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Developmental and Individual Differences in Understanding of Fractions

Robert S. Siegler and Aryn A. Pyke
Carnegie Mellon University

We examined developmental and individual differences in 6th and 8th graders' fraction arithmetic and overall mathematics achievement and related them to differences in understanding of fraction magnitudes, whole number division, executive functioning, and metacognitive judgments within a cross-sectional design. Results indicated that the difference between low achieving and higher achieving children's fraction arithmetic knowledge, already substantial in 6th grade, was much greater in 8th grade. The fraction arithmetic knowledge of low achieving children was similar in the 2 grades, whereas higher achieving children showed much greater knowledge in 8th than 6th grade, despite both groups having been in the same classrooms, using the same textbooks, and having the same teachers and classmates. Individual differences in both fraction arithmetic and mathematics achievement test scores were predicted by differences in fraction magnitude knowledge and whole number division, even after the contributions of reading achievement and executive functioning were statistically controlled. Instructional implications of the findings are discussed.

Keywords: fractions, numerical magnitude representations, arithmetic, mathematics difficulties, mathematics achievement

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Research on numerical development has focused on the development of whole numbers. This knowledge is obviously important, but numerical development also involves understanding of other types of numbers, such as fractions. Fractions are crucial not only because they express values that cannot be expressed with whole numbers but also because they provide most children's first opportunity to learn that many properties of whole numbers are not properties of numbers in general. All whole numbers can be represented by a single numeral, have unique successors, never decrease with multiplication, never increase with division, and so on. None of these properties, however, is true of fractions or of numbers in general. Instead, the only property that all real numbers have in common is that they have magnitudes that can be located and ordered on number lines.

Current Understanding of the Development of Fraction Knowledge

Many children have great difficulty acquiring fraction knowledge. Although children receive substantial fraction instruction beginning in third or fourth grade (National Council of Teachers of Mathematics, 2006), a recent National Assessment of Educational Progress revealed that 50% of eighth graders could not correctly order three fractions (2/7, 1/12, and 5/9) from least to greatest. The difficulty continues in high school and college; for example, on another National Assessment of Educational Progress item, fewer than 30% of 11th graders translated .029 into the correct fraction (Kloosterman, 2010). The pattern is longitudinally stable; children who have early difficulties with fractions tend to have later difficulties as well (Hecht & Vagi, 2010; Mazzocco & Devlin, 2008).

Failure to master fractions has large consequences. It impedes acquisition of more advanced mathematics and precludes participation in many occupations (McCloskey, 2007; National Mathematics Advisory Panel, 2008). A nationally representative sample of 1,000 U.S. Algebra 1 teachers rated fractions as one of the two largest weaknesses in students' preparation for their course (Hoffer, Venkataraman, Hedberg, & Shagle, 2007). Consistent with these informed opinions, analyses of two large, longitudinal data sets, one from the United States and one from the United Kingdom, indicated that fifth graders' knowledge of fractions uniquely predicted those students' knowledge of algebra and overall mathematics achievement in high school 5–6 years later, even after statistically controlling for other types of mathematical knowledge, verbal and nonverbal IQ, reading comprehension, working memory, and family income and education (Siegler et al., 2012).

Understanding fractions requires not only knowledge of procedures for solving fraction arithmetic problems but considerable conceptual knowledge as well. Conceptual knowledge of fractions involves knowing what fractions are: that they are numbers that stretch from negative to positive infinity; that between any two fractions are an infinite number of other fractions; that the numerator–denominator relation, rather than either number alone, determines fraction magnitudes; that fraction magnitudes increase

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Correspondence concerning this article should be addressed to Robert S. Siegler or Aryn A. Pyke, Psychology Department, Carnegie Mellon University, Baker Hall 342c, Pittsburgh, PA 15213. E-mail: rs7k@andrew.cmu.edu or apyke@andrew.cmu.edu.

with numerator size and decrease with denominator size; that fractions can be represented as points on number lines; and so on.

Understanding magnitudes appears to be a particularly important aspect of conceptual understanding of fractions. Several measures of fraction magnitude representations, including number line estimation, magnitude comparison, and ordering of multiple fractions, correlate highly with knowledge of fraction arithmetic and overall mathematics achievement from fifth to eighth grades (Bailey, Hoard, Nugent, & Geary, 2012; Mazzocco & Devlin, 2008; Siegler, Thompson, & Schneider, 2011). This relation between fraction magnitude knowledge and overall math achievement remains strongly present even when fraction arithmetic competence is statistically controlled.

Acquiring conceptual and procedural knowledge of fractions poses several serious challenges. One challenge in acquiring conceptual knowledge is that children's massive prior experience with integers leads to a whole number bias, in which properties of positive integers are assumed to extend to fractions (Ni & Zhou, 2005). For example, even high school students often claim that there are no numbers between fractions such as 5/7 and 6/7 (as there are no integers between 5 and 6) and that multiplication cannot yield products smaller than both multiplicands (Vamvakoussi & Vosniadou, 2004).

Acquiring procedural knowledge of fraction arithmetic poses different difficulties. Components of different fraction arithmetic procedures are complexly related. Addition and subtraction of fractions require common denominators, but multiplication and division of fractions do not. Fraction multiplication problems can be solved by applying the arithmetic operation independently to the numerators and to the denominators, but fraction addition and subtraction problems cannot be solved in this way. In fraction division, but in no other operation, answers can be obtained through the invert-and-multiply procedure.

This analysis suggests that conceptual and procedural knowledge of fractions should be related, because without conceptual understanding, the procedures are highly confusable and difficult to remember. Consistent with this reasoning, conceptual and procedural knowledge of fractions correlate substantially over a wide age range (Hecht, 1998; Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010; Siegler et al., 2011).

Goals of the Present Study

Describing Fraction Development Among Low and Higher Achieving Children

The present study was designed to provide a more nuanced description of developmental and individual differences in fraction arithmetic knowledge than currently available. To pursue this goal, we investigated differences between sixth and eighth graders in conceptual and procedural knowledge of fractions among low achieving (LA) and higher achieving (HA) children. We examined three tasks to assess conceptual knowledge of fractions (magnitude comparison and 0–1 and 0–5 number line estimation), five aspects of fraction arithmetic (accuracy, strategy use, strategy variability, confidence, and calculation errors), and two types of processes that seemed likely to underlie fraction proficiency (whole number division and executive functioning).

This design allowed comparison of HA and LA children on all of the measures, as well as comparison of the types of errors made by the two groups, which have been found to vary in previous studies (Mazzocco & Devlin, 2008). Of special interest was whether increases in fraction knowledge between sixth and eighth grades would differ for HA and LA children.

One plausible position was that LA children generally benefit less from instruction than do typically achieving children (Fuchs et al., 2005); they might be expected to make less progress in this domain, too. Consistent with this argument, Hecht and Vagi (2010) and Mazzocco and Devlin (2008) found poorer fraction learning among LA than HA children between fourth and fifth grades and between seventh and eighth grades, respectively. The LA and HA children in our sample were drawn from the same classrooms; thus, they were exposed to the same teachers, classmates, and textbooks and received very similar instruction. However, this does not imply that their learning was similar.

Another logic yielded the opposite prediction. Having learned less from initial instruction in fractions, LA children would have more to learn later. Such patterns of change have been observed at times with whole number procedures (Chong & Siegel, 2008). Instruction during middle school might help LA children catch up to classmates whose fraction understanding was already relatively good.

A third possibility was that patterns for conceptual and procedural knowledge would differ in ways partially consistent with both of the other hypotheses. Conceptual knowledge of fractions is usually taught before fraction arithmetic (National Council of Teachers of Mathematics, 2006). If HA children leave elementary school with greater conceptual understanding of fractions, they might make greater strides in learning fraction arithmetic during middle school. In contrast, LA children might make as great or greater progress in conceptual understanding during middle school, because of their having more to learn in that area, but their arithmetic learning might be poorer, because of their not initially possessing adequate conceptual understanding to understand the arithmetic procedures. The present design allowed us to determine which of these scenarios most accurately described changes in fraction knowledge during middle school.

Identifying Sources of Individual Differences in Fraction Knowledge

A second goal of the study was to examine the contributions of four potential sources of individual differences in fraction knowledge: weak executive functioning, inadequate understanding of whole number division, limited knowledge of fraction magnitudes, and inaccurate metacognitive assessments of one's fraction arithmetic knowledge.

Contributions of executive functioning. Executive functioning, which involves control of working memory and attention, was one likely contributor to individual differences in fraction arithmetic and mathematics achievement. Inhibition and updating, two processes identified as central to executive functioning by Miyake, Friedman, Emerson, Witzki, and Howerter (2000), have been found to be related to whole number arithmetic; moreover, children with LA tend to exhibit executive functioning deficits relative to typically achieving peers (Bull & Scerif, 2001; Mazzocco &

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Kover, 2007; Passolunghi & Pazzaglia, 2004, 2005; Swanson & Beebe-Frankenberger, 2004).

Although most research relating executive functioning to math knowledge has been done with whole numbers, executive functioning might be at least as strongly related to fraction knowledge. Conceptual knowledge of fractions requires inhibiting the tendency to view fractions as two independent whole numbers and instead focusing on the relation between numerator and denominator. Procedural knowledge of fractions requires frequent updating of working memory to keep track of completed and yet to-be-completed subgoals and procedures (mentally add 2/3 + 4/5 to get a sense of the demands of fractions arithmetic on updating).

Contributions of fraction magnitude knowledge. Logically, fraction magnitude knowledge could be unrelated to fraction arithmetic. Children could learn such procedures through rote memorization, without reference to the numerical magnitudes that the fractions represent. However, consistent relations have emerged, probably for several reasons. Fraction magnitude knowledge allows children to check the plausibility of their answers and to reject procedures that generate implausible answers (Hiebert & LeFevre, 1986). Magnitude knowledge also might inhibit the use of flawed arithmetic procedures that treat numerators and denominators as independent whole numbers rather than as parts of a single magnitude that can be combined with another magnitude. A third reason involves motivation. Without knowledge of fraction magnitudes, fraction arithmetic, pre-algebra equation solving, and other aspects of middle school mathematics become arbitrary procedures that generate meaningless outcomes. When material is meaningless, people have little motivation to learn it (Reyna, Chapman, Dougherty, & Confrey, 2011). For example, without understanding what 1/3 means, there is no particular reason why on $1/3 \times X = Y$, X must be larger than Y. As this example illustrates, fraction knowledge is necessary for subsequent mathematics, including pre-algebra and algebra, to make sense. Thus, fraction magnitude knowledge was expected to be related to overall math achievement as well as to fraction arithmetic, even after other aspects of intellectual functioning and mathematical knowledge were statistically controlled.

Contributions of whole number division. From one perspective, the relation of whole number division to fractions is obvious. Any fraction can be viewed as a division problem (N/M); in this sense, fractions are division. However, this logical relation does not imply that the two are closely related psychologically. Indeed, knowledge of whole number division and of fractions in fifth grade independently predicted the children's mathematics achievement in 10th grade; neither predictive relation explained the other (Siegler et al., 2012).

Whole number division might be important for understanding fractions for several reasons. Alone among whole number operations, division often yields fractional answers (e.g., $14 \div 3 = 4\ 2/3$). Experience with whole number division also could provide knowledge of the magnitude of some fractions (e.g., 8/2) and a rough sense of the magnitudes of others (e.g., 10/3 is between 3 and 4). In contrast, whole number addition, subtraction, and multiplication never yield fraction answers and thus do not present opportunities to learn about them.

Contributions of metacognition. A fourth potential source of individual differences involves assessments of one's own knowledge. Inadequate metacognitive knowledge might produce either overconfidence in arithmetic answers across the board or lack of differences in confidence in correct and incorrect answers. If children are confident that incorrect procedures and the answers they yield are correct, they might not check their work or consider whether the answers are plausible (Mazzocco, 2007).

Previous findings indicate that children who are more accurate on a given math task tend to be more confident in the correctness of their answers and that typically achieving children are more confident in their answers than are children with math difficulties (Desoete, Roeyers, & Buysse, 2001; Garrett, Mazzocco, & Baker, 2006; Luncangeli & Cornoldi, 1997). However, the children in the current study were considerably older than those in most previous studies (middle school vs. early elementary school). At these older ages, both HA and LA children might accurately evaluate the correctness of their answers.

Thus, our main goals in the present study were to describe the development of fraction knowledge during middle school among LA and HA children and to test four potential sources of individual differences in fraction arithmetic and overall math achievement: knowledge of fraction magnitudes, knowledge of whole number division, executive functioning, and metacognitive awareness of one's own fraction competence.

Method

Participants

Participants were 60 sixth graders (M age = 12.04 years, SD = 0.61; 50% girls; 68% Caucasian, 25% African American, 2% Asian, 5% biracial) and 60 eighth graders (M age = 14.08 years, SD = 0.49; 53% girls; 67% Caucasian, 30% African American, 2% Asian, 1% biracial) recruited from three low-income public school districts near Pittsburgh, Pennsylvania. All students who returned consent forms were included in the study. Testing was done by two female research associates and the second author (one Asian, two Caucasians).

The math section of the previous year's Pennsylvania System of School Assessments (PSSA), the standardized achievement test used in Pennsylvania, provided a measure of overall mathematics knowledge. Children whose scores were in the bottom 35% were classified as LA, and those in the top 65% were classified as HA. This threshold has been used to good effect in previous studies (Hanich, Jordan, Kaplan, & Dick, 2001; Jordan, Hanich, & Kaplan, 2003). PSSA results for eight children were unavailable because they had transferred from another state or because their parents did not release their scores. Among children whose scores were known, 33% (15 sixth graders and 22 eighth graders) were LA. Test scores of sixth and eighth graders were comparable: For HA students, Ms = 66th and 67th percentiles, $p = .916$; for LA students, Ms = 18th and 17th percentiles, $p = .894$.

Tasks

Number line estimation. On the 0–1 task, children were presented number lines on a computer screen, each with 0 at the left end, 1 at the right, and the fraction to be estimated above the line. Children responded by moving the cursor to the position on the line that they thought corresponded to the number being estimated and clicking the track pad. Then, a new number line with a different fraction appeared, and the process repeated. To acquaint children with the track pad, the experimenter asked them to first locate the practice fraction 1/4; no feedback was given.

After the practice trial, children estimated the positions of the 10 fractions that provided the experimental data: from smallest to largest, 1/19, 2/13, 1/5, 1/3, 3/7, 7/12, 5/8, 3/4, 7/8, and 13/14. Two of these fractions were drawn from each fifth of the number line. Here, as on all experimental tasks, presentation order of items was random and no feedback was provided.

The 0–5 number line task was identical, except that the right endpoint was labeled *5,* the practice fraction was 7/2, and the 10 fractions to be estimated were 1/5, 7/8, 11/7, 9/5, 13/6, 7/3, 13/4, 10/3, 9/2, and 19/4.

Magnitude comparison. Children were asked to compare 3/5 to a series of other fractions shown one at a time in random order on a computer screen: 2/7, 1/3, 5/11, 4/7, 2/3, 3/4, 7/9, and 5/6. The task was to press the *a* key if the fraction was smaller than 3/5 and to press the *l* key if the fraction was larger than 3/5.

Fraction arithmetic. Children were presented with 16 problems, four for each of the four arithmetic operations, one at a time on a computer screen. For each operation, the four problems combined the operand 3/5 with 1/5, 1/4, 2/3, and 4/5. Thus, half of the problems for each operation had operands with equal denominators (e.g., 3/5 + 1/5; 4/5 + 3/5) and half had operands with unequal denominators (e.g., 3/5 + 1/4; 3/5 + 2/3). For subtraction and division problems, the larger operand was always listed first. On each item, children received a sheet of paper that stated the problem at the top and afforded ample space for paper-and-pencil calculations. After generating an answer, children were asked to rate their confidence in it on a 5-point scale (1 = *not confident at all,* 5 = *extremely confident*) and then to describe how they solved the problem. These explanations were audio recorded for later coding.

Whole number division. Children were presented three whole number division problems: 56 ÷ 8, 306 ÷ 9, and 91÷ 4. Pencil and paper for calculations were provided. For the two problems with whole number quotients, children received 1 point for being correct; for the problem that had a quotient including a fraction, children received 1 point for a completely correct answer (e.g., 22.75 or 22 3/4), 0.75 point for an answer with a correct remainder (e.g., 22, R 3), and 0.5 point for answers where only the whole number was correct (e.g., 22 or 22.3).

Standardized math and reading achievement tests. We obtained children's scores from the PSSA mathematics and reading sections given toward the end of fifth and seventh grades, about half a year before the study began. The PSSA math section examined a range of skills, including knowledge of whole number and fraction arithmetic; probability and statistics; interpretation of tables, graphs, and figures; pre-algebra; geometry; and series extrapolation. The PSSA reading section examined vocabulary, general comprehension, and inferential reasoning skills. For example, children read about a baker who argued that a man enjoying smells from his shop should have to pay for the pleasure; the children then were asked if the baker lacks (a) talent, (b) popularity, (c) success, or (d) generosity.

Executive function (EF). Children were presented two EF tasks adapted from Miyake et al. (2000), one measuring updating of working memory and one measuring inhibition.

Working memory updating. Letters were presented one at a time on a computer screen for 2,750 ms each. The task was to recall, in order, the three most recently presented letters when "???" appeared on the screen. For example, after seeing "Q, D, X, R, M, ???" the child was to respond "XRM." Children were encouraged to rehearse aloud the three most recent letters as each new letter appeared. This task required updating one's memory by adding the most recently presented letter and dropping the fourth most recently presented. After three practice items, children performed nine trials on which five, seven, or nine letters appeared before the recall request. The nine trials included three of each length and were ordered randomly. The dependent measure was percentage of recall of the 27 letters from the nine trials.

Inhibition. A fixation cross was presented at the center of the computer screen for a period varying from 1,500 to 3,500 ms. A black square (0.4° wide) then appeared for 200 ms on a randomly determined side of the screen. Next, a target letter appeared, either in the square's previous location (prosaccade trials) or on the opposite side of the screen (antisaccade trials); the task was to identify the letter. Because the letter was present for only 150 ms before being masked by a gray square, initially looking toward the wrong side generally ruled out correct letter identification.

Participants first completed 26 practice trials: 18 prosaccade trials followed by eight antisaccade trials. Then they completed the 36 antisaccade trials that provided the experimental data. These antisaccade trials were the trials of interest, because they required participants to inhibit the inclination to first look toward the cue that appeared in their periphery (the black square) and instead to attend to the opposite side of the screen. The dependent measure was the percentage of antisaccade target trials on which children correctly identified the letter.

Design and Procedure

Children were tested individually in a quiet room in their school and completed all tasks on a laptop computer during a single 45-min session. Both the items within each mathematics task and the tasks themselves were ordered randomly. The two executive function tasks (updating and inhibition) were presented in random order after the math tasks.

Results

We describe, in order, findings on development of fraction magnitude representations (number line and magnitude comparison tasks), fraction arithmetic, whole number division, executive functioning, and predictors of individual differences in fraction arithmetic and overall math achievement. Pairwise comparisons using the Bonferroni correction were used to interpret statistical interactions. Mauchly's test indicated that the assumption of sphericity was violated in a few cases; when that occurred, Greenhouse–Geisser corrections to the degrees of freedom were used. Performance of the eight children whose PSSA scores were unavailable was excluded from all analyses that included LA or HA status as a variable.

Fraction Magnitude Tasks

Number line estimation. Accuracy of number line estimation was indexed by percentage of absolute error (PAE), defined as

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$$(|\text{Answer} - \text{Correct Answer}|)/\text{Numerical Range} \times 100\%.$$

For example, if a child was asked to locate 9/2 on a 0–5 number line and the child marked the location corresponding to 7/2, the PAE would be 20%: (|3.5 – 4.5|)/5 × 100%. PAE is an index of error; the higher the PAE, the less accurate the estimate.

A 2 (grade: sixth or eighth) × 2 (achievement status: HA or LA) × 2 (number line: 0–1 or 0–5) analysis of variance (ANOVA) revealed that eighth graders' estimates were more accurate than those of sixth graders (PAEs = 17% and 22%), $F(1, 108) = 7.31$, $p = .008$, $\eta_p^2 = .06$; HA students' estimates were more accurate than those of LA students (PAEs = 13% and 26%), $F(1, 108) = 63.62$, $p < .001$, $\eta_p^2 = .37$; and estimates were more accurate on 0–1 than 0–5 number lines (PAEs = 16% and 23%), $F(1, 108) = 23.17$, $p < .001$, $\eta_p^2 = .18$. No interactions were significant.

Magnitude comparison. A Grade × Achievement Status ANOVA indicated that comparison accuracy increased marginally from sixth to eighth grades (69% vs. 75% correct), $F(1, 108) = 3.17$, $p = .078$, $\eta_p^2 = .03$, and was higher among HA than LA students (81% vs. 63%), $F(1, 108) = 24.05$, $p < .001$, $\eta_p^2 = .18$. In both grades, the greater the distance of the comparison fraction from 3/5, the greater the number of correct comparisons: $r(6) = .73$, $p = .039$, for sixth graders and $r(6) = .66$, $p = .076$, for eighth graders.

Fraction Arithmetic

To obtain a nuanced understanding of fraction arithmetic development, we examined accuracy, strategy use, strategic variability, calculation errors, and confidence ratings.

Accuracy. A 2 (grade) × 2 (achievement status) × 4 (arithmetic operation) × 2 (denominator: equal or unequal) ANOVA on number of correct answers yielded main effects for all four variables: grade (41% correct for sixth graders vs. 57% for eighth graders), $F(1, 108) = 15.57$, $p < .001$, $\eta_p^2 = .13$; achievement status (36% correct for LA students vs. 61% for HA students), $F(1, 108) = 37.97$, $p < .001$, $\eta_p^2 = .26$; arithmetic operation (60% correct for addition, 68% for subtraction, 48% for multiplication and 20% for division), $F(1.93, 208.3) = 45.82$, $p < .001$, $\eta_p^2 = .30$; and denominator equality (55% correct for problems with equal denominators vs. 43% for those with unequal ones), $F(1, 108) = 48.87$, $p < .001$, $\eta_p^2 = .31$. Interactions were present for Grade × Achievement Status, $F(1, 108) = 4.24$, $p = .042$, $\eta_p^2 = .04$; Achievement Status × Arithmetic Operation, $F(1.93, 208.3) = 8.89$, $p < .001$, $\eta_p^2 = .08$; Achievement Status × Denominator Equality, $F(1, 108) = 34.34$, $p < .001$, $\eta_p^2 = .24$; Denominator Equality × Arithmetic Operation, $F(2.13, 230.0) = 39.15$, $p < .001$, $\eta_p^2 = .27$; and among arithmetic operation, denominator equality, and achievement status, $F(2.13, 230.0) = 5.85$, $p = .003$, $\eta_p^2 = .05$.

The three-way interaction and several of the two-way interactions reflected differences between HA and LA children in the effects of denominator equality on the four arithmetic operations (see Table 1). Among HA children, accuracy was higher on equal than on unequal denominator problems for addition (87% correct on equal vs. 74% on unequal denominator problems, $p = .013$), subtraction (91% vs. 77%, $p = .002$), and division problems (36% vs. 27%, $p = .002$). In contrast, HA children's multiplication accuracy was lower on equal than on unequal denominator problems (34% vs. 61%, $p < .001$). This difference reflected children often extending equal denominators in the operands to the answer (e.g., 1/5 × 3/5 = 3/5), an approach that was correct for addition and subtraction but incorrect for multiplication.

Table 1
Percentage Correct on Fraction Arithmetic by Problem Type for Low Achieving (LA) and Higher Achieving (HA) Students

Denominator	Achievement status LA	HA
Addition		
Equal	63	87
Unequal	12	74
Subtraction		
Equal	75	91
Unequal	28	77
Multiplication		
Equal	40	34
Unequal	55	61
Division		
Equal	12	36
Unequal	6	27

LA children exhibited effects of denominator equality that were similar but much larger than those of their HA peers for both addition (63% correct on equal denominators problems vs. 12%, on unequal denominator ones, $p < .001$) and subtraction (75% vs. 28%, $p < .001$). For multiplication, the effect of denominator equality was somewhat smaller among children with LA than among their HA peers but in the same direction (40% on equal vs. 55% on unequal, $p = .050$). For division, the accuracy of children with LA was low regardless of whether denominators were equal or unequal (12% and 6% correct, respectively, $p = .114$).

The forest and the trees. It is easy to get lost in the details of these data and to miss the most striking finding: the differing pattern of change during middle school in fraction arithmetic knowledge of HA and LA children. As shown in Table 2, increases in percentage correct between sixth and eighth grades were greater for HA than LA children on all eight types of fractions arithmetic problems. HA children's accuracy was significantly higher among eighth than sixth graders on 6 of 8 types of problems, compared with 0 of 8 for LA children. The LA children were already less accurate as sixth graders, but the gap between their and the HA children's accuracy widened considerably between sixth and eighth grades.

Strategies. Children's fraction arithmetic strategies fell into four categories. *Correct strategies* (54% of trials) were generally the standard fraction arithmetic algorithms. *Independent whole numbers strategies* (14%) involved performing the arithmetic operation on numerators and denominators separately, as if they were unrelated whole numbers (e.g., 3/5 + 1/4 = 4/9). *Wrong fraction operation strategies* (26%) involved importing components of procedures that are correct for another fraction arithmetic operation but inappropriate for the requested operation (e.g., leaving a common denominator unchanged on a multiplication problem, as in 3/5 × 4/5 = 12/5). Finally, the *none/other* classification (7%) included trials on which children used idiosyncratic strategies (e.g., inverting the wrong fraction on a division problem) or refused to try any approach.

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Table 2
Gains in Fraction Arithmetic Accuracy From Sixth to Eighth Grades Among Low Achieving (LA) and Higher Achieving (HA) Children

	% accuracy gain	
Denominator	LA	HA
Addition		
Equal	6	12
Unequal	4	22*
Subtraction		
Equal	10	13
Unequal	3	20*
Multiplication		
Equal	6	20*
Unequal	9	31*
Division		
Equal	11	36*
Unequal	11	40*

Note. Accuracy gains are the difference between eighth grade and sixth grade percentages correct.
* $p < .05$.

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The analysis of frequency of correct strategies yielded highly similar results to the analysis of frequency of correct answers, despite correct strategies yielding incorrect answers on 9% of trials, usually because of whole number arithmetic errors (e.g., 5 × 3 = 20). Therefore, we only report analyses of frequencies of the two common incorrect strategies.

Independent whole numbers strategy. Treating numerators and denominators as independent whole numbers leads to incorrect answers for addition and subtraction and to unnecessarily complex answers for division. We excluded multiplication from analyses of this strategy, because it and the correct strategy were identical.

A Grade × Achievement Status × Arithmetic Operation × Denominator Equality ANOVA indicated that the independent whole numbers strategy was more frequent among sixth than eighth graders (25% vs. 16%), $F(1, 108) = 3.09$, $p = .082$, $\eta_p^2 = .03$; among LA than HA children (32% vs. 10%), $F(1, 108) = 17.66$, $p < .001$, $\eta_p^2 = .14$; on addition than on subtraction or division problems (26%, 20%, and 15%, respectively), $F(1.79, 193.4) = 10.37$, $p < .001$, $\eta_p^2 = .09$; and on problems with unequal than equal denominators (25% vs. 16%), $F(1, 108) = 16.58$, $p < .001$, $\eta_p^2 = .13$.

These effects were qualified by interactions between arithmetic operation and denominator equality, $F(1.74, 188.1) = 11.48$, $p < .001$, $\eta_p^2 = .10$; between arithmetic operation and achievement status, $F(1.79, 193.4) = 9.49$, $p < .001$, $\eta_p^2 = .08$; and among arithmetic operation, denominator equality, and achievement status, $F(1.74, 188.1) = 4.04$, $p = .024$, $\eta_p^2 = .04$. The three-way interaction and one of the two-way interactions reflected differing strategy use by HA and LA children. Among HA children, use of the independent whole numbers strategy was comparable for equal and unequal denominator problems for addition (10% and 12%, $p = .643$), subtraction (6% and 10%, $p = .209$), and division (12% and 8%, $p = .306$). Among LA students, the same was true for division (20% and 21%, $p = .735$), but these children used the independent whole numbers strategy twice as often on problems with unequal as equal denominators for addition (55% vs. 28%, $p < .001$) and subtraction (45% vs. 21%, $p < .001$). This suggests that LA children sometimes used the whole number strategy because they did not know how to generate a common denominator rather than because they thought it was correct.

Wrong fraction operation strategies. Children often applied components of procedures that were correct for another fraction arithmetic operation but incorrect for the operation specified in the problem. A Grade × Achievement Status × Arithmetic Operation × Denominator Equality ANOVA indicated that such wrong fraction operation strategies were more common among sixth than eighth graders (30% vs. 22% of problems), $F(1, 108) = 5.31$, $p = .023$, $\eta_p^2 = .05$, and much more common on division and multiplication than on addition and subtraction problems (55%, 46%, 1%, and 1%, respectively), $F(1.77, 190.0) = 130.98$, $p < .001$, $\eta_p^2 = .55$. The disproportionate frequency of these errors on multiplication and division problems seems attributable to children transferring components of earlier learned fraction arithmetic operations (addition and subtraction) to later learned ones.

Interactions were present for achievement status and grade, $F(1, 108) = 5.55$, $p = .020$, $\eta_p^2 = .05$; achievement status and arithmetic operation, $F(1.77, 191.0) = 3.30$, $p = .045$, $\eta_p^2 = .03$; and achievement status, grade, and arithmetic operation, $F(1.77, 191.0) = 5.12$, $p = .009$, $\eta_p^2 = .05$. The three-way interaction reflected different patterns of differences between sixth and eighth grades for HA and LA children. Among HA children, wrong fraction operation strategies were more common in sixth than eighth grade children on multiplication and division problems (65% to 34% and 65% to 29%, ps $< .001$). Similar decreases on addition and subtraction were impossible, because wrong fraction operation strategies were already uncommon in sixth grade (0% and 3%). In contrast, among LA children, use of the strategy was comparable for sixth and eighth graders for multiplication (47% and 40%, $p = .596$) and division (57% and 68%, $p = .385$). As with HA children, wrong operation errors were already rare in sixth grade (3% and 1%), so no decrease was possible. Thus, another way that the arithmetic performance of HA children improved between sixth and eighth grades and that of LA children did not was in reduction of use of wrong fraction operation errors for multiplication and division.

Variability of fraction arithmetic strategies. Highly consistent strategy use might have been expected within the pairs of virtually identical problems (e.g., multiplication problems with common denominators, e.g., 3/5 × 1/5 and 3/5 × 4/5). However, most participants used different strategies on at least one pair of highly similar problems. This variability was present among a greater percentage of LA than HA children (79% vs. 51%), $F(1, 108) = 7.94$, $p = .006$, $\eta_p^2 = .07$. LA children also generated a greater number of pairs of trials that showed such strategic variability (21% for LA children and 11% for HA children), $F(1, 108) = 13.10$, $p < .001$, $\eta_p^2 = .11$.

This variability often did not reflect children lacking knowledge of the correct strategy and therefore vacillating between different incorrect strategies. Children used one correct and one incorrect strategy on 65% of the inconsistent pairs of trials (67% for HA children, 64% for LA children, *ns*).

Calculation errors during fraction arithmetic strategies. Incorrect answers sometimes resulted from whole number fact errors, such as writing "5 × 3 = 20." A Grade × Achievement Status ANOVA indicated that sixth graders made more such errors than did eighth graders (25% vs. 16%), $F(1, 108) = 9.76$, $p = .002$, $\eta_p^2 = .08$, and that LA children made more of them than did HA children (28% vs. 13% of trials), $F(1, 108) = 28.60$, $p < .001$, $\eta_p^2 = .21$.

Confidence in fraction arithmetic answers. A Grade × Achievement Status × Arithmetic Operation × Denominator Equality ANOVA indicated that confidence in the correctness of answers was greater among HA than LA children (means of 4.1 vs. 3.4 on the 1–5 rating scale), $F(1, 108) = 20.17$, $p < .001$, $\eta_p^2 = .16$; on addition than subtraction, subtraction than multiplication, and multiplication than division problems (*Ms* for the four arithmetic operations = 4.2, 3.9, 3.7, and 3.1, respectively), $F(2.23, 240.7) = 75.56$, $p < .001$, $\eta_p^2 = .41$; and on problems with equal than unequal denominators (3.8 vs. 3.6), $F(1, 108) = 19.63$, $p < .001$, $\eta_p^2 = .15$.

Denominator equality and arithmetic operation interacted, $F(3, 324) = 20.20$, $p < .001$, $\eta_p^2 = .16$. Confidence was higher on equal than unequal denominator problems for addition (4.4 vs. 4.1, $p < .001$), subtraction (4.1 vs. 3.7, $p < .001$), and division (3.3 vs. 2.9, $p < .001$), but lower on equal than unequal denominator multiplication problems (3.6 vs. 3.8, $p = .002$). This interaction for confidence ratings paralleled the one for accuracy; just as children erred more often when multiplication problems had equal denominators, they were also less confident in their answers on such problems. A three-way interaction of denominator equality, arithmetic operation, and grade was also present, $F(3, 324) = 4.19$, $p = .006$, $\eta_p^2 = .04$, although the reasons for it were unclear.

Among the 92% of children who generated at least one correct and one incorrect answer on the 16 arithmetic problems, a within-subjects analysis revealed that children were more confident in their correct answers than incorrect answers (mean confidence rating = 4.2 vs. 3.5), $F(1, 109) = 126.61$, $p < .001$, $\eta_p^2 = .54$. Mean confidence in correct versus incorrect answers was higher among most children (79%-87%) in all four Grade × Achievement Status groups.

Whole Number Division

A Grade × Achievement Status ANOVA revealed that HA students were much more accurate than students with LA at whole number division (89% vs. 53% correct), $F(1, 108) = 43.26$, $p < .001$, $\eta_p^2 = .29$. No other effects were significant ($Fs \leq 1$).

Executive Functioning

A Grade × Achievement Status ANOVA indicated higher memory updating scores for HA than LA children (78% vs. 69% correct), $F(1, 108) = 8.54$, $p = .004$, $\eta_p^2 = .07$. A Grade × Achievement Status interaction, $F(1, 108) = 5.26$, $p = .024$, $\eta_p^2 = .05$, reflected higher updating scores among eighth than sixth grade HA children (82% vs. 74%, $p = .042$) but not among LA peers (65% for eighth graders, 73% for sixth graders, $p = .176$).

A Grade × Achievement Status ANOVA indicated that HA children were more successful than LA ones in inhibiting inappropriate responses (40% vs. 34% correct), $F(1, 108) = 4.07$, $p = .046$, $\eta_p^2 = .04$. No effect for age or interaction was present.

Individual Differences

Predictors of fraction arithmetic performance. In both grades, almost all measures were related to fraction arithmetic accuracy (see Table 3). To examine predictors of fraction arithmetic accuracy in a more structured way, we constructed a hierarchical regression model. We first entered the child's PSSA reading score, to control for general intellectual ability. Next, we entered measures of performance on the experimental tasks in the order in which the competencies seem likely to be acquired during the developmental/educational process: (a) executive functions, (b) whole number division, and (c) fraction magnitude measures. The two EF measures were entered in a single step of the hierarchical regression, as were the three measures of fraction magnitude knowledge.

Reading scores accounted for 14% of the variance in sixth graders' fraction arithmetic scores, $F(1, 54) = 8.53$, $p = .005$, and 33% of the variance in eighth graders' scores, $F(1, 54) = 26.98$, $p < .001$. The two executive function measures explained an additional 2% of variance for sixth graders (*ns*), and 1% for eighth graders (*ns*). Whole number division accounted for an additional 8% of variance for sixth graders, $F(1, 51) = 5.43$, $p = .024$, and 6% for eighth graders, $F(1, 51) = 5.31$, $p = .025$. The fraction

Table 3
Correlations Among Experimental Tasks and Math Achievement Tests for Grade 6 (Above Diagonal) and Grade 8 (Below Diagonal)

Measure	1	2	3	4	5	6	7	8	9	10	11
1. Math Achievement (PSSA)	—	.600**	.456**	−.477**	−.499**	.164	.234	−.470**	.466**	.010	.345**
2. Reading Achievement (PSSA)	.745**	—	.460**	−.315*	−.361**	.351**	.369**	−.405**	.299*	.206	.149
3. Fraction Magnitude Comparison	.567**	.427**	—	−.493**	−.318*	.524**	.559**	−.519**	.418**	.088	.123
4. Number line 0–1 (PAE)	−.763**	−.651**	−.568**	—	.527**	−.266*	−.295*	.271*	−.341**	−.295*	−.109
5. Number line 0–5 (PAE)	−.628**	−.507**	−.476**	.457**	—	−.325*	−.343**	.368**	−.484**	−.357**	−.190
6. Fraction arithmetic correct strategy	.621**	.579**	.455**	−.483**	−.507**	—	.960**	−.586**	.393**	.256*	.137
7. Fraction arithmetic answer accuracy	.615**	.577**	.405**	−.433**	−.540**	.917**	—	−.626**	.422**	.243	.076
8. Fraction arithmetic calculation errors	−.540**	−.561**	−.383**	.470**	.353**	−.493**	−.635**	—	−.348**	−.136	−.230
9. Whole number division	.703*	.603**	.421**	−.708**	−.488**	.656**	.628**	−.562**	—	.287*	.231
10. EF: Update (letter memory)	.507**	.569**	.306*	−.550**	−.457**	.363**	.446**	−.300*	.392**	—	.140
11. EF: Inhibit (antisaccade)	.306*	.277*	.228	−.381**	−.354**	.248	.265*	−.332**	.387**	.364**	—

Note. $N = 60$ per grade ($N = 56$ per grade for PSSA achievement correlations). PSSA = Pennsylvania System of School Assessments; PAE = percentage of absolute error; EF = executive function.
* $p < .05$. ** $p < .01$.

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magnitude tasks contributed an additional 12% of variance for sixth graders, $F(3, 48) = 3.04$, $p = .038$, and 8% for eighth graders, $F(3, 48) = 2.37$, $p = .082$. In all, these predictors accounted for 36% of variance in sixth graders' fraction arithmetic scores, $F(7, 48) = 3.78$, $p = .002$, and 48% of variance in the eighth graders' scores, $F(7, 48) = 6.36$, $p < .001$.

This ordering of variables might be viewed as underestimating the contribution of executive functioning, given that it shares variance with reading achievement. To evaluate this hypothesis, we reordered the predictors so that executive functioning was entered before reading achievement (the order of the other variables and their contribution remained constant). In this analysis, EF accounted for 4% of variance in sixth graders' fraction arithmetic accuracy, $F(2, 53) = 1.10$, $p = .342$ (*ns*) but 17% of the variance in eighth graders' accuracy, $F(2, 53) = 5.28$, $p = .008$. Reading achievement accounted for an additional 11% of variance in sixth graders' fraction arithmetic accuracy, $F(1, 52) = 6.83$, $p = .012$, and 18% of variance in eighth graders' performance, $F(1, 52) = 13.91$, $p < .001$.

Predictors of mathematics achievement. Bivariate correlations indicated that among both sixth and eighth graders, all three fraction magnitude tasks, whole number division, calculation errors, and at least one EF measure were related to overall math achievement test scores. The relations tended to be stronger for the older children (see Table 3). The relations of PSSA scores to fraction performance were not due to fraction questions being major parts of the test. Analysis of test guidelines suggested that only 10%–15% of PSSA mathematics test items are dedicated to fractions for the grades in question (Pennsylvania Department of Education, n.d.).

To examine predictors of individual children's PSSA math scores more systematically, we conducted a hierarchical regression analysis that paralleled the one for fraction arithmetic. Because instruction in fraction arithmetic generally comes after instruction in fraction magnitudes, we entered fraction arithmetic after fraction magnitudes.

PSSA reading scores accounted for 36% of the variance in sixth graders' PSSA math scores, $F(1, 54) = 30.40$, $p < .001$, and 56% of the variance in eighth graders' scores, $F(1, 54) = 67.44$, $p < .001$. The two executive function measures explained 8% of variance beyond that explained by the reading scores for sixth graders, $F(2, 52) = 3.93$, $p = .026$, and 2% for eighth graders (*ns*). Whole number division accounted for an additional 9% of variance for both sixth graders, $F(1, 51) = 9.38$, $p = .004$, and eighth graders, $F(1, 51) = 14.13$, $p < .001$. Fraction magnitude tasks contributed an additional 10% of variance for sixth graders, $F(3, 48) = 4.10$, $p = .011$, and 9% for eighth graders, $F(3, 48) = 5.50$, $p < .002$. The fraction arithmetic measure added a nonsignificant 1% to the variance explained by the other measures for both sixth and eighth graders. Together, these predictors accounted for 64% of variance in sixth graders' PSSA math scores, $F(8, 47) = 10.33$, $p < .001$, and 76% of variance in eighth graders' scores, $F(8, 47) = 18.83$, $p < .001$.

When entered first, EF accounted for 12% of variance in sixth graders' math achievement scores, $F(2, 53) = 3.59$, $p = .035$, and 28% in eighth graders' scores, $F(2, 53) = 10.05$, $p < .001$. Reading achievement explained 33% additional variance in sixth graders' math achievement scores, $F(1, 52) = 30.40$, $p < .001$, and 30% in eighth graders' scores, $F(1, 52) = 36.10$, $p < .001$.

Fraction magnitudes, fraction arithmetic, and mathematics achievement. Fazio, Bailey, Thompson, and Siegler (2012) found that children who based number line estimates predominantly on fraction magnitudes did better on other mathematics tasks and on a math achievement test than did children whose number line estimation was based predominantly on numerators or denominators alone. To examine whether this relation was again present, we followed Fazio et al.'s procedure of computing gamma correlations between each child's 0–1 number line estimates and the order of the fractions if ranked by numerator, denominator, or fraction magnitude. The predictor corresponding to the largest of these three correlations was interpreted as the child's predominant approach, unless all relations were weak ($|\gamma| < .25$), which resulted in a strategy classification of *unknown*. Two thirds of children (69%) relied predominantly on fraction magnitudes, 17% on numerators, and 12% on denominators.

As in Fazio et al. (2012), children who relied mainly on fraction magnitudes on the 0–1 number line task generated much better performance on all measures of mathematical knowledge than children who did not. This was true on 0–5 number line estimation among sixth graders (PAE = 18 vs. 30), $t(58) = -3.73$, $p < .001$, $d = -1.03$, and eighth graders (PAE = 17 vs. 25), $t(58) = -2.75$, $p = .008$, $d = -0.82$; on magnitude comparison among sixth graders (82% vs. 60% correct), $t(58) = 4.64$, $p < .001$, $d = 1.28$, and eighth graders (83% vs. 55%), $t(58) = 5.48$, $p < .001$, $d = 1.63$; on fraction arithmetic among sixth graders (50% vs. 39% correct), $t(58) = 1.88$, $p = .066$, $d = 0.52$, and eighth graders (64% vs. 40%), $t(58) = 3.13$, $p = .003$, $d = 0.93$; on whole number division among sixth graders (83% vs. 62% correct), $t(30.81) = 2.55$, $p = .016$, $d = 0.78$, and eighth graders (87% vs. 39%), $t(58) = 5.83$, $p < .001$, $d = 1.73$; and on overall math achievement test scores among sixth graders (PSSA = 1,597 vs. 1,320), $t(54) = 4.39$, $p < .001$, $d = 1.26$, and eighth graders (1,547 vs. 1,213), $t(54) = 5.91$, $p < .001$, $d = 1.86$. Thus, the differentiation based on 0–1 number line estimation was strong not only on other measures of numerical magnitude knowledge but also on fraction arithmetic, whole number division, and mathematics achievement.

Focusing on the numerator alone or the denominator alone seemed likely to be related to LA/HA status, and it was. Reliance on fraction magnitudes was much higher among HA than LA children in both sixth grade (81% vs. 27%) and in eighth grade (94% vs. 45%).

Discussion

Development of Fraction Knowledge in LA and HA Children

The most striking finding from the study was that differences between the fraction arithmetic knowledge of HA and LA children widen substantially between sixth and eighth grade. HA children's fraction arithmetic accuracy was much higher in eighth grade than sixth grade, but LA children's accuracy was similarly low in both grades.

At least three processes contributed to this pattern. One was differentiating between properties of whole numbers and fractions.

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Among LA children, whole number strategies were equally common on fraction arithmetic problems in sixth and eighth grades; in contrast, among HA children, whole number strategies were fairly common in sixth grade but almost nonexistent in eighth grade. A second likely contributor to the differential improvement involved establishing common denominators. On operations that require equal denominators (addition and subtraction), LA sixth and eighth graders showed equally poor performance, but HA eighth graders were considerably more successful than HA sixth graders. A third likely contributor involved variability of strategies. Both sixth and eighth grade LA children and sixth grade HA children fairly often used a correct and an incorrect procedure to solve pairs of virtually identical problems, but HA eighth graders consistently used correct procedures.

Sources of Individual Differences in Fraction Arithmetic and Overall Math Achievement

Fraction magnitude knowledge. As found previously (Siegler et al., 2011), individual differences in fraction magnitude knowledge were closely related to individual differences in fraction arithmetic and mathematics achievement. The present results showed that this relation remains present even after controlling for reading achievement and executive functioning.

The relation to math achievement test scores seems likely to be due in part to the importance of fractions for solving pre-algebra and algebra problems. Consistent with this interpretation, in both Siegler et al. (2011) and the present study, the relations of fraction magnitude knowledge to mathematics achievement test scores were stronger in eighth grade, when children would have taken or be taking pre-algebra and algebra courses, than in sixth grade. The fact that in both studies the relation was also present, albeit weaker, in sixth grade suggests that fraction magnitude knowledge also facilitates learning of less advanced mathematics.

Whole number division. Whole number division accounted for substantial variance in sixth and eighth graders' fraction arithmetic and overall math achievement test scores, even after controlling for reading achievement and executive function. This predictive relation probably had several sources: Whole number division and fractions are logically equivalent (N/M); whole number division is assessed on middle school mathematics achievement tests; and it is essential to solving many pre-algebra, algebra, and statistics problems that also appear on the tests. Consistent with this analysis, HA and LA children's whole number division accuracy differed greatly (89% vs. 53% correct.)

In contrast to this large difference between LA and HA children's division accuracy, no difference was present between sixth and eighth graders in either group. A likely reason is that whole number division is not a focus of middle school math instruction. The lack of improvement during middle school helps explain why fifth graders' whole number division uniquely predicts 10th graders' mathematics achievement (Siegler et al., 2012). If children do not learn whole number division when it is taught, they may be unlikely to learn it later.

Executive functioning. Our findings regarding relations of executive functioning to fraction arithmetic and overall mathematics achievement were complex. Eighth graders' working memory updating and inhibitory skills correlated with their fraction arithmetic accuracy and math achievement. These relations made sense: Success at mixed series of problems, like the present ones, requires inhibiting inappropriate procedures, such as maintaining equal denominators in the product during multiplication, and it also requires continuously updating working memory contents, such as which steps in a computation have been completed and which remain. In contrast, sixth graders' fraction arithmetic accuracy was not related to either executive function, and their mathematics achievement was related to inhibitory skill but not working memory updating. Adding to the complexity, when reading achievement was statistically controlled, the relation between executive functioning and fraction arithmetic disappeared at both grade levels, and the relation of executive functioning to overall math achievement was present for sixth but not eighth graders. These results were less consistent than those obtained in some previous studies (e.g., Bailey et al., 2012) but similar to those obtained in others (e.g., Mazzocco & Kover, 2007). The reasons for this inconsistent pattern, both within the present study and in the literature in general, remain to be established.

Metacognitive judgments. Previous studies have not tested whether children who use inappropriate arithmetic procedures believe those procedures are correct or whether they suspect that the procedures are incorrect but use them anyway because they cannot recall a correct strategy. The present study provided evidence for both interpretations.

Consistent with the view that children often believe that incorrect procedures are correct, HA children were wrong on 26% and LA children were wrong on 46% of answers in which they indicated maximal confidence. Such overconfidence may have negative consequences for learning. Children who are confident that their incorrect answers are correct might not check them or attend to subsequent instruction.

However, the present results also reflected considerable metacognitive knowledge. In both grades, children were more confident in their correct than in their incorrect answers, and their mean confidence on each problem correlated positively with their overall accuracy on the problem. The overall pattern suggests that both overconfidence in incorrect strategies and uncertainty about the correct strategy contribute to use of incorrect approaches.

Several limitations of this study should be noted. The study was cross-sectional, so the stability of individual differences could not be examined. The sample was relatively small and included few Hispanic or Asian students. Results regarding unique contributions of executive functioning did not yield clear conclusions.

Instructional Implications

A general instructional implication of these findings is that skills that are prerequisite for subsequent mathematics need to be taught much more effectively than they are at present. Illustratively, the recently adopted U.S. Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, 2010) prescribe that whole number division be taught and learned in fourth grade, but the present results demonstrate that many eighth graders have not learned it. The strong relations of whole number division to both fraction arithmetic and math achievement test scores found in this study, as well as the inherently foundational role of division for later mathematics, show the necessity of improving teaching and learning in this area.

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Another instructional implication of the present findings involves a target for early intervention that might avoid many subsequent difficulties that children have with fractions. Roughly one third of children in the present study and a similar percentage in Fazio et al. (2012) based their fraction magnitude estimates primarily on the numerator alone or the denominator alone. This tendency was associated with many negative outcomes, including poor performance on fraction arithmetic, whole number division, and overall mathematics achievement test scores. Relying on the numerator or denominator alone also overlapped with LA classification and with the tendency to treat numerators and denominators in fraction arithmetic problems as independent whole numbers.

Viewing fraction magnitudes as independent whole numbers is understandable, because fractions with larger numerators do tend to be larger, as do fractions with smaller denominators. Moreover, the approach is appropriate for fraction multiplication and yields correct answers to some other classes of problems that are often presented in textbooks, for example, comparing fraction magnitudes with equal denominators (e.g., 1/5 vs. 3/5) or equal numerators (e.g., 3/5 vs. 3/7). However, it is not the right way to view fractions and does not yield consistently correct performance across unrestricted sets of fraction comparison or arithmetic problems. Placing greater instructional emphasis on the need to view each fraction as an integrated magnitude that expresses the relation between its numerator and its denominator might avoid subsequent difficulties not only in fraction arithmetic but in learning of mathematics more generally.

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New Editors Appointed, 2015–2020

The Publications and Communications Board of the American Psychological Association announces the appointment of 6 new editors for 6-year terms beginning in 2015. As of January 1, 2014, manuscripts should be directed as follows:

- *Behavioral Neuroscience* (http://www.apa.org/pubs/journals/bne/), **Rebecca Burwell, PhD,** Brown University
- *Journal of Applied Psychology* (http://www.apa.org/pubs/journals/apl/), **Gilad Chen, PhD**, University of Maryland
- *Journal of Educational Psychology* (http://www.apa.org/pubs/journals/edu/), **Steve Graham, EdD,** Arizona State University
- *JPSP: Interpersonal Relations and Group Processes* (http://www.apa.org/pubs/journals/psp/), **Kerry Kawakami, PhD,** York University, Toronto, Ontario, Canada
- *Psychological Bulletin* (http://www.apa.org/pubs/journals/bul/), **Dolores Albarracín, PhD,** University of Pennsylvania
- *Psychology of Addictive Behaviors* (http://www.apa.org/pubs/journals/adb/), **Nancy M. Petry, PhD,** University of Connecticut School of Medicine

Electronic manuscript submission: As of January 1, 2014, manuscripts should be submitted electronically to the new editors via the journal's Manuscript Submission Portal (see the website listed above with each journal title).

Current editors Mark Blumberg, PhD, Steve Kozlowski, PhD, Arthur Graesser, PhD, Jeffry Simpson, PhD, Stephen Hinshaw, PhD, and Stephen Maisto, PhD, will receive and consider new manuscripts through December 31, 2013.