to demonstrate the process, not just the product. In that sense, she considers that the animation in GSP can develop a better understanding of the GTs. The animated process in GSP can make the processes of GTs visible and dynamic. It also helps in the measurement and development of concepts of GTs once procedures are done. For Cathy, GSP seems to skip the steps, but it is making steps faster rather than making them shorter. She expresses, "I think that procedurally it (GSP) skips steps, but not really skipping steps. It is just quickening the steps."

On the contrary to Cathy, Jack seems to consider that the use of GSP could enhance both procedural as well as conceptual understanding of GTs. For him, what we see happening in relation to GTs with GSP is procedural. The animation can show what is going on, as in spinning. At the same time, he considers that GSP helps in understanding the concepts of change in angles with different GTs, especially rotation. For him, an understanding of GTs with GSP is related to one's ability to see what's happening. He claims, "That it gets cool. It's because I can see exactly what's happening." He expresses his thought that students' ability to construct lines and image of an object under a GT and measure the angles (and possibly sides) helps them in making sense of what is happening. He reiterates the act of seeing in relation to understanding. He claims, "…when you draw something like this, where it is seen, which angle which, and what's happening. I think that helps with the concept of a rotation." Hence, he seems to believe that GSP is helpful to develop both procedural and conceptual understanding of a GT through ability to visualize and measure the properties (See protocol 5).

Hence, Cathy and Jack expressed their beliefs in relation to understanding of GTs with the use of GSP at four levels – indirect process (quickening steps), direct process (animation), spatial understanding (visualization), and reasoning (measurement). These levels have been further re-interpreted in relation to the relevant literature and theory of teacher beliefs in the third-order interpretation.

Third-order interpretation

Some key aspects of reasoning about GTs with GSP from this discussion are- skipping versus quickening steps, animation to see what happens, visualization is part of understanding, and measurement as part of reasoning. Many researchers and authors (e.g., De Villiers, 1999;

Flores, 2010; Khairiree, 2006; Saunders, 1998) highlighted the different aspects of understanding geometry with GSP. De Villiers (1999) talks about five aspects of understanding geometric proofs- explanation, discovery, verification, challenge, and systematization that help in developing an understanding of geometrical relationships, in general. It seems that these aspects are equally valid for understanding different GTs with GSP. Hollebrands (2004) discusses intuitive understanding of geometric transformations with GSP. She highlights key aspects of understanding GTs in terms of knowledge of transformations (both procedural and conceptual) – reflection, translations, and rotation. Her notion of understanding GTs is related to students' ability to construct object and image with appropriate reference. These references are – line of reflection, point of rotation, and vector of translation. She expresses her thought that students' understanding of these transformations is related to their ability to construct, conjecture, measure, verify, and explain (Hollebrands, 2004).

Skemp (1976) introduced two different forms of understanding – instrumental understanding and relational understanding. Usiskin (2012) thinks that instrumental understanding is related to procedural understanding, and relational understanding is related to conceptual understanding. For Usiskin (2012), mathematical understanding may include both instrumental understanding (i.e., procedural understanding) and relational understanding (i.e., conceptual understanding). Many other researchers and authors (e.g., De Villiers, 1999; Flores, 2010; Khairiree, 2006; Saunders, 1998) have discussed the role of the dynamic geometry environment of GSP for developing a deep understanding of geometry in general. It is extremely important to conceptualize different understanding of geometry in terms of visual understanding (understanding from observation), real understanding (understanding at depth), shallow understanding (surface level understanding), differential understanding (understanding that helps in differentiation of properties), computational understanding (understanding from calculations), and spatial understanding (understanding of space in terms of distance, area, and volume). Cathy and Jack did not make these different understandings explicit in their narratives; however, their accounts of future teaching of GTs with GSP have many of these elements of understanding (See protocols 5, 8, 9, 13, 18, & 16). It seems that GSP can help teachers and students to develop both procedural and conceptual understanding. One needs to unfold the nature of both kinds of understanding with the use of GSP as a tool (for the concept) and an artifact (for both the concept and procedure).

Beliefs Associated with the Environment

The beliefs associated with the environment have the sub-categories- classroom context: recognition and contradiction; student-content relationship: making another point of connection and real life application; student-student relationship: glorious conversation, hindrance, and helping out; and student-teacher relationship: being independent (Fig. 11). I discussed each of these sub-categories under separate sub-headings with protocols as the first-order analysis, discussion on the narrative as the second-order interpretation and connection to literature as the third-order interpretation.

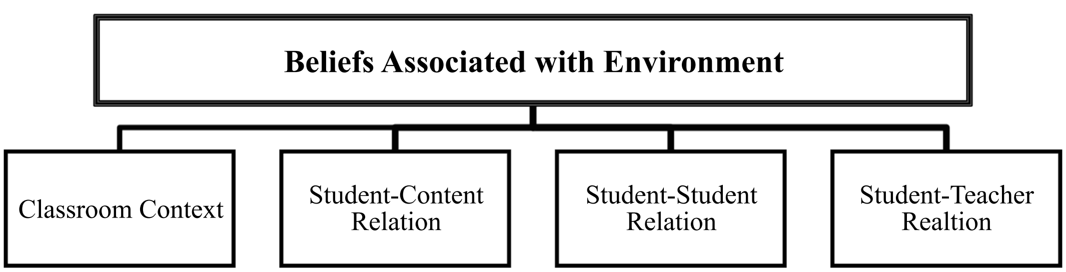


Figure 11. Beliefs associated with the environment

Classroom context: Recognition and contradiction

The participants expressed their beliefs in terms of using GSP and developing a classroom context for teaching GTs with GSP. The following protocol 15 shows an exampl of their beliefs about classroom as a context.

Protocol-15

R: What kind of environment is good for teaching and learning of geometric transformations?

C: Um, like a classroom set up, that I need a positive learning environment. I think that they need the ability to recognize their own and in groups, like simultaneously. So you could work at a table, the table would lend itself to both of those really well. And, like discussion being an important part of the classrooms like your students can't be afraid to talk during the talking time. You have to be able to distinguish between 'here is your personal work time to make your own cluster of thoughts'. And 'here is partner time or whatever you can share with others' and make them grow into whatever direction. [4]

J: More technology would be better, I think. You could have just the Sketchpad, a computer lab, or every kid has a computer, that's key. Other time you are sharing the computers you get a kid off-task, or he doesn't learn it. So I think of an environment where every kid has a computer, or even if like I have this up on the screen they can see it and everybody will watch it. I think it's like a normal classroom. [4]

R: Do you think that GSP creates a better teaching and learning environment?

C: I don't know about that. I don't think I would say it does. I think you and your class develop environment. So, I don't know if GSP would...I mean it would affect the environment in some way. But, I think you could make it great even without GSP. [5]

J: I don't think it's better, it's just different. [5]

R: Do you think that GSP enriches classroom environment?

C: Yes. It would make your work more efficient and effective. Efficient, you are not gonna take twenty minutes for construction, you just gonna get it. And, you are gonna get more ideas to yourself faster. When you do have time to share it, you have something go back to and share instead of like- 'ok I made one construction this is what I was thinking about the different construction. You have it; you can share your thoughts because you make them'. [4]

J: I do. I think I like the visualization. How you can visualize the angles, like you can put the angles in here so that they can see it right out there, and side lengths. I think that's cool. I think it really can help build on it. [4]

Second-order interpretation

Cathy expresses her thought that the classroom setup with a positive learning environment is important for teaching GTs with GSP. She also considers students recognizing self (who they are) and others (in relation to them) as part of the classroom environment. When students recognize self and others, they can participate in group activities in a responsible way. They can collaborate in learning with each other. Cathy seems to believe that the classroom discussion in groups creates a positive learning environment. She claims, "….discussion being an important part of the classrooms like your students can't be afraid to talk during the talking time." However, students should be able to distinguish personal working-time and group working-time in the class as part of the environment. Hence, she appears to believe that the students' fearless talks about their learning during talk-time may help them in making cluster of thoughts. Cathy seems to consider that being aware of technology is not a part of building the environment. She expresses, "I think you and your class develop environment. So, I don't know if GSP would...I mean it would affect the environment in some way." She considers technology (e.g., GSP) as a tool that students and teachers can use in teaching and learning. However, for her, teacher and student create the environment, but not the technology such as GSP. Hence, she seems to believe that GSP may affect the environment, but does not create it. She expresses her thought that teaching and learning environment could be created even without GSP.

On the other hand, Jack focuses on technology in a classroom as a part of the environment. He expresses his thought that varieties of technological tools may create a better classroom environment. For him, even students' sharing of computers in the classroom creates a collaborative environment. He claims, "More technology would be better, I think. You could have just the Sketchpad, a computer lab, or every kid has a computer, that's key." He even seems to believe that all kids having their computers in the classroom may help them in creating a better learning environment. However, sharing a computer among kids may take them off-task.

He says, “Other time you are sharing the computers you get a kid off-task or he doesn’t learn it.” Jack accepts that GSP enriches the teaching and learning environment when every kid has a computer. For him, GSP makes work of constructions efficient, and it helps students and the teacher build on it. Also, GSP helps students (and teacher) for quick constructions of GTs. Moreover, for him, GSP supports in creating environment through visualization (See protocol 5 & 14). Hence, he appears to believe that GSP aids in building a different (not necessarily better) teaching-learning environment for GTs through construction and visualization.

Hence, Cathy and Jack seem to believe that a positive learning environment is a key for teaching and learning of GTs. For this, the students as members in that environment should respect the individual and group potential in the classroom. They have to respect for the personal and group working time. They may participate in the construction and visualization of GT processes by using the dynamic environment of GSP. These key points of their beliefs can be further re-interpreted in the third-order interpretation in relation to literature and theory of teacher beliefs.

Third-order interpretation

Some key points from the above discussion are – positive learning environment, recognition of individual and group ability, personal and group working time, construction and visualization, and technological environment. There are different things that GSP can offer to create a classroom context for teaching GTs - it brings cohesiveness to the content and pedagogy, connects concepts across geometry-measurement-algebra, helps students in understanding the meaning of what they are doing, helps in exploring relationships, makes invisible (abstract) mathematics visible to the students, helps the teacher to grow up (by improving pedagogy), teachers get empowered, builds a connection between constructions and concepts, helps students in making and testing conjectures, and extends pedagogical ability of the teacher (Key Curriculum Press, 2014).

There could be three key elements that GSP adds into this environment, for example, input-environment, process-environment, and product-environment. In terms of the input-environment, the use of GSP helps students to begin either independent or guided discovery of various procedures and concepts associated with GTs. For example, some students who are proficient in using GSP can directly jump into the tools of transformations and explore reflection, rotation, and translation. For those students who are not proficient in the use of GSP, the teacher can provide support in terms of step-by-step instruction or a directed activity that can lead them toward practice of GSP tools before exploring the properties of GTs. All the students have their efficient entry points for learning GTs with GSP at their pace. Hence, GSP can be a part of the input-environment that may influence pedagogical goals within three dimensions - "affective, cognitive, and reflective" (Dastgoshadeh, Javanmardi, Nadali, & Jalilzadeh, 2011, p. 336). In terms of process-environment students are engaged in making sense of different GTs with the use of GSP. Students investigate different relationships between the object and image under a GT. The teacher and students use GSP for optimal learning. The teacher may differentiate his or her instruction based on students' ability. He or she may provide student support for a better procedural, conceptual, and other pedagogical needs. The output-environment is related to how students assess their learning and how the teacher uses the assessment for designing further teaching and learning activities. Hence, in every step there is a role that GSP can play in creating a smooth teaching and learning environment.

Finzer and Jackiw (1998) and Finzer and Bennett (1995) highlighted three key roles that GSP can play in creating a classroom context for teaching and learning geometry in general and GTs in particular. It creates an environment for 'direct manipulation, continuous motion, and complete immersion.' Students are engaged in direct manipulation of properties of GTs without going further details of constructions. The 'transform' tools directly take student toward the construction of the image of an object under a GT. The students can make any changes by holding a vertex and dragging it in any direction and distance and can observe the corresponding effect on the image. When the students are completely immersed into the dynamic environment of GSP while working on constructions, measurements, conjectures, and verifications, they see the technological world reflecting the real world. The hands-on mathematics can be translated into minds-on and vice-versa.

Gray (2008) claims that teachers can transform their mathematics teaching and learning environment in the classroom from a teacher-centered to a student-centered one with the use of the technological tools, for example GSP. Gray (2008) further iterates that students may have a personal control in their learning of GTs with GSP and a have greater fun and ownership of what they learn. In such environment students respect the incorrect solutions to a problem and they seek the meaning of the process, not just the correct answers. In this sense, the use of GSP for teaching GTs may help in conjecturing, experimenting, testing, and proving various relationships about GT processes. Cathy and Jack seem to be aligned with these views to some extent in an implicit way. More research needs to be done in order to explore the dimensions of classroom environment and the influence of GSP in that environment while teaching GTs.

Student-content relationship: Making another point of connection and real life application

The participants reflected their beliefs in terms of student-content relationship in teaching GTs with GSP. The following protocol 16 is an example.

Protocol-16

R: What about relationship between students and content (with GSP)?

C: I think it gets into another level and another way of getting into the contents. [5]

I think it would enhance it quite a bit and make it stronger because you have another point of connection with the content. So, again if you have only one, some of them gain nothing, but when you get three or four into something that is so interactive. Hopefully, every single student will have built relationship with the content and GSP will only gonna further that relationship. It's another way to interact with it. [4]

J: I like that because that's where you get into the real life applications. And, they are building on it. 'Hey, I want to be an engineer. Look, what am I doing?' You know, or do anything that they can see it. 'Hey, I want to be a video game tester'. That's what all the kids want to do. They can see it. This is where the foundation is. You can't (do it) if you don't understand transformations. You are not gonna build video games. [4]

Student-content, yea because it's gonna be visualizing and seeing what they are doing. [5]

Second-order interpretation

Cathy expresses her thought that GSP provides another level of student-content relationship. She expresses, "I think it gets into another level and another way of getting into the contents." She considers that use of GSP is a way of getting students into content. She considers that GSP enhances student-content relationship by providing an alternative resource to interact with content in GTs. She explains, "So, again if you have only one, some of them gain nothing, but when you get three or four into something that is so interactive." This explanation shows that Cathy seems to consider group-work as interactive. Such interaction may "further that relationship". Hence, she seems to believe that every single student may have the opportunity to build a positive relationship with the content through the use of GSP.

In the same line, Jack seems to believe that GSP bridges contents with the real life applications. He reveals his thought that students can build on real life application of GTs using tools in GSP. For him, the use of GSP helps students to see the connection between contents and what they want to learn. He uses the metaphors of 'engineer and video game tester' to make sense of what students can see in terms of applications of dynamic features of the tools in GSP. The video game examples may motivate students toward learning GTs with GSP. However, he accepts that a foundation is necessary to build up a connection between students and contents. He claims, "You can't (do it) if you don't understand transformations." Hence, he seems to believe that the students can see the processes of GTs to build a foundation. He also relates the doing-side (action of construction) with visualization through animation and understanding of GTs with the use of GSP. That means he appears to believe that the student-content relationship should be a positive one for the effective learning and that could be possible through the use of GSP.

Hence, Cathy and Jack seem to believe that GSP provides students and teachers another way to getting into the content of GTs. It helps them in building on what they already know about geometry of points, lines, and angles. They also seem to accept that use of GSP can be instrumental in making connection to real life applications, for example construction of a Ferris wheel and animations in video games. These aspects of their beliefs can be further re-interpreted in the third-order interpretation by making connection to the literature and theory of teacher beliefs.

Third-order interpretation

Some key points from this discussion are – another way to getting into the content, building on what students already know, and making connection to applications. It seems that student and content relationship can be developed at three levels – foreign relationship, familiar relationship, and ownership. Here, the foreign relationship between student and content could be understood as the relationship being distant, disconnected, and with a lot of misconceptions. At this level of relationship, students may not feel mathematics in general and GTs in particular are in their domain of interest. They might not see the connection of mathematics to their future (or career). They may not have experience in using GSP to learn GTs (or other contents). There can be a negative image of mathematics in general in the minds of the students (Belbase, 2013; Tall & Vinner, 1981). This kind of image may be the result of an "anxiety or failure" or "extremely negative attitudes" (Maxwell, 1989 as cited in Belbase, 2013 d, p. 232).

In the next level, familiar relationship, students begin to build up a positive relationship with the content. They may think of learning mathematics (or GTs) is important. They can see and understand the value of learning GTs with the use of GSP. They can make sense of most of the GTs with GSP, but they may not be yet well convinced that it is their world. They may still have doubts, confusions, misconceptions, and anxieties of learning GTs and also the use of GSP to learn it. At this level, the students may have a positive image toward mathematics, however there might be a low motivation to learn it due to low trait anxiety (Miller & Michsel, 2004). At the third level, students may have built up a good relationship with the mathematics content in general and GTs in particular. They may feel that mathematics is a potential career. They have a deeper interest in learning GTs with the use of GSP. They seem to love constructions, explorations, conjectures, verifications, and challenges in investigations of GTs with GSP. They may be completely immersed in the world of GTs through the dynamic environment of GSP. The students may have an image of mathematics as infallible, no negative anxiety but a high selfesteem, and a positive attitude towards mathematics (Belbase, 2013 d). At this level, they have an ownership of learning and understanding GTs with GSP.

Only a few researchers and authors (e.g., Phillips, 2009) have mentioned this kind of relationship as an imporntant aspect in teaching and learning. Phillips (2009) noted that "…teacher-student-content-context relationship that are necessary for matheamtics to be embraced and learned" (p. 279). McLaughlin, McGrath, & Burian-Fitzgerald et al. (2005) highlighted the importance of student-conent engagement as a key factor for successful teaching and learning. For them, it is a form of "cognitive interaction" (p. 4) between the students and the subject matter they are learning. There are there key elements in this relationship- the students, the subject matter in the instruction and the instruction itself. When it is related to the use of GSP for teaching GTs, then the relationship may extend to the students, subject matter or content (GTs), the instruction (or interaction), and the technological environment of GSP. However, there is still a lack of literature in this field and further research is needed for more knowledge about the student-content relationship (McLaughlin, McGrath, & Burian-Fitzgerald et al., 2005).

Student-student relationship: Glorious conversation, hindrance, and helping out

The participants expressed their beliefs on student-student relationship for teaching GTs with GSP with me. The relationship between student to student in the teaching and learning of GTs has been discussed in terms of their experience and anticipation of potential benefits and issues of using GSP. The following protocol 17 is an example.

Protocol-17

R: Do you think that GSP helps in building relationship among the students?

C: I don't think GSP itself would do that. I don't think that you working alone and GSP does that. [5]

I think you could use it too. I think it depend on the teacher. You could have them completely focused on GSP and never talk to other students. Cause you don't really need to talk to another student when you have everything in front of you. I personally think that the conversation is glorious so I would not let that happen cause I care about what is on computer screen not what they talk about other than what is in computer screen. So, I think in my classroom you have to talk to another person, especially about what you have done. So, yes but I would also have them talk with paper and pencil. So I don't think it enhances further their relationships, I think it could hinder it if you don't do it right. [4]

J: Yea. I do. I think if you have a kid who is struggling, and another kid who is good at it. You can have them help each other and work together. Maybe that eliminates some of our own time while you are trying to walk around and help every kid. You get some kids that really want to help their friends, just like help out. I think that helps. [4]

Yea. I mean student to student if they are doing the jigsaw. [5]

Second-order interpretation

For Cathy, the use of GSP as a tool does not make student-student connection. She says, “I don't think GSP itself would do that.” She seems to consider that one can work on GSP without making connection to others. She iterates that making student-student connection is a matter of teacher’s choice. She appears to believe that a teacher can completely focus on GSP without making student-student connection. Hence, she seems to believe that talking to each other may not be a part of working with GSP while teaching and learning GTs. Also, she expresses her thought that the conversation can be a glorious, but it may not be helpful in student learning. However, she accepts that conversation after doing some tasks can be helpful. She even seems to believe that GSP may not enhance relationship, rather it can hinder student-student relationship. She claims, “…I don't think it enhances further their relationships, I think it could hinder it if you don't do it right.”

Jack, on the other hand, accepts that the use of GSP helps in building student-student relationship. He expresses, “I think if you have a kid who is struggling, and another kid who is good at it (GSP). You can have them help each other and work together.” He appears to think that students can help each other while working with GSP and GTs and this actually may save teacher's time to help other students. He seems to believe that the student-student relationship could be enhanced by the collaborative use of GSP in learning GTs. He says, "You get some kids that really want to help their friends, just like help out. I think that helps." He appears to believe that student-student relationship could be even better when they are doing a jigsaw activity.

Hence some key points that can extracted from the beliefs expressed by Cathy and Jack in relation to student-student relationship in teaching GTs with GSP are – making personal connections, collaborating each other, hindering or supporting relationship by the use of GSP, and eliminating some of teacher's time. These key aspects of their beliefs have a theoretical importance that can be further re-interpreted in the third-order interpretation making connection to the relevant literature.

Third-order interpretation

Some key points extracted from the above discussion are – GSP does not make personal connections, students may collaborate each other not GSP, and students' collaboration may eliminate some of teacher's time to help them. According to the American Council on Education, "What happens in the classroom, the interaction between teachers and students, the curriculum, pedagogy, and human relationships, is the core of the academic experience" (Green, 1989, p. 131). Computer technology in general and GSP in particular can help students to collaborate at different levels while learning GTs. The student-student collaboration can be of three kinds - mutualistic, commensal, and parasitic. The first kind of collaboration benefits all the collaborators. The students may work in peer or group. All the students may take responsibility to teach each other and learn from each other (Chew & Andrews, 2010). The second kind of collaboration (commensalism) can benefit one student without benefitting the other. However, the non-benefitted student has nothing to lose with the collaboration. The third kind (parasitic) collaboration can be beneficial for one and harmful for the other. For example, if a weak student and a good student in mathematics are paired to work together for problem solving, then the one who is weak may need step-by-step guidance from the other. The other student who is relatively good in mathematics wants to solve challenging problems rather than going through step-by-step tutorials to teach his or her peer. He or she may think teaching the other as a waste of time and effort, which is a loss for him or her.

In these relationships, students have different interest, ability, and identity that make them unique persons. A symbiosis within the classroom could be an interesting area for further study. It seems that cooperative learning through the jigsaw method could be a possible way to develop a positive relationship among students while teaching and learning about GTs with GSP. The jigsaw approach may increase the accessibility, cohesion, and interest toward GTs with the use of GSP (Cleaves, 2008). Cleaves (2008) further highlights some key features of the jigsaw for multiple representations in mathematics – comparing strategies with peers, providing tools to develop different perspectives, independence within interdependence, flexibility of organizing thoughts, willingness to take risks, different ways to present, opportunity to become expert member, and building confidence. Teaching of GTs with GSP could apply the jigsaw cooperative learning in which the group members set a goal, divide tasks, organize their tasks, and share what they learned in expert groups to the home groups. When students do not want to participate in large group-work, they can be engaged in peer works (Cleaves, 2008).

Hence, one can synthesize from Cathy's view about the uses of GSP for teaching GTs that it may not make personal connections. However, it may hinder or enhance student-student collaboration with each other depending on how GSP is used in the team or peer works. In the contrary, Jack thinks that student-student collaboration may eliminate some of teacher's time to help the students if there is mutualism among them. This shows that Cathy's view about use of GSP for mutual collaboration in teaching GTs contradicts with the literature and theory of collaborative learning with technological tools.

Student-teacher relationship: Being independent and feeling proud

The participants reflected on their beliefs about student-teacher relationships for teaching GTs with the use of GSP. Their beliefs about student-teacher relationship was related to reciprocal roles and responsibilities in the process of classroom teaching and learning of mathematics in general and teaching and learning GTs with GSP in particular (Protocol 18).

Protocol-18

R: What about the relationships between the students and teacher?

C: It might develop teacher and student, but that's I care about, and they don't care about that. Students are students. [5]

Um, so I think that students need to become independent. So I think that the relationship would be stronger because they become more independent. Personally, like if I need to lean on all the time I think that's a good relationship. But, I think that this (GSP) would make more independent into their thought. So, the relationship would be stronger cause they are not totally dependent. [4]

J: Yea. I mean as long as they can explain what they are doing. Cause the downside with procedural thing is that they just do it, maybe they don't know how. But, if they can tell you why or what they are doing, that helps. Everything a kid does is related to relationship cause they can tell you what he is doing. If they go over the screen, maybe they are proud of it. [4]

Student to teacher I think so, cause you are gonna show them real life applications of GTs with GSP, that's the key. Everyday, I ask the kids about it. [5]

Second-order interpretation

Cathy expresses her thinking that there is the possibility of enhancing student-teacher relationship with the use of GSP. In this context, she considers students' independence with the use of GSP for learning GTs as part of building a positive relationship. She accepts that the more the students are independent, the stronger the relationship is. She claims, “I think that students need to become independent. So I think that the relationship would be stronger because they become more independent.” Hence, she seems to believe that the students’ independent work is a part of positive relationship between students and the teacher.

In the same line, Jack considers that the use of GSP enhances student-teacher relationship. However, he expresses his thinking that procedural learning may not build student-teacher relationships. The procedures may not help students learn concepts. The use of GSP helps in conceptual learning and enhancing student-teacher interaction. He expresses, “Everything a kid does is related to relationship cause they can tell you what he is doing.” Therefore, for him, what students do is related student-teacher relationship while working on GTs with GSP, and the relationship is affected whether they are using GSP for a procedure or a concept. Also, he considers that a student-teacher relation could be enhanced by the use of GSP for teaching GTs “…cause you are gonna show them real life applications of GTs with GSP, that's the key.” He focuses on showing students real-life applications of the tool that may help them make sense of what they learn about GTs with GSP. When students are proud of what they could accomplish with GSP in the class that may enhance student-teacher relationships.

Hence, Cathy and Jack’s beliefs related to environment in relation to student-teacher relationship while teaching GTs with GSP can be summarized with a few key points – students’ independence with the use of GSP enhances the relation, everything a student does with GSP for learning GTs is related to this relationship, and the students’ can be proud of what they do with GSP leading to a stronger student-teacher relationship. These views have some relevancy in the literature of teacher beliefs and hence further re-interpreted in the third-oreder interpretation.

Third-order interpretation

Some key points from the above discussion cane be summarized with key points related to students' independence, students'creativity, and pride in their works with GTs by using GSP. Britt (2013) discussed student-teacher relationship to develop a classroom teaching-learning environment. She highlighted positive interpersonal relationships, trust between teachers and students, dominance in the classroom, freedom to students, independent learning environment, complexity in student-teacher relationship, reshuffling of school time in the name of remediation, respect to each other, and instructional decision making as some of the key elements to build up student-teacher relationship in the classroom. She discussed these issues in the context of a general classroom; however, they seem to be equally important for teaching GTs with the use of GSP.

Sands (2011) discussed six different stages of student–teacher relationship. The first stage is transactional model in which students are the passive receivers of the mathematical contents, and the teachers are the transmitters of information (mathematics knowledge). The student-teacher relationship is dominated by teachers' active role through lecturing. The second stage is a little more flexible than stage one in the sense that the students can ask the teachers questions. There is not much going on in terms of student-teacher interaction. At the third stage, there is some level of interaction between the students and the teachers. Teachers prompt the students and students try to collaborate with the teacher by answering the prompts. Hence, there is questioning and answering from both sides. At the fourth level, the students and the teachers engage in conversation. They establish trust and mutual respect in the classroom environment. However, the teacher plays a dominant role to control the class. At the fifth stage, the teachers play the role of guide or facilitator in the class. Students begin to collaborate among the team members. This stage is more democratic than the earlier stages. The sixth and final stage is a transformative stage in which students and teachers collaborate to create a better teaching and learning environment. They work together on constructing, exploring, conjecturing, verifying, and making decisions. Hence, they share responsibility for learning from each other. The classroom becomes a learning community (Sands, 2011). It seems that technology in general and use of GSP for teaching GTs may enhance student-teacher relationships through caring, sharing, and building trust among each other through an independent learning environment (Hackenberg, 2010).

Some key points from this discussion are related to how GSP creates a classroom environment, and the use of GSP makes work efficient and effective. Technological tools for mathematics in general and GSP for teaching GTs in particular can help students to construct knowledge, model real-world tasks, visualize concepts, design new problems, manipulate variables, get immediate feedback, represent relationships, integrate mathematical and non-mathematical ideas, promote collaboration, make new connections to contents, and empower learning (Mouza & Lavigne, 2013). The use of GSP as a technological context for teaching and learning of GTs can have these advantages for students and teachers. Moreover, GSP may have some specific affordances to the teaching and learning GTs in terms of a pedagogical tool.

As a pedagogical tool, GSP can influence "interaction between students as well as between students and teacher, particularly whole-class and small-group pedagogical modes" (White, 2013, p. 82). Application of GSP in teaching and learning GTs can offer a "pedagogical shift toward social and collaborative forms of learning" (Slotta & Najafi, 2013, p. 94). That means the use of GSP may lead students and teacher toward the "zone of pedagogical extension" (Belbase, 2013 a, p. 33) by expanding the scope of content, pedagogy, and technology. Hence, GSP can offer a possible "transformation of teaching and learning" (Mouza & Lavigne, 2013, p. 7) through the dynamic representations, visualizations, inquiry of GTs to the depth, and modeling of sophisticated functions within GTs.

It seems that the use of GSP for teaching and learning of GTs may create an environment conducive for the students' development at three levels – affective, cognitive, and pedagogical. The use of GSP may influence students and teachers' interest, feeling, and engagement toward teaching and learning of GTs (Wilson, 2011). The application of GSP in teaching and learning of GTs can enhance students' active participation in the construction and exploration. It also may promote their reflective thinking and reasoning in the context (Roschelle, Pea, Hoadley, Gordin, & Means, 2000). Although teachers and students together create an environment in the class for teaching and learning of GTs, it seems that GSP can enhance the environment toward more conducive, flexible, cohesive, and constructive one. Then students may gain independence, what students accomplish may lead to better student-teacher relationship, and hence students' can be proud of what they do with GSP for learning GTs.

Beliefs Associated with an Object

The beliefs associated with an object (e.g., GSP) have the sub-categories – interface between geometry and algebra: connecting hands-on and minds-on; semantics of GTs with GSP: constructions are not freehand and do not give a free thing; and syntactics of GTs with GSP: visualization debunks meaning and power (Fig. 12). I would like to discuss each of these sub-categories under a different sub-section.

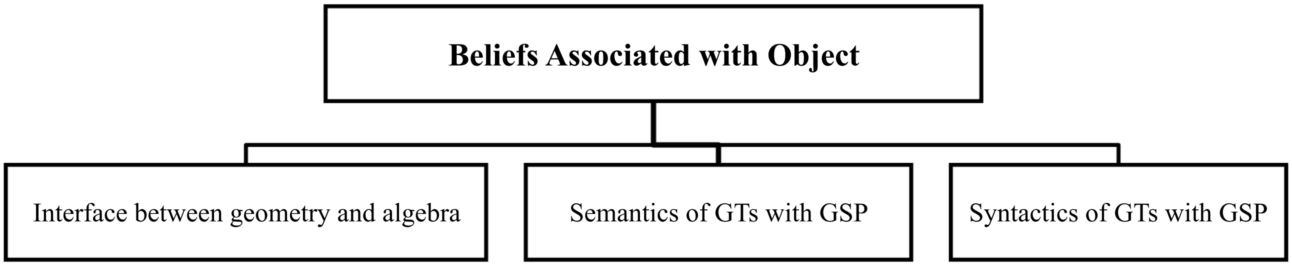


Figure 12. Beliefs associated with object

Interface between geometry and algebra: Connecting hands-on and minds-on

The participants expressed their beliefs about the interface between geometry and algebra in terms of hands-on side (geometry) and minds-on side (algebra) associated with teaching GTs with the use of GSP. The following protocol 19 is an example of belief narratives that reflect the participants' beliefs within this sub-category.

Protocol-19

R: Alright, so tell us something about GSP.

C: Something? It's awesome. It has abilities to show the picture side or the hands-on side of geometry. I haven't really explored algebra with it. I think it has the ability of the algebra. I don't know how far that goes. But, I think in geometry it has all of what we need, honestly, to teach geometry class. And, you can do like without a book. Instead of a book we tell them what you are doing. You don't even need the book because they can create every single thing. Or you can create and give them that document that they can play with it. Everything it would be in geometry class and possibly in algebra class. In a class, however, I have not discovered that far in algebra with GSP. [4]

J: Oh, I like GSP. Have you guys used GeoGebra? GSP is a lot cooler than GeoGebra. It costs more cause GeoGebra is free, but just the thing you can do with it, like overlaying the coordinate grid in GSP and visualizing everything right on it. I really liked that. I like how you can do the detail the constructing things like on the Ferris wheel. Just how you can show them, show how all those stuffs are constructive. Cause that's the one thing that really hard to cover is construction of shapes, like where did that circle come from. You can build actual square on here. All the detail behind it is so cool. You can build on it, hide it, and show it. [4]

R: Do you think you could explore some algebraic properties that are related to this example (Rotation)?

C: We didn't do algebraic. I think it would really be helpful. I have explored a ton of algebra except it is in GSP. I have done that more in GeoGebra. I think this (GSP) does have enough that you could be able to do that. [5]

J: You could. If I made right angles, you could do Pythagorean theorems or stuffs like that. But, you can just talk about distances, solving for a side, setting them equal to each other. You can do all sorts of different things. [3]

I mean the fact is that if you like to do a measurement and those stuffs, it's cool. You can see what that algebra means to without focusing much on the process so much. We can do a little conceptual understanding. That means they still need to take time to explore the process of where does that come from. They need to be comfortable with that before you give it to them. [5]

R: What properties could we explore from these examples (of the polygon and its image under rotation or composite GTs)?

C: Um, angles and side lengths. Orientation. Some of the properties change. So orientation and side lengths, I guess. Well, they always hold the same shape, I guess. [3]

There are same angles, same lengths, same perimeter, and the same area. Orientation changes every single time. Because we constructed all of these images, all of them move if we move the original construction. [4]

J: Um, you could talk about how the shapes are congruent. You can talk about distance between them, but I don't know if that's what we gonna do it in rotation. That's not relevant in rotation. [3]

You can, maybe do all the angles. Like A to A', B to B' and show that they are all 90 degrees. Measure each individual angle. You can even do that with exterior angles. That would be fun, just something different. Just new properties we explored last. It's pretty similar cause I do a kinda same way. You can explore similarity and congruence of triangles with it. [3]

Second-order interpretation

Cathy notices two aspects of GTs as interface between visualization and manipulation with the use of GSP. She claims, "It has abilities to show the picture side or the hands-on side of geometry." Her view of the picture side can be considered as a physical aspect of doing GTs with an ability of visualization and the hands-on side as a mental aspect with an ability of manipulation. She seems not quite sure about algebra related to GTs and how far it can go in the classroom teaching and learning. She accepts that GSP can have uses in the algebraic manipulation of different GTs. However, she doubts if she can make the connection. She appears to believe that she is not yet ready in using GSP for teaching concepts of a GT as an interface between geometry and algebra.

Cathy expresses that it could be possible to explore algebraic properties of GTs using GSP despite her lack of experience with algebraic manipulations. She expresses, "I think this (GSP) does have enough that you could be able to do that (algebraic manipulations)." She considers that various features of GSP can be helpful to study algebraic relationships of GTs. She did not use GSP for algebraic manipulation before the interview sessions. She explored coordinates and matrices of reflection and translation during the interview episodes 1 and 2. She related (x, y) coordinates to (-x, y) as a generalization of reflection under Y-axis. She considers that such exploration is possible for other transformations too. Cathy expresses that there are common geometric properties of GTs related to lengths, angles, and orientations in GSP. She points that some of these properties change with GTs and others preserved. She claims, "There are same angles, same lengths, same perimeter, and same area. Orientation changes every single time." That means she also seems to believe that lengths, perimeters, and areas remain the same when the orientations change. Hence, Cathy seems to believe that the geometrical properties related to points, sides, angles, perimeters, areas, shapes, and orientations are the ones that students can explore with GSP. Here, Cathy seems confused about the conservation of orientation in any GT. Orientation is changed only in reflection, but not in translation and rotation transformations.

Jack, on the other hand, seems to consider that GSP is a complete tool for teaching and learning of GTs. He accepts that the use of GSP can even replace the textbook. A teacher can use GSP for doing all kinds of constructions, manipulations, and reasoning about GTs without even opening a textbook. He expresses that GSP has a possibility of use in an algebra class too. In that sense, he even says that GSP can have uses in algebraic manipulation of GTs. He seems to feel lack of experience in exploring the algebra with GSP. He claims, “I like how you can do the detail of the constructing things, like on the Ferris wheel. Just how you can show them, show them how all those stuffs are constructive.” For him, GSP lends a constructive environment with options to show or hide all the details of constructions (interface between visual and mental). He appears to believe that overlaying constructions on the coordinate grid in GSP offers an interface between visualization and abstraction.

Jack anticipates that Pythagorean theorem can have uses as part of doing algebraic manipulation while working on GTs with GSP. He considers solving for distance or side lengths as algebraic manipulation with GSP. He further relates (x, y) coordinates to (-x, y) as a generalization of reflection under Y-axis. Hence, he seems to believe that the algebraic manipulation, for example, matrix operation, of GTs is possible with GSP. However, for him, this can be possible only when students have some background knowledge of matrices. Jack considers that moving to dynamic constructions in GSP is a huge geometric application. For him, it is important to manipulate with angles and sides (of polygons) in GSP while working with GTs. He expresses that he does not know if the distance is relevant for rotation. He then further contends that it could be, but it is more about angles. At this point, Jack seems confused about the distance as a property of rotation or not. He appears to believe that it is more about an angle.

Both Cathy and Jack seem to accept that properties of GTs and manipulating them with GSP can provide an interface between geometry and algebra. They contend that algebraic manipulation of GT processes could help students make sense of the geometric properties because it is not possible to do without having a sound background of coordinate geometry and matrices. Cathy does not believe that her highschool students can reach that far to algebraic manipulation of GTs. In the same vein, Jack also thinks that both geometric and algebraic properties of GTs can be explored with GSP. However, both of them feel lack of confidence in dealing with geometry-algebra interface of GTs by using GSP. Their beliefs in this regard can be further re-interpreted in the third-order interpretation by making connection to the literature and theory of teacher beliefs.

Third-order interpretation

Some key points from the above discussion are related to the picture and hands-on sides of geometry, ability of integrating algebra and geometry, and conceptualization and visualization. There are different ways that one can use GSP as a boundary between two comparable domains, for example, an interface between algebra and geometry, an interface between hands-on and minds-on activities, an interface between abstraction and visualization, an interface between conceptual and procedural learning, and an interface between artifact and tool for teaching GTs (and other contents). One of the key points emerged from the above discussion is that some properties may change and others do not change. For example, the perpendicularity, parallelism, concurrency, intersections (e.g., bisection, trisection), and the centrality may be preserved under a GT process (except in dilation) while orientation may or may not change depending upon type of transformation.

Other key point from this discussion is related to how GSP can be helpful for algebraic manipulation. The use of GSP can help in developing algebraic structures of a GT by setting problems around sides, angles, and others. A GT process can be generalized by using algebraic relationship, such as coordinates or matrices. This can offer an alternative approach to problem solving of GTs. One can see a two-way movement in the GT process- moving from geometry to algebra through coordinates to matrices and movement from algebra to geometry through matrices to the coordinates.

Many researchers or authors (e.g., Erbas, Ledford, Orrill, & Polly, 2005) offered different ideas about algebraic manipulations using GSP – showing how algebra and geometry overlap, producing multiple solutions, exploring the emerging patterns, visualizing and modeling problems, and immersing in algebra-geometry interface. Besides exploring different GTs, students can use GSP to explore linear relationships, properties of curves (parabola, ellipse, and hyperbola), and gradients (Meng & Sam, 2013). The students can learn about the interplay between geometrical and algebraic properties and representations by using GSP (Steketee & Scher, 2012). Some transformations, for example, translation can represent and preserve the algebraic properties of addition and multiplication of a variable with a number. Any variable in algebra can be represented graphically using GSP. Especially, the linear and curvilinear functions and their transformations would be interesting to observe and relate together (Steketee & Scher, 2012).

Schattschneider (1997) on "Visualization of group theory concepts with the dynamic geometry software" discusses how GSP can have uses to visualize the basics of group theory in algebra. One of the examples he presents is the three-point theorem and its proof using GSP. That means the use of GSP may help us see visually the abstract relationship of an object and its image under an isometry (Purdy, 2000).

It seems that teaching and learning of GTs allows students and teachers "to have a hands-on experience of the operational functions of Geometer's Sketchpad" (Stols, Mji, & Wessels, 2008, p. 17). These experiences include the constructions of geometrical figures; movement of any part of the figure using the dynamic tool; measurement of distance, angle, area, perimeter, slope, and other functional values; and manipulations of geometrical object or image in GSP environment. The minds-on activity associated with GTs could be conversions of one type of geometrical structure to another through a transformation, observation, and interpretation of area relationships, consolidation of results, debriefing the process and understanding sense of different operations (e.g., reflection, rotation, translation). Use of GSP may provide us an affordance to develop an understanding of abstraction of functional relationships among different variables on one side and it also offers visual manipulation of the function with click-and-drag. At the same time, it functions as an interface between concept and procedure. Hence, GSP functions as an interface between dualities of contrasting domains (e.g., hands-on and minds-on) within teaching and learning of GTs (Cuoco and Goldenberg (1997).

Hoong (2003) reports most of the US mathematics teachers (who use GSP) use the tool for teaching angle properties, transformations, locus, and coordinate geometry. Hoong (2003) discusses different uses of GSP for geometric manipulations in general and that can be applied to manipulations of any GT. Use of GSP for teaching GTs can help in the dynamic manipulation of geometric constructions just by using click-and-drag. The dynamic visualization offers one of the greatest benefits of developing concepts of transformational geometry (Gorini, 1997). While using the click-and-drag feature of GSP for doing any GT, their isometry is preserved. The use of GSP offers a lot of opportunities to relate geometrical properties and relations to GTs. The dynamic feature of GSP, not only visualizes the abstract concepts, but it also offers multiple ways to solve problems. That means use of GSP "can revolutionize the teaching of geometry" (Hoong, 2003, p. 87). This way, it seems that the theoretical geometry is transformed into experimental and practical geometry with the use of GSP. Hence, it makes sense of Cathy and Jack's notion of GSP as an interface between visualization and manipulation (picture-side and hands-on side) and transition between algebra and geometry.

Semantics of GTs with GSP: Constructions are not freehand and do not give a free thing

The participants expressed their beliefs about semantics of GTs with GSP. Here, semantics of GTs has been considered as both procedural, conceptual, and inter-connected meanings of the transformation process by using the dynamic feature of GSP. The following protocol 20 is an example of their narratives in which they expressed those beliefs.

Protocol-20

R: Do you like or dislike the features in GSP for teaching GTs?

C: I like them a lot. The only thing I don't want is that it does not tell you what you are doing really. It is just rotating, whatever that means, and it is just rotating. [5]

R: So what features about GSP do you like?

C: Um, I really like all of them. Honestly, the constructions are constructions, and they are not freehand. Something like you can go to the square, it's square. I can go and see that it is, in fact, a square. Like, you can't fool any. It's construction. That is probably the biggest thing that I like about it. I like that you can get immediate responses. So, I guess without having to go through it and find area of each triangle, that's not the point. You can get all that stuff. There is like a lot of built in software (program) in GSP. [4]

J: I love the animation feature in GSP. Because you can actually show them what's happening. I like how you also can construct. It's not just, oh I make a circle. There is a not a square button (tool). You know, you have to build on it. It does not eliminate and give them a free thing. They still have to learn the ideas. It is useful with the pencil and paper and then they can do more with it. So, I really like that. [4]

R: Where does it fit more either towards a procedural tool or it is a conceptual tool?

J: I think it's more conceptual. I mean procedural constructing, yea. You can do that with paper and pencil. The fact that they can see the concepts and like animation and moving, that's the bigger role I think. You can teach construction with a compass and a ruler. This is a lot you could speak it up, see it, and hide it. Not erase it. [5]

J: GSP is a tool for conceptual understanding. It skips a lot of steps. It has short-cuts. GSP is both a tool for problem solving and mathematical exploration, but I say more mathematical exploration if I have to pick between the two. I think that GSP makes teaching GTs meaningful cause they can see real life applications. [5]

C: I would only use it as a conceptual tool because I want the procedure to have been down (to be understood). Because it escapes the procedure really, it's short-cut. It's definitely the concept. Have the procedure down, now let's discover the concept. [4]

I would say conceptual because it is not doing the procedure. [5]

Second-order interpretation

Meaning of what we do in a GT with the use of GSP is a part of semantics. The spatial meaning of GT processes with GSP is implicit in the above protocol 20. Cathy makes this meaning in terms of conceptual tool. She seems to think that construction is a part of making meaning of GTs with GSP. For her, the built-in software in GSP is helpful for making sense of various processes in GTs. That means GSP shows what is happening in any GTs. She considers that building new tool is an additional feature in GSP that is not available in other similar programs, for example, GeoGebra. She accepts that GSP does not provide a freehand tool. Therefore, she appears to think that one has to construct or make sense of the available tools. Hence, she appears to believe that GSP may have a semantic domain (e.g., animation) that leads to conceptual understanding (making sense of a concept in GTs). But, on the other hand, she seems to contradict her earlier thinking about GSP. She claims, “The only thing I don't want is that it does not tell you what you are doing really. It is just rotating, whatever that means, and it is just rotating.” That means the tool in GSP does the function based on our operation. The meaning of rotation is something we have to abstract out of the process, and GSP does not tell us what’s going on.

On the other hand, Jack claims, “I love the animation feature in GSP. Because you can actually show them (students) what's happening.” He also appears to consider that some tools in GSP are constructions and not a free thing, for example, a tool for construction of a square. He says, “There is a not a square button (tool). You know, you have to build on it.” He further considers that GSP could be useful in conjunction with paper and pencil activities too. He seems to think that manipulating a concept of any GTs is better in GSP compared to other hands-on materials. Hence, he seems to believe that conceptual understanding of GTs is the biggest role of GSP. He claims, "GSP is a tool for conceptual understanding. It skips a lot of steps. It has short-cuts. GSP is both a tool for problem solving and mathematical exploration, but I say more mathematical exploration if I have to pick between the two." He accepts that GSP skips procedures and provides a short-cut to conceptual manipulations. He further says, "I think that GSP makes teaching GTs meaningful cause they can see real-life applications." That means he points to the semantic domain of GSP that helps in making sense of any GTs.

Both Cathy and Jack pointed to the semantics of GTs with the use of GSP in terms of spatial thinking, reasoning, and manipulating the properties of GTs. For Cathy, GSP as a tool is semantically neutral and it is 'we' who interprets the meaning of the GT processes. She claims that 'constructions are constructions' and she doesn't want that 'it (GSP) does not tell you what you are doing'. Whereas, Jack accepts that animation feature in GSP helps students to make sense of what is happening with a GT. That means for him the use of GSP may provide a greater semantic interpretation of the GT processes. These view are not explicit in their narratives, and I made sense of their narratives in terms of some sematics of GTs with the use of GSP. Their beliefs about semantics of GTs with the use of GSP have been further re-interpreted in the third-order interpretation in relation to the relevant literature and theory of teacher beliefs.

Third-order interpretation

Some key points from this discussion are related to cognitive ability associated with semantics to create meaning out of a GT process with GSP, ontic view about GSP and subjectivity of semantics in the construction and meaning, and the dynamic feature of GSP in the semantic domain. There might be some interesting semantic process in reflection, rotation, and translation in terms of “object of reference” (Hoong & Khoh, 2003, p. 46). The reference to a line, a point, and a vector makes sense of the GT processes. However, this aspect was not explicit in our discussion with the participants in terms of properties of GTs except when it was discussed in the last interview with backward thinking about the center of a rotation (See protocols 8 & 10, and also Fig. 6).

Another point, about ontic view and subjectivity, relates to Cathy’s expression “It is just rotating, whatever that means, and it is just rotating.” Here, Cathy’s view about ‘what is a rotation and what does GSP do in a rotation’ is associated with her ontic view of what is going on within the spatial structure. This also relates to her personal view of making sense of rotation within the dynamic environment of GSP. The third point about the dynamic feature of GSP provides a semantic interpretation of any GTs in Jack’s view. His expression, “Because you can actually show them what's happening” connotes his semantic interpretation of a GT within the dynamic environment of GSP. Hence, for him a GT process is associated with different attributes of points, lines, and angles; and these attributes are visible and students can see these concepts with animations. These features in a GT become explicit with the use of GSP. Hence a technological tool such as GSP may provide a greater sematic interpretation of GT processes (Alegre & Dellaert, 2004).

One can use GSP to construct and model different problems for students that allow them to use multiple approaches and use different semantic structures available with the tool (e.g., Bell, Greer, Grimson, & Mangan, 1989; Nesher, 1988; Vergnaud, 1988). While teaching and learning GTs with GSP one can use three layers of semantic structures - construction of object and image under a transformation, determine the transformation from the interpretation of object and image, and operation of multiple transformations (De Corte & Verschaffel, 1987; Wearne & Hiebert, 1988). The first layer is semantic input that begins with the construction of an object and image under a reflection, rotation, and translation. Without such constructions, one cannot visualize the transformation. The next layer could be related to re-interpretation of transformations in terms of references like the line of reflection, point of rotation, and vector of translation. The third point is about carrying out multiple transformations and making sense of them either as piecewise processes (like a rotation followed by a reflection) or a composite function. Cathy and Jack had the opportunity to work on reflection, rotation, translation, and composite transformations to visualize the process and the product of these transformations. Their expressed beliefs include the semantics of GTs with the use of GSP either as negative or positive way.

Syntactics of GTs with GSP: Visualization debunks meaning and power

The participants reflected on their beliefs about features and structures of the GTs with the use of GSP. Syntactics of GTs is associated with the features and structures inherent with the processes that the use of GSP may influence their organization. The layout of features of GTs within the environment of GSP allows the learners to visualize the dynamic relationship among various parts of objects, images, and reference points and lines. The syntactic of GTs with GSP may help the teachers and students to observe the nature and function of different organizational structures of GT processes. The following protocol 21 is an example of how the participants expressed their beliefs about syntactics of GTs within the environment of GSP.

Protocol-21

R: GSP is a tool for doing constructions or doing drawings?

C: It should be a tool for construction. [5]

J: Constructions. It actually forces you to construct things. [5]

R: Why?

C: Because it is totally possible that it is used for drawing too. Constructions are a lot more in geometry, and that's what it is gonna make it and work throughout. [5]

R: Do you think that the construction of an object and image under a GT with GSP is more visible, meaningful, and maybe powerful?

C: I think it might be more visible. But, I don't think it is more meaningful or more powerful. I think you constructing it with your hands with paper and pencil is pretty awesome fit, I guess. I think that would be more rewarding to the learner actually make this instead of click a button on the computer and make it. [5]

J: Yea. I mean they are, not a lot powerful, but it's definitely more visual. I don't know that it is meaningful. I think more powerful and more visual. You can hit meaning with a lot of different things. So, I don't know if GSP makes more on those, but it's definitely useful. [5]

Second-order interpretation

It seems that the structure with the internal programs and external tools in GSP has made it a tool for construction (not just for drawing). In this line, Cathy expresses her thought that the processes of GTs involve a lot of constructions (meaningful drawings) with GSP. She accepts that constructions of any GT with GSP make any concepts more visible. She considers that GSP may not change the meaning of a concept. It clarifies the meaning in a more visible way. This means that GSP does not lend syntactic differentiation of any GT process, and it just makes it more explicit. She finds hands-on with paper and pencil activity equally important and meaningful with what one does with GSP in relation to a GT. She finds it even more rewarding for a learner. She claims, “I think that (paper and pencil work) would be more rewarding to the learner, actually make this instead of click a button on the computer and make it.” Hence, she appears to believe that GSP is not a syntactical tool.

On the other hand, Jack seems to think that GSP is a lot cooler than GeoGebra. It costs, but GeoGebra is free (See protocol 19). He claims (in protocol 19), “Just the thing you can do with it, like overlaying the coordinate grid in GSP and visualizing everything right on it.” He appears to believe that GSP is a tool for construction and it forces one to construct things. He expresses, “It actually forces you to construct things.” He seems to think that one can build an actual square on GSP environment and save it as a tool. Then the next time, one does not have to reconstruct the square when he or she wants to construct a square (See protocol 19). He or she can use the tool to construct a new square. He appears to think that all the details behind it are so cool, that one can build it, hide it, and show it. For him, GSP as a tool for teaching GTs is not necessarily a powerful and meaningful, but it is more a visual tool. He claims, “I mean they are, not a lot powerful, but it's definitely more visual. I don't know that it is meaningful. I think more powerful and more visual.” That means he seems to believe that GSP is a syntactic tool that has options to change the structure, at least by hiding some features and functioning from the background giving a better visual sense to the learners.

Both Cathy and Jack seem to believe that the syntatic of GTs is more explicit with the use of GSP. Although Cathy thinks that GSP is a tool for construction, she does not believe that use of GSP is necessary to make sense of the geometry behind the GTs. She contends that the use of GSP provides a better meaning and power of the GTs. Jack also accepts that GSP is not a lot powerful and meaningful, but it is more visual. Hence, they seem to believe that the use of GSP offers a better visualization with syntactic differentiation of the process of the GTs, rather than power and meaning. These beliefs can be re-interpreted further in the third-order interpretation in relation to the literature and theory of teacher beliefs.

Third-order interpretation

Some key points from the above discussion are – The use of GSP can make syntatics of GTs explicit, but it could be done even without using the tool. Therefore, the use of GSP makes the GT processes more visual, but not necessarilty more meaningful and powerful. There is conflicting view that the use of GSP helps in understanding the syntactic processes behind the GT process. For some (e.g., Cathy), the use of GSP for teaching GTs does not add much except it is easy for constructions. And, for others (e.g., Jack), it is useful for visualizatoin of the GT processes. The use of GSP for GTs breaks down the phenomenon of any GT into steps and visual structures that can help students move from abstract to concrete visualization (Wearne & Hiebert, 1988). The structural feature of GSP as a dynamic tool allows the user to construct different GT processes. Due to the dynamic feature with click and drag it allows the learner to change the structure and visualize it from different perspectives. The dynamic feature within the tool also allows the learner to use it in a flexible way for solving geometry problem associated with constructions and interpretations. The tools available in GSP allow teachers and students to use the built-in program to do reflection, rotation, and translation instead of going through details of constructions. Although the details in the background (in the program and parameters) may not be visible to us, GSP provides us a conceptual structure of GTs (De Corte & Verschaffel, 1987) as a syntactic tool or artifact for teaching and learning GTs.

Holistic Findings and Discussion

After the completition of categorical analysis of data using the grounded theory approach of coding and categorizing, I re-analyzed and re-interpreted the data from a holistic perspective (Hall, 2008). From the holistic analysis and interpretation of the entire data of the participants' beliefs, new dimensions of beliefs emerged in terms of reflective and reflexive beliefs. The reflective beliefs are associated with objects, environment, and phenomena outside the 'self' of the research participants. The reflexive beliefs are associated with 'selves' of the two participants in relation to others (students, content, and context) and hence those beliefs are internal to them. Reflective beliefs are related to critical examination of actions, objects, and environment. Whereas, reflexive beliefs are related to critical examination of participants' self-awareness and consciousness to the self-other interface. These beliefs are more affective, attitudinal, and cognitive in nature. The directional and temporal dimensions of these beliefs indicated to the possibility of pre-, in-, and post- reflective and reflexive beliefs (Fig. 13). Each of these holistic holistic dimensions of beliefs is discussed in the following sub-section.

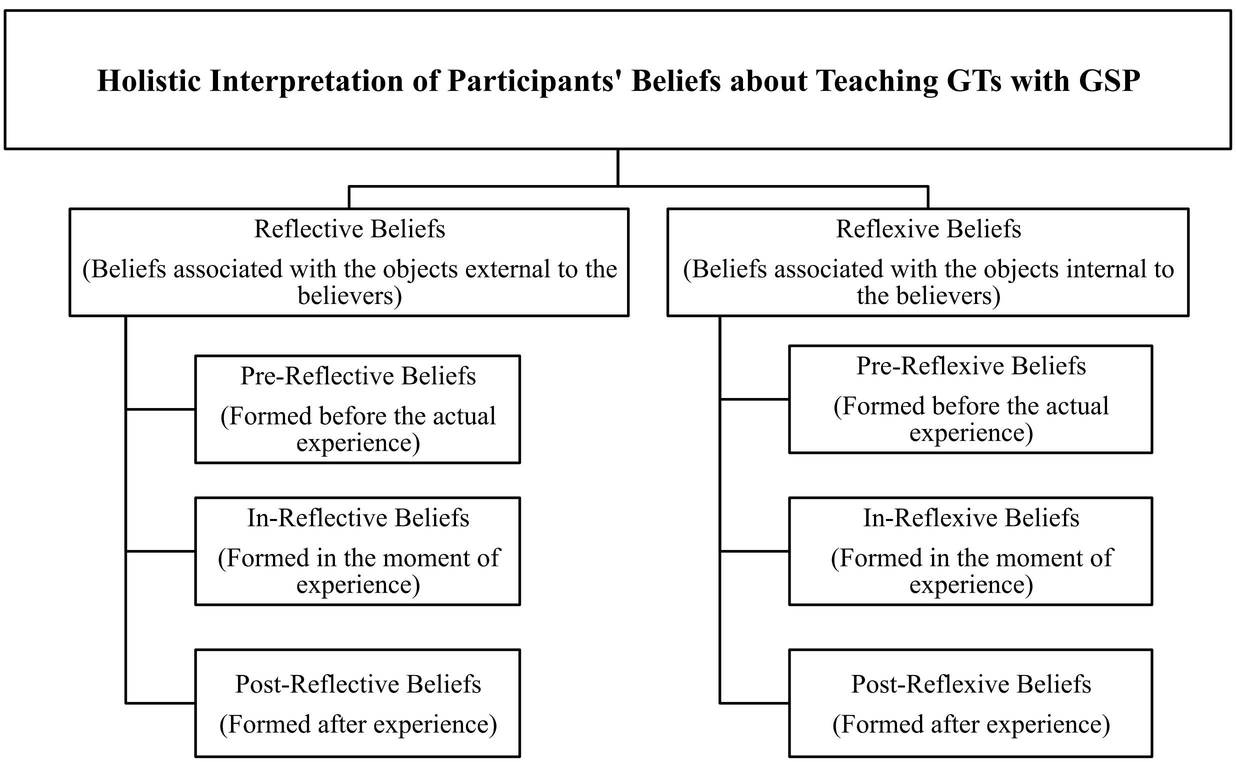


Figure 13. Holistic interpretation of beliefs in terms of reflective and reflexive beliefs

Reflective Beliefs

The observation of the entire interview transcript revealed that some beliefs the participants formed were related to the belief objects outside their personal 'selves'. These beliefs were found to be associated with the phenomena external to the participants' mental states. That means the participants were found to have those beliefs about objects, persons, and environment external to them. The belief narratives in the forms of reflective beliefs have been generated from the interview episodes by putting together bits and pieces of narratives to create a sensible belief-expression. The numbers in the braces at the end of narratives [ ] indicate the interview numbers from which the belief statements were taken to construct the narratives.

Cathy's reflective beliefs

The following belief narrative presents examples of Cathy's reflective beliefs extracted from the interview episodes. Cathy expresses her reflective beliefs in relation to the phenomenon of reflection transformation. These beliefs are also associated with GSP as a tool (object) to facilitate the study of GT processes. The narrative in protocol 22 is in the first person point of view where the narrator (speaker) is the research participant (Cathy), and it portrays the elements of her reflective beliefs.

Protocol-22

Reflection is about line, axis, and there has to be reflected across. You need to know here is a butterfly and how to reflect part of it. You need to know how to construct an image. Lines and points are important basic things to know. They (students) need to know the distance- yes. Um, congruency and similarity – yes. Ah, parallel, perpendicular, I don't know if you would need to know, but I think we need to know that shapes are similar. I would use it (GSP), but I wouldn't rule out the pencil and paper activity. I guess the title I would give it (GSP) as a discovery tool cause it's not doing the teaching, you are doing the learning. I want to know about their mistakes, first of all, cause their mistakes are good for learning. I think mistakes are fine, and they are the tools for learning. [1]

The properties of GTs related to angles and side lengths can be explored with GSP. Under a rotation, angles stay preserved and side lengths are also preserved. Under a rotation, orientation is not preserved, *but perimeters preserved, and areas preserved. I think that procedurally it (GSP) skips steps, but not really skipping steps; it is just quickening the steps. I think the individual activity would be the discussion. I think that in geometry it is incredibly important to discuss their ideas. Prompts would build their (students') confidence. [3]*

They (students) can explain to me like what they are doing, so like they decide they are gonna explore the area. I think that they need the ability to recognize their own and in groups, like simultaneously. They (students) could create like the Ferris wheel and then being able to kinda do animation of their own, which is meaningful to them. Yes, GSP helps in exploring properties of any transformation or even composite transformations. [4]

GSP is a tool for the mathematics exploration. The only thing I don't want is that it does not tell you what you are doing really. It is just rotating, whatever that means, and it is just rotating. GSP is not so much procedural, I guess. It is more of exploring, conceptual, visual, and dynamic. [5]

Jack's reflective beliefs

The following belief narrative is example of Jack's reflective beliefs in a cluster extracted from the interview episodes. His reflective beliefs are associated with the functions fo of GSP in explicating the GT processes. These beliefs are related to visualization of the GT processes within the dynamic environment of the GSP. The narrative in protocol 23 is in the first person point of view where the narrator (speaker) is the research participant (Jack), and it portrays the elements of his reflective beliefs.

Protocol-23

GSP has a lot of learning curve. They (students) have to know what they are trying to do. GSP is more visual. I think it's a great tool. I think we do the measurement of things and how it works. It (GSP) is like any computer program. I really like the simplicity with the sketchpad. They will be able to use the coordinates. Um, you know, think about how long I took just to count those tiny squares. Here (in GSP), you can go to the coordinates. Then all of a sudden you can get to the algebra. [1]

Properties, you could talk about how the shapes are congruent. The conjecture on rotation is gonna be pretty much the exploration or that kind of thing. I would choose angles, how they are related to each point and each object. Even you could talk about with the angles and sides and how they are the same. Like this here (with the rotation) that may not click right away, but if you can show them real life thing. Maybe even, you draw a satellite in the space and how it is orbiting. They can see that. So, that way it helps me in visualizing and explaining it. [3]

It is always cool about this (plotting of areas). Kids when build this now they are also learning about linear functions. Kids are actually ready to do with functions and go into linear transformations. You can totally compare that and build into a kind of lesson that is built on itself and then you can talk about linear functions. It is really cool. I think GSP helps you enrich this type of environment. I like the *visualization. How you can visualize angles, like you can put the angles in here so that they can see it right out there, and side lengths. I think that's cool. I think it (GSP) really can help build on it. [4]*

GSP is a tool for conceptual understanding. It skips a lot of steps. It has short-cuts. GSP is both a tool for problem solving and mathematical exploration, but I say more mathematical exploration if I have to pick between the two. I think that GSP makes teaching GTs meaningful cause they can see real life applications. That's what kids want. They want to know when they leave the classroom they may use it. I think I can help students in making and proving conjectures. Just the hands-on stuff and show them what they are looking at and making animations. I think that GSP can help in designing different instructional approaches. You could design different lessons and do hands-on. You can approach different demonstrations or student based learning where they do it themselves. [5]

Second-order interpretation

Some of the key points from the participant's reflective belief narratives are – nature of GSP as a discovery tool, building students' confidence, exploring properties of GTs with GSP, transition from geometry to algebra, and making implicit processes explicit.

Cathy things that GSP is a discovery tool in relation to teaching GTs. She explains, "I guess the title I would give it (GSP) as a discovery tool cause it's not doing the teaching, you are doing the learning." She further accepts that use of GSP in teaching GTs facilitates the teacher in prompting that can build students' confidence. She accepts that the use of GSP helps in exploring different properties of GTs. In this sense, GSP is not much procedural for her because it is more of exploring, conceptualizing, visualizing, and animating tool. For Jack, use of GSP helps students in making a transition from geometry to algebra. He expresses, "Here (in GSP), you can go to the coordinates. Then all of a sudden you can get to the algebra" and this clearly indicates his belief that the use of GSP in teaching GTs may facilitate teacher to make a smooth transition between geometry and algebra. While doing this students can visualize and explain the geometry-algebra interface of the GTs.

Third-order interpretation

I observed that these reflective belief narratives have temporal aspects associated with them. The time of actions or happening of something and forming beliefs based on them created three forms of reflective beliefs: pre-reflective, in-reflective, and post-reflective beliefs (Fig. 13). Each of them has been discussed under a separate sub-heading.

Pre-reflective beliefs

The above belief narratives included the following statements identified as examples of pre-reflective beliefs. Cathy and Jack expressed these beliefs about the actions that have not yet happened, but they already formed these beliefs. They seem to form these belief states in their mind even without having the actual experiences of actions. That means these beliefs are anticipatory. Following are the samples of their pre-reflective beliefs about teaching GTs with the use of GSP.

C: They will be able to move the vertices and see what's going on and what it (reflection) is doing. [1]

J: That (GSP) has a lot of learning curve. They (students) have to know what they are trying to do (with it). [1]

The first statement represents Cathy's belief about the dynamic nature of GSP that allows movement of vertices. Also, another relationship important for us to notice is the temporal dimension. Cathy is indicating toward her students' ability to move the vertices and see what is going on with the object and the image under reflection as the result of the movement. This clearly indicates that her belief is anticipatory, and hence it is pre-reflective. That means such beliefs are her reflections of future actions about teaching GTs with the use of GSP.

Again, the second statement is a representation of Jack's belief about GSP and his students' knowledge of this tool. The first part about the learning curve seems to be nontemporal. However, the second part (the dominant one) is about his student's anticipated knowledge about what to do with GSP, which forms an anticipatory belief and hence it seems to be his pre-reflective belief. These beliefs seem to associate with pre-reflective being. Pre-reflective beliefs are non-representational (does not represent ontic world) and formed at the level of perception and future action (Romdenh-Romluc, 2007). "The notion of pre-reflective belief also account for the paradoxical situation where psychological breakdown has taken place, but has not been experienced" (Groarke, 2014, p. 35). Teachers form pre-reflective beliefs about their future actions with common sense beliefs. Such beliefs are formed with anticipatory and preactive reflections (Van Manen, 1991). These beliefs may influence the planned actions and anticipation of acting in a certain way to achieve a goal.

Hence, the characteristics of pre-reflective beliefs are – intuitive (but not experienced), common sense, anticipatory, and non-representational. Cathy and Jack's beliefs about the use of the GSP in their future teaching have these features and hence they are pre-reflective and pro-active beliefs.

In-reflective beliefs

The belief narratives included the statements identified as examples of in-reflective beliefs. Cathy and Jack expressed these beliefs about the actions that are on going and they formed these beliefs in the moment of experience. They seem to form these belief states in their mind at the time of actual experiences of actions. That means these beliefs are participatory or beliefs within the moment of participation in the action. Following are the samples of their in-reflective beliefs about teaching GTs with the use of GSP.

C: The coordinate (x, y) changes to (-x, y) under a reflection on Y- axis. [1]

J: GSP helps in the visualizing (of any GTs). I think that's the biggest thing, I could see that. [3]

In the moment of construction and discussion about the nature of transformation, Cathy observes the coordinates and expresses her belief based on the generalization of the coordinates that (x, y) changes to (-x, y) under a reflection in the Y-axis. Here, the first belief statement represents Cathy's belief about the nature of the change of coordinates of vertices of the object triangle into the image triangle. Likewise, the second belief statement represents Jack's belief about the nature of a GT in terms of visualization. Both of these belief statements are related to immediate action or operations about GTs with GSP as validating context. If belief has a validating context (through experience and observations) and can be represented in a form (a symbol) is an in-reflective belief (Sperber, 1997). Such beliefs, according to Tremlin (2006), "probably require a demonstration before it is accepted" (p. 138). That means in-reflective belief has a context to validate it. A belief is represented and expressed by symbol means it can be codified and de-codified within a field of practice that generates a belief or is influenced by a belief. These beliefs act on moment of performing an action and also may modify with immediate confrontation of problems. Hence, such beliefs are head-on beliefs. Normally, one may not form a belief within the situation of confrontation or when reflecting on an action at the moment (Van Manen, 1991). There is proximity of belief object and the moment of forming the belief itself within that experience.

Hence, in-reflective beliefs have the characteristics of – contextual, immediate, and within the flow of an action and expereince. The properties of in-reflective belief resonate with what Cathy and Jack expressed in the above examples. Hence, some of the beliefs expressed by both Cathy and Jack (during the interviews) seem to be in-reflective beliefs.

Post-reflective beliefs

The belief narratives also included examples of post-reflective beliefs. Cathy and Jack expressed these beliefs about the actions that already happened, and they formed beliefs as a consequece of their experience and reflection after the actions. They seem to form these belief states in their mind after having the actual experiences of actions. That means these beliefs are experience oriented. Following are the samples of their post-reflective beliefs about teaching GTs with the use of GSP.

C: I think it (construction with GSP) might be more visible, but I don't think it is more meaningful or more powerful. [5]

J: I think that the constructions of an object and image under a GT by using GSP are not a lot powerful, but it's definitely more visual. I don't know that it is meaningful. I think it is more powerful and more visual. You can hit meaning with a lot of different things. So, I don't know if GSP makes more on those, but it's definitely useful. [5]

When I presented some activities of geometric transformations using GSP to Cathy and Jack, they already had some experiences of working with GSP. Before the final interview, they already had gone through some activities with GSP, especially reflection, rotation, translation, and composite transformations during the earlier interviews 1, 2, 3 and 4, respectively. These beliefs expressed in the above examples were no more momentary (immediate effect of operations or actions). That means they did not form these beliefs within the moment of actions. These beliefs are mental states after having some experiences with working on GTs with GSP (either in their classes or in the earlier interview sessions). Hence, these beliefs are post-experiential or retrospective beliefs.

Here, the first statement is a representation of Cathy's retrospective belief about the use of GSP and how it could make sense of teaching and learning of GTs. She expresses her thought that GSP provides visualization of GTs. However, she does not seem to believe that it is meaningful or powerful. In the same line, Jack also seems to believe that GSP is more a visual tool, but not necessarily meaningful and powerful. After having some experience with the tool, they appeared to form these beliefs. Cathy has a longer experience with GSP than Jack beside the interview sessions. Since these beliefs formed or retained as a result of their experience with GSP in the past, then one can say that they are historical or post-experiential. They may not be truly representational as the ones that at the moment (in-reflective). These beliefs may not be non-representational as the one that has not been experienced. Since they experienced using GSP (at least as learners and future teachers) and the psychological states actually perpetuated through their experience in the past. Hence, they present here a set of pseudo-representational or historical beliefs or post-historical or ex-post-facto beliefs. These beliefs have root actions in the past, but psychological or mental effect perpetuate until now and may transcend further to the future. Therefore, they can be considered as post-reflective beliefs.

Kwanvig (2013, p. 234) mentioned about post-reflective belief in relation to epistemic principles and theory of rationality. He did not explicitly discuss the term 'post-reflective belief'. Bouma (1997) mentioned about John B. Cobb Jr.'s contributions in theology stating that "…psychic wholeness leads us to match post-reflective belief systems with pre-reflective experience" (para 4, sub-title: Anthropology). It appears that philosophers have some sense of post-reflective beliefs, but none of them has spelled it out clearly. Post-reflective beliefs are associated with recollective or retroactive reflections on experiences (Van Manen, 1991). There is a temporal and spatial distance between the object of belief and the time of forming or sustaining a belief about an action or a phenomenon (Van Manen, 1991). Hence, post-reflective beliefs have the characteristics - recollective, retroactive, distanced, and experience oriented. Cathy and Jack's beliefs about attributes of the GSP in terms of power, use, visibility of process, and meaning are post-reflective beliefs.

The reflective beliefs expressed by Cathy and Jack seem to have roots to their reflexive beliefs that are related to their consciousness of self, other, and the reciprocal relationships. I observed that there were many instances of beliefs in their narratives that focused on the participants' personal 'selves' in the forms of efficacy, awareness, and consciousness toward the phenomenon of teaching GTs with the use of GSP. These reflexive beliefs are discussed in the following sub-section.

Reflexive Beliefs

The observation of the entire interview transcript revealed that some beliefs the participants formed were related to the belief objects inside their personal 'selves'. These beliefs were found to be associated with the phenomena internal to the participants' mental states. That means the participants were found to have those beliefs about their actions, perceptions, and cognitions internal to them. These beliefs are about un-observable mental constructs related to self-awareness and self-consciousness. These beliefs cannot be represented with any external source object or a phenomenon. These are self-referential beliefs that the participants formed about themselves in terms of ability, cognition, affect, and attitude. The belief narratives in the forms of reflexive beliefs have been generated from the interview episodes by putting together bits and pieces of narratives to create a sensible belief-expression.

I would like to discuss the participants' reflexive beliefs under separate sub-headings with examples of belief narratives followed by interpretation of such beliefs.

Cathy's reflexive beliefs

The belief statements in protocol 24 are a few examples of Cathy's reflexive beliefs extracted from the different interview episodes. The protocol shows that her reflexive beliefs are related to her consciousness and aswareness to the self-other relation. The narrative in the protocol is in the first person point of view where the narrator (speaker) is the research participant (Cathy).

Protocol-24

I guess a bunch of students really liked the tutorial, but I didn't because I like to make my mistakes and learn from them. That's where I learned GSP. I think I learned it really well, and I still remember how to do most of the stuffs because I learned it so. In my own classroom, I think I would still start with like folding something (e.g., a paper) before they come to this (GSP). If you can't see the computers, they (students) can go off-tasks a lot more. How I would go off-tasks is much different than most students would. I strongly believe in as a teacher I am just there to spark their interest into show you things that are interesting. [1]

We can have them get a new do an activity completely, and they will open a new window, they will need to recreate a picture or something and send them to rotate. That way (with new construction) I can see they can rotate it that they know what rotation means, and they are not just following the procedure that they have just to follow. I guess, so having GSP on with it reaches more students, and I think more students would have a deeper understanding, but I can teach the concept in a different way. [3]

You have to be able to distinguish between here is your personal work time to make your own cluster of thoughts, and here is your partner time or whatever you can share with others and make them grow into whatever direction. If we use it (GSP), it needs to be a tool and not the sole way of expressing concepts. That (GSP) is not gonna reach so many students like me who needs the entire semester to figure out, would be lost. [4]

Once we know what rotation is I don't want them (students) to draw a rotation every single time. I want them to go quickly to it (GSP) and then explore with it. I have to get honestly more comfortable with the idea of teaching with it because I have to figure out the procedural side. If I have figured out the procedural side, I would feel more comfortable in teaching (with GSP). I think you (the teacher) and your class (students) develop the environment. So, I don't know if GSP would. I mean it would affect the environment in some way. That moment right there (finding the center of rotation), that made me really think about using GSP while teaching GTs. [5]

Jack's reflexive beliefs

The following belief statements in the protocol 25 are a few examples of Jack's reflexive beliefs extracted from the interview episodes. The narrative in the protocol is in the first person point of view where the narrator (speaker) is the research participant (Jack).

Protocol-25

I think it (matrix of reflection) will be interesting to them. Some people are really interested in that, but some people don't. It just confuses. I like them, but I have a little more experience. It depends on where they are in matrices. They may have a problem with that. Um, if they are starting it (GTs) algebraically, they can derive this stuff probably. Probably they need to understand what does Y = X mean, and if you go from there, another problem is some students don't visualize things. I think GSP is gonna add to it (teaching and learning of GTs). It's really gonna help you out show what happens. Um, like you can move the line, and it moves the shapes. [1]

I need to refresh with GSP. I have spent two classes on GSP, very limited actual uses of GSP. Yea, with practice I can do it. I have such a limited thing. In methods class, we used it a little bit. Junior high does not have it. So, I haven't just messed up with it. I can just think about what's gonna interest them, what's gonna hold their attention. If you are just talking about rotation, you are just using formal terms, I think you are gonna lose their attention in ten minutes. I think you can gain their interest. You know maybe a kid is struggling with it (doing a rotation). All of a sudden the kid sees it in the computer what's happening. Maybe that just turns around and builds his huge confidence of what he or she is seeing. [3]

I don't anticipate students explaining these (linear functions) themselves. It would be interesting to see if they can actually put it into words what they are seeing. It would be interesting to see if they can actually put it into words what they are doing. It would be interesting to hear their own language what they say. That would be a kind cool to hear. I think it's important to give somebody time to mess it up. As far as exploring, like I probably know two percent of what people do (with GSP). So, I wish I had more hands-on experience with it. In relation to use of GSP in the future teaching, I am half way there, but I am not getting there. The more I play with it (GSP), the more I like it. [4]

I am both yes or no in thinking if I am ready to teach GTs with GSP, probably more toward no. I just need to get more comfortable with it. I have sort of internalized the use of GSP for teaching GTs. I need more practice. I think that GSP makes teaching and learning more interesting than without using it. I mean, especially kids love technology. They can see that. You can relate it to building a video game. You can have them do the real thing they get interested. Maybe I can use the jigsaw method when applying GSP in teaching of GTs, if I have the technology, like the access. [5]

Second-order Interpretation

Some of the key points in the participants' reflexive beliefs are- awareness to the interest of self and others (students), a sense of understanding of understanding of others and own, and conciousness toward self-other relation. I would like to discuss each points in brief in relation to the excerpts of above narratives.

Cathy's epxression, "I strongly believe in as a teacher I am just there to spark their interest into show you things that are interesting" shows her awareness to the interest of her students. At the same time she is exhibiting her awarness to self as a teacher, and also her role as a teacher. Jack expresses that use of GSP makes teaching and learning more intersting (for both teacher and students) and the teacher can use this tool have the students do the real thing they are interested in with it.

Both Cathy and Jack expressed their beliefs in relation to how the use of GSP in teaching may help in developing student understanding of the processes inherent with GTs. Cathy expresses "I guess, so having GSP on with it (GTs) reaches more students, and I think more students would have a deeper understanding." On the other hand Jack says, "Probably they need to understand what does Y = X mean, and if you go from there, another problem is some students don't visualize things." These narratives of the participants indicate to their understanding of students' understanding of GTs with GSP.

There several instances in the narratives in which both Cathy and Jack expressed their beliefs related to their own consciousness toward self-other relation. For example, Cathy expresses, "I think you (the teacher) and your class (students) develop the environment. So, I don't know if GSP would." Here, she reveals her beliefs about self-other relation in the context of creating a classroom environment. She further reveals her belief that, "It (GSP) might develop teacher-student relation, but that's I care about, and they (students) don't care about that." She seems to be conscious of her role as a teacher to give meaning to that relation. Jack expresses, "Everything a kid does is related to relationship (student-teacher relationship) cause they can tell you what he or she is doing." That means Jack looks at the students' tasks from the viewpoint of self-other (teacher-student) relation. These views clearly show Cathy and Jack's pedagogical reflexivity and hence their reflexive beliefs.

Third-order Interpretation

I found that the reflexive beliefs discussed here have temporal aspects associated with them. The time of being aware of something and forming beliefs based on them created three forms of reflexive beliefs: pre-reflexive, in-reflexive, and post-reflexive beliefs. Each of them have been discussed under separate sub-sections.

Pre-reflexive beliefs

The belief narratives of Cathy and Jack have the elements of pre-reflexive beliefs. They expressed their personal beliefs toward self-awareness, consciousness, and dispositions before they experienced different phenomena associated with the GT processes. The following expressions are the examples of Cathy and Jack's pre-reflexive belief statements from their belief narrative.

C: I would have a different inquiry path, like look at the sides and angles. Look at the sides and angles, and look at the area. Maybe having them (students) present to each other, come together, and have a discussion of them about what a reflection is (using GSP). [1]

J: I can just think about what's gonna interest them, what's gonna hold their attention. If you are just talking about rotation, you are just using formal terms, I think you are gonna lose their attention in ten minutes. I think you can gain their interest…one of the cool things about GSP is that's where video games come from. You can show them that's where it is coming from. [3]

In the first statement, Cathy seems to express her belief about her inquiry path in which she anticipates engaging her students in a collaborative way of learning reflection transformation using GSP. She seems to have a sense of connectedness to the students. This connectedness in the inquiry path is visionary or anticipatory. She has not yet experienced the connectedness, exploration, and learning of GTs by the students through this inquiry path. However, she seems to be aware of her roles, relations to students, and to the teaching. The first statement portrays an example of Cathy's pre-reflexive belief that is primordial and pre-historic. Here, it is pre-historic in the sense that the belief formed before the happentence of phenomena (history).

In the next statement, Jack seems to express his beliefs about awareness toward his (own) thinking and toward students' ability and interest to learn GTs with GSP. He seems to be aware of his thinking about students' interest, attention, motivation, and connection. He appears to think that his students would be interested to see and use the dynamic features of GSP, possibly connected to the process of animation in the video games. Hence, his belief seems to be connected to his identity as a future teacher and awareness towards students' psychological or mental states, but still anticipatory. He seems to acquire this psychological/mental state without having actual experience in the classroom teaching of GTs with GSP. Hence, his belief statement in the above example appears to be his pre-reflexive belief that is primordial and pre-historic (before the happentance of phenomena).

Lizardo and Strand (2011) state, "….persons can form pre-theoretical, pre-conceptual beliefs about the world, and that they can reason and form expectations about the world at this pre-reflexive level" (p. 1). That means Cathy and Jack's pre-reflexive beliefs are pre-theoretical and pre-conceptual in the sense that they have not experienced the actual phenomena, but they already formed such beliefs. Eacott (2013) mentions 'pre-reflexive belief' in the context of leadership as a research object. There is not any description of the term except it brings a context of blurring "the boundaries between the epistemic and the empirical" (p. 227) sense of a person or agent in a social milleau. Zimmermann (2010, p. 185) states, “According to Bourdieu there is a kind of pre-reflexive belief, which is usually not questioned in the social practice.” This means pre-reflexive beliefs are sometimes latent as a part of habitus. It seems that habitus has a power to maintain an order through the pre-reflexive beliefs that the new members in a field have to adapt with in order to gain dominant role in the field (Bourdiew, 2001). Then here, pre-reflexive belief is a common sense belief as de-facto belief about one’s roles, position, and power in the field in which none generally questions. Both Bourdiew and Zimmermann’s views are related to social practice and field in terms of power relations and interactions among the social agents that maintains the social order.

Medeiros and Capela (2010, p. 43) mention about ‘pre-reflexive belief’ in the context of linking ethos to the construction of reality. They do not explicitly describe the term. However, it provides a glimpse of pre-reflexive beliefs within Bourdiew’s iedos and ethos. Emmerich (2014) also mentions about ‘pre-reflexive belief’ in the context of grounding eidos as instrument of construction and also the object constructed within “ethos and habitus” (p. 14). These examples from the literature show that pre-reflexive beliefs are matters of peronsal awareness, conciousness, and a habitus (personal dispositions).

Hence, pre-reflexive beliefs have the characteristics of – conditioned state of acceptance of something without actual experience, pre-theoretic, pre-conceptual, grounded in iedos, and anticipatory about self and other relation, awareness, and future course of actions. Both Cathy and Jack’s pre-reflexive beliefs about having a different inquiry path for teaching GTs with GSP, and gaining students’ interest to what they (Cathy and Jack) want them to do in learning GTs with the use of GSP have these elements. However, there is a wide crevice in the literatue in the area of pre-reflexive beliefs, and more research needs to be done in this area to understand the nature and functions of pre-reflexive beliefs of preservice mathematics teachers.

In-reflexive beliefs

The belief narratives of Cathy and Jack have the elements of in-reflexive beliefs. They expressed their personal beliefs toward self-awareness, consciousness, and dispositions at the moment they experienced different phenomena associated with the geometric transformations. The following excerpts from their narratives are the examples of in-reflexive beliefs.

C: The moment right there (while finding the center of rotation), that made me really think about using GSP while teaching GTs. How that point of rotation relates to the ….(the object and the image). I mean I would have gone through about it, but I didn't go that far. [5]

J: I have been able to perceive different roles of GSP, like teaching and learning, using it as a conceptual versus procedural tool. I don't know if I can think even beside those. You can use it in the classroom to teach concepts. That's the biggest thing. [5]

In the first statement, Cathy seems to express her belief about her existence and awareness at the moment when she is struggling to find the center of rotation. She is making connection to her own conscious thinking about the problem to find the center point. In the second statement, Jack seems to be aware of his current beliefs about GSP as a conceptual versus procedural or teaching versus learning tool. His belief is not just about the role of GSP as a tool, but his ability, intention, and perception. This kind of belief seems to have a relation with the direct experience and awareness of the person through the experience, not just anticipatory. Cathy and Jack's belief statements in the above examples seem to show their in the moment awareness and disposition and hence, they portray examples of their in-reflexive beliefs. These beliefs also have some relevance to the literature.

Richard (2013, p. 104) mentions about in-reflexive beliefs in terms of ascribing such beliefs within a sentence that has 'such and such' in conjunction with 'self' or 'his or her own'. Nozick (1983) mentions in-reflexive belief in terms of "disposition to behave" (p. 81) and "self-reference" (p. 79). Braude (1995, p. 72) mentions in-reflexive belief in relation to self-indexicality that refers to "one's states", by this he seems to refer to the mental states of self-awareness. However, the literature lacks a clear explication of one's beliefs in terms of at the moment belief or in-reflexive belief.

Post-reflexive beliefs

The belief narratives of Cathy and Jack have the elements of post-reflexive beliefs. They expressed their personal beliefs toward self-awareness, consciousness, and dispositions after they experienced different phenomena associated with the GT processes by using GSP. The following excerpts from their narratives are the examples of post-reflexive beliefs.

C: I guess a bunch of students really liked the tutorial, but I didn't because I like to make my mistakes and learn from them. [1]

J: Yea, with practice I can do it. I have such a limited thing. [3]

In the first statement, Cathy seems to make a connection of current belief to her past experience of learning GSP in the Foundation of Geometry class. She then connects that experience to her learning of GSP and using it for teaching GTs. This possibly shows her awareness of who she is. In the second statement, Jack seems to be aware of the situation that he did not have adequate practice on GSP and that limited his ability to use the tool for teaching. First, he anticipates his relationship to the content of GT and use of GSP. Then he seems to move further (mentally being more aware) of the consequence of his limited experience of using GSP. Hence, these examples show Cathy and Jack's post-reflexive beliefs with their awareness and anticipation after passing through situations of doing something with GSP. There is not explict theory and literature to address the explicit meaning and context of post-reflexive beliefs about teaching mathematics with technology. It shows that literature of teacher beliefs has not reached that far to analyze post-reflexive beliefs although we hold such belief as a result of an experience in our everyday life and professional practice.

Chapter Conclusion

I outlined the major findings of the study in terms of six major belief categories associated with teaching GTs with GSP. These categories were: beliefs about action, affect, attitude, cognition, environment, and object. The primary findings of participants' beliefs about teaching GTs with GSP were discussed in terms of their voice by constructing protocols under twenty-one subcategories followed by researcher's interpretations in terms of second and third-order interpretations. While doing this, I reconstructed the meaning of their voice and connected them to the literature. The secondary findings about participants' reflective and reflexive beliefs grounded on the data were discussed in terms of pre-, in-, and post- reflective and reflexive beliefs at three layers: the first layer represented their voice through the samples of belief statements, and the second layer represented my interpretation of their voice, and the third layer interconnected researcher's interpretation to the theories or literature. I summarized the beliefs expressed by the two cases (i.e., Cathy and Jack) in the following brief narratives.

Cathy's Expressed Beliefs

Cathy seems to believe that students' concrete work of manipulating a GT with constructions in GSP is helpful to know whether they learned it or not. She focuses on their creation and description as evidence of their learning of GTs. For this, she considers prompting as an approach to enhance students' learning of GTs with GSP. She seems to accept that sharing students' idea of GTs and use of GSP in the classroom is important because it helps them to solidify their understanding. During the paired sharing, one may do construction, and other can explain. In this context, she seems to accept that switching students' role in the group is important. For her, students' explanation may serve as an informal proof in GTs. She appears to think that GSP is neither a teaching nor a learning tool, but it is a practice or exploration tool. She expresses her belief that she would not allow students to begin with GSP. She seems to think that the first step in learning GTs is more procedural activity with paper and pencil. Once, the students are done with the procedures, she wishes them to use GSP for further explorations of different GT concepts.

Cathy seems to consider GSP as a great exploration tool that can help students in practicing different GT problems and developing conceptual understanding. For this, she appears to believe that a teacher needs to be confident enough to use the tool. She expresses her anxiety over not being able to figure out the center of rotation when she was asked to do it during the task-based interview session. She seems to consider that teacher's technological content knowledge has a great influence on teaching and learning of GTs with GSP. She appears to think that students may have limited ability for formal proof of different GT problems. However, she seems to accept that students might consider their construction with GSP as a proof in an informal way, perhaps through verbalization.

For Cathy, GSP might be helpful in building a good relation between teacher and students, but that may not be the case for student-student relation. She appears to consider that GSP could enhance positive learning and teaching environment, but it is not GSP that creates an environment for students. For her, both students and teachers create that environment. However, GSP provides another way to get into the contents of GTs. She seems to believe in promoting student autonomy through the use of GSP in teaching GTs. When students have a tool for practicing GT construction and problem solving, then they are more independent to the teacher and other students. This kind of independence, as Cathy appears to think, provides students opportunity to work on their own. Hence, for her, GSP may offer students some degree of autonomy in learning GTs.

Cathy appears to believe that the constructions in GSP environment are not just freehand, but students have to know what they are doing both procedurally and conceptually. She seems to assume that GSP as a practice and exploration tool that makes invisible processes visible and fast, but it is not necessarily more meaningful and powerful. She seems to believe that her past learning of GTs in a geometry course helped in building up her confidence towards use of GSP for own learning and also for future teaching.

Jack's Expressed Beliefs

Jack seems to believe that assessment in teaching GTs with GSP can be done by observing students' ability to explain what they have constructed. That means he considers their ability to draw something and explain to the class 'what it is and how it is' as part of teaching and learning of GTs with GSP. He seems to think that teaching GTs with GSP is much more than just using paper and pencil in the sense that students can visualize what's going on with the help of animation. He appears to believe that students can see what's happening under a GT in the dynamic environment of GSP. He appears to believe that GSP is both a teaching and learning tool. He seems to believe that it enhances teaching GTs by providing shortcuts to do procedures and visualizing concepts in the classroom. For him, it enhances students' learning of GTs because they can construct things related to properties of any GT, and they can visualize every step and relation. He seems to believe that when a student, grappling with the concept of GT, is able to construct it with GSP that may help him build his confidence. For him, use of GSP in teaching GTs would make it fun to learn for students. He seems to think that GSP even could be helpful to explore and generalize formulas and algebraic relationship or expression of GT properties. That means, to him, students can explore interrelation of points, lines, angles, areas, and perimeters. However, he seems to believe that excessive formalization of teaching GTs with GSP may cause students to lose their attention and interest.

He appears to believe that time management as an important aspect of using GSP for teaching GTs. He seems to consider that making a connection of teaching GTs with GSP to the real-life application is an important aspect for student motivation. He appears to believe that constructions such as Ferris-wheel and tilted-wheel could be helpful to motivate students because they can see multiple functions going on at the same time. One can construct, hide, and show things with GSP as necessary. He seems to believe that more access to technology such as GSP in the classroom would enhance student learning.

Jack seems to believe that use of GSP in teaching GTs may enhance the relationship between student-to-student, student-to-teacher, and student-to-content. For him, GSP offers both procedural and conceptual understanding of GTs. However, he seems to believe that GSP has a bigger conceptual role than the procedural. For him, the use of GSP for procedure of GTs may skip steps making it short-cut. He seems to believe that GSP is not a lot powerful and meaningful tool for teaching GTs, but it is more visual and useful. His limited experience appears to shape is belief that he is not yet ready to teach GTs using GSP effectively.

The next chapter deals with research questions, researcher's reflection, implications, and suggestions for future research.

CHAPTER 5: RESEARCH QUESTIONS, IMPLICATIONS, AND FUTURE DIRECTIONS

In this chapter, I addressed the two research questions based on the findings and discussion in Chapter 4. Key implications of the findings in terms of specific and general implications were presented followed by discussion of future directions.

Addressing Research Question 1

The first research question for this study was: What beliefs do preservice secondary mathematics teachers hold about teaching geometric transformations with Geometer's Sketchpad? This question aimed to explore preservice secondary mathematics teachers' beliefs in the anticipated contexts of teaching GTs with GSP. The findings and discussion of their beliefs (in Chapter 4) suggested some key categories of their beliefs that address this question of what beliefs do they hold about teaching GTs with GSP. These categories are associated with – action, affect, attitude, cognition, environment, and object. The research participants' beliefs in relation to research question 1 with respect to these categories are discussed in the following sub-sections.

Beliefs Associated with Action

The interview protocols 1, 2, and 3 and subsequent layers of interpretation highlighted Cathy and Jack's beliefs about student assessment, group activities, and individual activities in relation to teaching GTs with the use of GSP.

Cathy seems to believe documenting and manipulating to students' work, engaging them into a new activity, and asking them probing questions as key aspects of assessment. She expresses that group-work with GSP while teaching GTs is difficult. However, she appears to believe that switching the partner roles (for construction and description) and having conversations could be helpful to some extent. Nonetheless, she contends that partner work is not ideal with GSP. That means she is skeptic about effectiveness of group-work with GSP. For her, discussion between teacher and student and students' informal proofs by saying things out loud are some aspects of individual activities while teaching GTs with the tools in GSP.

Jack seems to be in favor of formal assessment as part of students' learning of GTs with GSP. He seems to be positive in letting students create GTs with GSP and having them explain it. He appears to believe that students can design GTs by working together which reflects their creativity and provides a better realization of what they have constructed. He thinks that different groups of students can work with different GTs with GSP and then teach each other, but that requires a lot of dynamism (i.e. movement) in the classroom. For him, student's using GSP to address real-life examples (e.g., Ferris wheel) could be good individual activities, but time could be an issue.

Beliefs Associated with Affect

The interview protocols 4, 5, 6, 7 and 8 and subsequent layers of interpretation highlighted Cathy and Jack's beliefs associated with experience with GSP, feeling and perception about the use of GSP, internalizing the use of GSP, readiness for using GSP, and worries and concerns about using GSP.

Cathy expresses that she learned geometric proofs in GSP without using tutorials. She struggled with doing proof all on her own and, therefore, she believes that she doesn't forget it because she did it herself. She believes that she was able to 'suck it up and do it'. However, she thinks that her students might not be able to do it. She doesn't like to teach GTs with GSP. For her, GSP could be an exploration tool, more of conceptual rather than procedural, visual and dynamic, and practice tool. She expresses her belief that she is confident in using GSP for teaching and learning GTs, but she does not intend to use it for teaching because she thinks that GSP is more conceptual than procedural, and she wants to teach the procedure first. Her concern is about the difficulty of developing conceptual understanding without mastering the procedures and students learning to use GSP commands but not understanding GTs and processes. She accepts that it enhances teaching, although she purports that GSP is neither a teaching nor a learning tool, but it is a practice and exploration tool. She reveals her sense of ability to explore GTs through depth of exploration of properties of GTs with GSP.

For Jack, his class presentation on GSP was not helpful in developing his skills and knowledge about the use of GSP for teaching GTs. To him, it was more procedural action with step-by-step instruction and less construction. He appears to believe that visual representations in GSP are engaging for students. He accepts that GSP is both a teaching and learning tool for GTs because it enhances teaching and learning. However, he believes that 'there is a fine line between wasting time and being productive'. He also asserts that using GSP could have a negative effect due to lack of resources, for example, computers in the classroom and availability of GSP for teaching and learning GTs. Lack of technological tools, like GSP, for teaching GTs may lead to frustration if students struggle with it. For this problem, he has to manage both 'hands-on time and one-on-one time' for students. Jack feels that his limited experience could be a hindrance in using the tool for teaching GTs. This might have influenced his sense of readiness and belief that he is not ready to use GSP for teaching until he gets more comfortable with the tool. This leads to his worries and concerns related to time, need for creating a foundational knowledge, and students not being able to understand the shortcuts with the use of GSP.

Beliefs Associated with Attitude

The interview protocols 9, 10, and 11 and subsequent interpretive layers portray Cathy and Jack's beliefs associated with developing confidence with GSP, efficiency of using GSP for teaching GTs, and exploring GTs with GSP. These beliefs seem to be similar to the beliefs associated with affect, but they are different in the sense that affective beliefs are more related to internal perceptions, feelings, experiences, anxieties, excitements, and values; whereas, attitudinal beliefs are consequences of self-confidence, sense of ownership and power, and ability to use the tool.

Cathy has a sense of high confidence in using GSP. She seems to believe that teacher confidence directly influences student confidence in using the tool. She expresses that prompting students helps them to increase their sense of confidence. For her, the Foundations of Geometry course was critical to developing self-confidence in using the tool. She appears to believe that seeing what is going on in GSP while working on GTs helps students in developing confidence. She seems to consider GSP as a practicing or exploring tool after mastering the procedures of GTs. For her, procedures precede the concepts in relation to teaching GTs with GSP. She seems to have a greater ability to explore GTs with GSP in depth, however she gets confused with some backward problem solving (e.g., finding the center of rotation when the object and image under a rotation are given).

Jack expresses his belief that he has a sense of getting there (i.e. being in a position to use GSP for teaching GTs), however he still has a lack of confidence. He feels necessity to spend more time on the tool (GSP) for teaching GTs. For him, both the doing (i.e. hands-on) and seeing (i.e. observation) of GT processes with GSP help his students in building confidence. His limited experience of using GSP seems to inhibit his sense of confidence. He seems to believe that efficiency of GSP as a tool depends on access, one's ability to see what's happening, and background knowledge. For him, GSP enables students to achieve on a better learning curve than without it. He seems to have positive beliefs about the usefulness of GSP for teaching GTs, however his beliefs about his own ability appear to be negative due to his limited experience with it.

Beliefs Associated with Cognition

The interview protocols 12, 13, and 14 and subsequent interpretive layers portray Cathy and Jack's beliefs associated with conjecturing about GTs using GSP, supporting and engaging students in learning GTs, and understanding GTs with GSP.

Cathy expresses her beliefs about conjecturing properties that are preserved (e.g., lengths, angles), explanation as key to understanding, deriving meaning, interrelated structures in GTs, and use of direct measurement for manipulating the properties of GTs by the use of built-in tools in GSP. She prefers to engage and support students in assessing their learning as a part of student engagement, exploring properties of GTs, helping students in constructing and describing what they have done by starting with easier things that students already know. She seems to believe that students don't understand formal proof and for them a construction is a proof. She further thinks that they can verbalize the constructions as a proof. She believes that students develop an understanding of GTs with paper and pencil before the use of GSP. That means she thinks that students' understanding of the procedure before concepts is important because, for her, the use of GSP keeps on skipping steps.

Jack seems to believe that he can help his students in exploring properties of GTs and find interrelations of different geometric features (e.g., angles, sides, areas, and perimeters). He accepts that he, as a teacher, should be able to help students express their work on GTs in words that develop their mathematical language, and also may enhance their understanding of the processes with GSP. He appears to think that some key aspects of student cognition of GT processes with GSP are related to their interest, attention, ability to see the application of GTs (e.g., in video games), and design of activities to engage them in class through hands-on constructions. Jack seems to feel that learning with GSP is cool and being able to see what's happening (visualization) provides students a deeper understanding of the process.

Beliefs Associated with the Environment

The interview protocols 15, 16, 17, and 18 followed by layered interpretive accounts in the earlier chapter portray Cathy and Jack's beliefs associated with classroom context, student-content relationship, student-student relationship, and student-teacher relationship.

Cathy appears to believe that classroom set up and positive learning environment are related to each other in which students recognize their potential and group ability. She purports for a classroom with no fear of talking with peers and students being able to distinguish personal and partner time. However, she is not sure if GSP creates such an environment, although it can affect the classroom environment. She accepts that GSP helps students get to another level because it is another way of getting into content (with GSP) by making a stronger student-content relation. She does not believe that GSP builds student-student relationship, but it may help in teacher-students relationship. For her, student-student conversation is glorious, but she believes that GSP does not enhance the relationship and may even hinder it.

For Jack, classroom context with more technology creates a teaching and learning environment. He seems to believe that students' access to computers may help in creating a learning environment for them. In this sense, GSP may create a different environment, but it may not be a better one. Here, he seems to contradict his own belief that more technology creates a better environment. He accepts that GSP helps students in getting into real life applications, building on the content, visualizing the process, and building a foundation for conceptual learning of the GT processes. At the same time, he expresses his belief that the use of GSP in teaching GTs can improve student-student relationships through collaboration. He seems to think that one student who is struggling can get help from another student who is good at using GSP for learning GTs, so that the teacher's time could be used to help students who are behind the class. Jack appears to believe that GSP might help in student-teacher relations and that this relation could be enhanced when students are able to explain GT processes with GSP. In addition, they may feel proud of what they construct and want to explain it to others using GSP.

Beliefs Associated with Object

The interview protocols 19, 20, and 21 followed by layered interpretive accounts in the earlier chapter portray Cathy and Jack's beliefs associated with interface between geometry and algebra, semantics of GTs with GSP, and syntactics of GTs with GSP.

Cathy accepts that the use of GSP provides an interface between static pictures and the dynamic hands-on geometry and an interface between geometry and algebra. She considers that GSP does not tell the users what they are doing really because for her constructions are constructions, but not freehand objects. That means, for her, it is the users (students and teacher) who make sense of what they are doing with GTs by using GSP. The use of GSP as a tool is neutral object (a programmatic tool) and it does not allow self-interpretation of the GT processes. She accepts that GSP implements procedures for students making it easier to discover the concept. For Cathy, GSP is useful as a construction tool, but not a free hand (drawing) tool because constructions follow rules (syntactical structures). She thinks that constructions in GTs are more rewarding with paper and pencil compared to GSP.

Jack accepts that GSP provides an interface between procedure and concept, and also an interface between algebra and geometry. His beliefs indicate the notion of interface of geometry and algebra of GTs with GSP are related to visual and mental abstractions of transformation processes during geometrical and algebraic operations of GTs. He seems to believe that GSP offers features that support dynamic geometry, provides built-in tools for easy constructions, and allows for visualization of the GT processes. For him, these constructions are not free tools because one has to make sense of them while doing them. Therefore, for him, manipulation of GT processes leads to conceptual learning and meaning making through different semantic interpretations. He expresses his beliefs that GSP leads to constructions. For him, GSP is a useful tool for visualization. He also accepts that meaning does not come from the constructions, but it comes from abstractions and the role of GSP is to show us what's going on so we make sense of them.

These categorical beliefs are primary (fundamental) beliefs expressed by the participants during the task-based interview episodes. These beliefs are not mutually exclusive and exhaustive. That means these belief categories and sub-categories are the ones that I found to be more explicit in the analysis and interpretation. These belief categories and sub-categories were used for further analysis and interpretation of the data leading to emergence of secondary or derivative beliefs in terms of directional beliefs that include reflective and reflexive beliefs. These directional beliefs are related to their anticipation of using GSP for teaching GTs in their future classes. These beliefs are associated with research question 2.

Addressing Research Question 2

The second research question in this study was: What beliefs do preservice secondary mathematics teachers hold about their future practices of teaching geometric transformations with Geometer's Sketchpad? This research question intended to characterize their holistic beliefs in terms of temporal dimensions and directions of beliefs about teaching GTs with GSP. The temporal dimensions are related to time of action or event and formation of beliefs. The findings and discussion related to beliefs of two preservice mathematics teachers in this study portrayed their nested beliefs in terms of reflective and reflexive beliefs as categories of directional beliefs. The further analysis of these beliefs revealed the nature of these beliefs within three temporal dimensions – pre-, in-, and post reflective and reflexive beliefs that could be associated with their anticipated practices of teaching GTs with GSP. The directional beliefs in terms of reflective and reflexive beliefs are derivative (secondary beliefs) of categorical beliefs discussed earlier. These beliefs have been considered derivative in the sense that these beliefs are inherently dependent on the six major categories and their sub-categories of beliefs. These beliefs characterize the entire domains of beliefs in terms of belief objects as external or internal phenomena to the believers (the research participants). In this sense, these beliefs are directional. The participants' holistic and directional beliefs are discussed in the following sub-sections.

Reflective Beliefs

The holistic analysis and interpretation of Cathy and Jack's beliefs revealed that some of their beliefs they consciously hold reflected a state of perplexity, hesitation, and doubt in a reflective mode.

Cathy expresses her belief that a reflection is about a line, axis, and there has to be an object reflected across the line. She thinks that one needs to know where the object is and how to reflect part of it. To do this he or she needs to know how to construct an image. She accepts that her students need to know the concepts of distance, congruency and similarity, parallel, and perpendicular. She appears to believe that she would use GSP, but she would not rule out the pencil and paper activity. The title she would give GSP is a discovery tool, because it's not doing the teaching, but the teacher does the teaching and the students do the learning. She reveals that she wants to know about their mistakes first of all, because their mistakes can be used to enhance their learning.

Cathy further expresses her beliefs that under a rotation, angles stay preserved, and side lengths are also preserved. For her, under a rotation, orientation is not preserved, but perimeters and areas are preserved. She seems confused in the case of orientation during rotation. She also thinks that procedurally it (GSP) skips steps, but not really skipping steps; it is just quickening the steps. She thinks that individual activity would be a discussion. She seems to believe that it is incredibly important to discuss their (students') ideas in geometry in general and GTs in particular. She considers prompts as an important tool for engaging students in creative thinking about properties of GTs. She accepts that students can explain to her what they are doing, and why they decide to explore the area or perimeter. She thinks that they need the ability to recognize their potential as an individual and in groups, like simultaneously. For her, GSP is a tool for mathematics exploration. She has concerns that 'it does not tell you what you are doing really with it'. She reveals her belief that GSP is not so much procedural. It is more of a dynamic tool for exploring, conceptualizing, and visualizing.

Jack appears to believe that GSP requires a lot of learning curves. For him, the students have to know what they are trying to do. In the case of nature of GSP, he thinks that it is more visual, and he reflects that it's a great tool. With the tools in GSP, he thinks, students can do measurement of lengths, angles, areas, and perimeters and they can observe how it works. He considers that students should be able to use the coordinates for algebraic manipulation of GT processes. He accepts that students can talk about how the shapes are congruent and how they would choose angles, how they are related to each point and each object. He suggests that students can talk about the angles and sides and how they are the same. For him, it may not click right away in the minds of the students, but if the teacher can show them real life applications, like asking them to draw a satellite in space and how it is orbiting within the dynamic environment of GSP, then understanding may develop. He accepts that they can see that (the construction), and it helps them in visualizing and explaining it.

Jack seems to believe that when students are engaged in plotting of areas of object (e.g., a triangle) and image (e.g., an image of the object triangle under a transformation) they are also learning about linear functions. For him, students are actually ready to deal with functions and go into linear transformations. He thinks that GSP helps a teacher to enrich this type of environment, for example, constructions, visualizations, and verbalizations. He appears to believe that GSP really can help students build in these environments. Jack considers that GSP is a tool for conceptual understanding. For him, it skips a lot of steps because it has shortcuts. He purports that GSP is both a tool for problem solving and mathematical exploration, preferring the second use. He appears to believe that GSP makes teaching GTs meaningful because students can see real life applications, and for him that's what they want.

Reflexive Beliefs

The analysis and interpretation of Cathy and Jack's beliefs revealed that some of their beliefs indicated their relational states with feelings, prejudice, suggestion, and restricted views of themselves and others that might have fixed their cognitive schemas with core beliefs about self, others, and the world. These beliefs are related to their awareness of self and others in terms of action tendencies of varied levels. With these beliefs, Cathy and Jack revealed their identity as students and future teachers and positioned themselves in the time and space with both cognitive and affective resources.

Cathy appears to believe that she learned the use of GSP with her self-struggles and making mistakes. She accepts that she likes to make mistakes and learn from them, and that's where she learned GSP. She also exclaims that she learned it really well, and she can still remember how to do most of the stuff because she learned it so. In her own classroom she thinks that she would still start with folding something (e.g., a paper) before her students come to GSP while teaching and learning GTs. She thinks that a teacher has to be careful about what students are doing on their computers while working on GTs with GSP. She seems to believe that if the teacher can't see the computers, the students can go off-task a lot more. She appears to believe as a teacher she is just there to spark their interest and show them the things that are interesting.

She would have her students recreate a picture and then rotate (or reflect or translate) it. That way (with new construction) she can see if they are able to rotate it which reveals if they know what rotation means, and they are not just following the procedure. She believes that using GSP reaches more students, and she thinks that more students would have a deeper understanding, allowing her to teach the concept in a different way. She expresses that her students should be able to distinguish between personal work time where they develop personal understanding, and partner time where they share with others and develop communal knowledge. For her, if they use GSP for learning GTs, it needs to be a tool and not the sole way of expressing concepts.

She claims that once her students know what rotation is, she doesn't want them to draw a rotation every single time. She wants them to use the tools in GSP and then explore with it. She accepts that she has to get more comfortable with the idea of teaching with GSP because she has to figure out the procedural side. If she has figured out the procedural side, she would feel more comfortable in teaching (with GSP). She further thinks that the teacher and her students develop the classroom environment. She seems not to be sure what role does GSP play in it. She agrees that it would affect the environment in some way. She reveals that finding the center of rotation makes her really think about using GSP while teaching GTs. She seems to accept that she is not able to think of how to reverse engineer the center of rotation from the object and image under a rotation. For this, she claims as a teacher, that she has to go through these processes before bringing it into the classroom.

During the task-based interviews 1 and 2, I engaged Jack in observing the process of reflection transformations from viewpoint of matrix operation. The algebraic manipulation of reflections on x and y-axes using coordinates of vertices of object and image polygons (e.g., triangles and quadrilaterals) lead to formulation of matrices for those reflections. Jack seems to believe that matrix of reflection could be interesting to his students. He accepts that some students might be really interested in that, but others would not be. For him, it just confuses them. He likes working with matrices of different transformations because he has a little more experience with it. He claims that it may depend on where they are in matrices. To him, if the students don't know about matrices, then they may have a problem with matrices of different transformations. For him, if they are starting GTs algebraically, they can probably derive this. He accepts that they need to understand what the line Y = X means, and if he goes from there, another problem is some students don't visualize things like this. He appears to believe that GSP adds to teaching and learning of GTs. For him, it's really going to help him show students what happens when they move the lines, and how that moves the shapes.

In relation to effective use of GSP for teaching GTs, he appears to believe that he does not have adequate knowledge and skill in GSP. He feels that he needs to refresh with GSP. He expresses that he has spent two classes on GSP with very limited actual uses of it for teaching/learning GTs. He seems to be optimistic that he can do it with a little more practice on the tool. He accepts that he has such a limited understanding about GSP. In methods class, he used it a little bit during a technology presentation, and during his practicum the school did not have it. He reveals his belief that he could have done more with GSP given more time to play with it. However, he still can think about what will interest his students, what will hold their attention, and what will motivate them. He claims that if he is just talking about rotation, he is just using formal terms and he is going to lose their attention in ten minutes. He thinks that if a kid is struggling with it (doing a rotation) and all of a sudden he or she sees what is happening on the computer, that builds his or her confidence.

He seems to believe that his students won't be able to explain the linear functions themselves. He claims that it would be interesting to see if they can actually put into words what they are seeing. For him, it would be interesting to see if they can actually put into words what they are doing. He accepts that it would be interesting to hear their language about what they have done or what's happening with a GT process. Also, he thinks that it's important to give his students time to mess up with GTs by using GSP. He believes that he needs more hands-on experience with GSP. In relation to using GSP in future teaching, he feels that he is just half way there, but he is not getting there.

Jack appears to waiver on if he is ready to teach GTs with GSP. He thinks that he is probably not ready. He seems to believe that he just needs to get more comfortable with it. He feels that he has internalized the uses of GSP for teaching GTs, nonetheless he accepts that he needs more practice. He thinks that GSP makes teaching and learning more interesting than without using it. He seems to believe that use of GSP in teaching may provide more opportunities for both the teacher and students to explore the properties of the GTs.

Implications of the Findings

I synthesized specific implications associated with the participants' beliefs about actions, affect, attitude, environment, and object. These implications have been derived from the categories of the participant's beliefs and hence they are specific. These implications have been further generalized in terms of psychological, pedagogical, epistemological, and practical implications. The rationale for the general implications is that participants' beliefs (in any form) have connection to the broader field of psychology, pedagogy, epistemology, and everyday practice of teaching and learning. Now I would like to discuss specific implications of the findings and discussion of participant's beliefs about teaching GTs with GSP.

Specific Implications

The implications synthesized from the categories of beliefs associated with actions, affect, attitude, cognition, environment, objects, and directions were specific in nature. These implications have been discussed under separate sub-titles.

Implications of beliefs associated with actions

The findings within category 'Beliefs Associated with Actions' have implications in the areas of assessment, personal values of individual and group activities, and informal proofs of students. Method of assessment could be accommodated with the different uses of GSP for teaching GTs. Personal values and beliefs about group activity might affect the classroom practice of using GSP for teaching GTs**.** Students' informal proof could involve constructions and verbalization in an open-ended environment for teaching GTs with GSP.

Implications of beliefs associated with affect

The findings within category 'Beliefs Associated with Affect' have psychological implications in the areas of personal experience, influence of feelings and perceptions on practice, internalization and practice, influence of readiness to practice, and worrisome problems that shape beliefs. Personal experiences of preservice teachers (whether as self-discovery or step-by-step procedure) might influence their anticipated practice of teaching GTs with GSP. Preservice teachers tend to do what they like, feel, or perceive in the situations of using GSP for teaching GTs. The way preservice teachers internalize the use of GSP for learning/teaching GTs may influence their actual practice. A sense of readiness may have a possible application of GSP for learning/teaching GTs at different levels. Worries and concerns about procedure or concept, foundational knowledge, and access could influence the proper use of GSP for teaching/learning GTs.

Implications of beliefs associated with attitude

The findings within category 'Beliefs Associated with Attitude' have psychological and pedagogical implications in the areas of sense of confidence, efficiency, access, order of use, ease of use, and knowledge to use GSP for teaching GTs. Preservice teachers' sense of confidence in the use of GSP for teaching/learning might influence their perceived usefulness and ease of application. Use of GSP may depend on preservice teachers' belief about the efficiency of the tool in terms of access, order of use, ease of use, and knowledge to use for teaching GTs.

Implications of beliefs associated with cognition

The findings within category 'Beliefs Associated with Cognition' have psychological and pedagogical implications in the areas of conjecturing and developing mathematical language by students, informal proofs by verbalization and constructions, and procedural versus conceptual understanding of GTs with GSP. Preservice teachers could help students to make conjectures and develop their mathematical language about interrelated structures and properties of GTs with GSP and test them. Their beliefs about how they think of engaging their students can influence their decisions to apply informal proofs through construction, visualization, and verbalization of GT processes with GSP. They have different beliefs about students' understanding (either as instrumental or relational) of GTs with GSP that may influence their anticipated practice.

Implications of beliefs associated with the environment

The findings within category 'Beliefs Associated with an Environment' have pedagogical implications in the areas of recognition (or identity), getting into content, building student-student relation, and enhancing student-teacher relation. Recognition of self-other relation, access, and ownership may create a positive learning environment. GSP provides another way to getting to the content and builds foundational understanding of GTs. Preservice teachers' beliefs about student-student relation (as positive or negative) through the use of GSP in teaching GTs might affect its applications. Preservice teachers' beliefs about role of GSP for building student-teacher relationship can help him or her in using the tool for student motivation and a pedagogical shift.

Implications of beliefs associated with an object

The findings and discussion within category 'Beliefs Associated with an Object' have pedagogical implications in the areas of moving across interfaces, semantic interpretations of GT processes, and structural organization of GT processes with GSP. Preservice teachers' beliefs about GSP having interface between picture side (constructions) and hands-on side (manipulation and abstraction), geometry and algebra, procedure and concept, and visual and abstract might help them in transitioning across the interfaces. Preservice teachers' beliefs about GSP for creating meaning of GT processes, providing layers of semantic interpretations, abstracting structural organization (syntactics) of GTs and role of GSP in this organization might affect how they apply GSP in teaching/learning GTs.

Implications of reflective and reflexive beliefs

The findings within category 'Beliefs Associated with Directions' have epistemological implications in the areas of forming and changing beliefs within temporal dimensions of pre-, in-, and post- reflective and reflexive beliefs. Preservice teachers' beliefs about the relationship among tool, process, and outcome in temporal dimensions in terms of pre-reflective, in-reflective, and post-reflective beliefs might influence their conscious/unconscious attitude toward the use of GSP in teaching GTs. Preservice teachers' beliefs about their relationship with the tools, processes, and outcomes in temporal dimensions in terms of pre-reflexive, in-reflexive, and post-reflexive beliefs might influence their awareness and anticipation of using GSP in teaching GTs.

General Implications

The findings from the study also have four broad implications – psychological, pedagogical, epistemological, and practical. However, these implications are not mutually exclusive and exhaustive. There are overlaps and also possibilities of drawing implications in terms of other than these.

Psychological implications

These implications have been synthesized from the findings within category of Beliefs Associated with Affect, Attitude, and Cognition. The affective factors are - personal experience, feelings and perceptions, internalization, readiness, and worrisome problems may influence one's beliefs both in the short and long term. The attitudinal factors that may influence one's beliefs about teaching GTs with GSP are- access, confidence, efficiency, order of use, ease of use, and knowledge to use. The cognitive factors that may affect in forming and shaping one's beliefs are- conjecturing, developing mathematical language, proofs by verbalization, proofs by constructions, and procedure versus concept of GTs with GSP. Hence, these factors that emerged from this study could be focal emphases in mathematics teacher education program to understand and influence their beliefs in a positive direction. Their preconceptions about using GSP for teaching GTs formed through their experiences with varied level of psychological variables. These variables tend to be strong forces that might inhibit or enhance the immediate change of beliefs. Therefore, keeping those psychological variables – affect, attitude, and cognition as the central aspects of forming or changing preservice mathematics teacher beliefs may serve as a strong basis to transform preservice and inservice mathematics teacher education.

Pedagogical implications

These implications have been synthesized from the findings within category of Beliefs Associated with Actions, Environment, and Objects. The factors associated with actions- assessment, individual and group activities, and negotiation with the informal proofs of students. The environmental factors that may influence one's beliefs about teaching GTs with GSP are - recognition (or identity), reaching the content, student-student relation, and student-teacher relation. The factors related to an object that may affect in forming and shaping one's beliefs are - interfaces, semantics, and syntactics of GT processes with GSP. Hence, these factors that emerged from this study have some pedagogical implications in mathematics teacher education program to understand and influence their beliefs in a positive direction. Their beliefs about assessment, classroom activities, and engaging students in conjecturing and proving may have a direct implication for their evolving pedagogy through the courses they take and experience in mathematics teacher education. These pedagogical variables including several others (e.g., experience and skill to use GSP, personal creativity, resources in the classroom, and school support system for technology and content integration) could be the central aspects of mathematics teacher education for forming and changing preservice teacher beliefs.

Epistemological implications

These implications have been synthesized from the findings within category 'Beliefs Associated with Directions'. The factors associated with this category are - pre-, in-, and post-reflective and reflexive beliefs. The findings associated with pre- reflective/reflexive beliefs show that preservice teachers may form their pre-theoretic a priori beliefs that are non-representational and have not yet experienced. However, these beliefs might influence the way they think and develop their confidence about what, how, and why they would use GSP for teaching GTs. Their beliefs in terms of in-reflective and reflexive beliefs were idiosyncratic and dynamic views they formed and expressed during the problem-solving episodes. These beliefs are based on their immediate experiences they had and therefore might be more flexible. These beliefs could be changed with further experiences. Hence, these beliefs could be the one that form their knowledge through actual experience. Their post-reflective/reflexive beliefs are related to past experiences of learning and using GSP, not necessarily for GTs. Their experiences in terms of how long they used the tool, how they learned to use the tool, and why they used the tool seem to have a big impact on their current beliefs about teaching GTs with GSP. Hence, post-reflective/reflexive beliefs could be a subject matter for studying beliefs that are historical (i.e. ex-post-facto) beliefs formed over time and could be more permanent compared to other beliefs.

The findings from the current study in terms of categorical beliefs in general and directional beliefs in terms of reflective and reflexive beliefs in particular can serve as an epistemological foundation for theory-practice assemblage of belief systems of preservice mathematics teachers about teaching GTs with GSP. The knowledge of preservice teachers' beliefs and the way these beliefs are evolved from the past, present, and future might help preservice mathematics teacher education programs to design and implement appropriate intervention strategies. These interventions could be designed and implemented in technological-pedagogical-content knowledge and practice that inform research on teacher beliefs and vice versa.

Practical implications

The findings revealed in this study have some practical issues related to the uses of GSP for teaching GTs. Some of these issues are – use of GSP as a teaching tool or learning tool, use of partner work, confidence in the use of GSP, post-master or pre-master application of GSP for teaching GTs, technology as part of classroom environment, GSP for student-student relation, and GSP as a syntactic tool for teaching GTs. The two participants in this study had contrasting beliefs in these areas that might be considered as key issues for practical implication of the findings. Cathy believes that GSP is neither a teaching nor a learning tool whereas Jack believes that it is both. These differing beliefs may have a direct influence on their practices of using the tool for teaching GTs. Cathy believes that partner work is not helpful for students because some students might do nothing in the group; whereas, for Jack collaborative work is helpful for students because they can teach and learn from each other. Cathy seems to be more confident in using GSP for teaching and learning, but Jack seems to be less confident in using the tool for teaching.

For Cathy, students should learn the procedures before they start using GSP for conceptual learning of GTs. That means students should master the procedures (without using GSP) before the concepts of GTs by using GSP. Whereas, Jack seems to think that GSP can be used at any time. Cathy seems to believe that GSP does not create an environment in the class, but it enhances the environment. Whereas, Jack seems to consider that GSP helps in creating a better environment for teaching GTs. Going further, Cathy seems to accept that GSP does not have any role in building student-student relation in the classroom. Nonetheless, Jack seems to contend that view and accepts that GSP may enhance student-student relationship because students can collaborate by using the tool and learn from each other. For Cathy, GSP does not add anything new to the structural organization of GTs and it is not a syntactic tool. On the other hand, Jack seems to consider that GSP is a great syntactic tool because it makes structural organization of GT process tacit to explicit.

The beliefs expressed by Cathy and Jack provided some contexts to draw some practical implications of the study. One can see the practical aspects of GSP for teaching GTs that can go either direction - a teaching tool or learning tool, post-master or pre-master application of GSP for teaching GTs, GSP as environment friendly or environment neutral, GSP as a relational tool or neutral to student-student relation, and GSP as a syntactic tool or neutral artifact for teaching GTs. Positive change in preservice teacher beliefs about practical use of GSP (or other technology) for teaching GTs (or other contents) might be associated with their direct involvement into classroom practices either through lesson activities in the mathematics methods and content classes or practicum or both.

Future Directions for Research

The findings and implications of this study encompass the potential future directions for research and practice on preservice mathematics teacher beliefs. These directions constructed from the findings, discussion, and implications are related to personal psychology, pedagogy, epistemology, and the future research agenda. Each of these potential directions is discussed separately. These directions are potential areas for the further study of preservice and inservice mathematics teachers' beliefs about teaching mathematics with technology in general and teaching GTs with GSP in particular.

Psychology of Mathematics Education

The discussions on psychological domains of findings in terms of preservice mathematics teachers' beliefs associated with affect, attitude, and cognition and related implications suggest that their personal mental construct is a critical factor in forming and changing their beliefs. Preservice mathematics teacher beliefs should be analyzed in depth from their mental states in order to have a depth of understanding of their beliefs that might directly influence their current or future practices of teaching GTs (or any other contents) with GSP (or other technology). Hence, the future research should focus on the qualitative analysis of psychological factors of forming and shaping beliefs of preservice mathematics teachers toward reform-oriented teaching and learning of mathematics in general and teaching GTs with GSP in particular. A short-term study does not inform the participants' beliefs within a sustained belief system. Therefore, researchers should focus on longitudinal studies of preservice teacher beliefs from the start of their preservice teacher education and follow-up studies even after they complete programs, and they are in their professional lives as mathematics teachers.

Pedagogy of Mathematics Education

The pedagogical domains of findings and related implications of this study in terms of pedagogy of action, relation, and environment suggest that future practice and research on preservice mathematics teacher beliefs should focus on both the integrated and diversified pedagogy of mathematics education with technology. The integrated pedagogy should put the technology and contents together as integral technological-pedagogical domain. The diversified pedagogy of practice should focus on the differentiated instructions by allowing the preservice teachers to adopt most suitable pedagogical actions to carry out in the areas of assessment, personal and group activities, constructing proofs, creating self-identity, getting to the content through technology, building relationships, moving across interfaces, and semantic and syntactic interpretations of the processes. These elements of personal pedagogy should be the object of the future research on preservice mathematics teacher beliefs in a greater depth through prolonged and sustained research programs.

Epistemology of Mathematics Education

Literature revealed that preservice teachers' personal epistemology might influence their future pedagogical practices in the classroom. Cathy and Jack's views on how students come to know mathematics (e.g., either as individual struggles or collaborative group-work) reflect their personal epistemology. These epistemological beliefs could be the subject of study in the future research with a focus on how knowledge and experience in the past, present, and anticipation of future knowledge influence their beliefs and subsequent practices. Preservice mathematics teacher beliefs in terms of pre-, in- and post- reflective and reflexive beliefs have not been studied in the past and this could be a new area of focus on their personal epistemological beliefs in future studies.

Research Agenda for the Future

The discussion of findings and implications of this study suggest a potential research agenda for the future in my professional life and possibly others with a similar research and pedagogical interest. The possible research agenda for the future nested to the key issues in this study are – (i) How personal differences of preservice teachers in terms of their pedagogical beliefs lead their professional life? (ii) What do preservice teachers believe about different semantic interpretations of mathematical operations with and without using a technological tool? Why do they believe the way they believe about different semantic interpretations of mathematical processes? (iii) How does the syntactic organization of GTs with GSP affect preservice teachers' beliefs about the semantic interpretations of GTs? And, (iv) How do the temporal dimensions of beliefs in terms of reflective and reflexive beliefs interact together in forming and shaping new beliefs of preservice mathematics teachers? These are not random questions for future research, but they emerged from the findings and implications of this study. With these questions and other related issues, I aim to continue research on student and teacher beliefs about mathematics, teaching and learning mathematics, and technology integration in teaching and learning mathematics in my career as a mathematics teacher and educator.

Chapter Conclusion

I explored the beliefs of two preservice mathematics teachers about teaching GTs with GSP with these two research questions- What beliefs do preservice secondary mathematics teachers reveal about teaching geometric transformations (GTs) with Geometer's Sketchpad (GSP)? And, what beliefs do preservice secondary mathematics teachers hold about their future practices of teaching geometric transformations with Geometer's Sketchpad? He discussed six categorical beliefs of the two participants to discuss the first question. These categories included beliefs associated with an action, affect, attitude, environment, and object. Then, he discussed the directional category of beliefs in terms of reflective and reflexive beliefs that are associated with the participants' anticipated practices of using GSP for teaching GTs. Their beliefs in terms of pre-, in-, and post-reflective and reflexive beliefs characterized their anticipatory actions of using GSP for teaching GTs historically, idiosyncratically, and intuitively. Based on the outcomes of the study, I synthesized specific implications of the study in terms of psychological, pedagogical, epistemological, and practical implications.

Finally, some directions for future research emerged from the categorical and holistic beliefs and their essences for a broader understanding of beliefs about teaching mathematics with technology in general and teaching GTs with GSP in particular in terms of personal psychology, pedagogy, epistemology, and future research agenda. Although studies on preservice or inservice mathematics teacher beliefs seem to receive much attention in the past and present, there is still need of further in-depth studies in the areas of teacher and student mathematical, pedagogical, and technological beliefs to construct more feasible models to deal with complexities of beliefs for the transformation of mathematics education.

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APPENDIX – A: TASK SITUATIONS AND QUESTIONS FOR INTERVIEWS

Setting up of Interview

One-on-one semi-structured task-based interviews will be setup in a designated place on the time and dates agreed by the interviewer and interviewees. The interviewer will begin the interview session being ready with video recording camera, a notebook and pencil to write important points, pencil and blank sheets for interviewee, computer or other technological tools available for use during interviews.

Beginning of Interview

The interviewer will ask a few informal questions to break the ice. "How are you doing today? What mathematics classes are you taking this semester besides Methods of Teaching Mathematics? Do you like mathematics? Why or why not?" Then the researcher will introduce the research purpose and purpose of the interview. The interviewer will remind the interviewee about his or her rights of not to answer any questions partly or wholly if he or she wishes not to do so. He or she will be informed about the interview process and time, and his or her right to take a break or withdraw from the interview any time he or she wishes. There will not be any negative consequences of withdrawing participation in the interview. Since each research participant selected for interviews will participate in weekly interview sessions, each session will be separately dedicated to teaching of geometric transformation using Geometer's Sketchpad.

Task Situation for Reflection

- At first, let's construct a polygon (e.g., a triangle, a quadrilateral, etc.) using GSP.
- Construct a line of reflection anywhere within the same plane.
- Select the line of reflection and mark it as a mirror.
- Construct an image of the polygon under reflection in the line you just marked as a mirror.

Let's stop here for a while and think back on what you have just constructed.
What properties can we explore from this construction (of the polygon and its image under reflection)? What one conjecture can we (or students') make about this reflection? Why did you make this conjecture? Do you think this conjecture is important in relation to teaching reflection transformation? Why do you think so? How will you help your students to test this conjecture?

Let's think further about the reflection. What another conjecture can we (or students) make about this reflection? Why did you make this conjecture? Do you think this conjecture is important in terms of teaching reflection transformation? Why do you think so? How will you help your students test this conjecture?

Task Situation for Translation

- Now, let's construct a polygon (e.g., a triangle, a quadrilateral, etc.) using GSP.
- Construct a line segment of any length and direction within the same plane.
- Select the line segment and mark it as a vector for translation.

- Construct an image of the polygon under translation about the vector you just marked.

Let's stop here for a while and think back on what you have just constructed.
What properties can we explore from this construction (of the polygon and its image under translation)? What conjecture can we (or students') make about translation? Why did you make this conjecture? Do you think this conjecture is important in relation to teaching translation transformation? Why do you think so? How will you help your students to test this conjecture?

Let's think further about the translation. What another conjecture can we (or students) make about translation? Why did you make this conjecture? Do you think this conjecture is important in terms of teaching translation transformation? Why do you think so? How will you help your students test this conjecture?

Task Situation for Rotation

- At this time, let's construct a polygon (e.g., a triangle, a quadrilateral, etc.) using GSP.
- Fix a point within the same plane as the center of rotation.
- Select the point and mark it as center of rotation.
- Construct an image of the polygon under rotation through an angle (e.g., 30, 40, or 60 degree) about the center of rotation at the point you just fixed.

Let's stop here for a while and think back on what you have just constructed.
What properties can we explore from this construction (of the polygon and its image under rotation)? What conjecture can we (or students') make about rotation? Why did you choose this conjecture? Do you think it is more important in relation to teaching rotation transformation? Why do you think so? How will you help your students to test and explore this conjecture?

Let's think further about rotation transformation. What other conjecture can we (or students) make about rotations? Why did you make this conjecture? Do you think this conjecture is important in terms of teaching rotation transformation? Why do you think so? How will you help your students test and explore this conjecture? How do you anticipate your students explaining this conjecture?

Task Situation for Composite Transformations

- Let's construct a polygon (e.g., a triangle, a quadrilateral, etc.) using GSP.
- Reflect the polygon about a line.
- Then translate the image after reflection with a vector.
- Finally, rotate the image you just constructed about a point in 60 degree.

Let's stop here for a while and think back on what you have just constructed.
What properties can we explore from this construction (of the polygon and its images under composite transformations)? What conjecture can we (or students') make about the composite transformation? Why did you make this conjecture? Do you think this conjecture is important in relation to teaching composite transformations compared to a single transformation? Why do you think so? How will you help your students to test and explore this conjecture?

Let's think further about composite transformations. What another conjecture can we (or students) make about the composite transformations? Why did you make this conjecture? Do you think this conjecture is important in terms of teaching composite transformations? Why do you think so? How will you help your students test this conjecture? How do you anticipate your students explaining this conjecture?

Further Questions and Prompts (Based on Codes and Categories from First Analysis)

1. How do you feel about using GSP in teaching reflection, translation, and rotation in a class? Do you think yourself mentally ready to teach such concepts using GSP?
2. Are you convinced that use of GSP in teaching reflection, rotation, and translation helps in conceptual understanding of these transformations? Tell us a little more what convinced you? Why?
3. How can you decide that students have learned something when you teach these transformations using GSP? How will you know you are convinced with their learning?
4. How do you compare GSP with other manipulatives (e.g., geoboard) in relation to teaching geometric transformations?
5. How do you balance between conceptual and procedural learning of geometric transformations using GSP?
6. Do you think you have internalized teaching geometric transformations using GSP? How? Why?
7. What worries you most when using GSP in teaching GTs? Why? …………………………………………………………………………………………
8. How do you describe or define reflection, translation, and rotation transformations?
9. How do you explain the process of reflection, translation, and rotation?
10. How do you distinguish reflection from translation, reflection from rotation, and translation from rotation? You can use GSP to demonstrate the distinctions.
11. How do you demonstrate that reflection, rotation, and translation preserve distance and area? You can use GSP if wish to show it.
12. How will you organize students' classroom activities related to construction of an object (e.g., a triangle) and its image under reflection, rotation, and translation?
13. How do you develop your strategy to engage students in reasoning in relation to reflection, rotation, and translation? What kind of reasoning do you expect them to develop by using GSP in relation to GTs? How would this reasoning be different with other manipulatives?
14. What kind of problems can you point out in relation to teaching geometric transformations using GSP? Why they are problems? How can you address them?
15. How do you differentiate your instruction for students of different abilities in relation to teaching GTs by using GSP? What would this look like for gifted students? What would this look like for struggling students?
16. How do you compare your ability to teach GTs with GSP with other contents in Geometry?

17. How do you justify the use of GSP in teaching GTs? Why?
18. Can you share with us three important things related to use of GSP while teaching GTs? Why they are important for you?

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19. What would be the best use of GSP while teaching GTs? How? Why do you think so?
20. Do you think GSP can help you teach GTs better than without using it? Why or why not?
21. Do you think GSP is a good teaching tool? Why or why not?
22. Do you think GSP is a good learning tool? Why or why not?
23. How does GSP help in procedural understanding of reflection, rotation, and translation?
24. How does GSP help in conceptual understanding of reflection, rotation, and translation?
25. Can GSP be a motivational tool during the teaching and learning of GTs? How or Why?
26. What kind of individual activity for students in the class would you organize while teaching GTs with GSP? Why would you select this activity? How does it help students in learning GTs?
27. How do you plan to organize a group activity for students in the class while teaching/learning GTs with GSP?
28. What properties of GTs do you think students can explore using GSP? How? Why do you think so?
29. Let's think back to what you just did (construction of reflection, rotation or translation). Do you think you are comfortable in using GSP while teaching GTs? Can you tell us more about what made you more comfortable in using GSP?

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30. You just demonstrated a construction of a polygon and its image under reflection, rotation, or translation. How do you feel about this process? What do you think about the construction? Why?
31. What else do you think can be done using GSP while teaching GTs? How or Why?
32. There can be static and dynamic features of construction related to GTs by using GSP. Can you construct one example each (one for static and the other for dynamic)?
33. How do you compare the static construction with the dynamic? (Note: some questions related to their field teaching experience)
34. Let's do a construction related to a reflection followed by a rotation or translation. You can use the one you already constructed. How does this construction support in understanding of the composite process? Why?
35. What kind of other constructive actions do you think are important for teaching of GTs using GSP? Why?
36. Will you demonstrate a construction of an object (e.g., a triangle), its image under reflection under X axis followed by rotation with -90 degree and then translation by a vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$? What do you think about this kind of construction? Why?
37. Do you think geometric constructions of GTs with GSP helps students in understanding the concepts? Why?

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38. Do you think GSP helps in developing concepts of reflection, rotation, and translation? How? Why?
39. Do you think GSP helps in bridging the concepts from one level to other in relation to teaching and learning of GTs? (Transition)
40. What skills do you think GSP can develop in yourself in relation to teaching GTs? What about students' skills? How? Why do you think so?
41. In your view, does GSP make teaching GTs easier or more complicated? How? Why?
42. In your experience, how does GSP help in developing one's confidence in learning GTs? Why do you think so?
43. Then, do you think that GSP is a developmental tool? Why or why not?

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44. Will you please show us, again, how you explore a property of reflection, rotation, or translation using GSP? Tell us something about your exploration.
45. Does your action lead you to explore other properties? How? Why do you think so?
46. Do you think you can explore some algebraic properties of reflection or rotation or translation by using GSP? How? Why do you think so?
47. Is it possible to explore matrix of reflection, rotation, or translation using GSP? How?
48. Do you think this kind of exploration is possible in the classroom teaching and learning GTs with GSP? How? Why do you think so?
49. What else do you think students can explore about GTs using GSP? How? Why do you think so?

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50. What kind of environment do you think is good for teaching and learning GTs? How? Why do you think so?
51. Do you think use of GSP will enrich such an environment? How? Why do you think so?
52. In your view, what makes teaching and learning of GTs meaningful? How? Why do you think so?
53. How would you use GSP to create a better teaching and learning environment? Why?
54. Is such an environment possible without use of GSP? How? Why do you think so?
55. What kind of environment within the class or school supports teaching and learning of GTs with GSP? How? Why do you think so?
56. Do you think nature of GSP is a part of such environment? How? Why do you think so?

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57. Tell us something about GSP. What do you think or believe about GSP?
58. Tell us something about your experience of using GSP. How do these experiences unfold? How these experiences connect to your beliefs?
59. What features in the GSP do you like? What features do you dislike or avoid? Why?
60. What uses of GSP you like in relation to teaching GTs? Why?
61. Does the use of GSP make teaching GTs easy for you? Why?
62. Does the use of GSP make the teaching of GTs faster than without using it? Why or why not?

63. Does the use of GSP make the lesson in GTs more interesting than without using it? Why or why not?
64. Does the use of GSP help in exploring the properties of reflection, rotation, and translation more meaningfully? Why or why not?
65. Does the use of GSP make the teaching and learning of GTs more flexible? Why or why not?
66. Does the use of GSP make it easy to observe the phenomena of reflection, rotation, and translation? Why or why not?
67. What other attributes can you add to the use of GSP while teaching GTs? How?
   …………………………………………………………………………………………………
68. Do you think teaching and learning of GTs is easier by using GSP? How? Why do you think so?
69. Do you think teaching and learning of GTs is more effective by using GSP? How? Why do you think so?
70. Do you think teaching and learning of GTs is more interesting by using GSP? How? Why do you think so?
71. Do you think GSP can be used in any stage of teaching and learning GTs? Why do you think so?
72. What else do you think GSP can add in teaching of GTs? Why do you think so?
73. What else do you think GSP can add in learning of GTs? Why do you think so?
74. What features GSP do you think makes its use more meaningful in teaching of GTs? How? Why do you think so?
75. Do you think use of GSP helps in building good relation among students in the class while teaching and learning GTs? What about between students and teacher? What about students and content? Why do you think so?
76. What kind of tool do you think GSP is in relation to teaching and learning of GTs? Is it a teaching tool? Is it a learning tool? Or any other? How?

[Note: The questions were not the same as listed here in the different interviews. They were modified based on the participants’ responses and activities during the task-situations.]

APPENDIX – B: LETTER FROM THE INSTITUTIONAL REVIEW BOARD (IRB)

UNIVERSITY
OF WYOMING

Vice President for Research & Economic Development
1000 E. University Avenue, Department 3355 • Room 305/308, Old Main • Laramie, WY 82071
(307) 766-5353 • (307) 766-5320 • fax (307) 766-2608 • www.uwyo.edu/research

November 18, 2013

Shashidhar Belbase
Graduate Student
Mathematics Education
University of Wyoming

Protocol #07032013SB00003

Re: IRB Proposal, *"Preservice Secondary Mathematics Teachers' Beliefs About Teaching Geometric Transformations with Geometer's Sketchpad"*

Dear Shashidhar:

The proposal referenced above qualifies for expedited review with a minor modification to a previously approved protocol (approved July 3, 2013) and is approved as one that would not involve more than minimal risk to participants. Our expedited review and approval will be reported to the IRB at their next convened meeting December 19, 2013.

IRB approval for the project/research is for a one-year period, from the date of the original approval. If this research project extends beyond ***July 3, 2014***, a request to extend the approval accompanied by a report on the status of the project (Annual Review Form) must be submitted to the IRB at least one month prior to the expiration date. Any significant change(s) in the research/project protocol(s) from what was approved should be submitted to the IRB (Protocol Update Form) for review and approval prior to initiating any change. Per recent policy and compliance requirements, any investigator with an active research protocol may be contacted by the recently convened Data Safety Monitoring Board (DSMB) for periodic review. The DSMB's charge (sections 7.3 and 7.4 of the IRB Policy and Procedures Manual) is to review active human subject(s) projects to assure that the procedures, data management, and protection of human participants follow approved protocols. Further information and the forms referenced above may be accessed at the "Human Subjects" link on the Office of Research and Economic Development website: http://www.uwyo.edu/research/human-subjects/index.html.

You may proceed with the project and we wish you luck in the endeavor. Please feel free to call me if you have any questions.

Sincerely,

Colette Kuhfuss
Colette Kuhfuss
IRB Coordinator
On behalf of the Chairman
Institutional Review Board