CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2012 ANNUAL MEETING /
RENCONTRE ANNUELLE 2012

CMESG
GCEDM

Université Laval
May 25 – May 29, 2012

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PROCEEDINGS OF THE 2012 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2012 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

36th Annual Meeting
Université Laval
May 25 – May 29, 2012

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INTRODUCTION

Elaine Simmt – President, CMESG/GCEDM
University of Alberta

Each year our membership comes together to interact with each other around a variety of mathematics education interests, challenges and (of course) problems (math problems, that is). This year's meeting was yet another wonderful opportunity to immerse ourselves in these important conversations. For me, highlights of the meeting emerged from the formal program, my time in a working group, dinner conversations and two excellent plenaries. It was with great pleasure that I sat with Paulus Gerdes (and my Canadian colleagues) at dinner, where we experienced stimulating conversation and absolutely wonderful wine and cheese. The evening with Gerdes left me inspired to think more about mathematics for the particular people I teach. He related the story of being charged with educating Mozambique's first cohort of mathematics teachers after independence and how he and his colleagues began by asking those first candidates what mathematics was for them. Margaret Walshaw asked a similar question of the teacher candidates she worked with, trying to understand mathematics identity. Both of our keynotes then, one explicitly and the other implicitly, reminded all of us of the reciprocal relationship between personal identity and cultural mathematics and how each shapes the other.

Although I attended the *Proof in Mathematics and in Schools* working group, and really appreciated the experience that our co-leaders facilitated, I know that there were a number of other working groups (*Numeracy: Goals, Affordances, and Challenges*; *Diversities in Mathematics and their Relation to Equity*; *Technology and Mathematics Teachers (K-16)*; *The Role of Text/books in the Mathematics Classroom*; and *Preparing Teachers to Develop Algebraic Thinking in Primary and Secondary School*) that I would have enjoyed and from which I would have learned just as much. Topic group sessions by Lovric and Charbonneau & Guillemette provided members with an opportunity to think more about teaching infinity in calculus and reading original texts in mathematics. As well, the panel discussion offered us a diverse set of perspectives from which we could think more about the content that should be in our mathematics curriculum in Kindergarten through Baccalaureate programs. Finally, the program was kept fresh and relevant with the new PhD presentations, the *ad hoc* sessions and the presentations in the Math Gallery. Thanks to all of you who contributed to our program.

The wine and cheese dinner was certainly a highlight for me, but I know that many of you enjoyed the other aspects of the local program as well. From the opening reception to our dinner out at Le Moulin de Saint-Laurent, our visit to Montmorency Falls and the night in Vieux-Québec, the local organizers knew just how to keep us fed and happy. On behalf of our executive and all of the conference participants, I would like to thank Frédéric Gourdeau and his incredibly helpful and pleasant math students, Marie-Pier Bédard, Simon B. Lavallé, and Sarah Mathieu-Soucy for their hospitality. Special thanks to Andréa Deschênes, Anick Lévesque-Gravel, Joannie Harvey, Malik Younsi, Jonathan Godin, Laurent Pelletier, and Emmanuelle Renauld for setting their mathematics aside long enough to serve at the opening reception and then again at the wine and cheese soirée. And, one last thank you to Lucie DeBlois who looked after the executive once the conference was over.

Horaire

Vendredi 25 mai	Samedi 26 mai	Dimanche 27 mai	Lundi 28 mai	Mardi 29 mai
	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Groupes de travail**	8:45 – 10:15 **Groupes de travail**	8:45 – 9:15 **N^{lles} thèses (3)** 9:20 – 9:50 **Séances *ad hoc***
	10:15 – 10:45 Pause café	10:30 – 11:00 Pause café	10:30 – 11:00 Pause café	10:00 – 11:30 **Panel**
	10:45 – 12:15 **Groupes de travail**	10:45 – 12:15 **Groupes de travail**	10:45 – 12:15 **Groupes de travail**	11:30 – 11:45 Pause café 11:45 – 12:30 **Séance de clôture**
	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	
14:30 – 17:00 **Inscription** *Pavillon De Koninck* 15:00 – 16:30 **Plénière GDM** 15:30 – 16:20 **Amis de FLM**	13:30 – 14:45 **Gallérie mathématique**	13:30 – 14:30 **Plénière 2**	13:30 – 14:00 **Petits groupes** 14:00 – 15:00 **Plénière 2 : Séance questions**	
	15:00 – 15:30 **Petits groupes** 15:30 – 16:00 Pause café	14:40 – 15:40 **Séances thématiques**	15:05 – 15:30 **Séances *ad hoc*** 15:30 – 16:00 Pause café	
	16:00 – 16:50 **Plénière 1 : Séance questions**	15:50 – 22:00 **Excursion** *Île d'Orléans* **Souper** *Le Moulin de Saint-Laurent*	16:00 – 16:45 **La parole aux anciens**	
17:00 – 18:45 **Cocktail Souper GDM-GCEDM** *Atrium Pavillon De Koninck*	17:00 – 17:30 **N^{lles} thèses (1)** 17:30 – 18:00 **N^{lles} thèses (2)**		16:45 – 18:00 **Assemblée générale annuelle**	
18:45 – 19:30 **Séance d'ouverture** 19:30 – 20:30 **Plénière 1** 20:30 – ? **Réception** *Atrium Pavillon De Koninck*	18:30 – ? **Vins et fromages** *Pavillon De Koninck*		**Souper libre** *Québec*	

Schedule

Friday May 25	Saturday May 26	Sunday May 27	Monday May 28	Tuesday May 29
14:30 – 17:00 **Registration** *Pavillon De Koninck* 15:00 – 16:30 **GDM Plenary** 15:30 – 16:20 **Friends of FLM**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 9:15 **New PhDs (3)**
17:00 – 18:45 **Cocktail Dinner GDM-GCEDM** *Atrium Pavillon De Koninck*	10:15 – 10:45 Coffee Break	10:30 – 11:00 Coffee Break	10:30 – 11:00 Coffee Break	9:20 – 9:50 ***Ad hoc***
18:45 – 19:30 **Opening Session**	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:00 – 11:30 **Panel**
19:30 – 20:30 **Plenary 1**	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	11:30 – 11:45 Coffee Break
20:30 – ? **Reception** *Atrium Pavillon De Koninck*	13:30 – 14:45 **Gallery Walk**	13:30 – 14:30 **Plenary 2**	13:30 – 14:00 **Small Groups**	11:45 – 12:30 **Closing Session**
	15:00 – 15:30 **Small Groups**	14:40 – 15:40 **Topic Sessions**	14:00 – 15:00 **Plenary 2: Q&A**	
	15:30 – 16:00 Coffee Break	15:50 – 22:00 **Excursion** *Île d'Orléans* **Dinner** *Le Moulin de Saint-Laurent*	15:05 – 15:30 ***Ad hoc***	
	16:00 – 16:50 **Plenary 1: Q&A**		15:30 – 16:00 Coffee Break	
	17:00 – 17:30 **New PhDs (1)**		16:00 – 16:45 **Elder Talk**	
	17:30 – 18:00 **New PhDs (2)**		16:45 – 18:00 **AGM**	
	18:30 – ? **Wine & Cheese** *Pavillon De Koninck*		**Dinner on your own** *Québec*	

Plenary Lectures

Conférences plénières

TOWARDS AN UNDERSTANDING OF ETHICAL PRACTICAL ACTION IN MATHEMATICS EDUCATION: INSIGHTS FROM CONTEMPORARY INQUIRIES

Margaret Walshaw
Massey University, New Zealand

This paper is an exploration into contemporary thinking about social justice within mathematics education. The paper first draws attention to a range of theoretical issues that are couched within conventional liberal democratic explanations of social justice. The move is then away from mechanisms offered by those explanations towards insights from contemporary social theory that seeks to explain how practices and identities are produced within discourses. Using examples from everyday life in mathematics classrooms, the paper not only confirms the potential of the radical democratic project, but also offers a way of understanding what we might do to effect change.

INTRODUCTION

Social justice is a recurring theme in many discussions centred on the provision of mathematics education. Each round of published international or national comparative datasets invites intense scrutiny of practices that are effective in terms of student proficiency outcomes. Closer inspection of the data highlights student experiences that are inequitable in the sense that for specific groups of students, mathematics presents as an impossible challenge (Anthony & Walshaw, 2007). Such datasets tend to run counter to claims made repeatedly within democratic societies that *all* students have right of access to mathematical knowledge. The starting point of this paper is that the issue of inequities surrounding access to mathematical knowledge is an extremely complex phenomenon. Conventional liberal democratic mechanisms that provide a means to explain social justice, it is argued, do not get to the heart of the issue. An equitable mathematical experience will not be achievable for specific groups of students if we cannot begin to unravel that complexity. Such an exploration is the aim of this paper.

Contemporary debates about democratic mathematics provision have drawn out attention to the problems confronted by teachers, schools, and district boards, with respect to student disaffection and disengagement with mathematics. These discussions provide an arresting reminder of the trend of systemic achievement. Concerns over poor students' achievements, relative to international benchmarks, have been a key rallying point for administrators and policy makers, putting huge political pressure on education systems in a number of Western nations. Stakeholders tend to blame sitting governments at both state and national levels for not doing enough to demonstrate good or internationally-comparable student performance. In my own country, the current response to systemic underachievement consists of a suite of proposals including: the passing of a policy on National Standards, more public-private partnerships, increased class size, and the commensuration of teachers' salaries with competence.

There are important synergies between the educational climate New Zealand currently experiences and that in Canada, where policy's most recent engagement with the issue of social justice has arisen, in part, through reflections on the changing nature of our mathematics classrooms, which increasingly cater to diverse groups of learners. Increased socioeconomic disparity, as well as cultural and linguistic diversity, have put responsive engagement with the marginalized and underprivileged sharply into focus. The solution often offered to address underachievement pivots around the conflation of equity with equality, in which unequal approaches, unequal access, and unequal opportunities are deemed to fully explain why specific groups of students do not succeed with mathematics. Equality, in these explanations, is privileged over any other advocacy, based on the understanding that equal outcomes, approaches and access, taken together, yield a comprehensive picture of equitable pedagogical practice for students, irrespective of any social determinations.

This kind of approach has been seriously undermined by people like Foucault (e.g., 1969/1972). However helpful the concept of equality might be in enhancing students' engagement with mathematical ideas, in trying to paint a picture of equitable arrangements in mathematics education, issues of structures, as well as interactions between contexts and people, cannot fail to intervene. A number of researchers (e.g., Appelbaum, 2008; Brown, 2011; Roth & Radford, 2011; Sfard & Prusak, 2005; Walkerdine, 1989) have provided analyses that take into account the wider context. For all these researchers, the teacher's and the student's classroom practice is always situated within a web of wider influences. In their expanded views, social and political factors that impact on learning are hugely significant. Hence, an approach that conceives of mathematics as constructed, situated within institutions, historical moments, as well as social, cultural and discursive spaces might signal how persistent inequities in students' mathematics education might be addressed.

This paper, then, is an exploration of contemporary thinking about social justice within mathematics education. It represents the culmination of conversations with many others, my reading of the literature, and a number of recent productive research collaborations. It establishes a direction for what we might do to effect change based on the potential of the radical democratic project. At the core of the discussion lies the question: Who are we, with respect to others, with respect to structures and with respect to history, and how does that understanding play into social justice? In responding to that question, there are arguments put forward that are reconsidered through a lens that is different to the conventional liberal democratic explanations of social justice. I use that lens to analyse data from two of my own research projects. The aim in doing this is to keep the conversation going about social justice within teaching and learning settings.

CONNECTIONS AND DISCONNECTIONS

In a recent collaboration, Eva Jablonka, David Wagner and I (2013) investigated contemporary international theoretical trends in research in mathematics education. We attempted to map the field, paying particular attention to social, political and cultural dimensions, by investigating four recent proceedings of the conferences of the International Group for the Psychology of Mathematics Education (PME), namely, proceedings from the years 2007, 2008, 2009 and 2010. Our choice of PME was based around the understanding that PME is an established organisation with annual international conferences and hence is likely to reflect any changes from the mainstream. At the same time we were mindful of the limitations in confining our investigation to one source. We were also acutely aware of the restrictions imposed on PME authors for explaining their theoretical standpoints. In addition, we recognised that PME does not fully capture the research being undertaken world-wide. However, PME does provide a measure of theoretical choice.

In order to carry out the investigation we compiled a list of names of theories, frameworks and authors associated with socio-linguistic, socio-cultural, sociological and postmodern theories and searched the proceedings. From our interrogation, we noted a shift away from cognitive psychology towards a strong reliance on socio-cultural theory. We found that Vygotskian and neo-Vygotskian frameworks that build on the understanding of the prior necessity of social interactions for cognitive behaviour are currently highly influential in the field. We would go so far as suggesting that they have become mainstream theory within mathematics education. In much of this work, however, the take-up of Vygotskian ideas is eclectic: many theorists claim an allegiance to the social but nevertheless draw on terms and concepts frequently associated with constructivist leanings. In these perspectives, the construction of knowledge still remains the preserve of the individual stable mind, albeit influenced by social and cultural practices. Even as the importance of the social is acknowledged for knowledge construction, the social functions as a shaper of knowing.

In sociocultural formulations that maintain that the social functions as a constitutor rather than a shaper of knowing, teachers and students are both active participants in the learning process. In these various formulations, learning is embedded within a cultural and social context; it is a collaborative process rather than a function of the individual or the social setting; it is an apprenticeship that occurs through guided participation in social activities with a community that supports and extends understanding; and/or it is mediated by language and other cultural tools. Good teaching, then, is about building on student interests in a collaborative way; co-participating as a learner in a community of learners; engaging in dialogue between and with the students; and developing relationships between the teacher and student in less traditional ways. Crucially, then, in these formulations, teaching practices that are equitable involve transformative interactions between classroom contexts and the teacher and students. More specifically, equitable practice is characterized by an enhanced, integrated relationship between teachers' intentions and actions, on the one hand, and learners' dispositions towards mathematics learning and development, on the other. In other words, equity here means protection from, and resolution of, those processes or structures that serve to undermine a student's sense of self as a legitimate mathematical learner within the context of the practices of the mathematics classroom and within other communities in which the student participates.

In an attempt to sharpen the modalities that shape equitable practice, Roth and Radford (2011) have proposed a cultural-historical perspective in which equity is clearly not defined as protection from and resolution of unequal approaches, unequal access, or unequal opportunities. Rather, equity is a relation between settings and people, both of which have their own histories, at a specific time. Underscoring the relational nature of Vygotsky's (e.g., 1978) ontology, Roth and Radford emphasize the possibilities for ethical practical action. As Roth explains:

> *We need to look at activities that we want not only to understand but to transform. It is praxis that we want to understand and transform….It's a theory for action, it's not just a theory for understanding. It's one that's there to assist us, perhaps as a heuristic for going about transforming this world.* (Roth, Radford, & LaCroix, 2012, p. 9)

Equitable teaching practice, in the cultural-historical perspective, takes into account the ways of knowing and thinking, language, and discursive registers made available within the physical, social, cultural, and historical community of practice in which the teaching and learning is embedded. Thus, it follows that the identity of a student comes into being in relation to the negotiations that she undertakes, both in the past and the present, with other individuals and communities. Crucially, a change within a particular classroom context may also result in a change in a student's long-term assessment of his or herself as a mathematics learner. That is to say, an activity within the mathematics classroom has a direct bearing on the kinds of mathematical identities that students might take up and the kinds of proficiencies to which they might aspire. What students say and do within the discourses made available has the effect of contributing to the development of mathematics learners at a given time and place.

If identity is relational and is able to tell us about the nature of an equitable mathematical experience, then it needs to be expressed as something dynamic rather than as a static process or as a property of people as formulated in the conventional take-up of socio-cultural work. Situating themselves within the Vygotskian (e.g., 1978) intellectual spectrum as developed by Leont'ev (1978), Roth and Radford (Roth, Radford, & LaCroix, 2012) elaborate the category of *dialectics* to deal with the methodological difficulty in dealing with the "fundamental idea that life is something in motion" (p. 9). As they point out, "dialectics is the aspect of [the] theory that is the most difficult to understand and the least attended to because inherent in dialectical understanding is transformation" (p. 10). As Radford argues, "we've got something really wrong with our usual common understanding of the verb *to be*. *We are*, but we are always changing, so the phrase is always incomplete" (p. 9). Radford goes on to say that "[w]hen we say 'he is' or 'I am', the verb is always in transformation, and so is the subject" (p. 9).

The work of Roth and Radford (2011) acknowledges a point that many of us have felt instinctively: that, in many respects, our comfortable ways of making sense of the world from a familiar framework are becoming redundant. What Roth and Radford do is merge the cognitive, the social, the historical and the affective together. They do not stop at an analysis of the social context nor at an analysis of the individual. Their reading emphasises that culture and history are embodied drivers of thinking and being. From that reading we can map out new coordinates of identity. In differentiating themselves from the Cartesian effort to conjure a foundational status for the subject that is claimed to pertain to a rational 'man', Roth and Radford underscore the point that what mathematics education did in the past is now less predictive of its future.

There are significant points of convergence between the notion of the subject as conceived by Roth and Radford and that put forward by Foucault (e.g., 1966/1970). All promote a historically variable account of subject-constitution. For them, the subject is never fully constituted. When they talk about identity, they talk about identity as fluid in nature, forever in process.

In Foucauldian understanding, the idea is that the subject is, on the one hand, an agent and, on the other hand, has a connotation of being subjected to. The subject is internally contradictory since it has both the status of position of agency and the status of being acted upon. That is to say, the subject is embodied with a double valence: it is an ensemble, and never in one place only. That leads us to formulate subjectivity as a process rather than a state or a condition. Formulating the subject in this way lends itself to conceiving of people as 'verbs' rather than 'nouns'. A reading of the subject, like this, invested as it is in dynamism, requires the shift in language that Roth and Radford argue for in order to convey the point that the verb 'to be' is "always in transformation" (p. 9). However, if identities are constructed out of multiple and fluid layers how might we talk about equity and proceed with an emancipatory project?

RETHINKING EQUITY

So far we have noted that the argument for organizing practices to maximize equitable arrangements for all students has become a cornerstone in many contemporary approaches to equity within mathematics education. Sociocultural theory, in varying degrees, animates contemporary thinking in the field, expressing itself as a counterpoint to cognitive science's response to the issue of social justice. Cultural-historical approaches, presented more recently, amplify the contingent, embracing the post-human in preference to the fixed and unitary subject. Taking a postmodern turn, such approaches frame identity differently and in doing so, inspire a re-shaping of the equity imagery.

Cultural-historical theory becomes a point of interest for, and interrogation of, emancipation as depicted in the enlightenment project. Radford's (2012) reading of emancipation lifts it out of the narrower interpretation expressed within sociocultural theory and moves us towards appraising the enlightenment modern project as "a chimeric and unfulfillable dream" (p. 101). His view of the modern-day social justice project is aligned with the view of Foucault and other thinkers (e.g., Bingham & Biesta, 2010). For them, rethinking identity and the subject away from the modalities of autonomy and self-sufficiency invites a very divergent mobilization of the emancipation project and the role the project plays in everyday classroom practice.

For Radford, at the heart of the issue is the relationship between freedom and truth and, with it, the relationship between the individual and the social. By way of example, Radford discusses the student-centred approach, advocated in many official curriculum documents as the pivot of mainstream education policy and practice. In these documents the teacher is positioned within a learning community as a *guide-on-the-side*, facilitating interchanges, providing structured and purposeful activities and connecting mathematics to everyday contexts, all the while mindful of the cultural and mathematical knowledge the students bring to their learning. It is the teacher who is tasked with orchestrating thoughtful discussion around meanings and understandings. The difficulty arises when the ideas generated by students in the classroom are reconciled by the teacher with the conventional mathematical ideas as outlined in curriculum documents. As Radford notes, students' ideas are invited yet they can never be autonomous since they are "unavoidably engulfed in discourses and epistemes (i.e., systems of thinking) that are not [the students'] own" (p. 104). Radford asks, "How can the modern subject be the locus of meaning, feeling and intentionality if it has to talk, feel and intend through thoughts and words that are not its own?" (p. 106).

Lessons learned from Foucault (e.g., 1969/1972) show how bodies of knowledge, like school mathematics, are premised on a set of claims to truth. As such they are caught up in 'regimes of truth', and what comes to count as school mathematics does not pre-exist certain normalising and regulating practices. Knowledge about school mathematics is an *effect* of a primarily linguistic discursive formation. As in all the human sciences, particular rules of formation in school mathematics determine the spectrum of speech acts, and actions that can be taken seriously at any given historical moment. While these rules are often unknown to the actors involved, they circumscribe the possibility of thought concerning what exactly school mathematics is and set boundaries on what is taken as true. As Foucault (1980) has noted:

> *The problem does not consist in drawing the line between that in a discourse which falls under the category of scientificity or truth, and that which comes under some other category, but in seeing historically how effects of truth are produced within discourses which in themselves are neither true nor false.* (p.118)

The unmasking of school mathematics as intimately tied to the social organization of power then becomes crucial to our understanding of 'who can know mathematics?' Those who speak in the name of mathematics come to possess a certain power. During earlier times, access to the kind of intellectual development which mathematics promised was seen as the exclusive preserve of certain males. During the past few decades, interventionary measures in mathematics, aligned with wider practices of social inclusion, has remapped the school mathematics terrain, to the effect that mathematics schooling has become a site that embraces the participation of former marginalised groups. A space has opened for previously masked subjectivities.

However, creating a presence in the school mathematics classroom "occurs against the background of history and culture" (Radford, 2012, p. 110). Presence in the world implies engagement with a world "populated not only with material objects but also with systems of thought" (p. 110). Presence is never autonomous or fixed but is normalized and regulated by multiple and dynamic layerings. While it is important to recognize the harsh materiality of the marginalized, it is also important to acknowledge that presence is both excluded and oppressed throughout the entire social body in everyday social practices. The question then becomes: Is it possible to formulate equity in a way that does not rely on the autonomous individual, yet is still able to address the realities of inequitable experiences? Radford believes that it is possible, but that emancipation "can only occur in the common world when we come to recognize ourselves as historical and political beings and where we critically labor together to make this collective space better for all" (p. 111).

FOUCAULDIAN ETHICAL PRACTICAL ACTION

Foucault's investigation into the subject allows us to recognize that we are historically and culturally constituted. He denied that individuals were their own source of meaning, knowledge and action, claiming instead that subjectivities are produced within discourse. He suggests that power circulates in practices and that it functions through micro-linkages within the social body in everyday social practices. In mathematics education, in Foucauldian understanding, teachers, learners, curriculum planners, researchers, and so forth, despite their stable appearance, are all merely productions of practices through which they are subjected. The identities these actors have of themselves are made in and through the activities, desires, interests, and investments of others. Hence, the truth about oneself is not something given, not something in our nature, and not something we have to discover for ourselves.

Crucially, Foucault is not merely arguing for the 'death of the modern subject' in favour of the discursive constitution of the subject. He is also providing the means by which we can 'undo' the subject by creating or modifying subjectivities. The radical potential for ethical practical action, then, can be linked to his notion of the fluidity of identity. It is the mobile space in which identity resides that is the key to ethical practical action. Indeed, ethical practice within mathematics education, in Foucauldian understanding, cannot proceed without the presumption that identity always fails to reach full determination. In Foucauldian work, the interrogation involves looking at who has what kind of access to and engagement with knowledge, by investigating into the operations of the complex and changing discursive processes by which identities are taken up, and by which they are subverted or resisted. The two short episodes that follow attempt to trace those processes to reveal how mathematical identities are tied to the social organization of power and, importantly, to highlight how inequities might be changed.

EPISODE 1

Rachel is a 'gifted' student enrolled in a senior school mathematics class in a large coeducational school in a New Zealand city and is learning calculus for the first time. In an interview with me Rachel tells me:

> *…I just seem to be good at maths…. I do a lot of study. Always read and do examples. Working out answers, checking them and making sure, and if I don't get it, I go back and try and figure it out …*

Later in the same interview she says:

> *…Mrs. S, she tends to go right over my head and I don't tend to ask questions from her because last time I did that she tried to explain and it just went ...Well, I sort of understood – I half understood when I asked the question and by the time she'd finished I understood none of it!…But I don't have a very good relationship with her, because we've had a few arguments in the past.*
>
> *…the guys, they know that I laugh really easily and they keep making me laugh in class and she just gets really frustrated with me because when I start laughing I can't stop….So it's a bit stressed there. I'm just trying very hard not to let the guys get to me now. Then I don't have to laugh.*

Foucault reminds us that people are the product of the discourses and practices through which they become subjected and with which they interact. He also tells us that subjectivity is produced and reproduced through practices and discourses. From her narrative we get a sense of how Rachel's identity changes in relation to discourses in circulation and in relation to other people. And importantly, it also changes internally, within herself. In that she says contradictory things, we can see how her meanings spiral back and forth as her investments and desires change during the telling. The transcript reveals how Rachel builds and rebuilds her identity as a gendered learner in relation to others; how it is never fixed, but constantly rearranged in relation to others. The point of difference of Foucault's work is that it provides a language that would help deal with constant change. In Rachel's case, instead of asking what her defining qualities were, we have to figure out what were the processes that constitute her as a learner in the classroom—both internally or in a relationship with others.

In the narrative there were processes in operation that made it easier for Rachel to be hard-working, capable, self-assured and confident in mathematics. At the same time, other internal processes were at play which contributed to her concern about the work and her lack of confidence during lessons. The very interesting thing was that as she slipped in and out of discourses from one moment to the other, her mathematical identity changed. What happened, too, was that the sort of mathematical knowledge and the depth of knowledge development clearly changed. This is an extremely important observation for social justice when we are talking about gendered subjectivity. It has major relevance for teaching and the way classroom learning is set up. It also has particular relevance for home practice and practices in other social institutions within which students are engaged.

EPISODE 2

Alicia is in interview with me during her year-long course and after her experiences during three practicum schools. She has observed other teachers in the schools but had already decided on the kinds of practices that she would like for herself.

Alicia says the following:

<u>School 1</u>

> *I would love to teach at this school! Top school. It's well equipped and well funded. On arrival the school feels like a professional place of learning, where the students seem fairly happy working within the school rules. The grounds are small but feel spacious and welcoming to everyone. The school principal has business and pedagogical ideals that mesh well together and work with my own ideas about how a principal should run a school.*

School 2

> *Some teaching not up to scratch, but other teachers really top of their game. Arrival at this school makes you aware that the school is in a working class town. On saying that, the students are open and welcoming. It has a relaxed feel about it….Certainly there's little importance put on completing homework and assignments compared with my first school….It's a challenging school to work in but when you get through to the less motivated students and have positive outcomes with them it sometimes seems more rewarding than getting excellence out of the students who achieve all the time.*

School 3

> *…[S]urprisingly this school is under-resourced….Learning in this school is very prescriptive. Teach X then Y and then we'll have a common test. Rote learning is more important than learning for knowledge. The girls I was teaching at Year 10 [age 14] had a variety of tests towards the end of my practicum….These girls had on-going testing. The pressure on them to achieve was immense. The scope for teachers to do their own thing in the classroom is very limited….There is no scope for inquiry learning—it takes too long.*

Foucault's ideas of normalization and of surveillance are useful to the analysis. Each of Alicia's practicum schools had its own rules governing beliefs about ways of operating. Not only did each school operate to normalize practice, but it also marked out social relations and created positionings between the classroom teacher and the prospective teacher under her care, as well as the students within the class. During her teaching for extended periods in classrooms at the three different schools, Alicia hinted at how her teaching identity was being shaped. She explained that the associate [co-operating] teachers at the first two schools developed a routine of sitting with her and discussing what the students needed to know. Then the teachers asked her to create either a lesson plan or unit plans which they sometimes discussed with her. These subtle practices of surveillance resulted in a construction of teaching for Alicia that appeared to her natural and inevitable. Her sense of self as a competent teacher rose and her passion for teaching began to flourish.

However, collegiality like this was not evident at the third school where the relationships with the associate teachers were fraught. The focus in the classroom was on facts and learning by rote. One associate teacher handed her a unit folder and said: "We will do X today and Y tomorrow." The other associate told her that she could find the unit plan on the computer. The two associates became very critical of Alicia's teaching precisely because they both wanted her to teach prescriptively in the way that they taught. By accepting as true only those practices that focused on rote learning, the two associates produced an understanding of what teaching competence looked like. In effect the practicum and the two associates operated to induce Alicia into a particular pedagogical identity which was at odds with what her course work and prior practicum experiences had offered. As a consequence, and after many attempts to teach an inquiry pedagogy, Alicia's confidence and sense-of-self as a teacher diminished. Regrettably, her determination to remain in the teaching profession fell dramatically.

THE TWO EPISODES

In the two analyses, the mobile constitution of identity in relation to mathematics learning and teaching comes directly to the fore. As in Episode 1, whether we choose to acknowledge it or not, truths about gendered subjectivity are produced in all classrooms. Rachel, in her classroom, provided an understanding of the political and strategic nature of gendered classroom life. Her classroom became a space that set cognitive limits on her. In the case of Alicia, we begin to see how the practicum imposes conditions and relationships in schools that shape teachers into a particular pedagogical identity.

A Foucauldian reading makes us aware of the constant tension experienced in confronting mathematics classrooms that are already populated with the meanings and intentions of others while simultaneously negotiating one's own everyday visions of being a teacher or a student. Rachel and Alicia are everyday people participating in mathematics classrooms but both were caught up in discourses through which they were unable to see a resolution. Reflecting on both, using Foucault's ideas, allows us to uncover how both become knowing, knowable and self-knowing subjects. Importantly, the approach allows us to see where there might be scope for change. It allows us to determine where Rachel and Alicia might constitute themselves as selves with agency within discourses and structural processes that seem so inflexible. Once we begin to do this, we can begin to work toward more equitable forms of organization.

CONCLUSION

Foucault's voice is one that unsettles the status quo. He reminds us that mainstream theory never fully captures the flow of what identity is. But the strength of Foucault's work, as opposed to the display of its individual concepts, is that it offers the tools for an analysis that is a way of invoking ethical deliberation. It allows us to see how mathematical identity is constituted within structures of power and where we might intervene for change. Specifically, it embraces the potential for creativity and agency within social constraints. As Foucault sees it, what we might become stands as the political, ethical, social, and philosophical problem of today.

A Foucauldian critical inquiry explores what we might become. It starts by making more visible the ways in which inequities are produced, tracking everyday small daily social relations. A critical ethos relevant for contemporary life, according to Foucault (1984/1986), involves actions situated at the level of an individual's daily practices. Tracking social relations involves investigating practices that "constitute, define, organise, instrumentalize the strategies which individuals in their liberty can have in regard to each other" (p. 19). Through that investigation, we discover where meanings and values are legitimated, whose investments are validated, and how those investments are sustained. We then begin to ask questions about which we have not previously thought to ask, such as: What formative events have brought this present situation about? How are individuals constituted as subjects of their own knowledge? How are they constituted as subjects who exercise and/or submit to power relations?

Asking questions like these is important because they allow us to discover why our interests are sometimes silenced and how we are caught up in conditions of constraint. At a deeper level, it also allows us to determine where we might find weak points to imagine a space for creative change. From the perspective of mathematics education, a new space for critical reflection on the scope and limits of freedom becomes available. We discover where we can envisage new possibilities and where we might be able to make them real. We, teacher educators, teachers, pre-service teachers, policy workers, students, and so on, can then begin to think of our work in mathematics education as a political resource for reflecting on what we are today, how we have come to be this way, and the consequences of our actions, and, importantly, what we might do to make a change.

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OLD AND NEW MATHEMATICAL IDEAS FROM AFRICA: CHALLENGES FOR REFLECTION

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Some challenges for reflection:

- *How can culture(s) be a source of inspiration for mathematics education?*
- *Who does mathematics? Who invents mathematics? What is 'mathematical thinking'? Who or which culture defines it?*
- *Can African and other cultures be a source of inspiration for the development of new mathematical ideas?*

AVANT PROPOS: SOME SIMILAR DESIGNS IN CANADA AND AFRICA

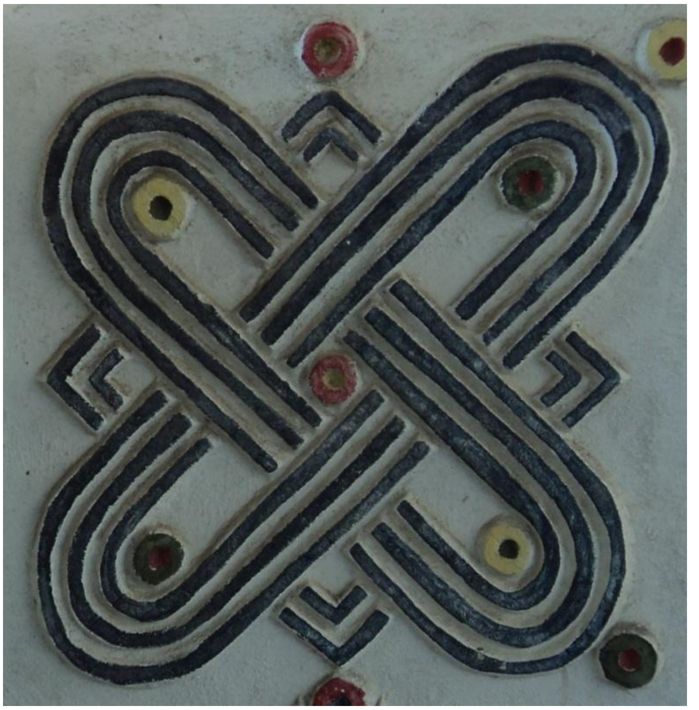


Figure 1. Detail of a wall decoration at the Université Laval, Quebec.

Walking around at the Université Laval, I observed interesting geometric patterns and shapes. Yesterday night, when leaving the atrium of the Pavillon De Koninck after the sympathetic Wine & Cheese welcoming session, I was surprised to see outside the building a decorated wall with the inscription "Québec", with four replicas, in different colours, of a design well-known from Africa in various variations (Figure 1). It appears, for instance, in the Cokwe culture from East Angola, drawn in the sand, to represent a tortoise (Figure 2). The drawing consists of a reference frame of dots marked in the sand around which three closed lines are traced (Figure 3); at the end the drawer adds the paws of the animal.

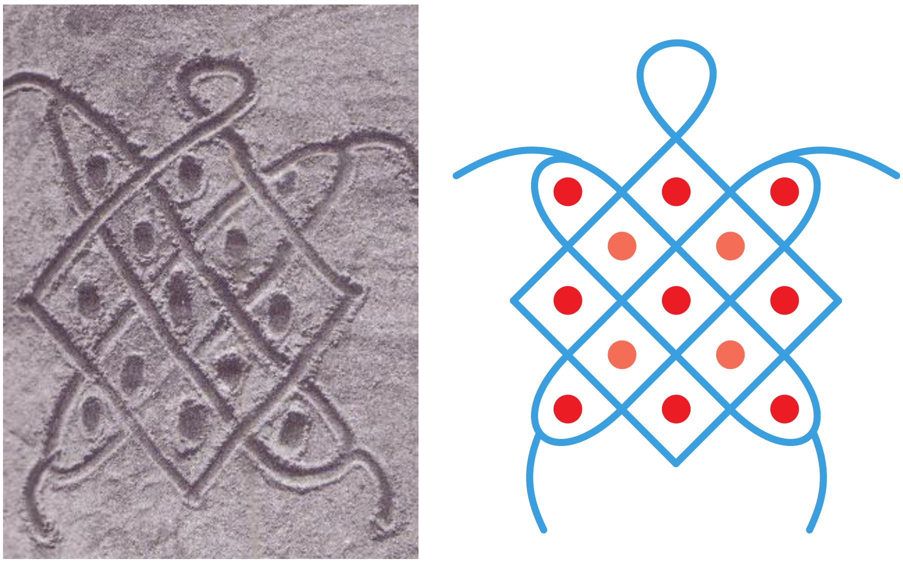


Figure 2. Cokwe representation of a tortoise.

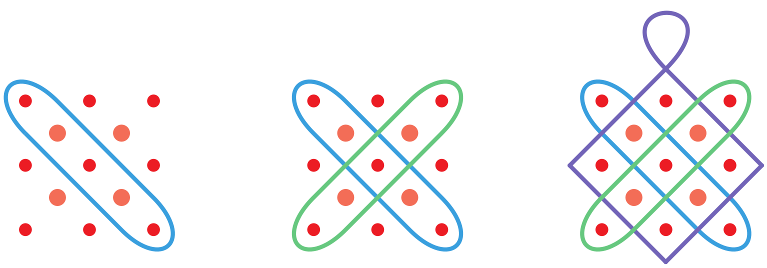


Figure 3. Execution of the drawing.

Before I continue, let me take the opportunity to thank the Canadian Mathematics Education Study Group for the invitation to give the 'mathematician' plenary at this year's conference at Laval University in Québec. My first contact with mathematics educators in Canada was when the late David Wheeler invited me to write papers for publication in *For the Learning of Mathematics*. I did so (Gerdes, 1985; 1986; 1988; 1990a; 1994a) and I am happy to see that the first paper was reprinted recently in Alan Bishop's (2010) book *Mathematics Education: Major Themes in Education*. It was also David Wheeler who, as a member of the international program committee, invited Alan Bishop, the late Peter Damerow, and me to organize the special 5th day of the 1988 International Congress of Mathematics Education in Budapest, dedicated to 'Mathematics, Education, and Society'. My 1988 paper in *FLM* was entitled "A widespread decorative motif and the Pythagorean Theorem" and dealt with the educational exploration of designs that appear both among Native Americans and in Africa. It included an infinite series of proofs of the theorem. My 1990 paper in *FLM* was on mathematical elements in the centuries old Cokwe *sona* sand-drawing tradition.

> *Challenge for reflection: How can culture(s) be a source of inspiration for mathematics education? Example: Theorem of Pythagoras. Other examples?*

'SONA' GEOMETRY

The tortoise design in Figure 2 is an example of a *lusona* (plural: *sona*). Formerly, Cokwe storytellers and educators used the *sona* as illustrations when teaching young boys. The colonial penetration and occupation contributed to the almost complete extinction of the knowledge about *sona*. I have been experimenting with the use of *sona* in mathematics education, both inside and outside the classroom. See, for instance, the children's book, *Drawings from Angola: Living Mathematics* (Gerdes, 2007a; 2012a), the book for high school pupils, *Lusona: Geometrical Recreations from Africa* (Gerdes, 1997; 2012b), the second volume on *Educational and Mathematical Explorations* (in English: Gerdes, 2013a; in French: Gerdes, 1995), and my trilogy *Sona Geometry from Angola* for use in mathematics teacher education and in the education of mathematicians. The first volume (Gerdes, 1995; 2006) of the trilogy deals with the reconstruction and analysis of mathematical ideas in the *sona* tradition and the third volume is a comparative study of *sona* with designs from Ancient Egypt, Ancient Mesopotamia, and India, and with Celtic knot patterns. As this is the 'mathematicians' plenary, I would like to present some new mathematical ideas that emerged in the attempt to analyze the mathematical potential of the (reconstructed) *sona* tradition.

> *A small question: Is Figure 1 positive or negative? In what sense? Why?*

FROM A PARTICULAR CLASS OF 'SONA' TO THE CONCEPTUALIZATION OF MIRROR CURVES

Figure 4 shows the Cokwe *sona* that represent the stomach of a lion and the path followed by a chicken being chased by a hunter.

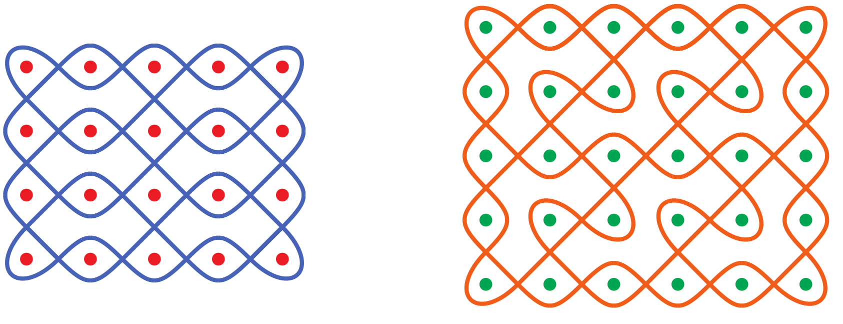


Figure 4. Lion's stomach and 'chased-chicken' path.

When I was analyzing *sona* like these two, I found that they can be generated in a particular way: Both are examples of what I call 'mirror curves', a concept I proposed for the first time in English in (Gerdes, 1990b). A mirror curve is

> *the smooth version of the polygonal path described by a light ray emitted from the starting place S at an angle of 45º to the rows of a grid (see Figure 5); and as the*

ray travels through the grid it is reflected by the sides of the rectangle and by the 'double-sided mirrors' it encounters on its path. The mirrors are placed horizontally or vertically, midway, between two neighboring grid points, as in Figure 6.

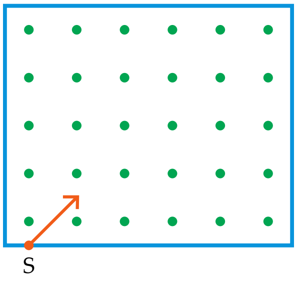


Figure 5. Light ray emitted from point S.

Figure 6. Possible positions of mirrors relative to neighboring grid points.

Figure 7 presents the position of the mirrors in the examples of the 'lion's stomach' and the 'chased-chicken' designs in Figure 4.

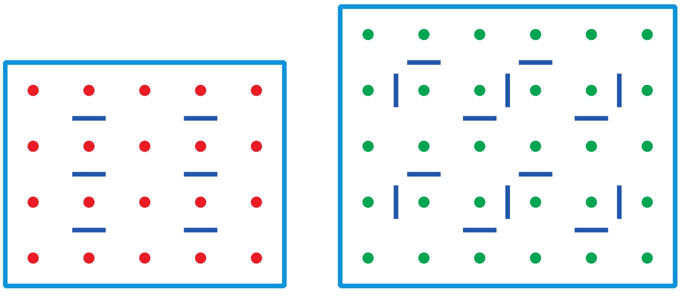


Figure 7. Position of mirrors in the case of the designs in Figure 4.

Once I had defined the concept of mirror curve in general, I started to look for the properties of mirror curves. To facilitate the execution of mirror curves, I used to draw them on squared paper with a distance of two units between two successive grid points. In this way, a line drawing such as the 'chased-chicken' path passes exactly once through each of the unit squares inside the rectangle surrounding the grid (see Figure 8).

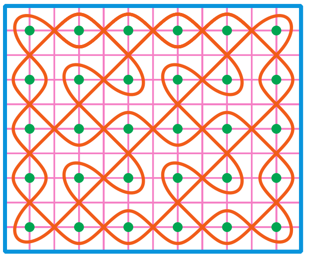


Figure 8. Line passing once through each of the unit squares.

This gives the possibility of enumerating the small squares 'modulo 2', with the number 1 being given to the unit square where one starts the line, and the number 0 to the second unit square through which the curve passes, and so on successively, 101010…, until the closed curve is complete. In this way a {0, 1}-matrix is produced. Colouring the unit squares numbered 1 black, and those numbered 0 white, a black-and-white design is obtained. As this type of black-and-white design generated by mirror curves was discovered in the context of analyzing *sona* from the Cokwe, who predominantly inhabit the Lunda region of Angola, I gave them the name of 'Lunda-designs'. Figure 9 presents two examples of Lunda-designs, using different colours.

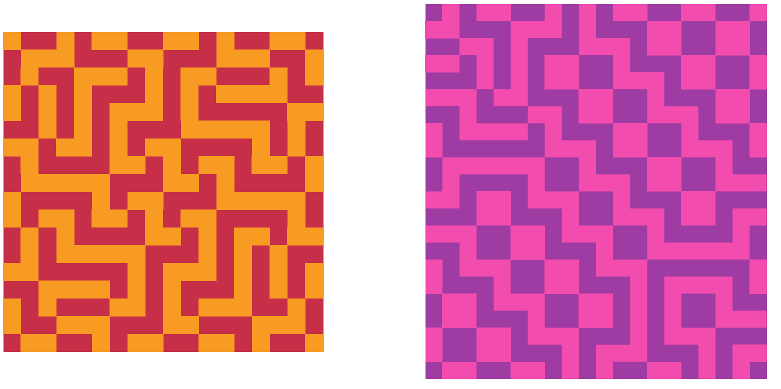


Figure 9. Two examples of Lunda-designs.

Lunda-designs have interesting symmetry properties, which often make them aesthetically attractive. For instance, in each row there are as many black unit squares as there are white unit squares. Also, in each column there are as many black unit squares as there are white unit squares. Furthermore, Lunda-designs have the following two characteristics:

1. Along the border each grid point always has exactly one black unit square associated with it (see Figure 10);

Figure 10. Symmetry situation along the border.

2. Of the four unit squares between two arbitrary (vertical or horizontal) neighboring grid points, two are always black (see Figure 11).

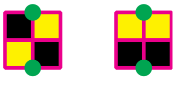


Figure 11. Symmetry situation inside the grid.

The concept of Lunda-design may be generalized in several ways. Circular and hexagonal Lunda-designs are some interesting possibilities (Gerdes, 1999; 2007b). The unit squares through which a mirror curve passes can be enumerated 'modulo *t*' instead of 'modulo 2', if *t* is a divisor of the total number of grid points. In this way *t*-valued matrices and *t*-Lunda-designs are created. Figure 12 gives two examples of 3-and 4-Lunda-designs.

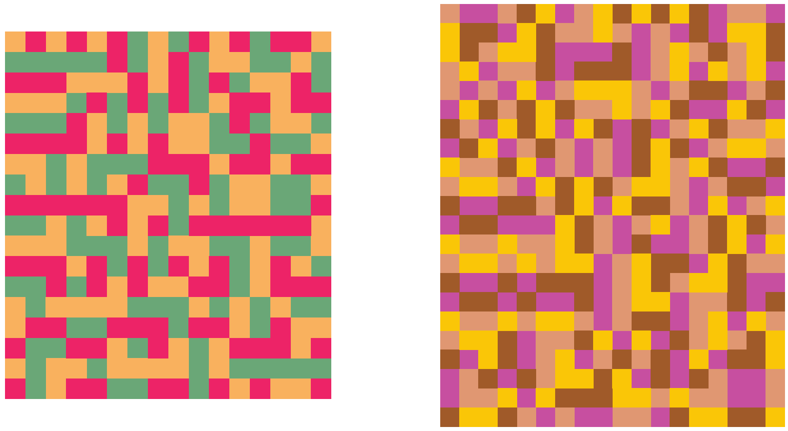


Figure 12. Examples of a 3-Lunda-design and a 4-Lunda-design.

PATH OF DISCOVERY: FROM LUNDA-DESIGNS TO LIKI-DESIGNS AND SPECIAL MATRICES

In 2001, on the eve of the 4th anniversary of my daughter Likilisa, I started to analyze a particular class of 2-Lunda-designs. As these designs turned out to have some interesting properties, I called them Liki-designs. In the case of Liki-designs, the second property is substituted by the following stronger condition. Consider the four unit squares between two vertically or horizontally neighboring grid points. Two of them that belong to different rows and different columns always should have different colours (Figure 13).

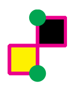
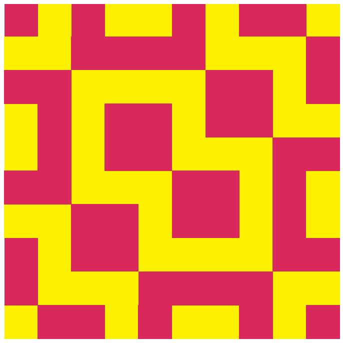


Figure 13. Situation inside the grid.

This property together with the border property (Figure 10) implies that a square Liki-design and its associated Liki-matrix are composed of cycles of alternating black and white unit squares, or of cycles of alternating 1's and 0's, respectively. Figure 14 presents an example of a square Liki-design and its corresponding Liki-matrix L. The matrix has five {0,1}-cycles. A question that naturally emerges is what will happen with the powers of Liki-matrices.



1	0	1	0	0	1	0	1	1	0
0	0	1	1	1	1	0	0	0	1
1	1	0	0	0	0	1	1	0	1
0	1	0	1	1	0	1	1	0	0
0	1	0	1	1	0	0	0	1	1
1	1	0	0	0	1	1	0	1	0
0	0	1	1	0	1	1	0	1	0
1	0	1	1	0	0	0	0	1	1
1	0	0	0	1	1	1	1	0	0
0	1	1	0	1	0	0	1	0	1

Figure 14. Example of a Liki-design (left) and its corresponding Liki-matrix (right).

5	2	2	1	1	3	3	3	3	2
2	5	1	2	3	1	3	3	2	3
2	1	5	3	2	3	1	2	3	3
1	2	3	5	3	2	2	1	3	3
1	3	2	3	5	2	2	3	1	3
3	1	3	2	2	5	3	2	3	1
3	3	1	2	2	3	5	3	2	1
3	3	2	1	3	2	3	5	1	2
3	2	3	3	1	3	2	1	5	2
2	3	3	3	3	1	1	2	2	5

L^2

16	9	15	10	9	16	12	13	15	10
9	10	16	16	15	13	9	10	12	15
15	16	9	9	12	10	15	16	10	13
10	16	9	13	16	10	15	15	9	12
9	15	12	16	15	9	10	10	13	16
16	13	10	10	9	15	16	12	15	9
12	9	15	15	10	16	13	9	16	10
13	10	16	15	10	12	9	9	16	15
15	12	10	9	13	15	16	16	10	9
10	15	13	12	16	9	10	15	9	16

L^3

75	60	60	53	53	68	68	66	66	56
60	75	53	60	68	53	66	68	56	66
60	53	75	68	60	66	53	56	68	66
53	60	68	75	66	60	56	53	66	68
53	68	60	66	75	56	60	66	53	68
68	53	66	60	56	75	66	60	68	53
68	66	53	56	60	66	75	68	60	53
66	68	56	53	66	60	68	75	53	60
66	56	68	66	53	68	60	53	75	60
56	66	66	68	68	53	53	60	60	75

L^4

Figure 15. Several powers of Liki-matrix L.

Figure 15 displays the first powers of Liki-matrix L. The third power has the same cycle structure as the first power: the first cycle of the third power is composed of alternating 16's and 9's, the second cycle of alternating 15's and 10's, etc. The even powers do not have the same cycle structure. Their diagonals are constant and they present other cycles, like the cycle of 2's in the second power. Figure 16 compares the cycle structures of the odd and even powers of the Liki-matrix L.

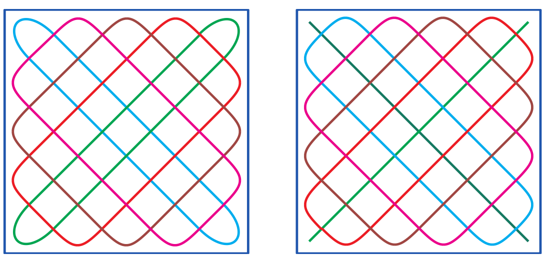


Figure 16. Cycle structures of odd and even powers.

The powers of a Liki-matrix, like the matrices L^2, L^3, etc., are themselves not Liki-matrices. Nevertheless, they display cycle structures. Let us call them *cycle matrices*. As the numbers on the cycles on the odd powers are alternating, we may say that these cycle matrices have period 2. As the numbers on the cycles on the even powers are constant, we say that these cycle matrices have period 1.

-1	3	-4	6	-2	0
3	6	-1	0	-4	-2
-4	-1	-2	3	0	6
6	0	3	-2	-1	-4
-2	-4	0	-1	6	3
0	-2	6	-4	3	-1

A

4	-2	3	5	-3	2
-2	5	4	2	3	-3
3	4	-3	-2	2	5
5	2	-2	-3	4	3
-3	3	2	4	5	-2
2	-3	5	3	-2	4

B

14	7	5	-17	18	-9
5	14	18	7	-9	-17
7	-17	14	-9	5	18
18	5	-9	14	-17	7
-17	-9	7	18	14	5
-9	18	-17	5	7	14

AB

14	5	7	18	-17	-9
7	14	-17	5	-9	18
5	18	14	-9	7	-17
-17	7	-9	14	18	5
18	-9	5	-17	14	7
-9	-17	18	7	5	14

BA

Figure 17. Cycle matrices A and B and their products.

Using the cycle structures, we may introduce the concept of a cycle matrix of period 2, independent of the context of Liki-designs in which I discovered the concept. Figure 17 displays two cycle matrices, A and B, of dimension 6x6, having period 2. Both A and B have the same cycle structure as the design at Laval University (Figure 1) and the basic design for the Cokwe tortoise (Figures 2 and 3). The products AB and BA have a different cycle structure, similar to the second cycle structure in Figure 16. Compare matrices AB and BA. Do you note something remarkable?

Later in my lecture, I will return to cycle matrices in a very different context. At this moment, I would like to underscore the newness of mathematical ideas arising from the analysis of the old Cokwe *sona* tradition and the multiple relationships of these ideas with other areas of mathematics. This reflects the profoundness and the mathematical fertility of the ideas of the Cokwe master drawers.

> *Challenge for reflection: Can African and other cultures become a source of inspiration for the development of new mathematical ideas? Example: 'Sona' designs from Angola. Other examples?*

After the elaborated example of the inspiration of *sona* geometry for developing new mathematical ideas, let me present a brief introductory overview of mathematics and mathematicians in African history and cultures, followed by some historic examples.

MATHEMATICS AND MATHEMATICIANS FROM AFRICA

From the earliest times onwards, humans in Africa and elsewhere have created and developed mathematical ideas. Mathematical reflections from Ancient Egypt, from Hellenistic Egypt, from Islamic Egypt and from the Maghreb during the Middle Ages found their way to Europe and have been contributing to the development of 'international' mathematics (Djebbar, 2001; 2005). Hundreds of mathematical manuscripts—written in Arabic and in various African languages—from Timbuktu, in today's Mali, remain to be analyzed. These should lift the veil from some of the mathematical connections between Africa South of the Sahara and the north of the continent (Djebbar & Moyon, 2011). The astronomer-mathematician Muhammed ibn Muhammed (c. 1740) from Katsina, in today's Nigeria, was well-known in Egypt and the Middle East. Thomas Fuller (1710-1790), brought in 1724 from West Africa as a slave to North America, became famous in the 'New World' for his mental calculations (Fauvel & Gerdes, 1990). Some *sona* geometrical knowledge has survived until the beginnings of the 20th century in the Mississippi area among people of African descent. During the second half of the 20th century, the African continent produced thousands of PhDs, of whom several hundreds have been working as researchers in Europe and North America [see the catalogue (Gerdes, 2007c)].

For an introductory overview of mathematical ideas in the history of Africa South of the Sahara, the reader may consult (Gerdes, 1994b) or the classic book (Zaslavsky, 1973). The study (Gerdes & Djebbar, 2007a, b) presents an annotated bibliography of mathematics in African history and cultures, containing over two thousand entries and indices by region, ethnic or linguistic group, mathematician, and mathematical topic. This study is one of the outcomes of the activities of the AMU Commission on the History of Mathematics (AMUCHMA), created in 1986 by the African Mathematical Union (AMU), to do research and disseminate research findings through lectures, conferences, and publications. Most of the newsletters that AMUCHMA produced in English are available at the following webpage: www.math.buffalo.edu/mad/AMU/amuchma_online.html

Recently, the thirty-seven newsletters were reprinted in two book volumes (Gerdes & Djebbar, 2011).

HISTORIC EXAMPLE: CRADLE OF MATHEMATICS

Very early on, humans living in Africa started to display an interest in symmetry, in constructing parallel lines, rectangles, and triangles, as attested by several objects made around 70,000-80,000 BC, and found during the last decade during excavations at the Blombos cave in the Eastern Cape region of South Africa. A counting rod, found in a cave in the Lebombo Mountains in the border area of South Africa, Swaziland, and Mozambique, dates from 33,000 BC. Better known are the bones found near Ishango in the East of today's Democratic Republic of Congo. The bones date from 20,000 BC. Figure 18 schematically displays one face of the first Ishango bone. The distribution of the number of engravings made into it give the impression that its marker was engaged, in one way or another, in duplication.

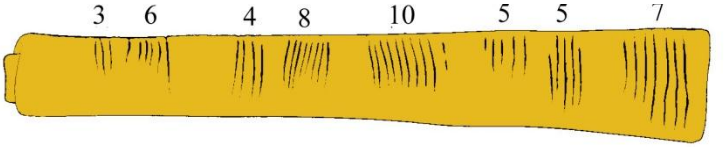


Figure 18. One face of the first Ishango bone.

Looking at the numbers of engravings in the first row of the second face (Figure 19), we see four odd numbers between 10 and 20; 15 is left out. Do they represent only odd numbers, or also prime numbers?

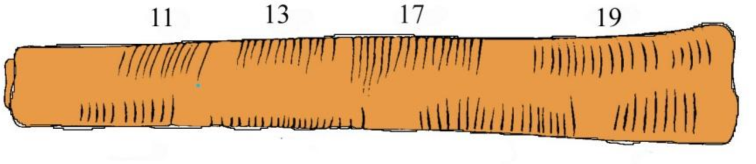


Figure 19. Quantities along the top side of the second face of the first Ishango bone.

Do the numbers at the lower row 11, 21, 19, and 9, that is, 10 + 1, 20 + 1, 20 – 1, and 10 – 1, reveal some special interest in multiples of 10 (Figure 20)? Did the maker use a spoken numeration system with base ten?

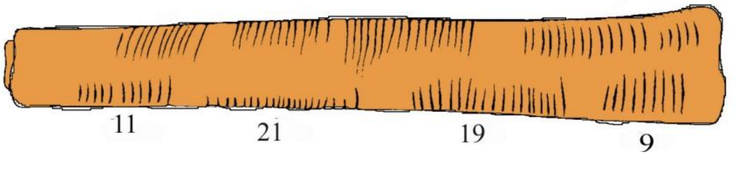


Figure 20. Quantities along the bottom side of the second face of the first Ishango bone.

Comparing the numbers in the two rows, one sees that the sums of both are equal:

$$11 + 13 + 17 + 19 = \mathbf{60} = 11 + 21 + 19 + 9$$

Does this reflect some early interest in the number 60? The Ishango bones have been the object of diverse attempts at interpretation ever since they were found in 1957. A special international conference dedicated to them took place in Brussels, entitled "*Ishango, 22000 and 50 Years Later: The Cradle of Mathematics?*" (February 28 – March 2, 2007) (cf. Huylebrouck, 2008). Considered a symbol of the birth of science in the world, a 7-meter high replica of the small, first Ishango bone was unveiled in 2010 as a statue in front of the Royal Theatre of Money in Brussels (Figure 21).

Figure 21. Unveiling of the replica of the first Ishango bone in Brussels (Belgium).

HISTORIC EXAMPLE: CONCEPTUALIZATION OF MATHEMATICS

The best-known mathematical text from Ancient Egypt is a papyrus written about 1,650 BC by the scribe Ahmes or Ahmose. It may be a copy of a text a couple of hundred years older. Unfortunately, the papyrus is often called the 'Rhind papyrus' after its 19th century buyer. In his book *Egyptian Geometry: Contribution of Ancient Africa to World Mathematics*, Théophile Obenga (1995) underlines that Ahmes' text is much more than a book of exercises with solutions. In particular, he draws our attention to the title of the papyrus: *Correct method of investigation of Nature in order to understand all that exists, each mystery, [and] all secrets*. Is this not a description or conceptualization of what mathematics is about?

Ahmes' title contains an early definition of mathematics. Even today it may stimulate a fruitful debate among mathematicians, philosophers, and mathematics educators about what is (the purpose of) mathematics (education).

Over the centuries, many other mathematicians in Egypt have contributed to the development of mathematics and have reflected about the nature of mathematics, like Euclid (4th century BC), Heron (1st – 2nd century), Diophant (3rd century), Theon and his daughter Hypathia (370-415), Abu Kamil (850-930), and Al-Haitham (965-1039), to name just a few mathematicians from the classic and medieval periods.

HISTORIC EXAMPLE: INTRODUCTION OF SYMBOLS INTO MATHEMATICS

The Maghreb (North-West Africa) played an important role in the internationalization of mathematics in medieval times. One contribution with a lasting influence on mathematics and mathematics education is the invention and dissemination of diverse symbols since the 12th century. As an example, Figure 22 presents a page of a manuscript of that time.

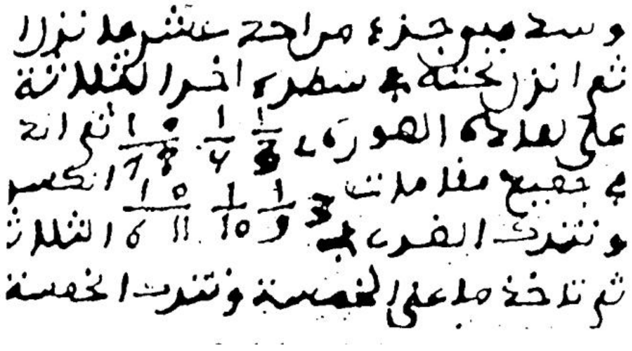


Figure 22. Page of a 12th century manuscript from the Maghreb.
(Reproduced with permission from Djebbar (2005), p. 93)

This text, written from the right to the left in Arabic, is at this moment the oldest text in which today's well-known notation for fractions appears that children all over the world learn to use. Symbols for arithmetical operations and extraction of roots were introduced about the same time. At present, it is not known for sure who the author of the text fragment in Figure 22 was, or who introduced other symbols. It may have been the mathematician and poet Ibn al-Yasamin (d. 1204)—'son of the jasmine flower'. His mother was a black slave from south of the Sahara, freed in agreement with the legal customs of the day after having given birth, the father being a Berber. Also as a mathematics educator, Ibn al-Yasamin has had a long-lasting influence in the Maghreb: for centuries his mathematical poems were used to teach, learn and memorize the basics of arithmetic.

As in the next part of the lecture, the Rule of Signs

$$(-) (-) = +$$
$$(-) (+) = -$$
$$(+) (-) = -$$
$$(+) (+) = +$$

will be referred to, it may be interesting to note here already that the Maghrebian geometer Ibn Al-Banna (13th C.) presented a proof of the Rule of Signs in one of his works.

Challenges for reflection:

- *What is 'positive'? What is 'negative'?*
- *Is the design in Figures 1 and 2 positive or negative? Why?*

HISTORIC EXAMPLE: INTERWEAVING ART AND MATHEMATICS

In African cultures, mathematical and artistic ideas are frequently interwoven. I open the book, *Geometry from Africa* (Gerdes, 1999), with the following sentence: "The peoples of Africa South of the Sahara desert constitute a vibrant cultural mosaic, extremely rich in its diversity." Among the peoples of the sub-Saharan region, interest in imagining, creating and exploring forms and shapes has blossomed in diverse cultural and social contexts with such an intensity that with reason, to paraphrase Claudia Zaslavsky's *Africa Counts* (1973), it may be said that "Africa Geometrizes".

The books, *Geometry from Africa* (Gerdes, 1999), *African Fractals, Modern Computing and Indigenous Design* (Eglash, 1998), *Women, Art and Geometry in Southern Africa* (Gerdes, 1998), and *African Basketry: A Gallery of Twill-Plaited Designs and Patterns* (Gerdes, 2007d), present regional and thematic overviews of geometrical ideas and practices in African cultures. Case studies of geometrical exploration in specific African cultures are presented in the books, *Sona Geometry from Angola: Mathematics of an African Tradition* (Gerdes, 2006), *Sipatsi: Basketry and Geometry in the Tonga Culture of Inhambane (Mozambique, Africa)* (Gerdes, 2009), *Otthava: Making Baskets and Doing Geometry in the Makhuwa Culture in the Northeast of Mozambique* (Gerdes, 2010a; 2012c), and *Tinhlèlò, Interweaving Art and Mathematics: Colourful Circular Basket Trays from the South of Mozambique* (Gerdes, 2010b).

As a historic and current example, I will present the decorated mats woven by Makwe women in the extreme northeast of Mozambique, near the border with Tanzania (Gerdes, 2007d). Figure 23 presents the Makwe master weaver, Idaia Amade, with some mats and bags during an exhibition in the Mozambican capital, Maputo.

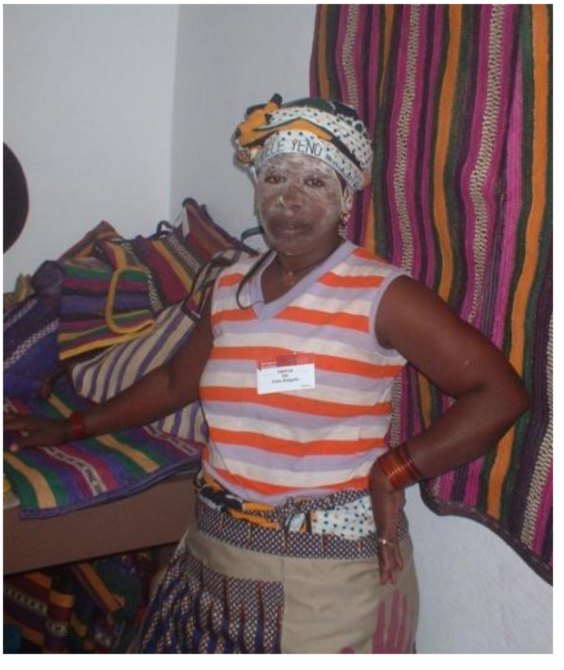


Figure 23. Master weaver Idaia Amade.

For centuries, Makwe women have been weaving their famous *luanvi* mats. In the 18th century these mats were among the most important products traded at Mozambique Island. The mats are made from brightly dyed palm fiber by sewing long plaited bands together. Mono-colour plain bands alternate with black-and-white ornamental bands. The (central parts of the) ornamental bands called *mpango* present all seven possible symmetry classes. Figure 24 presents one example of each class, with the international notation of each symmetry class indicated within brackets.

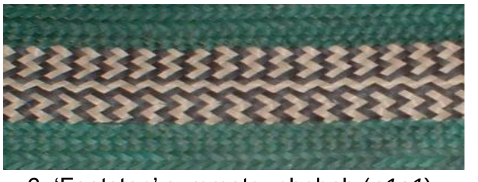
1: HHHHHH (*pmm2*)

2: pqbdpqbd (*pma2*)

3: VVVVVV (*pm11*)

4: DDDDDD (*p1m1*)

5: NNNNNN (*p112*)



6: 'Footstep' symmetry pbpbpb (*p1a1*)

7: LLLLLL (*p111*)

Figure 24. Examples of the seven symmetry classes.

In plaited and twilled basketry, front and backside of a woven band or mat display mostly the same image if they are made with black strips in one direction and with white strips in the other direction: only the colours are interchanged. In other words, one side is the 'photographic negative' of the other side. In Makwe weaving, the situation is different. Figure 25 presents an example: it displays both sides of a *mpango*.

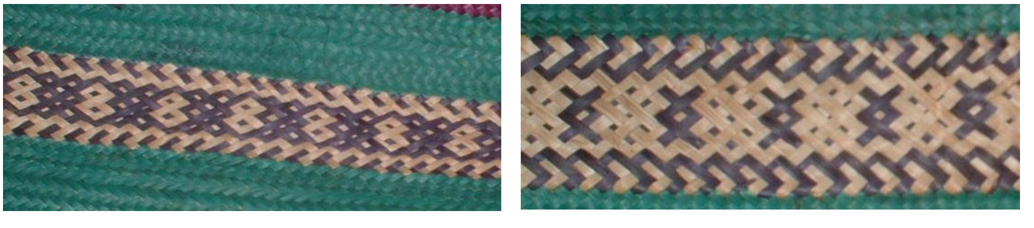


Figure 25. Front and back side of the same ornamental band.

The Makwe use a particular inversion of colour that is distinct from the photographic colour inversion. The black and white strands make angles of 45º with the borders of a decorative band. As, in both weaving directions, light coloured 'white' strands (0) and dark coloured 'black' strands (1) alternate (weaving code 01), we have the following situations:

1. Where a dark strand crosses with a dark strand, we see a dark unit square on both sides;
2. Where a natural strand crosses with a natural strand, we see a naturally coloured unit square on both sides;
3. Where a dark strand crosses with a natural strand, we have on one side a dark unit square but on the other side a natural unit square; the colours have been reversed.

As a consequence, under the Makwe colour inversion, half of the unit squares have the same colour on both sides of the mat (see the coloured unit squares in Figure 26), whereas the other half of the unit squares (white in Figure 26) have opposite colours on either side of the mat.

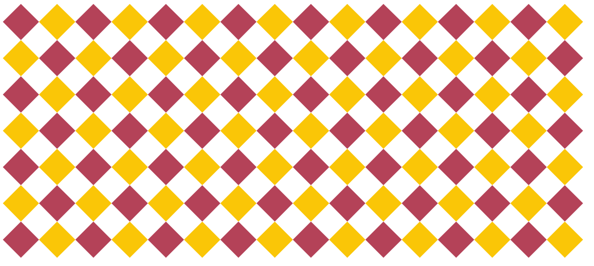


Figure 26. The coloured unit squares are invariant.

The design on a decorated band depends on the weaving algorithm used by the mat maker. Although the designs on both sides are normally distinct, they present the same symmetry. The weavers have invented various patterns with additional properties. For instance, both sides of the decorative band in Figure 27 display the same design but the colours are interchanged.

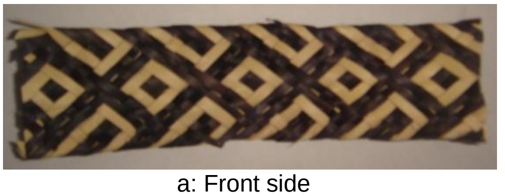
Figure 27. Special pattern with photographic colour inversion.

Makwe women have also explored weaving codes different from the 01-code, that is, they have explored other ways to alternate the colours in both weaving directions. For instance, they use the 011-code to produce the decorative band in Figure 28: each time one white strand is followed by two black strands. Figure 29 presents an example of the use of the 00111-code.

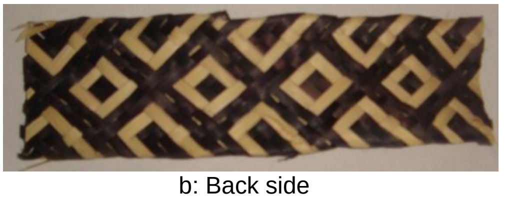
Figure 28. Example of the application of the 011-code.

Figure 29. Example of the application of the 00111-code.

A very interesting case of the use of the 011-code is the chicken's eye pattern (Gerdes, 2013b). Figure 30 presents the front and back side of a piece of a band decorated with the chicken's eye design.



a: Front side



b: Back side

Figure 30. Band with the chicken's eye design.

The pattern has period six: 011011. On the backside the same pattern appears as on the front side, however its orientation is inversed and it is slightly displaced. How could the inventor have imagined such an exceptional design? It is surely not the result of experimentation, as there are too many possibilities. The inventor, several centuries ago, had consciously constructed the weaving texture using some kind of careful mathematical analysis. Calculations and geometry-symmetry considerations were involved. Figure 31 displays the underlying number frieze of the weaving texture: a place marked by a 1 means that the descending strand passes over the mounting strand; a place marked by a 0 is one where the descending strand passes under the mounting strand. The number frieze has vertical axes of symmetry and a horizontal anti-symmetry axis (inversion of 0's and 1's not belonging to the horizontal axis). This chicken's eye design may lead to the study of new types of number friezes (cf. Gerdes, 2013b).

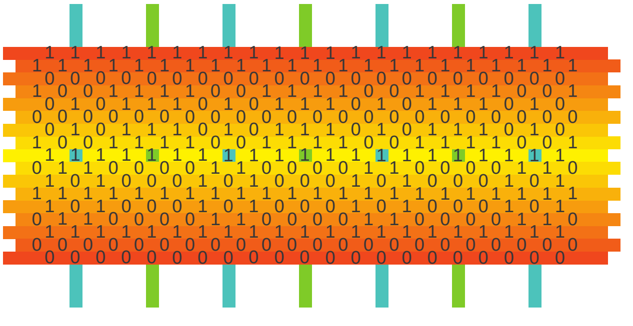


Figure 31. Underlying number frieze.

Challenges for reflection:

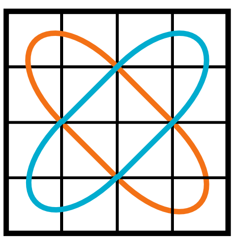
- *Who does mathematics? Who invents mathematics? What is 'mathematical thinking'? Who or which culture defines it?*
- *Can African and other cultures serve as a source of inspiration for the development of new mathematical ideas?*

Let me now return to the code 01 and explore a particular case of it. Figure 32 presents the front and back side of a Makwe design called the 'footprints of a lion'.

a: Front side b: Back side

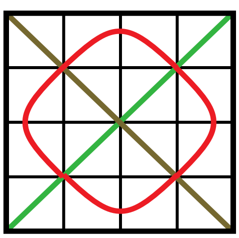
Figure 32. 'Footprints of a lion' design.

The designs on the front and back side are similar to the cycle structures we met earlier on (see Figure 16). The design on the front side corresponds to the cycle structure in Figure 33: we may attribute the number 1 to the unit squares through which the first cycle passes and the number 2 to those through which the second cycle passes. Analogously, the design on the back side corresponds to the cycle structure in Figure 34, composed of two straight segments (numbered 3 and 5) and one cycle (numbered 4). In this way, we constructed two cycle matrices of dimensions 4x4 and of period 1.



1	1	2	2
1	2	1	2
2	1	2	1
2	2	1	1

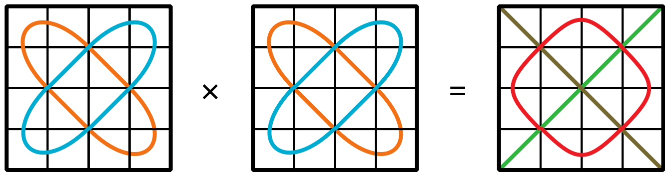
Figure 33. Front side design and corresponding matrix.



3	4	4	5
4	3	5	4
4	5	3	4
5	4	4	3

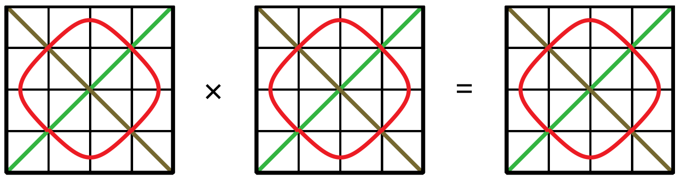
Figure 34. Back side design and corresponding matrix.

Let us multiply these two cycle matrices and see if something interesting happens. Figure 35 presents an example of the multiplication of two matrices with the first cycle structure; the result is a matrix with the second structure. Figure 36 presents an example of the multiplication of two matrices with the second cycle structure; the result is once more a matrix with the second structure. Figure 37 presents an example of the multiplication of a matrix with the second cycle structure with one with the first cycle structure; this time, however, the result is a matrix with the first structure.



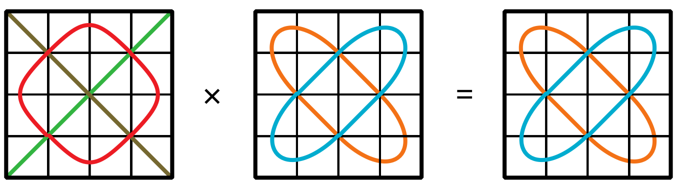
$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 9 & 9 & 8 \\ 9 & 10 & 8 & 9 \\ 9 & 8 & 10 & 9 \\ 8 & 9 & 9 & 10 \end{pmatrix}$$

Figure 35. Example of a multiplication.



$$\begin{pmatrix} 3 & 4 & 4 & 5 \\ 4 & 3 & 5 & 4 \\ 4 & 5 & 3 & 4 \\ 5 & 4 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 4 & 4 & 5 \\ 4 & 3 & 5 & 4 \\ 4 & 5 & 3 & 4 \\ 5 & 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 66 & 64 & 64 & 62 \\ 64 & 66 & 62 & 64 \\ 64 & 62 & 66 & 64 \\ 62 & 64 & 64 & 66 \end{pmatrix}$$

Figure 36. Example of a multiplication.



$$\begin{pmatrix} 3 & 4 & 4 & 5 \\ 4 & 3 & 5 & 4 \\ 4 & 5 & 3 & 4 \\ 5 & 4 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 25 & 25 & 23 & 23 \\ 25 & 23 & 25 & 23 \\ 23 & 25 & 23 & 25 \\ 23 & 23 & 25 & 25 \end{pmatrix}$$

Figure 37. Example of a multiplication.

These results hold in general. For matrices of dimension 4x4 of period 1, having the first cycle structure (Figure 33) or the second cycle structure (Figure 34), we have the multiplication table shown in Figure 38.

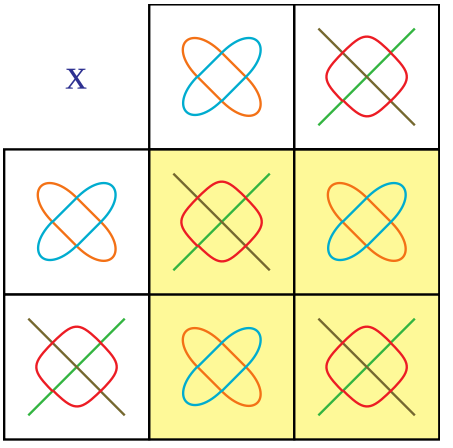


Figure 38. Multiplication table.

This multiplication table is similar to the aforementioned Rule of Signs for the multiplication of negative and positive numbers. The same holds for cycle matrices of any dimensions and any (admissible) period. Therefore, it is well justified to call designs and matrices with the first cycle structure 'negative', and designs and matrices with the second cycle structure 'positive'. The odd powers of Liki-matrix L (Figures 14 and 15) are negative, whereas the even powers are positive cycle matrices of dimensions 10x10 and period 2. Matrices A and B in Figure 17 are negative, while AB and BA are positive cycle matrices of dimensions 6x6 and period 2. The Laval University design in Figure 1 and the Cokwe design in Figure 2 are 'negative'!

Cycle matrices with their corresponding geometric designs have interesting properties, and may be applied in and outside mathematics. They are visually beautiful, like Lunda- and Liki-designs. They may be used as an attractive introduction to matrix theory, as I explain in my book, *Adventures in the World of Matrices* (Gerdes, 2008), written for high school students and undergraduates. Several proofs in the theory of cycle matrices may be given with geometric resources. Computer software may be explored to find properties of cycle matrices. From cycle matrices onwards it is possible to discover other types of matrices like helix and cylinder matrices (Gerdes, 2002a; 2002b).

CONCLUDING REMARKS

There exists an immense variety of 'Old and New Mathematical Ideas from Africa'. In my address I presented only a small selection, influenced by my personal research experience. During millennia, Africans have been developing mathematical ideas in diverse cultural contexts. The contributions of African professional mathematicians and mathematical practitioners, like artists, musicians, drawing masters, storytellers, and mat weavers, may serve as a source of inspiration for new generations.

Mathematical ideas from Africa may be explored in mathematics education at all levels. Traditions with mathematical ingredients, like the *sona* of the Cokwe drawing masters-educators and the *mpango* of the Makwe mat weavers, may serve, as shown, as a source of inspiration for the invention of attractive new mathematical ideas and new educational explorations.

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Elder Talk

La parole aux anciens

COODA, WOODA, DIDDA, SHOODA: TIME SERIES REFLECTIONS ON CMESG/GCEDM

William Higginson
Queen's University

[*Editorial note: The style of this paper has stayed in 'lecture' form. A slight modification to the spoken record comes with the addition of a few selected references.*]

Madame Chair: Thank you for your invitation and for the introduction. I confess, however, that there have been times over the past few months when I have identified with the American chap who was about to be tarred and feathered and run out of town on a rail, when he noted, "If it weren't for the honor of the thing, I'd rather walk."

Let me begin with a statement that, in some senses, is an abstract of the next half hour's observations: This 'Elder Talk' is dedicated to

- **Four Individuals**—the Visionaries from Year Zero: John Coleman, Claude Gaulin, Tom Kieren and David Wheeler;
- **Five Institutions**—consistent supporters of an unusual organization: The Universities of Alberta, Concordia, Laval, Queen's, and Simon Fraser;

and, finally, but perhaps most importantly, to

- **A large number of 'erics'**—an 'eric' being defined as an individual who works quietly, competently, consistently and dependably in roles such as treasurer, editor, secretary, president, and conference organizer, thereby making themselves absolutely critical to the success of an initiative. Those who have watched this phenomenon over the years within CMESG/GDESM will recognize that our most 'eric' of 'erics' has, not surprisingly, been Eric (Muller).

There comes a time in the preparation of an invited talk when a speaker has to come to grips with a fundamental question: "just what am I trying to achieve here?" An Oxford academic friend reported one interesting variation on this problem to me a number of years ago. He had invited the noted philosopher of science and mathematics, Imre Lakatos, to give a talk to his class. Lakatos reputedly had only one question: "What would upset them most?" I had occasion to be reminded of this as I began to think about this lecture. Just what am I trying to achieve here? Just what is an 'Elder Talk'? At the 1979 Kingston meeting, which we can think of as the group's third, our 'non math-ed' speaker was the feisty Israeli philosopher of science, Joseph Agassi. Agassi and Lakatos had been junior colleagues of Karl Popper at the

London School of Economics and in his opening remarks to the Study Group, Agassi cited another variation on Popper's idea that "the purpose of a professor is to provoke." He began his keynote address, *On Mathematics Education: The Lakatosian Revolution* (Agassi, 1981) by noting:

> *When a philosopher like me is invited to address a professional group like the present august audience, it is unreasonable to expect that he show expertise in their specialization of the level required from one addressing a conference of his peers in his own specialization. Hence, such an invitation must be based on a different expectation. Possibly the philosopher is expected to perform a ritual function, akin to that of a priest. I have been invited for ritual purposes many times in the past, but no more. One advantage of a reputation is that it prevents such understandable cross-purposes. Whatever the reason for my being invited, it is no longer to offer platitudes or homilies. I will neither soothe, nor preach, and by now this is known. …But there is one good reason for inviting a philosopher to any specialized conference; he may be able to make quite a lot of trouble in a short time.*

Stimulating as these thoughts are, elders are not necessarily philosophers and ritualistic as intentions may have been, one would be wise to pick and choose from Agassi's spectrum. Platitude and homily-free addresses would seem to be desirable and I feel no need to soothe. Not preaching might, however, be more of a problem, with the speaker's track record inclining hard to the hortatory. That having been said, we might go back to first principles and note (as will be seen to be pertinent shortly) that it is always a good idea to invoke Alfred North Whitehead early in a talk. Let us choose then his observation (recorded, as I recall, in Lucien Price's (1954) *Dialogues of Alfred North Whitehead*) that he "seldom disagreed with the last 90% of a book". It was only in the first few pages where, consciously, or unconsciously, the author states his or her assumptions, that Whitehead might disagree. So, duly reminded, let us state some assumptions—or, given the audience, 'axioms/ambitions/awarenesses' for the talk.

Axiom/Ambition/Awareness 1: The purpose of a lecture, if not quite as extreme as "to make quite a lot of trouble", should be, at least, to stimulate thought or to invite the reconsideration of long-held beliefs.

Axiom/Ambition/Awareness 2: Elder talks would seem to be largely about generations and transitions (I hesitate to add "within institutions"—a phrase which has bad connotations in senior's circles). I would hope that half an hour from now at least a few of you might have a deeper understanding of this group, at least in the sense of how it began and how it has evolved. [As something of a test of that, individuals might reconsider the earlier dedication and review it at the end of the talk.]

Axiom/Ambition/Awareness 3: (for enrichment) Individuals might consider how they might react to the hortatory conclusions of the talk [couched either in critical (math educators have been major, albeit mainly unwitting, accomplices of running-dog capitalism) or constructive (the next generation should strive harder to realize "Coleman's ratio") terms.]

Axiom/Ambition/Awareness 4: Given the nature of our topic, it is going to be hard to escape G. H. Hardy's (1941), "Good work is not done by 'humble' men", advice (in *A Mathematician's Apology*) that to be successful in any area, one must do two things: first, overestimate the importance of the area, and second, overestimate the importance of oneself in that area. [Aside: I first read that advice almost fifty years ago and in a fit of student enthusiasm repeated it to my university tutor, a rising star. It was coolly received.]

Axiom/Ambition/Awareness 5: Listeners who find the challenges posed by the prior entries on this list less than compelling might wish instead to ponder the following three questions (which will be addressed, at least in passing, later in the presentation):

1. What were the two factors that, above all others, shaped human history in the twentieth century?
2. If the answer is "Math is the answer", what is the question?
3. The hands of an accurate clock are 3 and 4. How far apart are their endpoints when they are moving most quickly away from each other?

With those fundamentals in place, let me move to a consideration of the origins of this group with an emphasis on some characteristics of its two main protagonists. The details of the lives of David Wheeler and John Coleman have been extensively recorded and can easily, with the help of the Googoli, be flushed out of bit-space. The elegant efforts of their principal epigoni (David Pimm and Peter Taylor) are particularly recommended.

David Harry Wheeler was born in London, England, in 1925 and attended educational institutions in that city. A teacher of secondary school mathematics, he was an early member of the Association for Teaching Aids in Mathematics. Founded in the early 1950's by the noted educator, Caleb Gattegno, ATAM morphed within a few years into ATM, The Association for the Teaching of Mathematics. Wheeler played key roles in the early years of this organization, serving an early stint as Secretary and as an active and influential editor of its journal, *Mathematics Teaching*. He worked for some time in the School of Education at the University of Leicester. From that position, he relocated to New York City in the late 1960's to join Gattegno who had started *Educational Solutions* there. He came to Montreal in 1975 to join the Department of Mathematics at Concordia University. He was the founding editor of *For the Learning of Mathematics* in 1980. On his retirement from Concordia in 1990, he moved to Vancouver where he maintained a close relationship with colleagues at Simon Fraser University until his death in 2000. He remained an active and highly respected member of the international mathematics education community during the later years of his life. His name is memorialized in the David Wheeler Centre for Mathematics Education at Simon Fraser University (http://blogs.sfu.ca/research/davidwheeler/about-david-wheeler/).

Albert John Coleman was born in Toronto, Ontario, in 1918 and was educated at local schools and the University of Toronto. He was a member of the University of Toronto team that won the first Putnam Competition in 1938. He worked for the World Student Christian Movement in Geneva after the Second World War. He was a member of the University of Toronto mathematics department from 1950 until 1960 when he came to Queen's University in Kingston to take over the headship of the Department of Mathematics, a position he was to hold for 20 years. He was an active supporter of curriculum reform in school mathematics in the late 1960's, in particular in his role of general editor of the series of texts published by W. J. Gage and Company. As an instructor he was famous for the extensive, impromptu deviations he could make from almost any technical starting point. These deviations were not universally appreciated by his undergraduate charges, especially on Saturday mornings. One story that John liked to recount was that of the forthright student who told him that "Dr. Coleman, your lectures are terrible, but your asides are fascinating." My own particular Saturday morning favourite was his observation to the class that "You are too young to understand infinity." He was a passionate proselytizer for Christianity, for his intellectual hero, Alfred North Whitehead, and for fine wines. He was president of the Canadian Mathematical Society (1971-73) and the senior author of Report #37 for the Science Council of Canada, *Mathematical Sciences in Canada* (1976). This document was coolly received by the mathematical community. In the 1980 federal election, he was the Liberal candidate for Kingston and the Islands, narrowly losing to the Conservative candidate, Flora MacDonald.

An internationally respected mathematical researcher, he maintained his intellectual interests into his nineties. He died at Kingston in October of 2010.

To complete this sketch of the three 'founding fathers' let me note a few things about my background and some insights into the ways in which I played a role in connecting these two sterling characters in 1977. I had been an undergraduate at Queen's from 1961 to 1965, doing a degree in Mathematics and Economics. [I had wanted to study Mathematics and English but that combination was seen as too radical for the period, so Economics it was]. By a rather circuitous route, I found myself in Coleman's first-year calculus class. In those more relaxed days, there were about six sections of first-year calculus and John had generated his slice by a fairly simple *droit de seigneur* exercise of taking the top sixth of the class by their average mark on the three Grade Thirteen departmental exams in Geometry, Algebra and Trigonometry. I was not in the top tranche, and Professor X, to whom I had (more or less randomly, I think) been assigned, very quickly failed to meet my 17 year-old stereotype of suave, debonair professor. By chance, John had been the chief examiner for the Department of Education's Geometry paper that year and my father (himself a Queen's graduate, class of 1936), simultaneously principal and geometry teacher in the small rural high school I attended, had been one of its 'markers'. [This was a fairly standard way for modestly remunerated teachers of the day to earn a bit more money.] My father had proudly told John that his son was coming to Queen's that Fall to study mathematics. John had muttered something in the nature of, "How nice—be sure to tell him to come to see me if he has any problems." So shortly thereafter, with problematic Prof X in my sights, I marched up the stairs to the Head of Department's Office and announced to his somewhat surprised-looking secretary that I needed to speak to Dr. Coleman. [Her surprise may have been partly related to the fact that I was hairless and wearing only a sheet: this was standard freshman orientation fashion at Queen's at the time.] John was gracious enough to see me immediately, didn't look surprised, and didn't reject outright my request to transfer to his class. He seemed, however, not to have any particular recollection of my father, or to be very impressed when I responded to his query about my Grade XIII grades. But then he paused and said, "And where did you say that that you went to high school?" When I told him he responded, "Well, for Sharbot Lake High School, those are good grades." Perhaps elitist, but probably true. The next day I slipped quietly into a bench in the Old Arts building to listen to John's asides. Probably half of the fifty or so students in the class would end up being academics, most of them in mathematics or in mathematically related fields like economics. Their number included George Elliott of C* algebra fame, David Dodge, later to become Governor of the Bank of Canada, and one Peter D. Taylor. I got an A. Not a high A, but an A. My father was happy and I was pleased to have met some interesting peers, several of whom became good friends.

Moving quickly forward: four very social years at Queen's successfully attempting to embody the concept 'callow', a year teaching secondary school in Kingston, two years with CUSO in East Africa, a year in Toronto; a year at Cambridge doing a Certificate in Education (BEd equivalent) followed by an MA in Mathematics Education from the University of Exeter. During my two years in England I was an active member of the Association of Teachers of Mathematics and met David Wheeler at several conferences. I started a PhD in Math Ed at the University of Alberta under the direction of Tom Kieren in September of 1971 and finished in August of 1973. In September of 1973, I joined the young Faculty of Education at Queen's. John had been quite impressed during some of his travels with work done out of the Polish Academy of Science on the professional inservice of mathematics teachers. So, with the 3rd ICME meeting looming in Karlsruhe in the summer of 1976, I was charged with the task of liaising with Prof. Semadini of the Polish Academy. It seemed all very LeCarré-esque to me, but Prof. Semadini turned out to be a charmer and the Polish initiative an impressive one, if not one that could serve as any sort of model for work in Canada. Working on my own, however, I uncovered another project that, at least in my opinion, was a much better fit. It was based—wait for it—out of Quebec City and was directed by Claude Gaulin. Who would have thought?! Two solitudes indeed. Cancel all those European plans!

Not so long after the ICME meeting, we have Wheeler's delicate letter of inquiry. New person in country, might there be merit in looking at possible ways of increasing communication across provinces, etc.? The letter is sent to a large number of senior mathematicians and mathematics educators across the country. The return rate is not terribly high and most of the responses express contentment with existing arrangements (NCTM, CMS, …). Among the outliers, two independent responses from Coleman (smarting a bit about the Science Council Study response, committed to education, having some unassigned resources from the Science Council Study) and Higginson (remembering how impressed he had been with Wheeler's work in the UK). An invitational meeting followed with Gaulin and Kieren as keynote speakers and the rest is Study Group history.

Time for a Colemanesque aside—this one political—which we might label: "Putnam whiz goes to Parliament". Recall first the somewhat unlikely pairing of Putnamist and Politician in the form of AJC. To do this let us look briefly at our third question, the one about the clock—number 3. All of you who got "root 7" raise your hands [pause]. "It's so tough to get good students these days". Now this question is of note today because of its source, the 1983 Putnam exam. Except for the glorious year of 1952 when Queen's won the event, 1983 was one of the best Putnam years for Canada with four teams in the top 10 from Canada. Joining the perennial (by that period) Waterloo (top five) were three other Canadian universities in the sixth to tenth place rankings—Alberta, Memorial and Queen's—worth noting also because the top scoring Queen's competitor (Honorable Mention) was the precocious 'Teddy' Hsu, recently (May, 2011) elected Liberal member of Parliament for Kingston and the Islands. Belatedly, Kingston gets its brilliant mathematical member. Also of note to the relentless researcher is the inclusion that year, in the *crème de la crème* (top 2%) list of honorable mentions, of one Frédéric Gourdeau of Université Laval. End of aside.

And so, by the late 70's the die had been cast—a set of circumstances and individuals had come together and collectively created something of lasting value. Without going into detail—both for lack of time and because I was less involved in later developments of the group—let me try to summarize what I feel had been accomplished. [This is the 'didda' section of my title.]

Summary: By the mid-nineteen eighties the Canadian mathematics and mathematics education communities had established a small, national organization that meets annually over a five-day period, usually late in May or early in June, in different parts of the country. Early on, the group established a standard meeting structure that has varied little over the years. This structure gives primacy of place to extended group discussions of a number of pre-identified themes. Each meeting features two keynote addresses from prominent scholars: one from mathematics and one from mathematics education. The proceedings of these gatherings have been comprehensively documented for more than three decades. The Group's leadership has, over the years, shown an impressive diversity over characteristics such as geography, language and academic specialization. It would appear to play an important professional role, perhaps something of a brief but regular return to graduate school, in the lives of a high percentage of individuals in the country who have educational interests in mathematics.

Another aside perhaps. DHW was not a man for excess. He did, however, particularly seem to openly enjoy my revelation at another CMESG occasion—that he had, in a wonderful Freudian slip, let an insightful typo into print with respect to our iconoclastic philosopher: not Agassi, "A- g- a- s- s- I", but "A- g- a- s- s- !" (exclamation mark!) A double 's' gass indeed! … [which brings to mind Northop Frye's confession that he lost much of his admiration for mathematics and mathematicians when he realized that by placing an exclamation mark after a number mathematicians were not expressing amazement and admiration—"7 WOW"—but something altogether more mundane]. But it is not fair to the coming generations to leave them without challenges, so let me move into the final ('shooda') section of this talk by adopting a considerably less charitable perspective.

Let's approach it through our remaining two questions. First, the query (number 1) about the factors that have shaped human history in the twentieth century. I have no doubt that our summative responses would be most interesting, but in light of time constraints let me give you the answer of the individual who posed the question:

> *One is the development of the natural sciences and technology, certainly the greatest success story of our time—to this, great and mounting attention has been paid from all quarters. The other, without doubt, consists in the great ideological storms that have altered the lives of virtually all mankind: the Russian Revolution and its aftermath—totalitarian tyrannies of both the right and left and the explosions of nationalism, racism, and, in places, of religious bigotry, which, interestingly enough, not one among the most perceptive social thinkers of the nineteenth century had ever predicted.*

Thus, Isaiah Berlin, the distinguished British political philosopher, in his essay, "The Pursuit of the Ideal", (included in the 1991 collection).

Moving directly to question two: "If the answer is 'Math is the answer', what is the question?"

The *Sunday New York Times* is the source of the correct response to this third question. There, Thomas L. Friedman (2012) had a column entitled "Lead, Follow, or Get Out of the Way". Part way through, he noted:

> *Now that the dictators are being swept away, Islamist parties are trying to fill the void. Who will tell the people that while Islam is a great and glorious faith it is not 'the answer' for Arab development today? Math is the answer.*

So, what can we take from these two observations? From Berlin I think that we can argue that math education is critically important to the historical development and to the future evolution of human society, and not just in the Friedmanesque sense. (And while it certainly might look like this is an obvious example of the aforementioned 'Hardy hazard', I do not think that this is the case). It can be seen to be a central discipline located at the intersection of Berlin's two fundamental factors. There is no question that contemporary natural science, and especially information science, and technology in general, are totally dependent on mathematics. That's the math bit. The strife part surely is at the heart of the reason we try to educate. So, mathematics education is, by its most fundamental nature, located at a critical nexus for our species.

The Friedman quote is just one recent example of how much of the world sees contemporary mathematics. There are many people in positions of responsibility, particularly from corporate or policy-making perspectives, claiming that mathematics is essential. But while on the surface we might be grateful for such powerful allies, it behooves us to dig deeper. There are, in my opinion, two realities underlying these statements. The first is that the view of mathematics they are endorsing (and pushing very hard) is exceptionally narrow. Secondly, it is narrow in a limiting and very dangerous way—a way in which the strife part of Berlin's two factors is not very far away, and one where human and environmental factors are seldom considered. Our schools have almost exclusively been teaching 'f*a*t' math. ('*a*' as in *a*ccountant/*a*bstract—math as almost exclusively *a*rithmetic—the math of simple comparisons—easily testable math.) Bottom line positive—good. Bottom line negative—austerity for others. As the old Russian saying goes, “The shortages will be shared among the peasants”. The junction of, and relation between, ‘*a*’ and ‘*e*’ is an interesting one culturally: in North America it is *A*rts and *E*ntertainment; in the British world, *A*ccidents and *E*mergencies. For mathematics education though, the richest consideration should come from the world of the theory of art—in particular, Wilhelm Worringer’s (1908/1997) 1903 doctoral thesis with its distinction between *A*bstraction and *E*mpathy. What math education badly needs is a balance between ‘f*a*t’ math and ‘f*e*t’ math—the math of the tactile, visual, human, empathetic, geometric and contextual. ‘F*e*t’ math has been alive and well at the Study Group since its inception. Think of the passion of the informal problem-solving sessions led over the years by Ralph Mason or Claude Gaulin. But like our bankster friends we’ve been happy just to be self-indulgent. For us, the richness of a glorious human construct. For the other 99%, what Nardi and Steward (2003) brilliantly recorded in their *British Educational Research Journal* paper, that is, “quiet disaffection” and T.I.R.E.D maths—characterized by Tedium, Isolation, Rote, Elitism, and Depersonalization. To succeed in changing fundamental perceptions is exceptionally difficult. But not to try to do so is a mark of moral cowardice. We have, in my opinion, exceptionally capable individuals in the Study Group for the consideration of this possibility. Would it not be wonderful if 35 years from now a wrinkled elder could say to the 70th meeting of the Study Group: “We have made terrific progress making mathematics learners realize what John Coleman knew in 1976, that, ‘to mathematize is to joy’.

Thank you for the opportunity to bring these views to your notice and also for your attention. I will follow your progress carefully from my elder armchair. Younger generation, over to you.

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Panel

PANEL INTRODUCTION: WHAT IS FUNDAMENTAL MATHEMATICS FOR LEARNERS?

Elaine Simmt
University of Alberta

In 2011 – 2012 mathematics curriculum and teaching methods became a topic of great interest in the nation's newspapers. One author in particular, writing for the *Globe and Mail*, triggered a great deal of debate among the public and mathematicians and mathematics educators. On September 29, 2011, Wente wrote that, "too many teachers can't do math, let alone teach it" and then again on December 15, 2011, she attacked the curriculum and current teaching methods, blaming them for why students "can no longer add or subtract…." At the heart of these conversations are beliefs about what constitutes a good mathematics curriculum.

Recognizing the responsibility we face for commenting on mathematics education, we felt it would be informative for CMESG members to consider the question: "What is fundamental mathematics for learners K-16 (primary school to university)?" To explore that question, we invited mathematics educators from a variety of perspectives to respond. Darien Allan, a current doctoral student and high school teacher from British Columbia, Ralph Mason, a former middle school teacher and current professor of mathematics education from Manitoba, Ruth Beatty, a former special education teacher and current assistant professor in mathematics education from Ontario, Peter Taylor, a professor of mathematics from Ontario, and Hélène Paradis, Manager of Educational Services with the Québec Ministry of Education, Leisure and Sport, responded to our question. What follows are the positions that each of these mathematics educators put forward in the panel discussion.

The CMESG executive, on behalf of the membership would like to thank each of these people for their insightful contributions to our 2012 meeting.

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WHAT IS FUNDAMENTAL IN MATHEMATICS?

Darien Allan
Graduate Student, Simon Fraser University
Secondary Mathematics Teacher, New Westminster Secondary

I struggled with how to respond to this question. In preparing for this panel I researched and read many articles about the math wars in the United States and about more recent assaults on the mathematics education community within western Canada. While I learned a great deal, I wasn't sure how I could use what I had learned to provide my fellow panellists and the audience with anything new to them.

I do not have the same level of experience, nor am I as well read as my co-panellists, and probably most of the audience, but I do feel that I bring a different perspective. As a practicing secondary school teacher I feel that I am closer to the heart of the matter. I have heard it referred to as being 'in the trenches'. It is from this central and unique perspective that I approached this panel question. Though I cannot tell you what it is that *all* secondary teachers believe is fundamental in mathematics, I did do a little research and I can report what *some* teachers feel is fundamental. I asked my colleagues and peers who teach secondary mathematics the following question:

> *Imagine every student comes into your classroom with three qualities or things; these could be attitudes, skills, or something else, but given they have these three things, you know that they will be successful in your class and in mathematics. What are these three things?*

I received a range of responses from a diverse group of twelve teachers. These teachers fall in almost a thirty-year age range, have greatly different teaching styles, teach in different districts with different socio-economic levels, come with varied experiences, pedagogical beliefs and practices, but despite all these differences there were some very clear commonalities in their responses.

Most noticeably, in response to my question, what one might have expected and what I did not get was a list of content or topics in mathematics. With one exception, there was no mention of trigonometry, logarithms, calculus, or the like. I was able to categorize the responses into two areas: skills, and attitudes or dispositions.

SKILLS

Every teacher asked mentioned the necessity of basic skills capability in some form. They wanted their students to have not only a facility with basic arithmetic operations on integers, but also a good *understanding* of them and thus a high level of confidence in their abilities. Teachers wanted students to not only have confidence but also a self-awareness of their own abilities and limitations. Some emphasized that students should be able to do these arithmetic operations in their head. Some respondents specified that students should be adept at operations with fractions. Having an idea of the size of numbers, having number sense, and being numerate were also key qualities mentioned.

The majority of the teachers stated that students should have logical thinking and reasoning skills, and problem-solving skills, including how to read a problem, how to understand a problem, how to plan or create a method for solving the problem, how to carry out that method, and how to run diagnostics.

ATTITUDES OR DISPOSITIONS

Usually the third, or second and third desired quality mentioned was a disposition towards or a belief about mathematics in particular, and/or school in general. These I have classified into things teachers want students to be willing to do, understandings teachers want students to have, and attitudes or beliefs teachers do *not* want students to have.

WILLINGNESS

Willingness came up often. My teaching colleagues want students to be willing to:

- try
- engage
- work
- persevere
- get stuck
- be frustrated (and accept this)
- take chances
- make mistakes
- learn from mistakes
- have fun
- make an effort to remember
- grow
- be open-minded
- …

UNDERSTANDING

Beyond an understanding of the concepts, there were some more general understandings that were desired: the need for practice (without an extrinsic focus on marks), the value of what we do (within the classroom), and the recognition that if they are unable to get the correct method or answer after their first attempt, they are not stupid. In many cases, this understanding precedes the willingness to do certain things.

UNWANTED QUALITIES

In many informal conversations over my teaching career, I have heard a lot of teachers indicate that they feel they are fighting a losing battle against the previously developed attitudes and dispositions students often bring with them to the classroom. My colleagues communicated this feeling as well. They stated that they want students who do not hate mathematics and who do not have the preconceived belief or attitude that they are incapable of doing mathematics.

SUMMARY

Though I surveyed only a minuscule proportion of the secondary mathematics teachers in the province, never mind the country, I feel that I would have obtained similar results with a sample a hundred times the size. Having the basic skills and a good understanding of these concepts helps students to develop confidence and fosters the desirable attitudes and dispositions that are foundational and fundamental to learning further mathematics. This is what my colleagues conveyed to me. The question that arises is, if this is what is fundamental, how can it be accomplished? And before we can begin to address this issue, it is first necessary to ask how we can develop and foster an understanding of the value of mathematics?

WHAT IS FUNDAMENTAL MATHEMATICS FOR LEARNERS?

Ralph Mason
Former Middle School Teacher
Professor of Mathematics Education, University of Manitoba

[Editor's Note: This submission was not available at the time of publishing.]

WHAT IS FUNDAMENTAL MATHEMATICS FOR PRIMARY/JUNIOR LEARNERS?

Ruth Beatty
Former Special Education Teacher
Assistant Professor of Mathematics Education, Lakehead University

This question is part of a larger conversation that gained a great deal of media attention across the country in 2011. A number of popular journalists began, once again, to muse about the fact that young students seem unable to carry out simple arithmetic operations.

> *Parents across Canada might be surprised to learn that the times tables are out. So are adding, subtracting, and dividing. Remember when you learned to add a column of numbers by carrying a number over to the next column, or learned to subtract by borrowing then practiced your skills until you could add and subtract automatically? Forget it.* (Wente, 2011)

Wente and other columnists conflate arithmetic and mathematics, and equate the quick recall of number facts with mathematical thinking. As mathematics educators, we have a different view of mathematics. We understand it as a human activity, a social phenomenon, part of human culture and historically evolved. We know that there is more to mathematical thinking than learning the "times tables".

What is the outcome for students who are primarily taught math using the traditional approach advocated by Wente, that is, through the memorization of standard algorithms? As an example, consider the 300 or so Bachelor of Education students who come to Lakehead University in Orillia each year. Most of these students primarily learned math via an instructional approach that emphasized rote memorization of algorithms. The B.Ed. students are required to write a math content exam that comprises 15 problem-solving questions at a grade 6-7 level. A pass is 75%, and students who do not pass are ineligible to graduate, and are not recommended to the Ontario College of Teachers until they do pass the exam. Every year 25% of students fail the first attempt. That translates to about 80 bright university graduates who cannot "do math" at an elementary level. And because they do not understand math, they actively dislike math, which they perceive as a boring, isolating activity that is based on sequences of disconnected arbitrary rules.

The experience of these preservice students is one small data point in a vast body of literature that questions the efficacy of traditional instructional methods. This approach may allow students to quickly carry out calculations and find the correct answer. However,

> *To give correct answers to questions within the range of the multiplication table is no doubt a useful accomplishment, but it is, in itself, no demonstration of mathematical knowledge. Mathematical knowledge cannot be reduced to a stock of retrievable "facts" but concerns the ability to compute new results. It is knowledge*

> *of what to do in order to produce an answer. It is constructive and, consequently, is best demonstrated in situations where something new is generated, something that was not already available to the operator.* (Von Glasersfeld, 1983, p. 51)

Rote learning does not support the development of mathematical thinking, in part, because it takes away the opportunity for students to do math. Although automaticity leads to an ability to perform quick calculations, the main goal of mathematics teaching should be the students' understanding of what they are doing, and why it is being done. A student may memorize the "times tables", and may perform quickly and accurately during Mad Minute Math but may still be unable to solve a problem that requires multiplicative thinking. In addition, traditional teaching places the ownership of the math with the teacher and the textbook and students are told whether they are correct or incorrect. But if the teacher assesses the correctness of a student's solution, it takes away the opportunity for a student to recognize and appreciate the logic of his or her own thinking – the wonderful "ah-ha!"

Current primary/junior math instruction, *teaching for understanding*, allows students to construct an understanding of mathematics that makes sense to them. As educators, we need to revisit our assumptions about what students are capable of, while at the same time re-visiting our assumptions about what they seem to understand. A case in point is Wente's example of "carrying" and "borrowing" in multi-digit addition and subtraction. Most students fail to understand that when adding or subtracting multi-digit numbers they are composing and decomposing groups of 10, 100, etc. and how this process is reflected in the traditional algorithm. In fact we know that many students have a tenuous grasp of place value, the foundation of our base-10 number system.

There are certainly important arithmetic concepts that are fundamental for primary and junior students because they are foundational for later mathematical thinking. These include number sense and the relationships among numbers (whole numbers extending to rational numbers), and the ability to determine the reasonableness of an answer. Also important are opportunities for students to make connections among the four operations through the use of students' invented methods, alternative algorithms and, eventually, the introduction of traditional algorithms. Emphasis is placed on teaching through problem solving, giving quantities and their operations meaning and context. In the past, numeric and symbolic representations were prioritized, but today we understand the importance of visual representations that allow young students to see and manipulate the mathematics in which they are engaged through exploration with manipulatives, visual models and computer-based interactive applications.

We also know that even very young children have the capacity to think about mathematics other than arithmetic. The development of proportional reasoning is critical. Many students who struggle in mathematics, particularly in the intermediate grades and beyond, lack true proportional reasoning. Indeed, some researchers contend that more than half the adult population cannot reason proportionally across situations (Lamon, 1999). Young students can also consider algebraic concepts at a very early age – for example, noticing what varies and what stays constant in a pattern in order to explore the notion of generalization in pattern rules.

Teaching mathematics is more than content instruction. Ball, Thames, and Phelps (2008) note that the goal of teaching mathematics is to help students become active participants in mathematics as a system of human thought. In today's math classrooms there is an emphasis on communication and building a math talk community of learners so that students can analyze, refine or discard mathematical ideas.

Mathematics should not become a gatekeeper of life opportunities for students, but a solid foundation of understanding on which they can continue to confidently and competently build. The goal of today's primary/junior math classes are to provide opportunities for students to explore, offer conjectures and hypotheses, test their conjectures, prove or disprove their solutions, refine their thinking, practice skills, expand computational fluency and develop mathematical communication. Students who are given an opportunity to build both conceptual knowledge and meaningful procedural knowledge view mathematics as dynamic, engaging, social and fun. As a result, students feel smart, not because the teacher tells them they are, but because they can reflect on their own mathematical thinking.

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WHAT IT IS IMPORTANT FOR THE STUDENTS TO LEARN—AT THE POST-SECONDARY LEVEL (AND ARGUABLY AT ANY LEVEL)

Peter Taylor
Professor of Mathematics, Queen's University

Queen's University has recently written a new Academic Plan. I was deeply involved in the process, being chair of the planning task force, and conducting interviews and town-hall meetings of many faculty, students, and staff.

The plan is focused on 4 pillars:

Pillar I. The Student Learning Experience
Pillar II. Disciplinarity and Interdisciplinarity
Pillar III. Reaching Beyond: Globalism, Diversity, and Inclusion at Queen's
Pillar IV. Health, Wellness, and Community

My remarks are focused on the Student Learning Experience. At the heart of this lie a number of what we call *Fundamental Learning Skills*:

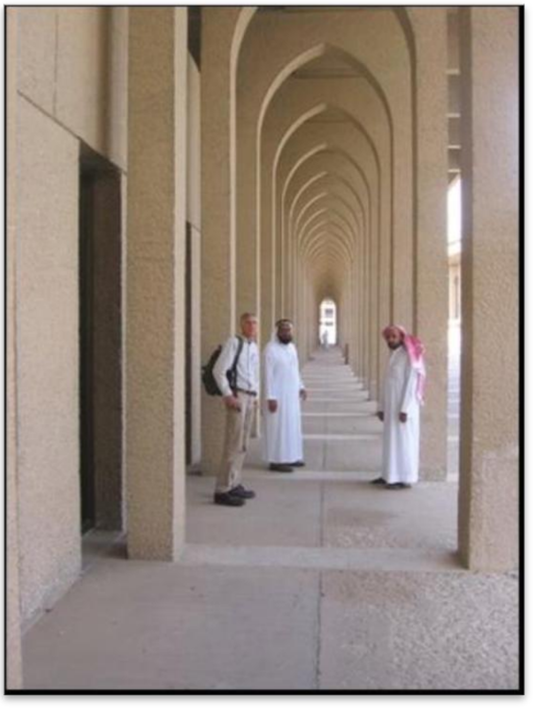
- critical reading
- effective writing and communication
- numeracy
- inquiry
- critical thinking
- problem solving
- information literacy
- academic integrity
- effective collaboration
- intercultural literacy

These are capacities or skills that are essential for the learning process in the sense that students who lack them will be ineffective learners. Two of these sit at the centre of the entire list, and these are:

- Simple clear powerful thinking
- Simple clear powerful writing

Students have always found these difficult, but I suggest that there are reasons to believe that many of today's students have a life style that has compounded these difficulties. For example, their dedication to the wide fast-moving world of online technology has given them a broad sophisticated view and proficiency at rapid multitasking, but has omitted to develop their slow, careful, critical and analytical skills. To remedy this, I will have to spend the three hours per week I have with the students of each of my courses in a different way. I will need to incorporate much more problem solving, and spend time with them individually or in small groups, at the moment they are struggling with the problem or attempting to write their solution down. Where will I find this extra time? By doing less—by not doing some of the things I used to do, and by doing others in another way. For example, I will teach much less specialized material (in favour of more elementary problem solving), and I will make much more use of blended learning, using technology to deliver many of the examples I used to solve in my lectures.

I end with an example of a problem that requires critical thinking. In the picture below, how far is the camera from the front of the first pillar, given that arches are spaced at intervals of 2.5 m going back? Students always find this difficult. The problem is that they throw everything they know about geometry at the problem without asking how a camera works.



QUELLES SONT LES MATHÉMATIQUES JUGÉES FONDAMENTALES POUR L'ÉLÈVE TOUT AU LONG DE SON PROCESSUS D'APPRENTISSAGE?

Hélène Paradis
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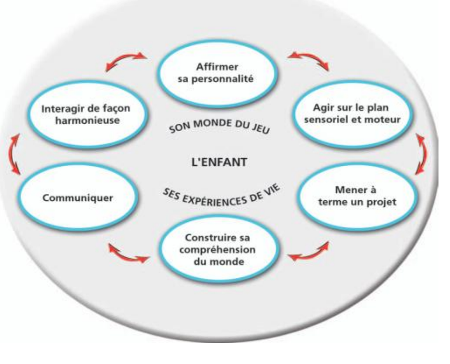
Au Québec, la réforme du curriculum scolaire a amené son lot de questions, notamment la discussion, en mathématique, entourant l'apprentissage des tables de multiplication. Ainsi, on se demande si, dans ces nouveaux programmes, il y a encore de « l'apprentissage par cœur ».

Les nouveaux programmes de mathématique favorisent le développement de compétences. Or, dans le Programme de formation de l'école québécoise, le concept de compétence retenu se définit comme suit : « un savoir-agir fondé sur la mobilisation et l'utilisation efficaces d'un ensemble de ressources » (Ministère de l'Éducation, du Loisir et du Sport, 2007, p. 11). D'emblée, cette définition amène l'enseignant à revoir sa conception de l'apprentissage.

ÉDUCATION PRÉSCOLAIRE

Le programme d'éducation préscolaire a une nature transdisciplinaire et vise le développement, chez l'enfant, de six compétences intimement liées qui s'insèrent dans un processus de développement global.

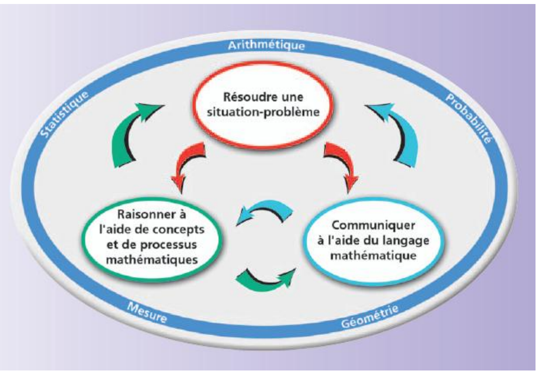


Source : *Programme de formation de l'école québécoise, Éducation préscolaire - Enseignement primaire*, p. 53.

La construction des concepts liés à la mathématique se réalise au quotidien dans les situations réelles et signifiantes de la vie de la classe. *Construire sa compréhension du monde* est la compétence ciblée, notamment lorsque le langage mathématique est sollicité. Cette compétence est étroitement associée au *développement cognitif* de l'enfant. Par ses actions et ses interactions, l'enfant développe des *stratégies* et acquiert des *connaissances* sans avoir une liste explicite de concepts mathématiques à maîtriser. Ces différents concepts sont, de manière implicite, au service du développement de toutes les compétences. Leur contexte de réalisation devient celui du monde de l'enfant, du jeu, de sa réalité et de ses expériences de vie. Ainsi, à la fin de la maternelle, l'enfant aura tout ce qu'il faut dans « son coffre » pour faire des apprentissages plus formels au 1^er^ cycle du primaire : une attitude positive envers les mathématiques, des comportements de « mathématicien », des démarches et des stratégies qui lui permettront de résoudre des situations-problèmes.

ENSEIGNEMENT PRIMAIRE

Un programme identique est offert à tous les élèves du primaire jusqu'à la fin de la 3^e^ secondaire. Le programme de mathématique est alors articulé autour de trois compétences : la première réfère à l'aptitude à résoudre des situations-problèmes; la seconde touche le raisonnement mathématique qui suppose l'appropriation de concepts et de processus propres à la discipline; la troisième est axée sur la communication à l'aide du langage mathématique. Le traitement de situations-problèmes est omniprésent dans les activités mathématiques, en tant que processus et en tant que modalité pédagogique.



Source : *Programme de formation de l'école québécoise, Éducation préscolaire - Enseignement primaire*, p. 125.

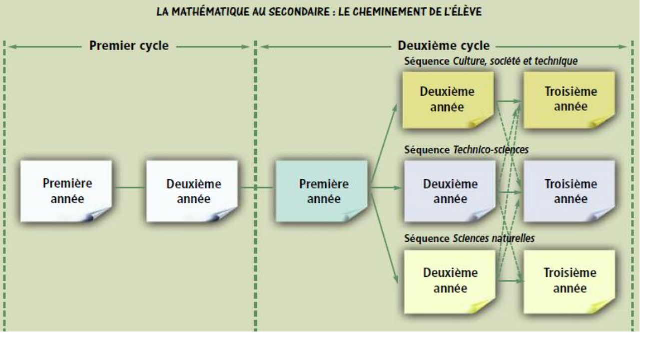
Les trois compétences du programme se développent en étroite relation avec l'acquisition de savoirs relatifs à l'arithmétique, à la géométrie, à la mesure, à la probabilité et à la statistique. Ainsi, connaissances et compétences ne s'opposent pas : elles se complètent. Les connaissances occupent une place aussi grande dans le Programme de formation que dans les anciens programmes, mais en plus de savoir et de savoir comment faire, l'élève doit développer un savoir-agir en contexte : savoir *QUOI faire*, *QUAND le faire* et *POURQUOI le faire*. De plus, les apprentissages en mathématique gagnent à prendre appui sur des situations ou des objets concrets, et à tirer profit du matériel de manipulation.

Les concepts et processus acquis et maîtrisés dans le champ de l'arithmétique constituent les éléments de base en mathématique puisqu'ils sont réinvestis dans tous les autres champs de la discipline. L'apprentissage « par cœur » a subi, en quelque sorte, une adaptation. Par exemple, l'élève doit toujours être capable de compter et de réciter la comptine des nombres naturels. Quant aux faits numériques, l'élève est appelé à développer le répertoire mémorisé, ce qui va au-delà de la seule « *mémorisation des tables* ». Mémoriser s'avère insuffisant. L'élève devra construire les faits numériques et développer *diverses stratégies* pour les maîtriser. En utilisant ces stratégies, il va rapidement réduire la quantité de faits à apprendre.

Les processus de calcul écrit conventionnels sont toujours au programme, mais, avant tout, les opérations arithmétiques peuvent être réalisées suivant des processus personnels plutôt intuitifs et relativement peu structurés.

ENSEIGNEMENT SECONDAIRE

Au cours de la 1re année du 2e cycle du secondaire, l'élève complète sa formation de base et choisit la séquence qu'il entamera l'année suivante. Ce choix doit correspondre le mieux possible à ses aspirations, à ses champs d'intérêt et à ses aptitudes.



Source : *Programme de formation de l'école québécoise, Enseignement secondaire, deuxième cycle, Domaine de la mathématique de la science et de la technologie, Mathématique*, p. 4.

Les mathématiques jugées fondamentales pour l'élève au primaire se poursuivent au secondaire. L'objectif premier du Programme de formation demeure le développement de compétences, et la plupart des concepts et processus doivent être *construits par l'élève* (et non appris par cœur), puis réinvestis dans des contextes diversifiés. Par exemple, l'élève est amené à construire les relations permettant de calculer l'aire de figures planes plutôt que de se contenter de les mémoriser. Au secondaire, les différents concepts et processus sont liés aux champs de l'arithmétique, de l'algèbre, de la probabilité, de la statistique et de la géométrie. Pour optimiser les apprentissages, les trois compétences se développent de façon synergique dans des situations contextualisées, signifiantes et complexes. Les situations d'apprentissage s'articulent autour des préoccupations sous-jacentes à l'activité mathématique : interpréter le réel, généraliser, anticiper et prendre des décisions. En plus du développement de compétences, tant disciplinaires que transversales, le contenu de formation favorise également :

- le développement de la pensée mathématique :
  - passage de la pensée arithmétique à la pensée algébrique,
  - pensée probabiliste et statistique;
- l'approfondissement du sens du nombre, des opérations et de la proportionnalité;
- le développement d'une habileté à modéliser des situations;
- le passage d'un raisonnement subjectif à un raisonnement basé sur différents calculs;
- le passage d'une géométrie intuitive, basée sur l'observation, à une géométrie déductive.

Ainsi, l'article du *Globe and Mail* « Why Alex can't add (or subtract, multiply or divide) », rapporte que les concepteurs des nouveaux programmes de mathématique avaient décidé que l'enseignement des apprentissages « par cœur » n'était pas une bonne idée. Comme les nouveaux programmes favorisent le développement de compétences, ce qui est fondamental pour l'élève qui apprend est de *SAVOIR QUOI faire*, *QUAND le faire* et *POURQUOI le faire*, tout en utilisant les stratégies pour y arriver.

♦♦♦♦♦♦♦♦♦

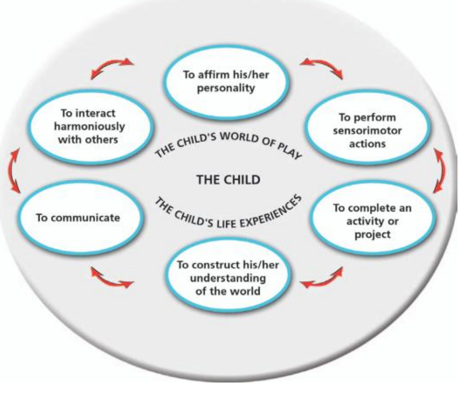
WHAT MATHEMATICS ARE CONSIDERED TO BE ESSENTIAL TO STUDENTS THROUGHOUT THEIR LEARNING PROCESS?

The reform of Québec's curriculum generated its share of questions that led, for instance, to discussions regarding the multiplication tables. People asked whether rote learning had any part in the new programs.

The new mathematics programs foster competency development. The concept of competency employed in the Québec Education Program is defined as involving "the ability to act based on the effective use and mobilization of a range of resources" (Ministère de l'Éducation, du Loisir et du Sport, 2007, p. 11). From the very outset, this definition has led teachers to review their concept of learning.

PRESCHOOL EDUCATION

The preschool education program is cross-curricular in nature. It is intended to help children develop six closely related competencies that are all part of a process of overall development.

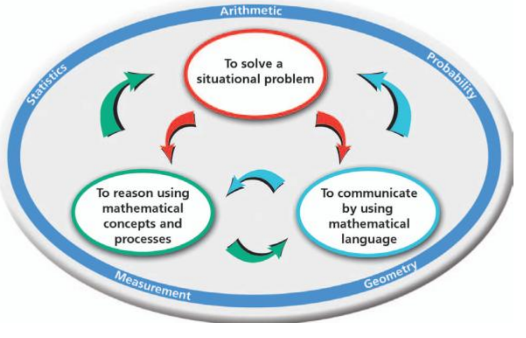


Source: *Québec Education Program, Preschool Education - Elementary Education*, p. 53.

The construction of mathematical concepts takes place daily in real and meaningful situations that are part of the students' classroom experience. Where mathematical language is concerned, the competency targeted is: *to construct his/her understanding of the world*. This competency is closely related to children's *cognitive development*. Through their actions and their interactions with others, children develop *strategies* and acquire *knowledge* without working from an explicit list of mathematical concepts to be mastered. These various concepts foster, in an implicit manner, the development of all the competencies in question. The context in which this achievement takes place is the world of the child, with its games, specific realities and life experiences. Thus, by the end of preschool, children should have all the tools they need to move on to more formal learning in Elementary Cycle One: a positive attitude toward mathematics, a 'mathematician's' behaviour, and approaches and strategies that will enable them to solve situational problems.

ELEMENTARY EDUCATION

All students take the same program from elementary school through to the end of Secondary III. The Mathematics program is organized around three competencies: the first refers to the ability to solve situational problems; the second pertains to mathematical reasoning, which implies familiarity with concepts and processes specific to mathematics; and the third focuses on communication using mathematical language. Mathematical activities always involve the examination of situational problems, both as a process in itself and as an instructional tool.



Source: *Québec Education Program, Preschool Education - Elementary Education*, p. 141.

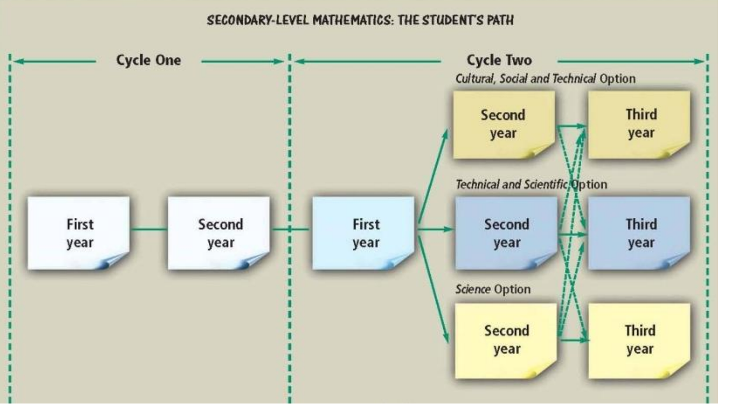
The program's three competencies develop in tandem with the acquisition of knowledge pertaining to arithmetic, geometry, measurement, probability and statistics. In other words, the knowledge and competencies involved are not contradictory. Knowledge plays as great a role in the new Québec Education Program as it did in the older programs but, in addition to acquiring knowledge and skills, students are now called upon to develop the ability to act effectively in particular contexts. This means knowing *WHAT to do*, *WHEN to do it*, and *WHY it should be done*. In addition, learning in mathematics benefits from the use of concrete situations or objects, and draws upon hands-on materials.

The concepts and processes acquired and mastered in the field of arithmetic constitute the basic elements of mathematics since they are applied in all other fields of the discipline. Rote learning has undergone an adaptation, so to speak. For example, students must always be able to count and recite the sequence of natural numbers. With regard to numbers, students are called upon to memorize in ways that go beyond '*memorizing the number tables*'. Memorization alone proves to be insufficient. Students are asked to construct numerical facts and to develop *various strategies* to master them. By using these strategies, they quickly reduce the number of facts they are required to remember.

The processes of conventional written calculation are still in the program but, first and foremost, arithmetical operations are carried out following mainly intuitive and relatively unstructured personal processes.

SECONDARY EDUCATION

During the first year of Secondary Cycle Two, students complete their basic mathematical training and select the sequence they will embark on in the following year. This choice must correspond as closely as possible to their aspirations, areas of interest and abilities.



Source: *Québec Education Program, Secondary School Education, Cycle Two, Mathematics, Science and Technology, Mathematics*, p. 4.

The mathematics considered essential at the elementary school level continues at the secondary level. Competency development is still the main objective of the Québec Education Program, and most concepts and processes must be *constructed by the student* (not learned by heart) and applied in a variety of contexts. For example, students are asked to *construct relations that can be used to calculate the area of plane figures*, rather than being content simply to memorize them. At the secondary level, the various concepts and processes are linked with the fields of arithmetic, algebra, probability, statistics and geometry. To optimize learning, the three competencies are developed in a synergistic manner in specific meaningful and complex contexts. The learning situations are structured around concerns that underpin mathematical activity: to interpret reality, to generalize, to anticipate, and to make decisions. In addition to the development of subject-specific and cross-curricular competencies, the program content fosters:

- the development of mathematical thinking:
  - transitioning from arithmetical thinking to algebraic thinking,
  - thinking associated with probability and statistical operations;
- a deeper understanding of the meaning of numbers, operations and proportionality;
- the development of the ability to model situations;
- the transition from subjective reasoning to reasoning based on various calculations;
- the transition from intuitive geometry based on observation to deductive geometry.

In “Why Alex can’t add (or subtract, multiply or divide)”, a *Globe and Mail* article recounted that the developers of the new mathematics programs decided that ‘*rote learning*’ was not a good idea. Since these new programs foster competency development, it is essential that students learn *WHAT to do*, *WHEN to do it*, and *WHY it should be done*, using the requisite strategies to do so.

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Working Groups

Groupes de travail

NUMERACY: GOALS, AFFORDANCES, AND CHALLENGES

France Caron, *Université de Montréal*
Peter Liljedahl, *Simon Fraser University*

PARTICIPANTS

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Stewart Craven	Geneviève Lessard	Susan Oesterle
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Taras Gula	Ami Mamolo	Sophie René de Cotret
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INTRODUCTION

In the last 10 years there has been an increasing awareness of and attention paid to the notion of numeracy in the K-16 system. But what is this thing we call numeracy and what is motivating the world-wide call for students to be more numerate? Given that mathematics has, for a long time, been part of the core compulsory curriculum, the rise of numeracy must somehow be different from, yet related to, mathematics. The purpose of this working group was to explore this tension between numeracy and mathematics and to come to an understanding of what numeracy is, what its role in society and in school is, and how to integrate it into our K-12 curriculum.

DAY 1

The first day of the working group was dedicated to the definition of numeracy. The session began with the participants engaging in two tasks – *Race Around the World* (see Figure 1) and *The Ski Trip Fundraiser* (see Figure 2).

Both of these tasks were designed by Peter Liljedahl, in conjunction with groups of teachers, in order to meet some of the numeracy goals set by schools and districts in British Columbia. They, along with other such tasks, can be found at
http://www.peterliljedahl.com/teachers/numeracy-tasks.

RACE AROUND THE WORLD

You have just entered a race around the world. The rules of the race are very simple:

- You must start and finish in Vancouver.
- You must visit one major city (marked) on each continent except Antarctica.
- Vancouver does not count as your North American city.
- Your airline ticket only allows you to travel east.

Your goal is to get back to Vancouver in the shortest amount of time.

To help you calculate your time please keep these simple rules in mind:

- Flight paths can be seen as straight lines between cities.
- 1 cm of travel on the map takes an airplane 2 hours to fly.
- Airplanes depart each city on every even hour local time. That is, they leave at 2:00, 4:00, 6:00, ...
- The dotted vertical lines on the map are time zones. Every time you cross one of these lines while travelling east you should advance your clock by one hour.

Good luck – and may the best team win.

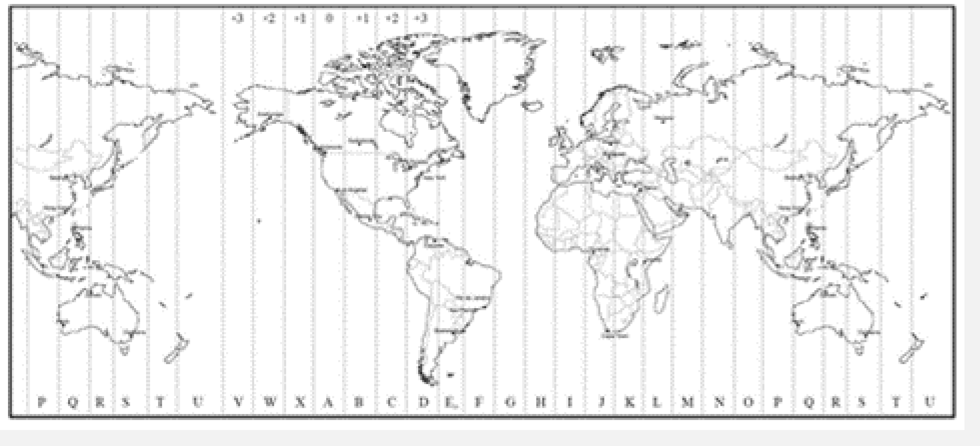

Figure 1. *Race Around the World*

SKI TRIP FUNDRAISER

The grade eight ski club is going to Grouse Mountain. Each person tried their best to raise money for their trip. Below is a chart that shows how much money each person raised, and their individual cost, depending on whether they need rentals or lessons.

Determine whether they have raised enough money for their trip. What would be a fair way to share the money that was fundraised among the people listed below? All of the money raised must be applied to the cost of the trip, and every person must go on the trip, even if it means that they may have to put in their own money to do it.

Figure 2. *The Ski Trip Fundraiser*

Name	Amount Raised	Rental Cost	Lift Ticket	Lesson Cost
Alex	75	20	40	40
Hilary	125	10	40	40
Danica	50	30	40	0
Kevin	10	40	40	40
Jane	25	0	40	0
Ramona	10	0	40	40
Terry	38	30	40	0
Steve	22	40	40	40
Sonia	200	20	40	0
Kate	60	25	40	0

Figure 2, continued. *The Ski Trip Fundraiser*

There was a fair bit of frustration among the different groups as they worked their way through the activities. This frustration, as revealed in the subsequent discussions, centred on the inherent ambiguity of the tasks. In particular, it was felt that the issue of *fairness* explicit in the *Ski Trip Fundraiser* task turned this problem into a moral or ethical activity more so than a mathematical one. The lack of information about the socio-economic status of the different children involved, the nature of the fundraising activities, and the school culture in which the tasks was set made it difficult for groups to begin. Once they did, however, a variety of interesting solutions emerged that took into consideration the inequities inherent in the data presented in the task.

The *Race Around the World* task, on the other hand, was initially seen as more mathematical in that the *rules* of the activity were well-defined in comparison to the *Ski Trip Fundraiser* task. That is, until it was pointed out by different members of the working group that there is a lot of ambiguity implicit in the task. When further discussed, it became apparent that there was no consensus around how long a competitor was required to stay on the ground between flights, what continent some cities were in, if it was permitted to visit cities not identified on the map, or even how many continents there was. This, it turned out, made the comparison of answers/results difficult.

Rather than work to resolve the issues raised, the participants were asked to accept, at least temporarily, that the tasks were, in fact, Numeracy Tasks. Based on this assumption they were then asked to construct a definition of Numeracy. The activity first led to the following definitions or characteristics: math ideas outside their symbolic representations, use of math in life to make decisions, analysis of the context, problem-solving strategies. The participants moved on to consider what skills were necessary in order to be successful in completing the aforementioned tasks. One group chose to consider, instead, what skills would need to be absent in order for a student not to be successful at these types of tasks. In essence, it was felt to be easier to examine, instead of what numeracy is, what it means for someone to be numerate (or innumerate as was the case for one group). Some of the definitions that emerged from these discussions were:

- A numerate individual is one who has an awareness, respect, appreciation, and understanding that mathematics is relevant and important, and has the ability to learn and do mathematics when needed within norms/standards.
- A numerate person is someone who can identify and understand mathematics in the world and be competent to use mathematics as needed.

- Numeracy should not be defined by a set of mathematical (quantitative) skills one has but by their sense of efficacy in significant life contexts with mathematical aspects.
- Numeracy is a constantly evolving blend of dispositions and skills that work in a situation of positive feedback loop and build a person's ability to comprehend numbers and their uses.

After presenting these definitions each group selected, at random, six definitions of numeracy or mathematical literacy from a collection of over 70 definitions culled from the literature (http://www.peterliljedahl.com/wp-content/uploads/Numeracy-Definitions.docx). The activity was to compare and contrast these six randomly selected definitions with the definitions that they had constructed. This was followed by a whole group discussion.

The resulting conclusion from these activities was that, although numeracy is difficult to define, it was uniformly accepted that numeracy is related to, yet distinct from, mathematics. As it appears to be linked to citizenship, numeracy conveys a political connotation. Its definition thus may very well depend on who the stakeholders are: governments, corporations, mathematics educators. While one participant positioned numeracy as an antidote to oppression, another raised the idea that mathematics can also be associated with some forms of oppression. The discussion paved the way for Day 2 of the working group, where we re-examined numeracy from a social perspective.

JOUR 2 – UNE PERSPECTIVE SOCIALE

Le deuxième jour, nous avons démarré la séance de travail avec un problème de Fermi :

« Combien y a-t-il d'hygiénistes dentaires au Canada » ?

Résoudre ce problème implique non seulement d'estimer notamment le nombre de personnes au Canada ayant recours à des soins dentaires, le nombre moyen de visites par année chez le dentiste pour ces personnes, le temps moyen passé avec l'hygiéniste par visite, mais aussi de faire l'hypothèse, plutôt forte, que le nombre d'hygiénistes au Canada permet de répondre adéquatement à la demande ainsi calculée, presque sans pénurie ou surplus, en comptant une année de travail régulière. De façon intéressante, l'acceptation de cette hypothèse n'a pas semblé poser problème à la plupart des équipes. Mais pour une participante, une telle hypothèse n'allait pas de soi; perplexe quant à l'enjeu de l'activité, elle s'est rabattue sur une recherche sur internet auprès des associations professionnelles. De telles observations renvoient au contrat implicite qui s'établit entre l'élève et l'enseignant, aux buts qu'on attribue à de telles situations d'apprentissage et aux normes de validation acceptées en classe pour les tâches construites sur des situations réelles.

Pour sortir momentanément de la classe et envisager la numératie dans une perspective sociale qui valorise le jugement critique et la prise de décision informée, les participants ont été conviés à faire l'analyse de nouvelles prises sur le web, portant sur différentes problématiques citoyennes.

- Un autre vieux satellite pourrait s'écraser sur Terre
  http://www.radio-canada.ca/nouvelles/science/2011/10/08/001-nasa-satellite-canada.shtml
- Another falling satellite may be heading to Canada
  http://www.cbc.ca/news/technology/story/2011/10/08/rosat-satellite.html
- Série l'argent: les analphabètes de la finance

http://www.lapresse.ca/actualites/quebec-canada/national/201202/18/01-4497491-serie-largent-les-analphabetes-de-la-finance.php

- Student Loan Debt: Wall Street's Next Bubble?
  http://occupiedchicagotribune.org/2012/04/student-loan-debt-wall-streets-next-bubble/
- Welcome to 2012: All debt, all the time
  http://www.theglobeandmail.com/commentary/welcome-to-2012-all-debt-all-the-time/article4181913/
- WTF: The federal budget and 50 years of Canadian debt
  http://news.nationalpost.com/2011/03/21/graphic-50-years-of-canadian-debt/
- Québec : brouillard statistique
  http://blogues.radio-canada.ca/geraldfillion/2012/02/17/quebec-brouillard-statistique/
- Un autre bilan routier encourageant pour le Québec
  http://www.radio-canada.ca/nouvelles/societe/2012/04/23/002-quebec-bilanroutier-2011.shtml
- Manif du 22 mars: combien étaient-ils?
  http://www.lapresse.ca/actualites/dossiers/conflit-etudiant/201204/21/01-4517612-manif-du-22-mars-combien-etaient-ils.php
- Plus fréquente en région qu'en ville
  http://www.radio-canada.ca/nouvelles/Science-Sante/2006/08/22/002-obesite-regions.shtml

Nous avions ainsi des articles témoignant du phénomène de l'endettement (des pays ou des individus), de l'appréciation de la taille d'une foule lors de manifestations, du risque de l'écrasement d'un satellite sur terre, et du lien possible entre l'obésité et le lieu de résidence (urbain ou rural). Comme plusieurs de ces articles étaient suivis de commentaires rédigés par des lecteurs, il devenait tout aussi intéressant d'analyser l'interprétation de l'article et le jugement de la situation chez ces lecteurs que le traitement proposé par le journaliste.

À travers cette analyse, nous cherchions à identifier des enjeux associés à la numératie qu'on gagnerait à considérer dans la définition du curriculum mathématique actuel. Les participants qui avaient lu un ou deux articles la veille étaient invités à en discuter en se joignant à différentes « conversations » traitant respectivement des idées suivantes :

- taille et ordre de grandeur ;
- espace ;
- relations, relativité et variabilité ;
- changement et évolution ;
- hasard, chance et risque.

L'objectif de ces discussions était de partir des conceptions et biais associés à ces idées et mis en évidence par les articles et les commentaires affichés, pour tenter de décliner le vaste enjeu de la numératie en termes de buts à viser dans le curriculum. Parmi les buts ainsi construits, mentionnons :

- développer une compréhension des grands nombres et des différentes façons de les exprimer ;
- apprendre à comparer, en utilisant le raisonnement proportionnel notamment ;
- interpréter les graphiques et leurs échelles ;
- développer le sens de très petites probabilités ;
- pouvoir déterminer quelles variables sont pertinentes ;
- s'interroger sur les hypothèses sous-jacentes et savoir les remettre en question;

- développer un sens de la complexité et de la variabilité ;
- reconnaître que les pourcentages sont relatifs ;
- comprendre les pourcentages dans un contexte d'augmentation ou de réduction ;
- savoir interpréter le vocabulaire utilisé pour décrire les évolutions en termes de croissance et de taux de variation ;
- ne pas confondre la corrélation avec la causalité;
- apprendre à distinguer la probabilité qui s'applique à un individu de celle qui a été construite à partir d'un échantillon ;
- ne pas généraliser sa propre expérience à toute une population ;
- apprendre à lier la probabilité à l'espace sur lequel elle s'applique ;
- être attentif aux biais méthodologiques.

Un participant a résumé de façon élégante (et quelque peu provocatrice…) sa vision de la numératie au terme de l'activité : "being numerate may be defined as having an understanding that numbers do not always give us the answer."

Comme en témoigne ce qui précède, les buts qui sont ressortis de l'activité ne renvoient pas qu'à des connaissances, mais aussi à des attitudes et à une capacité à mobiliser des connaissances qui peuvent se révéler utiles à l'analyse d'une situation donnée ; cela tendrait à rapprocher la numératie d'une forme de compétence au sens où l'entendent notamment Perrenoud (1997) et l'OCDE (2005). Prendre cette voie conduirait à examiner aussi l'ensemble des ressources que peut convoquer un individu pour répondre aux exigences d'une situation.

Sur ce plan, il convient de souligner que malgré quelques évocations ponctuelles, le rôle des technologies a été assez peu présent dans nos discussions autour de la numératie, autrement que pour mentionner l'accès élargi à l'information et à des boîtes noires de plus en plus grosses. Ces dernières semblent permettre à un individu « qui ne sait pas compter » de fonctionner dans une société de plus en plus numérique, qui n'en finit plus de compter pour lui… et peut-être même contre lui dans certains cas. Sans doute conviendrait-il d'enrichir ou de recadrer la vision de la numératie dans un tel contexte, en examinant de plus près les effets de ce recours croissant aux boîtes noires et la pertinence ou la nécessité d'apprendre à ouvrir certaines d'entre elles.

DAY 3

Having looked at the goals, stakes and social context of numeracy on the first two days, on Day 3 we tried to conceptualize how to embed numeracy in the K-16 curriculum.

The day began with another numeracy task (see Figure 3).

The WG participants were asked to work in groups and to engage in the task as though it was a school task. In debriefing the task, what emerged most prominently was that the social context of *crabbing* did not fit neatly into the context of school. That is, there was a general feeling that the openness and ambiguity of the task made it difficult to ensure that the goals of the task would be realized within a classroom setting. Some participants felt that it would never arise as a word problem, and that it becomes more of a numeracy task when a graph of the tides is provided.

This, of course, raised questions as to what the goals for such a task are. Clearly, the goal is not to train crab catchers. Rather, we want to use these types of tasks to operationalise numeracy. But, paradoxically, there may be a tension in using real-life situations when trying to develop numeracy in mathematics classes: in real life, we tend to use the least mathematics that we have to, and validation modes may be completely different than what would be valued from a mathematical perspective.

The Lost Crab Trap

Last week I went out crabbing with a friend. We took my canoe and paddled out to a point just off Belcarra Park and threw in our trap. It was 5:30 in the afternoon. As the trap went down ... down ... down, the rope fed itself out of the canoe until it got to the buoy tied to the end. When it went over the edge it too went down. The rope we had used was too short. We could see the buoy floating about 1metre below the surface of the water.

I noticed that the tide was pretty high, so I figured we just had to come back when the tide was lower and I'd be able to retrieve it. So, I went home and checked the tide charts. From this I learned that a high tide of 4.8 metres would occur at 22:00 that evening and a low tide of 1.2 metres would occur at 10:30 the next morning.

When should I have gone back to retrieve my trap?

Figure 3. *The Lost Crab Trap*

Having said this, the group felt strongly that unlike many tasks in the mathematics curriculum, the context of numeracy tasks is very important. These are not superfluous stories to be quickly discarded once the equation to be solved has been extracted. In these numeracy tasks, the context is important not only for posing the problem but also for thinking about the problem. To the issue raised by some participants that the numeracy status of a given task may depend on where you're from (e.g. Nanaimo vs. Winnipeg for the tide problem), other participants replied that the ability to question the context should not depend on where you're from, but that the answers may. Yet, this ability to question both the context and the adequacy of a model to capture it adequately may also be shaped by the curriculum. Tides are a good illustration of that, as they seem to have become a prescribed object of content in some of our math curricula in recent years, where they are wrongly assumed and taught to be governed by a single and pure sinusoidal function.

From this we moved to a thought experiment.

- Let N be our universally agreed upon definition of numeracy.
- Let L be the numeracy skills associated with N that we want each of our students to develop in the course of their K-16 experience.

The question then is, how do we develop L in the context of the current K-16 system which already has a well-developed mathematics curriculum (M) within it.

This led us back to our discussions from Day 1 regarding the relationship between M and N. From this, three possible models emerged (see Figure 4):

1. The first of these ($N \subset M$) is built on the assumption that the mathematical skills associated with most understandings of numeracy are a relatively small subset of what constitutes mathematics.
2. Contrasting this, the second model ($N \supset M$) is constructed from the perspective of contexts. Numeracy is concerned with a wide variety of social contexts, only a small subset of which, are normally seen in the K-12 curriculum.

3. The final model (N ∩ M) is a mixture of the first two. This model assumes that there is a subset of skills from mathematics that are relevant to numeracy, but that in order to be numerate a person must be able to operate flexibly and comfortably with these skills across a wide variety of contexts that go beyond the sort of experiences that one would normally encounter in K-12 mathematics curriculum. This fits well with one of the definitions encountered on day 1:

> *… efforts to intensify attention to the traditional mathematics curriculum do not necessarily lead to increased competency with quantitative data and numbers. While perhaps surprising to many in the public, this conclusion follows from a simple recognition—that is, unlike mathematics, numeracy does not so much lead upwards in an ascending pursuit of abstraction as it moves outward toward an ever richer engagement with life's diverse contexts and situations.* (Orrill, 2001)

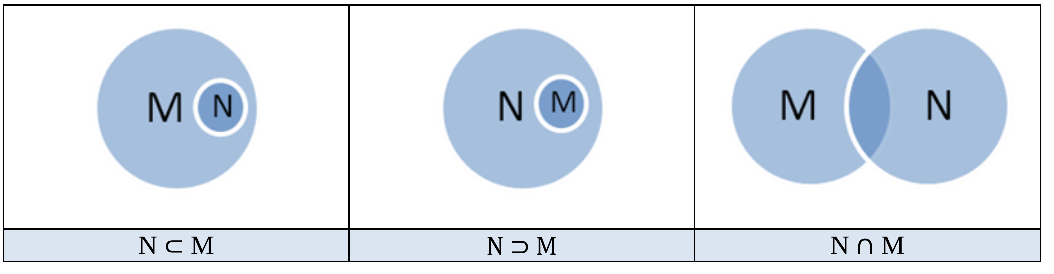

Figure 4

In the context of these discussions two analogies were offered as ways to think about the relationship between mathematics and numeracy within a K-12 or K-16 curriculum:

1. If mathematics is the trunk of a tree, then numeracy is the branches of the tree. Mathematics allows us to move up, while numeracy allows us to branch out.
2. If we think of real-life situations as animals in the wild, the transposition of these into the school mathematics curriculum tends to transform them into inanimate dead and stuffed animals on display—completely predictable and fully constrained. To reduce the gap and still contribute to the learning of mathematics and its application, would a good numeracy task in a math class resemble a zoo animal, still animate and unpredictable, but somewhat constrained?

The first of these analogies is in line with the third model and is very reflective of the Orrill (2001) quote above. The second analogy, on the other hand, is reflective of the second model, above, where the mathematics curriculum is a subset of the more unconstrained and animate numeracy curriculum.

These models and accompanying analogies are important as each has very different implications for how the integration of numeracy into an existing K-12 or K-16 curriculum could be envisioned. This may explain why in recent years, numeracy seems to have gone in two very different directions which either try to address the complexity of our world or promote a "back to basics" approach.

Aiming for a greater consensus on what numeracy truly entails could help us find an appropriate balance between skills and conceptual understanding, as these two aspects are not only mutually dependent, but also necessary to grow as learners and thinkers, as participating and critical citizens.

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DIVERSITIES IN MATHEMATICS AND THEIR RELATION TO EQUITY

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FRENCH ABSTRACT

Toute interaction humaine engage des personnes qui ont différentes histoires. Les interactions mathématiques sont de même nature de sorte que la mathématique, même comme champ d'études, se construit sur diverses pensées. Le discours mathématique possède des structures spéciales qui peuvent éclairer certaines différences et en rendre d'autres obscures. Nous explorons différents types de diversité qui apparaissent lorsque des personnes font des mathématiques ensemble. En reconnaissant la diversité comme ressources, les responsables, nous partageons ce que nous avons observé à propos des types de diversité dans une classe de mathématiques, incluant la diversité culturelle et linguistique. Toutefois, notre but est d'utiliser cette expérience de partage comme point de départ pour étendre la vision sur d'autres diversités et sur les relations entre elles. Par exemple, nous pourrions penser aux diverses formes pour représenter des idées mathématiques en les mettant en relation avec notre culture ou notre langage. Pour cela, nous nous demandons quels types de tâches mathématiques suscitent une diversité féconde.

Notre exploration de la diversité dans le discours mathématique est centrée sur la problématique d'équité et sur la façon de l'aborder dans une classe de mathématiques. Notre conception de l'équité inclut de multiples aspects qui se trouvent, entre autres, dans les travaux de Gutiérrez (2012). Ce dernier met l'accent sur la justice et les axes critiques du pouvoir et de l'identité en portant une attention spéciale aux élèves, surtout marginalisés, et à leur contribution potentielle au développement des mathématiques.

Nous posons des questions comme : Que nous apprend la diversité dans les rapports de pouvoir que nous observons dans les interactions mathématiques à l'école et ailleurs? Quel est l'apport d'une reconnaissance de la diversité comme ressource? Que peut-elle faire pour nous lorsque nous menons ou participons aux discours mathématiques? Comment cela est-il lié à l'équité? Dans un design d'environnement d'apprentissage, quelles activités mathématiques et quelles interactions mettent à profit des connaissances culturelles et linguistiques tout en développant les mathématiques? Enfin, que disent les pratiques actuelles en enseignement des mathématiques et les activités de développement professionnel par rapport à ces questions?

CONTEXT OF OUR WORKING GROUP DISCUSSIONS

This working group explored the various kinds of diversities that operate when people do mathematics together. We were interested in the way these diversities impact the mathematics that is done, and we looked at implications for equity both locally (in the mathematics classroom) and relatively globally (in society). For this exploration we did some mathematics together, watched others on video do and talk about their mathematics, and discussed our observations and reflections on our experiences.

With our perspective that diversity brings richness to any group, we recognize the importance of the unique context in which each member of a group is situated. As with any group, our working group was strongly influenced by the particularities of the people who came together and their stories, especially because our group was open to difference – open to hear, see, and otherwise experience the different stories and perspectives brought to the group from each individual. Thus it is important to us to cite the wonderful group that gathered to discuss diversities in mathematics and their relation to equity:

> Chris Brew, Déborah Nadeau, Jean-François Maheux, Jean-Phillipe Bélanger, Josianne Trudel, Khôi Mai Huy, Lisa Lunney Borden, Mary Stordy, Oana Radu, Paulus Gerdes, Ruth Beatty, Stephanie Rheaume, Tara Flynn, Tod Shockey, Veda Roodal Persad, Vincent Martin, and ourselves.

Our reflections, which follow, are built upon the insights and discussion from the group, but we know that the reflections do not do justice to the rich interactions of the group.

We opened our session by asking each group member to introduce her/himself by giving the name of her/his maternal grandmother and where she was from. This initial community building activity (which comes from the work of Lisa Delpit) helped to place our discussions in a context that included the historical and cultural background of our group members. In this way, we began to see diversities that would not otherwise be evident to us. We also learned over the course of the conference that one of our group members was a new parent and two would soon be parents. The student protest movement in the province was also part of our group's context. A number of our group members wore red squares, which were a symbol of this movement. All of these particularities of our context flavoured our experience together. It provided a foundation upon which to invite diverse ideas and contributions to equity in our mathematics classrooms. Our emphasis was on doing activities together and listening to each other, which meant that very early on our group decided to speak in both French and English (although many of the members were not bilingual).

EQUITY

In our first activity, we used sticky notes to record various completions to "Equity in math is …" or "L'équité en mathématiques est…". Working in table groups, we categorized our responses, and then placed the categories on chart paper on the wall, inviting others to add to them and rearrange initial categorizations over the next three days (see Figure 1). Gutiérrez' (2012) conception of equity became prominent in our discussion. She defined equity in mathematics in terms of access and achievement, which she places on a "dominant axis" (p. 20), and identity and power (see the middle of Figure 2) which form the "critical axis" (p. 20). In this vein, we discussed ways to address issues of power, affirm identity(ies), provide access to high quality mathematics, to respect diverse learners, to attend to difference, and recognize value in a multiplicity of approaches.

Equity in mathematics is …

- a start – un début
- a way to affirm identity
- respect des différents apprenants dans leur façon d'apprendre
- addressing issues of power
- rendre les math accessibles a tous quelles élèves aient l'impression qu'ils paissent s'impliquer.
- attends to differences and recognizes value in a multiplicity of approaches
- permettre, to permit, gives a chance to everyone to do mathematics
- respect
- resources
- supporting curiosity
- honouring cultural and linguistic knowledge that students bring
- intellectual security
- listening
- discuss in both ways: "talk" and "listen"
- la voie vers un plus grande justice sociale
- a way to empower people
- est faire les maths
- sharing one's math knowledge
- fairness
- est relative
- balance

Figure 1. Equity is...

Some group members lamented that typical high school mathematics no longer encourages exploration but rather focuses on the performance of mathematics procedures prescribed by the teacher. Others reported that "if you do not have access you cannot achieve anything" and "real access is achievement." As one group member put it, "Equity is finding ways for it [mathematics] to work for everyone." Throughout our time together, we shifted our thinking from accepting Gutiérrez' axis model to one that was more dynamic and cyclical in nature. We came to think of equity in mathematics as a start, *un début*, a kind of opening to what is possible, something in motion, rather than a fixed state. Figure 2 shows a representation of our ideas by our third day together.

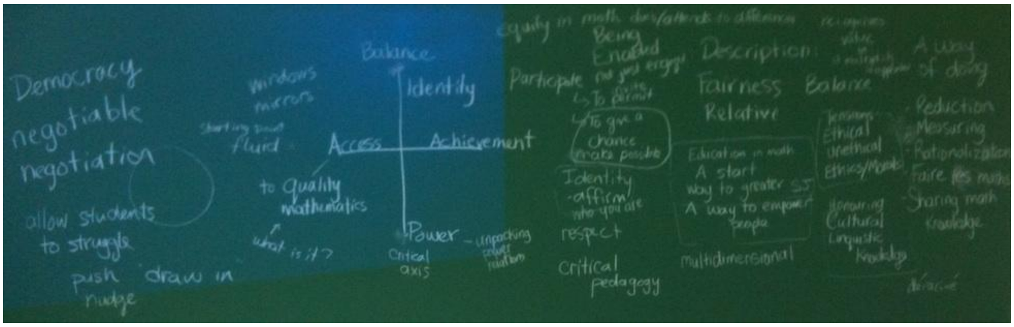

Figure 2. Revisioning equity.

EXPLORING DIVERSITIES

After discussing equity in mathematics, our working group engaged in some mathematics activities and viewed videos in which people were engaged in mathematical thought or activities that promoted diversities (e.g., special needs, linguistic/cultural diversity, different approaches to problems, ways of representing in different modes such as graphs, diagrams, verbal, gesture, positioning, etc.).

FOUR-CUBE CHALLENGE

Our first mathematics problem for attending to diversity had groups working with four interlocking cubes: “How many different arrangements can be made using four cubes?” We purposely offered no rules for the cube-arranging activity in order to see how the task might open up diversities and how people might attend to the diversities that arose as the problem was worked out. We hoped that the challenge might stimulate problem posing. Each group took a different approach to the challenge and reported back on mathematical results, and on the diverse approaches and perspectives they noted in their groups. Because the task did not require complex mathematics, we were able to focus group discussion on questions about diversity, as demonstrated in the groups’ posters in Figures 3a and 3b.

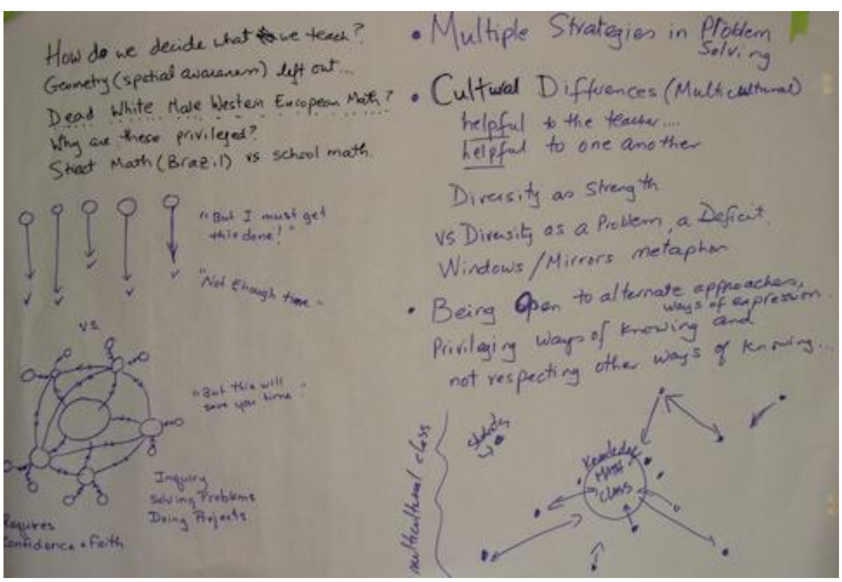

Figure 3a. Reflections on the four-cube challenge.

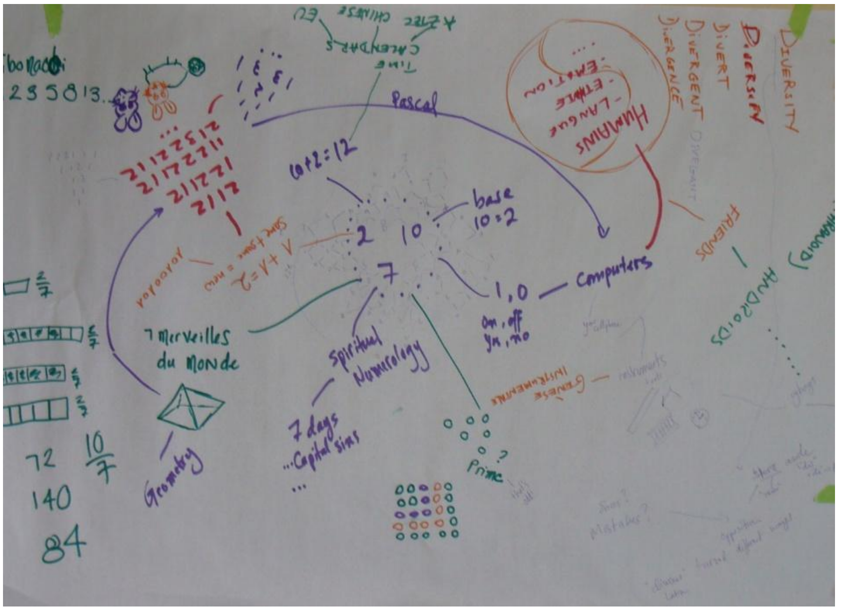

Figure 3b. Further reflections on the four-cube challenge.

One group designed a game in which they rolled the blocks as in a dice game to explore the probability of sitting on a certain face of the block. Another group focused on grid rotations by placing a block structure on a grid and rotating/rolling the block to reach an end point in certain positions or to follow certain restrictions of a path. The third group created a set of eight unique 3D structures using four cubes per structure. We noticed that limiting each group to four cubes pushed them toward relying on visual cues. Our discussion following the activity moved to an appreciation of rules and the idea of negotiable rules in the design of mathematical tasks to draw on diversities as resources. We also noted that curiosity sparks us to try to understand beyond our limits

Rule-making was deemed central to the connection between equity and diversity in mathematical tasks. We first saw rule-breaking as a way of opening up to new ideas that may be otherwise repressed. However, some people are by nature rule-followers; when there is good leadership, rules can set boundaries and directions that invite, or at least make space for, diversity. When diversity is too broad and open, there is a sense of unbearable chaos because it is human to want to control things to some extent. Limits are reassuring. Equity involves listening to the rule-breakers and the rule-followers. It is important that teachers be aware of who is making the rules and for what ends. Our experiences told us that teachers usually make the rules, but we reflected that a productive classroom ethos would have students making rules and setting bounds in large and small group work. Reflection on what bounds make the mathematics rich is perhaps the richest mathematical thinking. This should not be left to the teacher because we would want students doing such rich thinking.

Rules and bounds are similar, we said; rules are more explicit and bounds more implicit. For example, the grid pattern used in a game is not usually part of the explicit rules, but sets the bounds for the activity. This discussion suggested that tasks that have aspects that are intentionally vague invite students to set rules and bounds. When the students set rules and bounds, a teacher can lead discussion on the restrictions students chose and on how these restrictions were productive. From such discussion, students would learn to appreciate such restrictions and also learn that they are negotiable. When students are granted authority to make decisions, their self-confidence can blossom. This relates to Gutiérrez's ideas of equity in terms of the "critical axis" of student identity and issues of power (students having a voice in the mathematics classroom, distribution of authority, etc.), which work to increase student engagement and participation. When this authority distribution is discussed, students can develop understanding of the politics within mathematics.

BLIND MATHEMATIZING

Following our four-cube challenge and discussion, we viewed a video in which an interviewer presents the four-cube challenge to Eric, an adult who is blind. Eric is asked to use sets of four cubes to build as many unique 3D figures as he can. After Eric creates six arrangements (quite quickly) and states that he has completed the task, he seems surprised when the interviewer mentions that there are more arrangements possible. He seems stumped, but then begins to name and organize the arrangements he has completed: "We have Straight, we have T, we have this one that's weird looking, we have a more straight version of that one, we have Square, and we have the L."

The interviewer asks how he is able to hold all of the shapes in memory (naïvely supposing that the visual cues are necessary…). Eric responds, "It's the same thing as you looking at it…I mean I remember them all if I look at them kind of thing…it's exactly the same. You're just not using your eyes, that's all. You just do it all by touch. For me, it's just second nature. You just start manipulating [the cubes] until you have a shape."

He seems to struggle with the idea that there are more arrangements possible and expresses frustration. He then flips one of the figures into an upright position but quickly realizes that although it is flipped, it remains the same arrangement/figure.

He then works with one arrangement, using one cube to position and reposition until he makes the seventh arrangement and then remarks, "Hey, hold on…" and makes the eighth (which is the reflection of that particular figure).

The video disrupted our initial ideas that this was a visual task and showed how a person who is blind 'sees' with his/her hands. Eric even used visual metaphors to describe his sensing – e.g., "this one that's weird looking" quoted above. In addition to new insights into different ways of 'seeing', we noted from Eric's reflections that the naming of the cube arrangements can be a scaffold for working memory. We have seen young children use a similar naming process when building these figures.

This interview with Eric helps us see that drawing on diversities pushes the boundaries of understanding for teachers and students. Blindfolded students could work on mathematical tasks to develop their different senses and to experience different perspectives. Similar restrictions could promote new mathematics and representations – for example, banning speech.

LINGUISTIC VARIATION

Our second video prompt for discussion showed two Grade 1 English language learners who had been involved in the four-cube challenge activity in their classroom (i.e., like Eric, they had been asked to use sets of 4 cubes to build as many unique 3D figures as they could) as part of the Math for Young Children Project (M4YC).[1] The classroom teacher had commented that these particular two students remained quiet during the activity and

[1] The Math for Young Children (M4YC) project, under the direction of Dr. Cathy Bruce and Dr. Joan Moss, involves large-scale professional development and training of 40 public school teachers in lesson-study and inquiry-based teaching across 3 Ontario school boards (2011-2013).

subsequent classroom discussion, and she speculated that the students had not understood the activity. The video showed a researcher in the classroom (Sarah Naqvi) speaking with the two children in Urdu and asking them to show her how to make two 3D cube figures "look and go in the same way."

In the interview, the two girls were immediately able to flip and turn one of the figures to make it the same as the target. Naqvi repeated the process with other figures, and the students continued to demonstrate an understanding of the transformational concepts of 'flip' and 'turn'. When the students were asked to explain their thinking, they remained silent. When asked to explain "What was going on in your brain?" and given the opportunity to answer in Urdu, the students immediately spoke animatedly in Urdu, inserting English words 'flip' and 'turn' to describe their spatial reasoning. The video underscored the relationship between one's home language and engagement, and also between one's language repertoire and one's ability to explain ideas. Clearly, there are implications for equity. However, what those implications are is not so clear; the imperatives for teaching mathematics in multilingual environments remain complex. These complexities are described well by Setati (2012).

In addition to the implications of linguistic variation on mathematics and its teaching, this video prompted us to consider the place of spatial skills in Canadian curricula. Spatial concepts and skills are underrepresented in Canadian mathematics teaching, which favours number and number skills. Where there is attention to shape and space, there is an obsession with static objects. First Nations children in particular view space and shape as process. Many mathematical ideas that are described as nouns in English are described with verbs in Aboriginal languages. This is true in Canada (e.g., Lunney Borden, 2011) and elsewhere (e.g., Barton, 2008). For example, even numbers are verbs in Mi'kmaq, but the implications of viewing dynamic objects and ideas as static entities are likely to be greater for shape and space concepts than for number. All students would benefit both from working with objects that have been fixed in time and space and also from working with them in their dynamic forms (as they are in most life situations). We wondered what the implications are when school mathematics favours one way of seeing (static representations, in this case) and ignores others (dynamic representations).

Another symptom of school mathematics' obsession with static objects is the focus on results and not the processes through which children work to get results. One of the girls in the video said she 'flipped the structure' but she did much more. She visualized, she modeled with gestures, etcetera, and finally enacted the flip. The only process she recognized and the only one she could talk about is one that she probably had discussed in class – 'the flip'. If the girls' teachers had talked more with the students about their other processes, including visualization, modelling and gesturing, then this girl would have been more likely to recognize herself doing these things and also to value these things.

This video of the girls with their four-cube structures also got our group thinking about social interaction and its connection with the development of mathematical thinking. The girls did more than flips. They did more than visualizations, gestures, etcetera. The girls also did socio-relational things: they did what they were asked to do, they stayed in their seats, and they smiled. We wondered how we could invite such children (who aim to please) to act differently, to express their unique insights in the doing of their mathematics. One might ask "What happened?" instead of "What did you do?" We privilege the 'done' rather than the 'doing' when we focus on post hoc linguistic descriptions of what happened rather than engaging in action. Languaging our experiences is often taken for granted. It is very difficult to express certain experiences. We should focus on what children/students do, not only on what they say (about what they did). Furthermore, it is important to avoid assuming we know what happened by observing, because children's languaging can give insight into what happened in their thinking. Thinking, acting and describing are three domains that act together. We often assume that language describes experience and forget that the languaging changes our experience.

DIVERSITY IN THE HISTORY OF MATHEMATICS

Though mathematics is often characterized as a discipline that has one right answer to any question, it has a strong history of opening up new ways of seeing and analyzing the world, replete with examples of people introducing new perspectives that often turn past-knowledge and perspectives on their heads. We aimed to extend our working group's sense of what diversities are at play in mathematics. To get us going on this, we considered imaginary numbers and division by zero as examples of mathematicians extending the space of what conventional wisdom considered correct and possible. Here we consider the case of imaginary numbers.

A square number is the product of a natural number multiplied by itself – by arranging objects in rows and columns to make a square we generate square numbers. Mathematics allows us to extend this concrete idea to allow for squares of negative numbers, rational numbers and even irrational numbers by noting that one can find a product of any of these kinds of numbers multiplied by itself. These extensions can also be represented by a physical square but only with a greater stretch in imagination. Even so, all squares of these numbers are positive. An even greater extension of the idea of perfect squares comes with the decision to consider a space in which squares can be negative. This required the imagination of a new kind of number, which had no physical representation. Yet mathematicians have developed ways to represent such imaginary numbers physically.

As the protagonist said in the novel, *Smilla's Sense of Snow* (Høeg, 1993), mathematics "is like an open landscape. The horizons. You head toward them and they keep receding" (p. 113). Smilla's metaphor describes mathematics as a space that invites humans to move from one position to another and thus see objects and ideas from multiple perspectives. "Do you know what the mathematical expression is for longing? […] The negative numbers. The formalization of the feeling that you are missing something. And human consciousness expands and grows even more. […] It's a form of madness." (p. 112-113). Mathematics is a space that invites humans to move beyond their current limited vision of the world.

FURTHER DIVERSITIES IN MATHEMATICS

As exemplified above, mathematicians are characteristically interested in extending beyond conventional perspectives and assumptions. We asked what differences among people and their experiences can open up new perspectives and thus challenge assumptions and extend the possible mathematics. We identified the following diversities: special needs, linguistic and cultural diversity, different approaches to problems, representing in different modes (graphs, diagrams, verbal, gesture), social positioning, ways of representing the middle or centre, and learning disabilities.

REPRESENTATIONS OF NUMBER

As earlier, we used video prompts to help us consider difference and the impact on the development of mathematical thinking. We showed a video segment representing diversity in language. In this video, Diana (a Mandarin-speaker) describes how the concept of fractions is represented in her language and how the structure of the language itself supported her mathematical thinking. This video got our group thinking about the difference in the structure of a pictogram language rather than phonetic. We did not have sufficient experiential background in our group to take this discussion far. The fact that our group was lacking sufficient experience with pictogram language representation underscored the value that diverse experience has in the development of group thinking.

LEARNING DISABILITIES

In our final video prompt, we viewed a young adult with a learning disability reflect on his mathematics experience in secondary school. His story offers insights into the complexity of navigating the school system for a student with a learning disability. Silas reflects on his first day, having been assessed and placed into remedial math which was on the 6^{th} floor of a large high school: "Finding the 6^{th} floor is impossible at [this school], so the first four days I missed the class. I would go around asking different teachers where the 6^{th} floor was because only two staircases led to it. So on the fourth day, I found it but I was already behind. So that's where my high school math started off from and it never got better."

In terms of equity, he highlights the important role of others (especially teachers) in making mathematics accessible and in affirming his identity as a doer of mathematics. He spoke about failing Grade 9 Basic Math every year but he continued to take it (and fail) over the following two years at which point he almost dropped out of school. Along with his stubbornness, a new hope convinced Silas to try one more time – he found out that his friend's sister was going to be teaching math at his school and he was convinced to come back for one last semester. In the video, he talked about walking into class and heading for his usual spot at the back of the room. The new teacher had him sit at the front and she would notice when he stopped paying attention. She would focus the math problems on him. He described with gestures how the teacher drew a graph and wrote on the *x*-axis the amount of classes he attended, and on the *y*-axis the amount of sleep he got. He then smiled and gestured the slope of the line on the graph. Silas describes a teacher who was able to increase his engagement and subsequent participation – hallmarks of equitable teaching.

In our group discussion, we were oriented to the expectation that differences and diversities increase the space of the possible for developing mathematics. But do all differences contribute to the development of mathematics? Does a disability or a weakness make more or new mathematics possible? We noted that the people who took Silas seriously had to change the way they did and talked about mathematics in order to connect to his experience. Thus the mathematics for this teacher became different than it might have become. When we experience any human difference, we may marginalize it by calling it a disability or a weakness, or we may consider the person different from us as fully human, and address their ways of seeing and their ways of communicating. When we attend to the ways others see and communicate, we experience new perspectives ourselves, and new possibilities.

TASK DESIGN FOR DIVERSITIES AND EQUITY

We concluded our work together by considering answers to the following question: What are the characteristics of a mathematical task that opens the door for the diversity in a group to be generative in the development of mathematics? The question in French was posed as follows: *Quelles sont les caractéristiques d'une tâche mathématique qui permettent à la diversité de contribuer à des mathématiques fécondes?* Before answering the question, we discussed the best way to pose the question. We were conscious of the fact that the construction of the question (that is, our task) strongly influences the outcome of the response. Indeed, that reality is central to this particular question.

The French verb *permettre* was the object of considerable discussion because it is often translated as 'to permit'. We agreed that the English 'to open the door' approximated the meaning. We were interested in tasks that invite or encourage certain kinds of mathematical interaction by making space for it, by not restricting it. Our consideration of other possibilities for this verb helped us to realize the importance of agency in mathematical tasks. When a teacher 'directs' or 'draws out' particular activities in a task, the teacher is relatively active and the students are relatively passive or acquiescent. However, our group wanted to focus on the idea of opening space, not on forcing ideas.

Here are some raw responses to the question about the characteristics of a mathematical task that opens the door for the diversity in a group to be generative in the development of mathematics. We will elaborate on these below:

- Sufficiently open-ended to invite the people to define their own boundaries/premises.
- It has to have a problem that connects to the people's desires.
- It has to involve struggle for the people.
- It is powerful to push people into forms of representation, sensing or expression that are different from their natural/comfortable inclinations (but there should be permission for people to choose to do things differently).
- It should have a sufficiently different context from other tasks done by that group of people (e.g., social justice issues, pure mathematics, geometry, games, art, mathematizing nature, various cultural and linguistic contexts…).
- The starting point can be simple or complex.
- It has multiple starting points.
- It is in an environment in which people push further (investigate, pose further problems).
- It is part of something larger.
- It has sufficient variety to invite conjecture in extending the variety.
- It looks forward … "Where will you go?" *Quo vadis?* Ou vas-tu?

Before addressing the question directly, we assert the importance of recognizing that there is diversity in any group. Whether or not we recognize the diversity relates to our readiness to recognize the differences as significant. Differences may be ignored because of assumptions that a particular characteristic is normative, and thus other characteristics are repressed. Differences may likewise be ignored because of assumptions that they are not sufficiently different to be productive; one can believe that others are so similar that they cannot contribute a different perspective. The nature of a task is perhaps not as important as one's orientation working on the task. Thus, equitable relationships and good mathematics are supported with an expectation that one's group mates are different from oneself and thus able to bring a unique perspective, and an expectation that such different perspectives are by nature potentially powerful for generating insight. Even relatively homogenous groups working without constraints can be seen as an opportunity for contributions of diverse ideas.

Though our question focuses on the characteristics of good tasks, this first recognition about group dynamics reminds us that group composition is also part of a task. If ask a class to answer a particular question, we tend to think that the question is the task. Nevertheless, we are also saying explicitly or implicitly "you three, work together" or "work with the people at your tables." These too are tasks – to work together in particular ensembles. Furthermore, the ethos of the classroom is part of the task and part of the group composition. The sociomathematical norms (Yackel & Cobb, 1996) that govern or direct group interaction are part of the task as much as the groupings. When we give students a question to work on we are also saying explicitly or implicitly to work in a particular way with particular orientations. Thus an orientation to diversity, in which students expect insight from their peers who are different and unique is part of any mathematical task; and an orientation to tradition, with an expectation that there is really only one best way to approach the assigned question would also be part of the task. Though we are arguing that group composition and classroom ethos are part of a task, to avert ambiguity, for the remainder of this report we will use the term task to refer to the mathematical question or instructions in any given situation, distinct from the classroom norms and distinct from group composition.

Though group composition and classroom norms are significant, there are certain kinds of tasks and activities that have greater potential to open space for productive difference. So we repeat the question, with the acknowledgment in the back of our minds that it is not the task alone that is important. What are the characteristics of a mathematical task that opens the door for the diversity in a group to be generative in the development of mathematics?

Both relatively simple and complex tasks have the potential for inviting diversity. A problem-posing environment helps bring out difference because multiple questions are asked about the mathematical scenario. Though such an environment is partly a situational norm, tasks can explicitly ask for problem posing. Indeed, a problem-posing ethos is created through regular explicit requests for problem posing. With problem-posing experiences, an offered solution is not the end but rather the beginning of new lines of exploration. When students think they are 'done', or when a solution is offered, there is still room for further exploration and questioning. The struggle is part of the richness of exploration, and, with problem posing, students can struggle with either simple or complex tasks.

We open up the possibility for insights into mathematics and about mathematics when we invite people into forms of representation, sensing, and expression that are different from their natural inclination. As with an ethos of problem posing, people can look for diverse forms of representation, and can experiment with different sensual approaches and different forms of expression. The examples above illustrate such potential. For example, when one cannot see (or is restricted from using their sense of sight), it is easier to become aware of certain features of a physical object's shape. Again, like with problem posing, people can be oriented to the exploration of multiple forms of representation, sensing and expression no matter what the task is, but tasks can be constructed to ask explicitly for this kind of exploration. Furthermore, explicit requests for such exploration in tasks develop an ethos that encourages the exploration of diversity that can bring richness to any task.

We note that most of our answers to the question about what kinds of tasks build on diversity to generate good mathematics have focused on approaches to problems and tasks. However, it is imperative to notice also that these positive orientations are built with experience. And the experience that underpins the orientations we promote can be developed through tasks that explicitly ask for certain kinds of action. Teachers can ask explicitly for problem posing. Teachers can restrict certain senses to foreground others. Teachers can ask for different forms of representation. Once habits of mind are developed, and students experience the value of attending to diversity, any task they are given has greater potential. Open-ended tasks, in particular, invite difference. Ironically, students may attend to difference in open-ended task environments because of the necessity to define their own boundaries and premises.

Our working group also discussed the connection between tasks and students' experiences outside of school. It was suggested that tasks are richer when they address a greater need, one that extends beyond the classroom. When this is the case, students can more readily apply their experiences outside the classroom to the task at hand, and thus are more likely to bring an unexpected perspective to it. In this case, the task may be one in which students learn strategies for, or are motivated to learn strategies for, developing computational fluency in tandem with conceptual understanding. Here, there is potential for creating tasks of high cognitive demand, increasing student participation and engagement.

Different ways of addressing such greater experience include connections to culture (e.g. First Nations students working on mathematics that has been done by their ancestors), connections to current issues (e.g., analyzing and/or mapping local social justice issues, and perhaps working toward addressing the issues with policy-makers and others), and connections to nature (e.g., mathematizing a natural phenomenon). Work with cultural or linguistic artefacts can help students to see beyond their own culture and language (like looking through a window), or to see their own culture and language in a new way (like looking in a mirror).

Also, we were reminded that there are numerous resources developed for the express purpose of appreciating difference. For example, a member of our working group described a framework for differentiated instruction that had good suggestions for teaching to meet diverse needs (Commission Scolaire des Sommets, 2007). Though we think it is important to address the diverse needs of students in mathematics classrooms, it can be a different thing to envision difference and diversity as a resource for developing mathematical insights for the entire group. Both are important – addressing different needs, and building from diversity.

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- Bohm and the Rheomode: Verb-based language developed to help describe quantum theory: http://www.fdavidpeat.com/ideas/langling.htm
- Gloria Ladson Billings: Excerpt from an American Educational Research Association presidential address: http://www.nwp.org/cs/public/print/resource/2513
- Jean-François Maheux's film can be found on his website (in the middle, in the News section): http://www.math.uqam.ca/maheuxjf
- "Show Me Your Math" resources (Lisa Lunney Borden and David Wagner) at: http://showmeyourmath.ca/publications

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TECHNOLOGY AND MATHEMATICS TEACHERS (K-16)

LA TECHNOLOGIE ET L'ENSEIGNANT MATHÉMATIQUE (K-16)

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INTRODUCTION

At the *100th Anniversary of ICMI*, Colette Laborde (2008) discussed the evolution of the trends of research and of the integration of technology into real mathematics education practice. She stated:

> *For now more than 20 years, integration of technology is an issue debated in research as well as in the reality of the classrooms. Many countries support this integration at the institutional level but the everyday practice of a large part of teachers generally does not follow this institutional demand.* (p. 1)

Resulting from her analysis of the proceedings and programs of the ICMEs over the past decade, Laborde suggests a shift of research focus from ICME 8 to ICME 10. Whereas the point of view taken at ICME 8 was mainly addressing technology as a "catalyst for change", it had shifted at ICME 10 to stress "the need for more research that takes the teacher as central focus, and in particular the relationship between the teacher and technology" (p. 5). This grounds the focus of our working group about technology on the role of the mathematics teacher rather than on the mathematical task. The working group was organized around three topics: i) orchestration; ii) assessment; and iii) new technologies.

TECHNOLOGY AND MATHEMATICS CLASSROOM IMPLEMENTATION

The goal of the first session was to begin a discussion on how the teacher's role in the technology-based classroom could be described and/or theorized. We began, however, with a short presentation outlining the history of technology integration in mathematics education and chronicling the relatively slow emergence of a teacher focus after decades of work on student learning and task design. We began with an overview of different approaches described in the literature, with a particular focus on the emerging theories of *orchestration* (Drijvers, 2012) and *documentational genesis* (Gueudet & Trouche, 2009), that build on the student-focused theory of instrumental genesis. We then proposed three different tasks, at the primary, secondary and post-secondary levels, respectively, and asked participants to study the tasks and discuss how they would implement each given task in the classroom, and what the role of the teacher would be in such an implementation. The primary task actually involved a classroom video, so the focus was more on the role of the teacher, while the two other tasks were written on paper, but with a particular technology intended. Both in the small group discussion and in the plenary discussion, the participants noted that it was very difficult to focus on the teacher, and much more natural to question the design of the task or the mathematics involved. This experience provided the group with a palpable sense of why it might have taken so long to pursue research on the role of the teacher in a technology-based classroom.

TECHNOLOGY AND ASSESSMENT

The second session addressed issues of mathematics assessment with technology. For our working group, we decided to focus on students using technology *in assessment*, as opposed to the teacher using technology *to assess* (e.g., online assessment systems, possibly with feedback). Two activities were proposed to participants, each of them with the option of elementary, secondary, and/or tertiary level material.

The first activity involved solving practice exam questions (summative evaluation), found in a textbook or online resource, with the use of the free online computational knowledge engine, *Wolfram Alpha*. Overall there was a feeling of astonishment at *Wolfram Alpha*'s capacity, both in terms of detailed solutions (with explanations) and complementary information provided. Participants were invited to modify the problems, keeping in mind that students would solve them using *Wolfram Alpha*. In other words, the underlying rule was to suppose that students have access to the technology. This stressed the point of view of addressing the question of how one can use technology instead of avoiding it. Three strategies for problem design emerged from the whole group discussion while reporting and reflecting on the activity:

- Create questions requiring some mathematical modeling, i.e., in which the mathematical notation is not given, and as such, cannot be directly typed into the technology.
- Reverse one of the traditional problem formats: give properties and ask to find an example.
- Create questions involving comparing instead of identifying.

For an example of such questions, see a Grade 10 test in the Appendix about linear relations and systems for which students were allowed to use a graphing calculator.

The second activity involved creating an exam question using a technology of their choice, and testing the question from another group. During this task, diverse uses of technology were considered for the designed questions. For example, using the technology to validate an answer or using it to explore. In the latter case, some participants realized the great difficulty of creating an exploration/exam problem. Among many issues raised in discussions was the question: Would the problem evaluate mathematics understanding or would it instead evaluate technology-use skills?

Complementing the activities, a few concrete examples of technology use in assessment were briefly presented. For example, a 2001 compulsory Ontario Grade 9 exam using *The Geometer's Sketchpad* (EQAO, n.d.); a two-part university exam at USMA, one part with, and one part without technology (Heidenberg & Huber, n.d.); and, the US compulsory SAT exam for which "[s]ome questions on the Mathematics Level 1 and Level 2 Subject Tests cannot be solved without a scientific or graphing calculator" (College Board URL, n.d.). Also, statistics about some practices of assessment in Canada were presented: the use of graphing calculators in Quebec CEGEPs (Caron & Ben-El-Mechaiekh, 2010), and the use of Computer Algebra Systems in Canadian universities (Buteau, Jarvis, & Lavicza, in press) suggesting stronger technology use in assignments and projects than in in-class tests and exams.

NEW TECHNOLOGY/NEW AFFORDANCES

The third session began with a tour of the iPad applications currently available for mathematics, with an emphasis on those apps that are designed to take advantage of the new tactile and gestural interface. As a group, we discussed the mathematical potential of this interface and its relation to current research in embodied cognition, more broadly, including gestures. Following this, Nathalie described the design of the *TouchCounts* application, which is aimed at children aged 4-7, with a focus on number sense. The participants were shown the current state of the application, which includes Counting and Adding worlds, and which aims to leverage the tactile interface to promote mathematical understanding. They were then given the challenge of designing a world for subtraction and/or division. This provided a concrete context in which the group could discuss different conceptualisations of the four mathematical operations, and ways of ordering their introduction.

APPENDIX

Below is a Grade 10 Test (created by Paul Alves, Peel DSB) about linear relations and systems for which students are allowed a graphing calculator. This exemplifies the three strategies for question design that emerged from the discussion during the *Assessment* session.

1. Mr. Alves and Mrs. Charest are slightly competitive. On St. Patrick's Day, they each buy a box of *Smarties* that only has green and white candies. Each box has a total of 56 candies. They challenge each other by telling a clue about the number of green and white candies in the box, and the other player has to determine the actual numbers.

   Mrs. Charest stops Mr. Alves in the hall and says: "**Just so you know, the number of green *Smarties* in my box is 10 more than twice the white *Smarties*. Take that!**"

   Mr. Alves laughs and says "**That's completely impossible. You know, cheating doesn't make you win this game, Mrs. Charest!**"

   Is Mrs. Charest cheating or is Mr. Alves wrong? Mr. Alves and Mrs. Charest are competitive, so a detailed, efficient explanation is required to make either of them happy.

   (HINT: A linear system might be helpful here.)

2. Determine a **linear system** that has ALL of the following properties:

   - The solution to the system is (3, 6)
   - The slope of one of the lines is positive and the slope of the other line is negative
   - The lines cannot go through the origin

   Write down your thought process so that your teacher can understand what you did to get the linear system. You are being marked on your thought process. **Communicate clearly how you chose the two equations.**

3. Mr. Labrie just finished his lunch with Ms. Kongtakane and is expecting Mrs. Khodai to arrive. Mrs. Khodai is leaving school to go to Mr. Labrie's house, and Ms. Kongtakane is leaving Mr. Labrie's house at the same time to go back to school. The graph below describes their walk.

4. Who is walking the fastest? Show any necessary calculations that will **prove your case**.

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LA PREUVE EN MATHÉMATIQUES ET EN CLASSE

PROOF IN MATHEMATICS AND IN SCHOOLS

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[**NOTE:** Le groupe de travail était bilingue et le rapport reflète cette situation. The working group was bilingual and the report reflects this.**]**

INTRODUCTION

Dans le cadre de leur travail, les chercheurs en mathématiques doivent lire des preuves, énoncer des conjectures, chercher des preuves pour ces conjectures, écrire ces preuves et les faire paraître. Mais les mathématiques de la classe ne sont pas celles de la recherche et ne visent pas nécessairement à former des chercheurs en maths. Ainsi, la transposition de la preuve d'un contexte à l'autre ne va-t-elle pas de soi. Les écrits sur la preuve dans chacun de ces contextes abondent, mais ils soulèvent autant de questions qu'ils en résolvent. Dans ce groupe de travail, nous avons examiné quelques questions clés relatives à la transposition de la preuve, des pratiques de recherche à la classe. Parmi celles-là :

- Quelles activités de recherche liées à la preuve (lire des preuves, conjecturer, explorer exemples et contre-exemples, resserrer ou même changer la question, adapter les définitions, les théories, etc.) devraient faire partie des mathématiques scolaires ? Dans quel ordre, selon quelle importance, à quel niveau scolaire ? À quel type de coordination entre les connaissances pourraient-elles ou devraient-elles donner lieu ?

- *What proof related activities in research mathematics (e.g., reading proofs, conjecturing, exploring examples and counter-examples, narrowing or even changing the question, adapting definitions or theories, etc.) should be part of school mathematics? Which should come first, at which school levels, and what kind of coordination could or should be considered?*
- Quel devrait être le rôle de la preuve en classe ? Est-ce avant tout un moyen de vérification ? d'explication ? d'exploration ? Ou peut-être est-ce simplement un sujet où l'on apprend à « bien raisonner » ? Ou autre chose ?
- *What should the role of proof be in school mathematics? Should proof be primarily a means of verification, explanation, exploration? Or simply a ground for learning 'good logical thought'? Or something else?*
- Conjecturer constitue-t-il une motivation et une aide à la preuve, ou un obstacle à son élaboration ?
- *Is conjecturing a motivation and support for proving, or an obstacle to it?*
- À quels types de preuves les élèves devraient-ils être exposés ? Lesquels devraient-ils être capables de produire, selon les niveaux scolaires considérés ?
- *What kinds of proofs should students be exposed to, or expected to produce, at each school level?*
- Quel rôle devrait avoir la logique (formelle) dans l'apprentissage et l'enseignement de la preuve ? Le cas échéant, comment devrait-elle être enseignée ? Comme un sujet spécifique ? Ou transversalement, à travers tout le curriculum mathématique (post-primaire) ?
- *What role does (formal) logic have in proof teaching and learning? If logic has a role, how should it be taught? As a specific topic? Or throughout the entire (post-primary?) mathematics curriculum?*

THE TASK

To give our discussions a basis in a common experience, we explored, in small groups, the following task:

> Given a square *polymino*[1] of any size, with a "hole", for what positions of the hole can the *polymino* be tiled with dominoes? The hole can be anywhere, including an edge or a corner of the *polymino*. The figure shows a *polymino* of size 7 with a hole (shaded) (Figure 1).
>
> *Étant donné un* polymino[1] *(grille carrée) de taille quelconque avec un « trou » d'une case, pour quelles positions du trou est-il pavable par des dominos ? Le trou peut se situer n'importe où, y compris sur un bord ou un coin du* polymino. *Voici le dessin pour le* polymino *de taille 7 et un cas particulier de la position du trou (case hachurée) (Figure 1).*

The reader is urged to spend some time on this task before reading on.

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[1] The task was introduced in terms of *polyminos* because there were several related tasks planned. They were inspired by the work of the *Maths-à-modeler* group from Grenoble (http://mathsamodeler.ujf-grenoble.fr). In the end, the other related tasks were never used. We often referred to the square *polymino* as a 'grid' in our discussions.

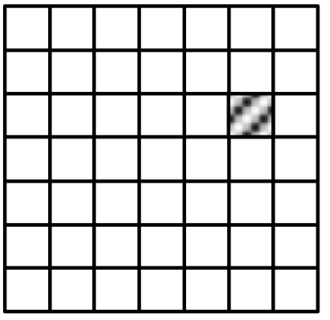


Figure 1

After spending some time on the task, the working group used this common experience to try to answer the questions raised above. Some participants provided written answers and comments on the activity, which are quoted below.

WHAT SHOULD PROOF BE IN SCHOOL?

Our first question concerned the *transposition didactique* from proving in research mathematics to proving in schools. But it makes sense to first consider another question:

WHY SHOULD PROOF BE IN SCHOOL?

> Le développement de preuves permet (ou plutôt devrait permettre) aux élèves de vivre (du moins en partie) ce que les mathématiciens vivent. Ainsi, les retours en arrière, les essais, les erreurs, les éléments exploités qui finissent par ne servir à rien, etc. font partie de l'activité mathématique. Le développement de preuves est associé de près au développement du raisonnement (raisonnements inductif, déductif, etc.). (Manon)

> I believe deeply that proof is essential to the understanding of maths. In fact, mathematics is not only a field in which we use numbers to calculate, it is also a way of thinking in which nothing can be taken as a result unless there is a convincing proof that the result is true. (Simon)

Having considered why, it is important to reflect on the differences between mathematicians and students. In other words, to ask:

WHO PROVES IN SCHOOLS?

> Une différence entre le mathématicien chercheur et l'élève est la capacité à rendre explicite le raisonnement suivi et à réécrire un processus visant à valider un énoncé final dans une suite d'autres énoncés intermédiaires servant à marquer les étapes de la validation. Ce travail exige une méta-conscience du travail fait dans la résolution d'un problème. Il exige donc une mémoire des étapes. Or, l'actualisation de cette mémoire exige un ralentissement du processus en cours, ralentissement dû à la nécessité d'expliciter les résultats obtenus ou conjectures faites tout au long du processus, souvent facilement un peu anarchique, de résolution d'un problème. Cette explicitation tout au long du processus a par ailleurs l'avantage d'aider à organiser le processus et de tester la validité de ses étapes. (Louis)

With this basis, it is possible to move to the issue of the *transposition didactique* of proof from research mathematics to school mathematics, and the question:

HOW SHOULD PROOF BE VIEWED IN SCHOOL?

> La question n'est donc pas tant de savoir quelles activités de recherche liées à la preuve devraient faire partie des mathématiques scolaires, que la façon dont ces activités de recherche sont exploitées pour correspondre le plus possible à l'activité mathématique vécue par le mathématicien. (Manon)

Lors de la discussion finale, nous avons réfléchi à la façon dont l'activité (les dominos) s'était présentée à nous, pour en dégager certaines caractéristiques et en même temps, regretter que ces caractéristiques ne soient pas plus souvent présentes dans les activités de preuve généralement proposées en classe :

- La preuve comme processus (proving), dans le cadre d'une activité de type « résolution de problème » relativement ouverte, où plusieurs énoncés peuvent se présenter, être dégagés, sans même qu'il soit nécessairement clair au départ qu'il y a quelque chose à prouver.
- On passe par une phase d'appropriation : il faut décoder l'information, la départager, la classifier, sortir du sentiment (inévitable au début !) de confusion, explorer sur des exemples pour comprendre la ou les questions et la (ou les) préciser.
- Identifier les « conditions » selon lesquelles les énoncés sont susceptibles d'être vrais ou faux. Reprendre la réflexion, explorer à nouveau. Ce n'est qu'à ce stade que la conjecture émerge et que le processus (proving) s'enclenche véritablement, en même temps que la conjecture.
- En ce sens, la conjecture en tant que processus n'est pas isolée de la preuve, elle en est simplement l'amorce. D'ailleurs, peut-être n'a-t-elle pas d'intérêt en tant qu'objet (isolé) ? Elle peut même être un obstacle si on la met trop en valeur en tant qu'objet ! (Voir plus loin la discussion sur la conjecture).

Finally, specific answers can be given to the question:

WHAT SHOULD PROOF BE IN SCHOOL?

> I believe that it would be useful for students to do some of the following:
>
> - 'Read proofs, conjecture, explore... counter-examples' (as mentioned in the question);
> - 'Explore examples' and the difference between demonstrating that something 'works' (for lack of a better word) in some particular cases and trying to prove that it 'works' in many cases, and proving that it holds in all cases;
> - 'Focussing or even changing questions' or posing new questions in the process of trying to answer a given question (e.g., related to our work on the task, asking whether an n by n grid might not be 'paveable' for some n and/or for some 'positions' of the hole);
> - Defining concepts that students use in their informal discussions of mathematical ideas (e.g. how several people spoke of 'paving') and then potentially re-naming this concept and/or adapting the definitions that they propose;
> - 'Adapt theories' and propose lemmas (whether students refer to them as such or not) when partial results can be proven that are seen as useful in proving a given theorem.
>
> (Nicolas)

> For me, the mentioned activities (conjecturing / exploring examples and counterexamples / changing the question / adapting a definition) seem to be part of an 'explorer's' activities. I believe these skills are related to *reflecting* on a problem possibly before even embarking on a solution (the problem could be a proof-related

problem or not); and that, furthermore, a 'reflection' or 'exploring' phase is an important phase in problem solving in general before choosing a definite solution path. Following this idea, I do believe that these activities should be a part of school mathematics, perhaps some at all levels (with an adaption to the age-level). (Dalia)

Ne pas oublier l'importance de la preuve pour développer chez les élèves un regard « méta » : s'interroger sur la généralité de l'argument, sur le type de certitude qu'il apporte, sur sa clarté, sa concision, son « élégance », etc. (Denis)

THE ROLE OF PROOF IN SCHOOL

Our second question is related to the role proofs play in school. Many possible roles have been identified (see Hanna, 2000) and advocated for proofs in schools, but the question is far from resolved.

Bien raisonner, c'est contrôler son raisonnement. Contrôler son raisonnement, c'est aussi pouvoir le communiquer à d'autres. (Louis)

Most students take 'mathematical facts' for granted so it is important to start asking questions: How do you know … ? Why is this true … ? Why should I believe that … ? at a very early stage in a student's education. This will prompt students to always reflect, wonder and question on their path of learning, especially mathematics. Elementary students are constantly making conjectures while they are discovering simple 'math truths'. Students need to be aware that these 'truths' were just conjectures at some point, and they should be encouraged to convince others (examples, counter-examples, writing convincing arguments, reading their peers' 'proofs', comparing, critiquing, analysing each other's 'proofs'). These are crucial components towards using clear mathematical language in communicating ideas in a convincing way. (Dorota)

Par définition, l'idée de preuve fait appel à une composante sociale (convaincre l'autre). J'aurais donc tendance à dire que la preuve va plus loin que la simple vérification. Certains diront toutefois qu'il est possible de développer une preuve pour soi (on cherche alors à se convaincre plutôt qu'à convaincre les autres). Peu importe la vision adoptée, ce qui semble important pour moi, c'est l'idée de convaincre. Bien souvent, nous présentons des problèmes aux élèves dont ils connaissent déjà la réponse (l'exemple de la somme des angles intérieurs d'un triangle en est un bon exemple). La preuve perd alors sa raison d'être, car ils sont convaincus avant de débuter le travail. L'importance placée sur l'idée de convaincre les autres permet alors le passage des preuves empiriques aux preuves intellectuelles. En effet, ce qui convainc des élèves du primaire ne convainc pas nécessairement des mathématiciens. (Manon)

School mathematics could have, for one, the role of helping students understand that mathematical statements are true in a given context and not absolutely (i.e. that they are proven from a set of axioms and definitions) and that they are not true or false simply because the teacher says so. (Nicolas)

Roles of Proof in School

- use mathematics;
- appreciate power of math;
- provide means for making sense. (Elaine)

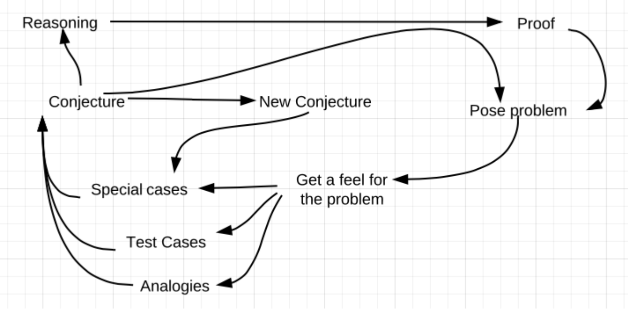
I think that the activities where the student has to participate in solving a problem in small groups must be part of students' math education. In fact, I think it would lead them to understand the epistemological aspect of proof: you need it to convince, just as the first mathematician needed it to convince! I would suggest proceeding in an activity of the kind presented in John Mason's (1994) book: *L'esprit mathématique*. It is a great collaboration with Leone Burton and Kaye Stacey. I think such activities could increase the understanding level of the reasoning process in which people are involved when they solve a problem! (Simon)

PROVING AND CONJECTURING

Mathematical activity is sometimes portrayed as having two phases. The first phase is one of discovery, conjecturing, and informal argumentation. The second phase is one of verification and proving. For some, the reasoning in these two phases is continuous, part of one "cognitive unity" (Garuti, Boero, & Lemut, 1998). For others, the argumentation involved in conjecturing is seen as an obstacle to proving (Duval, 1991). Our third question was posed with this debate in mind, but the group developed it into a much richer discussion.

> Je vois l'action de conjecturer comme une aide à la preuve, car cela permet de placer son attention sur un élément particulier (diriger, dans un certain sens, la réflexion). Dans quel sens cela peut-il être un obstacle? Parce que le fait de conjecturer nous limite dans notre réflexion? Parce qu'il est ensuite plus difficile de « penser autrement » ? J'aimerais bien avoir la chance de discuter de l'autre côté de la médaille, afin de mieux comprendre comment le fait de conjecturer pourrait constituer un obstacle. (Manon)

> After John Mason:



> (Elaine)

> Conjecturer est nécessaire à la découverte d'une preuve. Mais encore faut-il savoir quand un énoncé que nous faisons est une conjecture et non une affirmation qu'on va a priori considérer vraie. Une conjecture est en fait une hypothèse. On peut vouloir l'utiliser comme une affirmation dont on déduit autre chose, en la supposant vraie. Ou comme un énoncé à valider. Autrement dit, une conjecture peut aussi bien être vue comme une hypothèse dans un processus d'analyse (au sens des anciens) qu'une étape dans un processus déductif. Le contrôle de la conjecture est donc essentiel. Une conjecture dont la nature est mal comprise devient un obstacle. (Louis)

> Producing a proof was not the primary objective of our activity – that would be understanding the problem, defining the problem and convincing ourselves of certain things. Conjecturing was fundamental – maybe because David sat with us and restated our first observation as a conjecture. (Richard)

> I think conjectures are traps. It happened very often to me that what my intuition took for granted was a creation of my imagination and was far from the reality of the problem. At the same time, I must say these traps often lead to a new understanding of the problem. So, after having said all this, I realise that the conjectures are the key points of reasoning (at least of my reasoning) process. That means we have to be aware of that process that leads to establishing a conjecture, trying to find a proof of it, passing to a lemma or a theorem and using that theorem to solve, to dig deeper in our understanding of the problem. (Simon)

> Conjecturing involves creativity, intuition and pattern recognition. Proving involves reasoning, rationalising, and deduction. (Elaine)

PRACTICAL OBSTACLES TO PROOF IN SCHOOL

The working group did not limit itself to discussing the original questions. There was an ongoing interplay between discussion of theoretical issues and practical considerations.

> Mais est-il utopique de penser de cette façon ? Les contraintes du système scolaire permettent-elles réellement d'exploiter la preuve de cette façon ? Y a-t-il une incohérence entre ce qui est souhaité (dans les cadres théoriques des programmes d'études, on fait la promotion du développement de différents types de raisonnement) et ce qui est réalisable (dans les plans d'études des programmes d'études, on suggère peu de pistes pour développer les différents types de raisonnement chez les élèves et quand on le fait, peu de temps y est réservé) ? (Manon)
>
> Le problème, c'est que plusieurs auteurs s'entendent pour dire que le développement du raisonnement doit se faire sur le long terme. Or, les programmes d'études (du moins ceux du NB) réservent peu de temps à l'apprentissage de la preuve. Ainsi, se faire croire qu'on travaille réellement le développement de preuves quand les élèves n'ont pas vraiment le temps de penser (comme nous l'avons fait aujourd'hui dans le groupe de travail) et quand ils cherchent à arriver à « la » bonne réponse (souvent, celle qui contient le même nombre de pas de déduction que celle qui vient d'être réalisée au tableau par l'enseignant (!!!)), c'est un peu jouer à l'autruche et se faire croire qu'on aide les élèves à développer des habiletés liées à la preuve alors qu'en réalité, ils apprennent à faire des preuves comme ils apprennent à appliquer des algorithmes. (Manon)
>
> That leads to a question: if proof is an obstacle for high school students, will we avoid it and wait until college to talk about it? What if it represents an obstacle for college students as well? And for university students? Would that be the end of proof in education? Would it be the end of that kind of mathematics? (Simon)

INSIGHT IN PROOFS

While working on the task the participants made many conjectures, and proved some of them. As Richard put it:

> It was fascinating how different groups used different approaches and forms of representation (e.g., matrix notation, focussing on the hole #) but we all ended up being most challenged by the same issue (namely, proving the impossibility of some configurations).

The challenge was proving that if *n* is odd, it is only possible to tile an *n* by *n* grid with dominoes if there is a hole in one of the positions marked with a dot in the figure (see Figure 2).

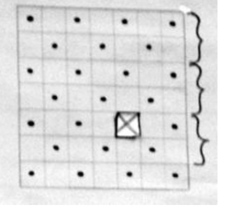


Figure 2

However, if one has a specific insight (*une astuce*) the proof is quite easy. This led to considerable discussion about role such 'tricky' proofs might play in schools. Should they be avoided as the students are unlikely to be able to generate these ideas themselves? Or should students see many such proofs in order to develop their own capacity for such insights? (See below).

SOME PROOFS

The working group produced some proofs related to the *polymino* paving task, of two kinds. Some were proofs we would expect students to provide given this task. Others were exemplary teacher proofs. Here is a sampling (see Figures 3, 4, and 5).

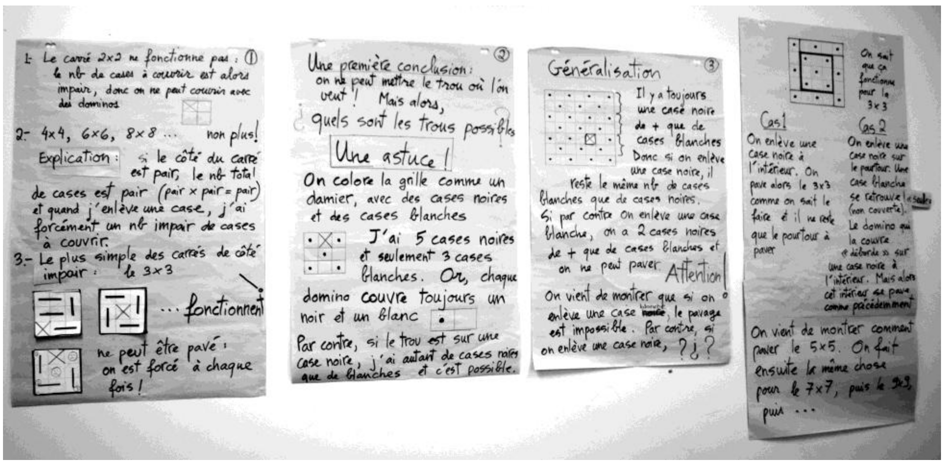


Figure 3

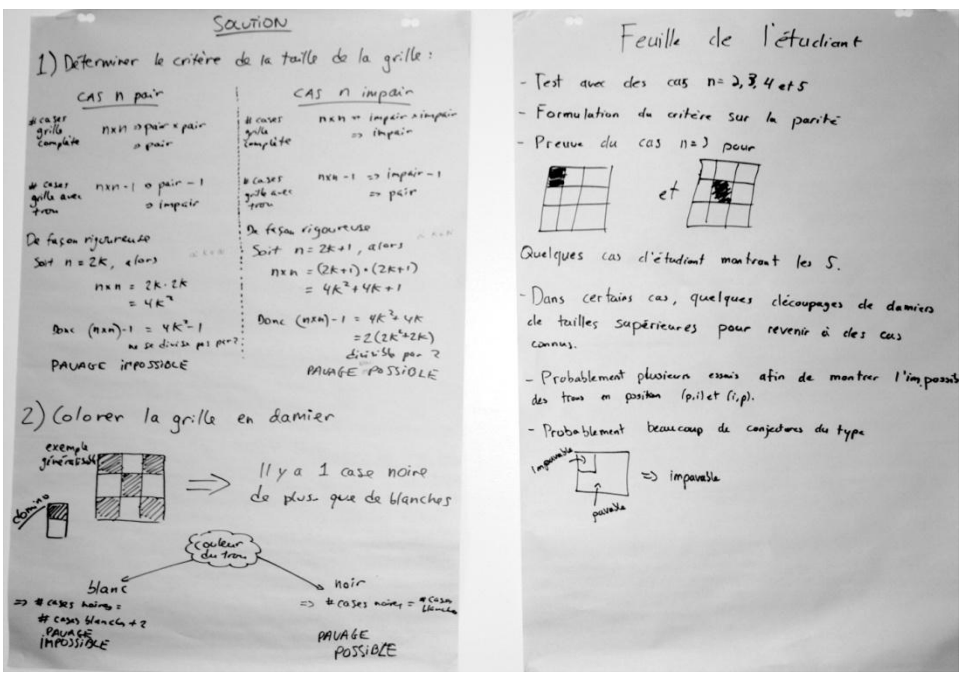


Figure 4

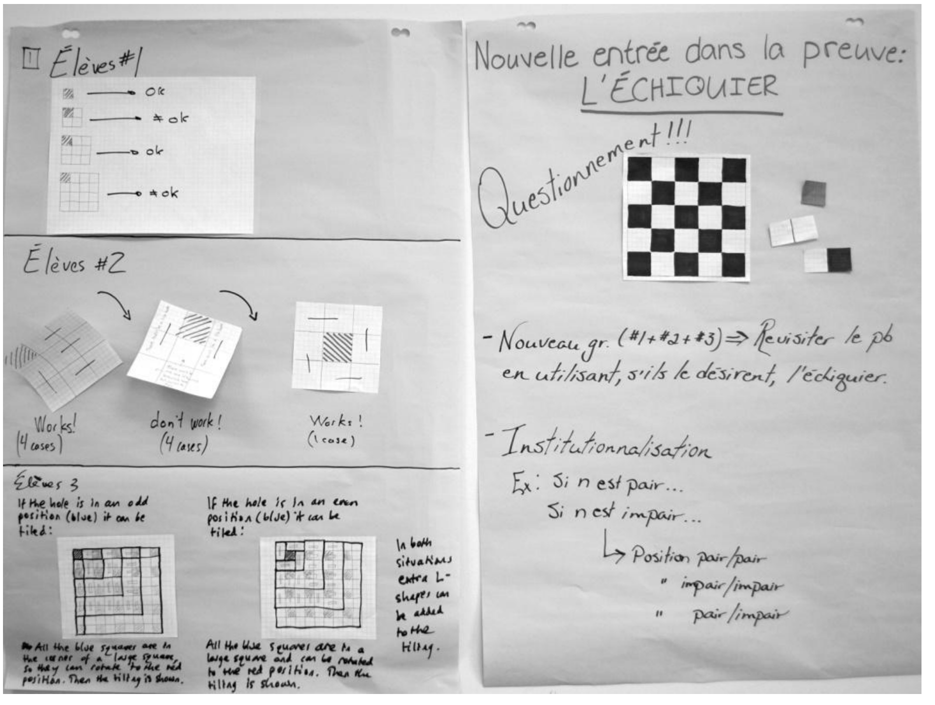


Figure 5

DISCUSSION

We finished off with a wide-ranging discussion, revisiting some of the themes from the first day, and introducing some new ones. The following dialogues attempt to capture (without reproducing exactly) the spirit and content of this discussion.

THE ROLE OF INSIGHT, CLEVER TRICKS, *LES ASTUCES*

Denis: How does this hint of the chessboard come in? As a hint? Then students get the idea that proof depends on a strike of genius. Could the chess board come right at the beginning?

Margo: You can explicitly say “here is an idea that might help”, or you can show a related problem where it might work. Like the mutilated chessboard. However, often learners need some time to explore the problem situation and exhaust their own ideas in a number of unsuccessful attempts before they can truly benefit from external hints and suggestions.

Nicolas: Could giving such hints block students from having insights; which we seem to agree are crucial to proving many theorems and, in particular, to justifying one’s solution to this problem?

Dalia: If we present to students several problems where having an insight is a key factor for solving the problem, then perhaps students will be encouraged to look for the *astuce* in future problems.

DavidG: The *astuce* is not a bad thing. Students also need to see that clever ideas are part of math.

Nicolas: The types of hints that we give or whether we give hints at all should perhaps depend on the age of the students we are teaching and on their previous experiences with mathematics. In any case, I don’t think that we want them to think that math is all about tricks.

Annette: Whose trick is it, the students' trick or the teachers' trick?

DavidG: It is hard for students to see the problem differently. It takes power, confidence and liberty. We want students to develop those qualities. So we need to not hide the *astuce* as it is part of learning that there are such possibilities.

Denis: Mais en même temps il y a un danger de décourager les élèves faibles, ou simplement moins sûrs d'eux, qui perçoivent « encore » l'astuce comme quelque chose qui est parachuté « d'en haut », quelque chose qu'ils n'arriveront jamais à trouver par eux-mêmes, comme d'ailleurs pour beaucoup d'autres choses en maths…

Manon: We are used to thinking in a certain way, which includes looking for tricks. Einstein said, "Information is not knowledge. The only source of knowledge is experience." But, if the curriculum says you have maximum of 2.5 hours to teach proof, that is an obstacle.

DavidR: Is this different in Québec where reasoning is a key competence in the programme of studies?

Louis: The *Programme d'Etudes* is good, but it is not taken up by teachers, or if it is, it is informal, implicit.

FORMALISM

Denis: That brings us to the question of formalism. Does it matter? Are the proofs on the posters formal proofs?

Annette: It depends what you mean by formalism. Is it a chain of justified statements, or does it require symbols?

DavidG: Historically it depends. Standards change.

DavidR: And it also changes by level of school, but there is not much explicitness about what those standards are in schools.

TEACHING PROOF

Nicolas: *(referring to the polymino task and opening his question to anyone in the group):* How would you teach students to solve a problem like this?

DavidR: We would all teach differently, and I see in the posters examples of several different teaching approaches: Providing *l'astuce*, offering a related problem that might let the students discover it, and exploring alternative (more difficult proofs) if the clever idea doesn't come from the students themselves.

Elaine: Multiple student solutions give rise to many possibilities, including a thorough exhaustive proof, a deductive proof, etc. Over time the teacher can develop a practice in the classroom that values more deductive reasons and more thorough reasons more highly.

Richard: You could show them three proofs that come from outside and ask them to choose.

Manon: Such a question lets the teacher know what they find convincing, what they value in the proof.

Elaine: Do we have to finish something in a single lesson? It could be that we come back and prove things later.

DEDUCTIVE REASONING VERSUS EMPIRICAL REASONS

Denis: How do we make students aware that a deductive reason is 'better' than an empirical reason? Comment faire découvrir aux élèves la satisfaction éprouvée à la réalisation d'un 'beau' raisonnement, à la réalisation d'une validation absolument générale et indubitable ?

Elaine: Can't you make a student feel more satisfied?

Valériane: Deductive reasoning is more convincing.

Annette: For whom?

Simon: Satisfaction comes from success, having the solution valued by the teacher.

DavidR: What about those who have inelegant or sketchy solutions? Do they feel success (even if their work is deductive)?

TEACHING FUTURE TEACHERS ABOUT PROOF

Simon: We are asking if it is important or not to teach proof. But another question is: Are the teachers interested and able to teach proof?

Denis: Some future teachers seem not to understand the difference between description and explanation.

Louis: They have the image that proof must be symbolic and formal, from their math courses. In didactics there is not so much pressure to make things explicit, and students are moreover told that informal arguments and explanations are important, with importance being put on 'verbalisation'. Arguments might be made orally, but if they're not written and formulated, then they can't be reflected on.

Denis: Right! Indeed, the *Programme d'Études* says teachers should emphasize oral arguments. But you have to find a good equilibrium…

Annette: With preservice teachers we do not have so much time, and they need to learn about proof if they are going to teach it.

DavidG: When future teachers have a math degree first (as in Switzerland), then they have a different experience.

DavidR: Often in English Canada, future secondary school teachers do a degree in mathematics first.

Margo: In my experience, even for teachers with math degrees, proofs often present a challenge. This is especially true when they need to find a right balance between rigorous treatment and intuitive presentation of their ideas for the students. Apparently, this ability does not come automatically through a mathematical training, and thus, it requires special emphasis in teacher training programs. One such possibility is to use problems with multiple proofs that support reasoning appropriate in various contexts and levels of difficulty (see e.g. Kondratieva, 2011).

Denis: Learning math is about learning a way of thinking that can be used in life.

DavidG: My future teachers are interested in motivating their students through using real world context, but their contexts are absurd. Perhaps better to say we do what we do because we do.

Louis: There is thinking in putting together *IKEA* plans, but is it deduction? In algebra there is no axiomatic system, the deduction is embedded in the calculations. There is no conscious deduction.

Manon: Yesterday you asked why are we teaching proof, you can't do anything with it. But maybe we need to give a reason like 'it's beautiful'.

DavidG: It is about the disposition to doubt and to acquire the power to know.

CONCLUSIONS

We began with questions, and we ended with questions. We will conclude here by posing some further questions, perhaps to be taken up in a future working group.

What is the difference between proof and proving? We talk about proofs, but not in the same moment about the process of proving. Is it important to distinguish between proof as an object, process, concept or tool? Similarly there is the process of conjecturing and the object: the conjecture. The relationship is complex. We began in the working group to explore all aspects: object, process, concept, tool and we saw some examples related to teaching, but there is much more to be done here.

Can we be more explicit about conjecturing and about our confidence in our conjectures? One can conjecture without making a conjecture, just as one can prove with making a proof. Maybe it helps to make students aware that a conjecture is to have a certainty score of 8, they are certain but not sure. Proving is looking for something that changes the 8 to 10. And maybe teachers should start by making clear to students that conjecturing is something other than mere guessing.

Out of the discussions of conjecturing came two more important themes. First, how might we teach the kind of creativity needed to arrive at *l'astuce,* at the clever insight? Second, the feeling of ownership of conjectures is important, and can sustain engagement in problem solving, even if the students do not arrive at the proof. They still engage in the process, they still need curiosity and critical thinking.

Is teaching proof related to teaching critical thinking more generally? History tells us that proof emerged at the same time as democracy, to control the discourse. And a link between teaching proving and participation in argument in social contexts is often claimed. Fawcett's (1938) work is an example where someone taught for transfer to general critical thinking and succeeded, but does typical proof teaching do this?

Finally, there is the deep challenge of teaching what is, in the end, a feeling. What makes a proof a proof, and makes proving a worthwhile activity, is the feeling of security it brings. This feeling is a personal one. How could this profound sense of mathematical truth be learned by school students? How can they become aware of this feeling?

Comment faire vivre à l'élève ce sentiment, cette satisfaction particulière et profonde apportée par la « vérité mathématique » comme elle l'est à travers la preuve ? Comment faire vivre à chaque élève ce que Georges Glaeser a appelé son « miracle grec » ? La distinction entre preuve formelle et preuve personnelle pourrait être liée à cette question. Insister sur le formalisme, ou même simplement sur un format spécifique de preuve, pourrait entrer en conflit avec cette appréhension personnelle de la preuve. On peut penser que la phase d'appropriation personnelle doit venir d'abord. Or, la phase publique doit elle aussi entrer en ligne de compte et là, le formalisme prend de l'importance.

But this public stage comes with conflicting requirements of conciseness and explanation that pull in opposite directions. Symbols allow concise communication, but can obscure meaning. Explanation is often a better means to convey the personal feeling of truth, but can be lengthy and lacking in precision. How can the interplay between the personal feeling of truth and the public requirements for conciseness, accuracy and rigour, be handled? From a slightly different perspective, as Durand-Guerrier et al. (2012) put it, is it possible to make students familiar with the openness of exploration which comes with flexible methods of validation such as (informal) argumentation, and at the same time make them aware of the strict usage they must ascribe to words, symbols and formulas when writing a deductive text to be shared? Moreover, how can teachers help students face these contradictory requirements, and make them aware of the way *when* is related to *how*?

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THE ROLE OF TEXT/BOOKS IN THE MATHEMATICS CLASSROOM

LE RÔLE DES MANUELS SCOLAIRES DANS LA CLASSE DE MATHÉMATIQUES

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INTRODUCTION

(en français en suite)

There is a North American tradition of 'leaning' heavily on the textbook in mathematics classes—for the sequencing and introduction of topics, and provision of examples and exercises. In some sense, the prescribed mathematics textbook is often taken to *be* the curriculum. This is in contrast to literature classes (which seldom have a single textbook), or art or music classes (rarely using textbooks at all).

Textbooks are changing though, and are already starting to look more like iPads than the familiar dog-eared, graffitied, duct-taped bundles of paper. This leads us to reconsider the nature and future of text/books in the mathematics classroom. Who might write them, and who read them? How do textbooks address their audience? How are they designed? What intended purposes do they serve, and what are their unintended effects? How have, and how will, textbooks change over time?

In this working group, we considered text/books broadly, including linguistic, multisensory, design and contextual aspects of texts. Our aim was to gain a deeper understanding of the cultural phenomenon of mathematics textbooks, their history and future. Topics included:

- ways to undertake discourse analysis, textual analysis and genre analysis of textbooks;

- the agency and positioning of text/books: how the text reads the reader as the reader reads the text;
- archaeology and genealogy of textbooks: considering historical textbooks (for example, Robert Recorde's books, and a math textbook from Franco's Spain) and proto-textbooks (the Passover *Haggadah* and other ancient works as teaching texts);
- sociocultural and political entailments of textbooks;
- teacher- and student-made open source electronic textbooks; and
- experiential texts (for example, a school garden as mathematics text).

♦♦♦♦♦♦♦♦♦♦

Il ya une tradition nord-américaine de « pencher » fortement sur le manuel scolaire dans les classes de mathématiques—pour le séquencement et l'introduction de sujets, et la fourniture d'exemples et exercices. Dans un certain sens, le manuel scolaire prescrit en mathématiques est souvent considéré équivalent au curriculum-même. Ceci est en contraste avec les classes de littérature (qui utilisent rarement un seul manuel), ou des cours d'art ou de musique (utilisant rarement les manuels du tout).

Les manuels changent cependant, et ont déjà commencé à ressembler davantage à des iPads que les familiers liasses de papier, écornés, couverts de graffitis et réparés avec du ruban adhésive. Cela nous amène à reconsidérer la nature et l'avenir des manuels scolaires à la classe de mathématiques. Qui doivent les écrire, et qui les lire ? Comment les manuels s'adressent à leurs lecteurs imaginés ? Comment sont-ils conçus et désignés ? Quelles sont leurs effets, intentionnels et non intentionnels ? Comment ont les manuels déjà changés, et comment vont-ils changer au futur ?

Dans ce groupe de travail, nous avons examiné les manuels des divers points de vu, incluant le linguistique, le multisensoriel, la conception et les aspects contextuels de textes. Notre objectif était de mieux comprendre le phénomène culturel des manuels de mathématiques, leur histoire et leur avenir. Les sujets abordés comprendront les suivants :

- les moyens d'entreprendre l'analyse du discours, l'analyse textuelle et l'analyse du genre des manuels scolaires ;
- l'agence et le positionnement des manuels : la façon dont le texte se lit le lecteur en même temps que le lecteur lit le texte ;
- l'archéologie et la généalogie des manuels scolaires : les manuels scolaires historiques (par exemple, de Robert Recorde, et un manuel de mathématiques de l'Espagne de Franco) et proto-manuels (la *Haggadah de Pesach* et autres vieux œuvres vus comme des textes d'enseignement) ;
- les inférences socioculturelles et politiques de manuels ;
- enseignants et élèves comme producteurs des manuels électroniques 'open source' ;
- les textes expérientiels (par exemple, un jardin de l'école considéré comme texte mathématique).

STRUCTURE OF THE THREE DAYS

We structured our sessions in roughly the following way:

<u>Day 1:</u> The **archaeology of mathematics textbooks**: Introduction and a consideration of the history of mathematics textbooks, with opportunities to analyze historical examples.

<u>Day 2:</u> A consideration of the **present and future of mathematics textbooks**, including printed books, ebooks, online 'books' and alternatives to the idea of textbook in mathematics classes.

<u>Day 3:</u> Trying our hand at **designing a textbook**, 'anti-textbook' or other resources for mathematics teachers and learners.

Working group leaders set up a blog for the working group at http://textbooksmanuelswg.blogspot.ca with sample pages from textbooks from other places and times, interesting articles on topics we would discuss, links and other resources. Throughout our sessions, participants and leaders added further posts that reflected the lively discussion of text/books in mathematics teaching and learning. In many ways, the resulting blog constitutes a report of the activities of our working group.

DAY 1: THE ARCHAEOLOGY OF MATHEMATICS TEXTBOOKS

Our discussion started with a consideration of sample pages from the following textbooks, posted to the working group blog:

- Robert Recorde, *The Grounde of Artes* (England, 1543/1632) and *The Whetstone of Witte* (1557), some of the first mathematics 'textbooks' in English
- Swetz & Katz's *Mathematical Treasures*, online catalogue of historical mathematics books (available at http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=2591)
- Legendre, *Elements of Geometry* (1835), English edition
- Wentworth, *First Steps in Algebra* (USA, 1894)
- Boyden, *A First Book in Algebra* (USA, 1895)
- Rivenburg, *A Review of Algebra* (USA, 1914)
- Shaw Business School's, *Common and Foreign Exchange Arithmetic* (Canada, 1920s)
- *Social Arithmetic* (USA, 1926)
- *Enciclopedia Practica del Parvulo* (Spain, 1954 during Franco's regime)
- *Problemas Salvatella* (Spain, 1950s)
- Sharp, *A Parent's Guide to the New Mathematics* (USA, 1964)
- Dolciani et al., *Modern Algebra and Trigonometry* (USA, 1963)
- Hamel & Lunkenbein, *La Mathématique à l'École Elémentaire Renouvelée* (Canada, 1972)
- Jacobs, *Mathematics: A Human Endeavour* (USA, 1970)
- Alexander & Klassen, *Mathquest 8* (Canada, 1988)
- Cossette-Marin & Trépannier-Moskal, *Mathématèque 5* (Canada, 1991)
- Stocker, *Math that Matters* (Canada, 2006)
- McAskill et al., *Precalculus 11* (Canada, 2011)

We also considered examples of online 'textbooks' (mostly at the undergraduate level), alternatives to traditional mathematics textbooks (for example, David Eugene Smith's *Number Stories of Long Ago*; Hans Magnus Enzenburger's *The Number Devil*; Anno's *Mysterious Multiplying Jar*; and other mathematical picture books, books of puzzles, and so on). We considered other non-traditional candidates that might act as 'textbooks' as well—for example, the Jewish Passover *Haggadah* as proto-textbook (viz. Alan Block's article, *Even if we were all scholars*), and the school garden as a mathematics textbook (as per Marta Civil's article on this topic (http://math.arizona.edu/~cemela/english/content/workingpapers/MCivil-CommunKnow-Equity-long.pdf)).

Based on our group's consideration of these diverse examples of mathematics textbooks and alternative textbooks from widely varied historical contexts, the two working group leaders began the discussion with the following burning questions:

PETER APPELBAUM'S BURNING QUESTIONS

i. **Identifying textbooks as opposed to something else**
What is a textbook, and how is that different from using something as a text with students?

ii. **The intransigent nature of the text as codification of the discipline**
Despite the documented successes of non-standard uses of textbooks (e.g., students critiquing rather than using, re-writing,...) and disruption of the authors and audiences for the texts (students making their own textbooks for themselves and others, different groups of students comparing different texts as resources for their own learning,...), the place of the textbook and its positioning of teachers and students persists. Why, how, and in whose interests?

iii. **Textbooks are sites of work-life struggle**
Text materials are commonly used to de-skill teachers (they become cogs or conduits rather than designers of experience). When teachers approach their work creatively, responding to the lives of their students, this is often considered inappropriate. What are the implications?

iv. **People need to see the textbook to know what the curriculum 'is'—or else?**
Isn't anything and everything in one's lifeworld a mathematics textbook? Why does this not so easily translate into understanding everyone as a mathematician whose 'inner mathematician' has been thwarted from expression by school mathematics?

Peter's questions were summarized by the working group as follows:

i. What is a textbook versus using something AS a text with students?
ii. Despite all the creative alternative ways one could use a textbook, the textbook persists in positioning teaching and students ... why? how? in whose interests?
iii. How can textbooks be constraining AND enabling for teachers?
iv. How might textbooks prevent us from seeing our life-world as text?

SUSAN GEROFSKY'S BURNING QUESTIONS

(related to 'textbooks as pedagogical genre')

i. **Intentionality and uptake of the genre**
How does the mathematics textbook genre format math classrooms, math students and teachers, math learning and teaching? What is the formatting power of the textbook?

How does the existence and use of textbooks affect the relationship between other mathematics 'resource books' and math textbooks?

How do students learn to read/not read math textbooks? (For example, does anyone read the preamble or introduction for the reader? Are students aware of the topics of units of work—do they read the table of contents or chapter headings? How do students read word problems?)

How do teachers learn to read/not read/ use mathematics textbooks? Is mathematics teaching 'meant to be' performing the textbook?

ii. **Addressivity of the genre**
What is the *addressivity* of a math textbook—how does it position its imagined audience?

Where is there space for learners and teachers to enter the world created by a mathematics textbook?

iii. **Chronotope of the genre**
Why do math textbooks tell us so much about their *chronotope* (i.e., setting in time, place, culture) rather than about mathematics?

iv. **Archaeology of the genre**
What other things are math textbooks like? What do they remind us of?

How did the math textbook genre develop/emerge historically? Do old math textbooks or those from differing cultural contexts seem the same or different?

Here are some of ideas and issues raised in our discussion of the archaeology of mathematics textbooks on our first day:

- Are textbooks controlling? Do they do a disservice to teachers and to teacher autonomy?
- Could textbooks be designed to be more like a novel, a movie or a work of art? Most good stories and movies start in the middle, in the midst of the action; most textbooks don't do this, but plod along sequentially, and thus do not hold readers' interest.
- Must textbooks be 'full of truth'? Is it possible for a textbook to be progressive and still be sold widely (and make money)?

Several members of the working group were textbook authors themselves. They commented variously that:

- It might not be fair to 'attack something that cannot defend itself' (i.e., the textbook). No one says that you have to read a textbook cover to cover, or that you have to cover everything in it. A huge amount of work and thought goes into writing textbooks, and this is at odds with the ways they are actually used—and not often really read.
- When a textbook is written by the person teaching the course, the relationship to the textbook is different (and more intimate).

Others commented that:

- Textbooks help create a confusion between teaching and learning, encouraging teachers to believe that once they have 'covered the textbook', the students have learned the mathematics. Isn't the ultimate textbook the 'mental textbook' that learners bring with them once the course is done—a personal 'textbook' that makes connections with their own experiences?
- Are math textbooks (and math courses) meant to be enjoyable, or simply the means to a goal—like all of schooling?
- Textbooks can be seen as the meditational body for the curriculum.
- 'Textbooks don't speak for themselves', and cannot substitute for teachers' own creativity, energy and choices of examples that ought to respond to the needs of a particular group of students.
- Are textbooks written primarily for teachers or for learners?

- When rich problems and mathematical activities are published in a textbook (or worksheet), do they become less compelling and more routine? Do they become 'worksheet-ized' or 'textbook-ized'? In other words, does the genre take over and remove a sense of surprise, energy or excitement?
- Could a textbook be used subversively, critically or artistically in a classroom? What could you do with a textbook other than what people usually do? For example, could a class rewrite the textbook, critique it, or turn it into the basis for artistic production (a puppet show? a story?) Could we resist the tendency of textbooks to deskill teachers (as educators, referring to De Certeau have pointed out) by engaging such tactics?

DAY 2: TEXTBOOKS, PRESENT AND FUTURE

In our second day's discussion, we spent some time trying to make distinctions between textbooks and other kinds of materials that might be used in mathematics classrooms—for example, reference books, mathematical novels and picture books, and other resource books like collections of mathematical puzzles.

The sense of 'textbook as genre' was complicated by a consideration of new media forms of 'textbooks', including ebooks produced by textbook publishers, ebooks produced by a particular teacher or group of teachers, interactive wikis with teachers, students, parents and others as contributors, and so on.

Here are some of the ideas that emerged as we struggled to characterize mathematics textbooks in relation to other related genres:

- **A textbook must carry pedagogical intentions or a pedagogical aim**. This is in contrast, for example, to reference books (that contain 'true facts', but may not be pedagogical) and children's books with mathematical themes (that have a primary intention of engagement, storytelling and entertainment, even though they may also teach).

- Typically, **this pedagogical aim is integrated into the textbook** by the author(s). However, if the author(s)' pedagogical aim is comprehensive, there may be no room for the teacher to insert her/his own pedagogical aims, and the textbook may serve to 'domesticate' and deskill teachers (as per De Certeau). Optimally, a published textbook ought to leave some space for the integration of the teacher's own pedagogical aims.

- We came the conclusion that **any book** (or work in another medium: a website, film, etc.) **may be used as a textbook by a teacher who brings pedagogical aims to the work**. For example, literature teachers use published novels, poems and other literary works as textbooks, and similarly, mathematics teachers may use mathematically-themed novels, films, puzzles or any works with rich mathematical content as textbooks.

- We noted **the power of genre to mold people's intentions to conform to the needs of the genre**. Even when we work consciously to disrupt and interrogate canonical textbooks, in our roles as teachers and textbook writers, there are certain kinds of typical expectations and uptake that may play out nonetheless. The genre itself has a formatting power, and the intentions encoded in the genre may take over and overpower the authors' intentions.

We left with a question: How might textbooks as a genre prevent us from treating our own lifeworlds as a text?

DAY 3: DESIGNING A TEXTBOOK (OR 'ANTI-TEXTBOOK')

On our third day, having looked at historical and contemporary print-medium and electronic textbooks (and having tried out the iBook textbook authoring software for the iPad), we took on the task of working in groups to design a sort of textbook or anti-textbook that we liked. We decided to take the mathematical theme of area as a common theme if needed.

As a preamble to our design work, we discussed the approaches we might take. Our discussion touched on these points:

- What is *not* included—the spaces and gaps—are as important as what is included in a textbook. One member of the group asserted that Wikipedia was the best textbook available, both in its inclusions and in what is missing or omitted. Wikipedia leaves gaps for the reader to fill in, and because it is a participatory project, always in progress, the reader may decide to add contributions to the wiki. Another group member recalled a mathematics textbook from the 1970s that left at least a third of each page blank for students to comment and fill in their remarks; the book with the student comments became the final book.
- We noted that online textbooks have the capacity to make a collection of meta-comments (in the form of reader commentary, reviews and links to other sites) part of what the textbook becomes over time. The accretion of online commentary recalled textual traditions of commentaries-upon-commentaries—for example, case law commentaries in legal traditions, or Talmudic traditions of scriptural commentary.
- Through wikis, commentaries, the addition of links, etc., online textbooks allow for collaboration among author(s), artists, teachers and students (and other readers). A group member suggested that in a situation like this, the teacher may act in the role of curator, selecting and controlling the abundant or even overwhelming resources available to students. Such a relationship to textbook resources would change the relationship among teachers, texts and students; teachers might need to loosen control, allowing others to collaborate on the huge task of curatorship with such a wealth of resources available.
- Electronic and online media allow for new resources to become part of the 'text'—for example, teacher-made (or commercially available) videos of explanations, lessons and worked examples can be used with, or in place of, explanatory written texts. In fact, the newly-popular idea of the 'flipped classroom' is built on the use of such videos as alternative texts.
- We noted that print textbooks do structure the power relationships within the mathematics classroom, and interactive online textbooks would/will/do restructure those power relationships. Interactivity allows students to talk with one another and the teacher online as well as in person and to connect to other sources of mathematical authority, and it may not be possible or necessary for the teacher to be privy to everything that is going on with the students' learning (if it ever was!). A group member drew an analogy to the way writing classes use peer editing processes, where teachers may not be part of much of the process of editing, corrections, etc.

As we started to work on designing our own 'text/books', we built on the following conceptualizations that arose from our group's discussion:

1. **Anti-textbooks:** On the model of the Anti-Colouring Books initiated in the 1970s (http://www.susanstriker.com), we conceived of Anti-textbooks, that offer some structure and guidance and ask interesting questions, but leave plenty of space for student questions, problem posing and problem solving, imagination and creativity. A particular Anti-textbook would play out somewhat differently with each class that used it.
2. **Non-commercial, teacher and student created textbooks:** We noted that there are already software applications available for teachers (and students, and parents, and…) to create collaborative e-textbooks that integrate graphics, photos, videos and links to other online resources. These make everything about textbooks negotiable: *what* they are and *why* use them, *who* makes and uses them and *how*, *where* they are made and stored, *when* they change, and so on. Commercially-produced print textbooks may already be obsolete in an environment of such changes.
3. **The form of a 'text/book':** There is no longer any reason to expect that an expensive, weighty, fixed bundle of printed pages needs to be the format for a textbook. Since the textbook genre has been closely associated with its physicality and the technologies that surround it, our consideration of new forms will call everything about textbooks into question.

Here are some of the suggestions that we came up with for new kinds of text/books for mathematics:

Textbook as an electronic and/or physical series of 'boxes' of resources

Using a container metaphor, educational *agents* (teachers, students, parents, etc.) could add resources to boxes organized by grade level and topic or theme. Resources could include guidelines to explorations, inquiries, activities, as well as problems, links to websites, videos, and so on.

This textbook would be dynamic and changing, as teachers and learners added to the boxes, included their findings, developed new thematic boxes and used (or didn't use) particular resources. Students would not be restricted to the boxes at their grade level, but could move back and forth among levels to work with a theme across grade levels.

We noted that the teachers' role could include curating those boxes, so that there is thought and attention given to the ever-changing contents. The role of curator or editor is an important one to avoid the accretion of outdated or unwanted materials.

We also noted that the container metaphor ('boxes') is only one possible metaphor that could be used to structure this kind of textbook. We considered an alternate rhizomatic or network structure, and talked about the idea of a gift or offering (on the model of Froebel's Kindergarten 'gifts').

Triangle(s) as textbook

A more radical approach to rethinking text/books as resources had a triangle (see Figure 1), or perhaps a set of different kinds of triangles, as textbook that would work for the whole K-12 mathematics curriculum!

It was suggested that a set of varied triangles (exemplifying right angle, isosceles, obtuse-angled, equilateral, etc.), made of a durable material, could serve as a mathematics kit for students learning topics from arithmetic to 2-D and 3-D geometry to algebra and trigonometry. With a triangle kit as textbook, teachers would not be deskilled (as is possible with our present conception of textbooks); rather, it could actually help to develop the skills of teachers. Taking this radical stance towards textbooks would certainly affect mathematics teacher education, and would encourage teachers to see themselves as makers and artists, rather than as mechanical implementers of a pre-set series of lessons.

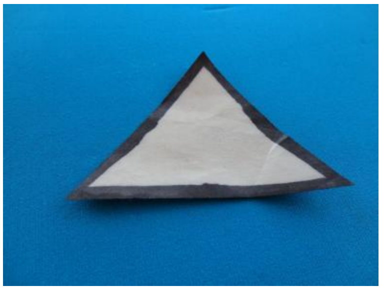

Figure 1

The Triangle Textbook could, of course, coexist with the use of online, multimedia resources and reference materials, but would necessitate a hands-on approach and a rethinking of the connections among topics, as well as a remaking of the teacher and students' roles.

CONCLUSION

At the end of our working group, we left several open questions:

- Does the idea of a fluid, interactive textbook sidestep the need for us as educators to take responsibility and make a commitment to our words and ideas, as we must do on the written page?
- Is there a role for experts or expertise in a participatory, wiki version of textbooks?
- What do we mean by *curation* of textbooks or other resources? How do curators make their decisions (in museums, art galleries, etc.), and on what basis? What can we learn from traditions of curating and editing for this new context?
- How would these ways of rethinking textbooks affect other power structures including the standardization of curriculum, tests and exams (and other forms of assessment and evaluation), and the form of K-12 schooling more generally?
- Are math teachers (novice and experienced) ready to be artists and co-authors? Are math students ready to be thinkers, questioners, inquirers and co-authors of their own course of education?

PREPARING TEACHERS TO DEVELOP ALGEBRAIC THINKING IN PRIMARY AND SECONDARY SCHOOL

PRÉPARER LES ENSEIGNANTS AU DÉVELOPPEMENT DE LA PENSÉE ALGÉBRIQUE AU PRIMAIRE ET AU SECONDAIRE

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[*The bilingual composition of our group allowed for rich discussions in both, English and French. Notre rapport est ainsi rédigé dans les deux langues, avec, tout d'abord, la synthèse en anglais, followed by one in French.*]

OVERVIEW

During the 1990s, an international movement, known as *Early Algebra* was created to change the teaching of algebra in school. Instead of preparing students to learn high school algebra, the *Early Algebra Group* developed new approaches to enriching mathematical content taught at the elementary level by creating opportunities for young learners to develop algebraic thinking based on specific mathematical notions and concepts, such as operation, equality, equation, pattern, formula, property, variable, and variation. The *Early Algebra* movement had an important influence on new curricula implemented in the early 2000s in many countries and Canadian provinces. For instance, in Ontario, from Grade 1 to Grade 8, *Patterning and Algebra* is one of five mathematical strands that frame the curriculum. The situation is similar in New Brunswick (French sector) where this domain is called *Patterns and Algebra* (*Régularités et algèbre*). While studying patterns, students develop their abilities to generalize, one of the central components of algebraic thinking. However, for many teachers this approach presents several challenges.

In our working group we generated a discussion by reflecting on the following questions:

1. How do we prepare teachers, pre-service and in-service, to meet this challenge? We recall Blanton and Kaput (2003) saying the elementary teacher must develop an 'algebraic eye and vision'. How could this be done within initial training and ongoing professional development of teachers?
2. What are the difficulties experienced by teachers who are trying to develop algebraic thinking in Grade 1-8 students (primary and middle school)? How can these teachers be supported in overcoming these difficulties?
3. In general, what can teachers draw from the research on the development of algebraic thinking to inform their practice connected to teaching and learning algebra?

The **first day** of our work was devoted to the clarification of the question: What do we mean by the terms *algebra* and *algebraic thinking*? We also did a tour of curricula of the provinces of our participants to examine the trends in the developments of algebraic thinking.

The **second day** was spent in discussing the question of the training, the initial and professional development of teachers, related to algebraic thinking.

During the **last day**, we focused on deepening our understanding of the issues and challenges related to the development of algebraic thinking in both teaching practice and teacher (pre- and in-service) education. Some promising paths in research and practice were formulated by participants.

SOME HISTORICAL BACKGROUND AND EMERGENCE OF *EARLY ALGEBRA*

Historically, algebra as a science appeared after arithmetic (in the 9th century with Al-Khawarizmi (see also in Smith, 1958; Rashed, 1984)). Furthermore, the idea of learning algebra after arithmetic has dominated the thinking of the mathematics education community for quite some time. Learning arithmetic as a prerequisite for learning algebra follows the logic of a didactical argument of continuity in mathematics teaching, the dialectic of building something new on the foundation of something known. But for students, passing from the stage of arithmetic to the stage of algebra appears to be far from an easy task (Vergnaud, Cortes, & Favres-Artigue, 1988; Kieran, 1992; Rojano, 1996). Also, spending too long learning arithmetic can be an additional obstacle for the students in learning algebra (Booth, 1988; Squalli, Dumont, & Tanguay, 2002). The idea of teaching algebra in early grades, in parallel to arithmetic, has been advanced by a number of researchers (Vergnaud, 1988; Davis, 1989, Bodanskii, 1991; Kaput, 1995). Despite being considered unrealistic by some and not reasonable by others, the idea is now adopted by a larger number of mathematics educators (Squalli, 2003).

Educators who have reservations about the early introduction of algebra argue that a solid foundation in arithmetic is necessary before tackling algebra. They believe that algebra can only be studied in secondary school once students have had sufficient experience of arithmetic, the foundation on which algebraic knowledge will be built. In any case, they argue, secondary students find algebra quite difficult so how can we think of having younger students manipulate *x* and *y*!!!

Those who are in favour of early algebra emphasize that algebra in early grades should not be perceived as a precocious version of the high school algebra; neither should it be seen as a preparation for high school algebra (pre-algebra). It is rather a strategy of enriching elementary school content with opportunities to deepen mathematical notions and concepts, including operation, equality, equation, pattern, formulae, property, variable, and variation, among others.

Lesley Lee reminded participants that in the 90s, Jim Kaput founded an informal group to reflect on teaching algebra (Kaput, Carraher, & Blanton, 2008). The group involved many researchers who drew their discussion from the two following premises:

- There are a significant number of students who fail secondary school algebra; however, there are studies demonstrating that even children in the beginning of their schooling have capacities for abstraction and pattern identification.
- School does not help students to develop algebraic thinking if it is focusing mainly on arithmetic based on doing calculations.

The idea was then to support the development of abstract reasoning starting in kindergarten, in order to avoid hitting an 'algebra wall' at the beginning of secondary school. But this idea met with a variety of objections:

1. Teaching must begin with the concrete and then move to the abstract. Primary students are not ready for abstraction.
2. One must begin with arithmetic.

However, there are experiences of teaching algebra prior to secondary school, even prior to arithmetic; for example, a Hawaiian group where teaching algebra without numbers started in kindergarten, or Davydov's work in Russia with teaching algebra in the primary grades, as mentioned by Viktor Freiman (Bodanskii, 1991; Carraher, Schliemann, Brizuela, & Earnest, 2006). We also saw evidence of young students developing algebraic reasoning through a research presentation by Ruth Beatty. She discussed her research in elementary classrooms and provided video data evidence of students in Kindergarten and Grades 4 and 6 working with coloured tiles and being able to generalize patterns, make predictions, and in the upper grades, graph the relationships found. The data clearly illustrated students' algebraic thinking, moving beyond discussing "what comes next?" to being able to generalize the pattern to describe the general term.

After having explored some activities, the participants arrived at a shared vision of what we mean by algebraic thinking, namely:

- **Learning to operate with an unknown quantity (analytical reasoning).** This aspect of algebraic thinking was underscored by many researchers.

  In a problem solving context, this algebraic reasoning involves considering an unknown quantity and operating with it as if it were known. This reasoning does not emerge spontaneously in students, who often opt for synthetic reasoning in order to find an unknown by operating only with known values.

- **Learning to generalize.** Generalization is an important process in mathematics and particularly in algebra. It may focus on the formulation and justification of different types of patterns of functional relationships, by using and navigating between diverse modes of representation: verbal, numeric, graphical, syncopated language. It can also consist of modelling concrete situations of co-variation (for example, by using manipulatives, like small cubes, students can build bigger cubes and question the relationship between a number of small cubes and the lateral surface area of a big cube while taking the face of a small cube as unit area), or more abstract ones (like a function machine). In this sense, many participants underlined the importance of using manipulatives which allow students to make sense of ideas of variable, variation/constancy, co-variation, functional rules, etc., and facilitate students' verbalisation, particularly expressing general rules and relationships.

SOME ISSUES RELATED TO TEACHER TRAINING AND PROFESSIONAL DEVELOPMENT

The development of algebraic thinking in students before secondary school is a challenge for teachers. The discussion pointed at some issues in preparing teachers:

- One of the issues is to support teachers in professional development, either initial or continuing, to go beyond their own school-years-related experiences with algebra to build a new vision of it. To do so, we need to have them live these new diverse algebraic experiences. Teachers' views and definitions of algebra will strongly influence how they teach algebra.
- Another issue consists in allowing students to extend their thinking through concrete manipulations with material and reflection. For example, tasks may be used in such a way that students anticipate or predict in situations where they cannot answer by counting objects (their number is insufficient); students can then reflect on algebraic representations in formulating the answer (in general form) and ask new questions arising from those reflections. Hence, the teacher helps students to reflect on their physical actions with manipulatives by moving towards more abstract representations of such actions.
- The teacher must use language that expresses abstract ideas in a concrete way, from general to specific, by helping students to move gradually to more formal language. The language issue seems to be crucial for the development of algebraic thinking. It is complex and does constitute an important issue for teachers. The teacher should propose different forms of expression (verbal and written) to students to serve at the same time as support to express their ideas and reasoning in the process of construction, but also as the object of reflection and mathematical reasoning. The language is introduced to help students' thinking and not to force them to think in language imposed explicitly.
- The teacher should be capable of making the trajectory of algebraic thinking clear and transparent. This trajectory may start with the study of concrete situations that lead to algebraic thought.

SOME IDEAS OF ACTIONS TO HELP TEACHERS DEAL WITH THESE ISSUES

Regarding the issues identified in the previous section, the group asked the question: What can we do as mathematics educators? Some of following ideas were put forth:

- The idea of having more challenging and enriched curricula, and thus extending challenges to all students in the mathematics classroom, has been articulated in the work of Linda Sheffield and emerged from an NCTM Task Force on mathematically promising students (NCTM, 1995; Sheffield, 1999, 2003). In Québec, in the context of the *Challenging Mathematics Curriculum* (*Défi mathématique*), Viktor Freiman has developed a situation approach which, aimed originally to identify and foster mathematical giftedness in early grades, can be made accessible to all students, as it is based on investigation of open-ended problems that lead students to generalization and abstraction (like, for example, the handshake problem) (Freiman, 2006, 2010).

- A new US-based project M^3, *Mentoring Mathematical Minds* (Gavin, Chapin, Dailey, & Sheffield, 2006-2008), is one of the latest initiatives which builds algebraic thinking on work with patterns. A special unit called *Awesome Algebra: Looking for Patterns and Generalizations* encourages students to study patterns and determine how they change, how they can be extended or repeated and/or how they grow. Students can then move beyond this to organize the information systematically and analyze it to develop generalizations about the mathematical relationships in the patterns with a strong focus on mathematical discourse revolving around how to verbalize a generalization. (See http://www.gifted.uconn.edu/projectm3/teachers_curriculum_3_unit4.htm; see also a paper by Lee and Freiman (2006)). A new series of books, *From Patterns to Algebra*, has been developed by Beatty and Bruce (2012) which provide activities to engage elementary students in algebraic thinking.

- In several provinces, algebra is explicit in the elementary curriculum, but not in all. In Québec, since algebra is not explicitly taught in the curriculum, we can talk about developing a *sense of operations*. In fact, primary teachers perceive the importance of activities that envisage the development of algebraic thinking when they, themselves, explore those types of activities. They see an opportunity to enrich the learning experiences of their students by moving beyond mere calculations towards the development of a *sense of operations*. However, some remain reluctant to use the terms *algebra* or *algebraic thinking*—algebra is not yet a part of the mathematical culture at the elementary level in Québec.

- Encourage collaboration between teachers. This collaboration can take different forms: collaborative work in planning lessons, in-classroom observations of co-planned lessons where one teacher teaches and the other makes observations, team teaching, or video-lesson study. It is important to open the walls of teachers' isolation and facilitate sharing their teaching.

- Think about models of team work that might include working across elementary and secondary boundaries, or across provincial boundaries.

- Make curriculum more explicit around big ideas in ways that bring teachers to understand what is meant by algebraic reasoning (beyond giving them a few examples).

In relation to research, some of following paths have been suggested:

- What is algebraic thinking? What are the connections with geometry or other strands of mathematics?
- Research on teachers who are working on developing algebraic thinking in their students in a regular classroom.
- Research on curricula, textbooks, and teachers' beliefs.
- Document how manipulatives are used by teachers.

♦♦♦♦♦♦♦♦♦♦

[*Ensuite, la synthèse de nos discussions en français (le texte n'est pas nécessairement une traduction directe de la partie en anglais).*]

DESCRIPTION INITIALE DU THÈME

Durant les années 1990, un mouvement international a eu lieu pour réformer l'enseignement de l'algèbre à l'école. Ce mouvement a donné lieu au courant *Early Algebra*. Il est vu non comme une préparation à l'algèbre enseignée au secondaire mais comme une stratégie pour enrichir les contenus mathématiques enseignés au primaire, en offrant aux élèves une opportunité de développer la pensée algébrique et approfondir davantage certaines notions et concepts mathématiques (les concepts d'opération, d'égalité, d'équation, de régularité, de formule, de propriété, de variable et de variation, entre autres). Ce courant international a influencé les curriculums de mathématiques mis en œuvre au début des années 2000 dans plusieurs pays et provinces canadiennes. Ainsi, par exemple, dans le programme-cadre de mathématiques de l'Ontario de la 1^{e} à la 8^{e} année, la modélisation et l'algèbre est un des 5 domaines d'étude qui structurent ce programme. La situation est similaire au Nouveau Brunswick (secteur francophone) où ce domaine est appelé *Régularités et algèbre*. L'étude des régularités est utilisée comme un contexte pour développer l'habileté des élèves à généraliser une composante centrale de la pensée algébrique. Au Québec, l'accent au primaire est mis sur l'enseignement de l'arithmétique. Les idées algébriques y émergent au fur et à mesure, également sur la base de généralisations de relations entre les nombres et une étude de patterns, sans toutefois entrer dans le langage formel (symbolisme) de l'algèbre. La dernière est introduite de manière plus explicite au 1er cycle du secondaire (7-8 années). Peu importe le contexte, le consensus entre les participants était que la mise en œuvre dans la pratique d'enseignement de cette approche est un grand défi, tout particulièrement pour les enseignants du primaire.

Dans ce groupe de travail nous avons organisé la réflexion autour des questions suivantes:

1. Comment préparer les enseignants, en formation initiale et continue, à relever ce défi? Selon Blanton et Kaput (2003), l'enseignant du primaire doit développer « un œil et un regard algébriques ». Comment cela pourrait-il se traduire dans la formation des enseignants?
2. Quelles sont les difficultés éprouvées par les enseignants lors de l'enseignement du développement de la pensée algébrique au primaire et au début du secondaire (1^{e} année à 8^{e} année)? Comment peut-on aider ces enseignants à surmonter ces difficultés?
3. D'une manière plus générale, quels enseignements tirer des travaux de recherche portant sur le développement de la pensée algébrique en lien avec l'enseignement et l'apprentissage de l'algèbre?

La **première journée** de travail a été consacrée à clarifier la question : L'algèbre, la pensée algébrique : de quoi parle-t-on?, ainsi que de relever les orientations curriculaires en lien avec le développement de la pensée algébrique dans les différentes provinces canadiennes représentées dans le groupe de travail (Alberta, Ontario, Nouveau-Brunswick et Québec).

La **seconde journée** a été consacrée à discuter la question de la formation, initiale et continue des enseignants, ainsi qu'au développement de la pensée algébrique chez les élèves du primaire et du secondaire.

La **dernière journée** était consacrée à examiner les enjeux et défis que pose le développement de la pensée algébrique au primaire et au secondaire, au plan de la pratique d'enseignement, ainsi que la formation initiale et continue des enseignants. Aussi, le groupe a identifié quelques pistes de recherche en lien avec les questions discutées.

PETIT RAPPEL DE L'HISTOIRE DE L'ÉMERGENCE DU MOUVEMENT *EARLY ALGEBRA*

Historiquement, l'algèbre est apparue après l'arithmétique. Aussi, l'idée que l'apprentissage de l'algèbre doit venir après celui de l'arithmétique a longtemps prévalue dans la communauté des éducateurs en mathématiques. Prendre l'arithmétique comme prérequis à l'apprentissage de l'algèbre est aussi un argument d'ordre didactique qui met en avant la dialectique de l'ancien et du nouveau, permettant ainsi de créer, espère-t-on, une certaine continuité dans l'enseignement des mathématiques. Mais le passage pour les élèves d'un stade arithmétique à un stade algébrique est loin d'être facile à réaliser et pose problème (Vergnaud, Cortes, & Favres-Artigue, 1988; Kieran, 1992; Rojano, 1996). De plus, il apparaît que les longs apprentissages qu'ont réalisés les élèves en arithmétique peuvent venir faire obstacle à leur apprentissage de l'algèbre (Booth, 1988; Squalli, Dumont, & Tanguay, 2002). L'idée d'initier les élèves à l'algèbre dès l'école primaire a été formulée avant les années 2000 par plusieurs chercheurs (Vergnaud, 1988; Davis, 1989; Bodanskii, 1991; Kaput, 1995) et, bien qu'elle soit considérée tout à fait irréaliste par les uns, non raisonnable par d'autres, elle est maintenant de plus en plus admise chez un nombre de plus en plus grand d'éducateurs en mathématiques (Squalli, 2003).

Le premier argument invoqué par les éducateurs qui sont réticents à cette idée est le suivant : avant d'aborder l'algèbre, il faut une bonne base en arithmétique; ainsi, l'algèbre ne pourrait être véritablement abordée qu'à l'école secondaire, une fois que les élèves ont acquis suffisamment d'expérience en arithmétique, une base sur laquelle va s'ériger la construction du savoir algébrique. De toute façon, renchérirent-ils, les élèves du secondaire trouvent l'algèbre bien difficile; comment alors concevoir faire apprendre à des élèves plus jeunes à manipuler des x et des y !!!! Probablement, ces éducateurs voient l'algèbre comme une manipulation formelle et non comme un mode de pensée.

Un autre groupe d'éducateurs, moins réticents que le premier, croit que l'algèbre au primaire peut être possible, mais que cela aura comme effet d'augmenter le corpus mathématique enseigné au primaire, déjà imposant, d'augmenter l'effort cognitif demandé aux élèves et d'augmenter la charge didactique et cognitive demandée aux enseignants. Ceux-ci ne voient pas qu'ils abordent, certains aspects déjà, de manière implicite.

Ceux qui sont en faveur de cette idée s'empressent à préciser que l'algèbre au primaire ne doit pas être perçue comme une version précoce de l'algèbre actuellement enseignée au secondaire ni comme une préparation à celle-ci (pré-algèbre). Elle est plutôt une stratégie pour enrichir les contenus mathématiques enseignés au primaire, en offrant une opportunité de développer la pensée algébrique chez les élèves et d'approfondir davantage certains notions et concepts mathématiques (le concept d'opération, d'égalité, d'équation, de régularité, de formule, de propriété, de variable et de variation, entre autres).

Lesley Lee rappelle aux participants que dans les années 90, Jim Kaput a mis sur pied un groupe informel de réflexion sur l'enseignement de l'algèbre. Ce groupe réunissait plusieurs chercheurs, la discussion partait des constats suivants :

- Il y a un grand taux d'échec en algèbre au secondaire, alors que des recherches montrent que les enfants même au début du primaire ont une capacité d'abstraction et d'identification de régularités;
- L'école n'aide pas les élèves à développer ce type de pensée, mais insiste sur l'arithmétique basée sur le calcul.

L'idée fut alors de soutenir le développement de ce raisonnement abstrait dès la maternelle pour réduire *le mur de l'algèbre* au début du secondaire.

Mais cette idée a rencontré de la résistance de différents types :

1. Il faut commencer par le concret avant l'abstrait, et les élèves du primaire ne sont pas encore capables d'abstraction.
2. Il faut d'abord commencer par l'arithmétique.

Pourtant, il existe des expériences d'enseignement de l'algèbre avant le secondaire. C'est le cas d'un groupe d'Hawaii où l'enseignement de l'algèbre sans les nombres commence à la maternelle. De même, sous l'influence des travaux de Davidov, Viktor Freiman rappelle l'expérience russe de l'enseignement de l'algèbre dès le primaire (Bodanskii, 1991; Carraher, Schliemann, Brizuela, & Earnest, 2006).

Après avoir exploré quelques activités, les participants de notre groupe de travail en sont venus à l'idée que développer la pensée algébrique c'est, entre autres :

- **Apprendre à opérer sur l'inconnue (raisonner de manière analytique).** Cet aspect de la pensé algébrique est souligné par plusieurs participants.

  Ce raisonnement algébrique consiste, dans la résolution d'un problème, à considérer l'inconnue comme si elle était connue et à opérer sur elle comme on opère sur les données connues. Ce raisonnement n'émerge pas spontanément chez les élèves, qui privilégient souvent des raisonnements de nature synthétique, consistant à trouver la valeur de l'inconnue en n'opérant que sur des valeurs connues.
- **Apprendre à généraliser.** La généralisation est un processus important en mathématiques et en particulier en algèbre. La généralisation peut porter sur la formulation et la justification de différents types de régularités, de relations fonctionnelles, en utilisant et en naviguant entre divers modes de représentation : verbale, numérique, graphique, langage syncopé. Elle peut aussi porter sur la modélisation de situations concrètes de covariation exploitant un matériel de manipulation (par exemple, en utilisant des petits cubes, former de grands cubes et s'intéresser à la relation entre le nombre de petits cubes et l'aire latérale du grand cube, la face d'un petit cube étant prise comme unité d'aire) ou non (comme la fonction machine : par exemple la chaîne d'opérateurs suivante, transforme un nombre de départ (input) en un nombre d'arrivée (output) : *multiplier le nombre d'entrée par deux et ajouter 10 au résultat*. Le but est de trouver la chaîne d'opérateurs). Dans ce sens, plusieurs participants ont souligné l'importance d'utiliser un matériel de manipulation qui permet de donner une signification aux notions de variable, de variation/constance, de covariation, de taux de variation, de règles fonctionnelles, etc., et offre un registre pour parler et exprimer le général.

QUELQUES ENJEUX POUR LA PRÉPARATION DES ENSEIGNANTS

Le développement de la pensée algébrique chez les élèves avant le secondaire est un défi pour les enseignants. La discussion a mis en évidence quelques enjeux pour la préparation des enseignants.

- Un des enjeux est d'amener les enseignants en formation, initiale ou continue, à dépasser leur vision de l'algèbre construite pendant leur parcours scolaire et les ouvrir à cette nouvelle vision de l'algèbre. Pour cela il est important de faire vivre aux enseignants diverses expériences algébriques.

- Un autre enjeu consiste à permettre aux élèves de se détacher du matériel de manipulation dans leur réflexion. Par exemple par des tâches d'anticipation, de prédiction dans lesquelles le matériel est insuffisant pour répondre, l'enseignant peut amener les élèves à réfléchir sur les représentations algébriques par la formulation et la réponse à de nouvelles questions qui émergent de ces réflexions. L'enseignant amène ainsi les élèves à réfléchir non sur les actions physiques réalisées à l'aide du matériel de manipulation mais sur des représentations abstraites des actions. C'est le prix à payer si l'on veut que les élèves dépassent le spécifique et prendre conscience du général.

- L'enseignant doit utiliser un langage exprimant des idées abstraites de manière concrète, parler du général à travers le particulier, tout en aidant les élèves à utiliser un langage de plus en plus formel. Cet enjeu du langage semble être crucial pour le développement de la pensée algébrique. Il est aussi complexe et constitue un enjeu important pour les enseignants. L'enseignant doit proposer aux élèves des formats d'expression (verbale et écrite) qui vont servir comme support pour exprimer leurs idées et raisonnements en construction, mais aussi pour devenir objet de réflexion et de raisonnements algébriques. Le langage est ainsi introduit comme nécessité pour soutenir la pensée de l'élève et non pour le forcer à penser avec un langage imposé a priori.

- L'enseignant doit être capable de permettre aux élèves de parcourir une trajectoire de développement de (d'un aspect de) la pensée algébrique. Cette trajectoire peut commencer avec l'étude de situations concrètes mais atteindre le monde de l'algèbre. L'enseignant doit, par exemple, aider les élèves à passer d'une vision de l'égalité comme un signal d'exécution de calculs, vers une vision de l'égalité comme relation d'équivalence; de représenter la chaîne d'opérations suivante

  Choisi un nombre
  Multiplie le par 10
  Divise le résultat par 5
  Divise le résultat par 2
  Soustrait ton nombre de départ
  Ajoute 7

  en une expression (nombre de départ) × 10 ÷ 5 ÷ 2 – (nombre de départ) + 7. L'enseignant peut alors amener l'élève à justifier la régularité (on obtient toujours le nombre 7) en exploitant la structure de l'expression et en utilisant des propriétés des opérations (puisque 10 ÷ 5 ÷ 2 vaut 1, alors (nombre de départ) × 10 ÷ 5 ÷ 2 est égale au nombre de départ, et ainsi de suite). L'élève apprend ainsi à laisser les opérations en suspens.

QUELQUES PISTES D'ACTIONS POUR AIDER LES ENSEIGNANTS À DÉPASSER CES ENJEUX

Face à ces enjeux, le groupe s'est posé la question : que peut-on faire en tant que didacticiens des mathématiques? En lien avec le volet formation, les pistes suivantes ont été proposées :

- À la différence du Québec, dans la plupart des provinces, l'algèbre fait explicitement partie des programmes d'études dès la maternelle. Au Québec, l'algèbre n'est pas dans la culture scolaire des mathématiques enseignées au primaire. La stratégie pourrait être de ne pas parler d'algèbre avec les enseignants du primaire, mais du développement du sens des opérations. En effet, les enseignants du primaire voient la pertinence d'activités visant le développement de la pensée algébrique, une fois qu'ils ont exploré ce type d'activités. Ils y voient des occasions pour enrichir les activités arithmétiques du primaire, souvent orientées vers la réalisation de calculs, par des activités visant à réfléchir sur le calcul (comme dans l'exemple précédent, l'enjeu est d'amener les élèves à justifier la constance du résultat non pas par des vérifications numériques, mais par un raisonnement portant sur la structure du calcul), de développer le sens des opérations des élèves. Cependant, certains sont réticents à l'emploi des mots algèbre ou de pensée algébrique. L'algèbre ne fait pas encore partie de la culture mathématique du primaire au Québec.

- Encourager la collaboration entre enseignants. Cette collaboration peut prendre différentes formes : travail en collaboration dans la planification de leçons; observations de classe, un des enseignants enseigne et l'autre observe la classe, *team-teaching*, visionnement d'enregistrements vidéo de leçons filmées. Il est important de briser l'isolement des enseignants et de leur faire accepter d'être observés par des collègues.

- Comment rendre le curriculum plus explicite autour de grandes idées de manière à amener les enseignants à comprendre ce qu'est le raisonnement algébrique (plus que quelques exemples)?

En lien avec le volet recherche, les quelques pistes suivantes ont été proposées :

- Que recouvre la pensée algébrique? Quelles sont ses connections avec la géométrie ainsi que dans les autres domaines des mathématiques?
- Comment les enseignants essayent-ils de développer la pensée algébrique chez leurs élèves?
- Comment l'algèbre est introduite dans les programmes et les manuels scolaires? Quelles sont les croyances et les attitudes des enseignants?
- Comment le matériel de manipulation est utilisé par les enseignants?

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Topic Sessions

Séances thématiques

COLLABORATION BETWEEN RESEARCH IN MATHEMATICS EDUCATION AND TEACHING MATHEMATICS: CASE STUDY OF TEACHING INFINITY IN CALCULUS

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INTRODUCTION

This paper studies approaches to designing instructional activities, *based on and informed by mathematics education research*, aimed at improving teaching of the concept of infinity in first-year university calculus classes. Talking about improving mathematics instruction, Artigue (2001) writes "[…] existing research can greatly help us today, if we make its results accessible to a large audience and make the necessary efforts to better link research and practice" (p. 207). That this endeavour is still far from reality is echoed in Burkhardt and Schoenfeld (2003): "In general, education research does not have much credibility—even among its intended clients, teachers and administrators. When they have problems, they rarely turn to research" (p. 3).

Although papers in mathematics education contain well-developed theoretical foundations, useful information, suggestions and insights, they stop short of discussing practical aspects of teaching—for instance, by providing content-specific suggestions or sketches of lesson plans. Case in point: there are many papers which address challenges and problems related to teaching infinity at a tertiary level—such as Tall (1981, 2001), Davis and Vinner (1986), Luis, Moreno, and Waldegg (1991), Fischbein (2001)—and yet none gives concrete suggestions which an instructor might be able to use in the classroom. Burkhardt and Schoenfeld (2003) write:

> *The research-based development of tools and processes for use by practitioners, common in other applied fields, is largely missing in education. Such "engineering research" is essential to building strong linkages between research-based insights and improved practice. It will also result in a much higher incidence of robust evidence-based recommendations for practice.* (p. 3)

The author's experience with the topic (in teaching and designing teaching activities about infinity, as well as in presenting the talk at the CMESG and in writing this paper) confirms in no uncertain terms that it is the lack of those linkages between researchers and practitioners that make it all so challenging. In Kim and Nakonechny (2012) we find an example of collaboration involving a mathematics instructor and a mathematics education researcher in designing a question on the Intermediate Value Theorem. Although the outcome (an actual test question) is less than satisfactory, the paper hints at certain difficulties involved in the collaboration between mathematics instructors and mathematics education researchers.

Needless to say, there is very little (if any) collaboration between mathematics education researchers and authors of tertiary level mathematics textbooks.

TEACHING ABOUT INFINITY: CHALLENGES

Of all flavours in which infinity appears in the tertiary curriculum, for this paper we choose the concept of a quantity approaching infinity (such as $x \to \infty$, $f(x) \to \infty$ or a sequence approaches infinity). A key component in designing learning activities on infinity (or any other math topic) is the understanding of students' cognitive models (which they start building in high school, or earlier), and the misconceptions they contain. Even after briefly scratching the surface, one realizes that this is a formidable task. Tall (1991) writes, "Advanced mathematics, *by its very nature*, includes concepts which are subtly at variance with naïve experience. Such ideas require an immense personal reconstruction to build the cognitive apparatus to handle them effectively" (p. 252).

No doubt—personal reconstruction is truly needed. High school students usually meet infinity in the context of calculating limits and asymptotes. They study rational functions with numerator 1, and use the correct fact that these functions have a vertical asymptote at every point where the denominator is zero. The incorrect modification of this fact (division by zero produces a vertical asymptote) enters the cognitive models of many students. The conflict occurs in a university calculus course, where students encounter functions such as $f(x) = \frac{x^2-1}{x-1}$ or $f(x) = \frac{e^x-1}{x}$ which do not have vertical asymptotes at the points where the denominators are zero.

Perhaps the most common misconception comes from the teaching practice by which infinity is rarely (in either high school or in university calculus) clearly defined and discussed in a *precise, well-defined context*. The following two facts are presented in high school: (1) there are infinitely many real numbers; (2) an irrational number has an infinite, non-periodic decimal representation. Not being aware of the distinction, some students qualify irrational numbers as infinite (and yet having a finite value).

Another obstacle to understanding infinity is the language, i.e., the myriad of meanings and metaphors associated with it (we mention some in the forthcoming section). Formal mathematics language, symbols and formulas, as well as their articulation, present further challenges. The fact that $\lim_{x\to 0^+}\left(\frac{1}{x}\right)=\infty$ is often shortened as $\frac{1}{0^+} = \infty$ or as $\frac{1}{0} = \infty$, and expressed as 'the division by zero gives infinity'. As well, writing $\infty + 3 = \infty$ or $4 \cdot \infty = \infty$ (without clear indication that these are statements about limits) might lead one to believe that one can use elementary algebraic operations with ∞, in the same way they are used for real numbers. Students might have a hard time trying to figure out if the symbol ∞ used in $\lim_{x\to 0^+}\left(\frac{1}{x}\right)=\infty$ is the same as the symbol ∞ used in the notation for the unbounded interval $(1,\infty)$.

One of the challenges in defining classroom activities is to precisely define learning outcomes. With this in mind, we ask ourselves—what does it mean that a student *understands* a concept or an idea, or is *proficient* in using an algorithm or applying a theorem? To make sense of this, we borrow the notion of "good enough understanding" (Reid & Zack, 2003, 2004). Although defined in the context of reading, it is not unknown in mathematics. Looking at research mathematicians (or practitioners), we realize that understanding is not a fixed state, but a process that develops over a long period of time (definitely much longer than the length of a calculus course). Understanding in mathematics includes understanding the limits of one's understanding in the process of learning. For instance, lots of effort has been put into models that try to make elementary school students 'understand' why $-(-a) = a$ for a real number a. The best that can be done then is to build a model (metaphor), which, at the time, is 'good enough'. It is not until one engages with the axioms for real numbers (in university) that one can truly see why this is true.

Students' understanding and conceptions about infinity are part of secondary to tertiary mathematics transition. In high school, infinity is presented and discussed on an intuitive level, whereas the calculations involving infinity are reduced to algorithms (e.g. finding vertical asymptotes). University instructors, however, assume that students are to an extent familiar with the concept of infinity (for instance in the context of limits).

INFINITY IN CALCULUS TEXTBOOKS

Numerous textbooks presently used in North America are heavily based on the so-called calculus reform, or reform-based learning, and so, besides definitions, we find numerous metaphors and 'explanations' which are supposed to help students develop intuitive understanding of concepts. In the case of infinity, it is these 'aids' to building an intuition that are worrisome, ineffective and sometimes make no sense. The space does not allow us to go into details, so we only briefly illustrate this point.

In the section on L'Hôpital's rule, in Anton, Bivens and Davis (2009), we read that a limit that leads to an expression involving ∞ and –∞

> *is called an indeterminate form of type ∞ – ∞. Such limits are indeterminate because the two terms exert conflicting influences on the expression; one pushes it in the positive direction and one pushes it in the negative direction.* (p. 225)

This narrative is misleading, at best too vague. For example, in the expression $\lim_{x\to 10}(x^3 - 10x^2)$, the two terms exert conflicting influences: x^3 pushes in the positive, and $-10x^2$ in the negative direction; however, the answer is zero (i.e., the limit does not lead to an indeterminate form). Hass, Weir and Thomas (2007) call the indeterminate limit $\frac{0}{0}$ (equivalent to $\frac{\infty}{\infty}$) a "meaningless expression, which we cannot evaluate" (p. 283) without supplying any rationale to explain what makes it "meaningless". Later, they use the useless term "ambiguous expression" when talking about other indeterminate forms.

TEACHING ABOUT INFINITY: OPPORTUNITIES

Both $\lim_{x\to\infty}\frac{x^2-4}{x^2-5}$ and $\lim_{x\to\infty}\frac{2x^2-4}{x^2-5}$ involve the division of ∞ by itself. However, the former is equal to 1 and the latter is equal to 2. The geometric series formula $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \ldots$ and the formula (derived from the power series expansion of *arctan*(*x*)), $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots\right)$, show that an infinite sum of rational numbers might, or might not be a rational number. These, and many other examples could be employed to convince students that the properties of infinity are not mere extensions of known properties of real numbers (such as $\frac{a}{a} = 1$ if $a \neq 0$, or the fact that a finite sum of rational numbers is a rational number). This is an ideal opportunity to face dealing with an abstract concept—we need to depart from our knowledge and intuition about real numbers and start building a new model. For instance, the fact that we can add infinitely many numbers without writing them all down is a powerful idea.

In order to clearly define what infinity we are dealing with, we need to make sure that the concepts are precisely *defined*. The necessity for, and the power of, a mathematical definition now become obvious. Students will see how the precise and clear language of a definition eliminates multitudes of meanings, inappropriate metaphors and ambiguities in their understanding. Moreover—because infinity does not behave like a real number, we need to *discover*, and then *prove* its properties. This seems to be a natural setting to start working with a mathematical proof.

BUILDING A MODEL FOR TEACHING INFINITY

The first task in building a model for teaching infinity is to separate the contexts within which infinity appears in mathematics (named "mathematical infinity") from all other contexts (named "non-mathematical infinity"); see Figure 1.

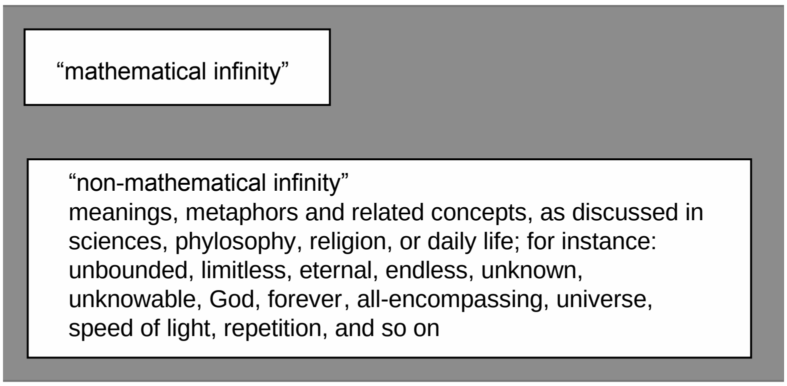

Figure 1. "Mathematical infinity" separated from "non-mathematical infinity".

Later, as needed, we will bring certain "non-mathematical" aspects or meanings of infinity into our mathematical model, in an attempt to achieve a good balance between presenting infinity as an abstract concept and in terms of appropriate representations and/or metaphors.

There is no single infinity in mathematics. Numerous misconceptions and perceptions of 'vagueness' of infinity come from thinking about it at the same time (or skipping from one interpretation to another within an argument) as a potential infinity, actual infinity, geometric infinity, and so on. Examples of such misconceptions include: infinity is the largest real number, i.e., it is the endpoint of real numbers, beyond which there are no more real numbers; infinity behaves like a real number under elementary algebraic operations; infinity is the last member of a divergent sequence; infinity is not precisely defined; it is a vague concept which cannot be fully comprehended. To address this problem, we split the mathematical infinity into its various meanings/appearances. Initially, it might make sense to think of those meanings/appearances as not having anything in common (Figure 2).

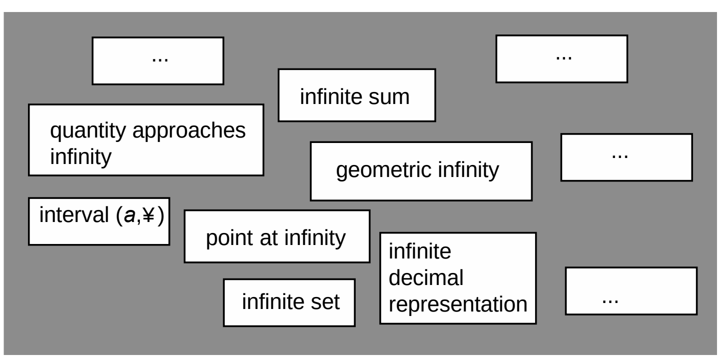

Figure 2. "Mathematical infinity" and its meanings/appearances.

Certain misconceptions and sources of confusion can now be dealt with. For instance, the fact that 0.99999… = 1 becomes clear once the infinite decimal representation of a number is given its precise meaning as an infinite series (in this case, geometric series) of numbers—and is kept away from other boxes. As well, once contexts are clearly separated and understood, we see that there is nothing paradoxical in the fact that we cannot 'reach' infinity (whatever that means; it's often heard in a classroom) but can add infinitely many numbers. Infinity is a "crystalline concept […] it is a thinkable concept, based on perception and action, increases in sophistication as we compress knowledge and link ideas" (Tall, 2011, p. 3). Thus, over time, the individual boxes/meanings grow, and we realize that they have points in common (Figure 3).

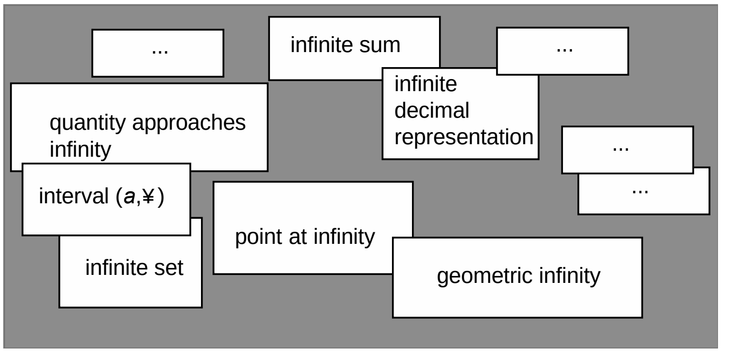

Figure 3. Concept of infinity growing up.

SUGGESTION FOR A TEACHING MODEL IN THE CONTEXT "QUANTITY APPROACHES INFINITY"

We now draft a teaching plan by outlining the steps in its development. It would take a lot more space than what we have here to fully develop each step in a complete lesson plan and to justify it based on math education research findings (when available). Exactly what format this should take on is not at all clear. This more ambitious objective is a theme for further reflection and a (definitely) longer paper.

Our strategy is to start by carefully building intuitive understanding of the concept of a *bounded function* using numeric, geometric and algebraic approaches. We assume that students are familiar with the material which, in a standard calculus course, precedes the introduction of limits (such as functions, graphs of functions, and working with inequalities). Once we establish basic intuitive notions, we embark on formalizing the language in order to write a precise definition of the statement "$f(x) \to \infty$ as $x \to \infty$". (This is where this paper will end.) Next, we would use this definition to prove certain properties of limits involving ∞. The benefits of this approach are two-fold: students immediately witness the importance of working with definitions in proving (somewhat non-intuitive) results about ∞. As well, the precision of a definition and our focus on one specific aspect of infinity will help eliminate potential sources of misconceptions in students' cognitive models of infinity.

To start, students are invited to investigate the pair of functions $f(x) = x^2$ and $g(x) = 1 - e^{-x}$. By drawing their graphs, they realize that both functions are increasing. A closer examination reveals that there is a significant difference in the way the two functions grow: it seems that the function $g(x)$ cannot grow arbitrarily large. This is confirmed algebraically: since $e^{-x} > 0$ for all x, we conclude that $g(x) = 1 - e^{-x} < 1$ for all x. The function $f(x) = x^2$ does not seem to be bounded in this way—as we follow the graph (for increasing values of x) we see that it grows larger than 1, larger than 100, larger than one million, and so on. This reasoning is confirmed by looking at the tables of values for the two functions. To further their understanding about bounded (or not) functions, students look again at the graphs of these two functions and think about relating the property of *boundedness* (not yet precisely formulated) to horizontal lines. As practice, students generate further examples of functions (represented as formulas, graphs and/or tables of values) to discuss their *boundedness* properties. We expect students to be able to articulate that *bounded* means that a "function cannot grow larger than some value", or that a "function does not increase beyond certain number".

Next, we introduce a formal definition: A function f is called *bounded from above* if there is a real number M such that $f(x) \leq M$ for all x in the domain of f The number M is called an *upper bound* for f.

Students go back to the examples of functions they identified as bounded in order to identify the value of M from the definition. This exercise will suggest that the value for M is not unique: for instance, the function $f(x) = 4\sin(x)$ is bounded from above, since $f(x) \leq 4$ for all real numbers x (so $M = 4$). However, since $f(x) \leq 5$ is true for all x as well, we conclude that $M = 5$ is another upper bound. We discover a new fact: a function which is bounded from above has many upper bounds. As a form of a 'break', we suggest a short discussion on why some upper bounds are 'better' than others. (There is no need for any kind of resolution, in the form of defining a *supremum* or otherwise). To further enforce the concept, students are given a homework assignment.

The next step presents a challenge: students are asked find examples of functions which are *not bounded from above*, and to generate a mathematically acceptable definition. (As most students are not familiar with the rules of formal logic, this is a non-trivial exercise. In the author's experience—even after introducing formal logic earlier in the course, it still takes a long time before students are able to apply it correctly in various contexts.) After examining several examples (and reviewing the rules of formal logic), we agree on the definition: A function f is not bounded from above if for every number M we can find a number x in the domain of f so that $f(x) > M$.

There are many ways (all found in textbooks) to say that $f(x)$ is not bounded from above—for instance: $f(x)$ approaches infinity, grows infinitely large, grows beyond any bound, increases without bound, can be made as large as desired, and so on. Next, students are invited to discuss why the following descriptions (again, found in textbooks), are not appropriate, or are incorrect: "$f(x)$ becomes infinitely large", or "$f(x)$ grows larger and larger". We emphasize that there is no mathematical object called "infinity" to which the function approaches ("becomes infinitely large"). Our definition of an unbounded function does not imply the existence of such an object. Instead, it communicates a property of a function which can be thought of as a dynamic process or property.

Only after we have come to this level of understanding, we introduce the notation: to say that a function f is not bounded from above, we write $f(x) \to \infty$. Next, by reiterating the way $f(x) \to \infty$ has been defined, we introduce the precise meaning of the symbol $x \to \infty$. (Again, this is a non-trivial task, as we now refer to the independent variable instead of a function.) Having spent some time working on this, we finally combine the concepts to write the definition: We say that $f(x) \to \infty$ as $x \to \infty$ if for every positive number M there is a positive number N such that $f(x) > M$ whenever $x > N$. Using notation for limits, we write $\lim_{x\to\infty} f(x) = \infty$.

Needless to say, what follows consists in designing activities which help students work with this definition. These involve both geometric approaches (identifying M and N in given graphs), and then numeric and algebraic (finding values of N, for a given value of M).

CONCLUSION

This is only a start. In the author's experience, designing a research-based lesson plan is clearly a challenging and demanding task. There are many issues one needs to address, many not even mentioned here (such as an obvious conflict between teaching experience and established classroom routines (whatever they are), on the one hand, and the models of teaching practice suggested by education research on the other). In spite of all difficulties, it is a task worth spending time and energy on.

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DIALOGUE SUR LA LECTURE DE TEXTES HISTORIQUES DANS LA CLASSE DE MATHÉMATIQUES

DIALOGUE ON READING ORIGINAL TEXTS IN THE MATHEMATICS CLASSROOM

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INTRODUCTION

Le texte qui suit tente de mettre en lumière les apports et le potentiel de la lecture de texte historiques dans le cadre de la formation des maîtres en mathématiques. Ce type d'activité a été couramment pensé et discuté dans la littérature, et ce, depuis plusieurs années. Aussi, les arguments qui supportent l'utilisation de sources premières sont nombreux et diversifiés (voir Furinghetti, 2007; Jahnke et al., 2000; Jankvist, 2009).

Sur le sujet, Évelyne Barbin (1994; 1997; 2006) est à l'origine d'un concept important, celui du *dépaysement épistémologique*.[1] Elle explique qu'introduire l'histoire des mathématiques remplace l'habituel par le différent et bouscule notre perspective coutumière des mathématiques en rendant le familier inusité. Comme cela survient lorsqu'une personne se trouve dans un contexte étranger, après une phase de confusion et de perplexité, il y a des tentatives de reconstruction de sens. Elle souligne que « l'histoire des mathématiques, et c'est peut-être sont principal attrait, a la vertu de nous permettre de nous étonner de ce qui va de soi » (Barbin, 1997, p. 21). Ce phénomène de *dépaysement épistémologique* a principalement été discuté dans le cadre de la formation des futurs enseignants ou des enseignants en service (voir Bagni, Furinghetti, & Spagnolo, 2004; Furinghetti, 2000; Lawrence, 2008).

Dans ce contexte, cet étonnement fondateur constituerait un véritable choc culturel devant mener à des compréhensions différentes des mathématiques et à un rapport nouveau à la discipline. En effet, Jahnke et al. (2000) et Barbin (1997; 2012) soulignent que ce *dépaysement épistémologique* peut mener à une « compréhension culturelle » des mathématiques, car « l'histoire inviterait à ancrer le développement des mathématiques à l'intérieur de contextes sociohistoriques et culturels » (Jahnke et al., 2000, p. 292, traduction libre). D'autre part, ce regard renouvelé amènerait un « repositionnement » des mathématiques, c'est-à-dire qu'« il permettrait de percevoir les mathématiques comme une véritable activité intellectuelle plutôt qu'un simple corpus de connaissances, qu'une simple collection d'outils disparates » (ibid., traduction libre). Ces deux éléments sont liés à un

[1] Dans la littérature anglo-saxonne, le concept de *dépaysement épistémologique* a été traduit par *reorientation* (réorientation) (voir Jahnke et al., 2000; Barbin, 2012).

besoin d'humaniser les mathématiques, d'en souligner l'historicité et de mettre en relief leur aspect évolutif. Il s'agit de dimensions fréquemment mentionnées dans la littérature (voir Furinghetti, 2004; Guillemette, 2009; Siu, 2000).

D'autres chercheurs partagent cette perspective sur l'utilisation de l'histoire et de son impact potentiel et accordent une importance considérable à cet argument du *dépaysement épistémologique*. En particulier, pour Radford, Furinghetti et Katz (2007), c'est précisément dans la mise en lumière du lien entre connaissances passées et actuelles que l'histoire des mathématiques apporte le plus à l'enrichissement de la perception de la discipline et à la compréhension des objets étudiés. Leur discours se fonde sur la pensée de Mikhaïl Bakhtine (1895-1975), philosophe, théoricien de la littérature et sémiologue russe à l'origine d'un cercle important de penseurs tels Voloshinov, Ilienkov et Vygotsky dont les travaux influencent profondément les théories socioculturelles contemporaines de la didactique. Pour Bakhtine, « le sens ne s'approfondit véritablement que par la rencontre et le contact avec un autre sens, une culture étrangère. Il s'engage alors une forme de dialogue qui surmonte la fermeture et la partialité » (Bakhtine, 1986, p. 7, traduction libre). Brièvement, l'histoire des mathématiques est donc un endroit où il est possible de surmonter la particularité de notre propre compréhension des objets mathématiques limitée à nos expériences personnelles. Elle « fournit les instruments de dialogues avec d'autres compréhensions […], avec celles de ceux qui nous ont précédés » (Radford et al., 2007, p. 109, traduction libre).

UNE PERSPECTIVE DIALOGIQUE

Dans le texte qui suit, un dialogue s'engage entre les auteurs sur ces différentes considérations théoriques. La discussion émerge à partir d'un extrait des *Éléments* d'Euclide issu de deux éditions différentes (Heath, 1956; Henrion, 1623). Ce dialogue souligne les réflexions et ambigüités qu'une telle démarche peut impliquer et tente de mettre en évidence les compréhensions des auteurs quant à cette perspective socioculturelle sur l'utilisation de l'histoire des mathématiques.

Louis : Voici la proposition 14 du livre 2 des *Éléments* d'Euclide selon deux traductions souvent utilisées. La première, anglaise, est celle de Heath. Je la lis en séparant le texte en parties encadrées que nous représentons par une suite de figures qui illustrent ce qu'Euclide décrit dans son texte.[2]

Il s'agit de construire un carré d'aire égale à celle d'une figure rectiligne A donnée (Figure 1). La démonstration débute avec l'encadré 1 qui est représenté (flèche 2) par le trapèze 3 qui se ramène au rectangle 5 (selon la proposition 45 du livre 1). On continue ainsi à lire le texte, étape par étape.

Nous arrivons, après plusieurs étapes, à la construction du point H sur le cercle dont le diamètre est le segment somme des côtés du rectangle d'aire égale au quadrilatère A (Figure 2). Dès lors, par la proposition 47 du livre 1, EH est moyenne proportionnelle des côtés de ce rectangle et donc le carré de côté EH a bien même aire que le rectangle et donc que le quadrilatère A. Mais pour arriver à cette conclusion, Euclide a dû utiliser entre autres la proposition 5 du livre 2. Celle-ci étant peu connue en général, la voici dans Figure 3.

[2] Nous reproduisons ici les acétates que nous avons projetés, avec les numéros d'actions successives. Ainsi 1 est l'apparition du rectangle et 7 est l'effacement de ce dernier, etc.

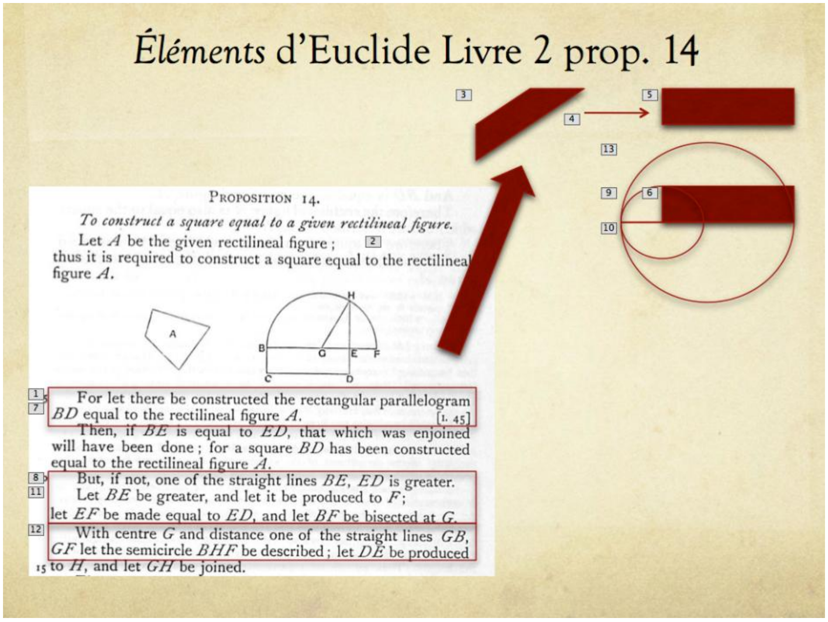

Figure 1

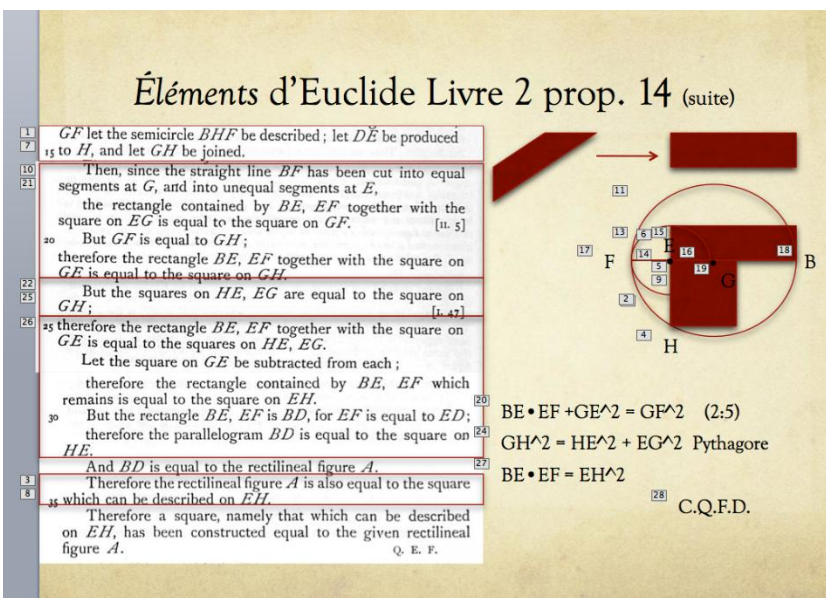

Figure 2

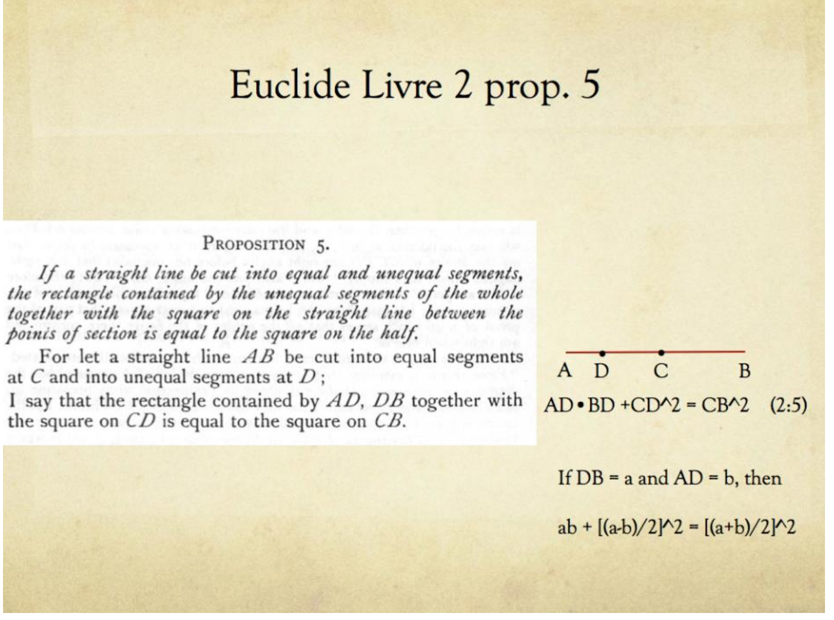

Figure 3

Louis : Revoyons, en trois acétates (Figures 4, 5 et 6), cette même proposition dans la traduction d'Henrion dans une édition de 1676.

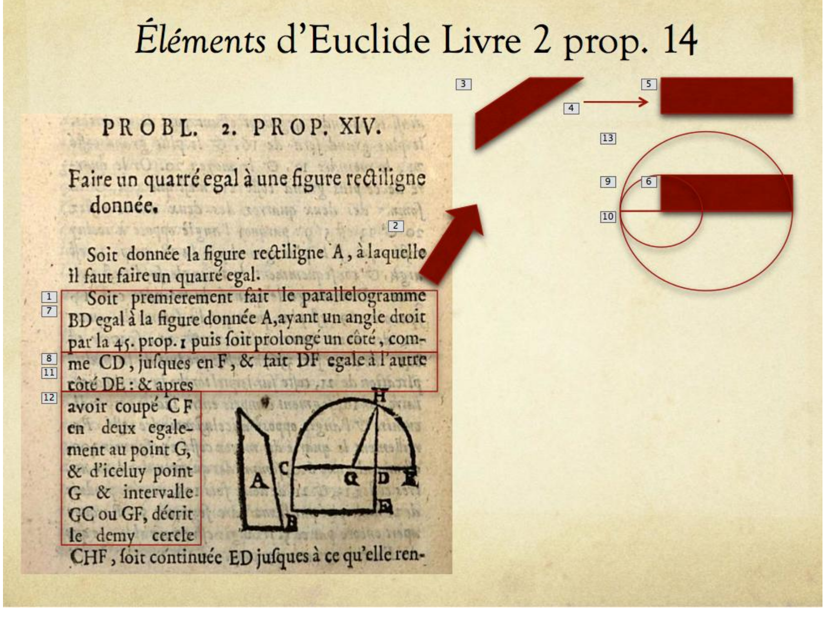

Figure 4

Louis : Puis :

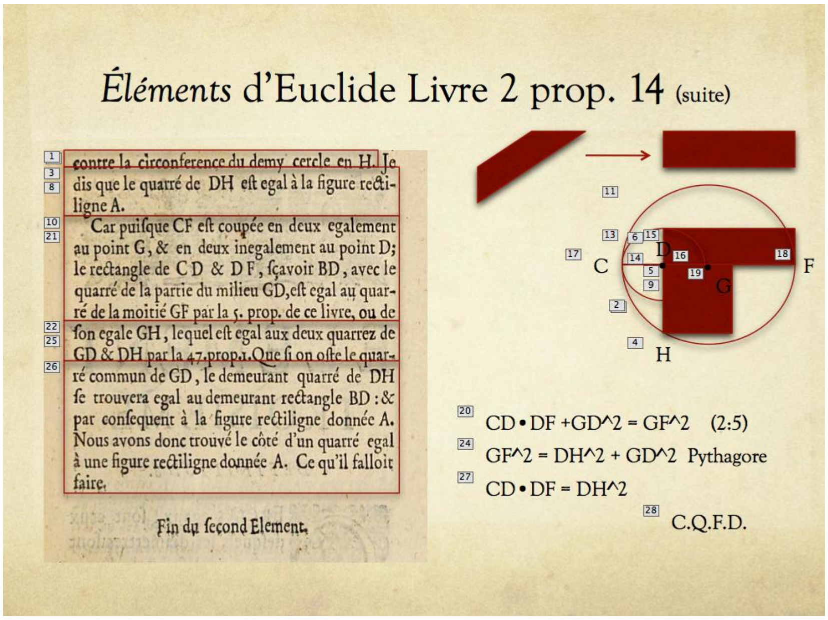

Figure 5

Louis : Et, enfin, la proposition 5 du livre 2.

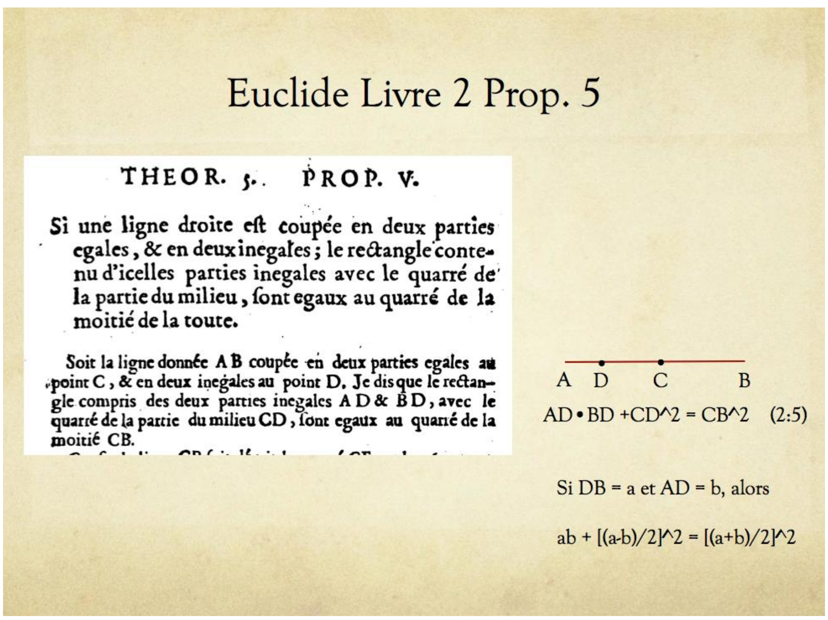

Figure 6

Louis : Ce genre de lecture, pas à pas, permet de comprendre relativement rapidement ce dont il est question dans la démonstration de cette proposition.

Que penses-tu David, de cette façon de lire un texte ancien?

David : C'est une transcription moderne, qu'y a-t-il d'historique là-dedans? N'est-ce pas un simple exercice de décodage, de translation, de traduction?

Louis : Euclide, n'est-ce pas un personnage historique?

David : Mais s'il n'y a que cela, c'est bien maigre. Quel est l'intérêt pour le lecteur amateur, le professeur, le mathématicien, l'étudiant ou même l'élève?

Louis : En fait, le texte français semble un peu plus historique, à cause de sa forme, la forme des lettres par exemple, la figure aux traits imprécis, etc. Le texte anglais de Heath est trop contemporain. Il pourrait être dans un manuel moderne.

David : Oui en effet, mais encore!

Louis : Que puis-je avoir de plus? Qu'as-tu à dire?

David : Je pense qu'il faut aller au-delà de la traduction. Il me semble y avoir, dans ces extraits, la possibilité de se plonger dans l'époque du mathématicien. D'en avoir une lecture plus diachronique que synchronique, c'est-à-dire en tentant d'établir un ancrage avec le contexte socioculturel du mathématicien et de mettre en évidence certaines idiosyncrasies qui lui donnent une certaine couleur. Il me semble possible de faire apparaître la dimension culturelle et historique de la discipline mathématique.

Louis : Mais pourquoi cette double lecture (mathématique et historique)? Je ne vois pas ce que ça peut donner pour l'apprentissage des maths?

David : Oui, pour la classe, il est difficile de dire comment ceci peut venir en aide à l'apprentissage de la preuve par exemple ou de l'identité remarquable sous-jacente, puisqu'il est question ici d'une démonstration géométrique. Mais je crois qu'on cherche d'autres types de réflexions à partir de ce genre d'artefact historique. Son potentiel en est un de réflexion épistémologique et métamathématique. L'aspect, le vocabulaire, la notation, le niveau de rigueur utilisé, tout ceci me semblent susceptibles de souligner l'éloignement historique et culturel de cet artefact. Et c'est cet éloignement, ce dépaysement, qui à mon sens pourrait amener une attitude ou une disposition différente face aux mathématiques et aux objets qui sont apparus à l'intérieur de la discipline. Attention, il s'agit plus que de la motivation ou d'une simple recherche du changement de la quotidienneté, mais vraiment un changement de disposition de manière d'être-en-mathématiques comme dirait Luis Radford (2009; 2011; 2012). Revenons en arrière. Lequel des deux textes t'impressionne le plus lors d'un premier regard?

Louis : Oui, la juxtaposition des deux textes peut être instructive. On voit que le premier est sous la forme construction-preuve-conclusion alors que le second est plutôt sur la forme construction-conclusion-preuve. Cela montre qu'au moins l'un des traducteurs n'est sans doute pas fidèle au texte d'Euclide. L'organisation spatiale des textes est aussi différente. Le texte anglais est divisé en plusieurs paragraphes, alors que le texte français est plus en blocs.

David : Mais, ce n'est pas une première impression, cela me semble déjà une analyse. Pourquoi cette question de l'impression première? C'est que je crois en la dimension entièrement incarnée et affective de l'apprentissage. Pour moi, le corps est l'ultime objet d'accès à la connaissance. C'est ma conscience sensible qui se noue au réel qui me permet l'apprentissage. Mais pour cela une disposition d'accueil face au réel est nécessaire. Que peut-on dire par rapport aux deux textes? Au niveau de l'attitude, de la disposition, de la manière d'être dans le monde?

Louis : Le texte français apparaît dès le premier coup d'œil comme un texte ancien. On le voit à la couleur du papier, à la forme des lettres, au fait que l'impression est moins précise. Le vocabulaire et le ton général du texte accentuent cette première impression. Le texte anglais laisse plutôt l'impression d'être récent.

David : Oui, la dimension esthétique me semble importante, l'étonnement fondateur devant un tel objet me semble primordial pour incliner la conscience sensible. Cela me rappelle l'argument du *dépaysement épistémologique* de Barbin qui souligne que le sentiment d'étonnement est au cœur de toute forme de démarche pédagogique ou didactique concernant l'utilisation de l'histoire. Dans ce sens, quel texte serait le plus intéressant pour la classe de mathématiques?

Louis : Mais quels sont nos objectifs précisément? Qu'est qu'on veut faire?

David : Je crois, comme le soulignent Radford, Furinghetti et Katz (2007), que ce genre de lecture amène une rencontre avec une autre forme de compréhension. Nos compréhensions des objets mathématiques sont limitées à nos propres expériences finies. C'est-à-dire, à nos champs culturels, à notre histoire civilisationnelle, à notre historicité personnelle, etc. L'histoire me permet de me percevoir comme un être qui fait des mathématiques dans son époque et sa culture. J'ai conscience de ma condition et je peux alors entrer en contact avec les compréhensions des autres, à plus petite échelle, de manière plus constructive.

Louis : C'est bien beau cette idée de l'historicité personnelle, mais ne sommes-nous pas prisonniers de cette historicité personnelle? Tout cela me semble plutôt paradoxal. Comment faire du neuf alors? Comment évoluer?

David : Pour véritablement élargir nos horizons, il nous faut faire la rencontre avec une forme véritablement étrangère et radicalement différente de compréhension. Expliciter ces différences culturelles permet de se mettre dans la peau de l'autre et de faire acte d'une certaine empathie. L'histoire rend donc explicite nos attachements culturels. Elle implique donc une certaine prise de pouvoir, une possibilité d'être autrement en mathématiques.

Louis : Mais que faire véritablement avec le texte? Comment le présenter? Qu'est-ce qu'on en fait?

David : Bonne question. Comment faire en effet? La question du « comment » est inévitable.

Louis : Oui, car il faut avoir accès au texte, et à cette lecture particulière. Quel environnement est nécessaire ou susceptible de nous mener à cette disposition de lecture et à ces réflexions?

David : On peut prendre à ce moment-là en compte les propositions de Fried (2008) qui met en relief l'importance d'une lecture d'une part diachronique et d'autre part synchronique. Plus spécifiquement, il mentionne que l'objectif de l'historien est de se plonger dans l'époque du mathématicien, de percevoir les idiosyncrasies de ce dernier et de situer l'ouvrage dans un continuum de développement des mathématiques. Le regard du mathématicien, quant à lui, tente de décoder les symboles désuets, de les restituer au langage moderne et de saisir l'aspect essentiellement mathématique des propos de l'auteur. Il qualifie de diachronique la lecture de l'historien et de synchronique la lecture du mathématicien, termes qu'il emprunte au sémiologue Ferdinand de Saussure (1967/2005). Aussi, le rôle de l'enseignant serait précisément de faire basculer l'apprenant constamment entre ces deux visions. C'est ce travail de va-et-vient continuel qui permettrait de faire émerger une certaine conscience de ses propres conceptions des mathématiques et de la possibilité pour lui de les confronter de façon constructive avec celles des autres. Cette double lecture mènerait « à une connaissance plus approfondie de lui-même, il se perçoit alors comme une sorte de créature qui fait des mathématiques, une espèce d'être-mathématique » (Fried, 2007, p. 218, traduction libre). Pour Fried, l'histoire devrait jouer un rôle central dans cette recherche de connaissance de soi et d'émancipation. Susciter le dépaysement donc implique de présenter à l'apprenant un artefact historique « éloigné » ou « qui éloigne », en terme historique et culturel.

Alain Bernard (2012) propose d'ailleurs dans ce sens différents types de lectures à faire (commentée, préparée, résumée, exposée, annotée, traduite, à haute voix, etc.).

Louis : Les propositions de Bernard concernent les sciences expérimentales. Or, il me semble que le rapport des étudiants en sciences à l'histoire des sciences est bien différent de celui des étudiants en mathématiques à l'histoire des mathématiques. En sciences, nous avons tous un certain nombre d'*a priori* sur la façon dont doivent se dérouler les phénomènes physiques, ne serait-ce que du fait que nous avons tous une expérience quotidienne de ces phénomènes. L'évolution de la pensée scientifique vient souvent bousculer ces *a priori*. C'est là un moteur pédagogique qu'il est facile à exploiter pour mettre en évidence justement cet « éloignement » cherché. En mathématiques, les objets mathématiques vivent dans notre tête. Il y a beaucoup moins d'*a priori*. Il est plus difficile de faire apparaître cet « éloignement ». Mais ceci dit, quels textes utilisés?

David : Regardons nos deux textes, qu'ont-ils de si différent? Ne sont-ils pas tous les deux des traces d'une pensée étant apparues il y a plus de 2000 ans? Laquelle de ces traces choisir? À mon sens celle qui apparaît la plus éloignée en terme historique et culturel.

Louis : Mais n'y a-t-il pas une contradiction en voulant susciter la connaissance de soi et l'empathie chez l'apprenant en lui présentant un objet qui est justement difficile pour eux à accueillir? De manière plus éloquente, comment est-il possible pour nous d'accueillir la culture et la pensée d'un autre? Ne sommes-nous pas englués, occupés par notre propre historicité, notre culture et nos champs culturels occidentaux? Comment crée le personnage d'Euclide? Comment en dessiner le portrait? Mais en même temps comment s'identifier à lui? Comment s'identifier à lui? Ceci nous amène à un nouveau paradoxe! Il faudrait à la fois s'approcher d'Euclide et, en même temps, il doit apparaître comme éloigné!

David : En effet, il nous faut élaborer le portrait d'Euclide avec les apprenants. Mais ce portrait ne peut être dressé qu'à partir de la perception d'une individualité. Une individualité qui est portée par les traces laissées par l'auteur.

Louis : Max Ernst nous a laissé un portrait d'Euclide. Le voici (Figure 7). Il semble que l'artiste a de son côté usé de sa fine sensibilité pour élaborer ce tableau qui rend compte de cette individualité. Cependant, ce n'est pas un tableau historique, un portrait authentique. Est-ce un problème? C'est la personnalité intellectuelle d'Euclide et non un portrait historique.

Figure 7

David : À nous alors de dresser le portrait d'Euclide avec les apprenants afin que tous nous puissions exprimer et développer notre sensibilité et notre manière d'être-en-mathématiques. Il faut que le texte fasse partie d'Euclide. Il faut que le texte imbibe le portrait... Et réciproquement. Un peu comme ce portrait modifié où apparaît en filigrane le texte de la traduction d'Henrion (Figure 8).

Figure 8

Louis : Mais s'il faut effectivement qu'Euclide soit profondément présent, ne serait-ce pas une bonne idée de commencer par montrer le portrait non historique d'Euclide et de demander aux élèves de dire ce qu'ils pensent de la personnalité de ce dernier?

David : Intéressant, je propose qu'on en débatte... tous ensemble.

CONCLUSION

La lecture de texte historique semble donc montrer un certain potentiel pour l'apprentissage des mathématiques. La rencontre avec une autre forme de compréhension, le développement de nouvelle manière d'être-en-mathématiques et la recherche d'un dépaysement épistémologique sont des éléments d'un discours émergeant dans le milieu de recherche. Seulement, les arguments qui supportent ce genre de pratique se doivent d'être étayés davantage par la recherche empirique et par leurs articulations aux cadres théoriques concernant l'enseignement-apprentissage des mathématiques.

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New PhD Reports

Présentations de thèses de doctorat

TEACHING TOWARD EQUITY IN MATHEMATICS

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RATIONALE

Since 1989, the mathematics reform movement has aimed to make mathematics equitably accessible to all students, characterized in part by the promotion of "constructivist" (Palincsar, 1998) experiences over rote learning. This pedagogical shift generated new teaching approaches designed to improve students' conceptual understanding, procedural fluency and strategic competence in mathematics (Donovan & Bransford, 2005). However, over twenty years later, reform mathematics has not resulted in educational equity for marginalized groups of students (Dixson & Rousseau, 2005; Ladson-Billings, 2006; Martin, 2003). When achievement in mathematics becomes a form of gate-keeping (Leonard, 2008; Nasir, 2007; Mason, 2006) and a barrier to higher learning and to earning potential and participation in society (Gutiérrez & Rogoff, 2003), this clearly becomes an equity issue, with equitable access to mathematics learning critical for all students.

Work in teacher professional development (PD) related to issues of equity in education (Cochran-Smith, 2004), equity in mathematics (Gutiérrez, 2007; Leonard, 2008; Nasir & Cobb, 2007; Gutstein, 2006), and culturally relevant teaching (Leonard, 2008; Ladson-Billings, 1995; Civil, 2007) shows promising ways to reach students who have been traditionally underserved by the educational system. In Canada, there is very little research in teacher professional development in mathematics education for teachers in a multicultural and multilingual urban context. Although many studies focus on the implementation and impact of single PD efforts, few studies have examined the multiple contexts of professional learning.

My study addresses this gap. It is an ethnographic case study of five elementary school teachers who are working toward a pedagogy of equity in their mathematics teaching in an urban inner city context. The purpose of my study is to reveal how mathematics and equity are conceptualized and practiced as teachers learn to teach for equity through their participation in professional learning communities. This study also targets a gap in the field of culturally relevant mathematics teaching and equity-focused teacher education, namely that in Canada, less research is being conducted in this area and what has been done focuses mainly on relatively homogenous, Aboriginal, and rural communities and classrooms (Lunney Borden, Wagner, & Johnson, 2011; Nicol & Brown, 2008) rather than diverse urban communities.

My research questions include:

1. How do teachers conceptualize equity in mathematics education in a Canadian urban multicultural context? How do they achieve equity through their instructional practices in their mathematics teaching?
2. How do these conceptualizations change over time when teachers are involved in a variety of professional learning communities that focus on mathematics education, student achievement, curriculum development, and culturally relevant and responsive pedagogy?
3. In the multiple contexts of professional learning, what ideas do participants take up, and which ideas do they reject, as they participate in the various PD opportunities?

Four main themes emerge in the literature related to teaching for equity in mathematics. The first theme is around the need to develop approaches to raise the *achievement* levels of marginalized students; the second theme is around providing *access* to high levels of mathematics for students who have been historically underrepresented in this area; the third theme examines the development of students' identities as 'doers of mathematics'; and the fourth theme examines issues of power imbalances in society as well as in mathematics classrooms.

THEORETICAL FRAMEWORK

My work is grounded in a sociocultural theoretical framework of situated learning and teacher change (Lave & Wenger, 1991) and Wenger's (1998) work in *communities of practice*. Communities of practice are groups of people who share a concern for understanding a problem, or a passion for something they do, and learn how to do it better as they interact on an ongoing basis. For this community of practice, equity served as a point of focus for teachers to organize and negotiate meaning.

RESEARCH METHODOLOGY

Five classroom teachers from Grades 1 to 5 volunteered to take part in my study.

I examined teachers' interactions with professional development contexts and their developing conceptualizations of equity. I investigated the major forms of professional development at their school and considered teachers' opportunities to learn in and across all of the various PD contexts. The school is located in a large, urban Canadian city. The school has approximately 450 students from kindergarten to Grade Five. Within the school and family population, there were 30 languages spoken and 35 countries represented. The school received extra funding from the provincial government and school board because it was identified as underperforming in provincial literacy and numeracy scores.

DATA COLLECTION

The study drew on ethnographic methods of observation, field notes, teacher interviews, and video-recordings of professional development (PD) sessions and classroom mathematics teaching. Over the year, I observed nineteen PD sessions (see Table 1) and documented these through video recordings and/or field notes. I visited each teacher's classroom approximately six times between September and June, which totalled 30 hours of observations. I also conducted three interviews with each teacher (at the beginning, middle and end of the year) to learn more about how they conceptualized equity in their classrooms, how they achieved equity in their teaching, and how the PD supported their learning to teach more equitably. A stimulated recall interview (mid-year) took place after videotaping a math lesson in each of the five teachers' classrooms. Teachers viewed videotapes of their teaching session as a catalyst to discuss the pedagogical decisions they made concerning equity in their teaching. The dissertation draws primarily on teacher interview data, which is triangulated with the data on the PD efforts themselves, and data from the classroom mathematics teaching sessions. Table 2 describes the data collection schedule.

Culturally Relevant and Responsive Pedagogy Seminar Series (CRRP)	A university-school partnership focusing on multicultural education, systems of inequity in society, and examining student demographics data
Participatory Action Research Project (PAR)	Participation in the process of inquiry, student-driven emergent curriculum, and knowledge production for democratic empowerment and social change
Junior Undiscovered Math Prodigies (JUMP)	Began as a tutoring program, use of incremental learning for success, emphasis on practice and praise, with computational fluency as prerequisite for access to higher level math
Teaching-Learning Critical Pathways (T-LCP)	Ministry mandated teacher inquiry process examining curriculum expectations and student work to collectively create assessment and evaluation criteria
Institute of Child Study (ICS)	Examining an inquiry-based approach to mathematics teaching through classroom observation and in conversation with Lab School teachers

Table 1. Professional development contexts.

Data Collection Schedule			
Time	**Classroom visits/observations conducted**	**PD Session observations conducted**	**Interviews conducted**
September/ October 2009	1 visit/teacher x 1hr = 1 hour x 5 visits = 5 hours	CRRP/PAR 12 hrs T-LCP 5 hrs	5 x 30 minutes = 150 minutes
November/ December 2009	1 visit/teacher x 1hr = 1 hour x 5 visits = 5 hours	CRRP/PAR 3 hrs JUMP 1 hr T-LCP 2.5 hrs	
January/February/ March 2010	2 visits/teacher x1hr = 1 hours x 10 visits = 10 hours	CRRP/PAR 9 hrs T-LCP 5.5 hrs JUMP 2 hrs	5 x 30 minutes = 150 minutes
April/May/ June 2010	2 visits/teacher x1hr = 1 hour x 10 visits = 10 hours	CRRP/PAR 6 hrs JUMP 6 hrs ICS 3.5 hrs T-LCP 1 hr	5 x 30 minutes = 150 minutes
TOTAL HOURS	6 visits/teacher = 30 hours	PD 46.5 hrs	7.5 hours

Table 2. Data collection schedule.

Figure 1 shows the multiple PD contexts in which teachers participated. Creating this grid was a pivotal moment in my study because it guided me to think about learning in specific contexts. It also became clear that this was only part of teachers' ongoing professional learning. As well, I used this grid as a prompt during the final interview to ask teachers about their experiences with the PD.


PROFESSIONAL LEARNING CONTEXTS

TPL	SEP	OCT	NOV	DEC	JAN	FEB	MAR	APR	MAY
CRRP	X X			X	X		X	X	
PAR		X	X			X			X
JUMP				X	X		X		
TLCP	X	X	X			X	X	X	
ICS									X

Figure 1. Professional learning contexts.

DATA ANALYSIS

Data analysis involved examining closely through their words and actions the pedagogical decisions teachers made as they tried to achieve equity in their teaching. Because of the theoretical framework of situated learning that frames this study, the unit of analysis was the group of teachers rather than individuals.

FINDINGS

The ideas that were taken up by teachers as a result of the professional development experiences included:

1. The importance of developing awareness of students and their communities/building curriculum on issues of social justice and students' lived experiences.
2. Teaching strategies to scaffold students' development of mathematical proficiency.
3. Strategies for structuring student-driven, inquiry-based learning for mathematics.

In the talk for CMESG, I examined changes in teachers' approaches to pedagogy and conceptions of equity in their mathematics teaching through their involvement in *participatory action research* (PAR) as a context for professional learning. Teachers in the study used PAR to collaboratively inquire into their practice and provided opportunities for students to 'map recess', first by having students visualize their playground and think about how they use it, then using maps to document play activities as well as areas of the playground that were most used and least used. Interesting issues emerged through this process. For instance, students in Grade Two reported that they didn't use one area of the playground "because the Grade Five's own that part". Teachers then embedded PAR in their data management units and inspired students to become "researchers, surveyors, and graph makers" to inquire into their own questions of interest about recess and the school playground. Students created questions and surveys to collect data and represent this data with graphs, which then lined the halls of the school.

In the PAR context, teachers reported on the number of mathematical possibilities that arose in the students' responses to the mathematics involved. They spoke about many opportunities to learn through curriculum designed by teachers and students and through encouraging students to ask and genuinely investigate their own questions about the world (e.g., the potential for students to develop number sense in tandem with examining issues of fairness and equity on the playground). Questions that students developed included: What games do children want at recess? Do students prefer wood chips, sand, or soft clay? How many times have students been bullied on the playground? Teachers noticed students (and themselves) moving beyond a focus on *conventions* of graphs to focussing on the *function* of graphs. Sally, a Grade 4 teacher, in the final interview explained:

> *You know, if we hadn't done the PAR, we would have graphed, I don't know, ice cream flavours or something that was just sort of, I don't know, more kid-friendly, let's say? But not to take it to where actually I might use this data I've got to present, like authentically present this data because that's what graphs are for. It's not just for getting marks. So actually I really have to present the data so that other people can interpret it.*

The same teacher described PAR as an inquiry-based approach:

> *Big scale…the inquiry, the questioning, the collecting data, the reflection upon the data and where do I go next, I mean those are really huge skills that you don't get a chance to cover with that curriculum which is to me very small scale.* (Sally, Grade 4 Teacher, final interview)

Although teachers' experiences with PAR resulted in what they described as an exciting new approach to mathematics teaching, when the PAR project was completed, they didn't transfer this inquiry-based approach to other areas of their mathematics teaching. Instead they focused on 'covering the curriculum' in the most efficient way they could to meet the requirements of reporting on student achievement through a 'teacher-as-purveyor-of-knowledge' approach. What emerged in the findings were a number of contradictions, which through Wenger's (1998) work I found to be dualities rather than opposites or strict dichotomies or tensions. These seeming contradictions in teaching led me to look specifically at Wenger's (1998) idea of duality or complementarity of reification and participation to look at the conditions and constraints teachers juggle as they facilitate students' development of mathematical knowledge. How did teachers make sense of the PD experiences within the structural conditions and constraints they faced? The following themes emerged:

1. Teachers wanted to pursue inquiry and to carefully structure and sequence lessons.
2. Teachers wanted to design instruction based on students' needs and interests and meet standardized curriculum expectations.
3. Teachers saw themselves as knowledge 'purveyors' and participants in a community of learners. For example, in the final interview, Stewart (Grade 5 teacher) said,

   > *I truly think that when you do this work [PAR], what happens in the process is that the students start taking more ownership so there's [sic] times where you step back from that role of teacher as purveyor of knowledge and you participate in the acquisition of the knowledge.*

   Later in the same interview, he reported, "OK I have to teach you [the students] this [curriculum expectation]. I have to actually purvey this knowledge before they get started because they need to have a base."
4. Teachers saw mathematics as a sequence of skills to be mastered *and* as a tool to analyze social justice issues.

THE TEACHERS' INTERACTION WITH PD CONTEXTS RELATED TO THEIR DEVELOPING CONCEPTUALIZATIONS OF EQUITY

Although I articulated clear stances of conceptions of equity in the literature review, these didn't play out as clearly separated categories in my study. Many of the conceptions overlapped with the four themes in the literature review and reflected the complexity of the topic. But for the purposes of reporting the findings, I reported on teachers' conceptions in clearly separated categories. See Table 3 below.

Teachers' conceptions of equity that remained constant across interviews:

1. Raising student achievement
2. Equity does not mean equal treatment
3. Providing access to mathematical concepts
4. Providing access to language in mathematics for English Language Learners (except for final interview)

By the mid- and final interview, there was a recognized shift in teacher talk around equity. I saw clear links between what they were saying, ideas they were trying in their classroom math teaching and the PD they had experienced.

New conceptions of equity to emerge in teacher talk during the Stimulated Recall Interview (SRI) and Final Interview:

5. Drawing on social justice issues to promote social change/exploring issues of social justice through mathematics (SRI and final interview)
6. Building on students' lived experiences (final interview)

DISCUSSION

Teaching abides in a complex system that is historically, politically, and socially situated. Participation in multiple contexts shapes what teachers teach, how they see themselves and how they make meaning of their work. Clearly, context matters. Teachers live in multiple contexts, each with its own rules, roles, artefacts, and goals. In this way, teachers' conceptions of equity and mathematics learning are context-dependent. Depending upon the context and its goals, teachers make pedagogical decisions (e.g., which ideas they take up or reject in the PD), design instructional strategies, and develop perceptions of students and perceptions of learning. School mathematics is shaped by the various contexts in which it is situated as much as it is shaped by the teachers who teach it. The teachers' negotiation of meaning and of structures or conditions in each context produced very different kinds of mathematics teaching, all for the purpose of creating viable ways of doing the work. That is why different kinds of mathematics teaching co-existed side by side and how various hybrids were created. For example, in nearly all of the professional development contexts, the explicit goal was to improve student achievement levels on provincial standardized tests. This deeply affected teachers' pedagogical decisions and choices. Improving achievement levels is a very different goal than revealing student thinking and learning about the mathematical ideas students bring with them to school through inquiry (or the PAR goals in which students are to have a sense of ownership in the design of curriculum).

Mapping Teachers' Conceptions of Equity across time with Professional Development (PD) across the school year									
	Time 1: Initial Interview (Sept/Oct)			Time 2: Stimulated Recall Interview (Jan/Feb)			Time 3: Final Interview (May/June)		
Teachers' conceptions of equity	***•Raising student achievement*** *Curriculum coverage* *Preparing students for standardized tests* ***• Equity does not mean equal treatment*** ***• Providing access to conceptual understanding*** *Computational fluency as a prerequisite for accessing higher level thinking in mathematics* ***• Providing access to language in mathematics for English Language Learners***			***•Raising student achievement*** *Curriculum coverage* *Preparing students for standardized tests* ***• Equity does not mean equal treatment*** ***• Providing access to conceptual understanding*** *Computational fluency as a prerequisite for accessing higher level thinking in mathematics* ***• Providing access to language in mathematics for English Language Learners*** - - - - - - - - - - - - - - ***• Drawing on social justice issues to promote positive social change*** ***• Inquiry as a form of equity***			***•Raising student achievement*** *Curriculum coverage* *Preparing students for standardized tests* ***• Equity does not mean equal treatment*** ***• Providing access to conceptual understanding*** *Computational fluency as a prerequisite for accessing higher level thinking in mathematics* - - - - - - - - - - - - - - - ***• Drawing on social justice issues to promote positive social change*** ***• Inquiry as a form of equity*** ***• Becoming aware of students' lived experiences/building students' cultural and linguistic knowledge***		
PD experiences	**September**	**October**	**November**	**December**	**January**	**February**	**March**	**April**	**May**
Teaching-Learning Critical Pathway (T-LCP)	Sept. 3: Reviewing CAT scores/Inquiry units worksheet *3 hours*	Oct. 13: Moderated marking *2 hours* (each grade team)	Nov. 2: Review EQAO scores Post-task *2.5 hours*			Feb. 8: Pre-task planning *2 hours*	March 10 & 25:Moderated marking, part one and two *3.5 hours*	April 15 and 16: evidence of Level 4 thinking *1 hour*	
Culturally Relevant and Responsive Pedagogy Seminar Series (CRRP &PAR)	Sept. 16: CRRP: systems of oppression in society/school *3 hours*	Oct. 16: PAR training *6 hours*	Nov. 2: PAR - Mapping Recess plan with teacher candidates *1 hour*	Dec. 9: Lesson planning/social justice theme: Africville *3 hours*	Jan. 21: Exploring issues of social justice through math *3 hours*	Feb. 26:Follow-up with PAR facilitator *3 hours*	March 24: Examining school board's demographic data *3 hours*	April 21: Systems of power + *Brown eyes, blue eyes* documentary *3 hours*	May 19: School groups share PAR projects *3 hours*
JUMP				Dec. 7: JUMP intro *1 hour*	Jan. 14:JUMP training *2 hours*			April 8: JUMP demo lessons *1 hour/session*	
Institute of Child Study Lab School									May 19: inquiry-based teaching/class-room observations

Table 3. Teachers' conceptions of equity over time in relation to Professional Development Contexts.

IMPLICATIONS

Teachers were not supported in making sense of the contradictions in the PD they were being offered, yet they managed to integrate new knowledge into their practice. Formal PD is only a slice of teachers' ongoing learning. PD that revealed inequities in society changed the way teachers talked about students and curriculum. In future work, it will be important to capitalize on the contradictions that arise during formal PD sessions and to support teachers in lessening the distance between what they learn in formal PD efforts and what they practice in their classroom mathematics teaching. To do this, teachers need to be given time and space to debate and discuss these contradictions explicitly and to be given a platform to critique or challenge policies that undermine teacher autonomy and creativity and quash the development of students' mathematical thinking. A community of practice approach suggests developing the learning potential of the school rather than simply "delivering courses" (Wenger, 1998). Future goals include the design of PD that builds on issues of concern for teachers and students, honours students' cultural and linguistic knowledge, allows space for discussion and debate, and uses mathematics to examine issues.

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INEQUALITIES IN THE HISTORY OF MATHEMATICS: FROM PECULIARITIES TO A HARD DISCIPLINE

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In this theoretical contribution, the history of inequalities is explored in the pursuit of understanding why inequalities are hard to meaningfully manipulate and understand. Memorable dates in the development of inequalities and the symbols for representing inequalities are highlighted. Then, well-known inequalities are presented and some novel proofs are shown. Finally, implications for the teaching of mathematics are identified.

WHY DO MATHEMATICS EDUCATORS LOOK AT THE HISTORY OF CONCEPTS?

It may certainly seem puzzling that researchers in mathematics education preoccupy themselves with the study of the *history* of mathematics. After all, we are not historians and, when analyzing a history book, we do not employ the lens of qualified historians. Moreover, we do not examine historical documents; instead, we prefer to search for answers in secondary sources of history. Radford (1997) argues that mathematics educators first look into the history of mathematics in a naïve way: to find anecdotes or old problems that render classes more interesting and to motivate study. However, there are approaches to the history of mathematics that can be legitimate and substantive.

For instance, the history of mathematics could be viewed as an "epistemological laboratory in which to explore the development of mathematical knowledge" (Radford, 1997, p. 1). In this type of research lab, which seeks to trace the evolution of a concept, the history of mathematics can inform researchers about the epistemological obstacles. From this knowledge, a parallel could be drawn between the obstacles encountered in the historical development of a concept and the problems that students have nowadays in understanding that concept (Sfard, 1995). Historical studies that explore the origins of a mathematical concept can, therefore, inform curriculum designers, teachers and instructors, as well as the epistemological theorists (Dennis, 2000).

THE SPECIAL CASE OF CHECKING THE HISTORY OF INEQUALITIES: WHERE TO LOOK

Research on inequalities reports mostly on students' misconceptions of inequalities or on the obstacles associated with understanding inequalities (Bazzini & Tsamir, 2004). For instance, Tsamir and Bazzini (2002) report that, often, "[r]esearchers witness students' and teachers' frustration with the difficulties encountered when dealing with inequalities" (p. 2).

Burn (2005) argues that the historical development of a mathematical concept "can reveal actual steps of success in learning [it]" (p. 271). This exploration could be applied when the research in education reveals that the understanding of a concept is not "consonant with students' intuitions" (p. 271), which seems to be the case with inequalities.

Checking the history of inequalities for periods of hardship in order to understand why the concept is difficult as a school subject is not an easy task. That may be in part due to the difficulty associated with finding references for this task. Although a web search brings forth almost fifty thousand results on the history of mathematical inequalities, there are only a few that can serve our purposes. One potential solution is to search for books in the library, but it is difficult to know exactly where to look. In school, mathematical inequalities are placed under Algebra. In undergraduate mathematics, as well, the algebra preview of Precalculus contains a section on inequalities. In Calculus, inequalities have no special treatment; they are tools for proving limits or analyzing functions. So, my first attempt to find references consisted of looking for inequalities in the history of algebra books.

IS ALGEBRA A GOOD PLACE TO SEARCH FOR THE HISTORY OF INEQUALITIES?

Material regarding inequalities in the history of algebra is scarce. Using a simple definition, *algebra* can be defined as the science of generalized computation.

In the history of algebra, three developmental stages are identified: rhetorical algebra, syncopated algebra, and symbolic algebra. This division is due to Nesselmann, based on the notion of mathematical abstraction (Radford, 1997). Rhetorical algebra is the algebra of words. Syncopated algebra uses a mixture of words and symbols to express generalities. These are the characteristics of the algebra of Pacioli, Cardan, and Diophantus. François Viète introduced the *species* (symbols) and made the distinction between a given quantity, which is constant but represented by a letter in an equation, and the variables in an equation. Viète was the first one to successfully solve parametric equations (Bagni, 2005; Sfard, 1995). Before Viète, algebra functioned at an operational level. After Viète's discoveries, equations became objects of higher order processes. Viète purified algebra from all the noise of words and presented it in abstract form, the encapsulation of a pure mathematical idea (Radford, 1997). From Viète on, it was time for structural algebra to make its appearance. The structure in algebra influenced geometry. The works of Descartes and Fermat, on the shoulders of Viète, helped geometry capture generality and express operational ideas. In the early years, algebra needed geometry for reification and verification; then, geometry used algebra for new reifications and development (Sfard, 1995).

Before the invention of symbols, algebra was the verbal interpretation of computational processes. It is important to ponder whether inequalities emerged from rhetoric or syncopated algebra, or whether the nature of inequalities is actually different from the essence of algebra. It is possible that the invention of a symbol for inequalities did help the manipulation of the known inequalities, but it took more than a mere symbol to make inequalities into a veritable discipline. In fact, it was the initiative of a great mathematician, Hardy, in the 20th century, that propelled inequalities to importance. Hardy carefully looked into the subject, collected, wrote proofs and published inequalities. The volume *Inequalities*, published in 1934, was the first monograph on inequalities. Moreover, the establishment of the *Journal of the London Mathematics Society* marks the most important date in the history of inequalities. The dates marked by Hardy, Littlewood and Polya (1934) in the history of inequalities are very recent, compared to the history of mathematics. By taking a closer look into the earlier history of mathematics, I explore whether we can trace inequalities even further back in time.

INEQUALITIES IN ANTIQUITY

The ancient mathematicians knew of "the triangle inequality as a geometric fact" (Fink, 2000). They also knew the arithmetic-geometric mean inequality, as well as the "isoperimetric inequality in the plane" (Fink, 2000). Furthermore, Euclid used words such as "alike exceed", "alike fall short", or "alike in excess of" to compare magnitudes (Kline, 1972, p. 69). The contemporary translation of Euclid's words uses the inequality symbols to help the reader understand the old text, but those symbols were foreign to Euclid. In the Pickering version of Euclid's *Elements*, the symbols introduced by Oughtred are used to write geometric inequalities. Working on calculations for approximating square roots of numbers, Archimedes was in fact manipulating inequalities arithmetically (Fink, 2000).

> *Using inequalities to measure awkward quantities dates back to Euclid and beyond. Archimedes in particular was skilled in using inequalities to deduce equalities, and after translating his method into algebra; such proofs were used by Fermat (1636) and are accessible to undergraduates today.* (Burn, 2005, p. 271)

INEQUALITIES IN GEOMETRY

Thus, in order to find inequalities in the history of old mathematics books, one needs an awareness of what one should be looking for: it seems that inequalities do not emerge from rhetorical algebra, but are actually embodied within geometry. We can also explore how inequalities looked in old geometry texts. The following figures represent inequalities that were well known in antiquity. Figure 1 represents a page form Byrne's (1847) *The First Six Books of the Elements of Euclid*. In this edition of Euclid's works, Byrne used colours to make the book attractive and appealing to students. The proofs were presented as pictures. Figure 1 (left) represents Proposition XXI from Book One. In plain language, the proposition reads:

> *If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.* (Joyce, 1996-1998)

Figure 1 (right) is the pictorial proof of proposition 21 from Euclid's Book 1. For inequalities, Oughtred's symbols were used to supply the pictures with inequality meaning without using too many words.

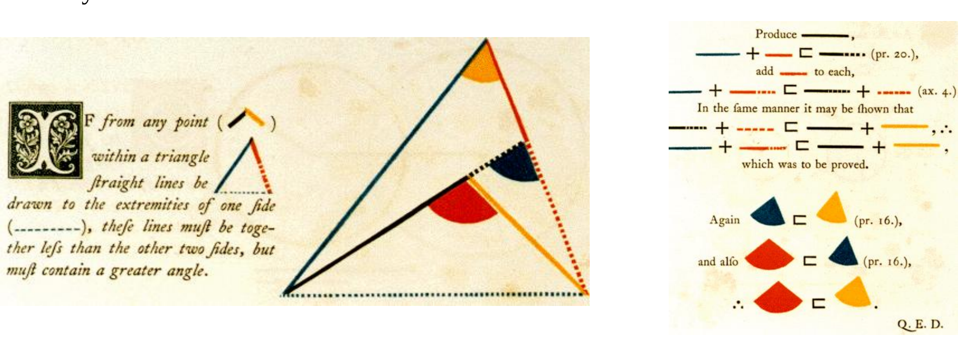
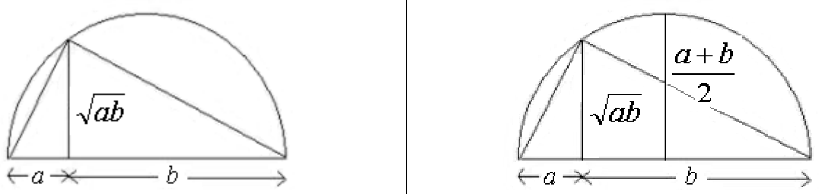


Figure 1. Proposition XXI [Source: Byrne, 1847, p. 21].

Another proof without words, or a geometrical proof, of the inequality of the means, $\sqrt{ab} \leq \frac{a+b}{2}$, can be seen in Figure 2, which represents a right triangle inscribed in a circle. The proof of the inequality is based on the result that the height of a right triangle is the geometric mean of the segments that it divides the hypotenuse into. This proof of the inequality of the means looks as Euclid would have imagined it (Steele, 2004).



a: The geometric mean

The height of the right triangle is the geometric mean of the projections of the legs over the hypotenuse: $h = \sqrt{ab}$

b: The arithmetic and the geometric means

The radius of the circle is the arithmetic mean of the projections of the legs over the hypotenuse: $\sqrt{ab} \leq \frac{a+b}{2}$

Figure 2. Inequality of the means.

From Figure 2b, it can be seen that the radius $\frac{a+b}{2}$ is the highest of all of the projections from points on the circle above the diameter, which proves the inequality.

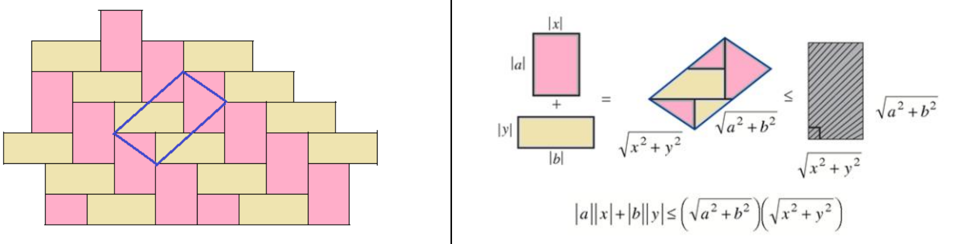
INEQUALITIES IN ARTEFACTS

Nelsen (1997) claims that he has seen the proof of famous theorems and inequalities on floor tilings and decorations. For example, the famous Cauchy-Swartz inequality is seen on the tiling found in *The Courtyard of a House in Delft*, a painting by Pieter de Hooch. It is quite a wonder to imagine that, when working, the painter or the tiling artist could have been aware of this inequality and of the mathematics we can see in the finished work. The tiles in the original picture are all of the same size. Choosing tiles of different sizes, Nelsen created a new tiling which shows a proof of the famous inequality of the means. To make it clear without words, Nelsen uses the fact that a parallelogram's area is smaller than the area of a rectangle whose sides are equal with the sides of the parallelogram to prove the Cauchy-Swartz inequality. Figure 3a represents Nelsen's tiling. Figure 3b shows the proof of the second of the two simultaneous inequalities comprising the Cauchy-Swartz inequality:

$$|ax+by| \leq |a||x| + |b||y| \leq \left(\sqrt{a^2+b^2}\right)\left(\sqrt{x^2+y^2}\right).$$

The first inequality can be proved using the triangle inequality and the properties of absolute value:

$$|ax+by| \leq |ax| + |by| \leq |a||x| + |b||y| .$$



a: The tiling as seen in the painting

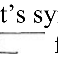
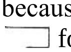
b: The decomposition of the tiling and the composition of the Cauchy-Swartz Inequality.

Figure 3. Proofs using tilings. [Adapted from Nelson, 2003, p. 8]

THE HISTORY OF THE INEQUALITY SYMBOL(S)

"It may be hard to believe, but for two millennia, up to the 16th century, mathematicians got by without a symbol for equality" (Lakoff & Núñez, 2000, p. 376).

The symbols < and > were first introduced in mathematics related texts by Thomas Harriot. He was a mathematician who worked for Sir Walter Raleigh as the cartographer of Virginia, present-day North Carolina. Harriot is considered to be the founder of "the English School of Algebraists" (Eves, 1983, p. 249). An anecdote says that Harriot was inspired by a symbol, (), he had seen on the arm of a Native American to 'invent' the symbols for inequalities (Johnson, 1994).

The mathematics community did not adopt Harriot's symbols immediately, because exactly at the same time, in 1631, Oughtred had suggested  for greater than and  for less than. Oughtred's *Clavis Mathematicae* was more popular than *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas* (*The Analytical Arts Applied to Solving Algebraic Equations*), Harriot's posthumously published work (cf. Eves, 1983). In 1734, the French geodesist Pierre Bouguer invented the symbols ≤ and ≥. These new symbols thus came to be used to represent inequalities on the continent (Smith, 1958).

One can only wonder how the 17th-18th centuries' symbols for inequalities would have managed the differences acknowledged nowadays when using < or ≤, for example. More precisely, the < symbol is used to represent quantities that are different, the first one being less than the second one. The ≤ symbol incorporates the equality as well; it allows the first magnitude to be equal with the second one. The ≤ symbol recognizes that the absolute extremum one quantity could touch is in the solution as well.

In conclusion, the inequality symbol allowed for the compression and aesthetic presentation of many old inequalities and permeated the development of a concept from a peculiarity.

It is well known that long before the appearance of symbolic algebra, people used to write arguments in longhand. There were no symbols to represent the unknowns and there were no symbols to represent the relationship between unknowns before Diophantus, during the 'rhetorical algebra' stage. There is nothing wrong with writing mathematical statements in plain language, but it may take several pages to describe a statement when in mathematical symbols the same job could be done, possibly, in one line. The use of symbols allows for more work to be performed in a shorter time. To the best of my knowledge, I am not aware of any research that would argue that even when symbols are well known as the best way to represent some piece of mathematics, one would use written or verbal plain language to describe the same idea. However, for getting meaning from a formal mathematical statement associated with symbolic notation, or to be able to reason about a mathematical statement, someone should 'read it' in plain language as well.

It is my perception that the symbol < is easier to assimilate than $\sqsupset$ to represent the inequality *less than*. Comparatively, when looking at the two symbols, one can be more successful in associating a metaphor with < than with Oughtred's symbol. It could be the case that this is only a cultural perception or habituation with a familiar symbol. The unfamiliar symbol $\sqsupset$ seems counterintuitive. All in all, it became more efficient to work on inequalities with the help of a special symbol. Then, efficiency helped algebra prosper, while Harriot's inequality signs stimulated further the proliferation of inequalities.

IS THERE A DISCIPLINE OF INEQUALITIES?

Geometry, arithmetic, and number theory are well-established disciplines from Antiquity. With the stage of symbolic algebra, new mathematical disciplines evolved, such as algebraic geometry. Sfard (1995) argues that geometry helped the reification of heavy computations in algebra, and then algebra helped geometry evolve and answer many of the problems that had been left unresolved from Antiquity. Initially, inequalities did not have a special status in mathematics; they were considered either mathematical peculiarities or tools for developing other theories. Two millennia and personal action changed the status of inequalities from that of support for mathematics to that of an actual discipline of study. Fink (2000) acknowledged that the history of inequalities had been written when Hardy et al. completed their 300 pages of inequalities with their proofs. Moreover, there are two journals of inequalities today – the *Journal of Inequalities and Applications* (JIA) and the *Journal of Inequalities in Pure and Applied Mathematics* (JIPAM) – as well as many other mathematics publications that print papers "whose sole purpose is to prove an inequality" (Fink, 2000).

Inequalities were tools at first, and when the circumstances became favourable, they flourished into a discipline. Embedded in geometry, they migrated to algebra to get the power of symbols from there, and then they settled for good in the theory of functions where they were enriched with new structures and philosophy. Embedded in functions, they grew omnipresent in many mathematical areas, from calculus to algebra, to statistics, to numerical analysis, to game theory. Paraphrasing Burn, I conclude this paper with a historical account of the concept of inequality: *Inequality* "encapsulates methods of proofs which originated in classical Greek mathematics, developed significantly during the 17th century and reached their modern form with [Hardy]" (Burn, 2005, p. 294).

IMPLICATIONS FOR MATHEMATICS EDUCATION

When the teaching, learning, or understanding of a concept runs into problems, there is a tradition in research in mathematics education to turn the search for the solution of the problem toward the history of the concept (Radford, 1997). In the development of the concept, one may find information about periods of slow development. There could be an indication somewhere that the concept had created problems for mathematicians first. For instance, Hippasus died for discovering irrational numbers. Even if mathematicians of his time had experienced incommensurability, they had problems accepting it. Such an incident informs us about the epistemological obstacles associated with that concept. Teaching a concept that has links to epistemological obstacles and being aware of those obstacles, the educator can plan when and how it would be more appropriate to introduce it to the students in order to avoid, if possible, students' cognitive conflicts.

A shallow search into the waters of history of inequalities shows that no apparent epistemological obstacles were encountered. However, it is recorded and documented that inequalities are not easy concepts to manipulate. Even Hardy, the man who can be called the father of inequalities, confessed:

> *There are, however, plenty of inequalities which are hard to prove; Littlewood and I have had any amount of practice during the last few years, and we have found quite a number of which there seems to be no really easy proof. It has been our unvarying experience that the real crux, the real difficulty of idea, is encountered at the very beginning.* (Hardy, 1929, p. 64)

Research on inequalities reports mostly on students' misconceptions on inequalities. Students encounter problems in the process of manipulation of inequalities, as well as at the level of interpretation of what an inequality is and what a solution of an inequality represents. Why are inequalities hard to manipulate? The answer to this question seems not to reside in the history of inequalities.

Burn (2005) sees a huge gap between a calculus student's mind and the mind ready to understand limits. As a solution to this, he proposes unconventional work with inequalities, for students to experience "the power of inequalities to obtain equality, when no direct path from equality to equality is available" (p. 271). Although the study does not report on epistemological obstacles related to inequalities, it can advise on the fact that it took almost two millennia for *inequalities* to become a discipline in itself. This could be a signal that learners might have conceptual or psychological difficulties when dealing with them. The development of a mathematical concept is extremely complex. Therefore, long before doing abstract work on inequalities, mathematics educators should empower students with experiences that will help the long-term development of their mathematics concepts.

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THE STUDY OF ON-LINE SITUATIONS OF VALIDATION EXPERIENCED BY 13- AND 14-YEAR-OLD STUDENTS WITH AND WITHOUT THE AID OF AN ELECTRONIC FORUM

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In spite of the emphasis placed on the development of different types of reasoning in the curricula in New Brunswick and in Québec, students are still encountering difficulties when developing or evaluating proofs. In this project, we are interested in the development of proofs using an electronic forum. This study first examines the impact of the use of an electronic forum on the development of algebraic validation skills among 13- and 14-year-old students from Québec and New Brunswick. Second, it focuses on the development of skills linked to the evaluation of proofs in algebra within the same group of students. The results seem to indicate that the use of the electronic forum fosters the transition from pragmatic proofs to intellectual proofs and facilitates an appropriate use of the rules of the mathematical debate.

PROBLEM

According to some members of the mathematical community, the development of reasoning abilities is one of the fundamental goals of learning mathematics, as reasoning is crucial to its understanding (Hanna, 2000; Martin & McCrone, 2001; Ministère de l'Éducation du Loisir et du Sport, 2005; National Council of Teachers of Mathematics, 2005). A second essential goal of learning mathematics is the development of proofs. Closely related to reasoning, the notion of proof enables students to derive meaningful understanding of concepts through the logical explanation of their work (Martin & McCrone, 2001; Miyazaki, 2000). The importance of reasoning in mathematics is also reflected in the orientations of the mathematics curricula in New Brunswick and the competences on which mathematics curricula in Québec are based. However, in spite of the emphasis placed on the development of different types of reasoning, both in the mathematics community and in the mathematics curricula in New Brunswick and Québec, students are still encountering difficulties when developing or evaluating proofs (Balacheff, 1987, 1999; Duval, 1991; Galbraith, 1981; Healy & Hoyles, 2000; Miyazaki, 2000; Sowder & Harel, 1998; Weber, 2001). Thus, to help teachers, it seems important to study the concept of proof in order to better understand the various elements that may be involved in the development and evaluation of proof.

In this project, we are interested in the development of proofs using an electronic forum, an on-line communication tool allowing interactions between students and teachers. This choice is based on research that shows that the use of this kind of tool sidesteps a number of constraints inherent in the school system (for example, the time factor) while taking into account characteristics considered important for the development of proofs (such as the social aspect of proofs).

THEORETICAL FRAMEWORK

TAXONOMY OF PROOFS

Several theoretical elements are necessary for the study of proofs developed by students when they find themselves in situations of validation. First, the different types of proofs developed by students are identified using Balacheff's (1987) *taxonomy of proofs*, which reflects the evolution of students' reasoning (Figure 1). In this taxonomy, proofs are divided in two main categories: pragmatic proofs, based on findings or measures, and intellectual proofs, where the idea of generality is predominant and where conclusions are based on properties or definitions rather than on one or more cases.

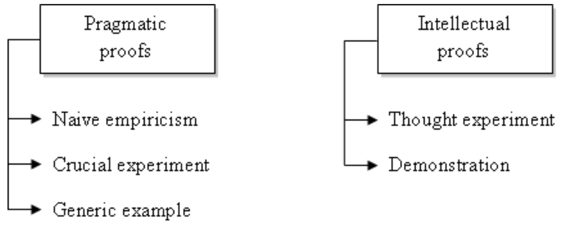

Figure 1. Taxonomy of proofs (Balacheff, 1987).

TYPES OF ARGUMENTS OF VALIDATION

Inspired by the works of Balacheff (1982, 1987, 1988) and Margolinas (1989), Mary (1999) developed an analytical grid in order to study the roles and functions of validation from the point of view of future secondary mathematics teachers. In this grid, we consider particularly the types of arguments (validation) that can be used in a situation of validation (Figure 2). It is important to note that the author adds two kinds of arguments that have not been mentioned by Balacheff or Miyazaki (2000): validation through an authority figure and validation through an audit of work (e.g., using a new method).

RULES OF THE MATHEMATICAL DEBATE

Finally, the proofs developed by students or the arguments given to explain the validity of different proofs are studied in order to target the rules of the mathematical debate that are properly used or infringed. These rules were developed by Arsac and his colleagues (1992) and read as follows:

- A mathematical statement is either true or false.
- A counter-example is sufficient to invalidate a statement.

In mathematics:

- to debate, we rely on a number of properties or definitions clearly stated on which we have agreed (axioms);
- examples that verify a statement are not sufficient to prove that it is true;
- a finding on a drawing is not enough to prove that a statement is true in geometry.

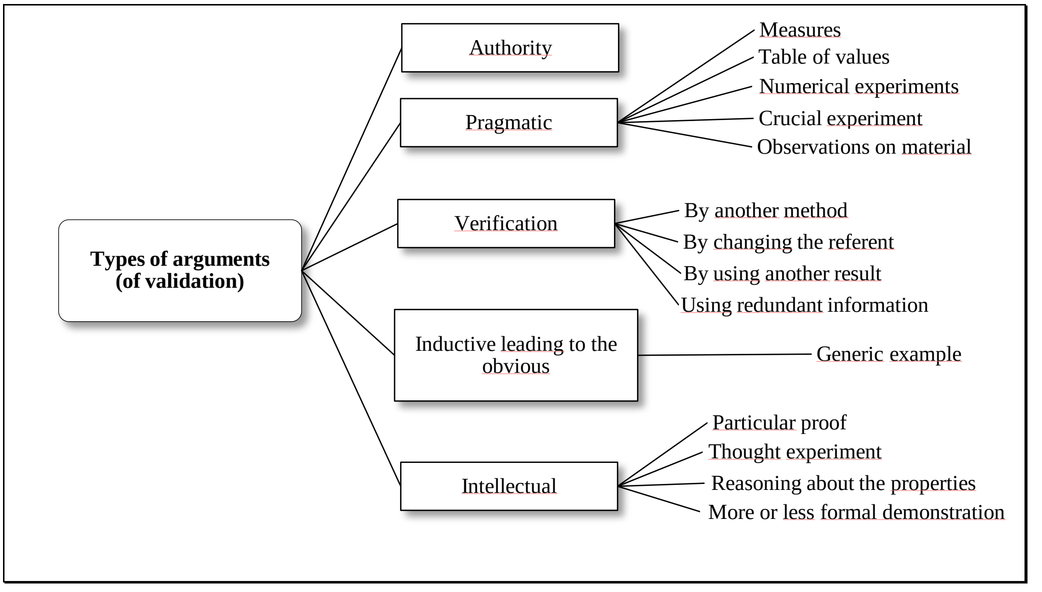

Figure 2. Types of arguments of validation (Mary, 1999).

METHODOLOGY

POPULATION

In this study, we work with four classes (two classes located in New Brunswick and two classes in Québec). A class from New Brunswick and a class from Québec form the experimental group (57 students and two teachers) and the two other classes form the control group (62 students and 2 teachers). Exchanges for the control group are strictly held in the classroom. Some exchanges in the experimental group take place in the classroom while the other part takes place in the electronic forum. The decision to work with classes from two provinces can be explained primarily by the desire to obtain a diversity of ideas during the mathematical activities. Although links can be made between the curricula of these provinces, the fact remains that they are distinct and that students may live different experiences through them. In addition, the resources to which students have access are not necessarily the same, because each province uses textbooks that are unique.

EXPERIMENTATION

This study is twofold. First, it examines the impact of the use of an electronic forum on the development of algebraic validation skills among 13- and 14-year-old students. Second, it focuses on the development of skills linked to the evaluation of proofs in algebra within the same group of students.

Different tools are used to collect data on proofs developed by students and the discourse adopted to validate or invalidate proofs. In the first place, a pre-test is given to the students from both groups. After the pre-test, four activities (each containing between four and six specific tasks) linked to algebra are completed over a period of four months. The work which students carry out focuses on the following three components: problem solving, the validation of solutions and the comparison of solutions in order to determine which ones are the most convincing. In a phrase, they have to ‘convince themselves and then convince others’. A post-test is given at the end of the four months. For each activity, students have to solve the problems individually and validate their answers with their teammates. That validation is solely done in the classroom for the control group, whereas it is done both in the classroom and in the electronic forum for the experimental group. The data collected in the pre-test, during the different activities, and in the post-test enable us to have a picture of the work done by the students in both groups during the experiment. Traces stored in the electronic forum allow us to follow the work of the experimental group throughout the implementation of the mathematical activities. Finally, data collected through semi-structured interviews conducted with the teachers as well as with a few students help us to better understand the way the mathematical activities are experienced by the students, both in the electronic forum and in the classroom.

During the experiment, the role of the teacher is crucial because he or she serves as a facilitator for the mathematical activities proposed. Among other things, it is essential that the teacher keeps pushing students by encouraging them to think differently. If students do not question the work of their peers, the teacher must guide them in their questioning. Once the activity is underway and students know what they have to do, the teacher becomes, somehow, a peripheral actor whose main task is to ensure that the activity runs smoothly. Decisions made by teachers can have an influence on the progress of the activities and thereby on the results of this research. In order to minimize this effect, various documents are presented to them in order to clarify the conditions for carrying out the experiment.

DATA ANALYSIS

1ST RESEARCH QUESTION: ALGEBRAIC VALIDATION SKILLS

The analysis of the data presented in Figure 3 leads to two main findings. First, in the pre-test, both groups appear more or less similar with respect to the production of proofs. However, this diagram only shows the types of proofs that are developed by the largest number of students in each group. Further analysis of the results shows that there is only one additional student in the experimental group that presents a proof using naïve empiricism. Thus, in the beginning of the experiment, students in both groups seem to show a certain willingness to use proofs by naïve empiricism. In addition, the evolution of the types of proofs differs between the two groups. In the control group, the changeover to the thought experiment in Activity 2 (Question 2) appears as an irregularity, for all other questions mainly generate proofs by naïve empiricism. In the experimental group, there seems to be more of a tendency for intellectual proofs, whereas only Activity 2 (Question 1) leads students to adopt a proof by naïve empiricism. Note that for both groups, there is a passage to intellectual proofs between the two questions in Activity 2. What is surprising is that these two questions are part of the same activity. There is therefore no teacher intervention or exchanges in the electronic forum between these questions.

In general, because they demonstrate a preference for proofs by naïve empiricism, students in the control group do not comply with the rule of mathematical debate that states that examples that verify a statement are not sufficient to prove that it is true. The solutions of the students of the experimental group, for their part, reflect mainly the proper use of the rule, which specifies that in mathematics, to debate, we rely on a number of properties or definitions clearly stated on which we have agreed.

2ND RESEARCH QUESTION: SKILLS LINKED TO THE EVALUATION OF PROOFS

Ranking of proofs

Figure 4 sets forth the types of proofs that rank first when calculating the average rank for the pre-test, Activity 1 and the post-test. In this case, the two groups have the same results: the thought experiment is preferred in the pre-test and Activity 1, while proofs by naïve empiricism rank first in the post-test. In the pre-test, students seem to prefer the proof that allows them to get the highest score (that is to say, the proofs preferred by their teachers and identified during institutionalization as the most mathematically powerful). In the post-test, instead, they seem to favour the proof that they personally prefer. Why aren't students influenced in the same way when they develop proofs? In fact, it is important to realize that reading a formal proof (especially a demonstration) may be difficult for students because all the elements implied are not given. They must, to really understand the proof, understand the innuendos. When they develop a proof, the situation is different, as they find themselves in a position that allows them to clearly formulate the elements that are sometimes removed in a demonstration.

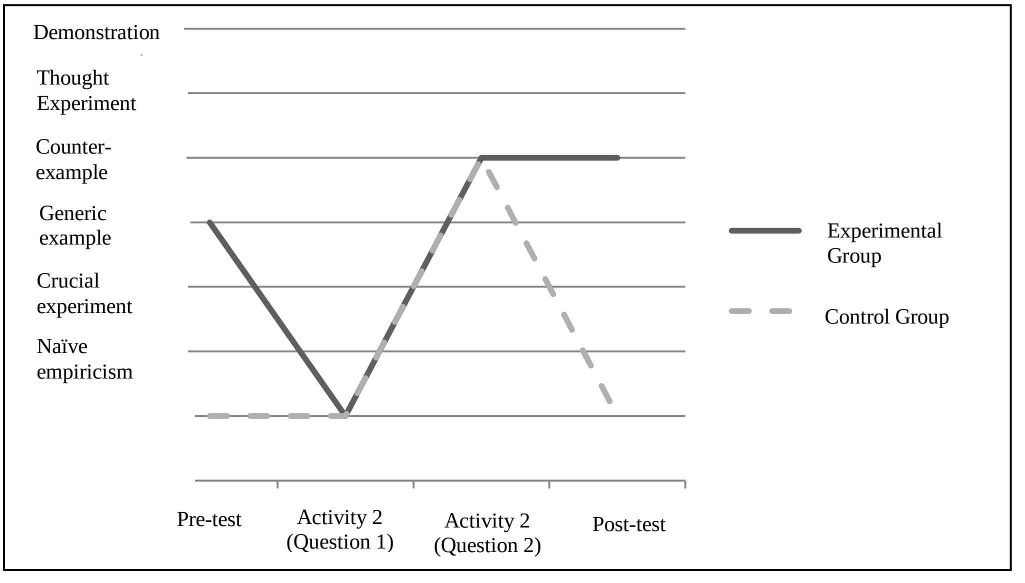

Figure 3. Evolution of the types of proofs that appear in a greater number in both groups in the pre-test, Activity 2 (questions 1 and 2) and the post-test.

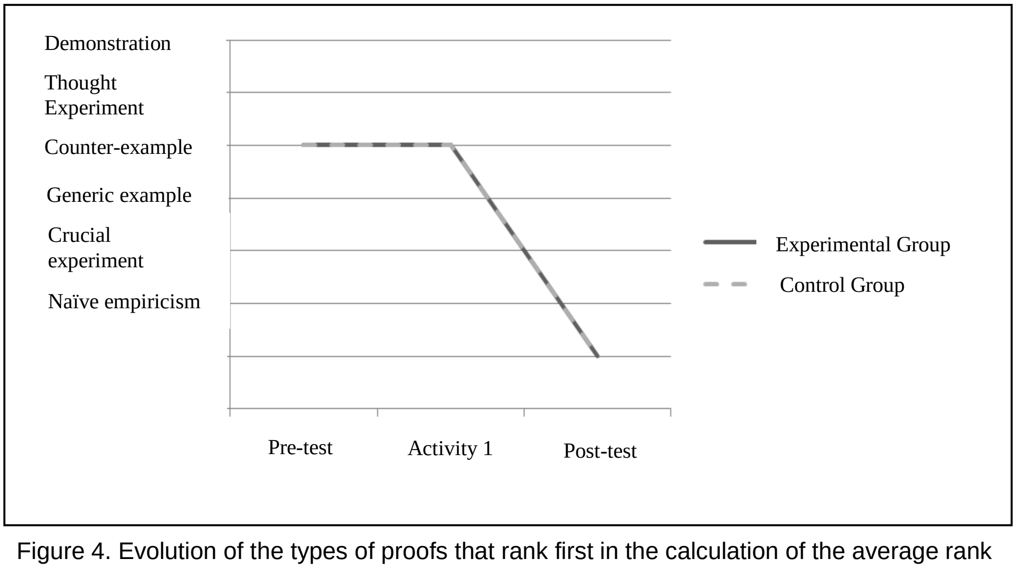

Figure 4. Evolution of the types of proofs that rank first in the calculation of the average rank for the two groups in the pre-test, Activity 1 and the post-test.

(In)validation of proofs

The analysis of the comments made by the students allows us to observe not only the rules mobilized, but also the accuracy with which these rules are used. In Activity 3, no student's comment in the control group mobilizes the rule of the mathematical debate associated with the use of a single counter-example to invalidate a statement. In the experimental group, many students recognize the power of the counter-example and demonstrate a proper use of this rule. The electronic forum may have an influence on the students' choice when they are faced with a counter-example. Indeed, in Activity 2, when working on paper, some students in the experimental group did not find a counter-example for the following statement: "If you replace n by any whole number in the expression $n \times n - n + 11$, you always get a prime number". On-line exchanges by this group of students, seen in the traces of the electronic forum, not only expose students to the existence of counter-examples, but also help them realize that a counter-example is sufficient to invalidate the statement. This awareness of the power of the counter-example carried out during Activity 2 is reflected in the students of the experimental group during Activity 3. Indeed, in this activity, almost three times more students of the control group (46%) than the experimental group (17%) say that a statement can be both true and false, because they don't recognize the power of the counter-example. Furthermore, a larger percentage of students in the experimental group (57%) than in the control group (37%) correctly find the right answer, which implies that the counter-example presented invalidates the statement involved.

In regards to the rules of the mathematical debate, the results allow us to note that the percentage of comments in which students mobilize these rules to justify their choice is higher in the control group than in the experimental group. However, a greater percentage of comments made by students in the experimental group than in the control group appropriately use these rules, especially the one related to the use of a counter-example to invalidate a statement.

THOUGHTS AND CONCLUSIONS

MAIN CONCLUSIONS

The results seem to indicate that the use of the electronic forum fosters the transition from pragmatic proofs to intellectual proofs and facilitates an appropriate use of the rules of the mathematical debate. In our opinion, this can be explained by the variety of messages that are posted on-line. Indeed, in on-line exchanges, students of the experimental group are exposed to various justifications that seek to validate or invalidate statements or evidence and these justifications are then themselves evaluated by students. Students thus learn to discern the arguments that are mathematically strong. It is also possible that the social pressure associated with the act of making one's comments public leads students to be more rigorous. On the other hand, the use of the electronic forum doesn't seem to have an influence on the proofs that students favour when they have to classify different types of proofs.

LIMITS OF THE RESEARCH

Certain limits are associated with this research project. First, the fact that we only work with four classes sometimes leads to a small number of participants, from which it is not possible to generalize the results. Problems encountered during the experiment (e.g., lack of time), which lead to some classes being unable to answer a few questions, also reduce the number of participants. The results therefore discuss the plausibility of the conjecture that the electronic forum has an impact on the work done by students more than it concludes in a sure way that the use of the forum has or does not have an influence on the work done by students. Second, we cannot ignore the limits associated with the use of an electronic forum. It is important to realize that any use of technology carries risks. For example, some students found themselves unable to post a message in the first activity.

ONE LAST WORD ON THE FORUM

Overall, the electronic forum seems to enrich the situation of formulation by promoting interactions. In some cases, it appears to be a lever for the social aspect. Exchanges that take place in the electronic forum also permit us to observe phenomena that escape us on paper. Indeed, when working on paper, strictly the results of class discussions are collected. In the electronic forum, interactions (which are not necessarily the final result) are recorded. In such cases, it seems that the forum can be an interesting engine of conceptualization. Therefore, it appears relevant to further research in order to continue the exploration of the possible influence that the use of an on-line communication tool, and more specifically the use of an electronic forum, can have on validation skills and algebraic skills related to the assessment of proofs.

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INSTITUTIONAL ACCULTURATION OF THE RESEARCHER, TEACHER, AND SECONDARY 1 STUDENTS WITH LEARNING DIFFICULTIES IN PROBLEM SITUATIONS INVOLVING RATIONAL NUMBER

LES EFFETS D'UNE DÉMARCHE D'ACCULTURATION SUR L'ACTION DIDACTIQUE CONJOINTE DE L'ENSEIGNANT, DES ÉLÈVES ET DU CHERCHEUR, DANS L'ENSEIGNEMENT/APPRENTISSAGE DES NOMBRES RATIONNELS AUPRÈS D'ÉLÈVES EN DIFFICULTÉS D'APPRENTISSAGE

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Our research focuses on the transformation of the way Secondary 1 students with learning difficulties deal with rational numbers. As shown by several studies, the major challenge is to avoid the vicious circle of reducing the learning issues and learning opportunities related to rational numbers for students with learning difficulties. To meet this challenge and rebuild a didactic memory that bears hope (Brousseau & Centeno, 1998), we focused on the 'ecological inscription' (de Rosnay, 1994) of rich and original situations, coordinated by 'learned knowledge' (connaissances) and 'taught knowledge' (savoirs). Our research aims to: 1) describe the progression of institutional processes of acculturation of the teacher, researcher and students and their effects on the process of development and management of teaching situations; and 2) specify the evolution of students' knowledge, habits and relationships with rational numbers. Our integration in the classroom for a period of six months, allowed us to assess the effects of the approach implemented in the joint didactical action of the teacher, student, and researcher. We noted significant changes in the topogenesis and chronogenesis of knowledge manifested particularly among students by means of: a) significant investment in complex situations; b) the adoption of mathematical practices more responsive to numerical data and relationships between these data; c) the appearance of 'unusual' and, at first look, 'useless' quite complex ideas. The results of our study support, therefore, the undeniable importance of considering students with learning difficulties as mathematically competent, as emphasized by Empson (2003), Houssart (2002) and Squalli, Mary and Theis (2011). The wealth of situations and didactic contracts that they solicit seem even more revealing in terms of the richness in individual variations than the only psychogenic characteristics of students as advocated by Fuchs et al. (2008).

♦♦♦♦♦♦♦♦♦

Notre recherche doctorale s'intéresse à la transformation des rapports aux nombres rationnels d'élèves de 1re secondaire présentant des difficultés d'apprentissage. Comme le montrent plusieurs recherches, le défi majeur est de ne pas s'enliser dans le cercle vicieux d'une réduction des enjeux de l'apprentissage des nombres rationnels et des possibilités d'apprentissage de l'élève en difficultés d'apprentissage. Afin de relever ce défi et de reconstruire une mémoire didactique porteuse d'espoirs (Brousseau & Centeno, 1998), nous avons misé, dans une démarche d'acculturation, sur l'inscription écologique (de Rosnay, 1994) de situations riches, originales et « défis ». Notre recherche vise à : 1) caractériser la progression des démarches d'acculturation institutionnelle de l'enseignant, du chercheur et des élèves et leurs effets sur les processus d'élaboration et de gestion des situations d'enseignement; 2) préciser l'évolution des connaissances, des habitus et des rapports des élèves aux nombres rationnels. Notre intégration en classe, d'une durée de 6 mois, nous a permis d'apprécier les effets de la démarche prônée sur l'action didactique conjointe de l'enseignant, des élèves et du chercheur. Nous avons noté des changements importants dans la topogénèse et la chronogénèse des savoirs, qui se sont manifestés, notamment chez les élèves, par : a) un investissement important lors de situations complexes; b) l'adoption de pratiques mathématiques plus attentives aux données numériques et aux relations entre ces données; c) l'apparition de conduites « inusitées ». Les résultats de notre recherche soutiennent donc l'importance indéniable de considérer les élèves en difficultés comme étant mathématiquement compétents, comme le soulignent Empson (2003), Houssart (2002) ainsi que Squalli, Mary et Theis (2011). La richesse des situations et des contrats didactiques que ces dernières sollicitent semblent plus à même de rendre compte des variations individuelles que les seules caractéristiques psychogénétiques de l'élève prôné par Fuchs et al. (2008).

DÉFIS DE L'ENSEIGNEMENT DES NOMBRES RATIONNELS ET ADAPTATIONS USUELLES

Depuis fort longtemps, les nombres rationnels attirent l'attention des chercheurs et des praticiens. Et pour cause! D'une part, plusieurs chercheurs (Biddlecomb, 2002; Bolon, 1996; Boulet, 1993; Brousseau et Brousseau, 1987; Chevallard & Julien, 1989; Grisvard & Léonard, 1981; Hackenberg & Tillema, 2009; Hasemann, 1981; Hart, 1980; Hiebert & Behr, 1988; Kieren, 1980, 1988; Lancup, 2005; Morissette, 2006; Moskal & Magone, 2001; Novillis, 1976; Perrin-Glorian, 1986; Lappan, 1987; Pressiat, 2003; Roditi, 2005) ont montré la *complexité* de leur apprentissage, notamment celle de surmonter de nombreux obstacles épistémologiques, de remettre en cause des modèles qui jusqu'à maintenant étaient valides avec les nombres naturels. Nommons à titre d'exemples, les conceptions selon lesquelles « plus un nombre contient de chiffres plus il est grand », « qu'il n'existe pas de nombre entre 3/4 et 5/6 », « que la multiplication permet d'obtenir une augmentation et la division d'obtenir une réduction ».

D'autre part, les nombres rationnels représentent un *objet riche et incontournable*. Comme le rappellent Kieren (1995) et Rouche (1998), ces nombres constituent un noyau fort important des mathématiques, modifiant en profondeur notre conception du nombre et servant de tremplin pour « penser » les nombres réels. Les difficultés des élèves relatives à cet objet d'enseignement entachent non seulement la poursuite de l'apprentissage de l'arithmétique, mais également des autres champs des mathématiques (algèbre, géométrie, probabilité).

Si l'enseignement/apprentissage des nombres rationnels représente un défi pour tout apprenant, celui-ci est amplifié lorsqu'il s'adresse à des élèves ayant des difficultés d'apprentissage. En effet, ceux-ci entretiennent fréquemment des rapports problématiques et des habitus[1] (Bourdieu, 1980) peu productifs à l'acte d'apprendre. Ce qui ne va pas sans altérer leur rapport aux nombres rationnels; ils désinvestissent les tâches, ils se réfugient dans des calculs qu'ils essaient de reproduire. Ces conduites ne sont pas le fruit du hasard : elles révèlent une fréquente confrontation à l'échec et sont empreintes des pratiques enseignantes.

Dans le même ordre d'idées, si l'enseignement/apprentissage des nombres rationnels auprès d'élèves en difficulté d'apprentissage représente un défi de taille, il est une fois de plus rehaussé dans l'enseignement de première secondaire. Pour les élèves faisant leur entrée dans une nouvelle institution scolaire, une rupture entre les mémoires didactiques (Brousseau & Centeno, 1991; Centeno, 1995) des enseignants et des élèves est prévisible. En effet, les enseignants n'ont pas accès, entre autres, aux pratiques et aux contextes associés aux objets de savoir qui ont donné sens à leurs connaissances sur les nombres rationnels. En revanche, les mémoires de leurs élèves sont façonnées de pratiques, de situations et d'événements didactiques qu'ils ont intégrés depuis leur entrée à l'école primaire. De plus, dans le programme de formation de l'école québécoise, le découpage conféré aux opérations impliquant des nombres rationnels lors de la transition primaire/secondaire nous montre le saut conceptuel fort important auquel seront confrontés les élèves (Stegen & Daro, 2007; Bednarz, 2009).

Dans ces conditions, il est difficile de faire abstraction d'un questionnement quant aux différents aménagements possibles et souhaitables auprès de cette population. Les adaptations fréquemment répertoriées (Giroux, 2007; Martin & Mary, 2010) telles les piétinements (Lemoyne & Lessard, 2003), les surinvestissements (Conne, 2003), la reprise d'activités (Sensevy, 1998) et le pilotage pas-à-pas de la situation d'apprentissage sont fort discutables. Elles représentent souvent un changement de la nature du savoir en jeu, contribuent à la baisse de désir et d'appétence d'apprendre des élèves et, plus encore, les contraignent, les restreignent dans « leurs » difficultés. Il nous est donc apparu nécessaire d'infléchir ce cercle vicieux.

REPENSER L'ENSEIGNEMENT POUR RELEVER LES DEFIS ET DESAMORCER LE CERCLE VICIEUX

Afin de reconstruire des habitus productifs et des rapports adéquats aux nombres rationnels, il nous a semblé nécessaire de leur soumettre des situations : 1) *riches,* pour revisiter des savoirs anciens problématiques tout en avançant; 2) *originales* (contrat didactique), pour ne pas que l'élève reconnaisse cette tâche, lui renvoie une image d'échec; 3) *défis,* pour exiger un engagement cognitif de la part de l'élève. En ce sens, la résolution de problème nous est apparue la voie privilégiée. Cependant, pour que les situations produisent les effets escomptés, il n'en demeure pas moins que leur dévolution auprès des élèves est indispensable. Ainsi, le besoin de procéder à l'inscription écologique des situations de résolution de problèmes a orienté notre démarche, démarche requérant l'acculturation de tous les acteurs (enseignant, élèves et chercheur) et qui positionne le chercheur tant qu'« écocitoyen [qui] doit mieux comprendre comment situer et insérer son action locale dans un système global » (de Rosnay, 1994, Ecologie et approche systémique, para.9). Pour l'ensemble des acteurs, le processus d'acculturation fait référence à un contact direct et prolongé permettant une

[1] L'habitus fait référence aux schèmes producteurs de pratiques; les actes qu'un élève pose et leurs résultats exercent une influence non négligeable sur sa perception des choses et sur ses dispositions à agir et à interpréter les événements qu'il rencontre.

modification dans les modèles culturels initiaux de l'un ou des groupes se traduisant, notamment par des appropriations, ou des réinterprétations d'éléments culturels.

OBJECTIFS ET MÉTHODOLOGIE

C'est dans l'esprit de repenser autrement l'enseignement des nombres rationnels auprès des élèves en difficultés que nous avons poursuivi les objectifs suivants : 1) caractériser la progression de la démarche d'acculturation institutionnelle de l'enseignant, du chercheur et des élèves sur les processus d'élaboration et de gestion des situations d'enseignement; 2) préciser l'évolution des rapports, des connaissances et habitus des élèves aux nombres rationnels au cours de la séquence d'enseignement. Notre recherche doctorale de méthodologie qualitative a été menée auprès de 17 élèves de 1re secondaire en difficultés graves d'apprentissage fréquentant une école spécialisée. Elle s'est étalée sur une période de six mois à raison de 46 périodes en classe. La Figure 1 ci-dessous illustre la démarche d'acculturation prônée que nous préciserons par la suite.

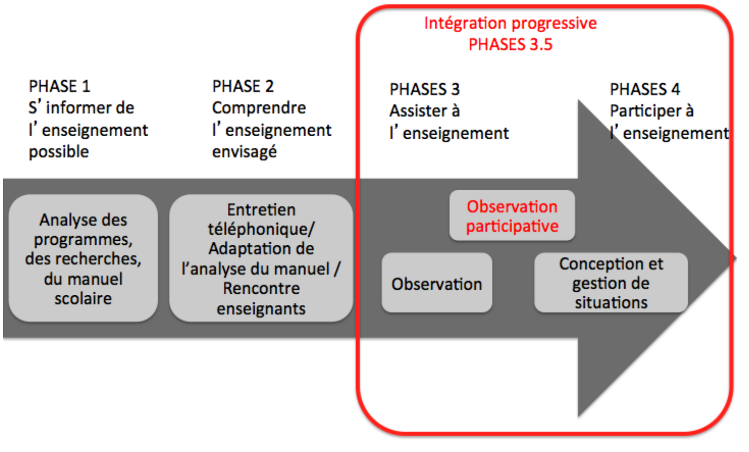

Figure 1. Démarche d'acculturation.

PHASE 1. S'INFORMER DE L'ENSEIGNEMENT POSSIBLE

La ressource matérielle principale de l'enseignant est le manuel scolaire. Il est donc tout à fait essentiel de l'examiner afin d'effectuer une entrée respectueuse, instruite et harmonieuse du chercheur dans l'institution scolaire. Ce travail nous permettra de mieux comprendre les contraintes et les possibilités didactiques qu'offre son usage ainsi que le projet de l'enseignant (ex. situations privilégiées).

PHASE 2. COMPRENDRE L'ENSEIGNEMENT ENVISAGE

Les premiers contacts du chercheur avec l'enseignant, tenant compte du fonctionnement de leur institution et des apports précieux qu'ils peuvent fournir aux chercheurs préoccupés par l'enseignement des mathématiques, sont fort importants. Ainsi, lors de cette rencontre, nous nous sommes intéressées à l'appréciation et à l'utilisation du manuel par l'enseignant, puis à son mode de fonctionnement. Nous avons également présenté quelques situations afin de connaître leur potentiel d'inscription au projet de l'enseignant.

PHASE 3. ASSISTER A L'ENSEIGNEMENT

Au cours d'une première phase d'intégration, nous avons assisté à l'enseignement dispensé par le titulaire de mathématiques et nous avons assumé, en tant que chercheur, les fonctions suivantes : prise de notes; échanges avec l'enseignant; interventions pendant l'enseignement, afin de soutenir le travail de certains élèves. Ces interventions s'avèrent particulièrement précieuses. D'une part, elles permettent de prendre acte des rapports de plusieurs élèves, de leurs habitus, de leurs représentations du contrat didactique (Brousseau, 1980). D'autres parts, elles sont des moments privilégiés pour une acculturation institutionnelle des élèves et du chercheur. Durant cette phase, l'enseignant assume ses rôles habituels. L'enseignant et le chercheur échangent leurs observations et formulent différentes questions. Un tel partage est une source d'informations pour le chercheur et l'enseignant. Le chercheur peut ainsi bénéficier d'observations sur les conduites des élèves effectuées par l'enseignant et de l'expertise de l'enseignant pour interpréter certaines observations qu'il a consignées. L'enseignant peut, de son côté, mieux appréhender les objectifs poursuivis par le chercheur. Soulignons enfin que cette première phase constitue une étape décisive du processus d'acculturation institutionnelle du chercheur, de l'enseignant et des élèves.

PHASE 4. PARTICIPER A L'ENSEIGNEMENT : CO-PLANIFICATION, CO-ENSEIGNEMENT, CO-EVALUATION

Les phases précédentes servent ainsi d'assises pour une inscription écologique de situations, situations empreintes de l'institution classe. Cette progression nous amènera à réagir avec plus de discernement et facilitera la conception ainsi que la gestion de diverses situations : a) situations issues de ressources du milieu comportant, au besoin, certaines adaptations; b) situations construites en référence aux études effectuées dans le domaine prenant appui sur l'analyse des conduites des élèves de la classe; c) situations découlant d'une analyse de besoins spécifiques exprimés par l'enseignant.

RESULTATS ET DISCUSSION

Dans le cadre de notre recherche doctorale, plus précisément des phases 3 à 4 de notre démarche d'acculturation (Figure 1), nous avons participé à 25 situations dont neuf ont été construites par l'enseignant, huit par les chercheurs et huit co-construites. Dans le cadre de cette communication, nous présenterons brièvement deux effets du processus d'acculturation. Le premier rend compte de l' « étonnante » participation des élèves à l'enseignement et de l'influence de cette modification de conduites sur la modulation des pratiques de l'enseignant. Le second exemple illustre l'influence du processus d'acculturation sur la conception, la gestion et l'impact des situations proposées.

EFFETS DU PROCESSUS D'ACCULTURATION

Participation des élèves à l'enseignement et modulation des pratiques enseignantes

La première activité[2] présentée le 30 janvier par le chercheur, *Dites-le avec des fleurs* (Bélisle, 1999), a suscité un intérêt particulier des élèves et de l'enseignant. Cet événement a conféré « une pertinence institutionnelle » au chercheur, a contribué à l'inscription écologique et à la dévolution de situations défis, originales et riches. Ayant trouvé, présenté et compris plus de cinq démarches différentes, les élèves se sont projetés dans un contrat « de plaisir, de mises au défi, de liberté d'action ». Nous avons tenté de tirer profit de ce contrat en leur

[2] La description de l'activité et des raisonnements des élèves sont décrits dans les Actes de Colloque du GDM, 2007, pp.169-182. En ligne : http://turing.scedu.umontreal.ca/gdm/html/actes.html

proposant, le 5 février, une nouvelle tâche[3] qui leur offre un espace non négligeable, plus précisément qui exige d'eux de produire le plus grand nombre de représentations possibles. Encore une fois, un élève nous (enseignant, élèves et chercheurs) a étonnées avec la représentation suivante de 0,3:

90 80 70 60

$$\frac{100\% - 0{,}100 - 0{,}10 - 10\% - 10\%}{2}$$

S'attarder et jouer avec les nombres, se mettre au défi et se donner une liberté d'action ne sont pas des habitus usuels chez les élèves en difficultés d'apprentissage. Or, diverses conduites ont abondé en ce sens :

- Il y a ces élèves qui, dans le cadre d'une situation problème de proportionnalité simple visant la construction de sens de la multiplication de nombres décimaux, s'attardent aux nombres et à leur relation afin de choisir une démarche économique. Il ne faut d'ailleurs pas négliger que ces élèves n'avaient pas eu d'enseignement formel d'algorithme de calcul.
    - Un élève décide de faire 15 × 175 au lieu de 150 × 17,5 malgré la proposition d'une intervenante de réaliser 150 × 175.
    - Un élève décide de se servir du nombre de gâteaux obtenus pour 10 boîtes et 1 boîte afin de générer celui pour 45/5 boîtes.
- De même, cet élève qui s'arrête et se questionne sur les coûts qu'il obtient pour ½ journée (21,30$) et 500 jours (21300$), à savoir la raison pour laquelle ses résultats sont composés des mêmes chiffres.

Un tel contrôle relève d'une quête de sens et d'un intérêt à réaliser la tâche pour elle-même. Ces élèves s'éloignent d'une « conceptualisation des opérations réduite à l'apprentissage de l'algorithme de calcul » relevée par Barallobres et Lemoyne (2006, p. 185).

Ce ne sont que quelques exemples de conduites qui nous ont amenées, enseignant et chercheurs, à modifier également nos pratiques. Nous avons noté des changements importants dans la topogénèse et la chronogénèse des savoirs (Mercier, 1995). Un de ces exemples concerne l'enseignant qui adoptait généralement la démarche d'enseignement suivante : effectuer un exposé des savoirs et des démarches que les élèves devaient consigner dans leurs notes de cours, afin de pouvoir par la suite s'y référer pour effectuer des exercices et, enfin, résoudre des problèmes. Elle a progressivement modifié cette démarche en proposant des problèmes qui permettraient aux élèves de coordonner diverses connaissances et de construire ainsi des savoirs auxquels ils pouvaient faire référence dans la construction de leurs notes de cours qu'ils pouvaient par la suite consulter pour effectuer divers exercices. L'une de ces illustrations est celle de la multiplication des nombres décimaux qui oppose, dans le Tableau 1 ci-dessous, ce qui était prévu à ce qui a finalement été réalisé avec les élèves le 12 février.

Nous avons effectivement soumis aux élèves des situations de plus en plus responsabilisantes et axées sur le sens pour délaisser un enseignement plus techniciste, ce que les élèves ont accepté. C'est dans cet esprit que le processus graduel d'acculturation nous a permis de concevoir et gérer des situations leur permettant de s'engager dans les tâches à première vue « menaçantes » telles que nous l'exposons dans la prochaine section.

[3] Pour une description de l'activité et des conduites plus détaillées, consultez les Actes de Colloque du CIEAEM61, pp.342-347. En ligne : http://math.unipa.it/~grim/cieaem/quaderno19_suppl_2.htm

théorie/prise de notes de cours/exercices/problèmes	problèmes/construction notes de cours/exercices
Notes de cours de dictées par l'enseignant : 1. Pour multiplier deux nombres décimaux, tu fais la multiplication sans tenir compte des virgules. 2. Tu comptes le nombre de chiffres après la virgule. 3. En commençant par la gauche, recule la virgule d'autant de positions qu'il y a de chiffres après la virgule. ***Tu peux t'aider en faisant l'estimation du produit sans la partie décimale.	Notes de cours construites par les élèves : Changer la multiplication pour n'avoir que des entiers. Ex_1 $6{,}35 \xrightarrow{\times 100} 635$ $\times\ \underline{12{,}6} \xrightarrow{\times 10} \times\ \underline{126}$ 3810 $+\ 12700$ $\underline{63500}$ $80{,}010 \xleftarrow{\div 1000} 80010$ Ex_2 $0{,}984 \xrightarrow{\times 1000} 984$ $\times\ \underline{0{,}195} \xrightarrow{\times 1000} \times\ \underline{195}$ 4920 $+\ 88560$ $\underline{098400}$ $0{,}191880 \xleftarrow{\div 1000000} 191880$

Tableau 1. Modification chronogénèse/topogénèse.

<u>Inscription écologique de situations riches, « défis » et originales</u>

Diversité et quantité des nombres proposés dans une tâche de comparaison

La phase 1 du processus d'acculturation (Figure 1) nous a permis d'identifier, en amont, des niches (Chevallard, 1994) potentielles et de cibler des conditions affectant la relation entre l'élève et la tâche. Par exemple, le rapport des élèves aux tâches de comparaison dépend, entre autres, de : a) la quantité de nombres à comparer (Mazzocco & Devlin, 2008); b) la présence de différentes représentations (fractions, pourcentages, décimaux) (Bednarz, 2009); c) la présence de nombres à virgule dont la partie décimale ne comporte pas le même nombre de chiffres (Grisvard & Léonard, 1981). Ainsi, souhaitant leur proposer des situations riches, « défis » et originales, nous avons décidé de les plonger dans de telles conditions. Il faut cependant comprendre qu'une telle proposition n'aurait pu être faite dès notre entrée dans le milieu, une acculturation des acteurs était d'abord nécessaire. En effet, la quantité et la complexité des nombres auraient rebuté les élèves. Cependant, la modification de divers habitus et conduites d'élèves, comme nous en avons fait été précédemment, nous ont porté à croire que la dévolution d'une telle situation serait possible.

Nous avons donc misé sur une diversité et une quantité de représentations plus importantes qu'habituellement afin d'obliger les élèves à une analyse des données. Pour exposer ce contraste et, par le fait même, l'influence du processus d'acculturation sur la conception de situations, nous présentons parallèlement les deux situations dans le Tableau 2 ci-dessous. La première est issue du manuel en usage et avait été « programmée » par l'enseignant en début d'année pour cet objet d'apprentissage et l'autre que nous avons finalement proposée aux élèves après trois (3) mois de présence en classe.

L'une des premières réactions d'une élève a été celle de dire : « C'est juste qu'on ne sait pas sur quoi le mettre sinon on serait capable ! » Elle venait de confirmer que nous avions eu raison de lui proposer la situation « modifiée ». Le choix de nos variables didactiques (quantité et la diversité des nombres) obligeait les élèves à prendre le temps de s'arrêter et à s'intéresser aux nombres avec lesquels ils travaillent afin de choisir la démarche la plus économique. Cette situation n'offrait pas le même potentiel d'apprentissage. Là, il était nécessaire de coordonner diverses connaissances et le simple recours au dénominateur commun en multipliant les dénominateurs pour comparer n'était plus suffisant.

Activité programmée au début de l'année	**Activité finalement proposée aux élèves (30 avril)**
1) 5/7; 3/7; 4/7; ...; a) Quelle est la fraction la plus petite ? b) Quelle est la fraction la plus grande ? c) Quelles fractions sont supérieures à ½ ? 2) 1/5; 1⁄4; 1⁄2; 1/6; ...; a) Quelle est la fraction la plus petite ? b) Quelle est la fraction la plus grande ? c) Quelles fractions sont supérieures à ½ ? 3) 3/5; 3⁄4; 3/2; 3/6; ...; a) Quelle est la fraction la plus petite ? b) Quelle est la fraction la plus grande ? c) Quelles fractions sont supérieures à ½ ? 4) 1/2; 2/3; 4/7; 2/9; 10/11; 3/6 et 3/8. a) Quelle est la fraction la plus petite ? b) Quelle est la fraction la plus grande ? c) Quelles fractions sont supérieures à ½ ?	Christine et Geneviève ont reçu chacune une tablette identique de chocolat noir Lindt très très mince qui présente l'allure d'une feuille quadrillée. Elles conviennent de donner une partie de leur tablette à 4 de leurs amies, mais elles ont des préférences. Voici ce que chacune recevra : **Christine a partagé ainsi sa tablette :** 0,125 de sa tablette à Chantale 31/124 de sa tablette à Élisa 9/48 de sa tablette à Yéran 100/320 de sa tablette à Karine et le reste ...pour elle! **Geneviève a donné :** 3/16 de sa tablette à Dan 3/24 de sa tablette à Judith 50/160 de sa tablette à Amélie 25% de sa tablette à Carole et le reste ...pour elle! Pour effectuer cette tâche, il est absolument indispensable que vous utilisiez le papier quadrillé et que vous soyez très très très précis dans vos mesures, de telle sorte qu'il soit facile de comparer la part de chacune. a) Qui est la plus généreuse? b) Quelle amie Christine préfère-t-elle? c) Quelle amie Geneviève préfère-t-elle ?

Tableau 2

Bien que l'activité n'ait pas permis à tous les élèves d'atteindre le même niveau de compréhension, ils ont su traiter conjointement diverses connaissances, connaissances impliquées dans les activités antérieurement réalisées comme le présentent ci-dessous les procédures connues et novatrices utilisées :

1. Procédures connues [Notes de cours : fractions équivalentes]

- division par un commun diviseur — ex. 50/160 [÷ 2/÷ 2] = 25/80 [÷ 5/÷ 5]= 5/16
- division par le plus grand commun diviseur — ex. 50/160 [÷ 10/÷ 10] = 5/16
- produit croisé — ex. 3/24 = ?/16...16 x 3 ÷ 24 = 2...2/16

2. Procédures novatrices

Sens rapport — ex. 3/24...24 ÷ 3 = 8 ...1/8

Sens quotient — ex. 3/24...3 ÷ 24 = 0,125 = 1/8; ex. ¼...0,250 ....1/8....0,125

Coordination de différents registres — ex.125 millièmes; 125/1000; 12,5%; 0,125

Ainsi, lors de la conversion ou le traitement des différents registres (Duval, 1995), diverses stratégies novatrices ont été mises de l'avant : 1) rechercher le nombre de fois que le numérateur entre dans le dénominateur (sens rapport); 2) représenter le nombre fractionnaire sous une forme décimale en divisant le numérateur par le dénominateur (sens quotient); 3) appliquer des procédures connues (contenues dans les notes de cours) comme diviser par un commun diviseur, diviser par le plus grand commun diviseur et effectuer un produit croisé; 4) coordonner différents registres sémiotiques, conversion et traitement. D'ailleurs, la majorité des élèves a su exploiter à bon escient le sens rapport de la fraction, afin de mettre en œuvre une démarche plus économique. Compte tenu de la persistance du sens partie-tout dans l'enseignement (Barallobres & Lemoyne, 2006; Blouin, 1993; Kieren, 1988), de son accessibilité dans cette activité et des gestes connus pouvant être exploités, ces conduites novatrices ne sont pas négligeables. Aussi, certains élèves ont su déployer des démarches contrôlées tout à fait inusitées. Citons, à titre d'exemple, une équipe qui exploitait une écriture intermédiaire pour la transformer par la suite en une écriture fractionnaire [31/124 = 0,25/1 = 0,25 = 1⁄4]. Bien que ce ne soit pas la plus efficace, elle a l'avantage d'attester des connaissances des élèves (fractions équivalentes, que la division par 1 ne change pas le résultat, etc.) et, surtout, d'exposer leur flexibilité ainsi que leur engagement dans la tâche. L'exposition des élèves à ce type de situation et leurs conduites qui en découlent nous illustrent bien les effets d'une démarche d'acculturation sur l'action didactique conjointe de l'enseignant et des élèves.

CONCLUSION

Notre recherche avait pour but de caractériser la progression de la démarche d'acculturation institutionnelle de l'enseignant, du chercheur et des élèves sur les processus d'élaboration et de gestion des situations d'enseignement et de préciser l'évolution des rapports, des connaissances et habitus des élèves aux nombres rationnels, au cours de la séquence d'enseignement. S'il est un aspect indéniable de cette étude, il s'agit bien de la réussite du processus d'acculturation, d'un maillage fécond des différentes cultures qui a permis de l'insertion écologique de situations originales, « défis », et riches dans la transformation des rapports et des habitus des élèves aux nombres rationnels. En effet, nous avons assisté à diverses modifications dans les modèles culturels initiaux des différents groupes qui se sont manifestées par un investissement de l'objet de la connaissance de la part des élèves (dévolution). La richesse des situations et des contrats didactiques que ces dernières sollicitent semblent plus à même de rendre compte des variations individuelles que les seules caractéristiques psychogénétiques de l'élève prônées par Fuchs et al. (2008). En effet, nous avons réussi à relever le défi de ne pas s'enliser dans le cercle vicieux de réduction des exigences auprès des élèves en difficultés et à créer une mémoire didactique porteuse d'espoir.

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MATHEMATICS EDUCATION: AN APORETIC OF EPISTEMOLOGY, LANGUAGE AND ETHICS

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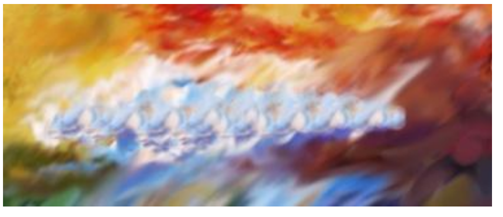
La poésie est une pipe—Magritte



Writing research in mathematics education, one is commonly asked to present questions, methods, concepts and findings, as if the research and the writing itself were mere intentional activities by means of which 'knowledge' is produced. Tout à l'opposé de cette vision volontariste, Derrida explique comment écrire (et je propose d'ajouter « faire de la recherche », mais également toute entreprise de « connaissance » y compris les explorations mathématiques des élèves) évoque plutôt une « descente hors de soi en soi du sens […] métaphore comme possibilité d'autrui […] où l'être doit se cacher si l'on veut que l'autre apparaisse ». From such a perspective, researching teaching and learning are moments/occasions in which knowing in mathematics education is always already knowing-with one another, and therefore constitutes an ethical relation… pour laquelle tout indique qu'il nous faut un nouveau langage.

Je propose de (re)lire ma thèse doctorale (essentiellement composée de pages tirées d'un carnet de voyages autour de la question « How Do We Know in the Day-to-Day, Moment-to-Moment of Researching, Teaching and Learning in Mathematics Education ») highlighting languaging issues (in both senses of the word) en lien avec l'activité de connaissance dans le monde de l'éducation mathématique vue comme rencontre avec l'autre. Abandonnant l'être de la chose (cherchée, enseignée, connue) for the becoming (in researching, teaching and learning), on verra (dans) l'activité des chercheurs, enseignants et élèves (se) (re)produire l'un l'autre à même chacun de leurs gestes. From there, one might then be expected to in-vite rather than (re)present, playing together passivity and intentionality, que ce soit au moment d'écrire ou de lire, de dire ou d'entendre.

INTRODUCTION

WHERE DID IT ALL BEGIN?

He arrived in Victoria (B.C.) to begin his doctoral studies. This was only a few days after he finished writing his Master's thesis. He was to work on a project pertaining to elementary-school children's mathematical understandings. Walking into the lab that very first day, he met a few graduate students and found on his desk, waiting for him, a pile of DVDs with classroom data. Soon enough, he was wholeheartedly engaged with recordings, reading whatever he could put his hands on, working out studies. Over and over again…

ÉCRIRE

Le problème de *l'écriture* lui apparaît très vite comme central pour la recherche autour de l'éducation mathématique. Non qu'il s'y pose de manière particulière (par rapport à d'autres disciplines), mais parce qu'il se présente véritablement *à lui*, dans le cadre du travail par lequel il tâche de s'*inscrire* dans une certaine communauté. Écrire en didactique des mathématiques semble généralement accessoire au travail de recherche lui-même. L'écrit est le « produit final » dans lequel on fera montre de questions de recherche bien justifiées et bien formulées, suivies d'éléments de méthodes et des balises théoriques permettant clairement de traiter ces questions (et idéalement « d'y répondre »)… du moins sur papier. Les sections suivantes présentent des analyses/résultats, sans doute suivies ou accompagnées d'une discussion, le tout repris en une conclusion où on voudra bien nous dire « quoi retenir » de tout ceci. Mais comment en viendra-t-il là?

He looks at the computer screen, where the dc1_feb12_mj.mov file had been playing for about 53 minutes. The camera shows a second grade classroom in which students sit at their desks, while an adult (he chooses to call her Rachel) stands at the blackboard. Seven sheets of paper, each presenting a rectangle the students drew using the same piece of string, are aligned up on the board as a result of a sorting activity (Figure 1). Rachel questions the students on the number of squares each rectangle contains (their area) before drawing their attention to the perimeter:

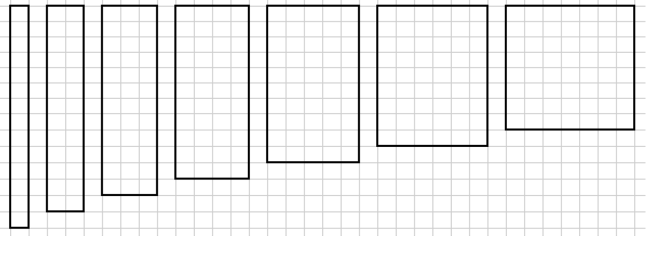


Figure 1. The 7 rectangles produced by the students.

Rachel: Tom, what was the same about all these rectangles? They all have…

Tom: These two ones….

Rachel: Oh, but we are talking about *all* these rectangles, all of them. They *all* have what?

Tom: They all... have… I don't know.

Luc: Hum... They are all rectangles because they are tinier than square, they are longer and skinnier.

Rachel: What else is the same about all of them? Melinda?

Melinda: They all have 4 edges and 4 vertices.

Rachel: They all have 4 edges or 4 sides, and they all have 4 vertices or 4 corners. And here is something else. They all use the same ... did we use different loops of string? Did you use a different loop of string for each rectangle?

Students: Noooo!

Rachel: They were all made with the... *same piece of string*! Yet, they made very different rectangles from the same piece of string. Charles?

Charles: This one has like one, this one has 2 squares at the bottom, this one has 3, and 4, and 5, and 6, and 7 squares.

Rachel: Yes. So you're saying you've think of 1, 2, 3, 4, 5, 6, 7. And you say… did you say you see a pattern?

Charles: Yes.

Rachel: That's very good of you to see that. There is a pattern there. You're getting wider and wider and wider ok! Conrad?

Conrad: I see 14, 13, 12, 11, 10, 9, and then 8, 7, 6, 5, 4, 3, 2, 1.

Rachel: That's a very interesting pattern as well. It goes 1 through 7 and the other way 9 10 11 12 13 14. Charles and Conrad see a number pattern going on. My question… talk to your partners and see if you have any ideas as to *why* this number pattern comes out of these rectangles.

What's happening in there? If he was to write about that, what are the questions? What is the method, what are the concepts, is this the data, how should it be analysed, what could be the result of such analysis, and what conclusions and implications should we derive from answering all these questions? So many things are taking place! How about writing on the way Rachel and the students *together* call upon mathematics as a "science of patterns" (Schoenfeld, 1992), and assert:

> ASSERTION: Enculturation into mathematics requires developing cognitive practices students cannot intend (Latin *ad* + *tendere*, to stretch toward), but nevertheless attend to (Latin *in* + *tendere*, stretch out). Learning mathematics is possible when students' 'tension' becomes contention, that is stretch-with (*com* + *tendere*) the socio-material environment (including the teacher).

Doesn't this episode provide a nice setup to think this idea, see what it could mean and where it might take us? He felt he could do that. But then again … why not rather use this episode to write the paper called "Mathematics for Love" he keeps thinking about since he read Maturana's striking:

> *Only love expands our intelligent behavior, because it expands our vision. Love is visionary, not blind. Accordingly, for the educational space to be a relational space of expansion of the intelligent behavior of the students and teachers, it must be lived in the biology of love. The biology of love are relational dynamics that conserves and fosters the self respect of the students, even when it seems necessary to correct their doings. […] But values, spirituality, honesty and justice cannot be taught as courses in a school, they must be lived at all moments as spontaneous aspects of daily life, and one should speak of them only as commentaries and reflections when they are momentarily lost due to errors and mistakes that we commit in our co-existence.* (Maturana & Rezepka, 1997, p. 19)

An assertion in that line of thinking could then look like:

> ASSERTION: For mathematics education to promote respecting oneself, others and the environment, researchers and educators must *recognize* the various means by

> which knowing/doing mathematics is produced, which includes all aspects of discursive, emotional, and sensorial experiences. Although necessary, this re-cognition (to know or to think again) *can* take place *with* students, and even be acknowledged as an integral part of mathematical activity (as a way of co-existing).

In any case, method, concepts, data, analyses and conclusions would come together *with/in* (with, in, and within) the writing, through back and forth movement between different parts of the 'study' built around the assertion. 'Reaching in' would help develop the assertion and sustain it by discussing gestures, silences, intonations, turn-taking, and so on. And at the same time, 'reaching out' would contribute to situating the piece, giving it a particular tone, through reading stuff, selecting a journal for publication, getting feedback from his supervisors/co-authors, and also from friends and colleagues (those who kindly ask "so, what are you working on?").

Tant de choses se donnent à voir, à penser. L'enjeu alors n'est plus en soi celui d'avoir quelque chose à dire, à rapporter. Il faut pouvoir dire, écrire, donner à lire. Tout est là, mais rien n'y est tant que la parole n'a pas tranché, tant qu'elle ne s'est pas mêlée à d'autres voix, tant qu'elle n'a pas été entendue.

THEORY

LA FORME ET LE FOND

C'est sans doute d'abord le *format* de la recherche entreprise qui mit si rapidement en lumière pour lui les difficultés de voir la recherche comme quelque chose d'existant en dehors de l'écriture, et que l'on projette ou rapporte au moment d'aligner les mots. Il s'engageait dans la rédaction d'une thèse par articles, thèse à réaliser dans le contexte d'un projet de recherche dont les bases étaient pour ainsi dire déjà jetées. Mais encore : dans une perspective ouverte, exploratoire. Il s'agissait donc de s'assoir et de lire, de regarder des vidéos, de faire des transcriptions, de formuler des hypothèses, de lire encore, de préciser des intuitions, de trouver un langage, des questions, des concepts, des arguments … et d'écrire : un article, un chapitre, un texte de colloque …. Écrire ce qu'il voulait, repartir en quête de données, changer de cadre théorique, s'intéresser à autre chose que ce qui avait été ciblé au départ, mais *écrire*. Écrire et réécrire (cent fois sur le métier … etc.) et par l'écriture donner corps à quelque chose « qui tienne » du point de vue de la recherche, pour l'auditoire et le médium qu'il se serait fixé. On voit encore les traces (vivantes) de ses échanges avec son superviseur principal : *What is your claim? Who are you writing this for? How is this relevant to mathematics education? What is the structure of your argument? What theory are you using? What does it mean, this concept you use? So what?* Ces questions, ne se posent-elles pas à tout moment et pour toutes recherches dans le monde de l'éducation mathématique? Ne sont-elles pas fondamentales à toutes approches de recherche dans le domaine? Elles apparaissent pourtant, souvent, fort tard dans la démarche … et semblent souvent de moindre importance. Il faut pouvoir écrire, certes. On attend de vous quelque chose d'écrit. C'est peut-être même tout ce qu'on attend, au bout du compte. Mais cela encore, il ne suffit pas de le penser, de le dire : il faudrait pouvoir l'écrire et « en faire quelque chose ». Sinon simplement répéter, re-dire, ré-écrire un certain Brown, un certain Mason, ou qui d'autre :

> *I hope that you may rightly accuse me of making more of my experience than is warranted for it is quite the opposite that typifies most conclusions of empirical research in mathematics education. They are at best safe, provide little resonance with our own experience and leave us with little desire to open our eyes and minds widely upon experiencing similar events in the future.* (Brown, 1981, p. 11)

> *…what is it that we are gaining by reporting on our studies? My radical response to such a question is that what matters most is educating awareness by alerting me to something worth noticing because it then opens the way to choosing to respond rather than react with a more creative action than would otherwise be the case.* (Mason, 2009, p. 12)

In other words, truly work with the idea that answers are always reactions, never solutions. How is this relevant to mathematics education? "Try to love the questions themselves like locked rooms and like books that are now written in a very foreign tongue" (Rilke, 1934, pp. 33-34).

Mais encore, écrire c'est d'abord *trouver les mots*. Trouver ou *créer?* La recherche, surtout quand on la vit si proche de l'écriture, c'est la quête, la question (du Latin *quaerere*) des termes (un mot dont l'origine est le Grec *horos,* 'frontière'), des formules, des expressions. Le choix de ce que les mots vont dire et ne diront pas, les lignes (les courbes, les contours) qu'ils vont dessiner, ce qu'on va pousser par devant soi et donner à lire. Ça aussi il l'expérimente très vite : les mots ne collent pas, les mots ne viennent pas, ils ne se donnent pas, il ne les « trouve » pas comme ça. Toujours il faut chercher, toujours en créer, les jouer entre eux (ces fameux jeux de mots!) pour *dire* au delà de ce qui est *dit*. D'où ce malin plaisir qu'il découvre à retourner les mots (souvent contre eux-mêmes) à la manière de Heidegger, qui avait si bien compris comment « le langage nous pense » au moment même où nous pensons avec lui. Il lui faut travailler des mots d'autant plus que cette thèse, il l'écrit dans une « langue étrangère », une langue « seconde ». Les mots ne sont alors que plus nouveaux et redoutables…. Parfois tellement libérateurs, aussi : en un tour de main (de langue), il s'était débarrassé du « didactique » qui l'avait toujours embêté, tellement plus à l'aise de parler de *mathematics education*. Mais le nœud de la langue et son étrangeté lui restent bien présents. Derrida (1996) explique : « Je n'ai qu'une langue, et ce n'est pas la mienne » (p. 13). C'est bien « son » regard qui s'écrivait, ses tournures à la française qui persistaient, ses fautes de grammaire, son côté verbeux, son sens du rythme, ses métaphores … mais rien de tout cela ne lui appartient vraiment : les mots sont toujours ceux de quelqu'un d'autre. Rien pourtant qui n'appartient pas qu'à lui. Il faut parler ce langage de la recherche pour être entendu par ceux qui s'y retrouvent. Redire assez de l'autre pour qu'il s'y reconnaisse sans pour autant s'y retrouver entièrement. Parler par exemple d'enculturation, de pratique cognitive, d'intentionnalité, d'attention, d'environnement socio-matériel en donnant l'impression d'avoir fait siens les écrits qui nous ont précédés, mais de dire aussi d'avantage. Dire plus tout en prenant appui sur ce que d'autres ont fait ou dit. Écrire pour faire parler, pour faire parler les données, les écrits.

Tom: They all ... have… I don't know.

Luc: Hum ... They are all rectangles because they are tinier than square, they are longer and skinnier.

Redire assez de l'autre pour qu'il s'y reconnaisse sans pour autant s'y retrouver entièrement … donner l'impression d'avoir fait siennes les observations qui nous ont précédés, mais de dire aussi d'avantage … dire plus tout en prenant appui … faire parler. Le problème de l'écriture, l'impossibilité d'enfermer dans un Dit tout ce qu'il y a à Dire, la rupture que la parole doit produire dans le tissus vivant de l'expérience : ne sont-ils pas *les mêmes*, qu'il s'agisse d'écrire une thèse ou de d-écrire ces figures?

RESEARCHING, TEACHING AND LEARNING IN THE DAY-TO-DAY, MOMENT-TO-MOMENT OF MATHEMATICS EDUCATION

Why? Why get into so much trouble? Why not just play the game, get to the point, write down the facts, and do 'as if' language and the nature of 'knowledge' produced therein was not such an issue? Work it out the way you want, add a footnote if you wish, but stop being such a *clown* .... Turns out he was also reading Bakhtin (1941) on the carnivalesque nature of human activities: we don't play roles, roles are playing us. There is no 'him' attempting to conform to one or another's idea of being a PhD student doing mathematics education research; no one taking up the 'supervisor' part. But socio-cultural historically evolved ways of being roles shape the situations, orienting everybody's actions as *responding* to them. And then again the computer screen, the video files playing, appeared to mirror his own struggles: there is no Rachel 'doing the teacher' and Tom or Charles or Conrad trying to be good students. No good or bad teaching (as if Rachel could be judged against some ideal teacher character she was responsible to play well), no good or bad students, even on the academic side (as if we could seriously consider students *following* 'learning trajectories' or even worse, set them up to one or another!). That 'roles are playing us' means, for one thing, that he, and Rachel and the students do not merely choose to do what they do, but 'find themselves' doing it as a result of what we all make mathematics education (and research) all about. And second, it implies that what they do is always to *differ* from what could be expected (otherwise they would be like puppets in an eternally unchanging play): being *grotesque* (incongruous, inappropriate) is how we respond to those roles, it is how we live. His work was about "taking into account this permanent infection" (to use Derrida's words), in which what I do is never only what 'I' do, but always 'my' differing.

Il écrivait cette thèse exactement comme ces enfants confrontés à une série de rectangles alignés au tableau et à qui Rachel demande ce qu'il y a de semblable et de différent à leur propos. Eux faisaient des maths, lui écrivait une thèse. On avait disposé devant lui un pile « d'évidences », et on lui demandait d'y trouver quelque chose : qu'est-ce qui est pareil, qu'est-ce qui se transforme, n'y avait-il pas quelque chose dans les conditions initiales qui pourrait nous avoir conduits à tel état final? Sauf que pour lui, ce « on » qui lui demandait tout ça n'avait pas à s'incarner en une Rachel donnant l'impression de chercher délibérément à le conduire « quelque part ». Personne pour lui dire « Avons-nous utilisé une corde différente? Non! Ils sont tous fait avec la même corde! ». Mais partout (et en lui-même!) des voix pour lui rappeler : *What's your point? Where are your evidences? So what?* Et s'il n'y a personne ici pour attendre de lui une réponse particulière (il n'y a pas *un point* à défendre, *une preuve* à exhiber, *une conséquence* à tirer), « prendre en compte cette infection permanente » c'est aussi vouloir travailler l'impossibilité wittgensteinienne d'attribuer à Rachel une intention précise : Que veut-elle entendre de la part des enfants? La non plus il n'y a personne à vouloir quelque chose de précis. Elle devra aussi, comme les enfants, *avec* les enfants, retravailler sans cesse les questions, *What's your point? Where are your evidences? So what?*

> To be fair, I think 'investigating' in mathematics education is the main issue behind my work. This is why I called the dissertation "How do we know?", and insisted on the 3 dimensions of researching, teaching and learning. I don't have 'a point', but a direction maybe. I don't have 'evidences', but traces, footprints, tracks. And there is no simple way to make this relevant, except to stress how I work from a researcher's perspective (my own unique place in existence, Bakthin would say), instead of working out from the students' or the teachers', as many might have.
>
> **Can you tell us more?**
>
> I really began to have problems with 'researching' when I completed my master's degree. I needed to better understand what I was doing as a researcher, and how it was connecting with teachers and students; especially in light of how students', teachers' or researchers' sayings or doings are legitimated in mathematics education. When I read Maturana and Varela (1998), I realized this is an *ontological* question with far reaching *ethical* considerations. They put it so brilliantly: "everything said is said by an observer" while "every act in language brings forth a world created with others in the act of coexistence which gives rise to what is human", which implies that "every human act has an ethical meaning because it is an act of constitution of the human world". Students, teachers and researchers are all in a very

similar situation then, all implicated in (re)production of mathematics education in and through their sayings, their doings, their beings.

I don't get it. Are we still talking about mathematics education?

You tell me.

ANALYSIS / DISCUSSION

MÉTAPHORE COMME POSSIBILITÉ D'AUTRUI

Faire de la recherche, comme tout geste d'écriture, *comme* geste d'écriture, ouvre à l'expérience un espace d'altérité. En recherche, on « dit des choses », et en disant on « *fait* sens » au delà de tout horizon d'intentionnalité. Le sens est cette plaie dans le tissu vivant du monde qui, sitôt qu'elle nous touche, nous rappelle que d'autres sont venus, que d'autres vont venir. Si ceci fait sens pour moi, c'est qu'il est humainement possible de le concevoir, d'y prendre appui, et ainsi de suite. Mais le sens est à jamais sens *pour moi*. Nous sommes assez familiers, aujourd'hui, avec la distance épistémologique que ce « pour moi » évoque (c'est la grande œuvre du constructivisme). Ce qui nous échappe davantage, et que Derrida évoque, c'est la destination, l'offrande, la donation de cette préposition. Si le sens est « pour moi », c'est qu'il m'est destiné, adressé par l'autre et pas simplement posé là.

Retournons à Rachel qui demande ce que « tous ces rectangles » ont en commun, et à Luc, par exemple, qui propose de les considérer comme « tous … plus minces qu'un carré, tous plus longs et plus minces » suivi de Melinda qui ajoute qu'ils « ont tous 4 côtés et 4 sommets ». Ces dessins, ce tableau, ces paroles sont des « faits » dans l'expérience que nous en faisons, que nous soyons élève, enseignant ou chercheur. Soit. Mais il y a plus. Parler de tous ces rectangles et les offrir au sens sous l'angle de la recherche d'aspects communs, comme le fait Rachel, c'est déjà *proposer* le monde, c'est déjà destiner aux élèves la présence de ces traces et leurs points communs comme lieu de rassemblement dans le sens. Pas un mot, pas un geste de Rachel qui ne soit rencontré par les élèves comme offert à leur attention *du moment où il fait sens pour eux*. Il n'y a donc pas de situation qui puisse être distinctivement marquée ou non d'une intention d'enseigner, où l'enseignant peut choisir ou refuser de se présenter comme « détenteur de savoir ». Il y a toujours destination, comme une main tendue, comme une manière d'être ensemble.

On the other hand, even when Rachel formulates the apparently rhetorical question, "Did you use a different loop of string for each rectangle?", her offering remains equivocal. How is being 'the same' in the case of strings clearer than in the case of the rectangles? Especially when considering that those rectangles were made by groups using their own version of "the same loop of string." What is 'for me' is not 'mine' yet. And reaching out and grasping it ("Noooooo!") does not make it 'mine' either. Only in the offering itself, only when giving it away, does something finally becomes mine.[1] Charles proffering about the rectangles that "this one has 2 squares at the bottom, this one has 3, and 4, and 5, and 6, and 7 squares" shows that the question is not 'rhetorical' at all. Instead of talking about the perimeter of the rectangle, he proposes observing that the different rectangles are characterized through a regular variation of "the bottom" length going from 2 to 7. He made this observation in response to Rachel's question *for him*, and made it *for her*, who in return recognizes it as *such* in the expression of what she heard (for her) from him (for him): "did you say you see a pattern?" (to which he agrees).

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[1] I should of course take time to explain how we always experience ourselves as another, with Ricoeur and others, but this would require too much time for today.

But then again, what does this mean to us? How is it a difference that makes a difference (to use Bateson's words), to focus on *relations* rather than entities? A change in words evoking to take 'entities', such as student, a teacher, an idea, as the *result* of a relation, instead of taking them as a starting point? And maybe go further and take them as *relations*, not entities in relation. To begin with the observation that for me, simply observing is a relation I came to 'have' (do 'I' really 'have' it? or does 'it' have 'me'?) thanks to other relations … and so on. He was sending out the dissertation, with yet nothing he could certainly (certainty!) pinpoint as *specific* to mathematics or mathematic education. Does it imply it should not concern us? And what if it was actually in the fact that they are *not* specific to our field that those questions and ideas bring *us* something specific? If one more question is allowed to surface here, let it be the question of ethics, for if the world is the word 'for me', the relation to the Other is then *first* in the way we talk about 'first philosophy'.

CONCLUSION / AUTHORING

Richer in meaning than he thought, 'for me' can mean 'for the purpose of myself' as much as 'in exchange of my self'…. An infinite play with words through which he clearly looks for something. Not your everyday version of what re-search in mathematics education looks like, but why not? If you want to learn something, go to the source: that is what they said. The source! Une des grandes contributions de Derrida au monde de la philosophie fut de problématiser l'écriture (au sens large, et incluant donc la parole entre autres) que l'on avait jusque là tenu pour acquis. Il est loin d'être certain que les leçons de la déconstruction qu'il nous offre ont été tirées pour notre domaine de recherche, et en particulier en ce qui concerne la recherche elle-même (toutes ces thèses, tous ces articles que l'on écrits), et pas seulement ses « sujets ». Mais voici que déjà d'autres préoccupations, de plus en plus pressantes, se bousculent.

S'attarder au « pour moi » est une manière de mettre à l'avant plan l'aspect *relationnel* de tout ce qui s'apparente à une démarche de connaissance, un de fondement de toute activité mathématique (incluant l'enseignement!) et de toute activité de recherche à son propos… que nous n'avons peut-être pas suffisamment problématisé. Is this really different from what others already said about *knowing* or *mathematics* or *learning mathematics* (and so on) as 'social' phenomena? In some places yes, in others no: work awaits for those similarities and differences to be told. To be authored. As in the Latin root of the word, *augere*, meaning "to increase, originate, promote." But if you ask me or him (yes, this was me all along!!), we'll tell you: it's like asking for the difference between love and friendship! We can try to explain it, speak of deepness and intensity and obviousness, big ideas and little nuances. Words make a difference (love/friendship) but this difference cannot be put in words. This does not mean there is nothing to say about this difference. We just can't tell. For the math lovers, let's call this the linguistic corollary of Gödel's incompleteness theorem.

A word of ethics …. Is research keeping education in sight, or wanting to be insightful about it? Is writing research about catching (and giving) a sight of what is going on "out there"? Is it about setting one's sight beyond sight-seeing? Could it be an even more protective watch, as in watching over? La relation à l'Autre qui aujourd'hui m'occupe, qui toute entière se dresse (se drape!) dans cette majuscule, je me donne pour tâche de l'inscrire dans notre domaine de recherche. Et de l'inscrire de sorte que dans cette écriture même se dresse et se drape à son tour cette « possibilité d'autrui » sans laquelle nos traces ne serait rien qu'un peu d'usure, accident d'encre et de papier. Écrire non pour rapporter, mais pour (faire) rencontrer l'Autre.

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DIVERSE PERSPECTIVES ON TEACHING 'MATH FOR TEACHERS': LIVING THE TENSIONS

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THE RESEARCH ISSUE

Concern over the mathematics preparation of elementary school teachers (e.g. Ball, 1990; Ma, 1999) has led to increasing calls for prospective teachers to take specialised mathematics content courses, i.e. Math for Teachers (MFT) courses, during their undergraduate programs (Greenburg & Walsh, 2008). While many post-secondary institutions offer such courses within their mathematics departments, very little is known about what happens in these courses or about the instructors who teach them. A review of the literature provides a wealth of recommendations on what should be done in order to help prepare effective elementary school mathematics teachers, but if we hope to improve this preparation, it is important that we have a sense of the current realities. This dissertation attempts to address this, at least in part, going beyond considerations of what MFT courses should be, to begin to build a picture of how they actually are. Through a study of the reported experiences of instructors of the MFT course, it offers a view of the course and its challenges from the instructors' point of view, a perspective that has been relatively neglected.

RESEARCH QUESTIONS

The aim of the study was to explore instructors' experiences of the MFT course, with special attention to diversity among the instructors' interpretations of the course and the tensions they face in teaching it. It addressed the following research questions:

1. How do instructors interpret and experience the teaching of MFT courses? What factors contribute to the diversity of these interpretations and experiences?
2. In particular, what are the major tensions they experience? What factors contribute to these tensions and how are they managed?

THEORETICAL FRAMEWORK

An extensive literature review, examining what research tells us about the knowledge, beliefs, and attitudes that elementary teachers need to teach mathematics effectively, formed the backdrop for this study, which was further influenced by constructivist grounded theory, hermeneutic phenomenology, positioning theory, activity theory, and prior research on tensions.

Constructivist grounded theory (Charmaz, 2006) provided a point of entry into the study, influencing the initial approach to data collection and analysis. Its philosophical orientation (symbolic interactionism (Blumer, 1969) and pragmatism) with its acknowledgement of the power and limitations of the subjectivity of the researcher's interpretation of data fit well with this study of instructor experience, a fluid, multi-dimensional phenomenon, embedded in cultural and temporal contexts.

The tradition of hermeneutic phenomenology (van Manen, 1997), which studies expressions of lived experience in order to uncover meaningful descriptions of that experience, confirmed face-to-face interviews as an appropriate methodology for data collection and influenced the approach to data analysis. In particular, it led to a heightened appreciation for the subtleties of language and ultimately to considerations of positioning. Positioning theory (Harré & van Langenhove, 1999) provides a way to study interaction between individuals and come to a deeper understanding of the potential meanings inherent in the situation. Sensitivity to positioning contributed to interpretation of the instructors' accounts.

Activity theory (Engeström, 1999) offered useful insights into the elements that might influence MFT instructors' experiences and raised my appreciation for the role of tensions, which became a central theme. Tensions, often expressed as "dilemmas", have been recognised as an integral part of teaching practice, dating back at least to the early 1980s. Lampert (1985) proposes the view that tensions in teaching are often "managed" rather than resolved. She characterises teachers as "dilemma managers" who find ways to cope with conflict between equally undesirable (or desirable but incompatible) options without necessarily coming to a resolution. In the face of a teaching dilemma, the teacher must take action, finding a way to respond to the particular situation, even while the "argument with oneself" (p. 182) that characterises the dilemma remains. For Lampert, the ongoing internal struggles presented by the tensions arise from and contribute to the developing identity of the teacher, and as such have value in themselves. Furthermore, she comments: "Our understanding of the work of teaching might be enhanced if we explored what teachers do when they choose to endure and make use of conflict" (p. 194). Adler (2001), building on Lampert's work, made use of the distinctions among practical, personal, and socio-cultural factors in describing tensions, a framework which also proved to be useful in understanding factors that contributed to the MFT instructors' experiences of tensions.

METHODOLOGY

The study involved a qualitative analysis of interviews with ten MFT instructors, all instructors in mathematics departments at post-secondary institutions in British Columbia. Theoretical sampling (Creswell, 2008) was used to achieve variety in type of institution, as well as varying degrees of experience in teaching MFT. The data gathered included single individual interviews with each of the instructors, and a further series of conversations with one of these instructors (referred to as "the case study").

The ten interviews were approximately one hour in duration and were semi-structured, beginning with a set of core questions but allowing for variations as needed. Such an open-ended ("clinical") approach is advocated by Ginsburg (1981) in situations where discovery or identification/description of a phenomenon is the objective. The questions sought to elicit the instructors' conceptions of the MFT course by asking them to examine their goals, describe the approaches they take, compare the teaching of MFT with teaching of other mathematics courses, and reflect on the challenges and the successes they experience.

For the case study, I conducted a series of 9 more-intensive conversations with one instructor throughout his first semester teaching the MFT course. This was followed by an interview one month later (one of the ten described above), and a final interview a full year after this. The focus of this final interview was to hear about changes he had implemented during his second offering of the course, and to see how (or whether) his tensions had changed over time.

The interviews were audio-recorded and transcribed, except for the 9 conversations for which detailed field notes were recorded. All data was analysed using constant comparative analysis (Creswell, 2008). An iterative coding process (Charmaz, 2006) was employed in order to allow concept codes and themes to be identified. Specific concept codes, including *priorities*, *wishes*, *doubts*, *barriers* and *resistance*, helped to locate instances of instructor tensions in the transcripts. After this, techniques from hermeneutic phenomenology, in particular, attention to language and discourse analysis, along with positioning theory (Harré & van Langenhove, 1999) allowed deeper investigation into the phenomenon of *tensions*.

OUTCOMES AND KEY FINDINGS

Analysis of the interview data confirmed that instructors' experience of teaching MFT differs considerably from teaching their other mathematics courses—the students are needier, the stakes are higher, the content is more 'elementary', but the objectives and standards are less clear. Instructors are faced with making decisions about these objectives and standards, yet the resources that might provide them with useful information (notably mathematics education research and communication with mathematics educators) are often not accessed. All of these factors contribute to both the diversity amongst the MFT instructors and the tensions that they experience.

What is offered in an MFT course goes far beyond the course syllabus and depends very much on the individuals teaching the course. In this study, four major points of divergence were identified, namely, differences in course priorities (specifically in the degree of emphasis on cognitive vs. affective goals), in the image of mathematics presented, in the degree of focus on preparation of elementary teachers, and in the classroom methods employed by the instructors. Practical, personal, and socio-cultural factors contribute to the diversity.

On a practical level, there is diversity among MFT courses because there can be. MFT is not a prerequisite for any other mathematics courses, allowing more leeway than for other standard university mathematics courses in interpretation of what content to deliver and how to deliver it. Furthermore, course syllabi characterised by topic lists and behavioural objectives leave affective goals and teaching approaches unspecified.

With respect to the personal, differences in priorities trace back to instructors' individual and diverse conceptions of the intended role of the course, their perceptions of the needs of their students, and beliefs about the relationship between cognition and affect. Classroom methods, treatment of mathematics, and orientation to teacher preparation are further influenced by instructors' personal inclinations and past experiences. Furthermore, desires to bring out different aspects of mathematics can be shaped by the characteristics of mathematics that instructors feel personally drawn to (e.g. structure/formalism or creativity/discovery). The degree of pedagogical content or connections to the elementary school context incorporated can be influenced by personal interest in, and life experience with, the mathematics learning of elementary school children.

Socio-cultural considerations also shape instructors' decisions on how to teach MFT, specifically their understandings of how the course fits into the larger *activity* (Engeström, 1999) of preparing elementary teachers to teach mathematics. There are varied understandings of the level of mathematics proficiency students should bring to the course and of the type and depth of mathematics knowledge that is to be addressed within it. The criteria for readiness to teach elementary school mathematics are not well defined, nor is it clear whether MFT is intended to act as a filter to screen out those not suited to elementary-school mathematics teaching.

Taken all together, these practical, personal and socio-cultural factors all contribute to diversity in their offerings, but they also contribute to the instructors' pervasive experience of tensions in the teaching of MFT.

During analysis of the interviews, six tensions were identified, two tied closely to instructor identity, experienced by only some of the instructors in the study, two referred to as internal tensions that reflect instructors immanent uncertainties with respect to course decisions, and two designated as systemic tensions, which are anticipated by the socio-cultural considerations above. These tensions can be summarised most simply in terms of questions from the instructors' point of view.

Personal Tensions:

- Should I approach teaching mathematics differently for MFT students compared with my other mathematics students?
- To what extent is my personal passion for mathematics, and my desire to share its abstraction and logic, at odds with what my MFT students want and are able to handle?

Internal Tensions:

- How should I set priorities for the MFT course? How can I address both the affective and cognitive needs of my students within the parameters of the MFT course?
- What level of mathematics proficiency should MFT students have when they leave the course, given considerations of their skills coming in to the course, and where I would like elementary teachers to be?

Systemic Tensions:

- To what extent should MFT courses address mathematics pedagogy and incorporate elementary school contexts?
- To what extent are MFT instructors responsible for ensuring the mathematics preparation of elementary teachers?

These tensions are not disjoint. Especially evident is the influence of systemic tensions on instructors' internal and personal tensions. Instructors are participants in the community that is wrestling with the larger questions of the preparation of teachers; the conceptions of the role of MFT courses that they construct for themselves as members of this community shape how they see the course, and in turn affect how they set their priorities and standards. The extent of the diversity between the instructors' offerings of the MFT course gives a measure of the magnitude of these systemic tensions, with greater diversity reflecting greater uncertainty within the community.

Clearly personal factors played a significant role in the *personal tensions* described, so much so that they were only explicitly evident in a few of the instructors' interviews, although it is conceivable that they play a role to a lesser extent in others. Specifically, these personal factors include personal preferences for particular styles of teaching, and prior experiences with both the teaching and learning of mathematics that might be at odds with current recommended education practice. Also, personal conceptions of mathematics, and what makes it interesting and relevant, can be misaligned with students' interests and abilities.

But personal factors also contribute to the *internal tensions*. Decisions about course priorities and standards are influenced by instructor beliefs about the students' immediate needs as well as their anticipated needs in the future as elementary teachers. These tensions seem strongest when the gap between these perceived needs is the greatest. Furthermore, these internal tensions can be accentuated by beliefs that the deficiencies in mathematics knowledge or negative attitudes that are not adequately addressed in MFT (or in a later mathematics course) will be passed on to future pupils, perpetuating a negative cycle. These beliefs are shaped by instructors' personal biographies, and rarely by encounters with the education research community.

The internal tensions are further influenced by practical factors. Ideals for priorities and standards within the MFT course are challenged by the skill levels of the students coming into the course and by the limited time available to cover the course material. Articulation agreements that dictate the course content and the use of particular textbooks were both described as limiting instructors' perceived ability to make the changes they would like to make to more closely address the perceived needs of their students.

Although the practical restrictions imposed by the articulation agreements and the textbooks were cited by the instructors, a closer examination revealed socio-cultural factors that contribute to instructors perceptions of these restrictions. In particular, there were indications of strong norms within the context of post-secondary mathematics instruction related to "covering the content", "following the text" and "assigning a grade". These norms limited the options instructors felt were available to them in making course decisions, and contributed to the persistence of these tensions over time, as was seen especially in the case study.

Within the description of tensions, a number of strategies for managing them could also be discerned. Some examples included instructors staying with what they are most comfortable with, resting in tradition, staying true to perceived institutional expectations and norms, deferring to experts, relying on personal experience, resigning oneself to not being able "to do it all", trusting the system, continuing to experiment, and more generally "doing one's best" (a common theme). All of these strategies represent ways of "living with the tensions", making the choices that need to be made on a day-to-day basis in the teaching of the MFT course, but not making those tensions disappear.

In discussing the *systemic tensions*, instructors offered a number of strong opinions on how the system for preparing elementary teachers could be improved (e.g. more math courses, higher prerequisites for MFT courses, early education mathematics specialists), but did not position themselves as having any power to make these changes happen. As confirmed by my own experience within our provincial system, responsibility for instigating change is passed up the line, with the post-secondary instructors looking to the universities who offer teacher certification programs to set appropriate mathematics prerequisites. At the universities, the mathematicians look to their education colleagues, who in turn, look to the College of Teachers, which ultimately sets the certification standards. In the meantime, on a local level, the instructors live with their tensions, doing the best they can to prepare their students mathematically to the extent they can in the time that they have.

The diversity and tensions exhibited in the instructor interviews translates to vastly different experiences for the prospective teachers in the MFT courses. Further research is needed to determine the eventual impact these differences might have on elementary teachers' classroom practice.

The dissertation concludes with four recommendations for changes to practice, or directions for research that are suggested by the results of this study and the research literature that supports it.

1. ***Prospective elementary teachers should be required to take more than one MFT course***. This recommendation is in line with recent policy documents from the United States (Greenburg & Walsh, 2008) which recommend that future elementary teachers take three MFT courses as part of their preparation. In British Columbia, the single course required falls far short of this, a situation that the local governing and accreditation bodies would be well advised to reconsider. As this study shows, the limited time that MFT instructors have with their students is a major aggravating factor in their experience of tensions in teaching the course, putting them frequently in a position of having to make difficult choices.
2. ***Closer lines of communication should be forged between MFT instructors within mathematics departments and mathematics education researchers***. This recommendation follows from the observation that MFT instructors are notably not accessing some of the resources that could support their teaching of the course. Exposure to the research would equip mathematicians who teach MFT with additional knowledge to support research-based decisions in the face of the many choices they must make in the teaching of MFT students. Two-way communication between these groups would also support the next recommendation.
3. ***Further research and negotiation between interested parties is needed to more clearly define the role of MFT in the preparation of elementary teachers.*** A major factor in the tensions experienced by the MFT instructors was the uncertainty around the place of the MFT course in the larger system. Policy documents, like the Conference Board (2001) report, make recommendations that make great strides towards the issue of more clearly delineating the object of teaching MFT courses, but these recommendations are often made in advance of the research to support them. There is an ongoing call for research into what makes an effective mathematics teacher and on priorities for development of these attributes, and beyond these there is a need for thoughtful consideration of where and how these attributes can be best developed.
4. ***Further research is needed to investigate specifically what mathematicians need to know in order to be effective teachers of MFT students.*** Given many of the challenges MFT instructors face in dealing with their students' cognitive and affective needs, there seems to be a need for research into 'mathematics-for-teaching-mathematics-teaching', or more broadly, pedagogical content knowledge for the mathematics preparation of teachers. Such work could help identify ways to support MFT instructors.

While these do not represent solutions, they offer the hope of some positive change.

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CONVERSATIONS HELD AND ROLES PLAYED DURING MATHEMATICS TEACHERS' COLLABORATIVE DESIGN: TWO DIMENSIONS OF INTERACTION

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The focus of this study is on interactions among teachers, and other educational stakeholders, in the collaborative design of mathematics teaching and learning artefacts. My purpose in conducting this study was to understand participants' interactions inside these teams of collaborative design. This research was conducted in three stages: (1) the study of a single case in which I participated as a member in a team of collaborative design, (2) interviews with participants in three other cases of collaborative design, and (3) the analysis of three pieces of literature reporting large-scale modes of teachers' collaborative design. The result of the study consists of a characterization for interactions in two dimensions: the conversations held during collaborative design, and the role and position played by each participant.

INTRODUCTION

The collaborative design of lessons, and other teaching artefacts, among teachers and other education stakeholders, has been increasingly employed around the world as a strategy for teacher professional development (Jaworski, 2006; Marton & Tsui, 2004; Slavit, Nelson, & Kennedy, 2009). This type of collaborative work typically focuses on students' thinking and their personal problem-solving strategies. As the designed artefacts are implemented in the classroom, this collaboration has a potential to impact both teachers' professional learning and the development of curricular material.

Research in mathematics education has provided strong and compelling evidence of the benefits of teachers' collaborative inquiry and task design as a means of teachers' learning (Hart, Alston, & Murata, 2011; Jaworski, 2006; Slavit, et al., 2009). Both *communities of practice* (Wenger, 1998) and *cultural-historical activity theory* (Engeström, 1998) are theoretical frameworks that have been commonly used for describing interactions among participants engaged in the collaborative design of mathematical teaching artefacts. These frameworks do not fully describe the interactions during the design process in teachers' collaborative design, nor how these interactions influence teachers' practice. Conceptual frameworks for the conversations and activities held in similar settings of collaborative work have been developed with a focus on pre-service teachers (Kaasila & Lauriala, 2010; Rowland, Huckstep, & Thwaites, 2005). Characterizing interactions among participants in this type of collaboration would shed light on our understanding of the social complexities of the interplays that take place in the particular context of the collaborative design of mathematics teaching artefacts.

COLLABORATIVE DESIGN OF TEACHING ARTEFACTS

The study described in this paper focused on a particular type of collaboration among teachers, and other education stakeholders, which I call *teachers' collaborative design*, or *collaborative design* for brevity. Three steps characterize this type of work: (1) the design of a teaching-learning artefact—such as lesson or unit plans, mathematical tasks, and assessment instruments—intended to approach previously selected goals; (2) the implementation of the artefact in a classroom, or several classrooms; and (3) a debriefing of the implementations and refinement of artefacts. Whereas a large variety of instances of this type of collaboration can be found in the literature, three particular modes—described in the following paragraphs—were of particular relevance for this research due to the involvement of a large number of teachers.

Lesson study is a popular model for professional development. It originated in Japan and extended to other parts of the world, including the US and Canada (Hart et al., 2011). In lesson study, as conducted in Japan, teachers design a lesson plan together, based on predefined learning goals. The lesson is taught by one of the teachers while the rest of the team members observe. After a debriefing of the results of the lesson and the refinement of the lesson plan, the lesson is taught again, and this time other members of the school faculty are invited—and possibly external people are involved as well. The lesson is debriefed with all the observers: the lesson plan and its results are published. Fernandez and Yoshida (2004) described one case in an elementary school, claiming that it is representative of Japanese lesson study.

Learning study (Marton & Tsui, 2004) is a model for collaboration among teachers and educators similar to lesson study. However, the main focus is on learning, as opposed to focusing on one lesson. A learning study cycle includes pre- and post-tests to measure students' learning, as well as the use of *variation theory* as a theoretical framework for the design for, and the analysis of, student learning. The Variation for the Improvement of Teaching And Learning (VITAL) project was conducted in Hong Kong from 2005 to 2008 using this mode of collaboration and involving a total of 120 elementary and secondary schools (Elliot & Yu, 2008).

Supported collaborative teacher inquiry is a model for collaboration among teachers proposed by Slavit et al. (2009). This model has an emphasis on the "complex layers of support" (p. 99) that are required for teachers to engage in inquiry in a topic related to mathematics teaching. Under this model of collaboration, a professional development project called Partnership for Reform in Secondary Science and Mathematics [PRiSSM] was conducted, lasting three years and involving 175 mathematics and science teachers in the north-western US (Slavit & Nelson, 2010).

The purpose of this study is to explore the interactions among team members in teachers' collaborative design in mathematics, and to conceptualize such interactions without using pre-established frameworks.

THEORETICAL CONSIDERATIONS

Symbolic interactionism informed the perspective on interaction during this study (Blumer, 1969). The artefacts designed in collaborative design are social objects produced, formed and transformed by social interaction and teachers act toward it "on the basis of the meaning of the object for them" (p. 68). The term *artefact* indicates a level of flexibility for the implementation of the artefacts in the classroom—as opposed to *instrument*, which entails a scheme of utilization (Gueudet & Trouche, 2009).

In a social context, people may not share the same meaning for a particular role or position. Langenhove and Harré (1999) described forms in which people position themselves and others through a conversation. In this sense, roles and positions depend on the perception of each individual, as well as the storyline developed by participants during and before the collaborative design. This perspective is consistent with symbolic interactionism's main premises (Blumer, 1969) in which the individual makes sense of situations and decides according to his or her own interpretations instead of just solely acting according to established norms or expectations.

THE STUDY

This research was conducted in three stages, differing in their sources of data (see Table 1). The first stage included only one case of collaborative design, named the *Lougheed Team*. Participants from three different groups of collaborative design were interviewed for the second stage. The third stage used the three aforementioned pieces of literature as second-hand data: the lower grade group as a case of lesson study in Japan; the Madrid group, representative of supported collaborative teacher inquiry; and the report of the VITAL project undertaken in Hong Kong, which includes transcripts of involved teachers and other stakeholders. The research questions during the first stage are presented as follows:

> RQ #1. *How can we characterize the participants' interactions during collaborative design in the case of the Lougheed Team? How can we identify factors that promote teacher professional growth in such interactions?*

The results of the first stage of the study consisted of a categorization for interaction among team members, informing the analysis of the second and third stages. During these stages the following research questions were approached:

> RQ #2. *Does the generated characterization from the Lougheed Team describe participants' interactions in other cases of collaborative design? What can be expanded from such a characterization by analysing other cases of collaborative design?*

The roles and positions held by team members was an emerging theme during the analysis. This theme in particular was approached through another research question:

> RQ #3. *What are the possible roles of participants in different cases of teachers' collaborative design and how do they influence the interactions within the teams?*

	RQ #1	**RQ #2 and RQ#3**	
	Fist Stage	**Second Stage**	**Third Stage**
Cases	(1) The Lougheed Team	(1) The professional independent project (2) The school district initiative (3) The independent Lesson Study group	(1) The lower grade group (2) The Madrid group (3) The VITAL project
Data sources	Video and audio recordings Interviews Field notes	Interviews Conversations Field notes	Literature

Table 1. Research stages.

METHOD

The method for this research was informed by constructivist grounded theory (Charmaz, 2006). During the first stage, I coded for categories and broader themes in an iterative process of constant comparative analysis and refinement of categories. Two broad themes emerged out of this analysis: (a) the focus of the conversations during the sessions of collaborative design, and (b) the roles and positions held by team members. During the second and third stages, I coded the data in a similar way, but with a focus on the emerging themes, continuing the process of refinement of categories as new data were analysed. For the sake of brevity, only data from the first stage will be presented in this paper. However, these data are presented using the resulting categorization from the overall study.

The Lougheed Team consisted of three lower high school mathematics teachers (see Table 2) and myself as an educator. Pseudonyms for the names of the teachers and the school are used as a measure to assure confidentiality and anonymity. I refer to myself as Armando in the transcripts and the data analysis.

Teacher	Years teaching mathematics	Years teaching	Also teaching	Undergraduate studies	Sex
Arnold	7	7	Science	Ecology	Female
Brad	15	20	Physics	Physics	Male
Sofia	6	6	-	Mathematics and Child Studies	Female

Table 2. Teachers' backgrounds.

The team met for two semesters during 2008 and 2009 to collaborate in a lesson study inspired project. We held weekly sessions during the morning at the Lougheed school, where participant teachers worked at the time of the study. These sessions were video recorded, including two group interviews. Exit interviews were conducted and audio-recorded. Table 3 contains descriptions of the data during this first stage.

First round September to December 2008	Second round January to April 2009	Post project June 2009
A grade 9 lesson plan (English)	A grade 8 lesson plan (French), a unit plan, and an assessment rubric.	
Meetings: 5 for design 1 for debriefing 2 for planning the next round	Meetings: 6 for design 1 general discussion about assessment 1 for debriefing 1 for a group interview April 3	
Group interview (November 18)	Group Interview (April 3)	Individual exit interviews

Table 3. Description of the data during the first stage.

RESULTS

The resulting categorization for interaction during collaborative design from this study consists of two dimensions, corresponding to the two emerging themes from the first stage.

CONVERSATIONS AND ACTIONS

During the first stage of the research it was clear that the conversations and actions during the design sessions often deviated from the original purpose of collaborative design. However, such conversations were, more often than not, focused on the educational context. I classified the conversations as *on-task* and *beyond-task*.

Dimensions	Categories	Examples of conversation topics
On-task	Anticipating Forecasting Commitment	Students' possible struggle during the task (forecasting), as well as the corresponding teacher's response (commitment)
	Achieving goals	Assuring the task serves to achieve the selected goals
	Pursuing coherence	Considering the connections of the task with the curriculum, the course, and particular students.
	Team organization	Scheduling further meetings and splitting labour
Beyond-task	Teachers' practice	Suggestions to approach particular topics in the classroom
	Mathematical context	Use and issues of mathematics outside school
	Collaborative work	Lesson study in Japan
	Casual conversation	Weather and food preferences

Table 4. Dimensions of the conversations held during collaborative design.

Three of the categories for *on-task* conversations were often interwoven in such a way that they could be observed simultaneously at some moments: anticipating, achieving goals and pursuing coherence. The following excerpt is an example of these moments:

> Brad: Maybe we can start with something easier, so that they [students] can come up with descriptions that you can easily write algebraic expressions for.
>
> Sofia: I do think it is a great problem [the cube problem], but I'm not sure that it is what is necessary, what we want for this lesson.

The goal for the lesson was for students to translate word problems into algebraic expressions. We can interpret Brad's comment as anticipating students' difficulties if asked to solve the problem that was being discussed during that moment: the cube problem. Brad not only commented that the problem could be difficult for students, but also suggested to use easier problems. This reflects the two subcategories of *anticipating*: *forecasting* and *commitment*. Starting with something easier also reflects the *pursuing coherence* category of the conversations; which in this case corresponds to the order of a series of tasks depending on the level of difficulty. Finally, *achieving goals* can be identified as Sofia wondered whether the cube problem was what the team wanted for the lesson. *Team organization* was an isolated part of the conversation in which the team distributed labour and defined the course of the project.

Categories for *beyond-task* conversations were isolated each from the other. The *teachers' practice* category could be use as an indicator of change of practice, at least during this instance of collaborative design, as we can read in the following excerpt:

> Brad: Last class we did toothpicks problems. … Today, having this discussion meeting here, I'm going to extend on that. Now using the toothpicks idea, if I make this pattern, and then another pattern and another pattern, ... but connecting with what they [Brad's students] have already done from the previous class.

The category of *mathematical context* includes conversations that went beyond the realm of the classroom. For instance, Brad was questioning the relevance of teaching certain topics of mathematics.

> Brad: We are always passing on information necessary ... to survival to the next generation. I don't feel that with algebra, I don't feel it with a lot of the math that we teach.

The category of *collaborative work* was mostly found in the Lougheed Team, probably because of the research nature of the project. The Lougheed Team was created with the purpose of conducting research on lesson-study-like collaboration. Finally, *casual conversations* were short and occurred mostly at the beginning of the sessions.

ROLES AND POSITIONS

It was clear, from the beginning of the study, that roles and positions were not perceived to be the same in the team. During the first group interview, participants described the role of each member of the team. Probably, the most striking difference was the perception of Armando's role, as portrayed in the following excerpt:

> Arnold: But, I think though there is a very special place as a researcher and as you [looking at Armando] become published, that always will set [you] outside of this community.
>
> Sofia: I don't think publishing gives any more respect or any more trust to what you are saying—just because you are published. Just because it is written doesn't mean it is any more true.
>
> Arnold: But it does exist in professional literature, and so is a privileged location.
>
> …
>
> Arnold: You [Armando] clearly have to have more authority on what you say. ... I would perhaps give more weight to what you say just because in theory you have more background knowledge. ... You are becoming a professional in this area. So, in theory you should know more.
>
> Brad: Like, you are the supervisor. You have your own supervisor and you are the supervisor of us—kind of.

For Sofia being a researcher did not represent more respect or truth to what Armando said, as opposed to Arnold who positioned the researcher as an authority. In contrast, Brad described Armando's role as a supervisor. Other examples of the individual perception of certain roles are evident in the following comment:

> Brad: I saw [Sofia], in this context, as the mathematics expert. She was coming with all this terminology .... So, I was learning new things from you [Sofia]. And you are always doing the puzzles. I would be sitting and watching you actually figuring out the patterns and coming up with the expressions. So, you were taking a much more active role in the sense that you were trying out and I just sat and watched.

Brad not only positioned Sofia as a mathematics expert, but also positioned himself as a novice, or non-expert. These examples suggest that roles and positions are not only perceived at an individual level, but also evolve in interaction.

Teachers' backgrounds and interests influenced the interactions in the Lougheed Team. One instance of this is Brad's perception of Sofia as a mathematics expert—a perception that is consistent with her background in mathematics and child studies. Another instance is the fact that Brad and Arnold often initiated conversations categorised as *collaborative work*. When asked about his motivations to participate in the study, Brad mentioned that he wanted to contribute to research in education, and Arnold indicated her interest in learning about lesson study and Japanese teaching style.

The analysis of other cases of collaborative design afforded more descriptions of roles and positions (Table 5). Contextual factors and the differences between modes of collaboration are reflected in the type of roles presented in each case. For example, *observer* is a very particular role that takes place in lesson study, but not in other modes of collaborative design.

First stage	Second stage	Third stage
Researcher Facilitator Promoter Sceptical voice Expert	Scribe Observer Implementer Designer Administrator Support seeker	Facilitator Disseminator Manipulative-designer Time-tracker

Table 5. Roles and positions added in each stage of the research.

CONCLUSION

The characterization for interaction developed in this study reflects the complexity of the social phenomena in teachers' collaboration design. This characterization was grounded in the particular context of mathematics education. The two dimensions, and their relationships, are barely present in frameworks used to describe social interaction in the literature of mathematics teacher education. For instance, the dynamic of switching between *on-task* and *beyond-task* conversations is usually neglected, as well as the potential importance of beyond-task engagement for teachers' learning. The characterization of different roles and positions developed in this study provides a language to talk about teachers' collaborative design, acknowledging differences between several modes.

Although the data used for this study were developed from a number of cases of collaborative design, I do not claim that every other case would fit into the categories described here. More research with other cases would extend the conceptualization and classification of roles and positions, as well as the relationship with the conversations, held during collaborative design.

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THE ORDINARY YET EXTRAORDINARY EMOTIONS AND MOTIVES OF PRESERVICE MATHEMATICS TEACHERS

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DESCRIPTION OF THE RESEARCH ISSUE

While considerable research has been devoted to independent studies of emotion and motivation, far less emphasis has been placed on the interconnectedness of the two (Dai & Sternberg, 2004; Schutz & Pekrun, 2007). The few exceptions link affect to motivation (Weiner, 1985), emotions to goal orientations (Seifert, 1995), and achievement goal theory to affect through an asymmetrical bidirectional model (Linnenbrink & Pintrich, 2002). The unidirectional (Boekaerts, 1993; Carver & Scheier, 1990) and bidirectional models between emotions and achievement goals illustrate the powerful yet developing links between emotion and motivation. While connections exist between students' emotions and achievement goals, further research would cement the understanding of the role of emotions on teachers' motivation as well as in comprehending the influence of motivation on teachers' emotions. Meyer and Turner (2002) put forth a call for a "comprehensive theoretical work that articulates how emotion, motivation, and cognition interact within classroom contexts" (p. 112). Missing from educational research are studies about the influence of emotions on teachers' motivation or about the types of meta-emotions of beginning teachers (Sutton & Wheatley, 2003).

Teaching has been gradually perceived and portrayed as deeply rooted in emotional experiences (Hargreaves, 1998, 2005; Nias, 1996). Emotional experiences have the potential to influence teachers' performances, identities, and even relationships with colleagues (Zembylas, 2002, 2003, 2005). Studies on teachers' emotions have examined the joy of noticing students' progress (Hargreaves, 1998), their pleasure and pride from interacting with students (Hargreaves, 1998; Sutton, 2000), and their frustration or anger when students misbehave (Emmer & Stough, 2001; Hargreaves, 2000; Jackson, 1968; Sutton, 2000). However, the correlation between teachers' emotions and goals, the collaboration between teachers and parents, and the relation between teachers and students were considered missing research pieces on teachers' emotions (Sutton & Wheatley, 2003). Therefore, this thesis filled a small gap within the research fields of motivation and emotion. The purpose of this research was to present the emotional experiences of pre-service mathematics teachers and to describe the similarities and differences existing between the emotional experiences of interns with different goal orientations.

RESEARCH QUESTIONS

The research described interns' emotional experiences, attributed causes, and thoughts accompanying various emotions. Emphasis was placed on the perceived effects of emotions on interns' confidence, time and effort, and on the teaching skills that ultimately foster positive emotional and motivational learning environments. It also described the bodily effects of emotions as well as the perceived effects of goals on emotions. Although this dissertation focused on pre-service teachers and not in-service teachers, it contributed to illustrating the complex relations between emotions and motivation within teaching practices.

What kind of emotional experiences do interns encounter while teaching mathematics? What similarities and differences exist between the emotional experiences of interns with different goal orientations?

THEORETICAL FRAMEWORK

This dissertation linked *achievement goal theory* (Ames, 1992) with Plutchik's (1980) *psychoevolutionary theory of emotions*. Plutchik's theory was the backbone of this research because it viewed emotions as complex feedback processes, as opposed to simple linear processes. Plutchik defines the emotion as "… an inferred complex sequence of reactions to a stimulus, and includes cognitive evaluations, subjective changes, autonomic and neural arousal, impulses to action, and behaviour designed to have an effect upon the stimulus that initiated the complex sequence" (p. 361). The theory consists of three interconnected models: structural, sequential, and derivatives. The structural model has a three-dimensional representation of emotions. The sequential model provides a detailed description of the multifaceted feedback systems of emotions. The derivatives model illustrates relationships between emotions and personalities, personality disorders, and coping styles.

Achievement goal theory (Ames, 1992) claims that individuals adopt performance and mastery goals within achievement settings. Elliot (1999) proposed a trichotomous achievement goal framework. Teachers adopt achievement goals toward instruction to facilitate their performance as well as to validate personal perceptions of competence and success in teaching. While definitions and various models have been put forth to research achievement goals for students and teachers, this thesis made use of Papaioannou and Christodoulidis' (2007) classification for teachers' achievement goals, namely, mastery oriented, performance approach, and performance avoidance. Mastery goals toward instruction include the desire to continue developing teaching competences, desire to attain teaching mastery, desire to learn new things within mathematics and mathematics teaching practices, or desire to learn new teaching models regardless of how difficult these may be (Butler, 2007). Performance approach goals toward instruction might include: the satisfaction that comes with looking more proficient than other teachers, the desire to demonstrate competence, a wish to outperform colleagues, or the importance of performing better than others (Butler, 2007; Papaioannou & Christodoulidis, 2007). Examples of teachers' performance avoidance goals toward instruction may include: the desire not to disclose lack of competence, to avoid solving problems in which they might look incapable, or to avoid teaching math topics in which they might look incapable (Butler & Shibaz, 2008; Papaioannou & Christodoulidis, 2007).

METHODOLOGY

Interviews and diaries were used in order to capture the essence of interns' emotional experiences. Furthermore, one case study provided an in-depth look into the intricacies of an intern's emotional experiences. Data were assessed using multiple methods of triangulation

(Merriam, 1988; Yin, 2006). Data captured the metamorphosis of emotional experiences and the perceived effects of emotions on achievement goals.

The participants were undergraduate students enrolled in an internship as part of their Bachelor of Education degree. They specialized in mathematics and completed the math methods course, *Teaching of Mathematics in the Intermediate and Secondary School*. During the internship, they taught mathematics courses at junior and senior high schools. Of the thirteen study participants, one intern offered to participate in the case study.

Data were collected via surveys, interviews, observations, and diaries. The survey classified achievement goals towards instruction as mastery, performance approach or performance avoidance (Papaioannou & Christodoulidis, 2007). The items were rated on a five-point Likert scale ranging from 1 (strongly disagree) to 5 (strongly agree), and were used to classify interns into three groups. The classification was not revisited and it was assumed that interns kept the achievement goal orientations for the duration of the internship. Interviews contained highly structured and open-ended questions about significant emotions encountered during the internship, attributed causes, thoughts, and perceived effects of emotions. Diaries collected data pertaining to the emotions associated with teaching mathematics. Daily non-participatory observations were conducted only in conjunction with the case study. These observations revealed attributed causes, perceived effects of emotions on the intern's actions, as well as the intern's gestures, facial expressions, posture, body language, voice, intonation, tonalities, pauses, silences, or eye contact (Babad, 2007). The *Facial Action Coding System* by Ekman, Friesen, and Hager (2002) was used to screen facial expressions, but coding charts were not used.

Data consisted of 407 pages of transcribed interviews, 81 pages of classroom observation notes, and 207 pages from 67 diaries. Transcribed interviews were checked for accuracy, completeness, and fairness. Data were catalogued according to mastery, performance approach, or performance avoidance goals. Six interns were mastery oriented, five were performance approach, and two were performance avoidance. Data were sorted and summarized, ranked and compared, weighted and combined. Meaning units used (Dupuis, Bloom, & Loughead, 2006) were about different emotions occurring while teaching math, such as satisfaction, happiness, excitement, frustration, or anxiety. Upon multiple readings, hidden patterns and categories took shape. The final categories included attributed causes, thoughts, and perceived effects. Data were scanned, coded, and analyzed by a second reader, blind to the initial data analysis. Findings were similar in both cases.

OUTCOMES AND KEY FINDINGS

Mastery oriented interns experienced happiness, satisfaction, or enthusiasm daily. Mastery and performance approach interns presented a number of similarities in the concepts and themes of the attributed causes of pleasant emotions. Similarities included: students understand mathematics, do their homework, do well on evaluations, pay attention, are engaged in classroom activities, behave well, work together, and lower achieving students succeed. Mastery oriented interns' descriptions of happiness revolved around students' understanding and reflected students' comprehension, aligning with the task-oriented descriptions of Ames and Ames (1984). For mastery oriented interns, being able to explain concepts in different ways represented their unrelenting effort in smoothing students' progress. When *self* was listed as an attributed cause of satisfaction or happiness, this was connected to students' progress. Salient particularities differentiated the two groups. Performance approach interns viewed students' understanding as a direct result of their teaching performances. Remarks about pride reflected a strong dependence on supervisors' comments (Seifert, 2004). Performance approach interns showed preference for demonstrating high ability (Ames, 1992; Dweck & Leggett, 1988; Urdan & Maehr, 1995). For performance approach interns, being able to explain concepts in various ways highlights one's own abilities to teach the same material differently. Their satisfaction is described in terms of domination over students' behaviour (Pelletier, Séguin-Lévesque, & Legault, 2002).

Mastery oriented interns thought of teaching performances through the lens of students' progress. Their happiness and satisfaction led them to think about their strong connections with students, their ability to make a difference in students' lives. Mastery oriented interns reflected on improving teaching skills and on devoting time and effort to create well-designed lessons such that students could benefit even more. This corresponds with Butler and Shibaz's (2008) research highlighting that mastery oriented teachers report having a good day when they learn something new. On the other hand, performance approach interns' thoughts in conjunction with pleasant emotions pointed towards doing a good job and being successful at it. Good teaching skills led them to think highly about self or that students' accomplishments were a direct result of their teaching skills. This meant that performance approach interns processed information in terms of self and others (Seifert, 2004).

Using Plutchik's derivatives model, the key components of the mastery oriented interns' sequence was described by this pattern. The gain of a valued object, i.e. their desire to learn more, led interns to think about students' understanding, about their excitement upon accomplishing something in math, or about their strong connection with math. Their overt behaviour was one of repetition, as part of an increased effort to experience happiness, satisfaction, and enthusiasm. The perceived effect of such emotions was one of acquiring techniques that would allow them to experience pleasant emotions again, and of exhibiting preference for engaging in challenging new tasks. Examples included learning about new teaching strategies, becoming more skilled as a teacher, or finding more teaching resources.

Performance approach interns experienced daily pleasant emotions such as happiness, pride, enthusiasm, and excitement. Plutchik's derivatives model follows this pattern. The gain of a valued object, their desire to appear talented at teaching, led interns to think about students' understanding as a direct result of their teaching performances, or to think highly about self. The overt behaviour was one of repetition, as part of an increased effort to experience happiness, pride, enthusiasm, and excitement. Repetition occurred because pride reinforced their social status and self-performance, and acted as a motivator for enhancing confidence. The perceived effect of such emotions was dreaming about becoming famous.

Mastery oriented interns experienced unpleasant emotions while teaching. These emotions appeared sporadically at the beginning of their internship. This finding aligned with research pointing out that mastery oriented students report more positive affect and less negative affect (Seifert, 2004). The stimulus events included: students don't pay attention, don't respect the effort put in by interns, interns' inability to get students to work, or deal with students' behaviours. The gain of a valued object, their desire to learn more, led interns to think about strategies that would help students to focus and progress, and into tactics that could be used to regain classroom control. Interns' overt behaviour was adaptive. It underlined mastery oriented interns' efforts to revert to experiencing pleasant emotions. The perceived effect of such emotions was of emotional regulation.

Performance approach interns experienced unpleasant emotions such as anxiety, frustration, nervousness, or disappointment. Within Plutchik's derivatives model, the stimulus events included students' behaviours, poor performances, and poor work ethics. The gain of a valued object, their desire to appear more talented at teaching than others, led interns to attribute such causes to students' inappropriate behaviour or to their lack of attention and cooperation. The overt behaviour underlined performance approach interns' efforts to refrain from expressing anxiety, frustration, nervousness, or disappointment. One perceived effect of such emotions was to consider avenues to further develop teaching skills and strategies. A perceived effect of frustration was a reinforcement mechanism, in an effort to appear more composed and in control than others.

Performance avoidance interns reported happiness or surprise daily due to the following: students understand math, do their homework, do well on examinations, or cooperate with interns. Other attributed causes included getting positive feedback from cooperating teachers, receiving positive feedback from students, or not encountering significant classroom disruptions. Without some happiness, they would quit teaching. Satisfaction made them realize that teaching can hold something rewarding. Relief made them feel less worried about teaching. Experiencing satisfaction, excitement, or happiness contributed to feeling able to handle the challenges of teaching. In this case, the gain of a valued object, their desire to avoid appearing unprepared while teaching, led interns to think about situations in which students understand math, do their homework, do well on examinations, and cooperate with interns, as well as about receiving positive feedback from cooperating teachers and students, or not encountering significant classroom disruptions. The overt behaviour was one of repetition, because relief contributed to making them feel less worried about teaching. The perceived effect of such emotions was that teaching was worthwhile.

Performance avoidance interns experience frustration daily. Accounts of frustration presented interns' sense of incompetence (Seifert & O'Keefe, 2001). Performance avoidance interns view students' failures as a threat to their self-worth (Seifert, 2004). Experiencing relief circled around avoiding manifestation of inferior teaching, supporting other research (Butler, 2007). For performance avoidance interns, the lack of math knowledge connected with insecurities about being good teachers. They expressed annoyance with the pressures of the internship and verbalized overt displeasure with the educational system. These interns felt unable to get students to work and to conduct lessons as desired. Within Plutchik's derivatives model, the stimulus events included interns' sense of incompetence, pressures of the internship, or insecurities about being good teachers. The gain of a valued object led performance avoidance interns to think about their inability to control the class, about their desire to avoid dealing with a rowdy class, and even question teaching as a full-time career. The overt behaviour underlined performance avoidance interns' efforts to escape teaching.

Furthermore, research suggests goals influence interns' emotions. The desire to learn new things motivated mastery oriented interns to maintain a positive outlook. These interns believed that without goals there would be nothing to accomplish and the drive to go to work would be completely gone. The desire to improve teaching skills and to learn new things kept frustration or disappointment in check. Goals contributed to raising their awareness of the short life span of emotions.

Goals kept performance approach interns grounded in reality, kept them calm and focused to get the class under control. Wanting to be a better teacher than others led these interns to work towards creating better lessons and classroom activities, and subsequently contributed to heightening their experiences of pride. Wishing to become a better teacher than others made them contemplate things in a positive way.

For performance avoidance interns, rational thoughts overcame the emotional aspects of teaching and contributed to their conscious attempts to hide their unpleasant emotions. The desire to avoid teaching ultimately contributed to increasing the frustration as these interns couldn't avoid teaching.

The case study revealed the internship as a place where Petra, a mastery oriented intern, could encounter both pleasant and unpleasant emotions. The case study described how disappointment, frustration, or anxiety help develop, strengthen, and shape behaviours, achievement goals, and teaching actions. It detailed modalities used to overcome unpleasant emotions and to convert them into pleasant emotions. Based on Petra's reactions in the face of unpleasant emotions, it is anticipated that if students were not successful, Petra would increase efforts to help their understanding. Such endeavors would take time, but Petra would eventually overcome such unpleasant emotions. Ultimately, unpleasant emotions would be converted to pleasant emotions, thus having the potential to improve Petra's skills and increase achievement motivation goals.

Petra encountered flow-like experiences, characterized by challenge-skill balance, clear goals, immediate feedback, perception of control, intense concentration, altered perception of time, loss of self-consciousness, and intrinsic reward (Csikszentmihalyi, 1975, 1990). Central to the flow experience was the balance between Petra's abilities and the challenges of teaching. Flow tied in with experiencing satisfaction, enjoyment, excitement, enthusiasm, or happiness. Flow offered intrinsic rewards and motivated repeats of the experience. As an autotelic experience (Csikszentmihalyi, 1990), flow aligned with Petra's experiences. For example, the link between emotions and the desire to repeat teaching experiences can be described through the lens of the derivatives model. Using Plutchik's (1980) derivatives model, the key elements of Petra's joy of sequence embrace the following pattern: the gain of a valued object, namely the teaching activity, lead Petra to think of students as her own, to think about possession and distribution of her knowledge. Her behaviour was repetitive, underlying her increased effort to re-experience joy and other comparable emotions. The perceived effect of joy was one of acquiring resources that would allow her to experience joy yet again. Such resources included but were not limited to finding out and learning new teaching strategies.

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ÉLABORATION ET ANALYSE D'UNE INTERVENTION DIDACTIQUE, CO-CONSTRUITE ENTRE CHERCHEUR ET ENSEIGNANT, VISANT LE DÉVELOPPEMENT D'UN CONTRÔLE SUR L'ACTIVITÉ MATHÉMATIQUE CHEZ LES ÉLÈVES DU SECONDAIRE

DEVELOPMENT AND ANALYSIS OF A DIDACTIC INTERVENTION, CO-CONSTRUCTED BETWEEN RESEARCHER AND TEACHER FOR THE DEVELOPMENT OF A CONTROL OF THE MATHEMATICAL ACTIVITY AMONG HIGH SCHOOL STUDENTS

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INTRODUCTION

Balacheff (1987) souligne l'importance d'une conduite rationnelle chez l'élève qui devrait occuper, dans l'enseignement, le même statut que la construction de savoirs :

Très tôt, disons dès la sixième[2], doit être posé le problème de l'évolution des fondements rationnels de l'activité mathématique des élèves en même temps, et avec le même statut, que celui de la construction des savoirs.

Ainsi, on devrait accorder la même place au processus de construction de connaissances mathématiques qu'au développement de la rationalité de l'élève. Dans notre travail nous interprétons cette rationalité comme le développement d'une activité de contrôle, liée à certaines composantes de l'activité mathématique de l'élève telles que la vérification du résultat obtenu, la justification d'un énoncé, d'une proposition ou de la démarche adoptée dans un problème, un engagement réfléchi dans la tâche, à la validation. Butlen, Lagrange et Perrin (1989) précisent que se donner les moyens de vérifier, de savoir si ce qu'on dit est vrai, si c'est juste ou non sans demander au professeur est à la base de la réussite des élèves. C'est le développement d'une activité réflexive, d'un *contrôle* chez l'élève qui est au cœur de notre travail.

Différentes recherches soulignent le peu de *contrôle* exercé par les élèves face à l'activité mathématique et ce, à tous les niveaux et dans différents domaines des mathématiques. Ce

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[2] Dans le système français, la sixième correspond à la première année du secondaire.

constat rejoint celui émis par les enseignants en exercice sous l'angle d'une vérification du résultat, de la démarche. Il émerge de ce constat partagé, de la pratique et de la recherche, un objet de recherche autour du nécessaire développement d'une activité de contrôle chez les élèves sous les différentes composantes relevées. Il s'agit là d'un enjeu important si l'on en croit plusieurs chercheurs en didactique des mathématiques et qui rejoint les préoccupations de plusieurs enseignants (Landry, 1999; Chevallard, 1989; Coppé, 1993). Cette préoccupation va se traduire pour les enseignants dans leurs propos dans des termes pratiques sous forme de moyens, de stratégies à mettre en place en ce sens pour aider les élèves. Sur un plan didactique, elle va se refléter pour le chercheur, en termes d'un questionnement sur des situations didactiques susceptibles de développer un tel contrôle. À la croisée de ce questionnement, un certain objet de recherche à investiguer prend forme autour de situations susceptibles de développer le contrôle chez les élèves. Dans ce texte, nous présentons les résultats de l'analyse d'une de ces situations mais avant, précisons les fondements du concept de *contrôle*.

LES FONDEMENTS DU CONCEPT DE CONTRÔLE

L'activité de contrôle se traduit par *une réflexion* de la part de l'élève, sur toute action, sur tout choix tout au long de la tâche : au début, en cours ou à la fin de la résolution; par la capacité *à prendre des décisions* de façon réfléchie, rationnelle; par une *prise de distance* par rapport à la résolution et le recours aux *fondements* sur lesquels on s'appuie pour valider. Il est présent tout au long de la résolution de la tâche.

En amont de la réalisation, le contrôle permet une anticipation, les élèves posent a priori une condition de validité du résultat avant de le connaître. Il assure une mobilisation des connaissances en jeu, il se manifeste par une relation entre les données et le but à atteindre. *En aval de la réalisation*, le contrôle assure un travail rétrospectif, une vérification, une validation du résultat pour dépasser le doute et acquérir une certitude. Si nécessaire, il permet un retour sur la tâche et contribue à une évaluation des décisions d'action. Il passe également par la perception des erreurs. *En début ou en cours de processus*, le contrôle se manifeste par des prises de décision sur la direction à prendre, la stratégie la plus efficace, la moins coûteuse en temps, par des évaluations périodiques tout au long de la résolution.

Une revue de la littérature nous a amené à relever six composantes du contrôle : l'anticipation, la vérification / validation, l'engagement réfléchi, le discernement / choix éclairé, le recours à des métaconnaissances, et la perception des erreurs / sensibilité à la contradiction / capacité de dépasser la contradiction. Dans cette partie, nous faisons le choix d'expliciter la composante *Discernement* / *Choix éclairé* qui est celle travaillée dans la situation présentée dans ce texte.

En résolution de problèmes, le discernement (choix) éclairé est relié à la capacité de voir différentes stratégies pour résoudre un problème et à la capacité de faire un choix pertinent d'une stratégie appropriée, efficace, peu coûteuse en temps en ayant préalablement écarté les stratégies qui sont inappropriées (Krustetskii, 1976; Schoenfeld, 1985). Dans un travail sur le symbolisme Schoenfeld (1985) donne un exemple d'une telle attitude dans le calcul de l'intégrale indéfinie $\int \frac{xdx}{x^2-9}$, du changement de variable $u = x^2 - 9$ qui est un choix plus efficace et économique que la factorisation $(x^2 - 9)$ ou le changement de variable $u = 3\sin\theta$.

Les situations qui requièrent un discernement, un choix éclairé sont celles où l'élève peut faire des choix, prendre des décisions. Ces situations doivent permettre plusieurs engagements possibles et favoriser l'idée d'un choix éclairé entre diverses stratégies possibles, pour discerner celle qui est la plus efficace, celle qui mène vers la solution le plus rapidement, sans trop de risques d'erreurs (Kargiotakis, 1996).

UN APERÇU DE LA MÉTHODOLOGIE

Notre intérêt pour l'élaboration d'une intervention didactique autour du contrôle, cherchant à prendre en compte le double point de vue des enseignants et des chercheurs, nous oriente vers une recherche collaborative : nous sommes en effet intéressés par la contribution que les enseignants peuvent apporter à la construction de ces situations ainsi qu'à leur exploitation en classe, le peu de contrôle exercé par les élèves étant, d'une part, une de leurs préoccupations, ils bénéficient donc à cet effet d'observations, de connaissances sur les élèves, sur ce qu'ils font, qui peuvent ici être mises à profit. Ils bénéficient d'autre part d'un savoir d'expérience développé dans l'action, in situ, qui peut là aussi s'avérer potentiellement riche dans l'élaboration de cette intervention. Les travaux qui viennent éclairer les ressources structurantes mobilisées par les enseignants dans la construction de situations d'enseignement en mathématiques montrent bien en effet l'apport possible d'un tel croisement entre didactique de recherche et didactique praticienne pour la mise au point d'interventions didactiques nouvelles (Bednarz, Poirier, Desgagné, & Couture, 2001; Bednarz, 2009; Barry, 2009). Une collaboration entre les enseignants et les chercheurs est ainsi au cœur de notre projet, cherchant à rejoindre les préoccupations partagées entre le monde de la pratique et celui de la recherche, et visant à travailler sur l'élaboration d'une intervention puisant aux compétences de ces deux mondes.

À travers une recherche collaborative menée avec une enseignante de secondaire 3 (élèves de 15-16 ans) nous avons investigué les situations susceptibles de développer le contrôle chez les élèves autour de l'algèbre et plus précisément sur les puissances de nombres et d'expressions algébriques.

RÉSULTATS AUTOUR D'UNE SITUATION TRAVAILLÉE EN COLLABORATION ENTRE UNE ENSEIGNANTE ET LA CHERCHEURE

Il a été convenu lors d'une des premières rencontres avec l'enseignante qu'une première version des situations allait être d'abord élaborée par la chercheure et ensuite envoyée à l'enseignante comme base de discussion pour finalement être retravaillée en commun lors d'une rencontre. *Le placement d'argent* est une de ces situations qui a été proposée par la chercheure à l'enseignante.

Placement d'argent

Je place 500$ à un taux annuel de 10%. Quelle est sa valeur après 3 ans? Après plusieurs années?

Figure 1. Version initiale du problème *Placement d'argent*.

En proposant cette tâche, la chercheure cherchait à pousser les élèves à trouver une façon rapide, efficace de calculer le montant obtenu après plusieurs années (en utilisant le fait que calculer le 10% d'un nombre revient à multiplier par 1,1), à passer à une méthode générale, plus efficace. La composante du contrôle visée est celle d'un discernement, d'un choix de stratégie plus efficace (entre différentes stratégies). Ce sont ces arguments qu'elle présente à l'enseignante :

> Ça les fait réfléchir sur leur quotidien et ils peuvent le faire en essayant de trouver le 10%, on calcule le 10% de 500 et ensuite j'ajoute à 500, ils se plantent parce que là ils ne peuvent pas le calculer sur un grand nombre d'années, il faut qu'ils calculent et calculent alors qu'il suffit de multiplier par 1,1 c'est pour ça que c'est intéressant. (28 février 2005, 370-374)

MODIFICATION DU PROBLÈME PENDANT LA DISCUSSION

Dans la discussion, l'enseignante précise que la question « après plusieurs années » ne pousse pas vraiment les élèves à généraliser, à trouver une « manière de faire » générale. Elle propose à cette fin de donner un nombre d'années qui va demander beaucoup de calculs, de changer l'année de départ, d'arrivée…

> Au lieu de dire plusieurs années on pourrait rajouter une petite question et dire « bon ok, je ne sais pas moi, Émilien avait 18 ans quand il a commencé ça là puis là l'espérance de vie d'un homme est de 78 ans… je ne sais pas, mettons, il prévoit de prendre sa retraite à 65 ans. » (28 février 2006, 457-461)

Placement d'argent

Émilien avait 18 ans quand il a commencé à placer son argent à un taux annuel de 10% par an. Combien d'argent va-t-il avoir quand il aura pris sa retraite?

Figure 2. Version du problème reformulé par l'enseignante.

La discussion entre la chercheure et l'enseignante les amène à aménager la situation. L'enseignante rentre à ce stade dans une analyse réflexive de la tâche en prenant en compte une anticipation de ce que les élèves vont faire dans le problème proposé (comment le modifier puisqu'ils ne généralisent pas, pour contrer les calculs à mesure…). Elle précise que les élèves vont avoir de la difficulté à voir qu'il suffit de multiplier par 1,1 :

> Ils vont te dire 500, 10% de 500 c'est 50, fait que là il a 550 fois 10% c'est 55… là il est rendu à 605 et là il va se dire « ben crime il doit y avoir une façon plus rapide de calculer ça. »[3] (28 février 2006, 463-470)

De plus, elle anticipe que les élèves vont avoir recours à une table de valeurs, soulignant que certains élèves y tiennent. La chercheure va dans le même sens notant que les élèves qui utilisent ce moyen ne vont pas être déstabilisés quelle que soit l'année de retraite choisie (qui va se rapprocher des 60 ans), les élèves qui ont fait une table de valeurs vont juste la rallonger de quelques lignes. L'enseignante prévoit alors un ajout dans l'énoncé de la situation pour déstabiliser les élèves qui font une table de valeurs, les amener à calculer pour un grand nombre d'années, 98 ans. Elle cherche ainsi à déstabiliser les élèves en les laissant s'engager dans une stratégie, l'introduction à ce moment d'un gros nombre va les dérouter et les questionner sur la stratégie qu'ils ont choisie.

Placement d'argent

Émilien avait 18 ans quand il a commencé à placer son argent à un taux annuel de 10% par an. Combien d'argent va-t-il avoir quand il aura pris sa retraite? Et s'il meurt à 98 ans et qu'il cotise jusque là quel héritage va-t-il laisser à ses enfants?

Figure 3. Dernière version du problème.

[3] Les élèves ne font pas le lien « montant + 10% du montant revient à calculer le 110% du montant. » Nous ne sommes pas sûrs qu'on prenne le soin de voir le raisonnement sous-jacent en contexte scolaire, d'expliciter aux élèves d'où vient le 110.

Un autre élément ressort de l'analyse qui a trait à une anticipation de la façon dont l'enseignante va gérer la classe autour du problème. En effet, au-delà des situations construites, un autre élément nous semble important à investiguer, les stratégies d'enseignement mises en place en classe, qui peuvent contribuer, au développement d'une activité de contrôle chez les élèves.

DES STRATÉGIES D'ENSEIGNEMENT SUSCEPTIBLES DE FAVORISER LE DÉVELOPPEMENT D'UNE ACTIVITÉ DE CONTRÔLE

Plusieurs chercheurs (Milhaud, 1997-98; Ngono, 2007; Peltier-Barbier, 2007) soulignent l'importance de se pencher sur les stratégies d'enseignement mises en place en classe qui sont déterminantes dans l'évolution, la progression des connaissances chez les élèves, et, dans notre cas, dans l'acquisition du contrôle qu'exerce l'élève sur son activité mathématique. Milhaud (1997-98) a remarqué que, dans certains cas, les enseignants ne laissent pas le temps à l'élève de s'engager dans la tâche, ils décortiquent à sa place l'énoncé du problème, le simplifiant ou changeant la consigne :

> *Par exemple, dans certaines classes, le professeur après avoir donné un problème, explique comment le résoudre, avant même que les élèves n'aient compris de quoi il s'agissait, et se soient engagés dans sa résolution.* (p. 64)

C'est le professeur « qui prend en charge une partie des transformations que devrait effectuer l'élève face à un problème » (Ngono, 2007).

Pendant la discussion, l'enseignante précise qu'elle va piquer la curiosité des élèves pour les amener à s'engager dans le problème. Elle prévoit de leur dire : « je serais gentille si je te disais combien d'années hein? Mais je ne te le dis pas. Puis je ne sais pas le gars, il a peut-être commencé ça à 18 ans et il a maintenant 65 ans, maintenant il a 5 ans »… « Si j'étais assez gentille pour te le donner le nombre d'années, qu'est-ce que tu ferais? » De plus, elle prévoit amener les élèves à voir la pertinence de généraliser, en piquant leur curiosité cette fois-ci pour les amener à aller plus loin, vers quelque chose de plus efficace :

> Oui moi je veux arriver et je dois refaire tous ces calculs vite là, tu imagines-tu là? Ok si le gars est mort dans un accident d'auto à 60 ans ok là mais si le gars veut savoir vite quand, tu ne vas calculer année après année surtout s'il a commencé à 18 ans. (…) Si tu dis dans trois ans c'est sûr qu'ils vont compter comme ça mais si tu leur dis dans 60 là…est-ce que tu vas calculer ça 60 fois? (….) Les élèves résolvent le problème puis là ils le font, puis là je suis avec ma calculatrice et je fais « ok, 65 moins 18 ça fait 47 ans, fait que je fais 1,1 exposant 47… et là le gars il va avoir tant » et là je leur demande « est-ce que vous avez fini? » « mais là c'est long! », « mais vous n'êtes pas vite ». Je leur fais toujours des affaires comme ça « mais vous n'êtes pas vite » et ça fait comme un clic et en plus je prends mon temps là. Je leur dis « l'avez-vous la réponse? Vous n'êtes pas vite! » [Rires.]

Cette mise en scène qui fait partie de la pratique de l'enseignante amène les élèves à trouver une façon plus rapide de calculer, elle suscite leur attention et leur engagement dans la tâche. L'enseignante précise que si les élèves ne sont pas convaincus par son calcul et croient qu'elle a calculé d'avance, elle va leur demander de lui donner un âge au départ et un âge à l'arrivée.

Dans cette analyse *a priori* du problème, des éléments importants ressortent du point de vue du contrôle, autour d'indicateurs de la part des élèves (les élèves n'iront pas spontanément vers une stratégie plus efficace) et de stratégies d'intervention susceptibles de contribuer au développement de ce contrôle (piquer la curiosité pour les amener à aller vers quelque chose de plus efficace; déstabiliser pour forcer un passage à quelque chose de plus efficace).

CONCLUSION

Le cadre de référence sur le contrôle nous a amené à choisir le problème du *Placement d'argent* pour travailler une des composantes du contrôle, *le discernement et choix éclairé*. Ce problème a été redéfini conjointement par l'enseignante et la chercheure (dans le cas de ce problème c'est essentiellement la voix de l'enseignante qui prend place). Il en ressort un éclairage sur les critères caractérisant les types de tâches qui favorisent le développement du contrôle (sous une de ses composantes) des situations forçant un *choix éclairé, un discernement* entre plusieurs stratégies pour choisir la plus efficace. L'élève est amené à avoir un discernement éclairé pour choisir la stratégie la plus efficace, celle qui permet une généralisation, le calcul du 10% du montant année après année amenant à de longs calculs fastidieux par rapport à une généralisation qui permet de multiplier par 1,1 le montant initial autant de fois qu'il y a d'années écoulées. L'enseignante décrit également des critères autour d'un jeu sur certaines variables didactiques, comme l'*ajout de grands nombres* pour contrer le calcul à mesure, forcer une réflexion et ainsi *amener* les élèves *vers une stratégie plus efficace*.

L'apport de l'enseignante est important en ce qui a trait aux stratégies d'enseignement susceptibles de développer une activité de contrôle. Dans cette situation, elle en explicite plusieurs : piquer la curiosité pour amener les élèves à s'engager dans le problème; piquer leur curiosité pour les amener à aller vers quelque chose de plus efficace (quitter le calcul un à un); déstabiliser pour forcer un passage à quelque chose de plus efficace; expliciter l'intention aux élèves, après avoir piqué leur curiosité; les amener à voir la pertinence de généraliser. Cette recherche ouvre la voie à une étude plus large sur la nature du *contrôle* à exercer autour d'autres notions mathématiques, dans le but d'enrichir le cadre de référence sur le *contrôle* sous différents aspects : ses composantes, les situations et les stratégies d'enseignement susceptibles de développer une telle activité. Les résultats obtenus soulignent l'importance de rendre les intervenants de différents niveaux scolaires (de la petite école à l'université) conscients de la nécessité de développer chez leurs élèves et étudiants une attitude de *contrôle* face à certaines activités mathématiques.

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Ad Hoc Sessions

Séances ad hoc

PUBLISHING IN THE *JOURNAL OF MATHEMATICS TEACHER EDUCATION*

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Although members of the Canadian Mathematics Education Study Group community engage in research on teacher education as reflected in working groups, topic groups, and new PhD research at recent annual meetings, few manuscripts are submitted to or get published in the *Journal of Mathematics Teacher Education* (JMTE). In this presentation, as editor-in-chief (Chapman) and associate editor (Walshaw) of JMTE, our goal was to provide information to encourage more interest in submitting appropriate manuscripts to the journal in order to share the research in this area being done in Canada with the international mathematics teacher education community.

JMTE is an international research journal of very high standing in the mathematics education research community that:

- seeks to improve the education of mathematics teachers and develop teaching methods that better enable mathematics students to learn;
- covers all stages of the professional development of mathematics teachers and teacher-educators; and
- examines institutional, societal, and cultural influences that impact on teachers' learning and their students' learning.

Manuscripts of research papers must reflect the main topics of the journal and go beyond local or national interest. Research questions should be directly related to studying subjects/participants who are teachers of mathematics and should address topics such as prospective or practicing teachers' thinking, knowledge (e.g., content, pedagogical, technological), beliefs, conceptions, identity, teaching, learning, and professional development.

As a research journal, JMTE requires manuscripts of original studies related to mathematics teacher education. Cursory descriptions of one's course with prospective teachers or professional development activities with practicing teachers are not appropriate. We offer the following as important factors to consider in preparing appropriate manuscripts for JMTE:

- Manuscripts should report on systematic, rigorous studies framed in an established or theoretically supported methodology.
- Research constructs (e.g., beliefs, identity) or interventions (e.g., teaching/learning approaches/experiences) being studied should be explicitly grounded in a theoretical perspective or framework.
- Findings should also be grounded in this theoretical perspective or framework and connected to the field of mathematics teacher education.
- The work should make a meaningful contribution to the field and this should be clearly articulated in the manuscript.
- Manuscripts should be framed as scholarly papers and follow established conventions unless some other approach can be justified theoretically. Such

> conventions include: an identified research problem and a clear research question(s); a synthesis of the relevant literature and close links with that body of knowledge; a discussion on how the work both contributes to and moves that literature base forward; and a coherent argument threaded through the discussion and directly related to the research problem.

We look forward to contributions from those participants at our session who indicated intent to make submissions and encourage others reading this paper to consider doing so.

IS IT POSSIBLE TO MEASURE THE EFFECTIVENESS OF A SPECIFIC APPROACH TO TEACHING FOUNDATIONS MATHEMATICS IN A POST-SECONDARY SETTING?

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The randomized control trial is the 'gold standard' of research methods in many disciplines. However, using this approach is particularly challenging in social science research generally, and particularly in education (Mosteller & Boruch, 2002). The springboard for the discussion was a Randomized Field Trial (RFT), which aims to measure the effectiveness of an intervention in the teaching of a foundational math course across 3 divisions of an Ontario college.

The session started with a quick presentation of preliminary results followed by three of the challenges facing research of this type:

<u>Challenge 1:</u> Measuring improvement in a foundations math class at a college level: Pre/post tests (Wechsler test of numerical operations) were used in the study, but capture only a small slice of what improvement would be. Discussion focused on the lack of a legitimate test; the possibility and challenges of creating one were touched on.

<u>Challenge 2:</u> All students are expected to improve their mathematics skills during a course in mathematics. When comparing an intervention group to a control group, how much extra improvement is sufficient in order to declare that a particular teaching approach is more 'effective' than the one used in the control group?

Of the three, this challenge elicited the most spirited discussion. A declaration of statistical significance is not enough, as it does not on its own tell the sceptical teacher how much extra improvement they should expect in students if they choose to teach using this 'new' approach.

Suggestions can be categorized into three groups:

- any extra improvement is worthy of consideration and adoption,
- the extra improvement should be dramatic and emphatic,
- any extra improvement in a competence score should be accompanied by changes in attitudes to mathematics.

<u>Challenge 3:</u> Ethical constraints and their effect on the generalizability of results. Studies in higher education in which informed consent is required (which include RFTs) will suffer from self-selection bias and low participation rates. This challenge was acknowledged but time ran out and it was not discussed.

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EST-IL POSSIBLE DE MESURER L'EFFICACITÉ D'UNE APPROCHE SPÉCIFIQUE À L'ENSEIGNEMENT MATHÉMATIQUES FONDATIONS DANS UN CADRE DE POST-SECONDAIRE ?

Taras Gula, *Collège George Brown*
[Traduction: Ariadna Gula, et Jean-Philippe Bélanger]

L'essai randomisé contrôlé est le « gold standard» des méthodes de recherche dans de nombreuses disciplines. Cependant, utiliser cette approche est particulièrement difficile dans la recherche en sciences sociales en général et en particulier dans l'éducation (Mosteller & Boruch, 2002). Le tremplin pour la discussion était un essai randomisé de terrain (Randomized Field Trial) qui vise à mesurer l'efficacité d'une intervention dans l'enseignement d'un cours de mathématiques fondamental dans trois divisions d'un collège de l'Ontario.

La séance a débuté par une présentation rapide des résultats préliminaires suivis de trois défis auxquels est confrontée le ou la chercheur(e) dans ce type d'approche :

Défi 1 : L'amélioration de mesure dans une classe de mathématiques fondations au niveau collégial : Pré / post-tests (test de Wechsler des opérations numériques) ont été utilisés dans l'étude, mais ne capturent qu'une petite tranche de ce que serait l'amélioration. La discussion a porté sur l'absence d'un test légitime; la possibilité et les défis de la création d'un test ont été abordés.

Défi 2 : Tous les élèves sont censés améliorer leurs compétences en mathématiques pendant un cours de mathématiques. Lorsque l'on compare un groupe d'intervention à un groupe témoin, combien d'amélioration supplémentaire est suffisante pour déclarer qu'une approche pédagogique particulière est plus « efficace » que celle utilisée dans le groupe témoin?

Parmi les trois, ce défi a suscité le débat le plus animé. Une déclaration de signification statistique ne suffit pas, parce que toute seule cette déclaration ne dit pas à l'enseignant sceptique combien d'amélioration supplémentaire il doit attendre des étudiants s'il choisit d'enseigner en utilisant cette « nouvelle » approche.

Les suggestions peuvent être classées en trois groupes :

- toute amélioration supplémentaire est digne d'examens et d'adoption,
- l'amélioration supplémentaire devrait être dramatique et emphatique,
- toute amélioration supplémentaire dans un score de compétence devrait être accompagnée par des changements dans les attitudes envers les mathématiques.

Défi 3 : Les contraintes éthiques et leur effet sur la généralisabilité des résultats. Études dans l'enseignement supérieur où le consentement éclairé est requis (qui comprennent RFT) vont souffrir de biais d'auto sélection et de faibles taux de participation. Ce défi a été reconnu, mais le temps a manqué et il n'a pas été discuté.

CHALLENGES IN SUPPORTING MATHEMATICS TEACHERS TO DEVELOP THEIR TEACHING PRACTICES

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This *ad hoc* session took the form of a discussion focused on challenges in supporting experienced teachers and teacher candidates to advance or reform their mathematics teaching practices. Collectively, the fifteen contributors drew upon a wide range of experiences related to mathematics teaching, mathematics teacher education, and teacher professional development. Ideas raised provided insight to the scope and complexity of this important topic in mathematics education.

A number of the ideas that were discussed are familiar to most mathematics educators. Prominent amongst these was the inertia of existing mathematics teaching practices and school culture, which, in turn, presents challenges for teachers—especially new teachers—when they want to do things differently. Included as part of this were: perceived constraints and politics of the school system reflected in mandated curriculum requirements, teachers' workloads, time constraints relating to 'covering' coursework, formal evaluation demands, and school/teacher accountability discourse; the strong tendency for teachers to default to teaching in the ways that they were taught; and a passive orientation to learning in general on the part of many students. The importance of teachers' understandings of mathematics, the importance of framing the mathematics of school curricula in terms of a small number of foundational concepts, and the need for students to have a strong grounding in these concepts were also considered.

A number of less familiar ideas were raised as well. These included the need to attend closely to the diversity of practicing teachers' and teacher candidates' experiences with mathematics and mathematics teaching and learning so that their professional development or training can be targeted accordingly. Differences in the needs of elementary and secondary level teachers and teacher candidates were highlighted in this part of the discussion. Another idea raised was the way that new approaches to teaching mathematics are presented and, at times, even imposed on teacher candidates and practicing teachers. The need to foster teachers' ownership of reform mathematics teaching through critical self-reflection on their beliefs and mathematics teaching practices was advocated, recognizing, in turn, the need for effective ways to engage teachers and teacher candidates in this kind of self assessment. The possibility that the individuals who pursue mathematics teaching as a career, as a cohort, may arrive to teacher education with strong commitments about what it means to be a mathematics teacher (i.e., what worked for them) and be reluctant to deconstruct their thinking as part of their teacher formation process was raised as well.

The final word of our session went to Ralph Mason who encouraged us to consider the task of changing mathematics teacher education in terms of making a fine wine, in stark contrast to making grape juice.

Thank you to everyone to who contributed to this rich discussion.

RAPPORT SUR LE *AD HOC* ÉTHIQUE ET ÉDUCATION MATHÉMATIQUE

REPORT ON THE ETHICS & MATHEMATICS EDUCATION *AD HOC*

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Je m'intéresse depuis quelques années à la question *éthique* dans le monde de l'éducation mathématique : je pense ici autant à l'apprentissage, à l'enseignement qu'à la recherche. J'ai cru entendre, durant le colloque du GCEDM, de nombreux échos à cette préoccupation pour l'éthique à travers les présentations et séances de travail auxquelles j'ai participé. Il m'a donc semblé intéressant d'en faire un thème explicite pour une session *ad hoc*, entre autres dans le but de discuter de l'intérêt d'organiser un groupe de travail sur cette question dans une prochaine rencontre du GCEDM.

I opened the *ad hoc* explaining how my personal interest in ethics stems from an early questioning on how *all* students' contributions in mathematics classroom can be received. I have come to think about this question in terms of *ethics*, a word I use with the distinction Paul Ricoeur (e.g. 1990) offers between *ethics* and *moral*. *Moral* (*la morale*), he suggests, is concerned with norms and rules, with obligations and deliberation. *Ethics*, however, is a bit more difficult to formulate: it has to do with a wish, a desire, a disposition which gives the foundation and the orientation for norms *and their transgression* in practical action, when encountering the other and responding to him/her.

Le groupe rassemblé *ad hoc* a alors discuté de différents aspects et de possibilités pour travailler autour de telles questions dans le cadre des *mathematics education*. Suite l'évocation de la perspective développée par Lévinas (e.g. 1985) pour qui il ne s'agit pas de définir l'éthique mais d'en explorer le sens, les travaux de Mary Boole ont été mentionnés. Nous avons également évoqué l'importance de prendre en considération le facteur *temps* (quand on réalise qu'une action a eu des conséquences non souhaitées). Il a aussi été question de tenter de faire *un lien fort et riche avec les mathématiques* elles-mêmes (par exemple dans la distinction souvent proposée entre transcendance et immanence des idées mathématiques).

The group also suggested that *different perspectives on ethics* could be brought into conversation, and not only the one I have come to adopt. Doing this, it seems important to consider various questions raised by ethics *without limiting ourselves to overtly specific situations* (e.g. teachers-students relation in the classroom): out-of-school mathematics or one-to-one math tutoring for example could be included in the discussion.

Face au défi que présente la question, récurrente, de la place des mathématiques dans de telles réflexions (e.g. en quoi ce que nous disons est particulier au monde de l'éducation mathématique?), nous avons retenu l'idée de réfléchir précisément à ce que nous pouvons *apprendre à propos des mathématiques* à partir d'une réflexion éthique et, inversement, ce que le travail mathématique lui-même pourrait *apporter à nos réflexions éthiques*.

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READING BIOGRAPHIES AND AUTOBIOGRAPHIES OF MATHEMATICIANS: WHAT DO THEY TELL US ABOUT THE SUBJECT OF MATHEMATICS?

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When mathematicians speak or write about their engagement with the discipline of mathematics, what lies beneath? What do these accounts tell us about mathematics and the mathematical endeavour? What are the stories of mathematics that give us glimpses of the human side of mathematicians? To what do they point? In particular, what themes are prevalent in biographies and autobiographies of mathematicians? I seek to map the landscape of engagement with mathematics.

In this *ad hoc* session, I began by describing this area of research, which was prompted by Leone Burton's (1995, 1999, 2004) study of mathematicians in the UK. I wondered whether mathematicians a continent away and a decade later would speak of the same influences and insights. It was striking to see the enthusiasm among the group as they reflected and shared their experiences with mathematics. There was a genuine desire to speak of the engagement with mathematics, along with a general sense of the dearth of opportunity for such reflection and voicing. One participant spoke of the concentration and rapture of being absorbed in doing mathematics at the dining table only to be brought to 'reality' by a scathing remark of a family member about time being spent on a pursuit deemed trivial in comparison to other tasks. Another spoke of the liberation offered by mathematics as a ground on which one can build an identity and be judged on criteria other than the usual considerations of social status.

The factors underpinning our engagement with mathematics are complex but are being increasingly recognized in the mathematics education literature. Beginning with McLeod's then state-of-the-art 1992 article on affect, researchers have sought to explore the role of emotions and other factors relating to the person and the self (beliefs, attitudes, dispositions, etc.) as opposed to the more usual considerations of sociocultural interactions with teachers, students, and classrooms. Another landmark is the February 1993 issue of the journal, *For the Learning of Mathematics*, on psychodynamics, teasing apart aspects of the self, conscious and unconscious, that we bring to bear on our responses to mathematics, such as defence mechanisms and transformations. In the last decade or so, by examining more closely the 'subject' who engages with the discipline, the research has arrived at psychoanalysis and subjectivity as a light by which to understand and describe the dis/en/gagement (Brown, 2011).

Does mathematics find us or do we find mathematics? What is involved in being in mathematics, in taking up mathematics? What are the demands/costs/rewards of the mathematical endeavour?

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ONLINE ENVIRONMENTS FOR MATHEMATICS SHARING AND COLLABORATION

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Our students interact online, sharing their thoughts and ideas in the form of short messages, images and videos (comScore, 2010, 2011). This move away from extended text-based communication and the arrival of *GeoGebra* (geogebra.org), an open-source mathematics tool, and *Jing* (*TechSmith*), a free screen-recording program, presents the opportunity for mathematics to join the world of online sharing and collaboration.

GeoGebra is written in Java and its Export function allows a student, while working on a problem, to capture code that will present the program in its present state. With this, the student can mount his/her work as an applet on a page in any wiki (*PBworks*, *Wikispaces*) or learning management system that permits the embedding of Java code. The image below (Figure 1) shows a *PBworks* wiki page on which Tom is displaying his *GeoGebra* work to build an algebraic model for light intensity vs. distance data. The lower half of this image shows *GeoGebra* as he left it after attempting to fit the model $y = 1/x$ to the points. The key point is that this is not just a picture of a *GeoGebra* screen, but rather a live applet that another student could pick up and work with to carry the exploration further. In fact, Tom's work began with using a *GeoGebra* applet left by another student working with a quadratic model.

Tom has also embedded a *Jing* video (top half of the image) that displays the *GeoGebra* window as he explored possible models and can replay his voice as he gave the reasoning behind his actions. Any student in the problem-solving group can follow the exploration progress by moving page to page, viewing the videos, listening to the recorded explanations, and studying the *GeoGebra* applets. Upon finding a point at which they can contribute, the student can pick up a *GeoGebra* applet as their starting point, work on the solution, and create a new page to display a similar record of their work.

All the online tools to support such interaction are readily available without cost to students or schools, and secondary school pupils, with coaching, can master the steps involved in recording their work and mounting it on wiki pages. More significant difficulties arise in convincing a class that mathematics study should be a collaborative enterprise and that progress will be enhanced by the free sharing of partial solutions and incomplete thoughts.

To see the above exploration in action, go to: http://collabmath.pbworks.com

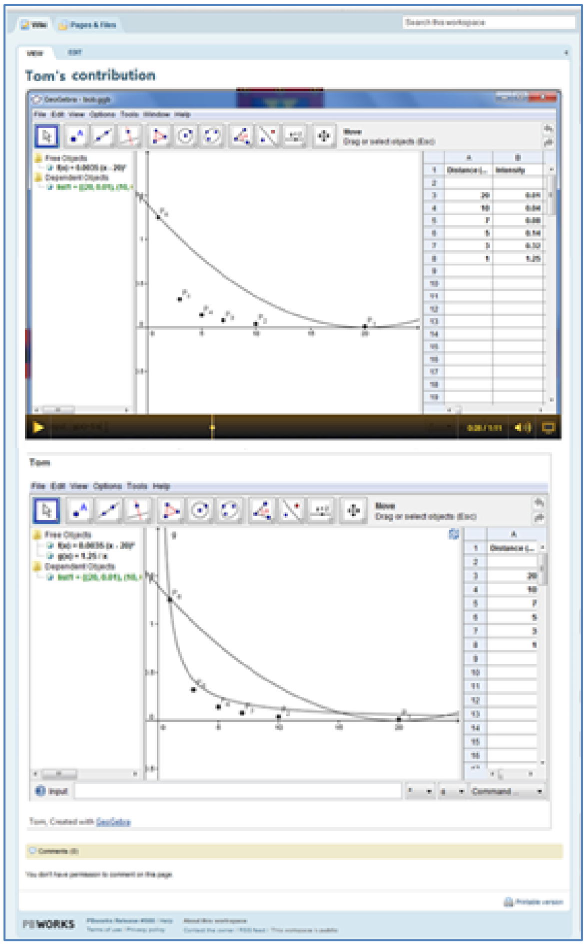

Figure 1

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AD HOC SESSION ON PLANNING FOR THE NEXT CANADIAN MATH EDUCATION FORUM (CMEF) TO BE HELD IN MAY 2014

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Informal discussions on the future of CMEF have taken place over the past winter among a few of the past co-organizers. It is felt that, having had 4 successful meetings, the time has come to clearly identify the role and function of CMEF in the Canadian Math Ed landscape. Its strength has certainly been in its broad participant base, as it brings mathematicians, math educators, teachers, publishers and admin/government folks together to talk about the place and face of mathematics in school, in college, in university, in the work-place and in Canadian society. It is clear that there is significant work to do in terms of reimagining and recreating the nature and character of mathematics education in both school and university. [Am I exaggerating here?—I think not.] We have such a wonderful and important (and central) subject and it is so misunderstood by so many. Anyway, CMEF would seem to be the ideal body to focus (and I do mean focus!) on this 'mission' and the question is, what sort of organization and meeting would best serve this purpose.

There was an *ad hoc* discussion of this question at the Québec CMESG meeting in May 2012. The proposal informing those discussions was that the Forum would meet every 4 years, except that in summer 2014 we would hold a smaller planning meeting to decide upon the shape of the organization, with the first meeting of the new 'format' to be held in summer 2018. The discussion at this session revolved mainly around the value of CMEF and the need to continue to incorporate the uniquely wide range of participants. At the beginning of June 2012, this matter was again discussed at the CMS Education Committee meeting in Regina. The first question asked was why we were waiting so long to do the planning. Could we not do enough of this planning in 2013 and hold the first 'new format' CMEF meeting in 2014? I now feel that the answer is yes!

I propose that the planning take place at two meetings, the first being the CMS Winter meeting in Montreal in December 2012 and the second being the CMESG meeting in May 2013 at Brock. Our objective, from both these meetings, would be to arrive at a good sense of the purpose and objectives of the Forum, and to outline a program structure that might be able to support and further those objectives. The CMS meeting dates are December 8-10, and the meeting will tentatively take place on the 9th. Anyone who wishes to be part of this is welcome to attend. Those who are unable to come are welcome to send me ideas or concerns. I will assemble those and make them available before the meeting. I will post further information later together with a procedure for attending the December meeting.

Mathematics Gallery

Gallérie Mathématique

PRESERVICE ELEMENTARY TEACHERS' BELIEFS TOWARD MATHEMATICS AND MATHEMATICS TEACHING

Sean Beaudette, Alexandra Penn, & Geoffrey Roulet
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Theoretical constructs linking teachers' beliefs concerning the nature of mathematics to their preferred instructional practices (Ernest, 1989) and studies revealing this beliefs-to-practice connection, particularly in the case of absolutist images of mathematics and traditional direct instruction (Raymond, 1997; Thompson, 1984), have appeared in the literature for over 25 years. More recently some researchers (Judson, 2006; Liljedahl, Rösken, & Rolka, 2006) have questioned the existence of a link between subject beliefs and instructional practice by identifying practicing and preservice teachers holding absolutist mathematics beliefs, but still employing, or planning to employ, instructional approaches associated with the current mathematics education reform movement (NCTM, 2000). At Queen's University we have recently completed a study to explore the alignment of mathematics subject and teaching views.

As they began their one-year teacher education program, 138 elementary school teacher candidates completed a questionnaire designed to measure their beliefs concerning the nature of mathematics, measured on a scale from absolutist to fallibilist, and their beliefs concerning effective mathematics instruction, measured on a scale from traditional to constructivist. Almost half (49%) of the participating preservice teachers held misaligned beliefs; having an absolutist view of mathematics, but intending to teach using constructivist techniques. Misalignment of beliefs occurred only in this direction, with no participants holding fallibilist beliefs and intending to teach traditionally. Two sub-studies were conducted to explore subject beliefs and teaching intentions in more depth. Interviews were conducted with volunteer questionnaire participants, with selection based on the questionnaire results.

Sub-study 1 (Beaudette, 2012) involved eight preservice teachers showing distinct absolutist or fallibilist views of mathematics. Individual interviews explored participants' beliefs concerning the use of information and communication technology, particularly interactive whiteboards (IWB), using a framework developed by Bruce, Flynn, Ladky, Mackenzie and Ross (2008). Participants with absolutist beliefs about the nature of mathematics were found to focus on the IWB as a presentation tool, while those with fallibilist beliefs appreciated the use of the IWB to support student exploration.

Sub-study 2 (Penn, 2012) involved eight preservice teachers with apparently misaligning absolutist beliefs concerning the nature of mathematics and constructivist beliefs concerning teaching. Interviews exploring participants' favoured instructional approaches, particularly those involving the use of manipulatives, showed that constructivist views involved essentially surface beliefs and that in fact manipulatives would be employed to support traditional direct instruction.

Our research suggests that the theoretical link between epistemological beliefs and teaching practice does hold at deeper levels, and that there is a danger in taking the presence of items from the mathematics education reform agenda as evidence that constructivist approaches have been fully adopted.

A PDF file of the poster may be accessed at: http://hdl.handle.net/1974/7572

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GEARING UP FOR GRADE 9: *A LEARNING OBJECT*

Laura Broley
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Students in the innovative Mathematics Integrated with Computers and Applications (MICA) program at Brock University are challenged to learn and explore mathematics through the use of technology. For instance, the final project in MICA I, a first-year mathematics course, requires students to create either what is referred to as an *Exploratory Object*, which they then use to investigate a mathematical concept or model a real-world situation, or a didactic *Learning Object* that "engages a learner through a game or activity and that guides him/her in a stepwise development towards an understanding of a mathematical concept" (Muller, Buteau, Ralph, & Mgombelo, 2009, p. 64). Over the span of one month, students must choose their topic, develop their ideas, design a program in VB.Net, create original content for their program and test their object.

My response to this challenge was *Gearing Up for Grade 9: Algebra Edition*. Complete with tips for success, lessons, worksheets and an interactive quest, the program was designed to aid elementary students in transitioning into a high school math environment and serve as a review tool for students in grade nine and beyond.

When I was introduced to the MICA I final project, I had had no formal training in didactics. Faced with this fact, I used my experiences as a student and as a personal tutor to guide the design and implementation of my *Learning Object*. Hence, the project opened my eyes on many levels. While creating the object, I learned about the programming and design aspects required to make an educational tool. In addition, I rediscovered concepts I had already mastered as a student from the point of view of being able to teach them. Specifically, I was able to investigate how to teach a mathematics topic using computers. By introducing the *Learning Object* to students within the intended audience and observing their reactions, I was also able to gain experience leading a class through computer-based lessons, reflect on my teaching ability and make changes based on what did and did not work.

The overall design of my *Learning Object* was driven principally by the fact that too many students today dislike math. The addition of game-like elements to the object, for example, stemmed from my belief that computer games can be used to positively change a student's attitude towards mathematics. One of the grade eight students who tested the program stated that, "It was fun and not really boring like you would think a math game would be." The success of my *Learning Object* with students and teachers alike suggests that integrating *Learning Objects* and computer games into the classroom may be beneficial. Such observations give rise to questions like: How can computer programs find their place in the average math teacher's lesson plan? How can students 'show their work' when using a computer to do math? And, would it be beneficial to continue to find ways to hide the math in math games? Is it even possible?

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STUDENTS' PERCEPTIONS OF THE ROLE OF THEORY AND EXAMPLES IN COLLEGE LEVEL MATHEMATICS

Dalia Challita and Nadia Hardy
Concordia University

Previous research has reported college level students' inabilities to deal with non-routine tasks and to justify in mathematical terms the ways in which they deal with routine tasks (e.g., Selden, Selden, Hauk, & Mason, 1999; Lithner, 2004; Hardy, 2009). Researchers have attributed these inabilities to the absence of theoretical content or its dissociation from tasks and corresponding techniques in the teaching approach to college mathematics.

What drives our ongoing research project is the question: "If we were to incorporate theory in our teaching approach to Calculus, how should we do it so that we provide students with discourses to justify in mathematical terms their approaches to problem solving and with tools and strategies to deal with non-routine tasks?" As a first step towards answering this question, we investigated students' perceptions and uses of theory and examples. To do so, we designed a teaching experiment in which students who had recently passed a pre-Calculus course were randomly assigned to one of four pre-recorded lectures that introduce the topic of limits at infinity; each through a specific teaching approach incorporating theory and/or examples in different ways. Subjects were met individually; they completed a pre-test, attended the videotaped lecture and engaged in a task-based interview.

In our poster we presented data corresponding to 30 subjects assigned to three of the lectures (10 to each). The results indicate that these students perceive theory exclusively as a validatory discourse and don't recognize the value of generalized examples. We showed instances of students' attempts to use particular examples as templates, in the same way we expect generalized examples to be utilized, and quoted students who explained that they are at a loss when presented with generalized constants. In particular, our results suggested that it would not suffice to include theoretical content in a teaching approach (and associate it with tasks and techniques); the roles of mathematical theory and generalized examples as generators of techniques and students' abilities to use them in problem solving have to be addressed if we want them to genuinely engage in mathematical activity.

We shared and discussed our findings with conference participants; in particular we discussed what teaching strategies could be put in place to change students' perceptions of the role of theory and examples in mathematics.

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UNE EXPÉRIMENTATION DE PRATIQUES GAGNANTES EN ENSEIGNEMENT DES MATHÉMATIQUES

AN EXPERIMENT WITH SUCCESSFUL PRACTICES IN MATHEMATICS TEACHING

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Dans le cadre d'une recherche formation, des enseignantes et des enseignants du primaire et du secondaire d'une commission scolaire ont été amenés à vivre une expérience qui les a conduits à réaliser un poster regroupant des « pratiques gagnantes ». Cet article rappelle brièvement les étapes ayant permis d'en arriver à cette production.

De concert avec la direction des Services éducatifs, une première journée de formation a eu lieu en 2010. Elle a permis à une douzaine de personnes de recevoir une journée de formation visant à interpréter de différentes façons les erreurs des élèves en mathématiques. Nous savons que les interprétations des enseignants influencent le choix de leurs interventions (DeBlois, 2006). À la suite de cette journée de formation, cette équipe d'intervenants a choisi de faire la promotion d'un projet de formation pour l'ensemble d'enseignants de la commission scolaire.

Des discussions entre la direction des services éducatifs et la chercheure ont permis de faire un choix quant à la façon de procéder pour réaliser cette formation professionnelle. D'un commun accord, les différents intervenants ont décidé d'offrir 2 journées de formation par groupe-cycle. Ainsi, les enseignantes et les enseignants de 1^er^ et 2^e^ année d'un groupe d'écoles ont reçu une journée de formation, le lendemain ceux de 3^e^ et 4^e^ année recevaient une formation semblable et enfin ceux de 5^e^, 6^e^ année et du 1^er^ cycle du secondaire. Ces journées se sont déroulé les unes à la suite des autres (en octobre 2011) durant la même semaine afin de faciliter les discussions dans les écoles respectives.

Au cours de la première journée de formation, les enseignantes et les enseignants ont dû analyser les caractéristiques d'une activité de leur manuel scolaire afin d'observer les liens et les obstacles des activités proposées pour les différents degrés scolaires. Cette analyse a permis d'anticiper les erreurs des élèves et de prévoir des interventions, ce qui a fait l'objet d'une plénière en fin de journée. Les enseignantes et les enseignants ont ensuite réalisé l'expérimentation de l'activité choisie dans leur manuel scolaire et apportée des productions de leurs élèves pour la réalisation de la deuxième journée de formation, un mois plus tard.

Lors de cette 2^e^ journée de formation, les mêmes enseignants et enseignantes sont à nouveau regroupés par cycle d'enseignement. Une discussion portant sur le résultat des expérimentations permet de dresser le portrait des interventions réellement expérimentées de même que les conditions qui les ont suscitées (DeBlois, 2012). L'ensemble des interventions discutées sont ensuite regroupées sur un poster qui est distribué à toutes les enseignantes et à tous les enseignants de la commission scolaire dans le but de favoriser les échanges. Le projet se poursuit durant l'année 2012-2013 afin de réutiliser ce poster en expérimentant d'autres types d'interventions que celles qui ont déjà été expérimentées.

♦♦♦♦♦♦♦♦♦♦

In the context of a teacher training project, primary and secondary teachers from one school board took part in an experience that led them to make a poster consisting of 'successful practices'. This article briefly reviews the steps leading to this production.

With the help of Educational Services, an initial training day was held in 2010. This allowed a dozen people to receive a day of training regarding different interpretations of student errors in mathematics. We know that teachers' interpretations influence their choice of interventions (DeBlois, 2006). Following this training day, those who were involved chose to promote a training project for all teachers of the school board.

Discussions between the management of Educational Services and the researcher allowed a decision to be made regarding how to proceed with this professional training. Through mutual agreement, the various stakeholders decided to offer two training days per division. Thus, 1st and 2nd grade teachers from a group of schools received a training day. The next day, similar training was given to 3rd and 4th grade teachers and then, finally, to teachers of grades 5 and 6, and junior high school. These days were held consecutively (in October of 2011) during the same week to facilitate discussions in the respective schools.

During the first day of training, the teachers had to analyze the characteristics of an activity from their textbook in order to observe the connections and constraints of the proposed activities for different school levels. This analysis allowed us to anticipate student errors and to foresee interventions, which was the subject of a plenary session at the end of the day. The teachers then experimented with the chosen activity from their textbook and brought examples of students' work to a second training day, one month later.

During this second day of training, the same teachers were regrouped according to their divisions. A discussion of the results from the attempted activities allowed us to document the interventions which were actually experienced as well as the conditions that gave rise to them (DeBlois, 2012). All interventions discussed were recorded on a poster that was distributed to all the teachers of the school board in order to facilitate exchange. The project will continue during the 2012-2013 year in order to reuse this poster to experiment with other types of interventions than those already experienced.

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BLENDED MATHEMATICAL COLLABORATION USING A WIKI, *GEOGEBRA* AND *JING*

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Geoffrey Roulet, *Queen's University*

The Ontario Mathematics Curriculum (Ontario Ministry of Education, 2005) identifies communication, reasoning, and representing as three of seven key processes in mathematics. We have been exploring the roles that ICT and online tools can play in supporting these processes. In this work with a Principles of Mathematics, Grade 10 Academic class we have combined the use of a wiki (PBworks), *GeoGebra* (GeoGebra.org), and *Jing* (TechSmith) videos.

The wiki serves to structure and organize online work, providing a place for the teacher to present course content and tasks, but more importantly it also provides space for students to share their work and respond to ideas presented by others. Using *GeoGebra* allows students to present their work without the tedious process of coding mathematics symbols, and in addition, since the *GeoGebra* postings are *Java* applets they are available to other students for manipulation and further exploration. Recording the computer screen along with an oral commentary while creating a *GeoGebra* product and posting the resulting *Jing* video further supports students' mathematics communication; a key process identified by research (Hufferd-Ackles, Fuson, & Gamoran-Sherin, 2004; Piccolo, Harbaugh, Carter, Capraro, & Capraro, 2008) and promoted by mathematics education associations (NCTM, 2000; OAME, 2012).

Although computer use and online sharing mesh with many of these students' lives outside of school (comScore, 2011), we have found a wide range of digital experience. Accounting for this range, and to build skills in a step-by-step manner, we have broken our project into stages: posting of mathematics in text on wiki pages, modelling using *GeoGebra*, embedding *GeoGebra* applets on wiki pages, commenting on each other's work, sharing mathematical reasoning via *Jing* videos on wiki pages, and finally a collaborative exploration combining these skills. Our work to date has revealed multiple related issues: accommodating the range of prior ICT experience, generating sufficient enthusiasm and engagement to encourage out of class participation, and technical problems that frustrate pupils and reduce participation. Students on the whole have embraced the tasks, appreciating the potential of the online tools to reduce the efforts involved in manipulating graphs and symbols and sharing explanations in writing, but we have had to move slowly; carefully building skills and providing sufficient time for task completion in the face of technical glitches.

A PDF file of the poster may be accessed at: http://hdl.handle.net/1974/7380

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SOME THINGS TECHNOLOGIES CAN TELL US ABOUT TECHNOLOGIES: AN INSTRUMENTED ANALYSIS OF TWO SUCCESSIVE MATHEMATICS CURRICULA

Jean-François Maheux et Fabienne Venant
Université du Québec à Montréal

DISCOURSES ON TECHNOLOGIES

Technology reaches all parts of society, including mathematics education...and research in the domain. Here, we explore discourses *on* technology as it appears in mathematics curricula, conducting this investigation using technology, thus also highlighting its potential for our field. Can we elicit distinctions in institutional discourse and observe changes therein? Doing so would provide us with resources to "reflect on the effect of discourses practices on individuals, institutions or opinions" (Magureanu & Paunescu, 2010), for such discourses are a privileged means of communication between, in this case, the Ministry and those involved in the actual realization of mathematics education, such as teachers and administrators. Analysing curricular discourses on technologies is thus a steppingstone to understand ongoing mutation in mathematics education and, for us, preliminary to developing a research program.

IMPLEMENTING AN INSTRUMENTED STUDY OF TWO CURRICULA

Many curricula recently became easily available in digital form through the Internet. This is the case for the two most recent programs in Québec, which makes them relatively easy to examine with text analysis software. More so, Québec's 1993/96 and 2000/04 curricula are interesting documents in which to trace evolutions in a community, for the people expected to use those documents directly took part in their redaction, especially in the latter (Bednarz, Maheux, & Proulx, 2012). With this corpus selected, we conducted a lexicometric analysis (Lebart & Salem, 1994), in which we used technology to approach written discourse in both a quantitative and a qualitative way (Pincemin, 2012). Statistical analyses enable us to visualize lexical entries related to technologies in the mathematics curricula, and compare documents in various ways; but we can also examine the textual context of the occurrences. Our analysis is thus *instrumented* (as opposed to being automatized), in that the software is used to highlight textual phenomena then further investigated (Bernard, 2011), pretty much in the way doctors use an MRI to guide a diagnostic. We settle here on three software programs, *Lexico3*, *Coocs* and *Wordle*, to characterize the discourses on technology in the various part of the 1993 and 2000 secondary school mathematics curricula (e.g. by year, or between 'profiles'), and also compare them.

A FEW RESULTS

As expected, the analysis demonstrates an evolution in the discourse: curricula speak more of technologies in 2000 than they did in 1993. We also notice significant changes in the lexicon, where an interest *in the instruments* (e.g. calculators) is replaced by concerns for students' *involvement with various technologies*, a shift interestingly mirrored in the change of foci from 'problems' to 'situations'. On the other hand, we also map contrasting areas in the curricula in which technologies are largely and very positively mentioned, or not. Regardless of the year, technology is celebrated when mathematics as a domain is introduced, but almost absent when it comes to pinpoint the actual mathematics (concepts, methods, etc.) to be taught and learned.

POSTER

The poster can be viewed at http://math.uqam.ca/maheuxjf/maheuxVenant.htm

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INVESTIGATING THE TEACHING PRACTICES OF A GROUP OF MATHEMATICS GRADUATE STUDENTS

Tod L. Shockey and Sibylle Weck-Schwarz
University of Toledo

This poster presentation shared our findings of working with a group of mathematics graduates as they begin their role as graduate-teaching assistants. These findings are from year one of a mathematics professional seminar that will continue into the 2012-2013 academic year. The first semester of the seminar attended to the details of getting ready for the first day of teaching, arriving on time, arriving prepared with a writing instrument, having prepared problems for students to consider and other minutiae often overlooked, but very important for success in teaching.

In semester two we focused our attention on the teaching practices of these graduate students. We had five objectives for this semester: 1) understand the needs of new teaching assistants as they prepare for their first teaching experience; 2) create a mentorship atmosphere for these teaching assistants; 3) allow the teaching assistants to critically review their peers teaching through live teaching episodes; 4) identify best practices for the teaching assistants; and 5) begin to develop the teaching assistants as reflective practitioners. The content focus was infinite series as it appears in Calculus II. Students devised a ten- to twelve-minute lesson on a specific aspect of infinite series. The student then 'taught' the lesson to their peer group of graduate students in mathematics. Some outcomes of these episodes included: difficulty in planning for such a short span of time; focusing the lesson and not doing too much; the anxiety of and anticipation of peer critique; sharing some mathematical 'tricks' to make some of the procedures more accessible; anxiety of being videotaped; and writing a reflective paper about the experience.

Most of the teaching episodes were focused on procedural knowledge with little to no attention to the conceptual knowledge (Hiebert & Lefevre, 1986). Participants were made aware of content trajectories (Ferrini-Mundy, Floden, & McCrory, 2005) and pedagogical content knowledge (Shulman, 1987). The collaboration between a mathematician, Schwarz, and a mathematics educator, Shockey, allowed for complementing views and discussions with these students about their classroom role. Our intention at CMESG 2012 was to gather feedback from participants. We were very pleased with the comments and suggestions and learned that this work is situated in the important research of Beisiegel and Simmt.

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MATHEMATICS FOR YOUNG CHILDREN: EXPLORING WHAT IS POSSIBLE IN EARLY MATHEMATICS EDUCATION?

Diane Tepylo[1], Joan Moss[1], Catherine Bruce[2], Tara Flynn[2], Diana Chang[2], & Zachary Hawes[1]
[1]*University of Toronto and* [2]*Trent University*

This gallery presentation introduced the *Mathematics for Young Children* (M4YC) project. Mathematics is a crucial area of learning for young children, yet in Canada, there has been limited research of how children aged 3 to 8 learn mathematics (Bruce, Flynn, & Moss, 2012; McGarvey & Moss, 2008). This knowledge requires immediate attention, given the importance of early mathematics in predicting later academic success (cf. Claessens, Duncan, & Engel, 2009), and that early mathematics intervention is important for lessening socio-economic gaps in mathematical achievement (Baroody, Lai, & Mix, 2006; Starkey & Klein, 2008).

M4YC explored what is possible in terms of early mathematics learning and teaching. Teachers and researchers from three school boards, three universities and one independent school closely worked together to carry out adaptations of Japanese Lesson Study. The lesson study teams examined research, designed and conducted clinical interviews with children from Junior Kindergarten to Grade 2, designed and enacted exploratory lessons, observed and theorized about student learning trajectories and the instructional trajectories that support them, and finally, invited educators observed and debriefed a public research lesson.

PRELIMINARY FINDINGS

Across all of the M4YC Lesson Study teams, educators found that young children are more capable of learning and demonstrating their understanding of mathematics than anticipated, and that young children benefited from child-initiated tasks, guided discovery, and explicit instruction. Analysis of the data continues on many fronts, and the dynamic learning resources developed by the teachers are being disseminated via web, conferences and symposia.

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MISE À L'ESSAI D'UNE SITUATION D'ENSEIGNEMENT-APPRENTISSAGE EN LIEN AVEC LE MÉTIER DU SCÉNOGRAPHE POUR FAVORISER L'ENGAGEMENT MATHÉMATIQUE DES ÉLÈVES DU 1ER CYCLE DU SECONDAIRE

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Université du Québec à Rimouski

PROBLÉMATIQUE

Afin de favoriser l'engagement des élèves en mathématiques, il faut proposer aux élèves des problèmes dont les contextes sont issus du monde réel (OCDE, 2010), favoriser la création de liens interdisciplinaires et encourager une approche orientante (Ministère de l'éducation, 2003). Certains travaux (Duatepe-Paksu & Ubuz, 2009; Saab,1987; Omniewski, 1999; Dorion, 2009) ont démontré que l'utilisation des arts dans l'enseignement pouvait avoir des effets positifs sur le développement des élèves. Pour la présente recherche, nous nous intéressons aux arts dramatiques et posons l'hypothèse suivante : la mise à l'essai d'une situation d'enseignement-apprentissage dont le contexte est issu du métier du scénographe et dont la résolution s'effectue en rôle favorisera l'engagement mathématique des élèves. Notre question de recherche est la suivante : Comment s'exprime l'engagement mathématique des élèves dans leur processus de résolution de la situation?

CADRE THÉORIQUE

Pour nous, l'engagement mathématique se manifeste dans l'activité (Leontiev, 1984), qui doit permettre de mobiliser et de développer des raisonnements mathématiques; c'est un processus dynamique dont les dimensions affective, cognitive et comportementale sont indissociables. La résolution en rôle permet aux élèves de résoudre un problème en se mettant dans la peau d'un professionnel mobilisant des raisonnements mathématiques au quotidien afin de favoriser l'exploration et l'échange avec les autres. Le professionnel choisi est le scénographe, qui doit réaliser les costumes et les décors au théâtre.

MÉTHODOLOGIE

Nous ferons une expérimentation de devis où le design sera la situation d'enseignement-apprentissage élaborée. L'échantillon sera une classe du premier cycle du secondaire. La collecte des données s'effectuera à l'aide d'entrevues semi-dirigées, d'observation directe. Nous nous intéresserons à l'expression de l'engagement mathématique dans cette situation, au rapport au savoir compte tenu du contexte utilisé et aux perceptions par rapport à la résolution en rôle.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
  Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe / Proof in mathematics and in schools
- The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES	Old and new mathematical ideas from Africa: Challenges for reflection
	M. WALSHAW	Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries
	W. HIGGINSON	Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242

Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	ED 529557
Proceedings of the 2003 Annual Meeting	ED 529558
Proceedings of the 2004 Annual Meeting	ED 529563
Proceedings of the 2005 Annual Meeting	ED 529560
Proceedings of the 2006 Annual Meeting	ED 529562
Proceedings of the 2007 Annual Meeting	ED 529556
Proceedings of the 2008 Annual Meeting	ED 529561
Proceedings of the 2009 Annual Meeting	ED 529559
Proceedings of the 2010 Annual Meeting	ED 529564
Proceedings of the 2011 Annual Meeting	*submitted*
Proceedings of the 2012 Annual Meeting	*submitted*

NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.