CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2011 ANNUAL MEETING / RENCONTRE ANNUELLE 2011

CMESG GCEDM

Memorial University of Newfoundland
June 10 – June 14, 2011

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PROCEEDINGS OF THE 2011 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2011 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

35th Annual Meeting
Memorial University of Newfoundland
June 10 – June 14, 2011

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INTRODUCTION

Elaine Simmt – President, CMESG/GCEDM
University of Alberta

It is my pleasure to take this opportunity to reflect on CMESG 2011. Without a doubt our colleagues at Memorial University and the people of St. John's, Newfoundland were wonderful hosts to us mathematics educators from across Canada. The hospitality of the people was most certainly one of the highlights of the experience for me. On behalf of our membership, I would like to officially thank the Dean of Education, Dr. Alice Collins for financial support and for welcoming us to Memorial University. Thanks also go out to Dr. Mark Abrahams, Dean of the Faculty of Science and Dr. Chris Radford, Head of the Department of Mathematics and Statistics, for their financial contribution to the conference. At this time we would also like to acknowledge the financial support for graduate students from AARMS. A very special thank you to Mary Stordy and Margo Kondratieva for agreeing to host CMESG 2011. Your local organizing team provided us with an exceptional level of support. Thank you to graduate student Sharon Facey, undergraduates, Christina and Mandy, and staff members Bernadette Powers, Cathy Madol, Glenda Goulding and Helen Manning, for their tremendous help. Lastly, we raise our glasses to Gene Power for sharing the traditional Newfoundland Screech-In Ceremony with us.

Having thanked the local organizing committee and its supporters I would like to turn our attention to the scientific program and extend our appreciation to all of the people who gave presentations and facilitated sessions. This year our plenaries both offered presentations that spoke directly to applications of mathematics and mathematics education. We had the pleasure of learning about the kinds of mathematics that underlies and is expressed in Chris Palmer's art-making with his talk, *Pattern Composition: Beyond the Basics*. Pessia Tsamir and Dina Tirosh demonstrated for us a strategy that they use in their team teaching practice, *The Pair-Dialogue Approach in Mathematics Teacher Education*. Working groups provided participants opportunities to work together on themes such as mathematics and climate change, procedural knowledge in mathematics learning, emergent methods for mathematics education research, using simulation to develop students' mathematical competencies, art and mathematics, and tasks for future mathematics teachers. Read on in the proceedings to learn more about the conversations that ensued in the working groups. The topic session speakers took on the challenge of how mathematics is represented in our teaching and experienced by learners. As well, we were fortunate to learn about the work of six new PhDs in our community. The conference was rounded out with opportunities for members to form *ad hoc* groups to have conversations about emerging ideas with colleagues. Finally, we added a new element, the Math Gallery Walk so that we could have the opportunity to catch up on the mathematics education work our members from across the country are doing.

As it always is for me, I fly home from CMESG rejuvenated and grateful because I am part of a vibrant and thoughtful mathematics education community in Canada.

Horaire

Vendredi 10 juin	**Samedi 11 juin**	**Dimanche 12 juin**	**Lundi 13 juin**	**Mardi 14 juin**
	8:00 – 9:00 **Inscription** Pavillon de l'Éducation			
	9:00 – 10:30 **Groupes de travail**	9:00 – 10:30 **Groupes de travail**	9:00 – 10:30 **Groupes de travail**	9:00 – 10:30 **Gallérie mathématique**
	10:30 – 11:00 Pause café	10:30 – 11:00 Pause café	10:30 – 11:00 Pause café	10:30 – 11:00 Pause café
	11:00 – 12:00 **Groupes de travail**	11:00 – 12:30 **Groupes de travail**	11:00 – 12:30 **Groupes de travail**	11:00 – 12:30 **Séance de clôture**
	12:00 – 13:00 Dîner	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	
	13:00 – 13:30 **Plénière 1 : disc.**			
	13:30 – 14:30 **Plénière 1 : Séance questions**	13:30 – 14:30 **Plénière 2**	13:30 – 14:00 **Plénière 2 : disc.**	
14:00 – 17:00 **Inscription** *Inco Innovation Centre*	-	-	-	
		14:45 – 15:15 **N$^{\text{lles}}$ thèses (3)**	14:15 – 15:15 **Plénière 2 : Séance questions**	
	15:00 **Excursion** Les Bus partent des résidences	15:15 – 15:45 Pause café	15:15 – 15:45 Pause café	
		15:45 – 16:15 **Séances *ad hoc***	15:45 – 16:45 **Séances thématiques**	
		-		
		16:30 – 17:00 **N$^{\text{lles}}$ thèses (3)**	-	
17:00 – 18:00 **Séance d'ouverture** *Inco Innovation Centre*		17:00 – 17:30 **Séances *ad hoc***	17:00 – 18:00 **Assemblée générale annuelle**	
		-		
18:00 – 19:00 **Plénière 1** *Inco Innovation Centre*		18:00 ***« Kitchen Party »*** Les Bus partent des résidences pour le centre-ville	-	
19:00 – 21:00 **Réception** *Inco Innovation Centre*			18:30 – 20:30 Repas dans la salle à dîner	

Schedule

Friday June 10	Saturday June 11	Sunday June 12	Monday June 13	Tuesday June 14
2:00 – 5:00 **Registration** *Inco Innovation Centre*	8:00 – 9:00 **Registration** Education Building	9:00 – 10:30 **Working Groups**	9:00 – 10:30 **Working Groups**	9:00 – 10:30 **Gallery Walk**
5:00 – 6:00 **Opening Session** *Inco Innovation Centre*	9:00 – 10:30 **Working Groups**	10:30 – 11:00 Coffee Break	10:30 – 11:00 Coffee Break	10:30 – 11:00 Coffee Break
6:00 – 7:00 **Plenary 1** *Inco Innovation Centre*	10:30 – 11:00 Coffee Break	11:00 – 12:30 **Working Groups**	11:00 – 12:30 **Working Groups**	11:00 – 12:30 **Closing Session**
7:00 – 9:00 **Reception** *Inco Innovation Centre*	11:00 – 12:00 **Working Groups**	12:30 – 1:30 Lunch	12:30 – 1:30 Lunch	
	12:00 – 1:00 Lunch	1:30 – 2:30 **Plenary 2**	1:30 – 2:00 **Plenary 2 – disc.**	
	1:00 – 1:30 **Plenary 1 – disc.**	-	-	
	1:30 – 2:30 **Plenary 1** **Q&A**	2:45 – 3:15 **New PhDs (3)**	2:15 – 3:15 **Plenary 2** **Q&A**	
	-	3:15 – 3:45 Coffee Break	3:15 – 3:45 Coffee Break	
	3:00 **Excursion** Buses leave residences	3:45 – 4:15 ***Ad Hoc***	3:45 – 4:45 **Topic Sessions**	
		-	-	
		4:30 – 5:00 **New PhDs (3)**	5:00 – 6:00 **AGM**	
		5:00 – 5:30 ***Ad Hoc***	-	
		-	6:30 – 8:30 Supper at dining hall	
		6:00 ***Kitchen Party*** Buses leave residences for downtown		

Plenary Lectures

Conférences plénières

PATTERN COMPOSITION: BEYOND THE BASICS

Chris K. Palmer
Shadowfolds

A principle of recursion developed by medieval artisans of the Middle East is demonstrated with sequential diagrams that reveal the steps of construction. Sequential diagrams with few abstract symbols are used to show how to make origami models. They can also be used to show paths to orderly complexity through small steps. These drawings attempt to show with solid colour and line weight the steps of the construction [Figure 1], the principle of substitution where different patterns can be inserted into the skeleton [Figures 2 and 3], and the recursive structure where a polygon is placed on its own corners in a ring [Figure 4] and the final composition [Figure 5] that combines two related tilings in a kind of checker pattern. This kind of tiling was described by Alicia Boole Stott as the expansion operation.[1]

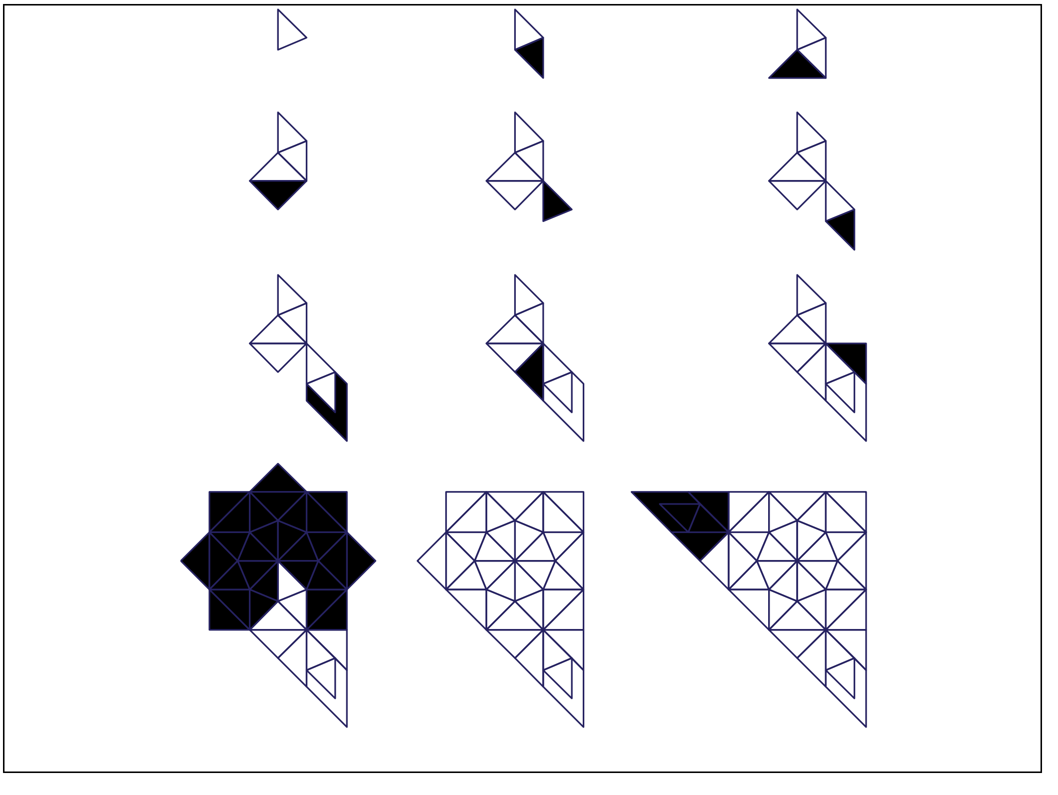


Figure 1

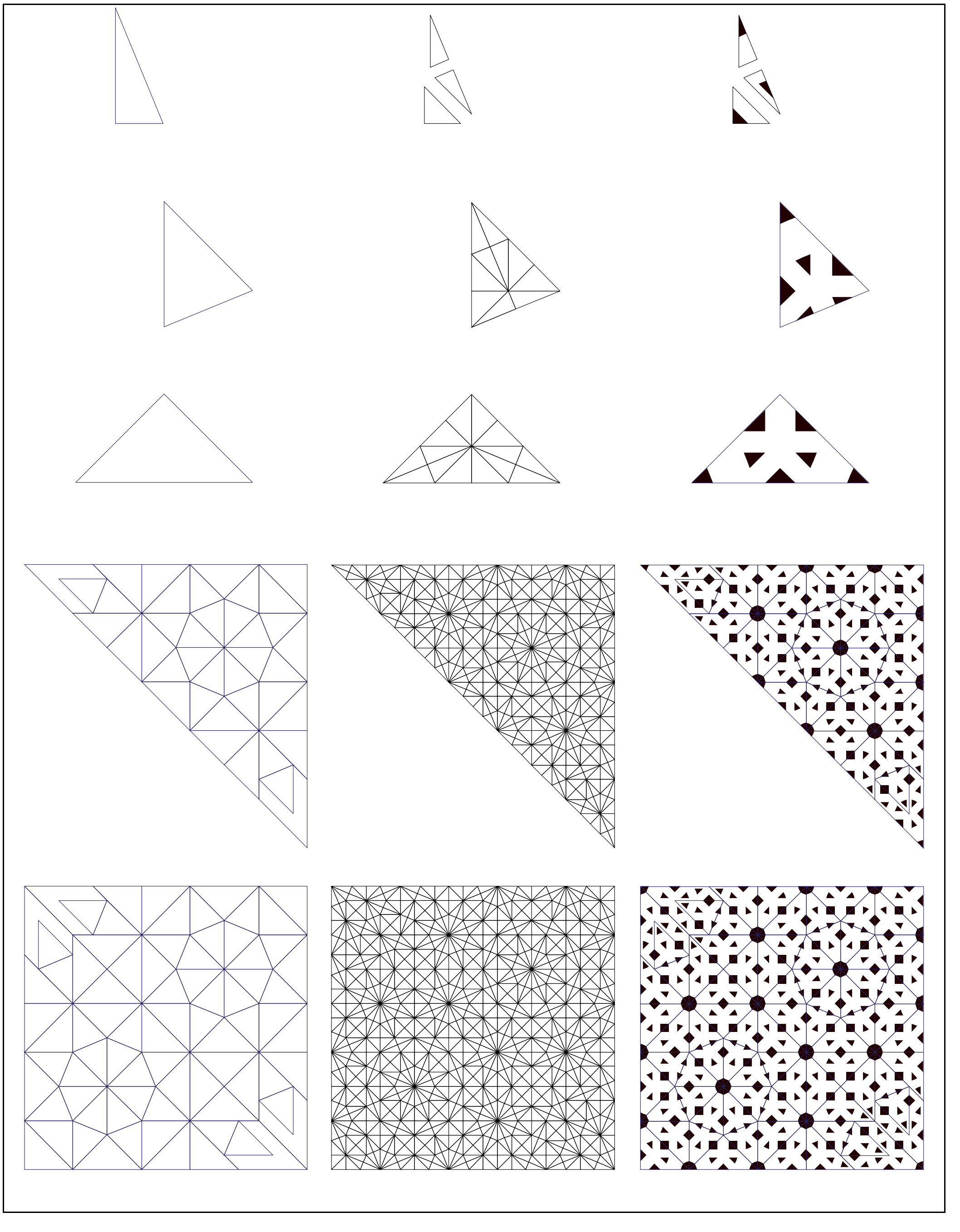


Figure 2

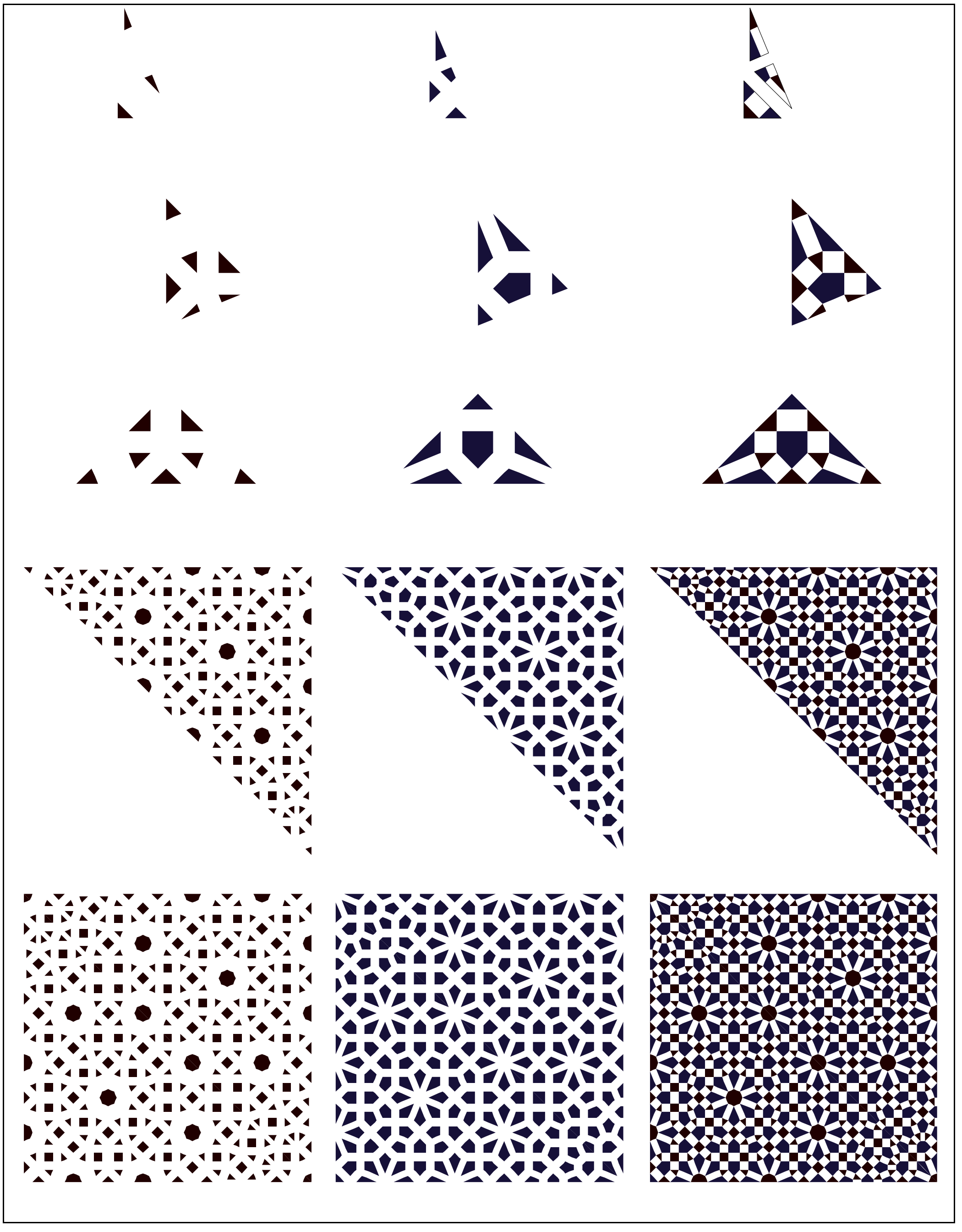


Figure 3

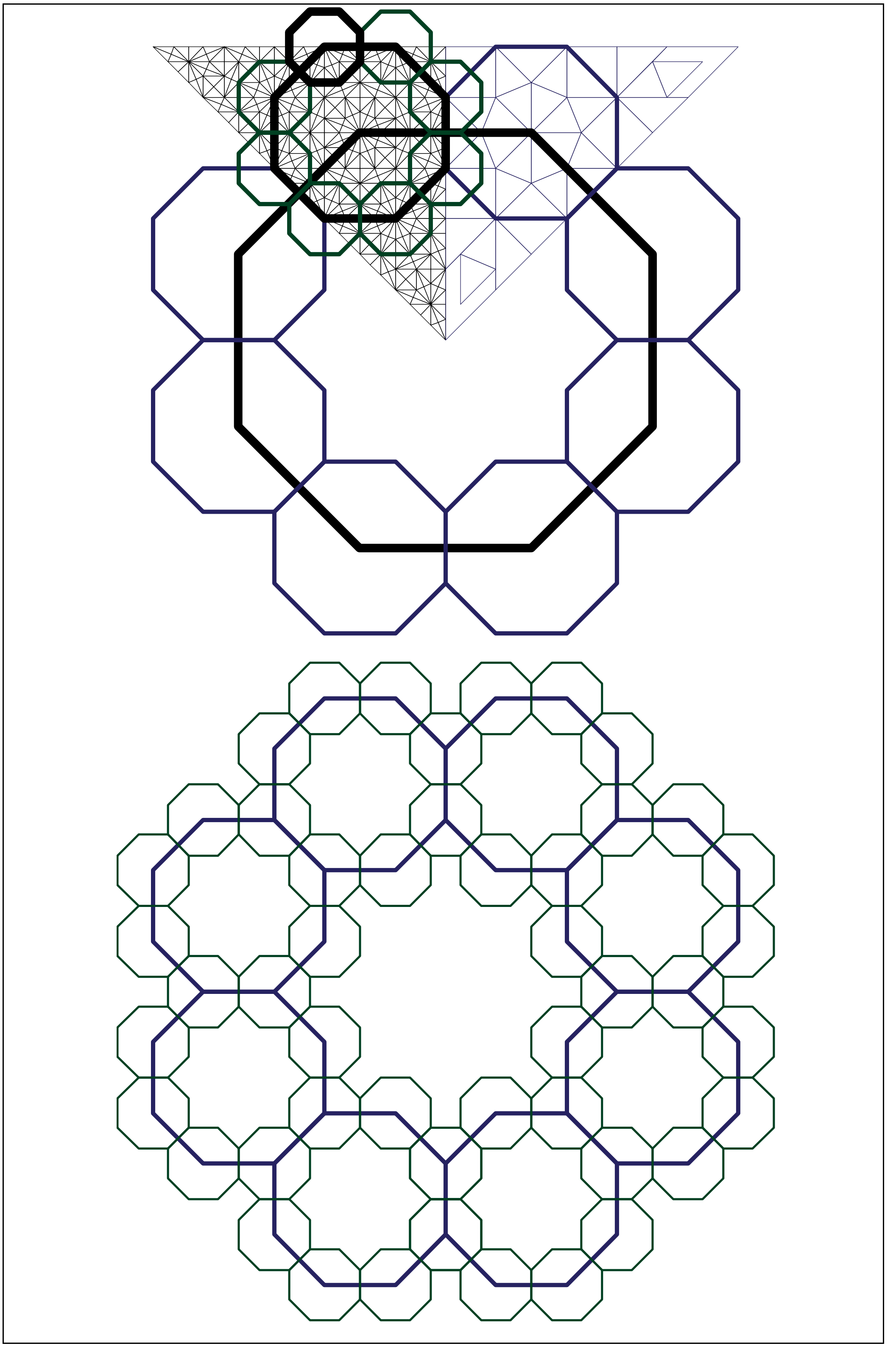


Figure 4

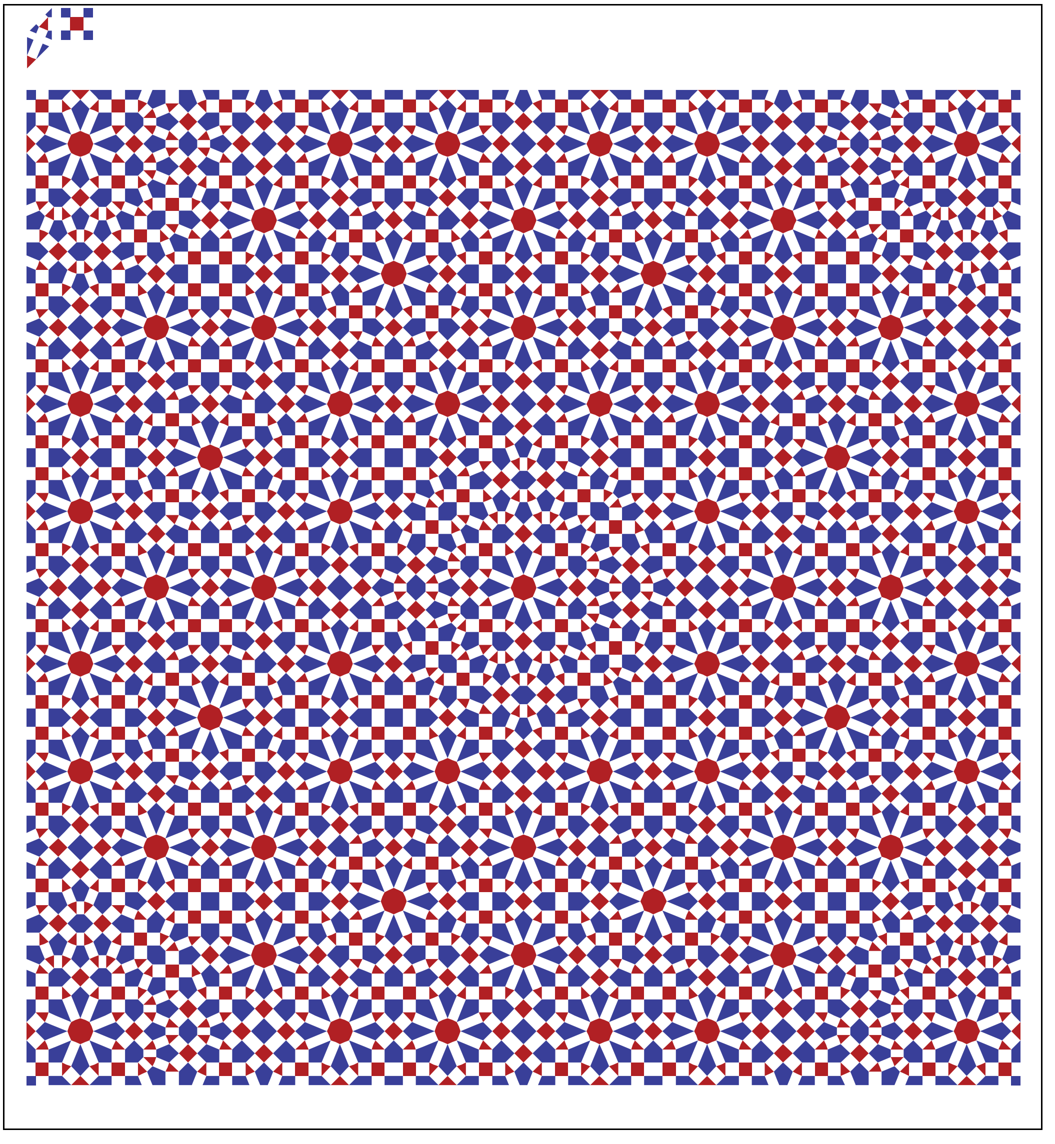


Figure 5

FURTHER READINGS

Bain, G. (1973). *Celtic art: The methods of construction*. New York: Dover.

Conway, J. H., Burgiel, H., & Goodman-Strauss, C. (2008). *The symmetry of things*. Wellesley, MA: A. K. Peters, Ltd.

[1] http://www.ams.org/samplings/feature-column/fcarc-boole
http://www.georgehart.com/virtual-polyhedra/conway_notation.html

THE PAIR-DIALOGUE APPROACH IN MATHEMATICS TEACHER EDUCATION

Pessia Tsamir and Dina Tirosh
Tel-Aviv University, Israel

In this paper, we describe the Pair-Dialogue (P-D) teaching approach. We then illustrate how we use this approach in professional development programs, working with kindergarten teachers on definitions of triangles and with secondary school teachers on validating and refuting elementary number theory statements. Our focus in these two examples is on issues related to mathematical content knowledge. Other P-D episodes pertain to pedagogical content and to curricular content knowledge issues. Special care is taken, in our interactions with prospective and with practicing teachers, to the sensitive, emotionally loaded situation which may result from working on incorrect responses.

There is a wide agreement that teachers may play a significant role in learners' mathematical development. Consequently, various attempts have been made to design, implement and evaluate professional programs that influence the nature and quality of teachers' knowledge and practice (Ball, Thames, & Phelps, 2008; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Cooney, 1994; Ebby, 2000; Hiebert, Morris, & Glass, 2003). We have devoted substantial attempts during the last two decades to promote teachers' mathematical knowledge needed for teaching. These attempts are accompanied with explicit discussions of the interplay between knowledge, reflective-practice and related affective issues. We have worked with individual teachers, small groups, as well as large groups of prospective and practicing teachers (e.g., Tirosh & Tsamir, 2004; Tsamir & Tirosh, 2005).

In the last decade, we have developed the Pair-Dialogue (P-D) approach and implemented it in various professional development programs in mathematics teacher education. The P-D approach is a specific form of team teaching in which we teach cooperatively. Team teaching approaches are forms of instruction in which at least two instructors work purposely, regularly and cooperatively to help a student or a group of students learn (Buckley, 2000). Various types of team teaching are described in the literature; a common type is that of a team of experts, each with a different expertise, sharing the responsibility for an interdisciplinary course (e.g., Gosetti-Murrayjohn & Schneider, 2009; Sandholtz, 2000; Shibley, 2006). A quite unique aspect of our approach is that the two of us have essentially the same expertise and status (professors of mathematics education with common fields of interest).

Our P-D approach is based on three major didactical components: (a) continuous, formative assessment of the participants' knowledge needed for teaching mathematics (b) teaching-learning interactions, addressing issues that are known to be challenging, i.e., errors or dilemmas, and (c) discussions of teachers' related, reflective practices. In our interactions with teachers we use a blend of pair performances (e.g., thought-provoking dialogue episodes) and discussions that involve the teachers (segments of "inviting the audience", prospective and participating teachers, to express their views on different ideas that are presented and to "help us out" in resolving the dilemmas that we raise). The dialogues are semi-structured, allowing for both prepared-in-advance and in-action adaptations to different populations of teachers. We employ various modus operandi of the P-D approach: sometimes, both of us offer correct (or erroneous or a mix of correct and erroneous) ideas. In other cases, one teacher educator acts as a "model learner", presenting students' dilemmas, and the other acts as the knowledgeable guide. The roles are altered occasionally, to refrain from creating a "clever, always-right" character and a "puzzled-erring" character, thereby avoiding irrelevant hints that may take away the mathematical essence of the situation. The activities have several modes of implementation: individual work (occasionally handed in to us), small group communications, and whole class discussions. A main gain of our approach is that the teachers are confronted, in a gentle and respectful manner, with their incorrect responses, and the P-D opening serves as a springboard for a thorough discussion of common errors.

The P-D approach has been implemented with prospective and practicing teachers from preschool to Grade 12. The durations of the professional development projects have ranged from one week to three years and have often engaged both the teachers as well as the children in their classes. In this paper we briefly describe and illustrate the P-D approach regarding two central, mathematical structures: definitions and proofs.

WHY DEFINITIONS AND PROOFS?

Definitions and proofs are two central constructs that play a crucial role in mathematics. Yet, studies have shown that learners often face difficulties when working with these mathematical entities and that intuitive obstacles are a main cause for these hurdles (e.g., Alibert & Thomas, 1991; Fischbein & Kedem, 1982; Harel & Sowder, 2007; Tall & Mejia-Ramos, 2006; Tall, 1999; Vinner, 1991). Many mathematics educators have recommended that developing a solid mathematical foundation, including references to definitions and proofs, should begin as early as possible. For example, according to the Principles and Standards for School Mathematics, mathematical definitions, reasoning and proofs may be and should be nurtured continuously from a young age: "*Instructional programs from prekindergarten through grade 12 should enable all students to recognize reasoning and proof as fundamental aspects of mathematics*" (National Council of Teachers of Mathematics, 2000, p. 122). Thus, classrooms performances should provide opportunities, even for very young children, to address definitions and proofs in a natural, systematic and coherent manner. Students should be encouraged to raise questions and assumptions, to suggest solutions, to provide acceptable justifications to explain their ideas and to consult definitions and proofs (e.g., Fischbein, 1993).

One may wonder what types of explanations, definitions and proofs are expected at different developmental stages. Evidently, the types of reasoning and justifications suitable for young children may differ from those appropriate for older children. At early ages we may focus on informal explanations, based on students' real-world experiences, rather than (or much more than) on formal explanations that consist of rigorous, symbolic representations. Levenson (2008) based on Koren (2004) differentiated between *Mathematically Based* (MB) explanations that employ only mathematical notions and rules, and *Practically Based* (PB) explanations that may also use daily references. Mathematics education researchers have illustrated how young children offer MB explanations in classroom discussions (e.g., Ball & Bass, 2000) and they have shown that many elementary and secondary school students understand, use and even prefer such explanations (e.g., Levenson, Tirosh, & Tsamir, 2006; Tsamir, Sheffer, & Tirosh, 2000).

Clearly, a major aim for mathematics educators is to promote learners' ability to produce and communicate MB and formal explanations. That is, to enhance students' capacity to justify and explain his/her mathematical solutions by using definitions and theorems. For instance, when being asked "Is this a …?" (e.g., "Is this a triangle?"), it is important to base one's answer, 'yes' or 'no', on the definition. Similarly, when being asked "Is this [mathematical statement] true?" (e.g., "Is the sum of three consecutive numbers divisible by three?"), the answer 'yes' or 'no' should be justified either by a validating or by a refuting proof. All in all, accepting that mathematical concepts are terminology-based entities and that mathematical theorems are rule-based entities are two pivotal constructs of the mathematical realm; consequently, concepts, definitions, theorems, and proofs play a central role in doing mathematics and in discussing mathematical issues.

In the following sections we describe and analyze episodes taken from two teacher professional development courses, one with preschool teachers regarding definitions of triangles, the other with secondary school teachers regarding proofs by validating or refuting Elementary Number Theory (ENT) statements. In this presentation we focus on several P-Ds relating to *Subject Matter Knowledge* (SMK) (Shulman, 1986).

WORKING WITH PRESCHOOL TEACHERS ON DEFINITIONS OF TRIANGLES

Recently, the issue of mathematics education for preschool children has come to the fore. A growing number of position papers, books and articles attest to the importance of early childhood mathematics education (e.g., Bartolini-Bussi, 2011; Clements & Sarama, 2011; Ginsburg, Inoue, & Seo, 1999; Levenson, Tirosh, & Tsamir, 2011; Tsamir & Tirosh, 2009). However, there is consistent evidence that many preschool teachers have limited knowledge of mathematics and of young children's mathematical reasoning, and that geometry is a major hurdle (e.g., Clements, 2003; Clements & Sarama, 2007; Lee & Ginsburg, 2007). Consequently, professional development programs for early childhood teachers that focus on the mathematics knowledge needed for teaching geometry have been initiated in various countries (e.g., Clements & Sarama, 2011; Levenson, Tirosh, & Tsamir, 2011; Pitta-Pantazi & Christou, 2011; Tirosh, Tsamir, Levenson, & Tabach, 2011). Yet, few studies have addressed the types of instruction that have the potential to enhance preschool teachers' geometric knowledge (e.g., Clements, Sarama, & DiBiase, 2004).

We have devoted, in the last decade, extensive efforts to working in low-income areas in Israel, in an attempt to meet the challenge of making geometry friendlier to preschool teachers. In one of these professional development courses, 17 preschool teachers participated in six, four-hour sessions. The participants stated that they were suffering from geometry anxiety, and that in their preschools geometry was commonly neglected.

In this paper we focus on the first session, in which we addressed the topic: triangles. The teachers were initially asked to reply in writing to a questionnaire (formative evaluation). Teachers' responses were then used in the design of later lessons. We begin by briefly describing the data that led to the formation of the questionnaire.

DESIGNING THE TASK: IS THIS A TRIANGLE?

The tasks that we formulated for the triangle-sessions and the related P-Ds were based on reported research findings from our past studies on students' and teachers' geometrical knowledge, and on the accumulated data that we collected from this specific group of 17 teachers.

The collected data led us to consider two dimensions when discussing examples and non-examples of triangles: the mathematical dimension and the psychological dimension (see Figure 1). The mathematical dimension is based on mathematical definitions, and therefore consists of two well-defined, disjoint sets of figures: examples and non-examples. The psychological dimension consists of two sets of figures: intuitive and non-intuitive, a distinction based on studies of children's and adults' conceptions and misconceptions when addressing each figure (e.g., Tsamir, Tirosh, & Levenson, 2008). Intuitive triangles are easily identified as such (e.g., the triangles that have one side parallel to the 'down edge' of the paper, see Figure 1, Cell 1), while non-intuitive triangles are commonly misjudged as non-triangles (e.g., upside down triangles, thin triangles, see Figure 1, Cell 2). In the same vein, intuitive non-examples of triangles are easily identified as not being triangles (e.g., circles or squares, figures for which learners tend to be familiar with their images and with their names, e.g., Figure 1, Cell 3). Non-intuitive non-examples of triangles are figures that are not triangles, but learners tend to identify them as triangles (e.g., a seemingly triangular shape with one bent side, see Figure 1, Cell 4).

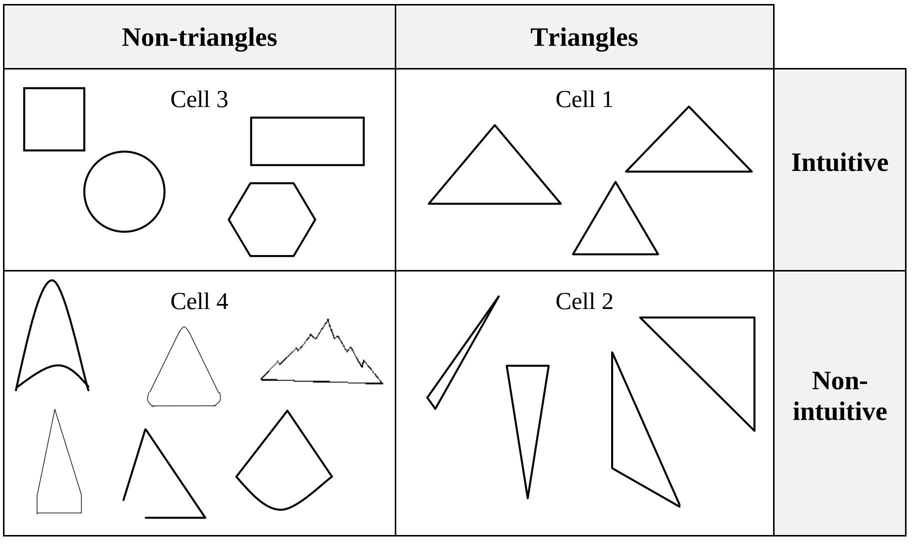


Figure 1. Examples of intuitive and non-intuitive triangles and non-triangles.

A worksheet that included intuitive and non-intuitive examples and non-examples of triangles was administered to the preschool teachers, and they were asked to determine if each of the figures was a triangle and to justify their assertion. Examples of the items that were included in the questionnaire are presented in Table 1. The questionnaire included additional items that are not reported on here (e.g., how confident they were in their answer).

This worksheet was designed to assess teachers' responses to identification-of-triangle tasks, as well as their tendency to refer to the critical attributes of a triangle in their justifications (van Hiele & van Hiele, 1958; Hershkowitz, 1990).

EVALUATING THE TEACHERS' CONCEPT IMAGES OF TRIANGLES

Table 1 indicates that all preschool teachers correctly identified the intuitive triangle and the two intuitive non-triangles (the circle and the hexagon). However, all 17 teachers incorrectly identified the "pizza-'triangle'" (Shape 7), and the "road-sign 'triangle'" (Shape 4), as triangles. There was also a tendency to incorrectly view the "arcs-'triangle'" (Shape 5) as a triangle, and some hesitations regarding the "open-'triangle'" (Shape 3). In the latter two cases, teachers further described entities as "sort of" triangles, and as "almost" triangles. Such expressions might suggest that they were unaware of the sharp mathematical distinction between examples and non-examples of triangles. After studying the data regarding the most frequent errors and the correct and incorrect responses of each participant, we conducted the following P-D.

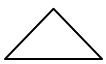
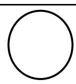
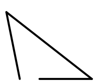
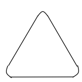
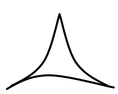
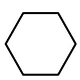
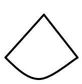
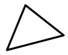
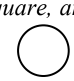
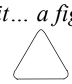
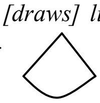
The Figure	**Triangle?** *Why?*			**Comments...**
1 	**Yes**	**17**		
	It has three sides		*15*	
	No explanation		*2*	
2 	**No**	**17**		
	It's a circle		*16*	
	It has no sides		*1*	
3 	**No**	**14**		
	It's missing a part		*14*	*I am not sure.*
	Yes	**2**		
	It has 3 sides		*1*	*One side is a bit too short.*
	No explanation		*1*	
	Almost	**1**		
	It's triangular with 3 sides		*1*	*One side is a bit broken.*
4 	**Yes**	**17**		
	It has three sides		*13*	*Like the triangular road-sign 5*
	It's the shape of a triangle		*2*	*1*
	No explanation		*2*	
5 	**Yes**	**9**		
	It has 3 sides		*3*	
	It has 3 bent sides		*3*	
	No explanation		*3*	
	No	**7**		
	No explanation		*7*	*I'm not sure. It may still be a triangle. 2*
	Sort of	**1**		
	It's triangular with 3 sides		*1*	*The sides should be more stretched.*
6 	**No**	**17**		
	It's a hexagon		*9*	
	It has 6 sides		*8*	
7 	**Yes**	**17**		
	It's like a pizza triangle		*15*	
	It has three sides		*1*	
	No explanation		*1*	

Table 1. Preschool teachers' responses to "*Is this a triangle? Why?*"

PAIR-DIALOGUE: HOW CAN I KNOW WHETHER THIS IS A TRIANGLE?

The aims of this session were (a) to challenge the preschool teachers' images of triangles, aiming at encouraging them to develop triangle-images that are consistent with the related mathematical definition (the notions concept images and concept definitions are taken from Tall & Vinner, 1981) and (b) to increase the teachers' awareness of the need to consult the definition when making decisions about the nature of the figures (whether it is an example or a non-example of a triangle). Several P-Ds were employed for this purpose. Here we present the first part of the dialogue "*How can I know whether this is a triangle?*" (In the dialogues, P and D stand for Pessia and Dina, respectively.)

> ***P:*** *I feel a bit confused about triangles… I mean… the identification of triangles, can you help me?*
> ***D:*** *Sure.*
> ***P:*** *Please draw a triangle.*
> ***D:*** *[draws]* 
>
> ***P:*** *This seems to be easy… I… kind of know that it is a triangle; I see it's a figure that has three sides. OK. It has to have THREE SIDES.*
> ***D:*** *Right. So, this [draws a square]*  *is not a triangle.*
>
> ***P:*** *Sure. It's a square.*
> ***D:*** *Yes. It's a square, and therefore, it has FOUR and NOT THREE sides. And this [draws a circle]*  *is also NOT a triangle.*
>
> ***P:*** *Sure.... It's a circle…*
> ***D:*** *It has NO SIDES.*
> ***P:*** *Ah… I believe I get it… a figure with THREE SIDES… right? Like this… [Draws a "road-sign" shape]* 
>
> ***D:*** *No… No… No… This is not a triangle.*
> ***P:*** *Why? It has three sides.*
> ***D:*** *But the corners, the vertices are round…*
> ***P:*** *So what? We said nothing about vertices… Do we need to?*
> ***D:*** *Yes. There should be three vertices…. Sharp corners…*
> ***P:*** *OK. OK. OK... If I get you right… you mean that a triangle is a figure with three sides and pointy vertices, right?*
> ***D:*** *Yes.*
> ***P:*** *OK. So the traffic-sign triangle is ALMOST a triangle.*
> ***D:*** *No. No. In geometry there is no "ALMOST". It is either YES… I mean a triangle... an example, or NO.*
> ***P:*** *[mumbles quietly as if to herself] either yes or no... [turns to D] I can surely draw a good example now… [draws] like this pizza triangle—It's even called [in Hebrew] a pizza TRIANGLE…* 
>
> ***D:*** *No… No… No… This is not a triangle. Not in geometry.*
> ***P:*** *WHY? It has three sides and three vertices... and EVERYBODY calls it a pizza TRIANGLE…*
> ***D:*** *But one side is not really a side… not geometrically… it is NOT STRAIGHT…*
> ***P:*** *Still… It's a side… I don't get it. Every time you add conditions… I'll never know what a triangle is…*
> ***D:*** *You need to address the definition… I mean ALL the critical attributes…*
> ***P:*** *ALL? What do you mean by ALL?? How do I know that I addressed ALL attributes? And suddenly you added another term… What is this CRITICAL thing that you mentioned? [Turns to the class] Can someone else help me? Do you agree with Dina? [P writes on the side of the blackboard, under the title: Dilemmas and Assumptions]:*
>
> 1. *How do we determine that a figure is a triangle?*
> 2. *What are critical attributes?*

This dialogue challenged the justification: "it has three sides" that most preschool teachers provided to justify their correct as well as their incorrect responses to their assertions that some of the shapes shown in Table 1 are triangles. The participants erroneously regarded this explanation as sufficient or as a definition, and many used the term 'side' in an everyday manner, employing the concept image of a wall or a fence that is not necessarily straight.

This episode illustrates one possible way of working with the preschool teachers on incorrect or incomplete responses. In this P-D one teacher educator (P) acted as a 'model learner', presenting students' opinions, dilemmas, and questions; the other (D) acted as a knowledgeable guide. A main gain is that the preschool teachers were confronted, in a gentle manner, with their incorrect responses.

This opening served as a springboard to a thorough discussion of the common errors. At this stage, Gal, one of the preschool teachers said:

> ***Gal:*** *I agree with you [P], the pizza triangle is DEFINITELY a triangle. It's even called so!*

Here we see a member of the 'audience' cutting into our pair-dialogue, expressing her position. Gal felt confident to interrupt us and to declare that "the pizza triangle is DEFINITELY a triangle". Her confidence in her erroneous solution is evident by her bursting into the dialogue, the terminology that she used ("definitely"), and her tone when voicing this word. The episode continued with one of the teacher educators (D) opening the discussion to the entire class, asking all participants to vote (triangle/not triangle) for each figure.

> ***D:*** *Wait a minute. [D smiles at P.] I see that we have some disagreements here. [Turns to the class] Let's do what my friend asked us to do... let's have another look at each of the figures and vote... Let's think about each of the figures [draws on the board the figures and the outline of Table 2]. You can vote for each figure only once – 'Yes' it is a triangle, 'No' it isn't, or 'I have not decided yet'.*
> ***P:*** *Why can't they vote twice, if they feel like... that it... I mean, if someone thinks that a certain figure in a way IS a triangle, but in another way it IS NOT?*
> ***D:*** *That's an important question. [D turns to the class.] What would you say?*
> *[Giggles and voices]: No. No it can't be. If it's a triangle then it's not a NOT triangle.*
> ***Galit:*** *But it can be SIMILAR to a triangle.*
> ***D:*** *If it's ONLY SIMILAR, please vote NO. We'll discuss it further later. OK. OK. So... let's vote.*

During this invitation (to vote), the other teacher educator (P) raised a substantial question: Can a figure simultaneously be a triangle and a non-triangle? And in general terms, can 'something' simultaneously be an example and a non-example of a mathematical concept? This encouraged Galit to use the problematic notion of "similar to". At this stage Dina guided the participants to vote 'no' when it's "only similar", exemplifying that a teacher in entitled to leave a discussion on some issues (in this case, the status of "similar to" in mathematical definitions) for later on. A profound discussion of this issue followed in a session that is not presented here.

DISCUSSING THE TEACHERS' CONCEPT IMAGES OF TRIANGLES

Table 2 shows that after this preliminary P-D, before a more profound discussion, eight and four preschool teachers, respectively, changed their minds (in the correct direction) regarding the 'rounded-edges' shape, and the 'pizza shape'. Two stated that the rounded-edges shape is NOT a triangle, and six confessed "I don't really know". One of the latter said that "it's almost a triangle, so by Dina's guidance I should vote that it is not, but I don't feel good about it".

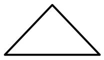
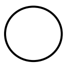
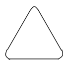
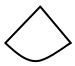
The Figure	It's a triangle	It's not a triangle	Don't know / almost
1 	17		
2 		17	
3 	9	2	6
4 	13	1	3

Table 2. The preschool teachers' vote on "Is this a triangle?"

IN BRIEF: THE TEACHERS' SMK AT THE END OF THE COURSE

In the final assessment, the preschool teachers were asked to address a rich collection of figures, to state, for each figure, whether it is a triangle, a quadrilateral, a pentagon or none of the above, and to justify their judgments. The 17 teachers correctly identified all of the triangles, and only one of them wrote that to her the 'pizza-triangle' feels like a triangle although she knows it is not. They also provided mathematical, correct, although not always full definitions to justify their answers. However, when addressing the pentagon, six of them incorrectly claimed that Item 4 in Table 1 is not a pentagon, because "it seems like a triangle" (2 teachers), "it does not look like a pentagon" (4 teachers); when addressing the quadrilaterals, nine participants argued that the square is not a quadrilateral "because it is a square" or "because it is called 'square'" (7 teachers). These findings indicate that the preschool teachers' concept images of polygons at the end of the course were: (a) more consistent with definitions than before the course, (b) still not always complete and not always consistent with the mathematical definitions, and (c) vulnerable when a figure could be labelled by more than one term (e.g., a square that is also a quadrilateral).

WORKING WITH HIGH SCHOOL TEACHERS ON *ENT* PROOFS

Proofs are often addressed in high school mathematics. Studies have shown that students often face various types of difficulties when requested "to prove". Various researchers have reported that students are not always aware of the necessity for a general, covering proof when proving the validity of a universal statement for an infinite number of cases (e.g., Bell, 1976) and that they tend to encounter difficulties in constructing a complete proof based on deductive reasoning (e.g., Healy & Hoyles, 1998; 2000). When refuting a statement, students tend to relate to a counter example as an exceptional case rather than as sufficient to refute a universal statement (e.g., Balacheff, 1991).

Several studies have focused on teachers' content knowledge of proofs (e.g., Knuth, 2002; Dreyfus, 2000), but only a few examined teachers' related knowledge with reference to "prove" tasks (i.e., produce a proof) versus "evaluate a proof" tasks (i.e., is the proof correct or incorrect?) (e.g., Barkai, Tsamir, & Tirosh, 2004). Here we briefly address the latter two issues with reference to ENT statements.

DESIGNING THE SESSIONS: PROOFS—VALIDATING AND REFUTING *ENT* STATEMENTS

The tasks that we formulated for the *Validating and Refuting* sessions were based on relevant publications, on our studies of students' and teachers' conceptions of proofs and on the data that we collected from the 23 secondary school teachers that participated in our program. The participants were first asked to answer a questionnaire consisting of six ENT statements (see Table 3: validity is determined by the combination of predicate and quantifier).

Predicate **Quantifier**	**Always true**	**Sometimes true**	**Never true**
Universal	**S1**: The sum of **any** 5 consecutive numbers is divisible by 5. *True*	**S2**: The sum of **any** 3 consecutive numbers is divisible by 6. *False*	**S3**: The sum of **any** 4 consecutive numbers is divisible by 4. *False*
Existential	**S4**: **There exist** 5 consecutive numbers so that their sum is divisible by 5. *True*	**S5**: **There exist** 3 consecutive numbers so that their sum is divisible by 6. *True*	**S6**: **There exist** 4 consecutive numbers so that their sum is divisible by 4. *False*

Table 3. Classification of statements.

The teachers were asked to determine, for each of the six statements, if they were true or false and to prove it in various ways (see also Tirosh & Vinner, 2004; Barkai, Tsamir, & Tirosh, 2004). All knew which statement was true and which was false, and all provided correct proofs to validate or refute the statements (frequently using only algebraic representations for proving the universal true statements). These findings are consistent with our findings in an extensive study that we carried out with the support of the Israeli Science Foundation (ISF, 900/06) with fifty secondary school teachers (e.g., Tsamir, Tirosh, Dreyfus, Barkai, & Tabach, 2008; Tabach, Barkai, Tsamir, Tirosh, Dreyfus, & Levenson, 2010; Tabach, Levenson, Barkai, Tsamir, Tirosh, & Dreyfus, 2010). Here we focus on a P-D that presented teachers with two attempts to prove the same statement, asking them to state their opinions regarding the correctness of each suggestion.

PAIR-DIALOGUE: LET'S PROVE THE STATEMENT IN DIFFERENT WAYS

The aim of this session was to challenge secondary school teachers' tendency to accept algebraic attempts to prove universal statements and to reject numeric ones. We provided two proofs that were written by students, for validating the statement "*the sum of any 5 consecutive numbers is divisible by 5*". The first proof was a numeric, valid, cover-proof and the second was an algebraic representation of an attempt to prove. In this latter attempt, no reference was made to the domain for *x*.

> **D:** *It might be interesting to find several proofs for a statement. For instance to prove that 'the sum of* ***any*** *5 consecutive numbers is divisible by 5'...*
> **P:** *I like that idea...*
> **D:** *I'd like to show you a nice numeric proof that a student once gave… The sum 1+2+3+4+5 is 15, right? So, it's divisible by five. To advance to the following five-consecutive-numbers you need to add one to each of the original numbers. So you have 2+3+4+5+6.*
> **P:** *That's 20 and it's divisible by 5.*
> **D:** *The idea is NOT to look at the 20, but at the process, when advancing from one 5-consecutive-numbers to the next 5-consecutive-numbers you add one to each number so all in all you add FIVE to the sum. So, the new sum is again divisible by 5, and so on. [P has a puzzled expression.] Let's call it "The Numeric 'Adding Five' Proof" [D writes on the board]:*
>
> *The Numeric 'Adding Five' Argument*
>
> *The sum of the first 5-consecutive-numbers is:*

1+2+3+4+5 is 15 and it's divisible by 5.
The sum of the next 5-consecutive-numbers is:
2+3+4+5+6 =
(1+1)+(2+1)+(3+1)+(4+1)+(5+1)=
(1+2+3+4+5)+(1+1+1+1+1)=
(Divisible by 5)+ 5=
Divisible by 5

P: *I n t e r e s t i n g… I'd rather have an algebraic proof; it gives a stronger sense of generality… This is also a solution that was once given by a student. Look: the first number is presented as 5x, then 5x+1… and so… the sum is [writes on the blackboard]:*

The Algebraic 5x+n argument

5x+(5x+1)+(5x+2)+(5x+3)+(5x+4)=
(5x+5x+5x+5x+5x) +(0+1+2+3+4)=
25x +10
Divisible by 5 + 10
Divisible by 5

D: *To me, NOT using algebra and still addressing the generality is stronger...*
P: *Perhaps we should consult our friends here [turns to the class]. What would you say? Is the numeric proof correct? Is the algebraic proof correct? Would you present and discuss both in class? Which one do you prefer?*

At this stage, the teachers were asked to write and submit their opinions regarding each of the suggested proofs. We report on the main findings.

EVALUATING THE TEACHERS' KNOWLEDGE REGARDING ARGUMENTATIONS AND REPRESENTATIONS

The teachers analyzed the two arguments according to their mode of argumentation and their mode of representation (Stylianides, 2007). Regarding the numeric representation, they expressed unease, and all but three stated that "it doesn't seem right". In response to the question: "Could a numeric representation be a correct proof?" ten teachers wrote "yes", eight of which added "but" ("not really", "not in high school", "I wouldn't use such a proof and/or I wouldn't like my students to use it"). Seven teachers wrote "no", explaining that "it isn't general", and occasionally adding comments like "we can't know about REALLY LARGE numbers...". Six teachers claimed that they could not state whether it is correct because "it's strange" or "I never use such methods".

When referring to the algebraic suggestion, all 23 participants stated that "algebra is the right way for proving that such statements are valid". Seven teachers praised the given algebraic proof: "it's good"/"interesting" because "it brings forward the divisibility by five, right from the first expression". Three of those teachers added "it is definitely better than the other [numeric] one". Five teachers wrote, "they are not sure" or "I never used such a sequence". The other 11 teachers referred to the presented "proof" as "partial", not covering *all* cases; yet five of them added that it is general and thus better than the numeric one.

By the end of the course all participants accepted numeric representations, which cover all cases, as valid proofs and rejected algebraic representations that failed to provide the needed cover. They were also very careful about the examination of the domain of algebraic representations.

A CONCISE SUMMARY

Mathematics education researchers who focus on the teaching of mathematics constantly search for promising, sensitive ways of enhancing teachers' mathematical knowledge needed for teaching. In this presentation we illustrated, via two examples, the application of the P-D teaching approach to mathematics teacher education.

There is still a long way to go with developing and implementing the P-D teaching approach with individuals, small groups and whole classes of prospective and practicing mathematics teachers. In this presentation we focused mainly on addressing some aspects of two general mathematical issues, namely, definitions and proofs. The research findings on students' and teachers' conceptions of definitions and proofs, and on students' and teachers' ways of thinking about the content topics that we addressed in these episodes (triangles, elementary number theory) served as a basis for developing these episodes. Such findings are essential for constructing episodes that address subject matter knowledge (SMK) and pedagogical content knowledge (PCK) issues. More generally, the growing body of knowledge on students' and teachers' ways of thinking about various mathematical concepts, operations and procedures is an asset for formulating P-Ds. Yet, when creating such episodes, affective issues are considered as well, and attempts are made to gently address typical, incorrect mathematical responses and to use such instances as springboards to enhance prospective and practicing teachers' SMK and PCK.

So far, we have related to the development and implementation of the P-D teaching approach. This approach, like any other approach to teacher education and to the professional development of teachers, should be evaluated. We are currently taking the first steps in this direction, attempting to identify productive ways to study the short-term and the long-term impact of this approach on the professional development of prospective teachers, practicing teachers and teacher educators.

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Working Groups

Groupes de travail

MATHEMATICS TEACHING AND CLIMATE CHANGE

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INTRODUCTION

This working-group report is based on text and notes contributed by Richard Barwell, Dave Lidstone, Stewart Craven, Dave Wagner, Doug Franks, France Caron and Brad deYoung.

Climate change is one of the most pressing issues facing the world today and will continue to be in the coming decades, although few people have more than a vague understanding of what it is, how it works and what might be done to reduce its effects. Mathematics is crucial for describing, understanding and predicting climate change, whether through observation and modelling of the climate system (including the atmosphere, oceans and ice fields), or through monitoring and modifying human behaviour (e.g., emissions, economics, population). The mathematics involved includes measurement, descriptive statistics, probability and mathematical modelling. The key role of mathematics in understanding and responding to climate change suggests a corresponding role for mathematics educators. What might this role be?

The aims of this working group were to explore this question and related issues:

- Helping students make sense of climate change through a mathematical lens is important, not only to build their understanding of the issue, but also to move them, or at least some of them, toward actively responding to the problem. As mathematics educators or teachers, what kinds of things can we do to build students' understanding of climate change and encourage their active response?

- Public discourse about climate change requires a degree of mathematical literacy in the general public, sufficient to understand various techniques or principles (e.g., averages and modelling) and interpret various forms of information (e.g., tables of data, graphs and charts). What implications might this situation have for mathematics education?

The main components of the working group included:

- a presentation by Brad deYoung, oceanographer, contributor to the Intergovernmental Panel on Climate Change, and Chair of the Department of Physics and Physical Oceanography, Memorial University, on the science of climate change;
- activities focusing on different aspects of the mathematics of climate change;
- development of applications (e.g. teaching activities) by participants.

STARTING POINTS: CLIMATE CHANGE TEXTS

When participants arrived at the start of the working group, they were invited to browse a wide range of texts that were displayed on tables, walls and laptops around the room. A selection of these texts is shown in the table, below.

Newspaper articles	Hanley, C. J. (2010, August 18). Climate scientists forecast more heat, fires and floods. *Globe & Mail*. Berners-Lee, M., & Clark, D. (2010, June 17). What's the carbon footprint of … a cup of tea or coffee? *The Guardian*. More than 42 million displaced by natural disasters in 2010. (2011, June 6). *Associated Press*. Retrieved from https://robertwmh.wordpress.com/2011/06/07/more-than-42-million-displaced-by-natural-disasters-in-2010-sympatico-ca-news/
Scientific articles	Excerpts from the IPCC *Climate Change 2007 Synthesis Report*, available from www.ipcc.ch Borken-Kleefeld, J., Berntsen, T., & Fuglestvedt, J. (2010). Specific climate impact of passenger and freight transport. *Environmental Science and Technology*, *44*, 5700-5706. Walter, M. E. (2010, October 12). Weatherquakes, earthquakes, mathematics and climate change. Retrieved from http://www.colorado.edu/math/earthmath
Policy texts	"The UNESCO Climate Change Initiative" available from http://www.unesco.org/new/en/natural-sciences/special-themes/global-climate-change/
Books	Weaver, A. (2008). *Keeping our cool: Canada in a warming world.* Toronto, ON: Penguin. Lovelock, J. (2009). *The vanishing face of Gaia: A final warning.* London, UK: Penguin. Hulme, M. (2009). *Why we disagree about climate change: Understanding controversy, inaction and opportunity.* Cambridge, UK: Cambridge University Press. McCandless, D. (2009). *The visual miscellaneum: A colorful guide to the world's most consequential trivia*. New York, NY: Harper Collins. Walter, M. (2011). *Mathematics for the environment*. Boca Raton, FL: CRC Press.
Website texts	Climate Modelling (n.d.). Retrieved from www.metoffice.gov.uk/climate-change/guide/science/modelling Environment Canada (n.d.). Calcul des normales climatiques au Canada de 1971 à 2000. Retrieved from http://climat.meteo.gc.ca/prods_servs/normals_documentation_f.html Environment Canada (n.d.). Calculation of the 1971 to 2000 climate normal

	for Canada. Retrieved from:http://climate.weatheroffice.gc.ca/prods_servs/normals_documentation_e.html Statcan (n.d.). Que représente une tonne d'émissions? Retrieved from www.statcan.gc.ca/pub/16-201-x/2007000/10542-fra.htm Statcan (n.d.) Chart 1.1: Global annual temperature anomalies computed from land and ocean data. Retrieved from www.statcan.gc.ca/pub/16-201-x/2007000/10542-fra.htm Mathematics and Climate. Retrieved from www.mathaware.org
Video clips	"America's Climate Choices: Advancing the Science of Climate Change" http://www.youtube.com/watch?v=FxaWVlzgkX4&feature=relmfu "Polar Bears Discussing Global Warming" http://www.youtube.com/watch?v=_Zo7wTOdc_M&NR=1 "Simulation – Human Land Use" http://www.ritholtz.com/blog/2011/03/global-land-cover-change-from-8000-bp-to-50-bp/ "The Most Terrifying Video You'll Ever See" http://www.youtube.com/watch?v=zORv8wwiadQ&NR=1&feature=fvwp "The Story of Stuff" http://www.storyofstuff.com/

In subsequent discussion of these texts, participants repeatedly expressed a feeling of being overwhelmed, both by the quantity of information available, as well as by the magnitude of the challenge of climate change. Several participants were concerned about the possible effects on future generations, including their own grandchildren or, more prosaically, their pensions. There was a clear recognition of the relevance of mathematics and mathematical literacy for interpreting the various texts. Some participants commented, for example, on the complexity of many of the graphs, issues of measurement, uncertainty, causality and the links with science. Finally, participants commented on the emotive nature of their response. We then generated the following questions:

- Comment analyser les données?
- Comment les choisir?
- Comment les représenter? (Afin d'établir un lien avec les activités humaines.)
- What are the implications of climate change for teaching math?
- How can we foster interdisciplinary work? E.g. in science, social science, French, English.
- Are we using climate change to make sense of math or vice versa? Which is the context?
- In what ways do the skills we teach in mathematics contribute to the communication of climate change?
- In what ways do the skills we teach help us to communicate relationships between humans, our actions, etc…. and the changing world?
- Should the outcome be a change in our behaviour or adaptation to climate change?
- How do / might / ought carbon credits / taxes work? Or not?
- How can we help students understand how people are using math to support their rhetoric?

THE SCIENCE OF CLIMATE CHANGE

The presentation by Brad deYoung had several inter-related themes. For this report, we will summarise the following: how the climate is changing; effects on sea level and glaciers; modelling future climate scenarios; and what might be done in response. The information below is based on Brad's presentation, slides and the discussion that followed.

HOW THE CLIMATE IS CHANGING

The Earth is several billion years old and the climate is always changing. Current changes can be considered in the light of the history of the climate at different time scales. Over the past 65 million years (i.e. since the time of the dinosaurs), the climate has cooled considerably, by as much as 15ºC. This cooling, however, is not smooth – there have been periods of warming and cooling, with abrupt changes of several degrees in the overall temperature of the planet. Over the past 5 million years, there has been an overall cooling trend amounting to several degrees, although with a great deal of variability. Some aspects of this variability show a cyclic pattern over periods of 41,000 and 100,000 years, related to cycles in the sun's activity.

Over the past 500,000 years, such cyclic patterns are more apparent (see http://climate.nasa.gov/evidence/ for a graph for this period). There is a strong relationship between levels of carbon dioxide (CO_2) and methane (CH_4) in the atmosphere and the temperature of the Earth. There is a much more complex relationship between the changes in temperature, CO_2 and CH_4, and the sun's activity. This relationship is complex because different parts of the climate system have different 'memories' – the time taken for energy to circulate. The atmosphere changes quite rapidly, over 10s of years; the oceans more slowly, over 1,000s of years; and icecaps and glaciers more slowly still, over 100,000s of years. This makes sense, since the atmosphere is very fluid, while glaciers typically take a long time to build up or to melt.

A graph of temperature change over the past 100,000 years shows in much more detail the variation that occurs from one millennium to the next. Abrupt changes mark shifts between ice-ages and warmer periods. While it is well known that astronomical forcing, leading to changes in solar insolation, causes these cycles of climate, it is quite clear from the time series that the response of the earth to the changes in the solar radiation, over long periods, is not linear. The graph also shows that for the past 10,000 years, the climate has been unusually stable, with much less variation and a relatively constant mean temperature. It is notable that this period coincides with the development of agriculture and complex human societies.

Finally, the trend in global temperature over the past 1,000 years has been largely stable, with a slight and gradual cooling – until the middle of the 19th century, since when there has been a sharp upward trend. The temperature difference between the warming experienced now, and that observed during the late Middle Ages is a subject of significant debate. The sharp upturn of the 19th century coincides with the industrial revolution and human production of greenhouse gases. Concentrations of CO_2 now exceed levels not seen for tens of millions of years. It is the rate of change of temperature, sea-level and greenhouse gases in the atmosphere, that is most striking in the context of the paleoclimatic record.

EFFECTS OF CLIMATE CHANGE ON SEA LEVEL AND GLACIERS

Global warming has effects on all aspects of the climate system. Brad spoke specifically about its effects on the oceans, since that is his area of expertise. Sea-level has been increasing throughout the past century. Over the past 50 years, it increased by about 1.8 mm per year. It is now increasing at around 3 mm per year. That may not sound like much, but when multiplied over decades, and when the immense volume of the oceans is considered, it is apparent that these changes are significant. Increasing sea-level is due to two main phenomena. One is thermal expansion: as the oceans get warmer because of global warming, they expand in the same way that other substances expand when they are heated. Thermal expansion accounts for about half of observed sea-level rise.

The other phenomenon he discussed is melting glaciers and ice-shelves. (Note that melting sea ice does not lead to an increase in sea-level). The Greenland ice sheet is 1.7 million km^2

and is 3 km thick. Some of the glaciers that make up the Greenland ice sheet are melting fairly rapidly. One glacier, for example, called Jakobshavn, is releasing 50 km^3 per year into the ocean. Worldwide, melting ice sheets account for about 1mm per year increase in sea-level, of which melting in Greenland contributes about 0.5 mm per year.

MODELLING FUTURE CLIMATE SCENARIOS

Climate models include separate components for the atmosphere, oceans, land, ice and biosphere. They model the Earth by dividing it up into squares and then taking vertical columns for the atmosphere and the ocean. The models themselves are based on the physics of the climate, and in particular, energy flows. There is only one external source of energy – radiation from the sun. Much of the solar radiation received by our planet is simply reflected back out into space by the upper atmosphere, clouds or ice and snow. This energy heats the atmosphere, the oceans and the land. Energy is also transferred between different parts of the climate system through, for example, evaporation of the oceans to form clouds and rain.

Climate models are run on large computers. Their efficacy is tested using historical climate data. Initial conditions are set at some point in the past for which we have sufficient data. The model is then run. The efficacy of the models is determined by how well they reproduce the observed climate since the starting point. Once a model is effective in reproducing observed climate from the past, it can be run further to project future climate. Existing models are now quite effective, although they include many uncertainties, such as those caused by the grid size used for the modelling, the representation of key aspects of ocean dynamics such as the Gulf Stream, cloud dynamics in the atmosphere and many other things. Larger grid sizes, for example, over-simplify both climate and topography – clouds or thunderstorms, for example, can be very localised.

Having developed reasonably effective models, they can be used to test a key hypothesis – that greenhouse gases trap additional energy than would otherwise be the case and hence are causing the observed global warming of recent decades. The test involves running established models with human-produced greenhouse gases either included or omitted. On this basis, a divergence appears at around 1970. After this point, models that do not include human-produced greenhouse gases show lower temperatures than those that do. The models that do include this element reproduce the observed warming, while those that do not, do not.

Using these models, various projections can be made about the future climate, from fairly general to fairly specific. In general terms, the Earth will get warmer. Such warming is not evenly distributed, however. Warming is likely to be much greater at higher latitudes than in equatorial regions. The extent of this warming depends on the quantity of greenhouse gases released into the atmosphere. In its most recent report, the IPCC produced various scenarios. Needless to say, the ‘business as usual’ scenario is projected to result in the greatest level of warming. More specific projections include the disappearance of summer sea-ice in the Arctic within 40 years.

DISCUSSION

Following the presentation, participants raised many questions reflecting their concerns as mathematics educators and as citizens. The following represents a selection of these questions and Brad’s responses.

> ***France:*** *There is a difference between data arising from models and data arising from direct observation.*
> ***Brad:*** *Yes, there are different sorts of data and every kind of data has an issue associated with it. Observations are not more right. Observations can be wrong and*

so can models. It isn't so simple as right and wrong. For example, when you change a temperature measurement technique, there is a jump in the record, so you have to make a correction.
Georges: *How do you convincingly choose a correct model…can you?*
Brad: *It's useful that people take different approaches. So, for example, for global temperature models, there are now 6 groups working independently, and their results converge.*

Greg: *Presenting these ideas for the public must be difficult.*
Brad: *One big problem for non-scientist audiences is the concept of non-linearity. It's hard to explain in a short time. I assume that the participants in this group understand, but for the general public, it doesn't mean anything. In* An Inconvenient Truth*, for example, Al Gore is trying to imply a dramatic linear change, when actually it could be much worse.*
Florence: *When you communicate with the public, or when you're teaching, how do you help your students be aware of non-linearity? I feel like we're really linear in our teaching of mathematics.*
Brad: *It's a real problem. Undergraduate mathematics is usually linear. I teach a course using* Mathematica*, which will solve differential equations, so students can vary parameters to see what happens. But the issue is the same at university level.*

Chris: *There's a connection here with critical mathematics education.*
Dave L: *Say more about that – can you give an example?*
Chris: *Rico Gutstein has reported on his mathematics teaching project in New York on the gentrification of the Bronx. The students work with real data on salaries and house prices.*
Stewart: *Who brings the questions – the students or the teachers?*
Dave W: *I've discussed this with Gutstein. He says students don't see the questions – they might see their friends moving away but he needs to formulate the question. In climate science it's a bit harder to do that – kids are more interested in their social world than in the climate.*
Brad: *Kids get an oversimplified message – they are often worried about a catastrophe.*
Dawn: *It depends where the kids are from: in Northern communities, permafrost is melting and they are encountering animals they've never seen before.*
Brad: *It's true there is a much greater connection with the environment and the climate in Greenland compared with, say, Toronto. In Greenland, there is a desire for science, a desire to understand more. Kids do see the connections locally.*
Doug: *Are kids asking 'what can I do?' or do they remain at a more passive level.*
Brad: *Anecdotally, I think it's both. Educators can take that hubbub of activity and guide it towards more critical consideration and thinking. I'm not so sure about activism in the classroom; critical thinking is the important thing. We need to be open to children thinking about the issues. In a physics class that I taught, we compared two films:* An Inconvenient Truth *and* The Great Global Warming Swindle *(a British film with a sceptical bias) and they found the latter more convincing.*

Wayne: *This is really interdisciplinary. I'm not sure that mathematics educators do that.*
Brad: *I teach Physics, Oceanography and Mathematics. My impression of high school mathematics is that connecting to things that are real for students is good, that this would stimulate both their interest and their understanding. It helps to bring the reality of other subjects. So you can explain what a Watt or a Joule is. Within climate change there are many problems, so pick one part and there's lots of mathematics inside.*
Wayne: *When students learn algebra and then transfer to trigonometry, it's a huge jump; so going outside of mathematics is hard.*
Brad: *We need to bring together mathematical expertise and problem-based thinking.*

Chris: *Maybe it's not about teaching skills and applying – I see problems first. Mathematics skills have meaning within contexts.*
France: *Students need to get their hands dirty with data, making their own assumptions to make sense of it.*
Peter: *At Queens we're trying to emphasise interdisciplinarity – at the moment we all work in silos. It's a great challenge to get out of that structure.*
Brad: *I think students would like thematic instruction drawing on different teachers. Mathematics becomes part of something else.*
Wayne: *But then you lose the essence of mathematics or physics, etc.*
Brad: *You need a bit of both.*
Chris: *Interdisciplinarity is important: in mathematics we may not know that much about the context, for example, we may not know that much about climate change.*
Greg: *But we're talking about mathematics and physics – is that really that interdisciplinary? It's the same frame of thinking. What about social issues – that would be a lot more interdisciplinary.*
Brad: *There's a danger of bringing in too many disciplines. You can spend the rest of your life looking at different things. I agree we can use social science, literature, etc. We live within the climate system; that's the planet we're on. But you can't do everything: pick something and work on that and make some of it locally relevant.*

WORKING ON MATHEMATICS IN THE CONTEXT OF CLIMATE CHANGE

In the next part of the working group, participants worked on three tasks prepared by the group co-ordinators. These tasks had three different mathematical foci: measurement and averages, stochastics, and proportional reasoning.

MEASUREMENT AND AVERAGES: IS THE WEATHER HERE ALWAYS LIKE THIS?

For the first task, participants were provided with a spreadsheet containing meteorological data recorded in St. John's, NL, over the past 30 years. The spreadsheet consisted of monthly mean temperatures (minimum, maximum and average), monthly maximum and minimum temperature recorded, and records of rainfall, snowfall and wind speed. Participants were invited simply to explore the data, prompted by the following questions:

- Is the weather here always like this? (During the conference, the weather was rather cool and unsettled.)
- What is 'normal' for St. Johns?
- How could you find out?
- Does it matter how you find out? E.g. If you calculate a mean over 20 years vs. 30 years?
- Has the climate in St. John's changed? How?
- What might you expect to happen over the next 10 years?
- Why?

Participants worked in groups and most produced graphs looking at trends in some of the data. As an example, a graph showing changes in mean temperatures in the month of December over the past 40 years is shown below (see Figure 1). The graph also includes a linear regression line and 5-year moving average for the same data.

Participants commented on various aspects of the task, including the authenticity of working with the data, which is quite complex, rather than preselected and hence rather decontextualized data. On the other hand, for most groups, the task did not progress beyond plotting a trend. This is a good starting point, but some participants wondered how such an activity could be taken further in a classroom situation. Other participants thought about the meaning of 'mean', as in 'mean monthly temperature': What does that represent? And how does one make the link from a locally observable trend of increasing temperature to the global phenomenon of climate change?

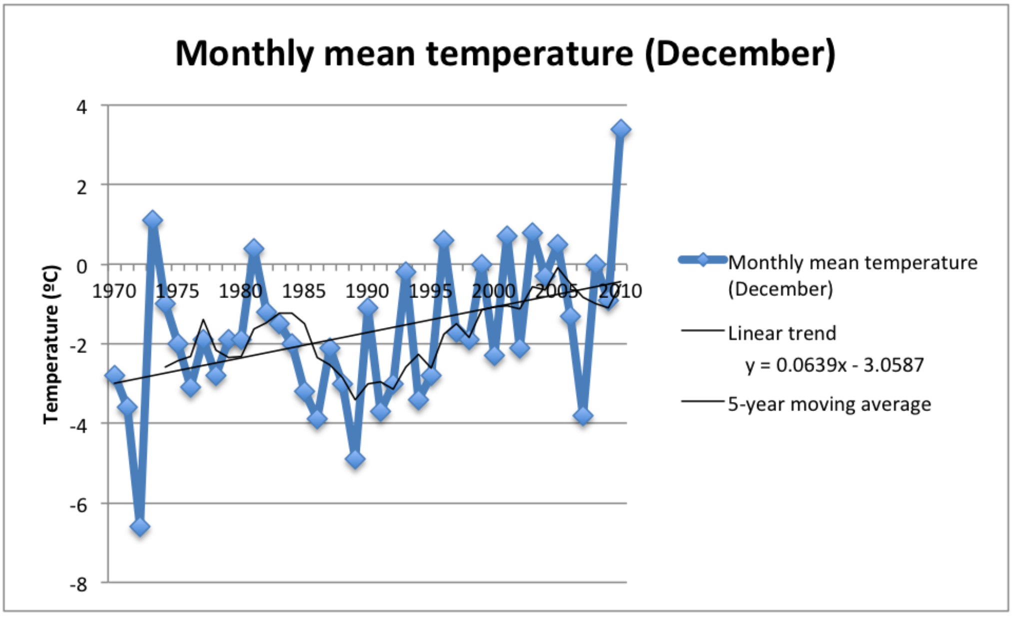


Figure 1. Changes in monthly mean temperature in December.

STOCHASTIC REASONING

The study and understanding of climate change is rife with stochastic issues. Indeed the very definition of global warming is statistical, based on a thirty-year average from a variety of data stations. A common metaphor to distinguish weather from climate is “weather is the outcome of a roll of dice and climate is the probability distribution for the roll of dice”. With this in mind, participants were asked to roll a pair of dice 30 times and record the sum of the faces on each roll. Some of the participants were given green standard dice and others were given red dice that had been altered so that the one of the fours was changed to a six. The intention of the activity was to promote a discussion of stochastic issues of climate change especially as they pertain to classrooms.

Notice that with the dice altered as they were there was no change in the sample space, which would have occurred had we changed say a one face to a three (see Figure 2). The probability distribution is, of course, different from what we would have for a standard pair of dice.

Sample space for the ‘red’ dice

	1	2	3	5	6	6
1	2	3	4	6	7	7
2	3	4	5	7	8	8
3	4	5	6	8	9	9
4	5	6	7	9	10	10
5	6	7	8	10	11	11
6	7	8	9	11	12	12

Sample space for the ‘green’ dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure 2. Sample spaces.

The results of our trials are tabulated below (see Figure 3) along the row labelled “ep” for empirical probability, and below that is a row labelled “tp” for theoretical probability.

Trial results for the ‘red’ dice

SUM	2	3	4	5	6	7	8	9	10	11	12	
	1	2	2	1	5	6	2	5	2	2	2	30
	0	2	1	6	5	6	3	2	3	1	1	30
	1	1	1	2	4	4	2	4	1	4	5	29
	2	1	2	2	2	8	3	5	1	2	2	30
total	4	6	6	11	16	24	10	16	7	9	10	119
ep	0.03	0.05	0.05	0.09	0.13	0.2	0.08	0.134	0.06	0.08	0.08	1
tp	0.03	0.056	0.08	0.08	0.11	0.17	0.14	0.111	0.08	0.08	0.06	1

Trial results for the “green” dice

SUM	2	3	4	5	6	7	8	9	10	11	12	
	1	0	2	4	8	6	2	0	3	4	2	32
	0	2	2	6	3	4	4	4	1	3	1	30
	1	1	4	3	2	6	5	4	1	2	1	30
	1	1	1	3	6	2	5	1	5	4	1	30
total	3	4	9	16	19	18	16	9	10	13	5	122
ep	0.02	0.033	0.07	0.13	0.16	0.15	0.13	0.074	0.08	0.11	0.04	1
tp	0.03	0.056	0.08	0.11	0.14	0.17	0.14	0.111	0.08	0.06	0.03	1

Figure 3. Trial results.

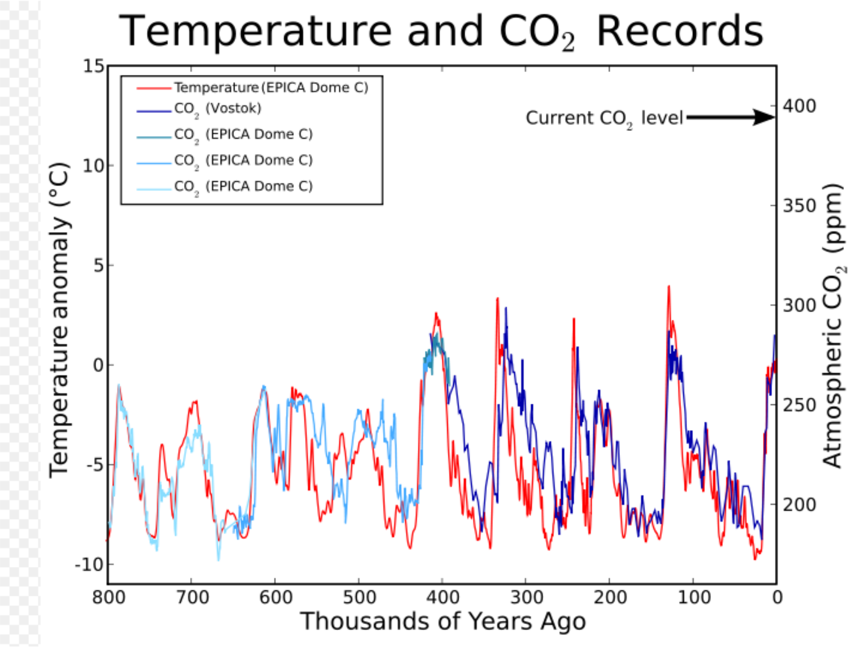
Among the points raised in discussion was that some participants had not noticed that their dice were out of the ordinary and their outcomes did not suggest this. We noted that we only had about 120 trials for each type and so the long run issues of probability have yet to be observed with such a data set. Indeed, determining the number of trials that would allow us to infer a change in the probability distribution is a standard, but not elementary, statistical problem. Although the activity does offer students an opportunity to distinguish empirical probability from theoretical probability, it does not address ‘subjective’ probability, the practice of daily weather forecasters. Nonetheless, experiencing random behaviour and confronting some of the complexities it entails seems to be an important part of understanding climate change.

PROPORTIONAL REASONING

In order to dramatize human contribution to climate change in the movie *An Inconvenient Truth*, Al Gore uses a huge graph depicting changes in atmospheric CO_2 concentrations over the last 800 000 years. He moves along the graph from left to right and then steps into a vertical lift that rises to illustrate the precipitous increase in atmospheric CO_2 concentrations over the last 150 years. The participants were provided with a copy of the graph (see Figure 4) as it was shown in the movie, plus one or two other graphical representations, and they were asked to:

1. Discuss the correctness of each of the representations in terms of proportionality.
2. Discuss the impact of the representations.
3. Suggest alternative but more powerful ways for conveying the information.

As a part of the discussion, in order to emphasize how selecting scales for both the horizontal and the vertical axes can affect the readers’ interpretations of information, we ‘redrew’ the ‘Al Gore’ graph (see Figure 4) such that 1 cm = 400 years. Using a cashier’s tape we rolled out the 'horizontal axis' to a length of 20 m. This begins to show the immensity of the time scale over which CO_2 concentrations fluctuated over a small well-defined range thus making the huge jump over the last 200 years (0.5 cm in our new graph) to be that much more surprising.



[Source: Leland McInnes, http://en.wikipedia.org/wiki/File:Co2-temperature-plot.svg]

Figure 4. The 'Al Gore' graph.

APPLICATIONS

The last portion of the working group was spent in groups developing ideas for classroom tasks or other applications. Brief reports of each group's work are provided.

Florence, Dawn and Samuel

This group worked on an interdisciplinary approach, starting with the topic on climate change in the Alberta science curriculum. They started with a radiation dose chart included in the texts in the opening session of the working group. The theme of the activity is nuclear energy as an alternative to carbon fuels, with a mathematical focus on understanding magnitude. Using the chart as a starting point, questions could be generated and then explored. Questions might include:

- How many radiation units (Sieverts) are in one banana?
- How many bananas would you have to eat to kill yourself?
- Would you have to eat them all at once? Could you?
- How long does it take to digest a banana?
- What if the banana was a unit of measurement?
- What is a Sievert? How are they calculated?
- Is the information in the chart reliable?

The text includes references to historical events, e.g. Three Mile Island.

Connections would be possible with local issues such as the installation of high-tension power lines. The statistics in the text are all US – it would be good to locate Canadian stats too.

Richard and Dave W.

This group worked at developing a game that models climate economies. They worked at varying the game *Free Trade*, which is a game David developed. It uses many dice to model free-market economies. Criteria they tried to incorporate into the climate economies game are:

1. The elements of randomness (the dicing) model unpredictability in one's decision making in the real financial and environmental economies.
2. There are two parallel economies – personal wealth and environment.
3. A healthy environment improves each player's ability (chances) to maintain and develop personal wealth.
4. Environmental degradation affects the poor more than it affects the rich.
5. Decisions that are good for the environment incur some sacrifice of personal wealth but everyone's wealth is dependent on sufficient environmental health.
6. The wealthier players' choices impact the environment more than the choices of the poor.
7. The wealthier players also have advantages in increasing their personal wealth.
8. The game should have the potential to sustain equilibrium in which every player stays alive while the environment remains stable and while reasonable fluctuations and differentiations in personal wealth occur.

Playing such a game would help one understand a complex system in which environment and personal wealth interrelate. Trying to come up with reasonably simple rules for the game involves a lot of mathematics. Perhaps we can give our students our initial ideas and ask them to fine tune the game (i.e. let them do the mathematics and let them think about matching the model to the way these interrelated economies work).

France and Georges

Deux pistes ont été explorées et présentées pour mieux apprécier et même comprendre les changements climatiques par le biais de la modélisation mathématique et statistique, assistée par la technologie.

Une première serait d'abord de distinguer entre météorologie et climatologie (http://accromath.uqam.ca/contents/pdf/climat.pdf), en saisissant le concept de variabilité à travers l'étude de l'erreur produite par une régression sinusoïdale appliquée aux températures mensuelles moyennes pour une même station météorologique sur une période de quelques années consécutives (http://climate.weatheroffice.gc.ca/climateData/canada_f.html); cette erreur est typiquement assimilable à une loi normale. On pourrait ensuite examiner si l'on observe un changement dans les paramètres du modèle entre deux périodes relativement éloignées.

Une seconde approche serait de chercher à modéliser le cycle de carbone, par une initiation des élèves à la modélisation compartimentale à l'aide d'un logiciel comme Stella (voir les *Actes du GCEDM 2004*, p. 75) ou par la simulation de tels modèles à l'aide de paramètres et de variables de contrôle. Des initiatives ont été menées en ce sens, et depuis quelques années déjà, comme en témoignent notamment les sites suivants:

- http://globecarboncycle.unh.edu/CarbonCycleActivities.shtml
- http://pedagogie.ac-montpellier.fr/disciplines/svt/stella/Presentation_Stella.htm
- http://www.unidata.ucar.edu/community/2006workshop/PresenterPowerPoint/Monday%20Afternoon/STELLA%20Models%20in%20the%20Classroom.pps

Laurent, Greg, Lucie and Dave L.

This group worked with a simplified arctic ice satellite image, with future ice extent forecast shown as a line. The image has been used as the basis for a grade 5-6 task to think about what the ice extent might be for 2020. The task is to find what proportion of ice is left and involves comparison of areas. The mathematics includes:

- calculating area of an irregular shape
- percentages
- measuring perimeter
- clarifying difference between perimeter and area

This activity is working on mathematics in the context of climate change. Other ideas could be:

- topographical maps looking at sea level rise
- decline in lake volume related to drinking water
- linking area and magnitude
- choosing a graphical representation and justifying the choice

Peter, Wayne, Stewart and Doug

This group looked at the background to the Stephan-Bolzman law, which relates CO_2 concentration to temperature and involves a 4^{th} power relation. In its analysis and derivation, which involve integral calculus, work with students on this topic would clearly be at the post-secondary math level. In its 'ideal radiator' form, the Stephan-Bolzman law is: $E = \sigma T^4$. In its non-ideal radiator form it has some additional factors.

Since this law is an important part of understanding global warming, this group talked about how the need to understand proportionality, and the non-linearity involved in this equation, would be crucial. Even without working with this equation directly, it would be important for students (and the larger community) to better understand the meaning and significance of non-linear relationships: What does it mean? What is implied by a linear relationship and therefore non-linear relationships? Where do these occur in life, nature, etc.? If there is any hope of understanding how one entity might vary as the fourth power of another, we have to spend time developing this understanding with 'simpler' relationships and models. It's useful to note that Brad emphasized in his talks the importance of non-linear relationships and how, in general, these are not well understood.

CONCLUSIONS

The working group ended with a profound sense of concern at what we learned about the climate of our planet. The work of the group was successful in exploring where mathematics plays a role in investigating and understanding climate change, as well as in the mathematical literacy needed to make sense of all the information available. One of the awarenesses that emerged from the task of creating activities relating to climate change was that the complexity and messiness of the topic was an opportunity. One participant commented that "if we make the tasks too elegant, we have nice discussions, but students find it abstract…having students or student teachers work on integrating different information or ideas is meaningful for them." That is, by engaging with the complexity, students can learn a lot, even if the outcome is not always tidy and complete. This awareness, in fact, applies just as much to the working group itself.

MEANINGFUL PROCEDURAL KNOWLEDGE IN MATHEMATICS LEARNING

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BACKGROUND

Mathematics is about more than knowing, it is also about doing. As educators, we acknowledge the importance of understanding the fundamental concepts that underpin mathematics, yet the assessments of our students are often procedure-based. If being able to do mathematics follows strictly from an understanding of mathematics, then procedural tests would be an accurate assessment of a student's understanding. Of course, this is not the case. The relationship between procedural and conceptual knowledge is far more complex and dynamic – it is possible that one can be obtained, to some degree, in isolation from the other. Recognizing the importance of both procedural and conceptual knowledge, how can both of these be best instilled in our students?

Much recent work in mathematics education has focused on the teaching of concepts. Less emphasis is placed on procedures, since it is often assumed that procedural ability will naturally arise and that procedural knowledge does not run as deep, and is less interesting from a research perspective, as conceptual knowledge. Our working group focused on procedural knowledge and found it to be complex and dynamic, worthy of far more attention than what it receives currently.

Some of the questions we considered in our working group are presented here along with brief introductions to relevant literature.

WHAT IS PROCEDURAL KNOWLEDGE? WHAT IS CONCEPTUAL KNOWLEDGE?

Many authors have attempted to discern and classify types of knowledge. Indeed, the history of these attempts stretches back to antiquity and will not be covered here. It must be emphasized that what is considered knowledge is not universal and unchanging; even what we take as scientific knowledge has its roots in 20th century philosophy. We focus our attention on those knowledge classifications that have appeared in the education, and related, literature.

The primary focus of this working group was on procedural knowledge, which is often defined alongside conceptual knowledge. Perhaps the most commonly accepted definitions of procedural and conceptual knowledge are due to Hiebert and Lefevre (1986):

> *[Conceptual knowledge is] knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network.* (p. 3)

In terms of procedural knowledge:

> *One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols.* (p. 3)

The connotations are clear: conceptual knowledge is somehow ‘better’ than procedural knowledge. This has led some authors to expand on these basic definitions, considering not just type of knowledge but depth. de Jong and Ferguson-Hessler (1996) review some of the knowledge constructs present in the literature and attempt to synthesize them. In the literature they review, they locate the following types of knowledge: generic (or general) and domain specific, concrete and abstract, formal and informal, declarative and proceduralized, conceptual and procedural, elaborated and compiled, unstructured and (highly) structured, tacit or inert, strategic, ‘knowledge acquisition’, situated, and meta knowledge. Understandably, this plethora of knowledge types can cause confusion for both researchers and practitioners. In an attempt to consolidate these constructs to “avoid the introduction of still more types of knowledge that do nothing more than to describe properties of generally accepted types of knowledge” (p. 105), the authors place the definitions in a two-dimensional array. One axis is *type*, the other, *quality*. Working from their field, physics, the authors identify four distinct types of knowledge, two of which — *conceptual* and *procedural* — are relevant to the current discussion. As for *quality*, the authors propose *deep*, associated with comprehension and abstraction, and critical judgement and evaluation, and *surface*, associated with reproduction and rote learning, trial and error, and a lack of critical judgement. This way of classifying knowledge has lead Star (2002) to enquire about *deep procedural knowledge*.

In a sense, the question Star (2002) poses is natural: we know what deep conceptual and superficial procedural knowledge look like, but what is deep procedural knowledge? Part of the problem in understanding deep procedural knowledge is we seldom look for it. Conceptual knowledge is measured verbally and through a variety of tasks, while procedures are measured in terms of task completion – is the answer correct or not? This binary assessment obscures the richness present in carrying out a procedure. Another issue is that mathematics education literature tends to draw from the same well of problems, those found in primary or early-secondary school. What are absent are higher-level problems where the richness in procedure execution is more transparent (Star, 2002). Any derivative in a calculus course, for example, can be computed in a great number of ways, all of which, if done correctly, will yield the same expression. It is while performing an operation like this that we encounter a rich mathematical performance by the student.

HOW DO PROCEDURAL AND CONCEPTUAL KNOWLEDGE RELATE?

It is natural to assume that the more one understands about a certain concept, addition, say, the easier it will be to perform an operation based on this concept. Although this may be the case for some concepts, it is almost certainly not a general phenomenon. This assumption is brought about by what Star (2002) identifies as a deficit of the mathematics education literature: the concepts typically examined are taken from primary and secondary school. Examples of higher grade-level concepts are needed to illustrate the dynamic interplay between knowing and doing. A good place to start is differentiation (Maciejewski & Mamolo, 2011). The derivative of a function is defined as the limit of a quotient of functions. There are many ways to conceptualize a derivative – as a rate of change, the slope of a tangent line, etc. – but no matter how deeply the concept of derivative is understood, the procedure for finding the derivative of a function remains opaque. Not only this, but the inability to compute derivatives inhibits how deeply the concept can be known. Furthermore, the acquisition of neither concepts nor procedures necessarily precedes the other (Rittle-Johnson & Siegler, 1998). A few studies have supported the notion that concepts and procedures can play off of each other, one reinforcing and strengthening the other (Byrnes & Wasik, 1991; Rittle-Johnson & Siegler, 1998). This may have profound implications for teaching practices: perhaps it is better to start with procedures with some students, concepts with others.

WHAT IS MEANINGFUL LEARNING?

All educators want their students to learn. But what does it mean to learn? Mayer (2002) identifies two components of learning: *retention* and *transfer*. *Retention* is the ability to recall a lesson, while *transfer* is the ability to apply the lesson to a novel situation. It is this second component of learning that Mayer identifies as an indicator of meaningful learning. According to Mayer,

> *Meaningful learning occurs when students build the knowledge and cognitive processes needed for successful problem solving. Problem solving involves devising a way of achieving a goal that one has never previously achieved; that is, figuring out how to change a situation from its given state into a goal state.* (p. 227)

Meaningful learning is harmonious with constructivist theories of learning. In both, students are actively engaging with the material at hand, devoting attention to incoming information, processing the relevant information into an appropriate mental representation, and coordinating this information with existing knowledge. In contrast, rote learning focuses on adding tidbits to existing memory. One of the central foci of our working group was the meaningful learning of procedures.

HOW IS PROCEDURAL KNOWLEDGE BEST LEARNED? CAN IT BE LEARNED IN A NON-ROTE, MEANINGFUL WAY?

How procedural knowledge is viewed, and the emphasis placed on its acquisition and execution, is not universal. The current emphasis in North American mathematics education on understanding concepts tends to downplay the execution of procedures. This is not shared with other regions; Asian cultures tend to place emphasis on procedural fluency. Shiqi (2006) identifies that

> *[I]n China, as well as East Asian countries, routine or manipulative practice is an important mathematics learning style. Practice Makes Perfect is the underlying belief [...] Through imitation and practice again and again, people will become highly skilled [...] Manipulation is the genetic place of mathematical thinking and the foundation of concept formation. It provides students with a necessary condition*

> *of concept formation and is the first step of mathematics comprehension.* (pp.129-130)

Shiqi also notes that Asian countries typically do quite well on international tests of mathematics ability, and this, in turn, may be attributable to the emphasis on manipulation work, a belief commonly held in China.

Shiqi examines possible sources of the Chinese perspective on mathematics learning. She notes that the classical Chinese text on mathematics, *The Nine Chapters on the Mathematical Art*, focuses on the solution of a collection of problems. In the work, a problem is stated, a solution is given, and an explanation of the algorithm used to generate the solution is provided. This work differs markedly from its Greek contemporaries, which sought general principles from a set of axioms. This work has had profound influence on mathematical thought in China; so much so that mathematics is known as the "subject of computation". The idea of mathematics as a tool of computation persists as many older adults in China refer to mathematics as arithmetic.

School mathematics in China, as in North America, is viewed as a difficult subject. But unlike North America, the Chinese believe that all students, regardless of ability can grasp and excel in mathematics, provided that they practice. This belief pervades Chinese education; indeed, the Chinese character for education may be translated as "young people grow and develop provided they make every endeavour to tackle difficult tasks". This notion is supported by Chinese idioms – *slow bird should fly earlier* – and by historical Chinese thinkers – Confucius says: *it is pleasant to learn and practice time and again*. The exam culture of China also contributes to the emphasis on practice. Examination is a millennium-old practice that still plays a vital role in Chinese society. Students face fierce competition for entrance to university and consider continual practice a good way to achieve well on these exams.

The Chinese word for *practice* is, however, far more dynamic than what it is in English or French. It goes beyond *practice* or *do* to encompass *familiarize with* and *be proficient at* (Shiqi, 2006). This leads to a dynamic way of practicing. Experienced teachers recognize the depth to practice, and structure the practice they give their students around promoting deep learning. This is often referred to as *variant manipulation* or *meaningful manipulation*.

Problems, of course, arise from excessive emphasis on excessive practice. Shiqi (2006) highlights a study that revealed that a group of math majors at a college in China could find the answer to a set of problems, but had essentially no idea why the answer should be true. It should be mentioned that it is of no benefit to students to place sole emphasis on either concepts or procedures; a balance must be struck between both.

HOW IS PROCEDURAL KNOWLEDGE BEST TAUGHT/LEARNED? CAN IT BE TAUGHT IN A NON-ROTE, MEANINGFUL WAY?

The learning and teaching of procedures has received comparatively little attention in the literature relative to that of conceptual knowledge. This is changing, however, as the importance of developing sound procedural knowledge is gaining recognition. Indeed, Kirschner, Sweller, and Clark (2006) acknowledge that

> *...expert problem solvers derive their skill by drawing on the extensive experience stored in their long-term memory and then quickly select and apply the best procedures for solving problems [...] We are skillful in an area because our long-term memory contains huge amounts of information concerning the area. That information permits us to quickly recognize the characteristics of a situation and indicates to us, often unconsciously, what to do and when to do it.* (p. 76)

In terms of classroom practices, research indicates that simple modifications to a course plan can have a profound impact on both conceptual and procedural knowledge. According to Rittle-Johnson and Koedinger (2009),

> *[k]nowledge of concepts and procedures may develop best in an iterative process, with improvements in one type of knowledge supporting improvements in the first. The current study converges with past research indicating that prior knowledge of one type can influence gains in the other type of knowledge [...] More importantly, it extends prior research by demonstrating that experimentally manipulating the order of instruction to follow an iterative sequence can improve learning, compared to a concepts-before-procedures sequence. This is particularly impressive given that all participants completed the same lessons; only the order of lessons differed.* (p. 496)

Research on how a student's procedural knowledge develops, and what experiences aid in this development, is in its infancy. In this regard, our working group was offered at an exciting time.

RETOUR SUR CHACUNE DES SIX SESSIONS

The working group discussions were organized into six sessions. Each of the six sessions had some specific questions to be addressed. For each of the sections, we will present the questions, an overview of the activities, and the discussions.

SESSION 1

- Quels sont les savoirs procéduraux?
- Quels sont les autres types de savoirs et comment peut-on les comparer?
- Est-ce que les savoirs procéduraux sont porteurs de sens?

Nous avons eu une discussion forte intéressante portant sur ces questions. Nous avons tout d'abord réfléchi sur les différents types de savoirs (entre autres, sur les savoirs factuels, intuitifs, pratiques, techniques, le sens commun, les savoirs spatiaux, sociaux, institutionnels, environnementaux historiques, la mémoire musculaire, les savoirs d'expérience, disciplinaires, les connaissances). Notre discussion nous a conduit à envisager les relations entre les savoirs procéduraux et les savoirs conceptuels. Nous nous sommes penchés sur des modèles ou bien des façons de représenter ces relations à l'aide d'un diagramme de Venn ou d'une corde. Nous avons convenu que le modèle des cinq filaments de la compétence mathématique ('the rope model of five strands of mathematical proficiency' (NCR, 2001) – see Figure 1) introduit par un participant était un modèle fort représentatif de ces relations :

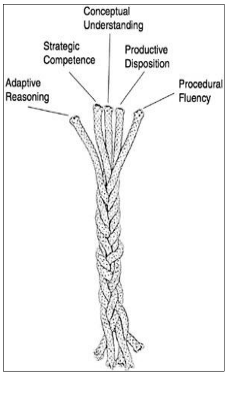

Figure 1. Mathematical Proficiency, NCR (2001).

- **Conceptual understanding** – implique une compréhension des concepts, des opérations et des relations entre eux. La compréhension conceptuelle se manifeste, entre autres, lorsque les étudiants comprennent les liens et les similarités entre des concepts et entre des opérations.
- **Procedural fluency** – implique flexibilité, précision et efficacité lors de l'utilisation appropriée de procédures. Les habiletés de la compétence comprennent savoir quand et comment utiliser ces procédures.
- **Strategic competence** – implique l'habileté de formuler, représenter et de résoudre des problèmes mathématiques.
- **Adaptive reasoning** – implique la capacité de penser de façon logique aux concepts et aux relations entre eux. Le raisonnement est nécessaire pour naviguer entre les différentes procédures, les faits et les concepts afin de trouver une solution.
- **Productive disposition** – réfère à une disposition positive envers les mathématiques. Les attitudes positives envers les mathématiques contribuent au succès mathématiques.

Nous avons conclu cette première session en reconnaissant que les savoirs procéduraux et conceptuels étaient complémentaires.

SESSION 2

- How do procedural and conceptual knowledge relate?
- Are these types of knowledge distinct?
- What are examples of procedural and conceptual knowledge?
- How is procedural knowledge viewed in different cultures?
- **Activity:** Participants solve an algebraic equation: Is knowing how to solve in different ways procedural or conceptual knowledge?

We watched a short video about the historical evolution of mathematics, focusing on how Egyptians did multiplication. We discussed the different meanings of multiplication and which procedures might be associated with them. Our discussion raised the question of whether the Egyptians were aware of the procedures and how we could know that they knew what they were doing. The discussion centred around Sfard's (1991) work on structural and operational descriptions of mathematics notions. The structure of a notion is related to its properties and the operation is understood as the possible actions we can take on this notion. This work helped us to see procedural and conceptual knowledge with other lenses. As a group we then solved a system of linear algebraic equations, similar to:

$$2x + y = -1$$
$$x - 3y = 1$$

in different ways. A variety of methods were proposed: substituting one equation in the other, representing the system as a matrix, graphing the equations, etc. We discussed whether knowing how to solve the algebraic equation in different ways entails conceptual knowledge or procedural knowledge.

In order to think about the depth of the procedural knowledge, we discussed DeBlois' (2003) model of interpretation of cognitive activities: *Modèle d'interprétation des activités cognitives* (see Figure 2). This model focuses on the representation of the situation by the learner, the procedures employed, and on the effects of the didactical contract (Brousseau, 1998). Coordination between them can create awareness toward the concept. The procedures employed are important because contemplating those procedures can lead to deep understanding of the concept. Procedural and conceptual knowledge feed each other; you need to understand procedures to advance conceptual understanding and vice versa.

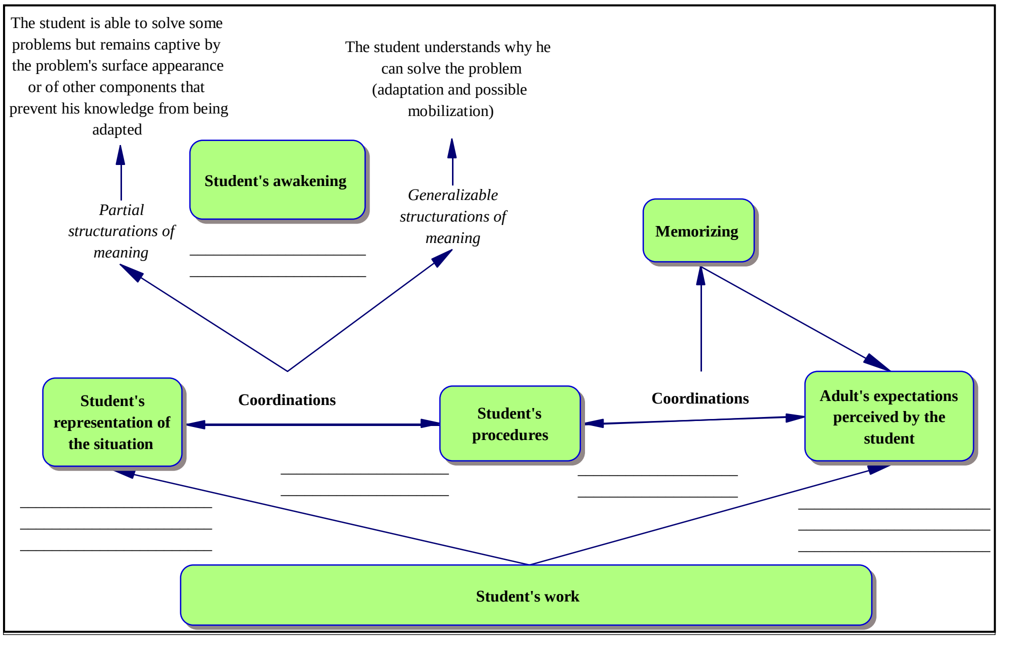

Figure 2. Interpretation of cognitive activities (DeBlois, 2003).

SESSION 3

- How is procedural knowledge viewed in other disciplines?
- **Activity:** Enumerate, for each of these topics, some examples of procedural and conceptual knowledge.

As a starting point of the session, we discussed procedures in music. Our primary focus was on a drum rudiment, the *paradiddle* – a rhythm created by making a sound with your right and left hands in the pattern RLRRLRLL. After mastering this procedure, we watched a video of Steve Gadd's *Paradiddle Shuffle*, in which Gadd transforms this basic rhythm into an elaborate percussion song.

Procedures and procedural knowledge pervade music. A musician learns a repertoire of brief musical segments that they draw upon when performing. This repertoire does not have to be consciously accessed by the musician; it becomes automatized. When asked about how he remembers so many songs, Keith Richards, of the Rolling Stones, replied, "I don't. My fingers do."

The discussion following the activity on music was rich. We came up with the idea that the process of learning a procedure implies perhaps more than we expect:

- Beginners require high cognitive investment.
- Practice diminishes the need for certain mental processes.
- Experts have the procedure fully automatized.

As a group, we felt that the further we explored the notion of procedural knowledge, the less we knew what procedures are. We tried to define *procedure*. We ended the session by discussing the following ideas and questions regarding the definition of a procedure: a procedure as something that can be broken down into steps; the difference between an algorithm and a procedure that is standardized procedure as an algorithm; the relationship between process and procedure and whether process entails conceptual knowledge or procedural knowledge, and finally, the relationship between computation and procedure.

SESSION 4

- How is it learned?
  - How is procedural knowledge best learned?
  - In terms of learning procedures and concepts, does one follow the other?
  - Can procedures be learned in a non-rote, meaningful way?

- **Activity**: Small group responses to the questions.

After watching a short video clip on 'a classroom check-in procedure' as a warm-up, we worked in five small groups in order to respond to the questions asked at the beginning of the session. In terms of the question of how procedural knowledge is best learned, participants offered some ideas, including: by practice and scaffolding; by incorporating prior knowledge; through mental math; by looking at different procedures; through embodied experience, using puzzles and games; through tasks that provide rationale for using a procedure; through the use of mathematics language and symbols; and through emphasis on the meaning of the step and the meaning of doing it. There were questions raised on whether we should use the word *best* in the question. As well, the question was raised about whether procedural knowledge involves more than knowledge of a specific procedure or knowledge of how to learn procedures generally.

With regards to the question of whether or not procedures and concepts follow each other in learning we found that it depended on the situations and tasks. We thought that concepts are necessary for meaningful procedures. As well, we noted that automaticity is more important than rote-learning of the procedures. We discussed whether or not procedures can be learned in a non-rote, meaningful way, including the difference between rote-learning and the development of automaticity in learning procedures. We noted that rote-learning involves learning procedures without understanding, while automaticity involves the development of automatic recall after learning the conceptual bases of the procedures.

SESSION 5

- How is it taught?
  - How is procedural knowledge best taught?
  - Can procedures be taught in a non-rote, meaningful way?

The discussion about how procedural knowledge is best taught turned around the idea of putting an emphasis on conceptual knowledge first and then procedural knowledge later. We also discussed using concrete materials, different tools, and/or cross-curricular competencies, in order to best teach procedures. We thought that procedures can best be taught in a context where meaning and reasons for doing the procedures can be provided. We noted that mental math plays a huge role by bridging different knowledge and by offering opportunity to play with mathematical ideas.

We agreed that we should develop conceptual understanding and procedural fluency simultaneously.

SESSION 6

In the last session, we revisited our original definitions of *procedural knowledge*. We wanted to know if anything had changed over the course of our working group and we wanted to think about how we use these definitions in everyday life. Our final conclusions for the working session are presented in Figure 3, which represents the role of procedural knowledge in understanding mathematics.

	Surface Knowledge	Deep Knowledge
Procedural	Algorithm	Capacity to choose and be flexible with algorithms/process

Figure 3

CONCLUDING REMARKS

During our discussions over the three days, organized into six sessions, we raised many issues and questions about meaningful procedural knowledge in mathematics learning. Our discussion was guided by the following questions: What is procedural knowledge? How do procedural and conceptual knowledge relate? Are these types of knowledge distinct? In terms of learning, does one follow the other? How is procedural knowledge best learned? Is procedural knowledge rote? Can procedures be learned in a non-rote, meaningful way? Is there depth to procedural knowledge? Throughout our discussions we engaged in activities that prompted our thinking about the topic and our responses to the questions.

Our working group discussions emphasized and highlighted the importance of teaching and learning procedures meaningfully in mathematics education. Participants shared ideas on how procedures could be taught and learned meaningfully. Most notably in our discussion was our observation that the distinction between procedural and conceptual knowledge becomes blurred in learning procedures meaningfully. As one small group put it:

> *Consider knowing a procedure (e.g. division of polynomials) as part of procedural knowledge. When it comes to knowing when that procedure can be applied in situations, is this procedural knowledge or conceptual knowledge? Is it both?*

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EMERGENT METHODS FOR MATHEMATICS EDUCATION RESEARCH: USING DATA TO DEVELOP THEORY

MÉTHODES ÉMERGENTES POUR LES RECHERCHES EN DIDACTIQUE DES MATHÉMATIQUES: PARTIR DES DONNÉES POUR DÉVELOPPER DES THÉORIES

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> *Qualitative researchers approach their studies with a certain paradigm or worldview, a basic set of beliefs or assumptions that guide their inquiries. These assumptions are related to the nature of reality (the ontology issue), the relationship of the researcher to that being researched (the epistemological issue), the role of values in a study (the axiological issue), and the process of research (the methodological issue).* (Creswell, 1998, p. 74)

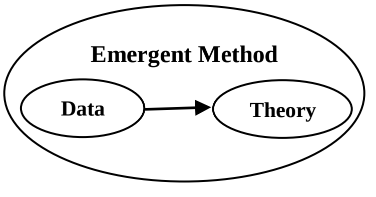
INTRODUCTION

Although qualitative research is common in mathematics education, increased clarity and transparency of its methods are of ongoing concern and interest to mathematics education researchers in order to keep enriching their work and contributions to the field. Adding complexity to the conversation space is a growing focus on constructing theories from within the field of mathematics education (e.g., Kieren, 1997). Thus, the goal of this working group was to provide participants with a space to begin or to continue to explore *emergent methods*, such as grounded theory (Morse et al., 2009), narrative inquiry (Clandinin, 2007), and design experiment (Lesh & Doerr, 2003), that offer possibilities of dealing with complexity in the context of using data to create theory.

A working group with this goal can be framed in different ways. For example, it could focus on how representing the interpretive act of listening to research participants in each of these methodological approaches may result in developing theories which seek to make sense of researched phenomena and generate further inquiry. It could focus on particular methodological issues related to: the above Creswell quote; what constitutes sufficient data; how data collection and analysis interact; data management challenges; research being both planned and emergent; strategies for data collection and analysis; and other aspects of interest to working-group participants. It could also consider specific questions, such as:

- How is theorizing by mathematics educators strengthened by beginning with data?
- How do we engage in the task of moving from data to building theory?
- How might research participants' voices be amplified through the development of theory?

In fact, the working-group leaders took all of these into consideration in planning the working-group activities. However, we decided to frame the activities in terms of three themes: *theory*, *coding data*, and *notions of emergence*. These interrelated themes address unique, central components of the emergent research perspective. On a simplistic level, as represented in the diagram below, the data and theory are nested in the emergent method that determines the path from data to theory. This path from data to theory is defined by coding techniques based on an inductive process, that is, one that allows the data to speak.



In considering these themes in the context of mathematics education research, our focus was on: What is meant by 'theory'? What does the coding of data looks like? What are key notions of emergence? We considered this to be a more meaningful way of engaging participants in the three sessions of the working group than moving chronologically through each aspect of the research process. Our approach to addressing these themes was to draw on relevant readings and the experiences of the working-group leaders and participants. Of importance was also engaging participants in a hands-on way with sample data based on the research of the working-group leaders and participants who wanted to contribute to this process.

This report of the working-group activities is organized based on the three themes and how they unfolded over the three, three-hour sessions of the working group. The sequencing of the themes was loosely chronological in relation to the three sessions in that *theory* was introduced as a focus for the first session, *coding data* for the second, and *notions of emergence* for the third. However, after the first session, movement back and forth between the first two themes, in particular, resulted in little time to address the third. Participants were fully engaged in the activities and contributed in ways that were insightful and helped to shape the way the working group unfolded. In the report, we try to capture the thinking and contribution of participants as examples of the nature of the discussions, but also as a way of offering something meaningful to the Canadian Mathematics Education Study Group/Groupe Canadien d'étude en didactique des mathématiques [CMESG/GCEDM] community and other readers of the proceedings.

THEME 1: THEORY

Theory, as a construct, is problematic in that it can mean different things. Depending on how it is viewed, one can question whether what results from data through an emergent approach such as grounded theory is really theory. For example, if a researcher using grounded theory concludes something about students' ways of experiencing mathematical problem solving, is this something *theory*? Our intention, then, in beginning with theory was to attempt to get to a shared understanding of it in mathematics education that could be used during our time in the working group to make sense of the emergent research method. This, however, would have been an unprecedented outcome for a CMESG/GCEDM working group if it had worked out as intended. The open discussion among participants in the first working-group session validated the position of *theory* being a problematic construct. There were more differences than similarities in how working-group participants conceptualized it based on their experience with it. Thus, instead of arriving at an understanding of it as a group, we gained insights of the diversity of ways in which one can view and use it.

REFLECTING ON QUOTES

In order to provoke particular thinking about theory, we presented participants with four quotes. Participants worked in pairs, then shared and discussed their thinking with the whole group. Each quote brought forward issues and ideas that continued to surface in all of the sessions of the working group. We highlight examples of these issues and ideas discussed for each quote.

> *One aim of mathematics education research should be towards generating models. … results from many research studies should contribute to the development of a theory (or model) that should have significant payoff over a prolonged period of time.* (Sriraman & English, 2005, p. 451)

This quote from Sriraman and English highlighted two particular issues that we discussed in our working group. The first issue we identified was related to the way in which the product of research is named, with specific focus on research in which theorizing is a predominant process of coming to understand a phenomenon. The suggestion in the quote that *theory* and *model* might be pointing to the same phenomenon elicited conversation about a contentious pairing. This pairing triggered suggestions and consideration of related constructs in addition to *model*, in particular, *frameworks* and *categories*, that raised similar concerns about how such terms are used with theorizing. Darien Allan, for example, explained her preference in pairing such terms: "I am much more comfortable discussing model or framework and talking about themes or theorizing rather than what I consider a more complex and abstract notion of theory." Claude Gaulin, on the other hand, reminded us of the contentious nature of the use of *model* in the context of theorizing by surmising that it is even more challenging to develop a shared understanding of *model*. Another suggestion offered was that the process of theorizing resulting in a model, framework or categories being published in reports of research could be representational, that is, the ways in which theories are represented may be through a structure-based metaphor languaging.

The second issue discussed was related to the type of theory that might be generated by a researcher in mathematics education. Veda Abu-Bakare drew our attention to the phrase "a theory" in the above quote, which seemed to signify the authors' support of a grand theory *of* or *in* mathematics education. This generated discussion of the idea of a grand theory. Claude pointed out how some researchers in the field of physics were attempting to develop a 'theory of everything'. In our attempt to think about a 'theory of everything' in mathematics education, Kate Mackrell aptly reminded us that what we seek to understand as researchers is much more complex, the teaching and learning of mathematics. This caused us to turn our conversation toward what it is mathematics educators might theorize about – as the Sriraman and English quote is vague in its prescription. Areas such as learning of mathematics, teaching of mathematics, the nature of mathematics, and the nature of mathematics education were offered as broad categories within which theorizing could be enacted.

> *I question how far mathematics educators have moved toward a theory, whether grand or otherwise.… I am happy to talk about theorizing, adopting a theoretical stance, or employing a theoretical framework, but I do not see extant theoretical constructions as warranting the label of theory.* (Kilpatrick, 2010, p. 4)

This quote helped our conversation to move from the notion of a singular theory in mathematics education toward sharing our varied perspectives on what theory might mean to each of us as researchers. The perspectives on the nature of theory were far ranging, including examples such as: *theory* as universal constructs, *theory* as a tool for explaining and predicting, dynamic quality of a theory changing over time as it is tested and revised, etymologically meaning a way of thinking and seeing, as absolute truths, interpretive ways of communicating understanding, a process, etc. Tony Pascuzzo noticed that within the multiplicity of understandings of what theory is, there is a liberating, rather than constraining, aspect to being explicit about and working from one's own perspective. Noticing that ways of thinking about what theory is within mathematics education reminded us of how our thinking has been impacted by our experiences in reading and engaging in research. The varied perspectives focused not only on the notion of theory, but also on the multiplicity of perspectives brought to working with data. This provided a rich opening for our work with coding data, described below. Beth Herbel-Eisenmann raised a question worth offering up to the broader community: *Does theory represent the phenomenon being studied or does theory represent the perspective of the researcher of the phenomenon?* Posing this question helped us as a working group to move forward in our conversation – with the acknowledgement that sharing a common understanding of theory is simultaneously elusive and unnecessary – to wonder about the reasons for engaging in theorizing in mathematics education.

> *Theories need to grow and aid growth. They need to enrich our understandings. If we prevent a theory from being questioned, articulated, justified, or from illustrating its power, then we don't grow as a field of inquiry.* (Proulx, 2010, p. 24)

This quote provided a different perspective of the notion of theory for our discussion, building on the momentum toward considering the purpose of theorizing. Reactions and conversation around the quote immediately recognized the use of a situated and context-specific theory. These types of theories are ones which individual researchers aim to develop through their ongoing work. In his final reflection, Lionel LaCroix emphasized an important idea connected to Proulx's quote when he wrote, "theories range considerably in their scope." So while the initial orientation in the quotes to theory in mathematics education was one of larger order, there was movement toward seeing theory as developing out of situated work. In fact, Glaser and Strauss (1967) in their early writing about grounded theory intended it to be a methodology that would support the development of mid-range theories.

Experiences of theorizing in mathematics education were also shared within the group, including by those who had developed "theories" or significantly shaped "theory" in their own research. As one example, Donna Kotsopoulos described the process of her doctoral dissertation research as employing the use of theoretical frames to come to understand her data. It was only later that through the experience of being in conversation with colleagues and reviewers of manuscripts that she began to consolidate her understanding toward a theory. Thus, our reflection on this quote allowed us to learn, from each of the examples shared by participants, the importance of dialogue with colleagues, the explicitness of working from particular epistemological and ontological orientations, and the long-term process that contributes to theorizing.

> *Theories are needed for several reasons – to guide the research process, to communicate with others, and to make the results useful. Developing and refining theories also provide the best evidence that we are making progress in our understanding.* (Hiebert, 1998, p. 144)

This quote highlighted for our discussion a tension between theories that "guide the research process" and the development of theories. Some doctoral students spoke of the challenges they faced in dissertation work where there was an expectation of a review/use of a theoretical framework and their desire to theorize in their work. This led to a discussion of this expectation in research reports for journal manuscripts or conference proposals. For example, mathematics education researchers were seen as sharing a common tension in the acceptance process of publishing academic manuscripts, especially in journals which disseminate a traditional research report genre. However, even for studies published that claim to be within grounded theory methodologically, there seems to be a myth of beginning the work *a*-theoretically. Instead, there are epistemological and ontological orientations within which a researcher works and approaches a particular study, but these may be a separate consideration from using a particular theoretical framework to force on the data. While a general approach is to describe the researcher's orientation, two specific examples given by group members were to use *sensitizing concepts* (Blumer, 1954) or *available concepts* (Desgagné, 1998) as ways to demonstrate the researcher's understanding of the field in which the research is conducted and the phenomena the researcher is drawn to attend to when interpreting data. Hiebert's quote was seen as an encouragement to mathematics educators to live within the tension by engaging in theorizing to make sense of the complex phenomena in mathematics education.

REFLECTING ON RESEARCH ARTICLES

In order to connect our working-group focus to how mathematics education researchers move from data toward the development of theory, we also inquired into specific instances of 'theories' developed from data and the use of grounded theory within our CMESG/GCEDM community. A sample of research articles were preselected by the working-group leaders for this activity. Small groups selected and worked with two articles paired according to the focus for discussion.

The articles were paired to exemplify two aspects of theorizing in mathematics education: one which used grounded theory as its methodological structure with a range of final products, and one which explicitly engaged in theorizing or representing a theory constructed by the authors. The pairs consisted of: Bruce (2007) and Zack and Reid (2003); Sinclair, Mamolo, and Whitely (2011) and Watson and Mason (2002); and Liljedahl (2010) and Pirie and Kieren (1994). The first article in each of the pairings used grounded theory as its methodological structure where the resulting products were "a model for examining preservice teacher efficacy" organized in a stage-based framework (Bruce, 2007, p. 3), "three main categories" (Sinclair, Mamolo, & Whitely, 2011, p. 155), and "five distinct mechanisms of change" organized as categories (Liljedahl, 2010, p. 422). However, in selecting these examples, we note that the use of grounded theory as a methodological framing does not necessitate the development of a theory as a researcher's final process of a project. Rather, the flexibility offered by formulations of grounded theory, such as constructivist grounded theory (Charmaz, 2006), encourage transparency in the data analysis and interpretation process. Each of the small groups attended to the description of research method as various formulations of grounded theory and the explicitness with which the authors described the research process that supported the development of their model, categories, or mechanisms.

As a way to contrast the use of grounded theory and the ways of communicating what researchers had learned, the second article in each pair did not explicitly use grounded theory but the authors named their work as 'theorizing' or the resulting product was a theory offered to the community. Zack and Reid (2003) point to their process of theorizing as engaging in "good-enough 'theories-for' [as] a worthy goal for research" (p. 43), highlighting the contingent nature of coming to understand as mathematics education researchers working with children. Watson and Mason (2002) engaged in "grounded theorizing" (p. 243), working from observations of students and themselves, and descriptions through analysis. Their product was named a "theory" (p. 237) and resulted in "a framework of five uses for asking learners to generate examples for themselves" (p. 242). Pirie and Kieren (1994) primarily describe and illustrate their theory of the "growth of mathematical understanding in the children that we observed in classrooms over time" (p. 165), represented by a model of overlapping circles. Of all the pieces used, they uniquely point to the use of their theory as a way to attend to students' dynamic mathematical understanding to inform a teacher's pedagogy. Compared to the explicit process of theorizing in grounded theory, these three examples wrestled with notions of theory in their presentation.

REFLECTING ON THE FIRST SESSION

At the beginning of the second session of the working group, we returned to consider our conversation about theory in mathematics education research. This provided an opportunity for participants to reflect and then express moments of thoughtfulness about the previous day. It was in this discussion that two key questions arose. The first question was concerned with the purposes of theory within mathematics education, that is, to what end do we engage in theorizing and how are theories taken up? The second question prompted an examination of research methods and their role in the building of theory through data-driven research, that is, how do research methodologies support theorizing? These two questions were posed as a desire to learn more and opportunities for each of us – and for the broader community of mathematics education – to consider the relationship of theory-building to the vibrancy of our field. Even at the end of our working group, Susan Oesterle reiterated the challenge of understanding theory in the context of mathematics education and recommended "that the field really should address" continually its goals and understanding of theorizing.

THEME 2: CODING DATA

After the theory activity, in the second session, the focus of our working group moved to the process of theorizing from data, in particular, the coding aspect of that endeavor. For many participants who were beginning to or were in the process of working on their theses, it was important to have the opportunity to become familiar with generating codes using grounded theory or any similar coding techniques or approaches. This was made possible by the mentoring provided through hands-on activities involving coding from a grounded theory and narrative inquiry perspective.

The working-group leaders provided sample data from their own research and led workshop-style small-group work on data coding. Two of the sample data were in English and one was in French. The working-group leaders presented their samples in terms of the larger research context, the research questions, and the research method involved. For example, Olive Chapman contributed a sample narrative from Chapman (2008) that was taken from a study that investigated prospective secondary mathematics teacher sense-making of "good mathematics teaching". This narrative was an example of one participant's initial story of "good teaching" of mathematics. It was written at the beginning of the second semester of her post-degree, four-semester Bachelor of Education program and before she had any exposure to theory on mathematics pedagogy, so it was based solely on her preconceptions of "good teaching". It was based on a Grade 10 lesson at the beginning of a trigonometry unit.

In order to further prepare for the coding, the following quote was presented and discussed: “Coding means naming segments of data with a label that simultaneously categorizes, summarizes, and accounts for each piece of data” (Charmaz, 2006, p. 43). Participants then selected to work with one leader. Most chose to work with the English data on grounded theory led by Janelle McFeetors. A large part of the session was spent in the small groups working on how to code the sample data. Participants worked in teams of two or more in the small groups to try and code the sample data on their own, based on the introduction provided.

The whole-group discussion that followed first dealt with the challenges and pitfalls of the exercise. The focus then shifted to the main coding strategies used by participants. For example, one strategy involved forming categories based on the research questions and trying to build codes related to those categories. Another strategy involved using a line-by-line coding, similar to the coding procedure suggested by Charmaz. These two coding strategies are highlighted because they reflect opposing perspectives. The coding/categorizing strategy driven by research questions is representative of a coding process driven by a predetermined list of categories. Here one can echo criticism addressed to that way of coding by grounded theory proponents who view this approach as not questioning existing concepts or theories, and merely seeking to exemplify such concepts/theory with data. The other coding strategy is representative of open coding that triggered discussions focused mainly on the requirements and limitations of this approach. For example, it was noted that this type of coding requires special attention to language used to generate codes which at the beginning of the process have to remain descriptive so as to better convey the research participants’ perspectives or voices. Also, that way of coding requires several trials and a constant return to data in order to come up with relevant codes. In this regard, one of the facilitators shared with participants his experience as a novice encoder having to rely on colleagues from a research team who helped to review, validate and refine initial codes. To stay close to data means to refrain from “jumping” too quickly to concepts. So, with regard to this open-ended coding approach, two related issues were raised by some participants: Does the researcher code from scratch? What role does his reference frame play? Two answers were offered. First, data as well as codes are constructed by the researcher (Charmaz, 2006) whose perspective or reference frame matters. Second, the researcher’s perspective and knowledge about a given research topic can lead him to the use of what Desgagné (1998) refers to as *available concepts*, meaning that few concepts are likely to help make sense of data, although they have to be somewhat “put on hold” pending analysis. In that respect, without falling into predetermined codes or categories (which would be contrary to the grounded theory approach), the researcher has to stay close to the data, thus allowing the emergence of codes, while not closing the door to *available concepts* when analyzing data. We have here a particular perspective on grounded theory.

Other issues related to data coding were raised. Does grounded theory always lead to things we did not know before? Ultimately, what is a theory? Our working group modestly aimed at giving participants an overview of the challenges of theorizing from data, an emergent but rigorous process. We hope to have achieved that goal, notwithstanding the many limitations of the experience we lived with the participants, who worked only with partial sample data and never came close to linking codes and generating categories (substantive and theoretical).

A final treat to participants to end the second session of the working group was a presentation by Susan Oesterle in which she shared an Excel file tracing craftily the network of codes and categories she built with data from her doctoral research. Susan shared some of the challenges she encountered while coding interview data. She found it useful to follow Charmaz (2006) and start initial coding with gerunds. This helped avoid theorizing the data (i.e., attaching concepts) too quickly. At the same time it posed challenges as coding a phrase with gerunds removes both subject and temporal cues. For example, the gerund code “struggling with fractions" does not indicate who the speaker is referring to (self or a student) or whether this struggle occurred in the past or more recently. As a result, context needed to be considered along with the gerund codes in moving to the next stage of concept coding. Susan was very aware of the subjectivity of this stage, finding that other readings she was doing and new ideas she was exposed to all influenced what codes she saw and was able to see in the data. She found it liberating to allow herself to code phrases with multiple codes. In fact, coding with multiple codes allowed her to better see connections between the codes and recognize themes. Another layer of analysis that she found useful was to create coding summaries. Phrases identified with each concept code were collected and sorted into sub-categories. This helped reveal the range of ideas captured within a single concept code and helped to increase consistency in the coding. The process was iterative and time-intensive, but in the end the codes became stable and clear themes emerged.

THEME 3: THE NOTION OF EMERGENCE IN MATHEMATICS EDUCATION RESEARCH

This theme did not receive much consideration because we ran out of time. The intent was to reflect on key notions of emergent methods and the relationship to mathematics education research. Some group members and the first author of the report felt that this theme should have been addressed earlier in the working group, but the other two group leaders felt strongly about having it last. So for completeness in addressing the three themes in the report, in this section we describe some notions of grounded theory and the use of narratives in mathematics education research.

GROUNDED THEORY

Based on the work of Glaser and Strauss (1967), grounded theory requires the ongoing interplay between action and reflection, that is, between data inquiry and data analysis and theory construction, respectively. It requires researchers to immerse themselves in the situation being studied to collect data of what is happening and the impact of the interactions. It uses an inductive approach to generate theory from data systematically obtained to insure that the theory will "fit and work" (Glaser & Strauss, 1967, p. 3). It represents a bottom-up method in which theory emerges from a process of data collection, coding and analysis. It, therefore, differs from qualitative methods that do not produce theory but focus on descriptions of the situations or phenomena being studied. Rather than the top-down hypothesis testing approach used in most qualitative methods, grounded theory assumes that theory is contained within the data collected. Uncovering the theory involves a process of writing memos in which the researcher articulates emerging ideas that become the basis of a theory. Thus, a central aspect of grounded theory is the constant comparative analytic procedure connected with the development of understandings of what is common among a set of data. For example, in the case of data involving mathematics teachers' experience and conduct in teaching mathematics, analysis can be directed toward theorizing or conceptualizing what they are, based on what emerge as common. In general, the method can lead to either the structure of the phenomenon or the processes entailed in it, or both.

NARRATIVE AS RESEARCH TOOL

Narrative inquiry is another type of emergent method. However, instead of reiterating the emergent qualities already described in the grounded theory section, here we focus on the use of narrative as a research tool in mathematics education research. This use is connected to the prospective of narrative as a way of knowing (e.g., Bruner, 1986). Chapman (2008) discussed studies dealing with the mathematics teacher that used narrative as a research tool to produce data from which to theorize about teacher knowledge and practice in order to inform teacher education. In this context, several topics have been researched using narratives in the form of stories. These include: prospective mathematics teachers' beliefs (Kaasila, 2007); prospective mathematics teachers' motivation (Harkness, D'Ambrosio, & Morrone 2007); prospective mathematics teachers' emerging identities (Lloyd, 2006); mathematics teachers' interpretations of a reform-oriented mathematics curriculum (Drake, 2006); and mathematics teachers' growth (Chapman, 2003).

As discussed in Chapman (2008), while there are similarities in the use of narrative in these studies, there are differences, not only in relation to the purpose, but also the methods of obtaining and analyzing the stories. In some of the studies, researchers used an interview approach to obtain the participating teachers' stories and participants were encouraged to tell stories of their teaching or learning experiences. In most of the studies, the narratives were obtained through writing, that is, the participants were required to write their stories with various degrees of guidance from the researchers. Most of the studies focused on identifying themes within and across stories as a central feature of the data analysis. In order to arrive at the themes, some studies used open coding of the stories as the analysis process, searching for characteristics that related to specific research questions. For other studies, the analysis process was guided by a predetermined model or specific categories based on theory.

Chapman (2008) further explained that narrative provides a means for researchers to obtain and unpack teaching and learning phenomena in order to understand mathematics teachers and their practice from a humanistic perspective. Studies framed in this perspective focus less on identifying deficiencies in teachers' behaviors and knowledge and more on understanding the nature of, and contexts that shape, their perception of their reality. This includes understanding teachers from their own perspective; how particular individual teachers understand their work (e.g., how do teachers make sense of implementing practices of mathematics reform). Thus, a focus of these studies is theorizing or conceptualizing the experiential knowledge of teachers and providing plausible explanations or "theories" emerging from data of teaching behaviors as they are for the teachers.

CONCLUSION

The working group brought together graduate students and professors interested in research methods involving an emergent approach that can be adopted in mathematics education. The themes of *theory* and *coding* were meaningful in engaging participants in two central aspects of an emergent research method with a focus on using data to develop theory. As the report suggests, the activities generated many questions and issues in relation to both themes. This suggests the need for ongoing discussion and exploration of emergent research methods in mathematics education. So we end our report the way we began it, by suggesting that

> *Although qualitative research is common in mathematics education, increased clarity and transparency of its methods is of ongoing concern and interest to mathematics education researchers in order to keep enriching their work and contributions to the field.* (Authors)

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USING SIMULATION TO DEVELOP STUDENTS' MATHEMATICAL COMPETENCIES— POST-SECONDARY AND TEACHER EDUCATION

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INTRODUCTION

Simulation has become one of the most important and widely used scientific methods for the analysis of complex systems. What evidence is there to support this statement? We employed a search engine, Academic Search Premier (http://search.ebscohost.com/), and looked for recent publications. We limited our search to the last ten years, and used the word *simulation* in the title. This generated an overwhelming list of close to 50,000 peer-reviewed publications. Do all these papers report the use of mathematics? The authors, in their choice of subject terms provide some insight into this question. We repeated the search but selected only those publications in which the authors' subject terms included one or more of the following: *mathematics, mathematical, numerical analysis, differential equations, statistics*, or *probability*. This new search produced an impressive list of just over 9000 publications. Surely this provides substantial evidence that mathematical simulation is extensively used. What is also impressive is that these papers address problems in a wide variety of disciplines including, Engineering, Biological and Medical Sciences, Economics, and Finance. Therefore, when undergraduate mathematics students are exposed to and use simulation they are connecting to an important scientific approach. Such an experience will also prepare them for future employment in one of the numerous disciplines that depend on simulation for analysis of complex systems.

In mathematics education at the post-secondary level and in teacher education, simulation can play additional roles. Here are some examples provided by faculty who integrate simulation in their mathematics courses: students use simulation to explore mathematical concepts, either prior to their introduction in class or after they have been exposed to the concept; students use simulation to analyse the truth or falsehood of conjectures; students use simulation to look for solutions to mathematical problems. Faculty also report integrating simulations in different learning environments, for example: a) games – where students problem solve using mathematics in order to achieve certain goals; b) review – where students are guided through a mathematical topic that has been covered previously and are challenged to solve problems; c) testing of mathematical knowledge – where students are asked to answer mathematical questions that are pseudo-randomly generated; d) in class demonstrations – where faculty demonstrate mathematical ideas by using simulations that allow for interactive interventions both in their graphical representation and through the variation of parameters; and e) exploration of mathematical concepts – where students use, in a structured environment, simulations to develop a preliminary understanding of a new concept. It is worth noting that when students program their own simulations, they often follow the ways mathematicians work; namely, they set up conjectures, they test them, they look for counterexamples and then they move to develop a proof or to refute it.

THE WORKING GROUP'S ACTIVITIES

The approach to the topic was practical. Participants began by selecting from six simulation activities that had been prepared by the co-leaders. A summary of each of these activities is provided in Appendix 1. These activities are also provided in more detail on the working group's website (http://www.cegep-rimouski.qc.ca/dep/maths/?page_id=148).

The co-leaders chose these activities to illustrate some of the many uses of simulation in mathematics education. Participants did not have time to explore all six activities and they selected some of them on the basis of the write-up that was provided on the website. In the time allotted for these explorations, the three activities most selected were: the Birthdays Problem (Activity 3), the Parabola (Activity 1) and Newton's Method (Activity 4). The common simulation experiences helped to focus the group's discussions as participants compared their reactions to simulations that they had all undertaken. The co-leaders had included, with each simulation, a number of questions and these were added to the pool of questions raised by the participants after their own simulation experiences. From this list the following questions were selected for discussion.

QUESTION 1: *What factors assist the student to make a transition from a simulation activity to the mathematics?*

For course-designed simulations their uses in class or lab normally highlight their mathematical purpose. Furthermore, these often include questions, problems, or applications specifically designed to motivate the transition to the mathematics. However, for stand-alone simulations, for example the Birthdays Problem, participants suggested that it would be necessary to set follow-up problems, questions, etc. to help the student to make the transition to the mathematical concepts. For stand-alone simulations, the group discussions duplicated most of the suggestions reported in the 2001 CMESG/GCEDM Proceedings of the Working Group "Where is the Mathematics?" (http://publish.edu.uwo.ca/cmesg/pdf/CMESG2001.pdf, pp. 53-57). It was noted that the question of transition applies equally to learning mathematics through the use of manipulatives.

QUESTION 2: *How should simulations be designed? For example, what software or programming languages are available to develop simulations? Is it possible to program simulations for mathematics courses that can be easily modified by another instructor to meet different course needs?*

There are many software packages that can be used for simulations, some examples are:

- *Cinderella* (with Java) (http://cinderella.de/tiki-index.php?page=Cinderella+1.4+laden&bl)
- *Maple* (http://www.maplesoft.com/)
- *GSView* (http://www.brothersoft.com/gsview-64548.html)
- *MatLab* (http://www.mathworks.com/products/matlab/)
- *Geometer's SketchPad*[1] (http://dynamicgeometry.com/)
- *Mathematica* (http://www.wolfram.com/)
- *GeoGebra* (http://www.geogebra.org/cms/)
- *Excel* (http://office.microsoft.com/en-us/excel/)
- *Minitab* (http://www.minitab.com/en-CA/default.aspx)
- *Fathom* (http://www.keypress.com/x5656.xml)

The programming of simulations can be time-consuming. In general, mathematics faculty program simulations for their own courses. Unfortunately these are rarely used by other faculty teaching the same course. Difficulties of compatibility are compounded when a mathematics department does not have a policy of using the same software in a given set of courses.

After working through the simulation experiments some participants would have liked to modify them to better match their philosophy of teaching and learning mathematics. Depending on the software used, and even when one has access to the code, this can be a complex problem, especially for those who are not familiar with the programming language. As an exercise to modify a simulation, group members were asked to explore the Optimization Problem (Activity 5) and to try to make modifications to the simple program to answer one of their 'What if?' questions. Participants found that, with a little help, it was relatively easy, in either *Geometer's SketchPad* or *GeoGebra*, to modify the simulation to address their questions.

QUESTION 3: *Why should simulations be used in the teaching and learning of mathematics at the university level? Given a simulation, how should it be used? When should it be used (before or after the presentation of the concept)? What support should be provided to the students?*

The group concluded that when simulation is used to explore or develop a mathematical concept, the order in which the definition and simulation takes place is dependent on the context and on the instructor's philosophy of mathematics teaching and learning. For example, the mathematical concept can be introduced first, followed by the simulation, or the students can use simulation to explore the concept before crystallizing their understanding of the concept. As specific examples, members of the group felt that simulating the Birthdays problem before it is analysed in class would be the preferred order, while in the example of the solution of certain differential equations, they would choose the reverse order. It was felt that intervention by a tutor or faculty would be beneficial and important.

The group was of the opinion that simulation procedures should be one of the mathematical tools that students have at their disposal for analysing and solving mathematical problems.

Appendix 1 provides a summary of the activities proposed to participants of the group. In Appendix 2, Jean-Philippe Villeneuve and Philippe Etchecopar describe the use of simulation in mathematics courses at the Cégep de Rimouski. In Appendix 3, Neil Marshall, a graduate

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[1] The Working Group's attention was drawn to Nickolas Jackiw's note to CMESG (see http://textsave.de/?p=27157).

of the Brock University core mathematics program MICA (Mathematics Integrated with Computer Applications), reflects on his learning of mathematics through the use of simulation. Finally in Appendix 4, Margo Kondratieva uses her experience of using simulation in teaching mathematics post-secondary and teacher education courses at the Memorial University of Newfoundland to address three questions raised in the group's discussions.

CONCLUSIONS

The working group identified a number of important conclusions:

1. Simulation is a useful tool in mathematics education at the post-secondary level, and it should be one of the approaches more frequently used in mathematics courses.
2. Simulation can assist the student's learning process in many different ways, as it provides means for exploration, visualisation, demonstration, illustration, problem solving, etc.
3. When students develop, design and implement their own simulations they work as many mathematicians do. As Neil Marshall says in Appendix 3 "Conjecturing, designing mathematical experiments, running simulations, gathering data, recognizing patterns and then drawing conclusions are things many modern mathematicians do as part of their research."
4. Future teachers should have experience with simulations as the various software packages available in their classrooms contain many applications of simulations; an example of such software is *Learning Objects*. In the classroom, teachers need to consider the same questions that were addressed by this working group, especially how to assist students to make the transition from the simulation to the mathematics.
5. There is much work being done in the development of simulation software, unfortunately the research in the use of the simulation software in mathematics education at the post-secondary level lags far behind. Some research publications were identified by the co-leaders; they were referenced in each activity and are listed in the references.

The working group's website (http://www.cegep-rimouski.qc.ca/dep/maths/?page_id=148) will continue to be available for those interested in the topic addressed by the working group.

APPENDIX 1: SIX SIMULATION ACTIVITIES

1. SIMULATION OF THE PARABOLA

<u>Why is this simulation proposed?</u>

The simulation activity has a number of components, the main ones being: an exploration of the role that parameters can play; interactive visual and graphical representations as the parameters are varied; and, a game environment.

Muller, Buteau, Ralph, and Mgombelo (2009) describe the mathematics core program at Brock University and how students learn mathematics as they design and implement Exploratory and Learning Objects that make extensive use of simulations. This activity uses one of the Learning Objects that focuses on the concept and properties of the parabola.

Activity:

Access the Brock site (http://www.brocku.ca/mathematics-science/departments-and-centres/mathematics/undergraduate-programs/mica/dept-learning-objects), choose the Parabola Games, and work through the Learning Object.

Questions:

a. What do the simulations provide that are not possible in a classroom with technological tools such as graphing calculators and computer algebra systems?
b. Can the games motivate students to ask and explore questions that mathematics curricula classify as 'too advanced' to be considered at this stage?

One of the fundamental questions provoked by this activity is:

Is there a role for simulation games in the learning of mathematics at the post secondary level and in the education of future teachers?

Research by Rieber and Noah (2008) provides a starting point for discussion on this question.

2. SIMULATION OF AN AUTONOMOUS ODE

Why is this simulation proposed?

This activity is typical of simulations that are integrated into post-secondary mathematics courses and that can be used either as an introduction or as a revision, and that contain additional problems for follow-up work.

This simulation activity is based on the work of the AIMP project at MIT. This is only one of many in the extensive library of simulation activities for mathematics courses available on this site: http://www-math.mit.edu/daimp

Activity:

Go to the MIT phase line Mathlet at http://math.mit.edu/mathlets/mathlets/phase-lines/. Click anywhere on the coloured rectangle to open an interactive page. In this new page and to the right of the small graph you will find seven different ODEs. On the bottom right you will be able to change the value of the parameter. By clicking in the Phase Line and Bifurcation Diagram boxes you will obtain their plots. See the working group website for the specific activity.

Questions:

Have you taught a differential equations course in the last ten years?

a. If yes, did you use simulation in the course? If yes, write down some of your class's experiences to share with the working group. If no, does the simulation suggest new approaches, for the teaching and learning of ODEs?
b. If no, does the simulation help you to understand how a parameter introduces a new 'dynamic' in the problem?

One of the fundamental questions provoked by this activity is:

When simulation activities are specifically designed for student use within a course, what properties should these activities possess and how and when should faculty use them?

Research by Miller and Upton (2008) provides a starting point for discussion on this question.

3. SIMULATION OF THE BIRTHDAYS PROBLEM

Why is this simulation proposed?

The advent of computers, with their ability to quickly generate large sets of data, has made simulation a natural fit in the teaching and learning of probability and statistics.

This simulation is taken from a paper by Relf and Almeida (1999). A common statement of the Birthdays Problem is: "In our class what is the probability that at least two of us will have the same birthdate?" The activity has been pre-programmed in *Excel*, using the approach suggested by the author. It is not difficult to change the program to explore 'what if?' questions.

Activity:

Open the spreadsheet entitled, "Birthdays Problem Simulation" (http://www.cegep-rimouski.qc.ca/dep/maths/?page_id=194), on the working group's website and follow the instructions given on the site.

Questions:

Have you solved this problem before using a probability calculation?

a. If yes, does the simulation bring new ideas, approaches, or questions about the problem? Write them down and, if possible, explain how these arose in your mind.
b. If no, does the simulation help you to develop a conjecture about the solution to this problem? Write it down and, if possible, explain how you developed it.

One of the fundamental questions provoked by this activity is:

For the student, the transition from a simulation activity to the development and learning of the analytical formulation can be challenging. Are there methods that are more successful than others to assist the student to make that transition?

Research by Relf and Almeida (1999) provides a starting point for discussion on this question.

4. SIMULATION OF NEWTON'S METHOD

Why is this simulation proposed?

Journey Through Calculus *(JTC) was developed by Ralph (1999) and is unique as it provides a complete package for learning calculus that is deeply rooted in simulation.*

This software is integrated into all calculus courses at Brock University.

Activity:

Anyone of the many modules could be explored; for participants of the working group, Newton's Method was suggested (see the working group website for details).

Questions:

a. The very specific order that JTC has instituted for students to work through and thereby learn a mathematical topic is reflected by the order of activities that you were asked to perform in Newton's Method. How does this order concur with, vary from, what you see as an optimum teaching and learning method?

b. In the simulations, graphs, and games, what incidences raised your interest, in the sense that you would find it challenging to provide similar experiences without technology? Can you visualize providing similar experiences with a different technology?

One of the fundamental questions provoked by this activity is:

Is there a role for simulation games in the learning of mathematics at the post-secondary level and in the education of teachers?

Research by Rieber and Noah (2008) provides a starting point for discussion on this question.

5. SIMULATION IN OPTIMIZATION

Why is this simulation proposed?

With a little experience, programming in ***Geometer's SketchPad*** *is quickly mastered, and this simulation is an example of a simulation that can be easily modified to answer 'what if?' questions.*

Activity:

Use either *Geometer's SketchPad* or *GeoGebra* to program the simulation of the problem provided by Barry McCrae (1998):

> *Kim is planning to walk from Ardale to Brushwood. The direct route, a distance of 14 km, will take her entirely through rugged bush country. However there is a large square clearing, of side length 7 km, situated as shown in Figure 1. The clearing has one corner C at the midpoint of the direct route and one diagonal along the perpendicular bisector of the direct route. Kim follows a route similar to the one shown in the figure, crossing the clearing from P to Q parallel to the direct route. One of the CAT (school-based Common Assessment Task) questions required the students to find and describe the route for which Kim's travelling time is the least, assuming she travels at an average speed of 1 km/h in the bush and 5 km/h through the clearing.* (p. 96)

Questions:

a. Does this simulation convince you that there is an optimal solution to this problem?
b. Using the simulation are you able to determine a solution to the problem?

One of the fundamental questions provoked by this activity is:

How can simulation be integrated into teaching and learning environments of post-secondary mathematics education and mathematics teacher education?

Research by Stroup (2005) provides a starting point for discussion on this question.

6. SIMULATION OF VOLTERRA'S MODEL

Why is this simulation proposed?

This simulation highlights the role of parameters.

This simulation is proposed at the Cégep de Rimouski in the first-year calculus course. In Volterra's Model, prey and predator populations, N(t) and P(t), respectively, are modelled by the following equations:

$$N'(t) = aN(t) - bN(t)P(t)$$
$$P'(t) = -cP(t) + dN(t)P(t)$$

The parameters a, b, c, and d have specific meanings, where a is the reproduction rate of prey, b is the probability of death for prey when they meet predators, c is the food needs of predators, and d is the probability of killing for predators when they meet prey.

Activity:

Euler's Method is used to explore solutions of this problem. Although *Maple* could be considered, *Excel* is used because it is available to all the students in the classes. Euler's Method is a discrete method used to solve differential equations by approximations:

$$P(t+1) = P(t) + dP, \text{ with } dP = P'(t)dt$$

Construct the simulation as detailed in the working group's website.

Questions:

a. Can we explain the impact of the variation of a parameter?
b. Are we allowed to choose any values for the parameters?

One of the fundamental questions provoked by this activity is:

How can simulation be integrated into teaching and learning environments of post-secondary mathematics education and mathematics teacher education?

Research by Stroup (2005) provides a starting point for discussion on this question.

APPENDIX 2: THE USE OF SIMULATIONS AT THE CÉGEP DE RIMOUSKI

Philippe Etchecopar and Jean-Philippe Villeneuve

A central objective of the mathematics programs at the Cégep de Rimouski is to educate our students in the scientific method and thereby enable them to:

- model and undertake a research activity;
- develop their power of reasoning and demonstrate their understanding;
- experiment and develop an algorithm;
- critically analyse a result or an argument;
- actively engage in mathematics, and explore alternative representations (graphic, numeric, analytic, algebraic, geometric);
- use technology appropriately when problem solving;
- communicate both in writing and orally.

Our students use simulation in two ways: one for solving problems, the other for learning mathematical concepts. In this Appendix we present examples of both of these uses.

1. THE MODELING-SIMULATING METHOD

This method of problem solving follows France Caron's (personal communication) reformulation of Blum's Modeling Cycle (Blum, 2002). When solving a problem, we ask our students to perform the following steps: Observation, Mathematization, Mathematical

Calculation, and Synthesis. An objective of this method is to solve a problem in as general a form as possible. One way to do this is to introduce parameters that are then used in the simulation.

To illustrate how students use simulation in modeling we present two examples.

In the first, students use simulation to explore the role of parameters. They use Verhulst's Model to approximate the evolution of the population of a species, without predators. In this model the population growth is given by:

$$\frac{dN}{dt} = r\left(\frac{K-N}{K}\right)N$$

where N is the size of the population, K, the maximum size, r, the reproduction rate, and t the time.

This model describes the population growth over time and can be solved analytically (in *Maple*) or discretely with Euler's Method. We recall that Euler's Method enables us to approximate a solution by the function:

$$f(x_1) = f(x_0) + dy = f(x_0) + f'(x_0)dx.$$

A graphical representation is shown in Figure 1.

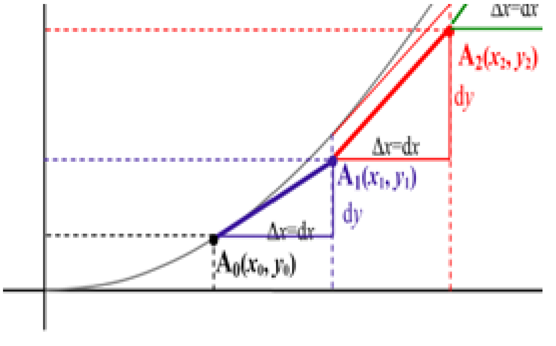


Figure 1

The implementation in *Excel* is similar to the one described in Simulation 6 of Appendix 1. Once implemented, students use the simulation to explore how changes in the parameters impact the model.

In the second example students use trial and error, within a simulation, to determine one or more values of a parameter in order that the resulting motion of a spring will perform in a prescribed manner. The second order differential equation for a spring had been previously developed in class and in the application the spring needed to be manufactured so as to meet certain specifications. In this simulation students find visual representations to be very helpful. For example the graphs, in Figure 2, for three different values of a parameter, b, provide insights into the behaviour of the spring, namely oscillations, damping, etc.

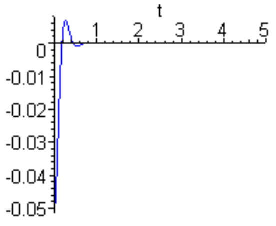
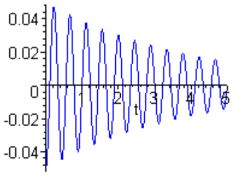
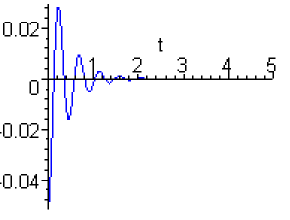
b = 30 000	*b* = 1 000	*b* = 10 000
		

Figure 2

2. SIMULATIONS WITH *MAPLE*

We now provide examples of how our students use simulations to learn mathematical concepts. In all cases the simulations have been done with *Maple*, some with the interactive components of *Maple*. In all these examples the simulations are pre-programmed.

<u>Simulations can be used to find a general formula</u>

After calculating the derivative of simple functions using the definition of the derivative, our aim is to introduce the derivative formulas. As part of this development we ask our students to choose values of the power of x to find the general formula of the derivative of x^n. The pre-programmed format is shown in Figure 3. The simulation enables the student to find a pattern in the derivatives calculated by *Maple*. The aim is for the student to come up with the right formula: nx^{n-1}. In this simulation students should experiment with different values of n: integer (positive and negative), rational, and irrational. These trials are part of the 'what if?' questions that can be generated by a simulation.

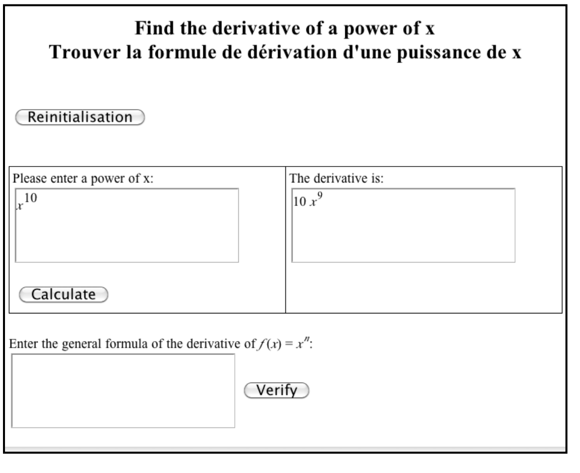


Figure 3

<u>Simulations can be used to give a geometrical interpretation of an analytic concept</u>

The pre-programmed environment for this example is shown in Figure 4. A student enters a function and an interval and is then able to use a cursor to move the secant. This simulation provides a geometrical interpretation of the derivative and helps the student visualize how the secant goes to the tangent as do their slopes.

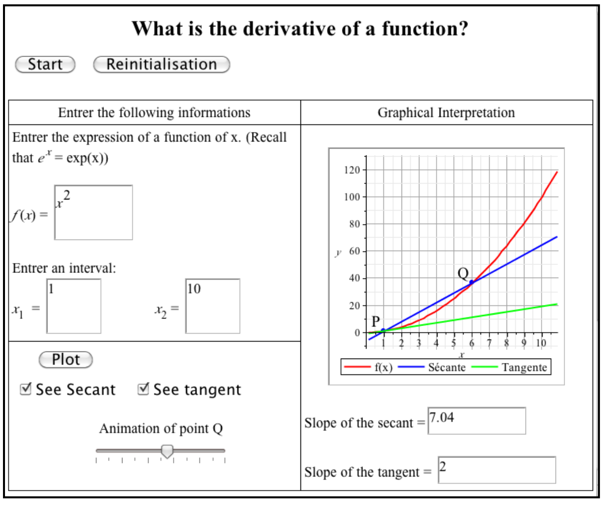


Figure 4

<u>Simulations can be used to develop a proof</u>

There are tutorials available in *Maple*. Examples are “Approximation Integral”, “Derivative”, “Differentiation Method”, and “Limit Method”. Our students use the last one to calculate the limit of a function using limit properties. Students enter a function and an *x* value, and then click on the different properties to determine the limit.

Hints are available, as well as the answer.

Sometimes students have difficulties following the steps used by *Maple* in solving a limit. Nevertheless, our students gain much insight using this simulation as they explore when and why they can replace *x* by its value and what to do otherwise. Using this process they are able to develop a proof.

APPENDIX 3: SIMULATION AND BROCK UNIVERSITY’S MICA PROGRAM—REFLECTIONS OF A GRADUATE

Neil Marshall

Brock University is home to an innovative and dynamic technology-enhanced undergraduate mathematics program, Mathematics Integrated with Computers and Applications (MICA). Technology is integrated throughout the entire program, with students being introduced to Maple in their introductory first-year linear algebra and calculus courses. At the core of this program are three[2] project-based courses that focus on mathematical exploration and simulation using technology. As a graduate of the MICA program (2006-2010), it is my hope that I can provide insight from a student’s perspective on how simulation is used at Brock by

[2] The program has been revised slightly since the author graduated, with the third-year MICA course being split into two courses and being optional for those pursuing a concentration in pure mathematics.

students to learn and do mathematics. For more information on the MICA program see Buteau and Muller (2006), Pead, Ralph, and Muller (2007) and Muller et al. (2009).

OVERVIEW OF THE PROGRAM

MICA I, the first course in the series, is a half-credit course that students typically take in the Winter term of their first year. There are two hours of lecture per week and two hours in a computer lab. The lectures mainly focus on mathematical theory used to structure the explorations conducted in lab and on assignments. Lab time is used to introduce the *Visual Basic* programming language and conduct investigations. The student population consists of mainly mathematics majors and concurrent education students aiming to be mathematics teachers. Topics visited in the first course include investigations on prime numbers and the hailstone sequence, RSA encryption, and exploration of the logistical function through a cobweb diagram. The course culminates in a final project where students chose their own topic and construct, depending on their academic goals, an exploratory object, a real-world simulation or a learning object for school-level students.

The second course, MICA II, is a full-year course, typically taken in a student's second year. The student population is mainly mathematics majors and concurrent education students. Both *Maple* and *Visual Basic* are used in this course. There are two projects, one at the end of each term. Topics covered include pseudo-randomness and chaos, Monte Carlo integration, Lotka-Volterra population dynamics and the Hénon map. The third MICA course is typically taken in a student's third year. It is an applied mathematics course, focussing on numerically solving partial differential equations using C++ in the Fall term and exploring the heat equation using *Maple* in the Winter term. This course is primarily taken by mathematics majors.

THE MICA SIMULATION PHILOSOPHY

Often simulation in the undergraduate mathematics setting is based on interactive objects designed by instructors or purchased commercially, where students can investigate a specific mathematical concept by changing parameters (see Chance & Rossman (2006) and Chae & Tall (2001) as examples). MICA students, however, are expected to build and modify the simulations themselves, with some instructor and teaching assistant support. The students themselves are in charge, controlling nearly every aspect of the simulation. Parameters cannot only be changed, but new parameters added and new mathematical questions formulated. Many first year students are uneasy at the open-ended nature of the first assignment in MICA I. Mathematics is liked by more than a few undergraduates, not least because there exist 'right' answers. Open-ended questions, conjectures and conducting mathematical experiments to gather data can seem distant to a student's conception of what doing mathematics is, no matter how relevant they are to modern mathematics. However by the end of the course, the final projects[3] constructed from scratch by the students contain a wide range of creative and innovative approaches to diverse areas of mathematics. Personally, I found a sense of accomplishment and pride in the creation and exploration of such simulations. Students identify strongly with such creations and take ownership of them that they wouldn't necessarily experience with solving a calculus or linear algebra problem.

A PERSONAL JOURNEY THROUGH ONE EXPLORATORY OBJECT

Throughout my undergraduate career, I developed several simulation objects including an exploration of vibrations in a one-dimensional atomic lattice and modelling the heat equation in a circular wire. To best illustrate the MICA process, I thought it best to discuss in detail one

[3] See http://www.brocku.ca/mathematics/studentprojects for examples of MICA student projects.

project that I am particularly proud of: Encryption Pseudorandomness Explorer, my first-year MICA final project.

One of the core topics in the first MICA course is RSA encryption. Students are expected in the second assignment to construct an application that encrypts and decrypts a number using public key RSA encryption and also generates these keys. As a first-year student, the obvious question that occurred to me was “how good was this encryption method?” I remembered that early code breakers had used the uneven distribution of letters in the English language as a means to break early encryption methods. I wondered how truly random a string of letters encrypted by RSA encryption using the means developed in class would appear.

The goal was to develop a mathematical tool that I could use to measure how random the character distribution of an encrypted body of text appeared to be. My goal was to combine the various tools I had built for each of the three assignments as a basis for the project, as well as implement new ideas.

The MICA approach can be extremely empowering for students. I had full control over my mathematical investigation. I designed the tool, chose which initial parameters I would use and how the data would be outputted, and tested my creations. As I tested and explored, I improved the tool that I had created, adding encryption algorithms to be used and parameters that could be tested. I started with a basic exploration of RSA encryption using two- and three-digit primes, adding Hill Cipher, varying the Hill Cipher matrix size, creating a feature that allowed for different input text to be encrypted to be compared with the same encryption key values, and improving the interface and output. I controlled the entire mathematical activity, from the initial idea to the actual exploratory task itself, attempting to reconcile the results of my exploration with the mathematical knowledge I had.

In my particular exploration, I was surprised that the Hill Cipher seemed to produce a more randomly distributed distribution of encrypted characters than RSA, even merely using a 2×2 matrix as the encryption key. To get a similar result with RSA, one had to use highly computationally- and time-expensive four-digit primes as a basis for the public and private keys. I was very excited when I realized that one major difference between Hill and RSA encryption is that the former is a private key encryption scheme, while the latter has a public key. I recognized that public key encryption is much more practical in large-scale use than private, and thus realised that there might be a trade-off for being more useful in commercial endeavours. I also realized that I had underestimated, not for the first nor last time, the complexity of matrix multiplication.

LOOKING BACK: REFLECTIONS OF A STUDENT AND A TEACHING ASSISTANT

I was an above average student in MICA. While we have had many exceptional students who have taken the courses, the experience is not limited merely to the very best of students. Technology allows our students to approach their mathematical questions from many different directions, and the instructors and TAs are there to guide the students and provide some assistance with coding, at the price of being very labour intensive even for the small class sizes. However, the payoff for such efforts is to be always surprised at the variety and creativity of student final projects. Speaking as someone who has not only been through the program, but also TAed for the first- and third-year courses, I am always amazed by the ideas and approaches by students, no matter what their mathematical and programming abilities are.

Consistency of technology is a key part of the MICA program. Since there is a commitment by the department to use technology where appropriate throughout a student’s entire undergraduate career, there has been a lot of thought put into what technologies are used where. Students are introduced to the computer algebra system *Maple* in their introductory linear algebra and calculus courses and use it in many courses throughout their undergrad. They are also introduced to and taught *Visual Basic.Net* in the first MICA course, which provides them the programming foundation to investigate more advanced topics in the second-year MICA II. This consistency saves valuable class time for instructors who can rely on students' previous programming knowledge, which is essential in making a multi-year MICA-like approach feasible.

One really nice feature of the *Visual Studio* development environment is that the creation of vivid and unique user interfaces is possible for even the weakest of students. This allows our students to personalize their assignments and projects, using colours, fonts and images. Students may thus have a personal connection to the objects they create, and may see it not as merely something they have to hand in, but as something that they made their own. You simply cannot have such an environment for all students with traditional assignments and hand calculations, and from personal experience it can make marking a much more enjoyable experience.

FINAL REMARKS

As with many things in life, it was only blind luck that I ended up at Brock University in the MICA program. I had studied Computer Science at the University of Waterloo, but became ill and had to leave school. It was several years (and several tries later) that I ended up in the Brock Department of Mathematics, mostly because it was close to home. I have often thought of how lucky I was to have such misfortune, because it introduced me to something that I view as innovative, inspirational, and most importantly, fun. Though I have always been good at simplifying equations, deriving proofs and using mathematical algorithms, through MICA I experienced something beyond traditional cookie-cutter hand calculations. I was introduced to modern mathematics such as encryption, population dynamics and mathematical models of complex physical behaviour. Not only that, but I was encouraged to explore and investigate these topics by designing and implementing my own tools to do so.

Mathematics is alive, vibrant and an integral part of our modern world. Often though, students have a misconception that mathematics essentially hasn't changed since the days of Fermat. Conjecturing, designing mathematical experiments, running simulations, gathering data, recognizing patterns and then drawing conclusions are things many modern mathematicians do as part of their research. MICA allowed me to see myself as doing mathematics in my own right and I discovered how much I enjoyed doing so. Not only did it teach me how to use technology as an effective problem-solving tool that I can use in appropriate situations, but it helped empower me to continue my studies beyond an undergraduate degree.

I am proud to be a graduate of such an innovative and unique program. It is my hope that by sharing my experiences, I might help foster new ideas and implementations that empower future students to 'do mathematics' through simulation, investigation and technology.

APPENDIX 4: USING SIMULATION TO DEVELOP STUDENTS' MATHEMATICAL COMPETENCIES—POST-SECONDARY AND TEACHER EDUCATION

Margo Kondratieva

I am teaching a course in Euclidean geometry at the University of Newfoundland. The course is taught in a traditional format with chalk-board lectures and handwritten assignments and tests. However, when I offer this course, my students also have an opportunity to use a dynamic geometry environment (DGE). There are no compulsory assignments to use simulations, but the students are given many examples of doing so. Some applets made in *GeoGebra* are used during lectures and are posted on the course website for students' possible use. A tutorial on how to work with *GeoGebra* helps students to create their own dynamic and static drawings. Such drawings can replace traditional drawings made on paper with a ruler and compass when students work on their assignments.

Although DGE is used in this course only for the second year and no extensive statistics are available yet, there are some observations that I am ready to share. I would like to speak to the following three questions, raised during the CMESG/GCEDM meeting.

QUESTION 1: *What is the role of simulation in the development of mathematical competencies at the post secondary level?*

Building dynamic drawings in *GeoGebra* and observing their behaviour helps my students to read the textbook and to understand better exactly what the statements are saying. When a drawing confirms that students' interpretation of a statement is valid, students report having greater confidence in what they are doing. It also gives more meaning to their actions and thus contributes in the development of their competency. With simulations related to certain problems or theorems, students have an opportunity to consider special cases and unite different particular cases by dragging. This helps them to see how ideas work across cases and how some of them can be transferred from a particular to a more general case. For example, some proofs found in the case of an acute triangle can be extended to the obtuse case.

Students use the 'Trace' function available in *GeoGebra* to better understand the idea of a locus of points with a given property. Seeing a locus drawn in a DGE sometimes gives then an insight into the explanation of the observed phenomenon. At the same time I found that in the majority of cases, simulations, as such, neither help students to generate proofs nor do they develop the need for a proof. Special care is required in order to move learners in the direction of rigorous thinking based on geometric simulations. Some researchers put forward the idea that "soft constructions" (Healy, 2000) and "maintaining dragging" (Baccaglini-Frank, & Mariotti, 2010) may help students to bridge the worlds of experimental and theoretical geometries. In my own practice, I emphasize 'basic geometric configurations', that is, drawing of basic geometric facts that contain elements of their proof. Enhanced by dynamic features in a DGE, such configurations assist in breaking complex geometrical drawings into more manageable parts and thus help to develop steps of a proof (Kondratieva, 2011).

QUESTION 2: *In pre-service and in-service courses, what simulation experiences and reflections are important for teachers of mathematics?*

Besides Euclidean geometry I also teach courses in mathematics education for pre-service and in-service secondary school teachers. Some of my students of geometry end up in my math education classes. Again with no statistical confirmation, I observe that students who had previous exposure to a DGE have a better understanding and are more responsive to the inquiry-based approach, which is suggested as an important component by recent documents regarding the secondary school mathematics curriculum.

If we expect our math teachers to help their students to mimic a scientific process, that is, to plan and carry out investigations, observe, conjecture, explain, question, assess and modify their actions if necessary, we need to ensure that the teachers themselves have gone through an appropriate training and are familiar first-hand with the process of scientific explorations. In this respect, I believe that simulations that give teachers an experience of mathematical discovery (even at a 'small scale' of an elementary but engaging project in planar geometry or number theory) are 'a must component' of a teacher education program.

QUESTION 3: *When students develop and program their own simulation activities, what mathematical and other competencies are necessary for them to succeed?*

Until they start to create their own applets in a DGE many students actually do not realize that they need to use geometrical knowledge and integrate the constraints specified in a problem or theorem. Precisely because it helps students to activate and apply their knowledge, creating a simulation in a DGE is an important and sometimes very challenging exercise in geometry. Students experience implicit learning while making robust constructions in *GeoGebra*. Some students do not feel that making applets helps them directly with finding a solution to a problem, but they report that it gives them a sense of accomplishment and satisfaction to a certain extent. I concur that asking students to reproduce drawings that preserve certain properties (or exhibit a certain behaviour) under dragging is an enriching activity that extends standard tasks on constructions with ruler and compass (see also the discussion in Laborde, 1998, p. 118).

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MAKING ART, DOING MATHEMATICS
CRÉER DE L'ART; FAIRE DES MATHS

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Tara Taylor, *St Francis Xavier University*

PARTICIPANTS

Ruth Beatty	Julie Long	Elaine Simmt
Valeen Chow	Chris Palmer	Kevin Wells
Michelle Davidson		

Figure 1. The participants, deeply engaged.
Right-to-left: Chris Palmer, Julie Long, Ruth Beatty, Michelle Davidson, Valeen Chow, Kevin Wells, Elaine Simmt

ABSTRACT

The connections between art and mathematics have a long tradition. This dates back to the time when knowledge disciplines were not as clearly segregated (as for example, the development of the laws of perspective during the Renaissance). In more recent times, the connections have been maintained both from the perspective of mathematicians who create aesthetically pleasing representations of their ideas, and from the perspective of artists making

La connexion entre l'art et les mathématiques a une tradition millénaire, datant d'une époque où les disciplines du savoir n'étaient pas aussi distinctes (prenons par exemple le développement de la perspective pendant la renaissance). À une époque plus récente, cette connexion a été maintenue aussi bien par la vision de mathématiciens qui créent des représentations esthétiquement satisfaisantes de leurs idées, que par des artistes faisant explicitement usage de

explicit use of mathematical concepts in their work. In this working group, we expect most participants to come with a primarily mathematical perspective and background. Thus we choose to take the antithetical position, and approach the connection from the point of view of artists. This connection can take a variety of forms. For example, and as members of the Concrete Movement believed, art should:

> *emerge from its own means and rules, without having to call upon external natural phenomena… By the act of modeling, art works take on a concrete form, they are translated from their mental form into reality; they become objects, with a visual and spiritual use.*[1] (Albrecht & Koella, 1982)

In consequence, "released from its attachments to natural phenomena and bound to natural laws, this art gives the feeling and shaping mind, the creative imagination, the greatest possible freedom" (Rotzler, 1989, p. 142). In the working group, we will explore and experience this freedom, focusing particularly on the ways in which mathematics can be integrated into the process of creating art. The three main (non-exclusive) ways are: the mathematics can simply be a tool for the creation of art, it can be the subject of the art piece, or it can be the source of inspiration. The focus of the working group is on mathematics as subject or inspiration for the creation of art.

concepts mathématiques dans leurs œuvres. Dans ce groupe de travail, nous anticipons que la plupart des participants se positionnent du point de vue en premier lieu mathématique. Nous optons donc pour la position antithétique et approchons la problématique du point de vue des artistes. Cette connexion peut prendre plusieurs formes. Par exemple, et comme le croyaient les membres du Mouvement Concret, l'art devrait :

> *[naître] de ses propres règles et moyens sans devoir les dériver ou les emprunter de phénomènes extérieurs naturels... Par l'acte de modélisation, les œuvres prennent une forme concrète, ils sont traduits de leur existence mentale en une réalité, ils deviennent des objets, avec un usage optique et spirituel.*[1] (Albrecht & Koella, 1982)

En conséquence, et « libéré de ses attachements à des phénomènes naturels et directement lié aux lois naturelles, cet art donne à l'esprit sentant et comprenant et à l'imagination créative, le plus de liberté possible » (Rotzler, 1989, page 142). Dans le groupe de travail, nous allons explorer et faire l'expérience de cette liberté, tout en convergeant notre attention particulièrement sur les moyens d'intégrer les mathématiques dans le processus de création d'œuvres d'art. Ces moyens peuvent prendre trois formes principales : les mathématiques peuvent simplement servir d'outil pour la création, elles peuvent être le sujet d'œuvres, ou elles peuvent servir de source d'inspiration.

THE WORKING GROUP'S ACTIVITIES

INTRODUCTION

Activities began with the letters game shown in Figure 2, below. This somewhat divergent undertaking sparked a fruitful discussion both regarding the exercise itself and regarding the pedagogy of such exercises. It was concluded that instead of asking where the missing letters do belong it would be more beneficial to ask the more subtle question of where they could belong. This proved to be a good introduction to the working group's activities because it raised the question of the contrast, frequently made between mathematics and art, that there is

[1] "*aus ihren eigenen Mitteln und Gesetzen entsteht, ohne diese aus äusseren Naturersheinungen ableiten oder entlehnen zu müssen.[...] Durch die Formung nehmen die entstehenden Werke konkrete Form an, sie warden aus ihrer rein geistigen Existenz in Tatsache umgesetzt, sie werden zu Gegenstände, zu optischen und geistigen Gebrauchsgegenstände.*" (Translation/traduction: Eva Knoll)

a single valid answer in the former and many in the latter discipline. (For an answer to the letters game question, please flip to the end of the report).

A EF HI KL N T VW YZ

B D G J OPQ S U

Where do C, M, R and X belong?

Figure 2. The Letters Game.

AN ART-HISTORICAL CONTEXT FOR THE WORKING GROUP

In the first half of the twentieth century, art movements were formed, including Constructivist and, in particular, Concrete Art, whose proponents claimed, amongst other things, that the subject of a work of art is the work itself. That is, art is not trying to represent any other object but itself. Examples of artists that worked in this way include Wassily Kandinsky, Theo van Doesburg, and Max Bill. More specifically in the case of Concrete Art, the means of creation include colour, space, light and movement. This allows for abstract ideas to be made concrete in visual form. To gain a better understanding of the perspective of Concrete artists, we discussed two manifesti while looking at specific examples of Concrete Art.

Some of the tenets that we discussed included:

- The work of art must be entirely conceived and formed by the mind before its execution. It must receive nothing from nature's given forms, or from sensuality, or sentimentality. We wish to exclude lyricism, dramatism, symbolism, etc.
- The picture must be entirely constructed from purely plastic elements, that is, planes and colours. A pictorial element has no other meaning than 'itself' and thus the picture has no other meaning other than 'itself'.
- "The construction of the picture, as well as its elements, must be simple and visually controllable" (van Doesburg, 1930, p. 1).
- "All the arts derive from the same and unique root" (Kandinsky, 1938, as cited in Caws, p. 520).

This discussion raised a marked resistance, on the part of some participants, who could not reconcile their reactions to the works with the claims of the manifesti. For example, the idea that the art receives nothing from sensuality was considered invalid because the art that was shown did provoke sensual responses in the observing participants. One participant described a specific piece as giving comfort because it gave an impression of being sheltering.

In order to gain a clearer, more authentic understanding of the artist's intentions, the group chose to attempt to avoid a 'presentist' perspective by trying to re-think the meaning of some of the terms used, within the context of the time. For example, the term 'mechanical', in the electronic age, suggests a different concept from what it would have been at the time of the manifesti.

This initial discussion led the group to a narrower conversation, focusing on the mathematical meaning as a possible layer in artwork. If the other layers are conceptually removed, a viewer is left with the underlying structure, which often has a strong mathematical element. This gave the group a chance to focus on the mathematical and aesthetic decisions involved in making art in general and Concrete Art in particular.

For this working group, the focus was deliberately placed on the perspective of the artist: the play, the creation, the open-endedness, and the unpredictability. We used mathematics to guide artistic choices and celebrated the aesthetic qualities of mathematics by playing with its structures and elements. Towards the end of his life, Max Bill explained the mathematical ways of thinking that he applied to his visual art:

> *In every work of art the basis of its composition is geometry…the means of determining the mutual relationship of its component parts either on plane or in space.*
>
> *As the artist has to forge into unity, his vision vouchsafes him a synthesis of what he sees which, though essential to his art, may not be necessarily mathematically accurate. This leads to shifting or blurring of boundaries where clear lines of division would be supposed. Hence abstract conceptions assume concrete and visible shape, and so become perceptible to our emotions.*
>
> *The art in question can, perhaps, be best defined as the building up of significant patterns from the ever-changing relations, rhythms and proportions of abstract forms…it presents some analogy to mathematics itself.* (1993, pp. 5-9)

In discussing and collaboratively constructing a piece of Concrete Art, the Working Group found that an emerging key theme was the way in which the mathematics and the aesthetics each contribute to guiding the successive decisions.

RETELLING THE PROCESS

The next stage of the working group was for the participants to experience the process of a collection of art pieces created by a contemporary Concrete Artist, the co-leader Eva Knoll.

The group spent some time watching as the artist walked through the steps involved in designing and creating a series of related art pieces, including the one in Figure 3 below. This gave further insight into the balance of mathematical and aesthetic decisions involved in the process.

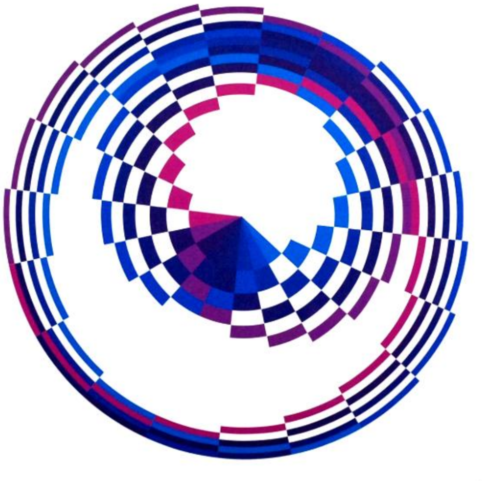

Figure 3. Circular Colourwave Base 20 - Step 8 (Ø 48”, 2008).

EXPERIENCING THE PROCESS COLLECTIVELY

The process of creating mathematical art is highly individual, and is greatly influenced by the artist's personal experience with mathematical ideas and concepts. This involves an individual artist's self-awareness in terms of the aesthetic appreciation of various concepts and ideas (mathematical or otherwise).

In this phase, the working group used some concepts and ideas that Eva Knoll had been exploring from just this perspective, and worked on a combination of tasks that involved getting acquainted with the concepts for themselves and discussing design decisions made both in advance by the working-group leaders (as anticipated by the manifesto) and later, as a group, based in part on how the pieces turned out at various stages. The purpose of this phase was to habituate the participants to think in terms of constraints and design decisions, both on their own terms and in terms of the opportunities they created to think about the aesthetics of mathematics and the mathematics of aesthetics.

The chosen starting point was the Hilbert space-filling curve. This curve was a good choice because of its aesthetic interest (its symmetry is not too simple or obvious). Another reason was because it is relatively easy to step up or down the complexity of the piece by increasing or decreasing the number of fractal iterations used, thereby finding a good middle ground between too complex (in which case the structure is submerged by the noise), or too simple (in which case it does not retain the viewer's interest). The process for this part of the working group session was deliberately designed to begin in a more constrained way, whereby the leaders gave the initial design constraints, and later, when they felt more confident, the participants were asked to contribute decisions. These decisions were also discussed in terms of their mathematical, aesthetic, or hybrid nature.

At first the participants developed their own individual understanding of the Hilbert curve by drawing successive iterations, thereby getting a 'feel' for its meandering path and the relationship between successive iterations.

This can be done in various ways. In the illustration below (see Figure 4), the upside-down "U" of iteration 1 (I-1) is replaced by the "Chalice" of iteration 2 (I-2), in the correct orientation to preserve the positions of the entry and exit points; then they are joined up. For I-3, again, the "U" is replaced by a copy of the "Chalice". The group remarked that going from I-1 to I-2 is the trickiest step because there is nothing to compare back to. One important aspect to notice is that in any given iteration, the instances of previous iterations (at any level) that touch the top edge are in the same orientation as the overall curve.

Some of the difficulties of this step derived from a property of the Hilbert Curve that distinguishes it from other fractals such as the Koch Snowflake. In the latter, each iteration proceeds by substituting various sections of the curve with copies of the whole curve from the previous iteration, with the start and end points not moving. In the Hilbert Curve, the start and end points drift closer and closer to the corners of the square that is being filled as the resolution increases with successive iterations. This difference took some getting used to as participants who were familiar with fractals like the Koch Snowflake were looking for a simple substitution rule. Different conceptual approaches were proposed to the participants struggling to understand the curve. A fair bit of time was needed for the participants to understand how to make sense of the curve for themselves.

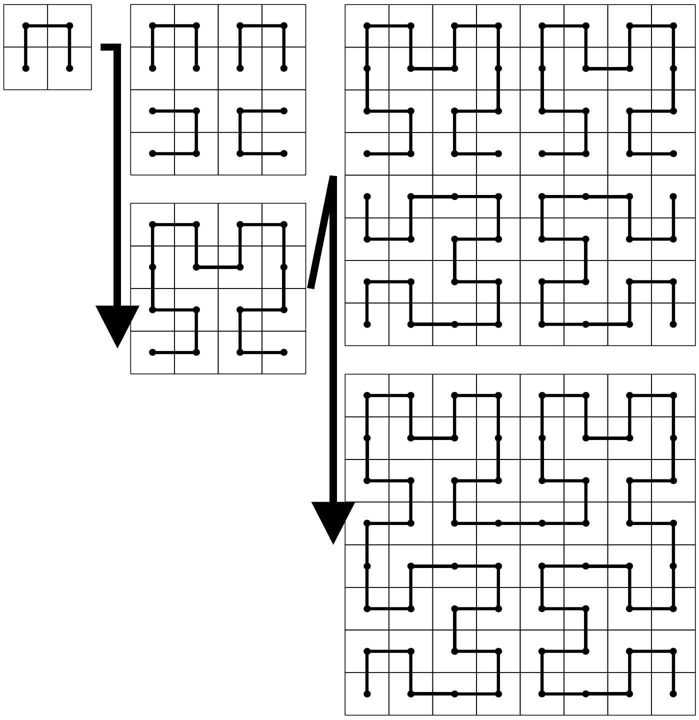

Figure 4. The first 3 iterations of the Hilbert Curve.

The next step in the process was to play with the curve to see if an idea or concept surfaced that could be combined with the curve:

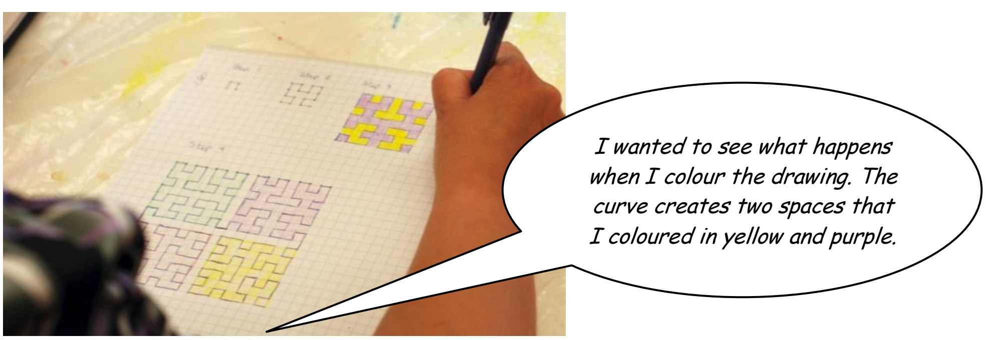

Figure 5. Playing with the curve.

The Hilbert curve fits into a square region and always connects the centres of squares on a grid of dimensions $2^n \times 2^n$ or 4^n squares, where n refers to the iteration. For the early stage where an artist is exploring what could be done with the pattern, I-3 (shown in the Figure 5 photo, top right) has a good complexity level, with its 64 unit squares.

At the leaders' instigation, the group generated 64-digit numbers by each writing an 8-digit number of their choice on a paper, then eight times picking one paper out of the pile and writing down the numbers sequentially. There were eight 8-digit numbers to select from, and it was possible to select the same number more than once.

At the leaders' instigation, again, the basic I-3 drawing was then combined with each participant's number as follows: the number is 'transposed' onto the square by following the path of the Hilbert Curve, sequentially, and by reinterpreting its digits according to completely determined rules: replace the individual digits by one of two square elements, each of which connects the four midpoints of the edges of the square, depending if the digit is even or odd (0 or 1 mod 2), as shown below.

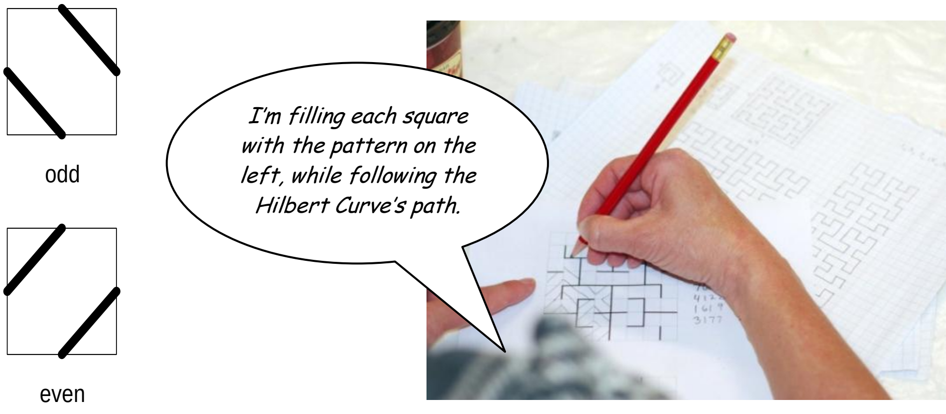

Figure 6. Integrating the number sequence.

Using a randomly generated 64-digit number is not entirely satisfying in terms of mathematical or aesthetic decisions if the aim is to bring out the harmonious, aesthetic structure and nature of mathematics. A suggestion was made to use, instead, prime numbers, which have aesthetic appeal for many mathematicians. Another option that was discussed was to use a number in its binary form, since it would then have only 1s and 0s, which are ready to be transposed into the two options above. A 64-binary-digit number lies between 9 223 372 036 854 775 808 and 18 446 744 073 709 551 616, i.e. between 9 and 18 quintillions. This seemed less practical, since an internet website yielded a huge list of 64-digit prime numbers, ready for the participants to select numbers for their new version.

It is at this point, also, that the group began working at a larger scale, both in terms of the 64-digit panels and in terms of the assembling of a larger piece. Because the Hilbert Curves can be concatenated to produce further, larger iterations, the group worked towards making 16 (2^{2n}) squares, about two per participant, making the assembled piece that is based on I-5.

The first step was to use specially prepared panels that were punched at each intersection of a 16 × 16 grid (that included the mid-points of edges for a more accurate tracing of the odd and even pattern).

This work took some time, which allowed for conversations about the choices that were made. For example, it is curious to note that in a given panel, lines never intersect or meet more than in pairs. Instead, all lines either close or start and end on an edge. Two colours are therefore always sufficient to colour the resulting drawing so that no two adjacent regions (sharing an edge) are of the same colour. In addition, it is always possible to colour a panel in such a way that the corners are all the same colour and panels can therefore always be joined so that the colouring continues across the edge.

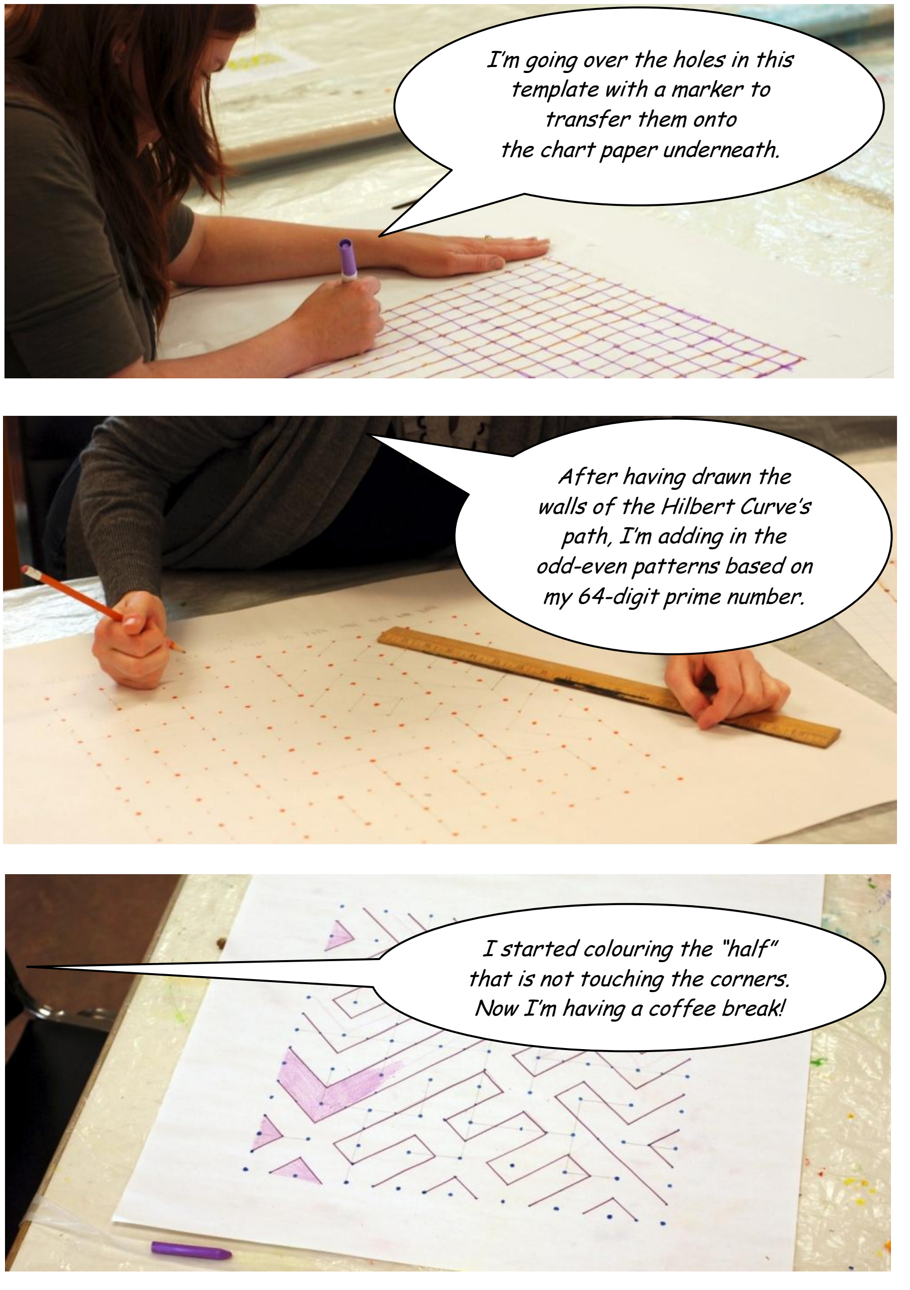

Figure 7. Making the larger squares.

The next step was to assemble the panels. There were of course a multitude of options on how to do this. Combinatorics tells us that if we count each panel's four possible orientations as distinct choices, there is a very large number of ways in which the panels can be combined. There is, however, an underlying structure to a Hilbert Curve, and the orientation of a panel is dictated by its position in the larger square. As noted previously and shown in the diagram (Figure 8), for example, in I-5, the I-3 modules of the first row all face the same way.

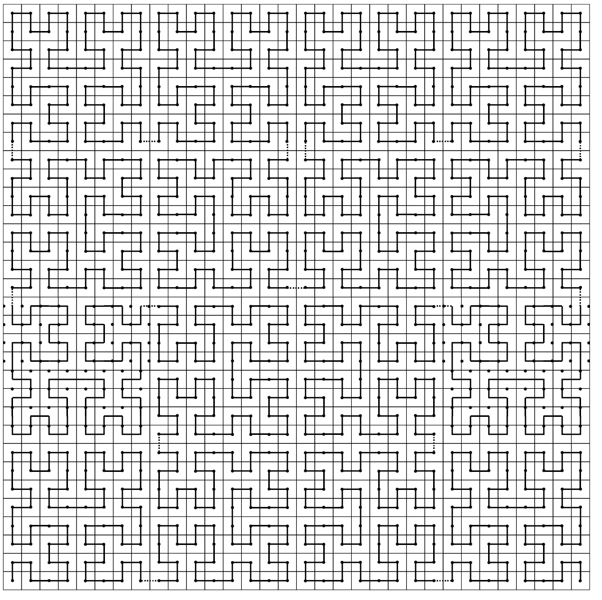

Figure 8. An I-5 Hilbert Curve.

The question, in this step, was to choose either the position or the orientation of each panel as the determining factor, since the two are linked. The group chose to consider position first and let it dictate orientation. Each panel had been coloured by its creator using a colour entirely of her/his choosing. The resulting colours were: two black, four different blues, gold, green, lilac, two orange, pink, red, teal, pale violet, and yellow. In a piece of art that was discussed previously in the working group, colour, and particularly the relative position of colours on the spectrum, were discussed as design elements, less in terms of their symbolism or the possibilities of harmonious or clashing juxtapositions than in terms of the information that can be encoded in the choices in terms of their distances along the spectrum. The eye slides more readily from colour to colour when these are near neighbours on the colour wheel. Excluding the two black panels, all the others could be placed in spectral order and this very order could be used to show the path travelled by the Hilbert Curve from panel to panel (i.e. at an I-2 level). But the spectrum is circular, which means that the group had to choose where to break the sequence.

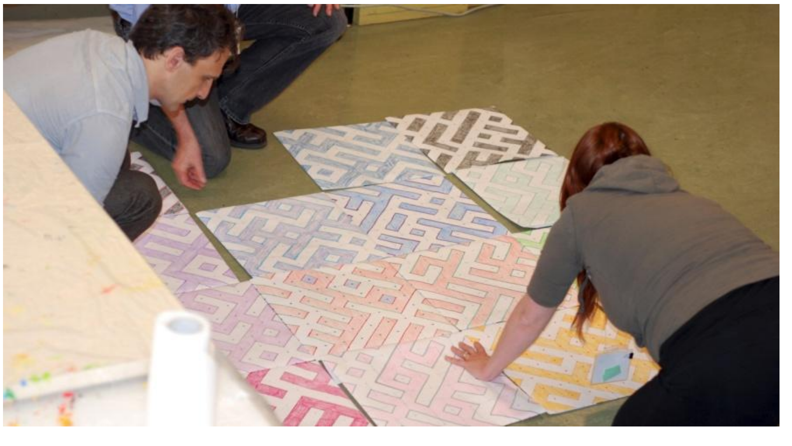

Figure 9. How do we assemble the larger piece?

The choice of the break in the spectrum was determined by two factors. Firstly, the curve passed near itself and where it did so, the two strands needed to be different enough that the eye would not jump across them. Secondly, the colours were neither unique (e.g. there were two orange), nor were they all equally spaced around the wheel (there were four blues that were very close together and there was a bigger distance between the yellow and the green). The solution involved using the two black panels as the start and end, then following the spectrum from yellow to green via all the remaining colours, *in spectrum order*.

Figure 10. The finished piece.

Following the spectrum along from (black to) yellow to gold, to orange etc., traces the 'chalice' that is the basic pattern of I-2. In addition, within each square, there is an I-3 path, so that overall, the underlying Hilbert Curve is an I-5.

Some mathematical questions were raised throughout the process that were not answered:

1. How does the number of odd digits compare with the number of even digits (in the case of the primes)? How does that affect the drawing?
2. Why are all the corners the same colour?
3. Why are two colours enough?
4. How does one decide how to orient the squares in the scaled up version?

Readers are invited to attempt these themselves.

MAKING OUR OWN ART; DOING OUR OWN MATH

In the final stage, a set of stations were laid out that showed starting points to math-art projects. Each participant chose one of the stations, with little hesitation, and spent the remaining time working on their individual ideas based on the station and the reaction of their internal mathematical artist, with its self-awareness and personal history.

There were 5 stations:

1. **Tartans and Peyote Stitch** — Tartans are made using the Twill pattern (similar to 2×1 herringbone tiling) and the Peyote Stitch is a beading method that creates a brick-row pattern. Both are topologically related to the hexagonal grid and so patterns can be transferred between them (Knoll, 2009).
2. **Paper-weaving polyominoes** — Using paper strips, it is possible to weave a multitude of designs by using one colour for each direction. What polyominoes can be woven into a monohedral tiling (only one type of tile) so that the same-coloured tiles don't touch along edges? (Knoll & Landry, 2011)
3. **Scaling the maze** — Taking the previous design with the Hilbert Curve to the next level, how can we create a life-sized maze in which people can walk around?
4. **More space-filling curves** — The Hilbert curve is only the first of many. What designs can be made with some of the others (Sagan, 1994)?
5. **Examples of Concrete Art** — A series of slides presented more examples of Concrete Art for more inspiration.

There was interest in specific projects that involved specific craft techniques, personal meanings (doing a tartan for one's clan), fractals, and technology.

> *I was immediately inspired by the work of the artist Max Bill and the possibilities of taking a highly organized geometric pattern and reducing its visibility to a functional aesthetic minimum…For my piece I chose to use the Geometer's Sketchpad to create a design. The starting idea was to divide a circle into sections based on a simple 'halving' principle of lines and angles. Once the grid was sufficiently dense, possibilities to select from the sections created began to suggest themselves. The underlying scaffold was 'hidden' to leave what is hopefully a work of concrete art which has both tension and visual appeal. I see the process as being interesting to students as well as highlighting how much of what we see has an underlying mathematical structure.* – Participant Kevin Wells

REFLECTING ON THE EXPERIENCE

While the participants worked on their own projects, there was much lively and interesting discussion about the process in general. One of the important themes that emerged was suggested by Elaine Simmt: the concept of using liberating constraints while teaching. The term 'liberating constraints' is used to draw a distinction between tasks that are proscriptive and those that are prescriptive (Davis & Simmt, 2003). Prescriptive tasks range from those that are too narrow to allow for varied interpretations to tasks that are too open to encourage focused interpretations. Thus it is useful to provide liberating constraints to allow for maximum benefit from learning activities. The discussion around liberating constraints in the contexts of creating art and in doing mathematics was a highlight of the working group.

There were various constraints that arose for specific projects, and this led to the necessity of making choices. There was not a clear dichotomy between aesthetic and mathematical preferences but rather a spectrum. Both facets contributed to the decision-making. For example, each participant chose a colour for their Hilbert curve. As shown in Figure 10, the colours range throughout the rainbow. This was not a result of a suggestion by the co-leaders, a collective decision, or even the availability of colours, since the participants were supplied with several boxes of colours to choose from. Once it was discovered that we *could* form a rainbow, the group decided to use this in the layout of the curves. The actual layout was mathematically inspired by the Hilbert curve itself. There were some aesthetic choices made in the specific order of the colours. A decision about the orientation of each Hilbert curve also needed to be made.

CONCLUSION

> *The apocryphal story goes that Picasso was sitting in a Paris café when an admirer approached and asked if he would do a quick sketch on a paper napkin. Picasso politely agreed, swiftly executed the work, and handed back the napkin — but not before asking for a rather significant amount of money. The admirer was shocked: "How can you ask for so much? It took you a minute to draw this!" "No", Picasso replied, "It took me 40 years."* [Public domain]

There was much discussion about 'what is art', and more specifically, how the artistic process can be analogous to the methods of a research mathematician. Both can consist of the development of new methods, recognition of patterns, and a search for aesthetically pleasing results. Both can be open-ended yet constrained by specific parameters. The participants each brought with them their own mathematical and aesthetic history, and this affected their art. In the words of one participant:

> *Having materials and opportunity to create going from mathematic to aesthetic experience surely set off thoughts about what one understands and can do...I don't often think that university mathematics academics have many experiences using visualization or considering hand-eye coordination. Certainly, elementary education is very strongly focused on use of such experiences in relation to mathematics. Inquiry about how we use this concept in teacher education might be interesting to consider and study.* – Participant Valeen Chow

And another:

> *[...] A working group called making math/making art was irresistible to me. I was fascinated to hear about the histories of various art movements that are explicitly dependent on mathematics and I welcomed the opportunity to make art that at once is constrained and enabled by mathematics. I have little to say about my particular piece. It was at its worst a mess (uninteresting in all the domains in which art is assessed) and at its best (that is in my other life) the first attempt in what could have been a piece of work that demonstrates the marks of Constructivist art. [...]*
> – Participant Elaine Simmt

Note: The letters game distinguished the letters that incorporate a curve from those that are made only of straight lines. This depends in part on the selected font. For example, "Russell Square" the font used on most digital clocks uses only straight lines.

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SELECTING TASKS FOR FUTURE TEACHERS IN MATHEMATICS EDUCATION

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INTRODUCTION

Prospective elementary teachers often encounter significant conceptual difficulties in mathematics as they enter teacher-training programmes (Morin, 2008). In order to help these students, one ponders what mathematics training should be provided to develop the knowledge required to effectively teach the subject. In this perspective, what knowledge should be emphasised for mathematics and for teaching? Which tasks should one select in order to better prepare future teachers of mathematics? Among these tasks one may consider *Problem Solving*, which readily brings students to employ concepts that they will later teach, *Role Play*, which fosters reflective analyses of mathematical concepts and the teaching and learning of these concepts (Lajoie, 2010), or additionally *Task Analysis* of student samples, which enables future teachers to see a variety of approaches and assessments of difficulties related to the targeted concepts. How do we, as mathematics educators, develop our pre-service teachers' "[…] understanding required to make explicit or reasonable the connections between students' current understandings (as exemplified with concrete experiences and examples) and the desired outcomes, such as generalization or a new method or procedure" (Kajander, 2010)? Using artefacts and concrete examples, participants in this working group were invited to discuss their practices and the different types of tasks they offer future primary teachers.

OUR GROUP'S TOPIC AND INTENTION

In general, we do not want tomorrow's elementary school teachers to teach mathematics as it was taught yesterday. We believe that mathematics in elementary schools can be a positive, enabling, and fulfilling experience for all of its participants. Yet few of the student-teachers in our pre-service classes feel that the mathematics they experienced had these qualities. The challenge we embrace in our pre-service courses is how to prepare future teachers to teach something that, in general, they resent and do not understand. In effect, we have 36 hours (or 72, or 180) to try to overcome our students' 1800 hours of experiences with school mathematics, and orient teachers toward a process of teaching the subject differently than what they themselves may have experienced. We could have talked for three days about the impossibility of such a mandate, and the significance of the factors stacked against such a challenge. We were, however, of the opinion that different parameters would possibly bear more palatable fruits. The barriers could not be ignored, any more than our lofty goals could be discarded. Our chosen task was to engage in inquiry into the selection of tasks for use in preparing future elementary school teachers to teach mathematics.

This report begins with a chronological account of our three-day endeavour. Each day was framed around a guiding question, intended to focus our actions on a particular intention. Activities with tasks led to discussions, often as a whole group, sometimes in smaller clusters. Were there substantive outcomes of our time together? We will share some of the qualities and intentions of tasks that emerged as personal priorities among our membership. Afin de bien rendre compte de l'esprit de notre groupe de travail, le rapport est tantôt en anglais, tantôt en français.

PREMIÈRE JOURNÉE

All readers who have led a working group know how difficult it is to start. In our planning, we quickly agreed on a place to start, and then we compiled a list of the things we really should do at the beginning *before* we actually start. Two things seemed to be worth stating, in the hopes of providing our working group with a *new* perspective:

- That we avoid deficit-based framing of our students, so that we can avoid replicating the thinking of the last thirty years regarding teaching elementary math methods, and postpone our engagement with the already well-mobilized topics of *pedagogical content knowledge* and *mathematics for teaching*.
- That we look to generate a fresh sense of possibility, or a fresh understanding of the challenges we face in teaching students to teach elementary mathematics – *imagining what might be possible*, rather than emphasizing what makes change seem *impossible*.

After a delightful introductory trip around the circle of participants, we introduced our guiding question for the day: *When we select our tasks, we portray to our students what we think math is, what we think math is for. So – what do we want our tasks to suggest for them, about the nature and purposes of mathematics?* Afin d'alimenter la discussion, nous avons proposé trois tâches et avons discuté de ces tâches, en relation avec la question. Ces discussions ont permis de préciser ce que nous recherchons lorsque nous proposons des tâches mathématiques aux élèves. Deux de ces tâches étaient *Fukushima Math* et *la calotte glaciaire*. Both tasks aimed at prompting discussions about mathematics being significant for anyone who wants to understand the world around them, and demonstrating that things like area are big ideas, not just a set of formulas to remember.

PREMIÈRE TÂCHE: FUKUSHIMA MATH

Fukushima Math was an activity-space for us to consider what value a task's context might offer: a promise of math's potency or relevance; or perhaps a suggestion of math's currency as a lens for viewing the world. It is based on the evacuation in 2010 around the nuclear reactor on the Japanese coast, which progressed rapidly from 1 km to 3, from 3 km to 6, and finally, despite calls for a 20-km evacuation zone, was established at 10 km by the end of the first week. In the activity, participants drew half-circles on centimetre grid paper to represent each evacuation zone before counting and/or calculating the square kilometres in the zone. Because pre-tsunami population figures were available for each phase, participants could check to see how the increase in evacuees aligns with the increase in land-area being evacuated. The matter of how quickly and how far to go with formal mathematics (graphing the relationships, linking back or forward to area formulas for circles or notions of density) was left open for discussion, but the notion of participants having visual experiences counting the areas of (half-) circles which they had constructed (the compasses didn't extend to a 20-cm radius, so everyone had to improvise to draw the largest suggested evacuation zone with string or a ruler) led toward a principle of students personally experiencing the use of mathematical ideas before they learn formulas.

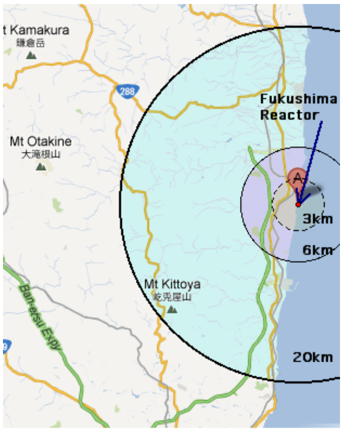

Figure 1. Sketchpad representation.

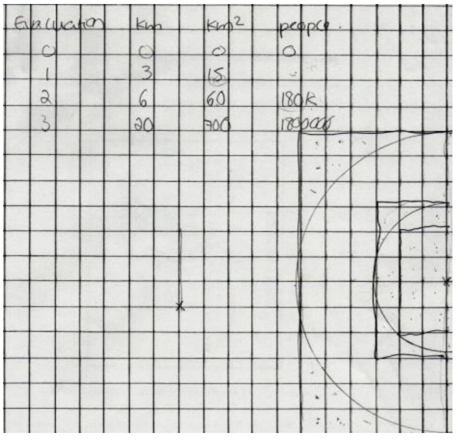

Figure 2. Work from one team.

DEUXIÈME TÂCHE: LA CALOTTE GLACIAIRE (ICE CAP)

Le point de départ de cette tâche, qui a été expérimentée avec des élèves du primaire, est un article relatant qu'une recherche prévoit que l'Arctique se réchauffera plus vite que le reste de la planète. (See Bruce, Lessard, & Theis, 2011.) L'article contenait des images qui montraient l'étendue prévue de la calotte glaciaire de l'Arctique en 2010, en 2040 et en 2070. Nous avons demandé aux participants de répondre à la même question qui a été posée aux élèves, à savoir: *Quelle sera l'étendue de la calotte glaciaire en 2010 par rapport à aujourd'hui?* Tout comme les élèves, les participants disposaient d'un agrandissement de la carte présentée à la figure 3 qui contient les prévisions de l'étendue de la calotte glaciaire pour 2010. Sur la figure 3, le trait indique les limites de la calotte glaciaire en 2003 et la surface blanche correspond à l'étendue prévue de la calotte pour 2010.

Derrière la question apparemment simple se cache une activité mathématique fort complexe pour des élèves du primaire: *Quelle stratégie vont-ils déployer pour déterminer l'aire de chacune des surfaces irrégulières? Quelles stratégies vont-ils mettre en place pour exprimer la différence sous forme de rapport, fraction ou pourcentage?* La richesse de cette activité provient entre autres de la possibilité d'avoir recours à plusieurs stratégies différentes, qu'il a été intéressant d'anticiper avec les participants. Par exemple, lors de l'expérimentation avec les élèves, plusieurs équipes ont utilisé un quadrillage, qu'ils ont superposé sur l'image afin de déterminer le nombre de carrés nécessaire pour recouvrir chacune des aires (figure 4). D'autres équipes ont eu recours à une stratégie similaire, mais ont entouré les surfaces à mesurer d'un rectangle dont ils ont calculé l'aire et en ont enlevé l'aire de la surface qui dépasse la calotte glaciaire. Enfin, des équipes ont aussi eu recours à une stratégie erronée qui consiste à calculer l'aire de la calotte glaciaire à partir de son périmètre. Toutes ces stratégies ont été discutées avec les participants et ont été l'occasion de préciser les qualités que l'on cherche à mettre de l'avant lorsqu'on propose une tâche mathématique à des élèves.

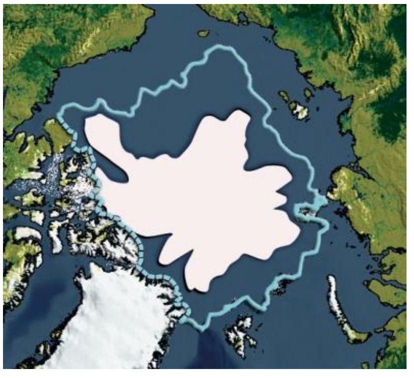

Figure 3. L'étendue prévue de la calotte glaciaire en 2010 (adapted from Hassol, 2004, p. 25).

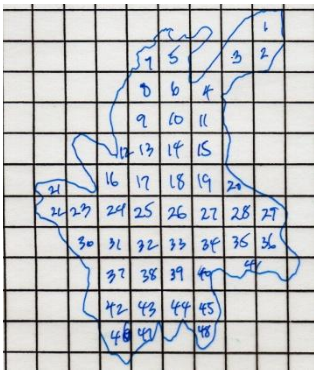

Figure 4. Travail d'un groupe.

As well, the notion of the utility of mathematics for looking carefully at the world and its crises proved more controversial than we leaders had anticipated. As one participant expressed in her day-two personal writing, "I feel like bringing the usefulness of every mathematical concept we teach strips out the mathematics of its beauty."

DEUXIÈME JOURNÉE

La question qui a guidé la deuxième journée était la suivante: *Do we want to prepare our students to use tasks as designed by others, or do we want to prepare our students to adapt tasks designed by others, or do we want to prepare our students to design tasks like those designed by others?* Pour préparer cette séance de travail, nous avons demandé aux participants de lire deux textes, un portant sur les *scripted lessons* (Commeyras, 2007) et l'autre sur l'enseignant musicien versus l'enseignant compositeur (Meyer, 2009). Ces deux textes, dans lesquels sont présentés des types d'enseignants très différents, ont donné lieu à des discussions fort intéressantes quant au type d'enseignant que l'on veut être et quant à la façon d'amener les futurs enseignants à faire des mathématiques. Le commentaire d'une participante montre bien cette idée: "I have enjoyed some of the discussions about how to engage our pre-service teachers in learning mathematics. This helps me to reflect on my own practice and think about how to improve my teacher education program."

Afin d'aller plus loin dans cette question, nous nous sommes demandés ce qui pourrait préparer nos étudiants à se voir comme des *créateurs* de leçons. Pour ce faire, nous avons exploré *les jeux de rôles*, tels que vécus à l'Université du Québec à Montréal (UQAM) et présentés par Lajoie (2010). Pour cette dernière, "Le jeu de rôles est la mise en scène d'une situation problématique impliquant des personnages ayant un rôle donné" (p. 103). Lajoie et ses collègues utilisent les jeux de rôles dans le but de contribuer au développement des compétences professionnelles des étudiants dans un contexte qui se rapproche du contexte réel de la classe. Brièvement, *un jeu de rôles* comprend une mise en situation, qui décrit une situation-problème impliquant des élèves et un enseignant, de même que des consignes à l'intention des étudiants qui, en équipe de quatre, auront à tenir le rôle de l'enseignant ou d'un élève. À titre d'exemple, voici une mise en situation, telle que proposée par Lajoie.

Les algorithmes personnels comme moyen pour respecter toute une diversité de stratégies de calculs

Vous enseignez dans une classe de deuxième cycle. À travers une situation problème que vous avez proposée à vos élèves, ceux-ci ont été amenés à développer des processus naturels de division de deux nombres entiers. Vous souhaitez maintenant faire une mise en commun collective du travail réalisé dans les équipes.

L'enseignant désigné aura quelques minutes pour demander à ses quatre élèves d'effectuer une division en utilisant chacun l'algorithme développé dans son équipe. Ensuite [même si au primaire ce qui suit est prévu seulement au troisième cycle], il devra présenter à ses élèves un algorithme plus traditionnel et les inviter à comparer cet algorithme aux leurs pour qu'ils puissent avoir une idée de différentes manières possibles d'effectuer la division de deux entiers.

(Lajoie, 2010, p. 104)

Après avoir présenté les jeux de rôles aux participants, nous avons amené comme exemple un jeu de rôles mené avec les élèves du primaire de la classe de Christian. L'idée de prendre exemple sur des élèves du primaire n'a pas été prise au hasard. Ce choix a permis de se décentrer des futurs enseignants et de vraiment discuter des jeux de rôles. En effet, si nous avions visionné un jeu de rôles tenu par des futurs enseignants, la discussion aurait facilement pu dévier.

ROLE PLAY WITH GRADE 6 PUPILS: PUPILS AS TEACHERS

During our pre-conference planning, Christian had been inspired by Marie-Pier's stories of the successes at UQAM, where their elementary mathematics education team introduced role-play around 1995 (Lajoie, 2010). Christian adapted the idea for use in his Grade 6 classroom before the conference. As is shown in Table 1, to do so, he had to modify the parameters which are used with the pre-service teachers at UQAM.

UQAM Parameters	Grade 6 Parameters
10 compulsory plays	2 entirely voluntary plays
No forewarning of participants' roles	Participants self-select the actors for a practice in Grade 6, and further select actors for the actual lesson in Grade 3
Instructor selects teaching situation	A Grade 3 teacher selects the concept to teach
Access to articles, manipulatives, previous lessons, instructor	Access to Grades 3 & 6 Math books, manipulatives, previous lessons, teacher
4 participants per team	2 teams of 4 participants
Students receive marks	Students receive feedback but no mark

Table 1. Comparing the two uses of role-play.

Both UQAM students and pupils were provided with time, resources and support during the planning phase. The pupils' task was to develop learning activities for the Grade 3 class that aimed to support a concept provided by the Grade 3 teacher. Pupils had the opportunity to practice in front of Grade 6 pupils who were pretending to be *very cooperative* Grade 3 pupils. After the two practices, participating pupils further selected two students, a 'teacher' and an 'assistant', to lead a lesson in the Grade 3 classroom.

In contrast, at UQAM the course instructor provided the learning/teaching situation, and the pre-service students were unaware of the role they would play until it was time for their presentation. Students therefore had to plan for both the teacher and the student roles.

After viewing a few video clips of the Grade 6 students, we discussed possibilities that such role-play could have for our own pre-service teachers. Many participants saw a resonance between the lesson delivery of pupils and that of their university students in peer lessons and in practicum. “The use of role-plays gives pre-service teachers a way into the ‘act’ of teaching rather than just ‘talking’ about teaching.” Il a également été intéressant de constater que la vidéo a permis de faire ressortir, chez les élèves, des conceptions que Lajoie (2010) a pu observer chez ses futurs enseignants. Par exemple, tout comme pour les futurs enseignants, lorsqu’un élève commet une erreur, l’élève-enseignant le lui dit « rapidement et clairement » (p. 109). Aussi, pour les élèves, tout comme pour les futurs enseignants, « enseigner, c’est expliquer » (p. 109).

Figure 5

TROISIÈME JOURNÉE

La question directrice de la troisième journée était: *Can our tasks convince our students that they can teach mathematics so that every student succeeds, and feels good about math (even if they themselves weren’t taught that way)?* Les tâches choisies pour cette dernière journée étaient *Penny-Flowers* et *“Which is larger?”*

PREMIÈRE TÂCHE: PENNY-FLOWERS (FLEURS DE SOUS)

Penny-Flowers is a collection of tasks designed to fill the void after pre-service teachers are told, when teaching multiplication facts, *not* just to teach for fluent recall, and what *not* to use (mad-minute speed drills, flash cards). The name came from grade three kids who took delight in the idea that it takes exactly six pennies to surround one penny, and happily converted all the pennies on a table at a counting station into penny-flowers. Our group’s engagement with this task began with a bag of pennies for each pair of people. For example, 23 pennies made three penny-flowers, with two left over (suggested names for the left-over pennies included ‘petals’, ‘seeds’, and ‘remainder’). The task sponsored conversation about the value of hands-on counting experiences as a foundation for multiplication facts and the multiplication concept – that students should be able to visualize 7×3 in various ways before they are expected to quickly state the answer. One participant worded that idea this way: “The difference between the concept and the representation of the concept was also interesting to me. How can we get pre-service students to think or talk about it?” On a second level, the conversation engaged with what it means to offer pre-service teachers task-space they had not experienced as students themselves. As one participant wrote, “Problem – we have tasks that we’d like them to use – but we can’t *tell* them to use them (unless we want them to *tell* their students to do math)! What tasks do we have that are *like* these tasks?”

Should our pre-service students be invited to be conduits for tasks designed by others – exercising their professional judgment in the selection of tasks to use, and the adaptation of the tasks to the needs of their classroom – or should they be creators of tasks themselves? To discuss whether we could expect teachers to develop extensions of the tasks that we provide, we shared a possible follow-up task. In *Les jardins de fleurs de sous*, each person was given a unique number of pennies – 47 perhaps, or 16 – and invited to make a penny-flower garden, on hexagons cut from coloured paper. First with pennies, and then by drawings supplemented with numbers and words, each of us made our own penny-flower garden. We were then invited to find our ‘penny-flower garden brother or sister’ (a person with the same number of left-over petals as we had), or our ‘ami(e) de fleurs de sous’ (a person with whom we could combine our gardens, and make one more penny-flower exactly from our extra petals). Imagine having a garden of 23 pennies: Which person, the one with 47 or the one with 16, would be your ‘penny-flower sister’ or ‘penny-flower friend’? Our conversation about this task included some attention to the idea of mathematics tasks needing to be socially engaging and personally rewarding, but still focusing on the multiple representations of quantities and operations on quantities. “The task is not just something to explore, to experience. There is the inner purpose to find and appreciate the math.”

Figure 6. Jardins de fleurs de sous.

Figure 7. Jardin d’amis de fleurs de sous (26 et 16).

DEUXIÈME TÂCHE: WHICH IS LARGER, 2 / 0.355 OR 0.355 / 2?

Our final task of the day began with the question in the title, a question that you might answer with arithmetic, perhaps with a calculator, or with general mathematical thinking. Our colleagues in the working group had no difficulty thinking either arithmetically, with estimation or calculation; or thinking with algebraic generalization: which is more, a big divided by a small, or a small divided by a big? Some people even formulated a simpler question with the same answer: which is more, 8 cookies shared by 2 people, or 2 cookies shared by 8? We teach division with decimals in grades 6 and 7. But what tasks could teach people to think about division with decimals – other than money?

Ralph pulled out one can of Coca-Cola and some glasses to demonstrate one possibility. He split the can, which held 355 ml or 0.355 litre of frothy pop, between two glasses. There it was, in one glass: 0.355 ÷ 2. He then pulled out an empty 2-litre Coke bottle, and, marking how high up on a glass the contents of the can had gone, proceeded to 'pour' imaginary pop into glasses: 0.355, and another 0.355, and another, and we were more than halfway done. The answer was clearly going to be something between 5 and 6 glasses.

We talked about how to compare the answers. One was in millilitres. The other was a number of glasses. We noticed that we were appreciating division itself as an operation more richly, having experienced it visually. In effect, the math question, which is more, was just an invitation to think about what we want to accomplish when we teach our students to teach arithmetic to their students. But this was just one little task, and it made us want more. One participant suggested that generating tasks might be a task for us to share: "How can we work together to build a bank of tasks/videos for use in our elementary math methods courses?"

DISCUSSION

It was never our intention to arrive at a shared answer: such an ambition would have been a denial of our working group's greatest asset – the individuality of each of our colleagues. We intended to put our ideas beside the ideas of our colleagues, and see each idea differently in the light of the company it kept. It's a different sense of task completion, more clearly appropriate for a CMESG working group's task than the tasks we offer to future mathematics teachers – and that's something else to think about! But for now, we offer a collection of the ideas that emerged from our group on the last day, grouped for easier access by each day's guiding question. We invite you to consider these ideas, and put your own ideas among them. There's always room for one more.

DAY 1: *When we select our tasks, we portray to our students what we think math is, what we think math is for. So – what do we want our tasks to suggest for them, about the nature and purposes of mathematics?*

- The task is not just something to explore, to experience, there is the inner purpose to find and appreciate the math.
- The language we use to describe tasks is important.
- How can we design / prepare tasks in such a way that our students can switch quickly between representations and contexts?
- The goal wasn't to have math tasks but to experience tasks that might preserve a tone.
- Tasks need to be based on a question that intrigues the students (not necessarily you).
- Tasks need richness: many possibilities (known and unknown) for branching off.
- Within a task, students can create multiple access points to mathematical knowledge.
- With great tasks, you are communicating your thinking in a way to foster reflecting, connection-making, solution vs. answer, and different mediums for doing math.

Math is thinking, talking, discussing. *Math makes sense.*

Math is a way to see the world. *Attribuer un sens.*

Real struggle and success. No learning without struggle. Perseverance.

Looking back and forward through and beyond subject matter.

Math ≠ solution (not always).

DAY 2: *Do we want to prepare our students to use tasks as designed by others, or do we want to prepare our students to adapt tasks designed by others, or do we want to prepare our students to design tasks like those designed by others?*

- I want more tasks like that [Fukushima Math], that let teachers share the interconnectedness of math, that tasks can "cover" multiple elements of a curriculum.
- What do we want to spend these 36 hours on?
  - Big ideas?
  - Let them know they need to go beyond what they experience from me.
  - Il y a autres choses à faire.
- Reflection is crucial – on how what we do affects our students and how we can improve our tasks for having more meaning for our students.
- The goal wasn't to have math tasks but to experience tasks that might preserve a tone.
- Reflecting on my favorite task: it's quick and I know my elementary students can learn a specific thing and recognize that tasks exist that do so. But there are not enough tasks like that, yet, and not enough time.
- [I realize I don't think the way my students do. I have been] struggling for 3 days to avoid putting expressions algébriques à chaque chose que j'écris.

Tasks should emerge from students' context or info or data.

Within a task, student should have opportunities to build connections.

Our students should be able to make each task their own – and be able to complete them and learn from them.

Curiosity inspires question posing.

DAY 3: *Can our tasks convince our students that they can teach mathematics so that every student succeeds, and feels good about math (even if they themselves weren't taught that way)?*

- [Teaching means] focusing not on the tasks but on the purposes of the task.
  - Where's the math?
  - Where's the pedagogy?
- What we do affects our students. [Through reflection] we can improve our tasks for having more meaning for our students.
- I want what we do to be fractal – what we do the pre-service teachers can do with their students.
- How can we design / prepare tasks in such a way that our students can switch quickly between representations and contexts?
- [Our students must be led to] have a range of contexts available. [They must] analyze a task in a way that it's not seen as a recipe but as an opportunity.
- To put the future teachers in the shoes of the learners, to feel what the students might from their lessons
- Shifting the power differential, creating tasks that let the doers feel the power of the math is theirs.
- A great task should lead to observable learning for both them and their future students.

Reflecting on the tasks,

The math and all that

Questioning our presumptions

Reflecting on experiencing of doing specific tasks.

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Topic Sessions

Séances thématiques

HOW TO PREPARE A PUBLIC TALK?

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You are invited to address a group of people on a mathematical topic of your choice. These people have come to listen to you willingly, but they do not necessarily share a common pool of mathematical knowledge. Your lecture should last about an hour. There will be no exam or grading involved. How do you orchestrate this talk?

INTRODUCTION

With the explosion of scientific research in the 20th century, it is rare that laymen know about recent progress in academic disciplines. Even university professors do not follow advances in disciplines other than their own. Would a professional mathematician be able to describe the key developments in genomics or to comment on the promising avenues in battery research? Over the last centuries, learned societies have felt responsibility to maintain public awareness of their field. In Canada, the Royal Canadian Institute in Toronto has held public lectures for more than a century. Their lectures touch upon all sciences and mathematics. But, to my knowledge, it is only recently that Canadian mathematical institutes have launched series devoted to mathematics: both the Centre de recherches mathématiques (CRM), in Québec, and the Fields Institute, in Ontario, now have their own public lecture series. Even though public lectures on mathematics have existed for a long time, they remain rare and not many mathematicians or mathematics educators have had the opportunity to explore this way of communicating mathematics.

The advantages of mathematical lecture series are numerous. They present mathematics as a living discipline, very much in development, and share the intellectual adventure of research with the public. By the range of problems covered, pure to concrete, they can show students and their parents that scientific activity may translate into career opportunities. They may also show that mathematics and science are useful to attack complex issues, but they do not necessarily provide definitive and clear cut answers. Public lectures are therefore a fruitful addition to "teaching mathematics" in its broadest sense, one that has an impact on citizens, governments and, of course, scientists.

The question raised above, "*How would you orchestrate this talk?*", is one facing professional mathematicians invited to give public lectures. But the question touches a much wider audience than this limited group. It is a question that has been faced by any person teaching mathematics at any level! Students in science and math teachers from high-school to university are asked on a regular basis to explain the purpose of their work. Their success might have an impact as diverse as fostering interest in science among teenagers or stopping the erosion of time devoted to mathematical activities in curricula.

I believe that the foremost challenge in designing a public talk is to properly assess the knowledge and interests of the future audience. This seems an easy enough task but the pitfalls are very real. Most of the other difficulties facing the future lecturer lie in her or his teaching habits. Mathematicians are almost always professors: they are able to teach good courses, they know how to cover the required material, they use the best practices in the classroom and their exams are fair. Or let us assume that they do have these abilities. How can these be put to work in preparing a public talk? A lecture for the larger public is not a course, there is no 'material' *per se*, blackboard talk is surely not the best way to engage a one-time audience and there will not be any exam. The next section will suggest ways to better meet these challenges.

Due to the abundance of public-talk series, sets of slides created for such lectures are now available on the internet, some by distinguished mathematicians. Public talks are very much a performing art. And you would not judge a concert by looking at the score. With this limitation in mind, the last section will (prudently) examine how the difficulties of preparing a public talk have been tackled by some courageous communicators. The three talks that I will discuss are related to the International Congress of Mathematicians 2010 held in Hyderabad, India: the first two were public talks given by Bill Barton and Günther Ziegler during the ICM 2010, and the third was given as part of the series *Grandes Conférences du CRM* in Montréal by Cédric Villani, one of the four young scientists who received a Fields Medal at the ICM 2010.

WHO ARE THEY? WHAT DO THEY KNOW? WHY ARE THEY THERE?

The first question that comes to a teacher's mind upon first contact with new students is about who they are. This is such a natural and obvious question that often it is raised and answered unconsciously. This is so because, with some experience, one knows the level of students in one's school. The real danger in preparing a public talk is not to raise this question about the public one is about to meet.

Who are they? This question is clearly the first to ask, but it is not an easy one. Some knowledge can usually be gathered from the organizers of the event where the public talk is scheduled. A professor might be invited to talk to high school students or at a science fair. In this case, the public is homogeneous and its common knowledge can be well circumscribed. Moreover one can assume that a quick reminder might be enough to bring back a concept seen in previous months. This almost ideal situation was faced by Bill Barton and Günther Ziegler at the ICM 2010 where the audience brought together thousands of (high-school) students and teachers from the state of Andhra Pradesh in India (Casselman, 2010). Of course, the attendees of the congress (professional mathematicians) were probably invited, but it is clear that Barton and Ziegler prepared for their talks with these students in mind. A less ideal situation is that of smaller professional meetings. The organizers of such events will be hoping that the public lecture reaches both the participants and their spouses and families. The lecturer is then faced with the challenge to entertain a very small portion of the audience, the spouses and families, while not boring his or her colleagues who will constitute the majority of the audience.

It is clear that most successful public lecturers consider assessing the public as a crucial step in their preparation. Barton's candid telling of his second experience in India is informative:

> *I was asked to give a second public lecture at rather short notice (three days), and, because it was to teachers, agreed. This was also a different experience. In the hour-long taxi ride through the chaotic streets of Hyderabad I chatted to my escort and became aware that I was not talking to secondary teachers, but to undergraduate college lecturers. Fortunately the trip was long, as I spent most of it adapting my PowerPoint.* (Casselman, 2010, p. 1276)

Even though the two groups, high-school teachers and undergraduate college lecturers, were not that different, the talk needed adjustments, at least in Barton's mind.

What do they know? Suppose that the audience is known and, if it is heterogeneous, that the lecturer has chosen to address mainly a given subgroup of it. The second step is clearly delimiting the mathematical ground upon which the lecture can be based. Again, a young homogeneous group, like primary or high-school students, is by far the easiest. The lecturer will be able to consult colleagues who teach them to check what has been learned and mastered. Of course, the difficulty will remain to build only upon this knowledge, but this is a manageable task, especially if one accepts limiting the scope of one's talk.

A more challenging situation is that of a public that has not visited a classroom for many years. In public lecture series, like the *Grandes Conférences du CRM*, the public is heterogeneous and includes pre-university and undergraduate students, but is also a public whose mathematical knowledge is difficult to circumscribe. Many in this audience will have a university degree, but not necessarily in mathematics or in science, and many might not be using science in their career. But they are interested in the scientific adventure! So, what do they know? Or, more precisely, what do they remember? A thought experiment is revealing. Suppose that you have not done any Euclidean geometry for the last ten years. Which of the following concepts and theorems would you remember: isosceles triangles, ellipses, Pythagorean Theorem, triangle medians and their meeting at one point? As an active practitioner in teaching and doing mathematics, you might think that nobody can ever forget these elementary facts! So, let us try instead questions at the same level but in a close discipline: What is a Faraday cage? What physical quantity has the ampere as its unit? What is a conservative potential? What is the speed of light? I will let you imagine questions in chemistry, history, geography. But this experiment suggests that such a public is likely to remember only the basics of its high-school mathematics.

Why are they there? And what do they expect? The public of talks organized at the pre-university level do not usually attend willingly. They are there because it is part of some compulsory activities. If a young student is scientifically inclined, she or he might not know what to expect, but some enthusiasm and curiosity can be taken for granted. Similarly among the public of lecture series, a curiosity in scientific endeavours is always present. The lecturer will have to create one or more magic mathematical moments during the talk, moments where a mathematical question is explained (and maybe solved) in a satisfying and elegant way. This is at the heart of the mathematical experience. But this might not be enough! Students and the public in general are likely to have been exposed to science-fiction (books, TV series or movies) and scientific popularization series. These productions have means and work teams well beyond those of a lone mathematician. Besides choosing an interesting subject matter, a lecturer will have to use all the basic techniques of oral exposition: setting the mathematical question in a concrete context, tying it to historical and human developments, providing metaphors to explain difficult details, etc., doing all these while being entertaining, humorous, and lively.

Obvious questions, difficult answers. The success of a public lecturer relies on her or his ability to keep these answers in mind during all preparation steps!

THREE EXAMPLES

I will now comment briefly on the three public talks listed in the introduction. Of the three, I only attended Villani's, but the slides of Barton and Ziegler reveal the efforts put into designing these talks. Each of the three extracts provides an example of a clear (and probably successful) attempt to communicate a mathematical fact within the public's mathematical grasp.

PROOFS FROM 'THE BOOK'

[*A public talk by Günther M. Ziegler (Freie Universität Berlin) in Hyderabad (2010)*]

Proofs are at the heart of mathematics and it seems natural to make an effort to describe to the public what they are and what they achieve. In fact, Ziegler wrote a book with Martin Aigner (Aigner & Ziegler, 2003) about some proofs that are so clear, direct and insightful, that they seem to have been written by God's hand. His book and lecture share the same title. We have seen that the public in Hyderabad comprised a large number of high-school students, brought in by the bus load (see again Casselman, 2010). It is clear that this fact is in Ziegler's mind.

The first few slides of his presentation are devoted to explaining the role of proofs in mathematics and for the mathematical community. This is done by formulating some simple questions that are very difficult to prove and also reproducing some quotes from famous mathematicians about proofs. Some are deep, others simply witty. Then Ziegler introduces 'THE BOOK', the one containing the 'definitive' proofs, and then he gives an example. It is his first real mathematical moment and it is likely to have been magical for many of the high-school students. Here it is.

A slide appears, totally blank but for one sentence and one drawing. The sentence is: "Theorem: The 'chessboard without corners' cannot be covered by dominos." And the drawing is the one on the left below (see Figure 1).

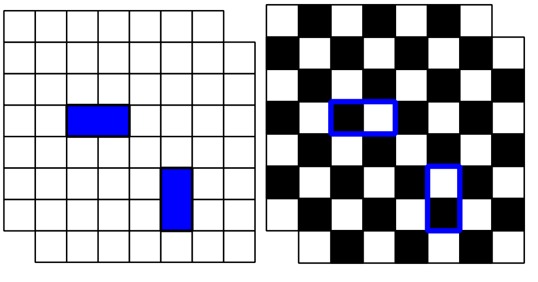

Figure 1

One can guess that Ziegler explains the statement that, even though it is possible to cover a whole chessboard with dominoes, each covering two neighbouring boxes, once two opposite corners of the chessboard are deleted, this task of filling the new chessboard by dominoes becomes impossible. A student who has not encountered this problem before might feel the urge to play with dominoes to see why this is impossible.

But a very simple argument proves the theorem, without any such attempt! It is given in Ziegler's next slide that contains only the drawing on the right above (see Figure 1) where the usual pattern of alternating black and white boxes has been added. There, in this single drawing, one is reminded that dominoes will always cover precisely one black and one white box, wherever they are placed. But one can also see at once that the removal of opposite corners of the chessboard has deleted two boxes of the same colour. In this case, the number of black boxes is now two less than the number of white ones and it is therefore impossible to accomplish the covering by dominoes.

The argument, by its simplicity and elegance, can be understood by most people and creates one of these magical mathematical moments in the mind of any scientifically inclined person. It is beautiful and, being placed at the beginning of the talk, captures all listeners in the experience. And it certainly belongs to 'THE BOOK'!

WHERE IS MATHEMATICS TAKING ME? AN EXCITING RIDE INTO THE FUTURE

[*A public talk by Bill Barton (The University of Auckland) in Hyderabad (2010)*]

Barton's talks present applications to problems of everyday life or sister disciplines. By doing so, his goal is clearly to show that mathematics can take scientists, and of course young students, on an exciting ride into the future. His talk presents many active domains of research: the probability of a celestial body falling on earth, the dynamics of the ice shelf break-up, the modeling of calcium in the human heart, the scheduling of air- and bus-lines, the swimming of knife fish that live in muddy water, the original algorithm of the *Google* search-engine, the modeling of the spotted owl, and the use of knot theory in molecular biology. In total, eight very different applications of mathematics are introduced and discussed. Of course, this number precludes going into much detail in any of these fields. But young high-schoolers might have been surprised to learn that their mathematics classes might lead them to such concrete tasks!

The original algorithm used by *Google* to order the webpages obtained from a search can certainly be explained, at least intuitively, to the public. It also lends itself to a more mathematical presentation that would be accessible to students entering a university program (see Rousseau & Saint-Aubin, 2008). Prior to any search, *Google* assigns a rank to each page in the World Wide Web, the PageRank ('*Page*' for Larry Page, one of the creators of the algorithm). Once a user sends a search, all the pages that contains the words requested will be presented to the user in order of decreasing rank. To obtain the rank of each page, *Google* studies the links between the pages in the web. Barton uses a simple example where the trillions of pages of the World Wide Web are replaced by only five (see Figure 2).

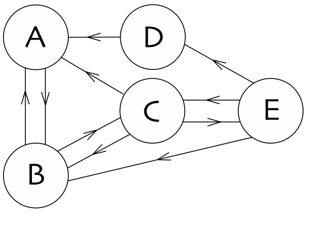

Figure 2

In this drawing the arrows represent links between the pages. For example, page C points to A, B, E; this means that page C about, say, 'growing lilies', invites its visitors to have a look at 'the proper care of lilies' on page A, 'the classification of hybrids' on page B and even points to 'a famous seller of lily seeds' on page E. The information on page C is well-known by the specialists and both the authors of the classification (page B) and the owners of the lily store also point to page C. One may wonder which of the five pages are the most popular. Of course popularity is a very subjective matter. A mathematical measure of this popularity needs to be defined. *Google*'s creators thought that the popularity of a given page should not depend on what its author thinks of it, but instead it should be measured by how other people rank this page. And the only information about this is how people link to the desired page. Therefore the *Google* algorithm measures the popularity using the arrows between the five pages in the drawing above or, in real life, between the trillions of pages in the World Wide Web.

A presentation of this algorithm, more detailed than the lines above, is definitely accessible to a wide public. It shows the power of abstraction (replacing the web by an oriented graph as in the above diagram) and that mathematical reasoning might lurk behind the most prosaic action of everyday life, like doing a computer search. The fact that the algorithm creators are now billionaires might also strike impressionable minds....

QUAND LA TERRE ÉTAIT TROP JEUNE POUR DARWIN

[*A public talk by Cédric Villani (Université de Lyon et Institut Henri Poincaré) in Montréal (2010)*]

Villani's talk, at the *Grandes Conférences du CRM* series, was well-attended. He had received one of the 2010 Fields Medals only weeks before and any person in Montreal with some mathematical curiosity was present. Of course many were university students in mathematics, but many others were older, and they were certainly not all in a math program! Villani had been invited to talk to the public and, fortunately, he did not change the level he had prepared for, despite the presence of many mathematics students.

Villani's aim was to present the historical debate around the age of the earth that took place at the end of the 19th century. On one side stood biologists, of whom Darwin is the most famous, together with the geologists. The evolution of species, put forward by Darwin in 1859, needed millennia for the cooling of the planet so that life could appear, and then many more years for species to multiply and diversify to what they are today. Biologists and geologists agreed that the planet needed to be much older than 10^8 years. On the other side stood Kelvin who, using new tools developed by the French mathematician Fourier, was able to get an estimate of the age of the planet that was of the order of 10^8 years. The two estimates were seen to disagree and an intellectual battle ensued.

Villani's talk presented the mathematical argument in a convincing way. But what first comes to my memory, and this might be very personal, is the drama and the breath-taking account of the clash where Darwin and Kelvin appear as intellectual Titans. Villani recounted the history of the great minds attempting to measure the age of the earth, first using biblical sources (Ussher), then scientific arguments (de Buffon, Fourier, Kelvin, Rutherford and then Perry). Villani's presentation remained elementary at each step.

Kelvin's computation rests on partial differential equations and Fourier analysis. These are surely not commonly mastered by the layman! Villani therefore spent some time recalling tools that intervene here: the trigonometric functions and the derivative of a function. The graphics he used, as seen below (see Figure 3), are likely to trigger a photographic response in the auditor's memory as they are universally used in textbooks. Never did he write down trigonometric identities or the definition of the derivative using the limit of $(f(x + \Delta x) - f(x))/\Delta x$. Only the concepts were recalled. (It is heartwarming to see a Fields Medalist patiently explaining the concept of derivative to a few hundred willing citizens...).

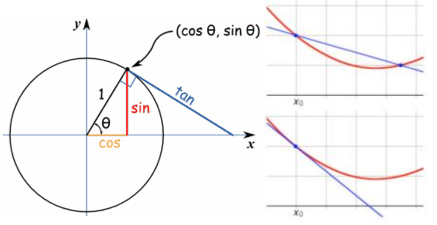

Figure 3

Clearly Villani chose to use graphical representations to convey mathematical ideas. There were several equations in his slides, but he explained only one, the equation describing the propagation of heat, and spent quite a bit of time doing so. His talk thus contained a single major mathematical idea; this idea was brought up with graphics and physical intuition, and was well explained. By limiting himself to a single equation, Villani was able to provide more mathematical details than on any good television programs of science popularization and he still had time to present the context of the mathematical breakthrough. Probably this was the optimal balance between historical developments, psychological analysis of the main characters, and mathematical arguments.

CONCLUSION

The mathematical levels of the three talks discussed in the previous section are different. At one extreme of the spectrum, we find Ziegler's which uses several equations to state the problems whose proofs are in 'THE BOOK'. He therefore takes for granted that some of his public can read and understand these formulae. His assumption might be valid for a fair portion of the public at the *International Congress of Mathematicians*. At the other extreme, Villani concentrates his argument on a single formula to which he devotes a considerable amount of time. The organizers of the series had described the public of the series to him and he aimed his talk to them. All three lecturers made a conscious effort to talk to what they felt was their public.

Villani also teaches us one more thing. A public talk does not need to be a marathon of mathematical results. His lecture is definitely exciting on the mathematical level (even if only a single equation is discussed in detail). But it is also interesting on another level: it exposes clearly that mathematicians belong to the intellectual community in its widest sense and, as such, their results might help understand other fields or, as in his talk, conflict with them. Mathematics, like other intellectual endeavours, develops in contact with other fields and, in return, nourishes them. It is a nice idea to send the public home with.

We are likely to hear more in coming years about pedagogical practices of public talks. These talks offer a scene totally different than that of the classroom. Despite these differences, I hope that they might help us reflect on our daily teaching practices.

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WARM HANDS TAKING COLD MATHEMATICS

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This written version of my presentation at the 2011 conference of the Canadian Mathematics Education Study Group focuses on humans doing mathematics, and is oriented around temperature metaphors often associated with humanity and mathematics. We humans are described as warm when we are seen to care for people (and for animals and the environment). We are seen to be cold when we make decisions without care. Mathematics is taken as cold because it is taken as abstract and disinterested about context.

I begin with an overview of the context, embedded in my worldview. From there I consider the conceptions of mathematics that position it as cold, and arguments against such conceptions of mathematics. From there I turn to mathematics classrooms, asking how students' experiences of mathematics could be different.

I recognize that my presentation on this topic is oriented around my worldview, so I describe that first to help readers identify any differences between our worldviews. Skovsmose's (1994) claim is helpful – that one's background and foreground are important contextual factors. Background refers to the culture and experiences one brings to a situation, and foreground refers to one's hopes and expectations for the future. I summarize my work as being oriented around developing mathematics education that supports my more fundamental hopes, which include a desire for respectful and peaceful human relationships in classrooms and elsewhere. I wish for myself and others to value and learn from diversity, to respond to tensions without violence, and to pursue lifestyles that are sustainable. Through my work in mathematics education, I have come to believe that mathematics teaching that aims for these things also supports the development of mathematical skill and understanding.

People with differing worldviews may arrive at conclusions similar to mine. Imre Lakatos, renowned philosopher of mathematics, was a young adult in Hungary during and shortly after World War II. He hid from the Nazis, taught Marxism in the underground movement, helped communism establish power and eventually fled the regime he helped establish. He knew authoritarianism intimately. As described by Long (2002), in his efforts to combat authoritarianism, Lakatos was known to use violence to pursue his ends. By contrast, I have enjoyed relatively peaceful political and social situations in a stable, relatively wealthy country, growing up in a Mennonite tradition known for rejecting the idea that wars and violence can be legitimate: there is no context that can justify killing. My relatively sheltered life makes it easier to speak against violence as a means to a better end, but my recent ancestors carried this stance against violence through circumstances likely more challenging than Lakatos'.

Despite the differences in background and foregrounds, I have found that Lakatos' (1976) reflection on mathematics education is similar to mine. He wrote, "It has not yet been sufficiently realized that present mathematical and scientific education is a hotbed of authoritarianism and is the worst enemy of independent and critical thought" (pp. 142-143). Like Lakatos, I have come to the conclusion that mathematics is implicated in the development of authoritarian regimes, which can operate on a large scale, such as a dictatorship, or in subtle ways within a democratic environment.

Two questions underpin much of what I do. What is the role of mathematics in violence? How can mathematicians and mathematics educators work for peace and against violence? If I were asked to make a choice between developing good mathematicians or good citizens who respect and care for one another, there would be no question. I value non-violence over mathematics. However, I believe that mathematicians and mathematics educators can work for peace and against violence, just as we can support violent worldviews in our work.

MATHEMATICS

Mathematics is powerful. It enables us to model and thus visualize phenomena that physical tools cannot access. It enables our imagination to explore spaces that conventional wisdom scorns as unreal, impossible or insignificant. It facilitates the management and arrangement of data that exceeds human ability to sense. Because of its power, mathematics can be used both to expose social injustices and to support resolutions of these injustices. Likewise the power of mathematics can underpin and sustain violence. Even mathematics that seems at first to work with imaginary spaces often proves relevant to real applications, which may be wonderful or terrible.

Imagery to represent mathematics often characterizes it as cold. Perhaps this imagery relates to the abstraction and generalization that is central to mathematics – the de-contextualisation, de-personalisation, and de-temporalisation (Balacheff, 1988). With this abstraction we humans can use mathematics to make decisions insulated from the pain or pleasure that may be connected to our choices. This is an important technology in democracies, in which decisions are supposed to be free from human bias (Porter, 1995). This is true for large social decisions and for local decisions. For example, when we add sums of money, the result should be dependable. The result should not depend on the culture or position of the person doing the adding. Mathematics is supposed to be dependable and non-discriminatory. The stability of mathematics and its disregard for context is a source of protection for people in particular contexts. We can use mathematics to reason with others and to convince them of a more fair way of doing things. But we can also use mathematics to ignore the pain and suffering related to certain decisions. For example, when doing cost benefit analyses, governments often create or sustain conditions that cause certain people to suffer. The cold mathematics may be seen to justify the decision while the effects are hidden. Nevertheless, the benefactors of such decisions may see the mathematics as a tool for security and fairness, just as vendors and customers at a market may see the dependable and stable approaches to measurement and counting as a tool for fairness.

The Chandler Davis poem "Cold Comfort" exposes the irony that there can be security in a cold, heartless mathematics. With an interest in using mathematics to understand his world, he identified the sense of security that mathematics can support, yet he struggled with his realization as a mathematician that this image of mathematics is flawed. His opening lines describe his refuge in mathematics in an unpredictable world (Davis, Senechal, & Zwicky, 2008, p. 52):

> *If I took alarm at the prospect*
>
> *of things spinning out of control*
>
> *(and I might*
>
> *for they are*
>
> *oh, I well might)*
>
> *this refuge would tempt me.*

Tobias Dantzig (1930/2005), in his history of number, similarly plays against the common imagery of mathematics being cold. He argued against the imagery.

> *For here, it seems, is a structure that was erected without a scaffold: it simply rose in its frozen majesty, layer by layer! Its architecture is faultless because it is founded on pure reason, and its walls are impregnable because they were reared without blunder, error or even hesitancy, for here human intuition had no part! In short the structure of mathematics appears to the layman as erected not by the erring mind of man but by the infallible spirit of God. The history of mathematics reveals the fallacy of such a notion*. (p. 188)

Dantzig identified the humanity in our mathematics by placing its development in cultural contexts. The recognition of the cultural contexts of the development and uses of mathematics may seem to be at odds with the abstract nature of mathematics, with its characteristic move to establish truths that are neither contingent on the person nor on the person's historical, geographical, cultural or disciplinary place. However, abstraction is a human action, performed for particular reasons that relate to the person's current place in their world. Similarly, applications of mathematical abstraction (applied mathematics) are human moves to bring context-independent knowledge into contexts. Generalization and abstraction are features of mathematical thinking that have their place in thoughtful human problem solving.

There is value in asking what is always true regardless of context, but there is also value in asking how results drawn from such generalization and abstraction can be applied or not applied to any given human problem. Nevertheless, the acts of abstraction and generalization are the results of human choices motivated by values and hopes, even though the final expressions of such abstraction and generalization are represented as free from context. I suggest that many uses of mathematics are in fact motivated by the desire to appear objective, not biased by particular cultures.

For this reason, the ethnomathematics program is important; it recognizes an aspect of mathematics that is often hidden. Much ethnomathematics research is focused on identifying mathematics that is not reflected in mainstream academic traditions. However, it is important to note that ethnomathematicians claim that all mathematical ideas arise from humans addressing their issues or problems in particular cultural milieu. It is not only non-academic mathematics that is cultural. We might enjoy experiencing cultures with travel, but we should not forget that our home context is also a culture, equally strange to others. When we live in a dominant culture it is easy to forget that. Similarly, academic mathematics traditions are so dominant that it may be easy to overlook the fact that they are culturally situated.

Though characteristics of mathematics inspire the imagery that depicts it as a cold, hard discipline, I aim to promote representations of the human choices that are part of mathematics. The human choices that are indeed part of any mathematics ought to be recognized in mathematics classrooms and in society at large. Yes, mathematics may be cold, but it is a cold tool (or set of tools) designed and used by warm human hands. Perhaps the humans involved may have 'warm hearts' with concern for the needs of others, but whether or not this is the case, the hands are warm; mathematics is embedded in living, warm-blooded human bodies (including human minds), which live in particular environments (including cultural and physical aspects of these environments).

One reason for promoting the recognition of the warm hands at work in creating and using mathematics is because it is recognizing the truth that humans are at work in and with mathematics. However, I am suspicious of most claims of truth because so often such claims are eventually revealed to be expressions of power – the assertions of dominant people and dominant cultures. Thus, I am more interested in the effects of recognizing humans at work in mathematics, and how these effects relate to my views on social justice.

I claim here that it is dangerous for the people in a society to see mathematics as cold and abstract when the people do not recognize the human hands at work in and with the mathematics. Mathematicians, educators and other users of mathematics who suggest that mathematics is values-free or independent of culture tacitly render rhetoric that uses mathematics as being above reproach. It is possible to make this suggestion explicitly – to argue that one's claim is above question – but I believe that the message is even more powerful when it is subtle, when the human choices that are part of mathematics are obscured. If I make a claim explicitly, I invite debate; if I say "mathematics is above critique" I tacitly raise the question "Is it in fact above critique?" But when we all talk about mathematics as if it is sure, secure, predictable and free from human particularities, others are unlikely to think about the alternative, namely that mathematicians regularly challenge each other and regularly develop new ideas that seem to break the old rules, and that people use mathematics for their particular agendas.

Why is it dangerous to develop the sense that mathematics is above critique? If it is taken as above critique it can be a powerful tool for manipulating people. Leaders of social change in politics, critics of politics, advertisers, social justice advocates and any others who want to convince people of something can and do use mathematical tools to press their points. Often, such rhetoric is used to justify decisions with extreme effects on the lives of many people. But the public is ill-equipped to recognize that mathematics is being abused because of the perception that it cannot be abused. If mathematics appears secure and perfect, claims resting on mathematics are beyond critique. Present mathematics education practices repress critique by giving students experiences with mathematics that have them making few decisions and developing skills outside the context of human problems. I believe this is why Lakatos, as quoted above, concluded that mathematics education practices undermined free thought and supported authoritarianism.

MATHEMATICS CLASSROOM INTERACTION

How then can we lead people to see the humans at work in and with mathematics? I will suggest two ways to do this in classrooms. The view of mathematics shaped in classrooms will in turn shape society because people carry their views of mathematics from the classroom into the rest of their experiences. We can draw attention to the cultural contexts of mathematics and we can get students doing the kind of mathematics that has them making decisions and discussing each other's choices.

The ethnomathematical program, mentioned above, is one way of drawing attention to the cultural contexts of mathematics. We can tell students about ethnomathematical work; we can show them how mathematics is done differently in different cultures. I suggest that our accounts of ethnomathematics will be more meaningful to students if we have them doing ethnomathematics themselves. Teachers in the Mi'kmaw schools on Cape Breton Island have been doing this for four consecutive years with the “Show Me Your Math” event. Lisa Lunney Borden and I have given an account of this annual event in a brief article in *CMS Notes* (Lunney Borden & Wagner, 2011), and more elaborate accounts in some forthcoming book chapters (including Wagner & Lunney Borden, 2012).

Another way of drawing attention to the cultural contexts of mathematics is to engage students in the history of mainstream mathematics. (I say ‘mainstream’ here because ethnomathematical work is the investigation of the history of mathematics, but is usually focused on the history and practice of mathematics that is not mainstream.) Students can be told stories from the history, and, like with ethnomathematics, they can be asked to investigate parts of the history themselves. I would suggest that even speculation about the motivations and rationales for particular mathematical ideas could be similarly powerful for drawing attention to the human problems addressed by mathematics, but I would caution that teachers make clear the distinction between speculation and evidence-based historical accounts. Perhaps such speculation could be followed up with investigation of the history.

In addition to these ways of drawing attention to the cultural contexts of mathematics, I promote the importance of having students be active decision-makers in mathematics. My work with Beth Herbel-Eisenmann, in which we have analyzed mathematics classroom discourse (e.g. Herbel-Eisenmann & Wagner, 2010), has revealed the way classroom talk masks the recognition of people making choices in mathematics. The grammar used in class encodes this lack of choice. Alternatively, if mathematics teachers give students tasks that have them making choices and creating their own solution approaches, classroom discourse would change to embody the discussion of diverse points of view. Pure mathematics investigation tasks (e.g., Booth & Grant McLoughlin, 1995; Mason, 1982; Mason & Watson, 1998; Morgan, 1998) and tasks that use mathematics to investigate local social issues (e.g., Frankenstein, 1989; Gutstein & Peterson, 2005; Stocker, 2006) would support this kind of discourse.

If students could experience mathematics as doers and decision makers, and also come to know stories of people inventing mathematics ideas to respond to real problems in their cultural contexts (i.e. ethnomathematics and the history of mathematics), then they will be equipped to critique the mathematics they see in society. They will know that the mathematics might be done differently. In addition to being equipped for critique, I believe that this kind of experience of mathematics can help students appreciate diversity. They will have experiences of receiving insight from their peers and from the mathematics of different cultures. The appreciation of diversity is a fundamental characteristic of a vibrant, peaceful society.

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New PhD Reports

Présentations de thèses de doctorat

OPPORTUNITIES TO LEARN IN AND THROUGH PROFESSIONAL DEVELOPMENT: AN ANALYSIS OF CURRICULUM MATERIALS

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Though professional development (PD) is seen as key to improving mathematics instruction, little is known in the research literature about what teachers learn in and through PD, nor about what those who conduct the PD might be learning as well. In my dissertation study, I address this knowledge gap by analysing a selection of commonly-used PD curriculum materials to ascertain the opportunities they provide for middle-school teachers to learn ideas central to improving their instructional practice. Additionally, I examine the extent to which the curriculum materials are designed to also support the learning of professional developers. This paper provides a brief overview of the dissertation study and its main findings. The full study can be found online at jenny.badee.net.

INTRODUCTION

It is a widely held belief that the professional development (PD) of teachers is paramount to improving mathematics instruction. However, in the United States, teachers have generally experienced a fragmented PD system with few opportunities for rich or sustained learning (Wilson & Berne, 1999). Such PD offerings seem inadequate to improve mathematics instruction and this inadequacy is a serious unsolved problem for policy, practice, and research in American education (Borko, 2004). Though millions of dollars have been invested in PD programs and there is a body of research on PD, “surprisingly little attention has been given to what teachers actually learn in professional development activities” (Garet, Porter, Desimone, Birman, & Yoon, 2001, p. 923). Since understanding teacher learning in PD requires attention to both the curriculum and the pedagogy employed (Ball & Cohen, 1999), this lack of attention to what teachers learn represents a problematic knowledge gap.

My dissertation study addresses this gap in the research by analyzing a selection of commonly used PD curriculum materials. These curriculum materials were analyzed to ascertain the opportunities they provide to middle-school (MS) mathematics teachers to learn ideas central to improving their instructional practice. The study addresses the question: To what extent and in what ways do PD curriculum materials provide opportunities for MS teachers to learn about mathematical content and pedagogy? In particular, it focuses on teachers’ opportunities to learn about: (1) MS mathematical content, specifically the key topics of proportionality, rational numbers, and linear functions; (2) using multiple representations of mathematical ideas; and (3) using cognitively demanding mathematical tasks in instruction. Additionally, due to the importance of facilitation in PD, the scarcity of continued training opportunities for professional developers (Ball & Cohen, 1999), and the lack of attention in research to the learning of professional developers (Elliott et al., 2009), the study also attends to the learning opportunities provided to these individuals. The fourth research question explores the ways in which the curriculum materials are designed to provide professional developers with opportunities to learn how to support teacher learning – that is, the extent to which they appear to be educative.

OPPORTUNITIES TO LEARN IN PROFESSIONAL DEVELOPMENT CURRICULUM MATERIALS

In this study teacher learning is conceptualized as the development of *teacher capacity* (Grossman, McDonald, Hammerness, & Ronfeldt, 2008) to do the work of mathematics teaching. Teachers can develop this in PD settings by engaging in collective inquiry into mathematics instruction, and learning in and from practice by analyzing and reflecting upon the mathematics teaching of others and themselves. In PD curriculum materials, *professional learning tasks* (PLTs) provide a context for reflecting and learning about mathematics instruction. PLTs are "activities that are situated in and organized around components and artifacts of instructional practice that replicate or resemble the work of teaching" (Silver, 2009, p. 245). In the study they serve as a unit of analysis within the curriculum materials and are conceived as having two main components: (1) a mathematical task and (2) a 'link to practice' component which portrays the ways students and/or teachers interact with the mathematics and each other in classrooms. The mathematical task provides teachers with opportunities to revisit the mathematics they teach and to learn more about the conceptual underpinnings of the topics addressed (Ferrini-Mundy, Burrill, & Schmidt, 2007). The 'link to practice' component, through the use of artifacts of practice, such as narrative cases or student work samples, links discussion to the work of mathematics teaching (Borko, 2004) and allows teachers to develop their knowledge of mathematics, pedagogy, and student learning simultaneously (Ponte et al., 2009). The PLTs in PD curriculum materials were analyzed to determine the opportunities they provide teachers to study, reflect upon, and learn to use particular mathematical topics and instructional practices.

In the K-12 setting, curriculum materials that support both student learning and the learning of teachers are referred to as *educative curriculum materials* (Schneider & Krajcik, 2002). Some of the features of such materials are that they are designed to support teachers' learning of subject matter, make transparent the authors' pedagogical judgements, and develop teachers' design capacity so they can adapt the materials to achieve specific instructional aims (Davis & Krajcik, 2005). In this study I have extended the concept of educative curriculum materials from the classroom into the PD space, defining *educative PD curriculum materials* as curriculum materials designed to support the learning of both teachers and professional developers in PD settings. Since curriculum materials can provide learning opportunities on a large scale (Schneider & Krajcik, 2002), educative PD curriculum materials have the potential to provide much needed learning opportunities to large numbers of professional developers.

METHODOLOGY

The study is focused on curricula commonly used with large numbers of MS mathematics teachers. A survey was conducted with the purpose of identifying, based on empirical evidence, a sample of publicly available PD curriculum materials that are being extensively used across the United States. The survey was administered to the principal investigators of 32 large-scale PD projects funded by the National Science Foundation. These projects operated in a variety of contexts, rural to urban, across 21 states and Puerto Rico. The collected survey data illustrate the usage of PD curriculum materials in projects working with 6203 MS mathematics teachers. Based on the survey results, four sets of curriculum materials were identified as having been used by 60% of all teachers involved in the surveyed PD projects. These four make up the sample under analysis in the study: (1) *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development* (Stein, Smith, Henningsen, & Silver, 2000) [ISBI]; (2) *Improving Instruction in Rational Numbers and Proportionality: Using Cases to Transform Mathematics Teaching and Learning, Volume 1* (Smith, Silver, & Stein, 2005) [IIRP]; (3) *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Strategies for Teachers* (Lamon, 2005) [TFRU]; and (4) *Developing Mathematical Ideas, Number and Operations Part 2: Making Meaning for Operations* (Schifter, Bastable, & Russell, 1999a, 1999b) [DMIMMO].

In order to analyze the sample PD curricula, I developed two analytic frameworks. The *opportunity to learn* (OTL) framework was designed to answer the study's first three research questions by directing attention towards how the curriculum materials, and individual PLTs within them, present teachers with opportunities to learn specific mathematical content and pedagogy. The *educative features* (EF) framework was developed, using the schema of Davis and Krajcik (2005), to consider the challenges of professional developers' work with teachers and curricular features which would lend support in meeting them. It consists of 20 specific educative features that would provide learning opportunities to professional developers. These two frameworks were used to analyze the four PD curricula in the sample.

FINDINGS AND DISCUSSION

OPPORTUNITIES FOR TEACHER LEARNING IN THE FOUR CURRICULA

The four curricula have different foci, but were all widely chosen for use in PD programs because they provide opportunities for MS teachers to learn mathematical content and pedagogy. Over the four curricula, there are 110 identified PLTs that consisted of sets of mathematical tasks, samples of student work, narrative cases of mathematics instruction, or prompts for classroom activities or general reflection on mathematical or pedagogical issues. Within the PLTs, 284 individual mathematical tasks provide opportunities for teachers to individually revisit and grapple with the mathematical content they teach. Through the narrative cases, classroom activities, and student work samples, teachers are provided with opportunities to compare and reflect upon how other learners understand and use these ideas.

Teachers' Opportunities to Learn About Middle Grade Mathematics

While all three topics of proportionality, rational numbers, and linear functions are central topics in the middle grades (National Council of Teachers of Mathematics, 2006), they are addressed to different degrees in the PD curriculum materials. Proportionality is addressed in 65% of the PLTs across the four curricula. Rational numbers are addressed in 38% of the PLTs across the sample. The third topic of interest, linear functions, which is also a central topic of focus within middle grade mathematics, is only addressed once across the four curricula. Thus, while teachers are provided with many opportunities to learn more about multiplicative reasoning and proportionality, they are provided with scarce opportunities to learn that proportionality forms a conceptual foundation for linear equations and functions.

All four of the PD curricula focus on mathematical content, providing opportunities for teachers to move beyond procedural knowledge and deepen their understanding of the concepts underlying mathematical topics. Given that proportionality is an overarching concept in MS mathematics that presents many challenges to teachers (Cramer, Post, & Currier, 1993), it is appropriate that it is the topic most focused upon in the sample curricula. Teachers were also given many opportunities to learn about rational numbers, especially fractions, which have been found to present a challenge to teachers (Ma, 1999). Though linear functions is one of the key topics in middle grade mathematics and is a topic that teachers have an underdeveloped conceptual understanding of (Even, 1993), it is basically overlooked in all four curricula. It is problematic that commonly used curriculum materials do not address this important topic and that curriculum materials that do so are often not chosen for use. As MS teachers are the ones to formally introduce this topic to students, there is a need to ensure that they have opportunities at some point in their PD experience to learn more about linear functions.

Teachers' Opportunities to Learn About Using Multiple Representations of Mathematical Ideas

Many representations are used in each of the four curricula and opportunities to learn about how multiple representations can be used in concert in classroom instruction are provided. As words are our primary mode of communication, verbal descriptions were present in every PLT. Aside from verbal descriptions, the two prevalent representational forms used across the four curricula were visual diagrams and symbols. I analyzed not only which types of representations were used, but also the connections made between representations. As seen from Figure 1, strong connections are made between verbal descriptions, visual diagrams, and symbols within each curriculum. Graphs are used in only 3 PLTs across the entire sample. Due to this scarce use, teachers are provided with limited opportunities to connect graphs to visual diagrams and tables, and no opportunity to connect graphical representations of mathematical ideas to symbolic ones.

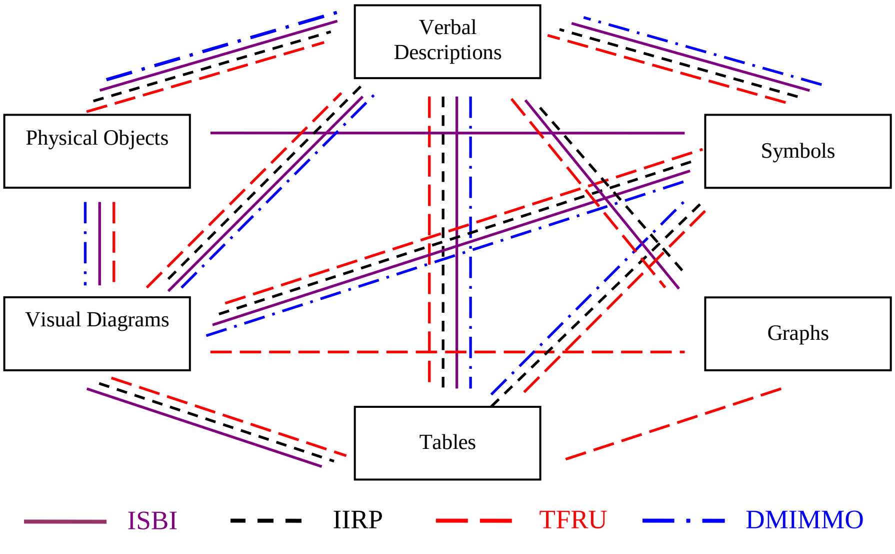

Figure 1. Degree and strength of connectivity between representations in the four curricula.

The flexible use of multiple representations is core to communication in mathematics (Elliot & Kenney, 1996) and a key component of students' competent mathematical thinking and problem solving (Brenner, Herman, Ho, & Zimmer, 1999). With extensive opportunities to create and use diagrams, symbols, and verbal descriptions in combination, the PD curriculum materials provide teachers with many opportunities to learn about using multiple representations of mathematical ideas in their mathematics instruction. However, the scarce use of graphs and the missing connection between graphs and symbols is problematic, as making connections between graphs, symbols, and tables is an important practice in teaching functions and algebra (Kieran, 2006). In order to better prepare MS mathematics teachers to teach algebra it is important to ensure that they have opportunities to learn to use various combinations of representations, especially the combination of symbols, graphs, and tables.

Teachers' Opportunities to Learn About Using Cognitively Demanding Tasks in Instruction

The vast majority of mathematical tasks presented in the four curricula (99%) can be classified as *cognitively demanding* (Stein & Smith, 1998) for the students for which their use was designated – whether elementary or MS students. Teachers were provided with extensive opportunities to use such cognitively demanding tasks themselves. The 38 PLTs featuring narrative cases provided teachers with opportunities to reflect upon examples of how such tasks could be used by another teacher in his/her classroom instruction. These cases showcased the classroom-based factors that are associated with the maintenance or decline of the cognitive demand of mathematical tasks, such as sustaining pressure for justification or failing to hold students accountable for high level products/processes (Henningsen & Stein, 1997).

Cognitively demanding tasks often do not have a specified solution path, requiring students to engage in problem solving and often resulting in a set of diverse and unexpected student solutions that teachers find challenging to manage (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). While the majority of mathematical tasks in the PD curricula could be categorized as cognitively demanding, most were designed for use with students in 7th grade or below. This, no doubt, is linked to the mathematical topics being focused upon – rational numbers and proportionality. However, it is important for MS teachers to have opportunities to reflect upon images of mathematics instruction using cognitively demanding mathematical tasks at the 8th grade level. The opportunity to solve and analyze the use of cognitively demanding mathematical tasks has been shown to support teacher learning (Koellner et al., 2007). Thus, it would be important to supplement teachers' learning opportunities provided in the sample curricula with additional opportunities to explore the use of such tasks at higher grade levels.

EDUCATIVE FEATURES OF THE FOUR CURRICULA: OPPORTUNITIES FOR PROFESSIONAL DEVELOPERS' LEARNING

Educative PD curriculum materials are designed to support the learning of both professional developers and teachers in PD settings. Using the EF framework, I found that the four sample curricula differ significantly in the extent to which they appear to be educative – ranging from the TFRU curriculum which provides a scarce 5 educative features, to the IIRP curriculum which offers a wide offering of 340 instances of educative features to support the learning of professional developers. From the survey data, it is apparent that professional developers often use the four sets of curriculum materials in conjunction with each other. Used in combination, the sample curricula provide professional developers with ample opportunities (more than 99 instances) to learn: (a) how to facilitate discourse during collective inquiry into mathematics instruction; (b) when to expect common teacher ideas to emerge; (c) why the activities were designed in a particular way and what teachers are expected to learn from them; and (d) how to focus teachers' attention on specific aspects of pedagogy laid open for investigation in PLTs.

In the United States, professional developers vary greatly in their preparation to work with mathematics teachers (Banilower, Boyd, Pasley, & Weiss, 2006) and have few opportunities for continued training (Ball & Cohen, 1999). Educative PD curriculum materials can offer one avenue of training support to professional developers by providing opportunities to learn how to support teacher learning around specific PLTs in curricula and in learning activities generally. The educative features most frequently presented in the four curricula relate to supporting collective inquiry and common discourse around mathematics instruction, and to planning for facilitation. As they relate to activities core to facilitating PD (Ball & Cohen, 1999), it seems appropriate that these are the features most commonly provided. The EF framework identifies the many features that can be incorporated into PD curricula to support the learning of professional developers. The four curricula showcase the different degrees to which curriculum materials contain these features and are educative for professional developers.

CONCLUSIONS

The PD landscape in the United States is relatively uncharted. Districts operate largely independently of each other, the educational community does not have a clear picture of what occurs, and there is a gap in the existing literature about the content being addressed in PD. By analyzing PD curriculum materials commonly used with large numbers of teachers across the country and, being publicly available, having the potential to be used with thousands more around the world, this study charts out the landscape of PD by identifying what teachers using these curricula would have opportunities to learn.

The results of the study have many practical and theoretical implications for teacher education and PD. The identification of teachers' specific learning opportunities in the curricula can support professional developers to select and sequence PD curricula to provide teachers with rich and sustained learning in PD. The study not only explored teachers' learning opportunities but also how PD curricula can support the learning of professional developers. My extension of the concept of educative curriculum materials to the PD space is an important contribution to the small, but growing, body of research on learning to lead PD. The analytic frameworks developed in the study can be used to further explore PD curriculum materials and their connection to teacher learning outcomes in future research studies.

To develop our understanding of teacher learning in PD settings we need to better understand both *what* teachers learn and *how* they are taught (Wilson & Berne, 1999). I have identified *what* large numbers of teachers using the four curricula have opportunities to learn in PD. The dissertation study, therefore, has contributed to a better understanding of MS mathematics teachers' learning in PD and has set the stage for future studies that can research *how* these learning opportunities unfold in PD. It has contributed to a better understanding of teacher learning *in* and *through* professional development.

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CHALLENGING OUR BELIEFS AND PRACTICES IN SECONDARY MATHEMATICS EDUCATION

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INTRODUCTION

This paper addresses the portion of my dissertation that describes how I worked with practicing teachers to examine and challenge their beliefs and practices, and how our critical action research experience helped to support the participating teachers in their quest for a productive change in their practice. This collaborative research study was set in a small high school that enrolled approximately 700 students. The participants were seven teachers. There were three experienced male teachers, two less experienced female teachers, and two male teacher candidates. The school was in a rural area outside a larger city centre. This mathematics department responded positively to my invitation to participate in research, and were willing to investigate their practice.

It has been argued in the literature, both locally and internationally, that mathematics reform movements have had little impact on classroom practice. There is also evidence that there is a lack of consistency between what teachers believe and what they practice. Moreover, there is a paucity of studies that address the pedagogical needs of practicing secondary mathematics teachers, and a lack of studies that apply a critical research approach to challenge teachers' institutional discourses and assumptions.

During my presentation at the CMESG, I addressed the following research question: To what extent do teachers benefit from examining barriers, power relations, and institutional assumptions implicit to their practice? This short paper focuses on the second phase of my dissertation, which was framed using Brian Fay's (1975) second phase of enlightenment. Fay predicted that, after an enlightenment period, participants would experience empowerment as they learned to become critical of their beliefs and practices. Since this discussion begins in the middle of a study, it is important to point out that, by this stage, participants trusted the facilitator. They described the facilitator as having integrity, generosity, and competence. Without the skills and philosophical framework that the facilitator brought to this context, it would not have been possible to challenge teachers with respect to their beliefs and practices and expect a productive outcome.

The purpose of this phase of the research was to ask questions that would elicit conversations about larger social and political structures in the education system, address specific power relations that exist in teachers' work, and examine barriers and other institutional narratives that might limit participants' ability to imagine a change in their practice. During this phase, the role of the facilitator was to challenge, support, and encourage voice.

CRITICAL AND EMPOWERING RESEARCH PEDAGOGY

Teacher education that is empowering "should provide practitioners with the tools and resources they need to recognize, analyze, and address the contradictions, and in so doing, open-up the possibility that conditions in schools can be different" (Smyth, 1989b, p. 4). Freire (2000) described the ideas of human agency and empowerment as being essential to worthwhile learning and research practice, and he generalized it to anyone who plays a leadership role in a learning situation. The commitment of participants includes more than just their presence in the study; rather, participation involves a deeper understanding and commitment to the research project. Specifically, with respect to the interpretivist view that knowledge is co-created, Stinson, Bidwell, Powell and Thurman (2008) added:

> *In the most general sense, critical pedagogy supports pedagogical theories and practices that encourage both teachers and students to develop an understanding of the interconnecting relationship among ideology, power, and culture and rejects any claim to universal foundations for truth and culture, as well as any claim to objectivity.* (p. 3)

The work of educational leaders and participants in critical research occurs within a cultural context that must itself be acknowledged. The critical perspective holds that teachers have been *normalized* by the organizations and teaching culture (Walshaw, 2004). Foucault was concerned about how, as members of society, we often limit ourselves by ascribing to the rules we have created. Foucault called this *governmentality* – the way in which we, as a society, organize our language, perceptions, values and practices to be truths that we simultaneously impose upon ourselves. He maintained that our reality defines for us what is or is not possible, and affirmed the importance and value of critical reflection by emphasizing that "the critical perspective in itself is sufficient because it opens up possibilities" (Allan, 2006, p. 292).

Fay (1975) first wrote about critical social theory in 1975 and described it with three characteristics. Firstly, the critical theorist must come to understand the participants and seek to find out how they feel. Secondly, the critical perspective must seek to uncover the assumptions and unconscious knowledge that are held. Lastly, and most importantly, theory and practice should be merged for the work to be useful (Fay, 1975). In 1987, Fay developed a model that is commonly referred to in the emancipatory literature. Fay (1987) viewed critically reflective practice as moving through three stages of *enlightenment*, *empowerment* and *emancipation*. Fay's model requires the articulation of barriers such as social norms, power relationships and previous learning. He argued that the reified barriers can become so embodied that it does not seem possible to question their existence, let alone work to transform them. These norms are responsible for the maintenance of the status quo in societies and organizations, and they tend to remain unquestioned. These norms are re-created through discourses or narratives that *reconfirm* their truth. Through critical reflection and understanding, individuals can learn to question what they might have originally believed to be unalterable realities. Fay also suggested that an atmosphere should be created in which research participants can exercise their voice and develop a sense of human agency. Fay argues that, when his model is applied, there will be evidence that the participants have become empowered through their actions, and that they will learn who they are as practitioners.

INSTITUTIONAL DISCOURSES

This study also intended to provide the reflective space for teachers to explore and understand their beliefs as well as the "interpretations of [their] social institutions and traditions" (Stinson et al., 2008, p. 619). It is argued that people "are bound by social norms (tradition), by power relations with others (authority), and by previous learning that has become embodied (embodiment)" (Johns, 2004, p. 8). This provides the rationale for examining barriers that "blind and bind...[and that] limit the practitioner's ability to respond differently to practice situations even when they know there is a better way of responding to situations in tune with desirable practice" (p. 8).

An awareness of the barriers was necessary in the process of this study because barriers tend to block teacher professional growth if they are not acknowledged. As Kemmis and McTaggart (2005) described: "[p]eople not only are hemmed in by material institutional conditions, they frequently are trapped in institutional discourses that channel, deter, or muffle critique" (p. 571). Facilitators who ask tough questions and elicit barriers can predict that their research discussions might result in institutional discourses that claim certain truths about the world of teaching. "Institutional discourses are made up of the assumptions, concerns, and vocabularies of members of socially organized settings, and the ways in which they interact" (Miller, 1994, p. 280). These are the stories and assumptions about schooling that are told and retold and are accepted as the norm in schools. However, this pedagogic discourse is not always a neutral entity. As Bernstein (2000) described, pedagogic discourse is a "carrier of power relations external to the school and a carrier of patterns of dominance with respect to class, patriarchy, race" (p. 4). That is, the conversations that repeat themselves in teachers' discourses reflect the surrounding culture, a culture that necessarily includes issues of power. What is interesting about institutional discourses is that they are repeated and rarely challenged. They create and sustain unconscious assumptions that nothing can be done about these *facts*, because the stories just describe *the way it is*.

Generally, teachers are not aware of the assumptions they make until these are challenged. This limits teachers' potential to learn and make pedagogical changes. Teachers cannot be agents of change unless they understand that *things can change*. It is possible to help adults learn to recognize the assumptions they are making, and some research is designed to address them. As Kegan and Lahey (2007) described:

> *[p]eople often form big assumptions early in life and then seldom, if ever, examine them. They're woven into the very fabric of our lives. But only by bringing them into the light can people finally challenge their deepest beliefs and recognize why they're engaging in seemingly contradictory behavior.* (p. 50)

Kegan and Lahey call these narratives that impede our growth and learning *subconscious competing commitments*, and describe how it is important to help people make those barriers explicit, and to challenge those subconscious competing beliefs or discourses.

Critical action research promises that participants can be empowered through greater self-knowledge and self-development, through an examination of barriers and assumptions about teaching culture. The three stages of enlightenment-empowerment-emancipation, as described by Fay (1987), are necessary for teachers so that they may come to understand their personal knowledge, beliefs and attitudes, and make sense of any tensions between them. Fay described how reflective practitioners will experience a *liberation* – or a "state of reflective clarity" (p. 205) bringing harmony to those tensions.

As Smyth (1989b) noted, it is important to ask explicit questions about power and influence. The following questions were asked during the CMESG presentation as well as orally and through journaling during the research study:

1. In my job, who (or what) has power or influence over my work?
2. In my job, over whom (or what) do I have power or influence?
3. What are the biggest barriers that stop me from implementing the best lessons?

4. What are the factors that facilitate my best work as a teacher?

These questions elicited many institutional discourses from the teachers. Barriers that were named included not having enough time to cover the curriculum, not having enough preparation time, a concern about government bodies and their accountability practices distracting teachers from instruction, low government funding, and issues of low student motivation.

A typical institutional discourse would be one that describes how mathematics should be taught. For example, Raymond, one of the experienced teachers, described how he felt that students had to learn and master basic mathematics skills in order to solve higher order problems. Raymond believed that an approach that applied too much emphasis on problem solving and not enough on the practice of basic mathematics skills would not be successful for students. He also felt that group-work was not productive because only a few of those students in groups would actually do the work. Having worked with teachers for over 20 years, I have heard the statements before. These ideas needed to be recognized and heard prior to this teacher moving forward in his practice. Also, through the implementation of the four stages of critical participatory action research, I was able to reflect on this teacher's needs, and implement actions so that we were able to learn and practice structures for working well and collaboratively in groups. Once this idea was presented, it removed a barrier for this participant. Later, during one of Raymond's reflective journaling exercises, his transformed perspective was revealed:

> *To tell you the truth I find writing reflections a little bit annoying because it forces you to look in the mirror and sometimes I don't always like what I see. Although making me feel guilty about myself, it makes me want to change, and that might be a good thing*. (Raymond's Journal - Phase V)

In his journal, Raymond described the moment that he became willing to step out of his comfort zone and test or implement a new teaching practice. Because his beliefs were challenged, Raymond was propelled into Fay's emancipation phase. He was later able to take action in his classroom, and, moreover to collaborate and communicate with his colleagues in the public sphere. This is discussed further in a paper that addresses the third phase of the research.

ALL VOICES MATTER

External to the work of teachers in the study, the critical action research practitioner must continuously reflect on equity in group discussions. Keeping this in mind, it was important to maintain an equilibrium between the voices of the participants. Although the facilitator had made this condition explicit, and had asked participants to make sure that all were included equally in our conversations, it proved to be difficult for individuals in this group to notice that they were not participating equally in the group. It was evident from the first day that, without intervention, some voices in the group would remain weaker. According to the participants, a very impactful practice was the use of a *Wordle* (Feinberg, 2009). The facilitator used the text transcribed from the audio data in the first session, which always listed who was speaking by introducing the speaker with her or his name followed by a colon. For example, when the facilitator spoke, the audio text that was transcribed read: "*Lorraine: What we...*". That meant that, every time someone spoke, her or his name preceded what he or she said. Since the online *Wordle*™ software uploads text and transforms it into a graphic that represents the frequency of a word by the size of its font, this was an appropriate and concise way to show the highlights of our conversation, and to show evidence of who spoke most often. The participants appreciated the use of this graphic to illustrate the nature of our conversations. In the *Wordle* (see Figure 1), it is easy to see by the size of the font that

Lorraine and Alex spoke the most, followed by Julian and Grant. Lecia and Cara hardly contributed at all, and Raymond only seized a small bit of the conversation time.

After the use of the *Wordle*, the balance of voices was more evident. Raymond even underlined it as an important factor of the study that the facilitator worked to include everyone's voice in the discourse. His and others' comments are presented in a future paper on facilitating critical action research.

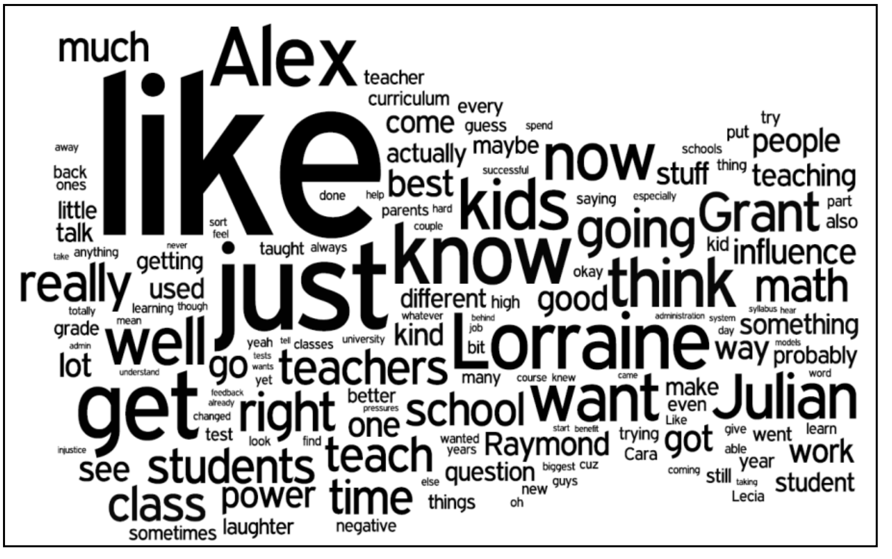

Figure 1. *Wordle* (Feinberg, 2009) showing the frequency of the words we used during our first session and displaying how often each member of the group spoke during the first phase.

TOWARDS A NEW MODEL FOR CRITICAL PRACTICE

This study showed that with certain supports and conditions, and through the process of completing the three phases of this research study, it was possible to impact the practice of a group of teachers with varying levels of loyalty to a traditional practice. By proceeding through certain processes, and by ascribing to a critical practitioner's stance, the facilitator of this research was able to take the participants through Fay's (1987) phases of enlightenment, empowerment, and emancipation to a point where the teachers felt they understood themselves better as educators. These teachers felt aware and confident enough to enact practices that they deemed to be somewhat outside their comfort zone, and still more student-empowering.

The figure on the following page (Figure 2) gives a summary of what were the most impactful practices for the participants. The processes that made a difference for the teachers in this study are listed in three columns, and the supporting perspectives and skills are given in the section at the bottom of the diagram. Further reflections on each of the elements are given in the sections that follow.

The discussions in this paper focus on the column in the middle of the figure below. These describe the most impactful practices found in the design of this study, in particular those that engaged participants in critical thought and in the investigation of their assumptions. The discussion must include the complex role of the facilitator while she asked difficult questions and worked to support all teachers equally throughout the process.

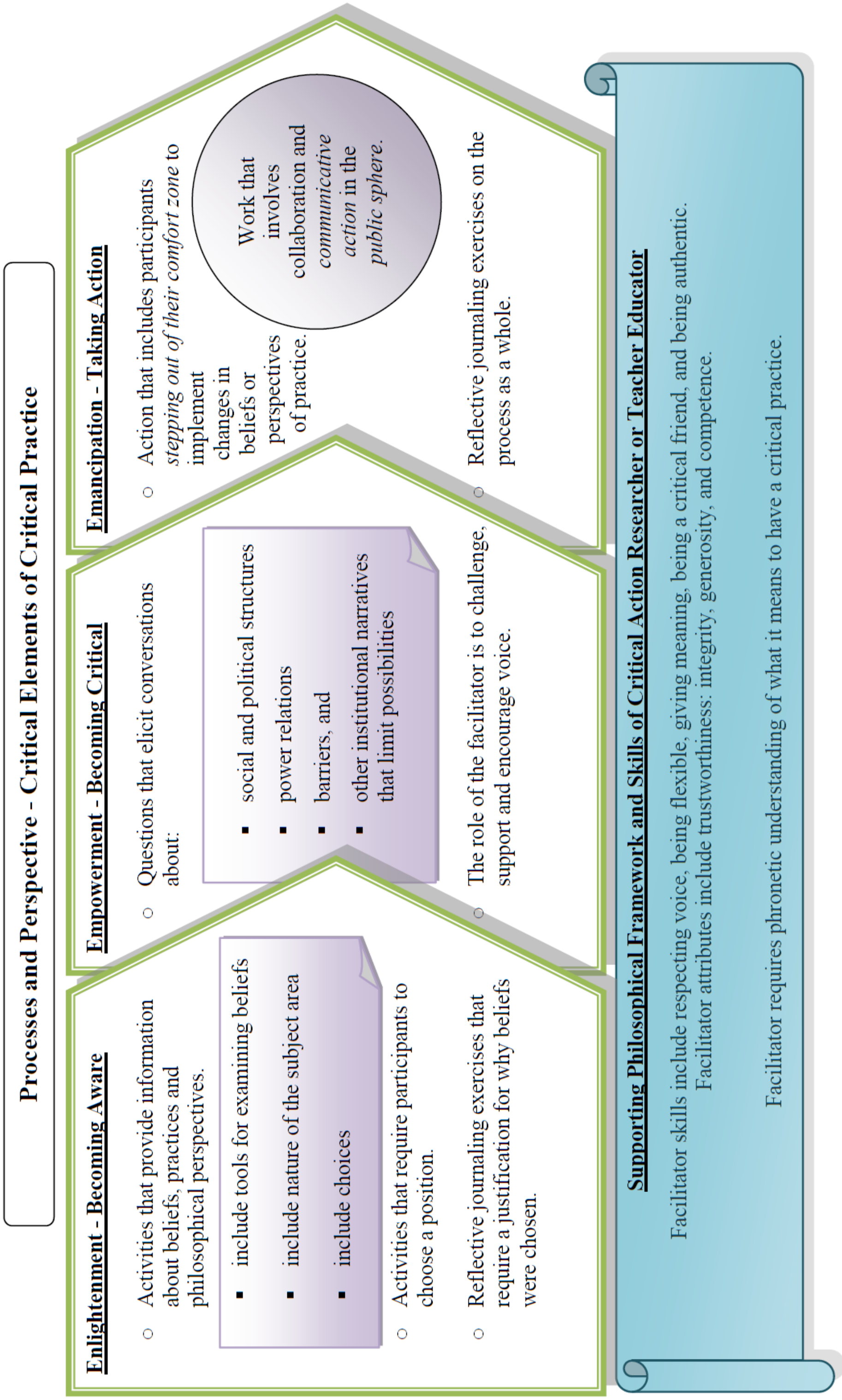

Figure 2. Processes and perspective: Critical elements for critical practice

IMPLICATIONS FOR PRACTICE AND RESEARCH

Participatory action research takes the stance that research and practice are viewed as integrated. It is certainly the case in this study that role of the teacher educator and critical action researcher was the same (Kemmis, 2006). Therefore, the following two implications are presented as ideas that can be applied to research or to practice.

APPLYING THE SUGGESTED PROCESSES AND PERSPECTIVE FRAMEWORK IN TEACHER EDUCATION OR OTHER SIMILAR SETTINGS

An implication for research and practice is that this study, or one that follows its philosophical perspective and its processes of enlightenment, empowerment, and emancipation, could be repeated in similar environments such as pre-service settings, elementary teacher groups, or graduate-level courses in order to ascertain its generalisability and its applicability. These settings are similar to the one in the study because they usually hold comparable time-lines and parallel purposes, and because the participants would likely have corresponding needs. As Hofer (2006) expressed:

> *[w]e do need to know more....about how teachers resolve the cognitive dissonance that presumably arises from strongly endorsing a worldview that appears incongruent with the practices of the educational systems in which they are placed, and we need to do more in our teacher education programs to prepare teachers to address such inconsistencies.* (p. 90)

IMPLICATION FOR PRACTICE

An implication for practice is that questions such as the ones in this study, or others such as those recommended by Smyth (1989a, 1989b) could be used to elicit assumptions and to challenge teachers to notice the social constructs that define and may limit their work. These questions can be asked no matter what the participants' teaching level or field. Whether working with teachers in practice or in post-secondary settings, it is important to discuss and challenge the barriers they perceive, and to ask them difficult questions about power relations and about other assumptions they make in their work that serve to define what they can and cannot do. Furthermore, these questions are not limited to work with teachers. There are surely other professions where social and political pressures, power relations and other barriers limit people's ability to do their best work. Examining questions such as these would be a benefit and could be applied to professions such as health care or law.

An important implication that will be discussed in the fourth paper of this series is that of how it is effective, or even possible, for teacher educators to challenge teachers' beliefs. This is not done in a vacuum. As seen in the model, *Processes and perspective: Critical elements for critical practice*, the facilitator must practice her research and teaching consistent with a set of values that exhibit attributes of integrity, generosity, and competence. Without this perspective, the task of challenging teachers would not only be ineffective, but may be damaging to the participants, and most certainly to the process. The process of developing and sustaining the important qualities of the facilitator are discussed in detail in the fourth and last paper of this series.

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PATTERN RULES, PATTERNS AND GRAPHS: ANALYZING GRADE 6 STUDENTS' LEARNING OF LINEAR FUNCTIONS THROUGH THE PROCESSES OF WEBBING, SITUATED ABSTRACTIONS, AND CONVERGENT CONCEPTUAL CHANGE

Ruth Beatty
Lakehead University

CONTEXT

In recent years, increasing numbers of mathematics educators, policy makers and researchers have proposed that the learning of algebra become included in the elementary curriculum as part of the "algebra for all" and "early algebra" movements (Warren & Cooper, 2006; Blanton & Kaput, 2004; Carpenter, Franke, & Levi, 2003; Kieran, 1990, 1991, 1992; Kieran & Chalough, 1993; Greenes, Chang, & Ben-Chaim, 2007). The rationale for introducing algebra into the elementary mathematics curriculum is to develop young students' abilities to think algebraically with the hope of diminishing the abrupt and often difficult transition to formal algebra in high school (Kieran, 1992). Further, researchers propose that an early introduction to algebra would help to provide *all* students with equitable opportunity for success in later mathematics learning, ultimately broadening their educational and career choices (Greenes, Cavanagh, Dacey, Findell, & Small, 2001; Kaput, 2007). Algebra plays a critical role as a gatekeeper in school mathematics and in society beyond school years – particularly for minority students and for those from lower socioeconomic status backgrounds. Preparing elementary students for the increasingly complex mathematics of the 21st century requires extensive research to identify learning experiences that best support early algebraic thinking. However, while the inclusion of algebraic concepts in teaching prior to high school is becoming a wider practice, research to support both the assumed benefits and curricula development is in early stages. Although there is a body of research that looks at students' development of algebraic concepts (e.g., Kaput & Blanton, 2003; Carpenter et al., 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006), this work investigates students' use of symbols and equations based on generalizations found in arithmetic.

In light of this new international focus on *early algebra*, I spent three years of my doctoral studies conducting research on the potential of elementary students to think algebraically. My thesis constitutes the third year of this larger study and reports on the design, implementation, and analysis of an innovative approach to teaching linear relationships and negative numbers, two historically difficult areas of mathematics to teach. The approach was designed to support Grade 6 students' understanding of linear relationships by prioritizing visual representations in the form of linear growing patterns and graphical representations of linear relationships. The lessons were also designed to introduce students to working with negative numbers by anchoring these within the quadrants of the Cartesian graphing space. The teaching sequence

– seven lessons – was implemented in a classroom of ten Grade 6 students over the course of four months. The design of the study can be considered an "instructional experiment" (Freudenthal, 1991), defined as a research design that incorporates an intervention in the form of an instructional sequence as a way of broadening students' insights into a particular mathematical construct, while simultaneously providing the researcher with a greater understanding of students' learning processes.

ALGEBRAIC CONTENT LEARNING

This study built on my previous two years researching an experimental approach to algebraic instruction. When considering the study of linear relationships, mathematics educators recommend that students be introduced to various representational forms of linear relationships in order to develop the ability to effectively use these representations as a means of considering quantitative relationships (e.g., Janvier, 1987a, 1987b; Moschkovich, Schoenfeld, & Arcavi, 1993). However, in practice, most students are formally taught linear relationships using only equations.

My previous studies had prioritized visual representations of linear relationships, merged with the numeric expressions of the mathematical structure. A number of researchers note that when visual and numeric representations are integrated, students can construct a deeper understanding of linear relationships (Moss & Case, 1999; Yerushalmy & Sternberg, 2001; Noss, Healy, & Hoyles, 1997; Mason, 1996). Students first worked with linear patterns and discovered the relationship between the position number of each iteration of the pattern and the number of tiles in that position. The pattern below (Figure 1) is a representation of the linear relationship "the number of tiles = position number times 2 plus 3", which can be written as the pattern rule: tiles = position × 2 + 3. If we substitute y to stand for the number of tiles (dependent variable), and x to stand for the position number (independent variable), we have $y = 2x + 3$, a linear algebraic expression. The $2x$ is represented in the pattern by the tiles that increase by 2 at each successive position. The +3 is represented by the three tiles that stay the same at each position. By understanding the relationship between the position number and the number of tiles, it is possible to predict how many tiles would be in a far position of the pattern (e.g., the 100th) or in any position of the pattern.

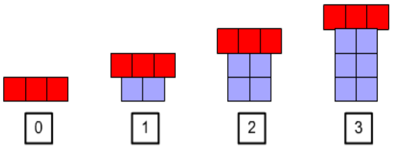


Figure 1

Researchers also stress that it is the ability to make connections among different representations, particularly equations and graphs, that allow students to develop insights for constructing the concept of a linear relationship (e.g., Evan, 1998; Bloch, 2003). Numerous studies have documented the difficulties students have when exploring the connections between equations and graphs (e.g., Evan, 1998; Moschkovich, 1996, 1998, 1999; Brassel & Rowe, 1993; Yerushalmy, 1991). A related concern is the emphasis placed on procedural knowledge when teaching students how to solve linear equations of the form ax + b = cx + d. Students are taught a standard algorithm for solving this equation. Although students who learn the algorithm can generate correct solutions, it is generally accepted that the learning of any mathematical procedure must be connected with conceptual knowledge to foster the development of understanding (Hiebert & Carpenter, 1992).

THESIS INSTRUCTIONAL SEQUENCE

The first goal of this study was to see whether an understanding of linear patterns could support an understanding of linear graphs, and whether this could potentially be connected to solving equations. Based on my previous work, for this thesis I developed an instructional sequence grounded in linear growing patterns and used these patterns to introduce graphs. As students engaged in the activities, it was my hypothesis that they would discover the connections among the pattern rule, the pattern, and the graph. Students could also compare the trend lines of two patterns, which leads to an understanding that the point of intersection on the graph represents the position number at which both patterns would have the same number of tiles. This is precursory understanding for solving $ax + b = cx + d$.

My approach to instruction departs significantly from traditional approaches in three ways: 1) by introducing graphing through patterning; 2) by introducing graphing prior to teaching formal algebraic notation; and 3) by focusing on the graph as a representation of a linear relationship, not as a representation of ordered pairs (i.e., students are asked to graph the pattern *rule*, not to graph a series of coordinates). See Figure 2.

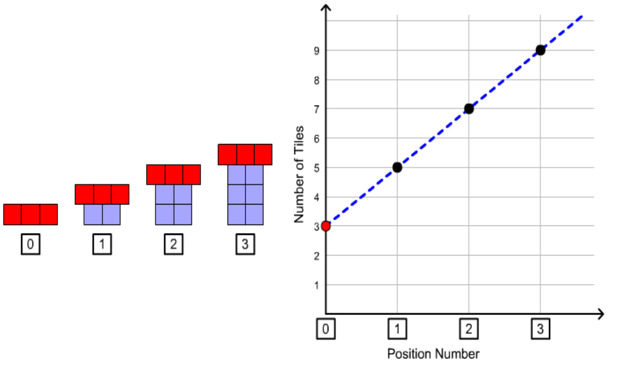
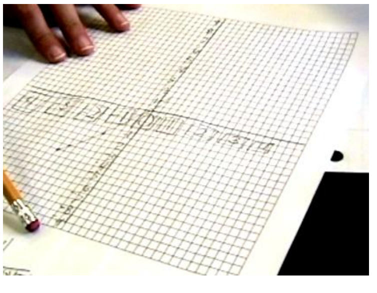


Figure 2



I then used this understanding of graphical representations to introduce working with negative numbers. Since a four quadrant graphing space is essentially two perpendicular number lines, I hypothesized that this would give students both a visual anchor for working with negative values, and meaning to operations with values less than 0. Students could explore the outcome of operations with negative numbers in a two-dimensional space.

THEORIES OF LEARNING

The second goal of the study was more theoretical – to assess the potential utility of combining two complementary frameworks in order to meticulously document and assess the development of algebraic understanding – both individual student understanding and the collective understanding of the group, and the interactions between the two. I was interested in discovering how individuals and groups construct knowledge, how groups work together, how individuals function within a group, and how students construct convergent and divergent theories.

One framework was based on Noss and Hoyles' notions of *webbing* and *situated abstractions*, which can be defined as the development of successive approximations of formal mathematical knowledge in individual students (e.g., Hoyles, Noss, & Kent, 2004; Hoyles & Noss, 2003; Noss & Hoyles, 1996, 2006). *Webbing* is defined as the interconnection between abstract mathematical concepts and the concrete tools and models used to construct and reconstruct mathematical knowledge through experience. Learning is therefore defined as the construction of a web of connections "between classes of problems, mathematical objects and relationships, real entities and personal situation-specific experiences" (Noss & Hoyles, 1996, p. 105). A *situated abstraction* is a particular form of mathematical understanding that emerges through sense-making activity. The conception of the *situated abstraction* recognizes that the abstraction of mathematical properties is situated and shaped by the tools/artifacts being used when working on particular activities designed with the intent of supporting mathematical thinking. Because the approach I developed was new, I wanted to explicitly document the situations in which the learning took place so that I could better understand the mathematical concepts students developed.

The other framework came from Roschelle's (1992) work on collaborative conceptual change, which allowed me to examine and document mathematical understanding that developed at the whole-class level. "Situated abstractions by their nature are diverse and interlinked with the tools in use, so the question is how can meanings be shared in the classroom and interconnect with each other?" (Hoyles, Noss, & Kent, 2004, p. 317). How can the various 'bits' of learning taking place within individuals be shared among the larger group? According to Roschelle, the basis of collaboration is the convergence of meaning – two or more people constructing shared meanings for concepts and experiences. Students engage in an iterative cycle of displaying informal understandings and constructing a common understanding within the context of situated actions as they seek to refine their minimal abstractions into increasingly integrated sophisticated concepts.

I used the two analytical frameworks to make sense of individual and group level data in order to pinpoint the interplay between individual and collective actions and understanding. I used a two-level case study approach in order to simultaneously analyze both individual learning and the collective learning of the group.

RESEARCH QUESTIONS

1. What situated abstractions are forged at the group level and how are shared abstractions constructed?
2. For each individual student, what situated abstractions are forged through the webbing of internal resources (intuitions, past experiences) and external resources (classroom tasks, tools, discourse experiences)?
3. How do individual students' situated abstractions converge/diverge as students participate in this lesson sequence?
4. To what extent does this third-year lesson sequence support students in developing an understanding of graphical and numerical representations of linear relationships? To what extent does this third-year lesson sequence support students in developing an understanding of negative numbers in the context of graphical representations?

ANALYSES

To answer research questions 1 and 2, I utilized a two-level case study design. I developed the case studies (one whole-class and ten individual) using a variety of qualitative data, primarily videotape data. In order to analyze the data, I utilized Powell, Francisco, and Maher's (2003) framework of seven interacting nonlinear phases of videotape analysis for studying the development of learners' mathematical ideas. These are: viewing the video data attentively; describing the data; identifying critical events; transcribing; coding; constructing a storyline; and composing a narrative.

The process of analysis occurred both during the lesson implementation and during subsequent videotape analyses after the intervention had ended. As the intervention was being conducted, I analyzed data for preliminary categories (identified both *a priori*, and based on the literature review, and *a posteriori* as categories emerged), and then collected additional data as the intervention progressed. After the intervention, I continued to analyze the video data and transcripts. Initial data reduction was accomplished by using an inductive, line-by-line categorizing coding strategy (Padgett, 1998) in order to 1) identify actions and conversations that pertained to the task from those that did not; and 2) identify actions and conversations in order to track the learning path taken and the situated abstractions articulated both at the group and individual levels.

The result was the creation of eleven case studies. Chapter Six of my dissertation presents the whole class case study and Chapter Seven presents the case studies of each of the ten students. Every lesson is described, and key points of learning identified. Each episode of interest is interspersed with an interpretation of the student learning demonstrated, based on both the research literature about the mathematics the students were learning, and also based on the frameworks of *webbing/situated abstractions* and *collaborative conceptual change*. The learning trajectory of the class as a whole, and of each student are documented in terms of the interplay between activities engaged in, tools and techniques students utilized, and the resulting student understandings developed by the class as a whole and by each individual student.

To answer question 3, I compared each individual's learning trajectory with that of the group and coded individual students' understandings as convergent or divergent. Divergent understandings were further coded as graphically or numerically based, that is, whether the primary site for problem solving was the Cartesian graphing space, or the use of linear equations. I then conducted frequency counts and used descriptive statistics to summarize and compare the frequency counts.

To answer question 4, assessing the innovative approach to teaching linear relationships and negative numbers, I carried out a qualitative analysis of pre-post student interviews, and also a detailed analysis of student responses on a pre-post pencil and paper mathematics survey. I developed both the interview protocol and the survey based on a comprehensive review of the literature and on the results of my previous two years of research. In addition, I conducted an analysis of pre-post test scores using non-parametric descriptive statistics, and determined the effects of the lessons on student learning as a factor of demonstrated student achievement level by comparing mean gain scores for high-, mid-, and low-achieving groups of students.

RESULTS

One goal of the study was to determine whether this lesson sequence would support students' understanding of linear relationships and negative numbers. The results were unprecedented. The participating Grade 6 students were all able to engage in the kinds of algebraic thinking that have shown to be difficult for high-school students. The students were able to make connections among different representations of linear relationships (figural, graphical, and numeric) and, based on these connections, developed sophisticated methods for solving equations of the form $ax + b = cx + d$. They also developed an understanding of working with negative numbers and, by using their understanding of plotting trend lines on a graph, were able to conceptualize adding, subtracting and multiplying with negative numbers. This instructional approach, which prioritized visual as well as numeric representations of linear relationships, seemed to have alleviated many of the difficulties outlined in the research literature.

The second goal of the study was to analyze the algebraic understandings constructed by the whole class, and by individuals within the class. When comparing the tables of situated abstractions for individual students, approximately two thirds of the situated abstractions listed were those that were constructed at the group level. This emphasizes the importance of communication and collaboration as the ideas generated and modified at the class level were then internalized and incorporated into individual's developing understanding.

Documenting the external resources of the learning situation (collaboration in the classroom and tool use) and the internal understandings (situated abstractions) allowed me to learn about the nature of learning for the group, and for each individual student. The result is an extensive documentation of the building up of the layers of intuitions that underlie the development of mathematical abstractions, and the interplay between situations, actions through tools, and developing intuitions as a way of constructing mathematical meaning.

CONTRIBUTIONS OF THE STUDY

With respect to designing and assessing a new learning sequence, this study makes two important contributions. The first is the learning sequence itself, which supported Grade 6 students in developing sophisticated understandings of linear relationships and negative numbers, and alleviated some of the well-known problems in students' understanding of both linear graphs and negative numbers. By the end of the sequence, these Grade 6 students had developed an understanding of how to solve linear equations of the form $ax + b = cx + d$, and how to meaningfully carry out operations with negative numbers. Because the field of *early algebra* is relatively new, this adds to our understanding of the potential of children's algebraic reasoning. When offered a carefully sequenced series of tasks it is evident that young students can grapple with difficult algebraic content, and that these are concepts that can successfully be integrated in elementary curricula.

Related to this is a detailed analysis of the multiple kinds of representations students preferred to work with. The idea of multiple representations of mathematical concepts now permeates all research in the field of mathematics education. In my study I analyzed students' emergent understanding of linear relationships mediated by their reliance on figural visual representations (patterns and graphs) versus more numeric representations (equations). These analyses are relevant given that in recent years many scholars (Becker & Rivera, 2005; Carraher et al., 2006; Blanton & Kaput, 2004; Warren & Cooper, 2006) have been investigating the affordances of figural versus numeric approaches to linear relationships.

The other main contribution of this study is a model of utilizing two complementary analytical frameworks in order to gain a broad understanding of the kinds of student understanding this instructional sequence supports. Because the instructional approach was new, the aim was to get an overview of student learning through the lens of convergent conceptual change, and also through the related lenses of situated abstraction and webbing. I chose these two frameworks because they both emphasize the situated nature of learning, that is, the need to take into account actions and communications in relation to specific situations in order to understand the kind of learning taking place.

As the implementation of the learning sequence progressed, extensive data was collected and interpreted with reference to collaborative process based on Roschelle's theoretical framework, and simultaneously with respect to the development of individual students' understandings based on Noss and Hoyle's ideas of *webbing* and *situated abstraction*. The resulting learning trajectories list the development of situation abstractions as they became more refined for the group, and for each individual within the group. Adapting complementary analytical frameworks allowed for an analysis of algebraic content understanding. It also allowed for an analysis of the pathways of understanding that developed and how these pathways converged and diverged for the participating students.

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LEARNING MATHEMATICS FOR THE WORKPLACE: AN ACTIVITY THEORY STUDY OF PIPE TRADES TRAINING[1,2]

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INTRODUCTION

Mathematics in workplace training has often been a source of difficulty, if not an obstacle, for many students seeking better jobs and advancement (Fownes, Thompson, & Evetts, 2002; Zevenbergen & Zevenbergen, 2004). Yet relatively little attention has been paid to workplace-training mathematics in education research. In an effort to provide insights into this important area, this study examines mathematics practice and learning within an eight week pre-apprenticeship training program for the skilled trades of plumbing, steamfitting, sprinkler fitting, and gasfitting conducted at a trade union run school in British Columbia. Sociocultural theory, specifically activity theory (Engeström, 1999; Leont'ev, 1978) and Radford's (2008b) recent elaboration, the cultural-semiotic theory of knowledge objectification, serve as theoretical lenses to draw attention to the unique features of this form of mathematics activity in the present analysis. These complementary perspectives draw attention to mathematics learning as a culturally and historically situated, multilayered, and goal-directed social process mediated by artifacts, including semiotic resources and norms or conventions of workplace training. Mathematics learning from here is regarded as a process of objectification – a process in which one becomes progressively aware and conversant, through one's actions and interpretations, of a cultural logic of mathematical objects. Findings from this study will inform both the teaching and design of workplace training as well as school mathematics programs intended for students who will move on to workplace training afterwards.

RELATED RESEARCH

To date, the research related to workplace or vocational mathematics has focused, largely, on three things. First, mathematics *content* used, from a perspective of school mathematics, in specific workplaces or vocations for the purpose of informing mathematics curriculum design and the teaching of *the* mathematics needed for work (e.g., FitzSimons, 2005; Zevenbergen & Zevenbergen, 2004, 2009); second, particular features of mathematics practices within specific workplaces, including the use of workplace-specific conventions and artifacts, or the connection between local knowledge and these practices (e.g., Martin & LaCroix, 2008;

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Williams & Wake, 2007b); and, very much related to both of these, the problematic notion of transfer of school mathematics competencies to the workplace (e.g., Williams & Wake, 2007a). Zevenbergen and Zevenbergen (2009) provide the following poignant critique of existing research literature on workplace mathematics:

> *Much of the work undertaken in mathematics education that explores workplace numeracies is premised on seeking to identify the [formal] mathematics of the workplace . . . . Applying a mathematical lens to observe workplace activity means that the activity can be overridden by the mathematical imperative. Skovsmose (1994) argued that this phenomenon can be seen as the formatting power of mathematics, so that it is often difficult to see events for their activity, but rather to subjugate the activity for mathematics. Such an approach preserves the hegemony of particular forms of knowing and doing. However, it fails to recognize and validate the processes employed by workers as they undertake their tasks and how they go about solving problems.* (p. 184)

A relatively small number of mathematics education researchers have conducted research for the purpose of understanding the mathematics within the workplace as culturally situated forms of practice making explicit use of cultural-historical activity theory (e.g., FitzSimons, Mlcek, Hull, & Wright, 2005; Noss, Bakker, Hoyles, & Kent, 2007; Williams & Wake, 2007a; Zevenbergen & Zevenbergen, 2004, 2009), drawing on the work of the contemporary activity theorist Yrjö Engeström (e.g., 1999, 2001). A significant limitation of Engeström's popular take on activity theory, however, is that while acknowledging the prominent mediating role of semiotic resources in activity, such as workplace mathematical activity, it does not provide for a detailed account of this (Noss et al., 2007).

THEORETICAL PERSPECTIVE

Activity theory provides a framework for examining how humans purposefully transform natural and social reality (including themselves) as a materially and socially mediated, and culturally and historically situated, process. Originating in the dialectical sociocultural psychology of Vygotsky, this perspective was developed into a theory of activity by his student and colleague, A. N. Leont'ev (cf., 1978) and others. Today in the Western research literature, this perspective is referred to as cultural-historical activity theory, or the acronym CHAT, emphasizing the essential situated nature of activity. Central to CHAT is the view that an *activity system* comprised of a subject, community, tools (including signs and artifacts), rules or norms, and division of labour, all oriented towards the object and outcome of the activity, constitutes the minimum unit of analysis (cf., Engeström, 1987, 1993, 2001). Furthermore, for any activity system, it is the meeting of a human need that serves as its motive.

Radford developed the theory of knowledge objectification to unpack processes and nuances of the mathematics activity and learning of individuals from a cultural-semiotic activity perspective based on his reading of Vygotsky's semiotics, Leont'ev's activity theory, and the more recent work of Felix Mikhailov and Evald Ilyenkov (cf., Radford, 2006, 2007, 2008c). Radford's cultural-semiotic activity theory is distinguished from other developments in CHAT by its foci on specific aspects of the consciousness, learning, and being of individuals, as well as the semiotic and social dimensions of mathematics activity. From this perspective, learning is conceptualized as an interactive and creative acquisition of historically constituted forms of thinking. This theorization emphasizes:

1. the intimate dialectical relationship between human thinking – including mathematical thinking – and the material and cultural world;
2. the central role of semiotic resources and social interaction in mathematical activity and learning; and

3. the reciprocal processes of *objectification* – making sense of and becoming critically conversant with the cultural-historical logic with which systems of thought, such as mathematics, have been endowed – and *subjectification* – the process of becoming (see also Radford, 2008a).

Radford's concept of objectification emphasizes the dialectical way that the subject of mathematics learning activity and the cultural object being attended to are related. *Semiotic means of objectification* refers to enacted reflections of this process. This refers to the use of semiotic means to draw and sustain the attention of others and one's own attention to particular aspects of mathematical objects in an effort to achieve stable forms of awareness, to make apparent one's intentions, and/or to carry out actions to attain the goal of one's activity. Radford has identified the following three forms of semiotic means of objectification from his empirical classroom research of collaborative mathematical problem solving and learning:

1. *Iconicity* – the process of noticing and re-enacting or re-voicing significant parts of previous semiotic activity for the purpose of orienting one's actions and deepening one's own objectification (Radford, personal communication, September 29, 2008);

2. *Semiotic nodes* – places in mathematical activity where multiple semiotic resources are used together and in a coordinated manner to achieve knowledge objectification (Radford, 2005);

3. *Semiotic contraction* – the process of coming to recognize and attend to the essential elements within an evolving mathematical experience and making one's semiotic actions compact, simplified, and routine as a result of this acquaintance with conceptual traits of the objects under objectification and their stabilization in consciousness (Radford, 2008a).

The process of mathematics learning or objectification is accounted for readily by the theory of knowledge objectification through analysis of social interactions and the semiotic means of objectification used within learning activity (Radford, 2008c).

The theory of knowledge objectification addresses squarely Noss et al.'s (2007) call for an activity-theoretical approach to account for the semiotic dimensions of mathematics activity mentioned earlier and, furthermore, it provides a basis for making clear the distinction between mathematics activity in workplace training and that in primary and secondary school. Radford accomplishes this by introducing the concept of *the territory of artifactual thought* to highlight the integral role of material artifacts in human thinking and *semiotic systems of cultural signification*. This concept positions beliefs about conceptual systems and conceptions about truth (ontology), knowability (epistemology), methods of inquiry (methodology), and legitimate knowledge representation (semiotic systems), as essential elements of any form of mathematics activity. These concepts, in turn, call attention to mathematics within different historical or cultural contexts, including various forms of training and workplace activity, as distinct and entirely legitimate forms of semiotically- and artifactually-mediated mathematics practice. From this perspective, academic or school mathematics as we know it is only one of a number of different and legitimate forms of mathematics. While, to my knowledge, Radford's theory of knowledge objectification has been applied only to the analysis of mathematics learning within school classrooms to date, it is ideally suited to the task of analyzing mathematics activity and learning in workplace training. In turn, the study of mathematics activity and learning within the context of workplace training provides an ideal setting in which to ground the theory of knowledge objectification.

RESEARCH CONTEXT AND RESEARCH QUESTIONS

This study has two foci. The first is the mathematics activity of the pre-apprenticeship class as a whole over its 8-week duration. The second is the activity of a single pre-apprentice (who will be referred to here as C) learning to read fractions-of-an-inch on a measuring tape during a 33-minute one-on-one impromptu tutoring session with the researcher serving as tutor (referred to here as L). The pre-apprentice who was the focus of this investigation was a high school graduate.

The specific research questions that framed this analysis were:

1. What is the nature of the mathematics activity within this pre-apprenticeship program?
2. Within the context of a tutoring session, what constitutes the mathematics activity of learning to read fractions-of-an-inch on a measuring tape?
3. What other significant processes within this mathematics activity are not yet addressed by activity theory or the theory of knowledge objectification, and in what ways do they inform these theories?

METHODOLOGY

The analysis draws on video recordings, copies of the various print artifacts used, including course handouts and students' written work, and field notes taken by the researcher throughout the pre-apprenticeship course. Analysis of the mathematics work done by the students was based upon a systematic review of the print materials used in the course. Analysis of the tutoring session entailed slow-motion and frame-by-frame analysis of the video to assess the role and coordination of various semiotic systems, actions, and artifacts. Particular attention was paid to: the semiotic system of cultural signification, norms of practice, contradictions or conflicts that serve to motivate this activity, specific objectives of or subgoals in the learning process for this student, semiotic processes used both by the student and tutor in the objectification process, as well as changes in the subjectification of both the pre-apprentice and researcher-as-tutor during this process.

SUMMARY OF FINDINGS

This training program was designed to prepare pre-apprentices for entry into formal pipe-trades apprenticeship programs in one of four pipe-trades specializations following completion of this course. The mathematics addressed reflected mathematics workplace production applications that would be addressed in the early years of the in-school components of these programs. More specifically, the pre-apprentices were required to think mathematically in historically and culturally constituted ways that led them to interpret various technical documents, read an imperial ruler or measuring tape, and perform calculations needed for a well-defined set of pipe-trades production activities efficiently, reliably, and to within acceptable tolerances. In its practical dimension, these findings were consistent with those cited in the workplace mathematics research literature (e.g., FitzSimons et al., 2005).

However, the present analysis departs from the existing reports of mathematics in workplace training by providing a detailed and nuanced view of distinctive features of this activity as a whole. The use of the theory of knowledge objectification, specifically the semiotic system of cultural significations, highlights ways in which the mathematics in this activity was a distinct and a legitimate form of mathematics with its own ways of doing things, not merely a sub-set of school or academic mathematics. Unlike school mathematics, for example, mathematics within the context of the pipe trades uses discrete numbers (i.e., all linear measurements are made to the nearest 16^{th} of an inch) and empirical methods of validation or *fit* as a legitimate basis for establishing mathematical fact or truth.

It is a fundamental assumption of activity theory that all elements within an activity system serve to mediate it. The activity system within which C learned to read fractions-of-an-inch on his measuring tape involved the semiotic resources that C and L employed including:

- spoken language including mathematics vocabulary;
- voice inflection and changes in volume;
- mathematics notation;
- three forms of gesture – pointing or indexical, sweeping, and chopping;
- a line drawn to represent 5/8 of an inch;
- indexical inscriptions such as circling or underlining existing inscriptions;
- counting;
- written text;
- rhythm in speaking or gestures; and
- the position, orientation, alignment of physical objects.

In addition, artifacts that C and L used included: the imperial measuring tape from C's toolbox, a pencil, a paper, and a set of rulers on transparencies. The workplace conventions, or norms, that were enacted included the use of binary fractions to the nearest 16^{th} of an inch for measuring, as well as norms shared with school mathematics, such as expressing fraction results in lowest terms. Last, the division of labour that mediated the learning activity during the tutorial included the manner in which C and L each participated in leading the discourse. (C started by playing a passive role in this regard and became a more active contributor as the session progressed, and L's role throughout the discourse was as the sole arbiter of the correctness of C's work.)

While it was not possible to determine precise mediating roles of these elements throughout the activity, we can see evidence of each playing a dynamic role in shaping the course of events. The various semiotic systems employed, for example, served to draw C's attention to particular aspects of the object of the activity and to deepen his understanding. The design of the particular measuring tape used (marked in 32nds of an inch up to 12 inches, and in 16ths thereafter) necessitated that this difference be attended to explicitly and negotiated during the activity. And the conventional design of the measuring tape with the endpoints of subintervals of an inch indicated by a system of signs necessitated that L draw C's attention explicitly to the intervals between these divisions rather than the division markings themselves as the object of their discussion in the process of learning to measure.

A number of processes, identified within the theory of knowledge objectification that shaped and reflected C's and L's understandings within the activity of learning to read the measuring tape, figured prominently within their exchanges. To summarize, C repeated or re-enacted what L had just said or done relating to the task-at-hand on 60 separate occasions during the 33-minute tutoring session. These actions reflected C's effort to deepen his sense of – literally, to deepen his sensory experience of – these statements or actions using the same means of semiotic expression that L had used, or other means. On one occasion C re-enacted a unique form of semiotic expression, a novel form of gesture, that he had just used himself; on a few occasions L re-enacted or repeated what C had done or said earlier; and on another occasion L created a zone of proximal development for C to help bring coherence to his understanding of the division pattern on the measuring tape by inviting C to explain what he (L) had said earlier and then by providing him with verbal prompts to help him along. These examples of repeating or re-enacting what another had said correspond to the process of iconicity, identified by Radford (2008c) as a significant part of the process of attaining a cultural logic of thinking or knowledge objectification.

The ways that L and C used multiple semiotic systems together (semiotic nodes) throughout the tutoring session and, in C's case, the further enactment of semiotic contractions, reflected their understandings of the object of the activity. These included L's frequent use of various combinations of words, pointing and sweeping gestures, fractions written using digits and words, the fractions-of-an-inch division pattern on the measuring tape and transparency rulers, along with other semiotic resources in a coordinated manner to draw and maintain C's attention to/on various aspects of the system of binary fractions-of-an-inch on the measuring tape. Given L's extensive experience working with the system of binary fractions-of-an-inch extending back to his own elementary school days, it is not surprising that his use of various semiotic systems remained relatively consistent during his explanations to C throughout the tutoring session, reflecting little or no change in his understanding in the process.

In contrast, there was a marked shift over the duration of the tutoring session in the way that C expressed his understanding using various combinations of semiotic systems as he communicated with L and brought clarity to his own thinking. Early on, when C responded to L's request for him to explain what difference he noticed in the patterns of divisions below 12 inches on the measuring tape, where it was marked to 32nds of an inch, and above 12 inches where it was marked only to 16ths, C's response was predominantly gestural, accompanied by only a single sentence and two sentence fragments. As the tutoring session progressed, C's means of expressing himself shifted completely, at times, to the clear and succinct use of words alone – an ultimate form of a semiotic contraction.

The last process from the theory of knowledge objectification to be summarized here is that of C's and L's subjectification. Over the 33 minutes of the tutoring session, C became more active in the way in which he participated within the activity. This is evidenced by the collective changes in the patterns of his gaze and attentiveness, his role in the dialogue, his affective responses, and his own expressions of agency and self-reliance regarding his use of the measuring tape. C also nodded his head or said "okay" or "yeah" on numerous occasions throughout the session, acknowledging to L that he was following what L was saying. This also reflected part of C's process of subjectivity within the activity. L changed during the tutoring session as well, but in a less obvious way. Specifically, L changed in his approach to teaching C how to read the measuring tape from a more generalizable approach (intended for reading any form of binary measuring tape or ruler marked to any binary subdivision of an inch), typical of school mathematics teaching, to a much more practical one tailored specifically to the workplace demands within the pipe trades, specifically reading fractions-of-an-inch only to 16ths.

This analysis revealed a number of features of mathematics learning activity that are new to activity theory generally, and the theory of knowledge objectification in particular. A new form of iconicity was identified (that of re-enacting a form of gesture that appeared initially as a novel form of gesture enacted by oneself) as was the social process of *semiotic extraction* – the process of making a conscious, systematic, and sustained effort to make apparent to another the mathematical meaning of a conventional semiotic contraction used in practice – a complement to the process of semiotic contraction identified by Radford. These contributions are new to the theory of knowledge objectification and the field of social cognition. The identification of particular categories of actions and operations that provided evidence of a subjectification during mathematics learning elaborates existing research as well.

DISCUSSION

This analysis of the mathematics activity within the pre-apprenticeship training program as a whole portrays ways in which this is a distinct form of cultural practice. The analysis of the pre-apprentice learning to read fractions-of-an-inch on a measuring tape informs Radford's theory of knowledge objectification by showing, through fine-grained analysis, relevant aspects of its dynamics and by calling attention to a new form of iconicity and a process of semiotic extraction, both original contributions to research. It also shows various ways in which a learner's subjectification is evident in the process of learning mathematics. Together these results have a number of practical implications for the teaching of mathematics generally, and mathematics for the workplace in particular, by drawing attention to the social, cultural, historical, and mediated dimensions and dynamics of mathematics learning activity and the need to address these in mathematics training for the workplace.

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TRANSFORMING MATHEMATICS EDUCATION FOR MI'KMAW STUDENTS THROUGH *MAWIKINUTIMATIMK*

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INTRODUCTION

I am frequently asked by teachers for good resources that they can use to include more Indigenous perspectives in mathematics. One such moment occurred at a high school mathematics workshop I was leading for teachers of Aboriginal students in Regina in 2007. A teacher entered the room and immediately told me, "My administrator told me to bring back three good ideas, so I need three good ideas." The need for three good ideas was sparked by a recent move within her province to infuse indigenous perspectives in all areas of curriculum. She seemed somewhat let down when I suggested, "Know your students, know their communities, and believe that they can learn." She was looking for pre-packaged lesson ideas or perhaps a checklist of steps that would guarantee success in mathematics for her Aboriginal students. But there are no shortcuts to authentically understanding and addressing the complexities associated with mathematics learning for Aboriginal students.

I began my teaching career in a Mi'kmaw school in Nova Scotia in 1995. When I was hired to teach mathematics for students from Grades 7 through 12, I committed myself to ensuring that my teaching supported the cultural identity development of my students. In those early years, I might have hoped for three good ideas as well. I do recall searching for culturally appropriate lessons and trying to make lessons connect to student lives, which proved to be moderately helpful, but I quickly realized that culturally responsive teaching was far more complex than just three good ideas. My doctoral work emerged from ten years of learning with colleagues and community members, ten years of questioning what might work well for my students and what might create conflicts, ten years of listening to my students and learning from them. In this paper I share some of this journey of discovery.

RESEARCH CONTEXT

The Mi'kmaw communities have a stated goal of decolonizing education by incorporating indigenous knowledge, culture, and values in their curricular and pedagogical practices. Such a decolonized approach to education that allows for the inclusion of indigenous worldviews has been advocated as a necessity to meet the needs of Mi'kmaw students (Orr, Paul, & Paul, 2002; Battiste, 1998, 2000). Yet Mi'kmaw communities are also bound by the agreement to offer provincially transferrable curriculum and to demonstrate measures of success based on provincially developed assessments. As such, teachers regularly grapple with ways to negotiate the space between school-based mathematics and Mi'kmaw ways of reasoning about things seen as mathematical.

It has been argued that disengagement from mathematics emerges a result of the conflict between Aboriginal culture and the cultural values embedded in school-based mathematics programs (Cajete, 1994; Secada, Hankes, & Fast, 2002). Gutiérrez (2007) has shown that the lack of windows and mirrors in mathematics curriculum can result in disengagement for many students from marginalized groups who feel their identity is being denied and they lack power to influence curriculum. She has argued that for these students the cost of participation means denying self and community to participate in the dominant view of mathematics. Often times these costs are seen as too great and children choose not to participate. Doolittle (2006) echoed this idea, cautioning that in learning mathematics, "as something is gained, something might be lost too. We have some idea of the benefit, but do we know anything at all about the cost?" (p. 19).

My doctoral research examined the tensions in mathematics teaching identified by teachers in these Mi'kmaw communities. The journey of this research project was an attempt to uncover key issues that must be attended to in transforming mathematics education for Mi'kmaw students. This research project addresses the following key research question: *How can curricula and pedagogy be transformed to support Mi'kmaw students as they negotiate their position between Aboriginal and school-based concepts of mathematics?*

METHODOLOGY

An indigenist methodology was used for this research. The indigenist perspective emerged as a response to a need for a new paradigm of decolonizing research (Denzin, 2005) and is seen as a way to "research back to power" (Smith, 2005, p. 90). This approach to research "is formed around the three principles of resistance, political integrity, and privileging indigenous voices" (Smith, 2005, p. 89) and has a "purposeful agenda for transforming the institution of research, the deep underlying structures and taken-for-granted ways of organizing, conducting, and disseminating research and knowledge" (p. 88). There is an underlying "commitment to moral praxis, to issues of self-determination, empowerment, healing, love, community solidarity, respect for the earth, and respect for elders" (Denzin, 2005, p. 943). Such paradigms create space to privilege indigenous knowledge (Denzin, 2005; Smith, 2005) and acknowledge that knowledge production must happen in a relational context (Denzin, 2005).

In search of an appropriate indigenist paradigm, I sought the advice of many community elders. I searched for a way to describe the activity of people coming together to discuss an issue or solve a problem. During an informal conversation with one community leader, it was suggested that I use the word *mawikinutimatimk* which means 'coming together to learn together'. It implies that everyone has something to share and everyone has something that they can learn. Thus *mawikinutimatimk* became the methodology for the doctoral research project and will continue to be the methodology for this new phase of the research.

The project was conducted in two Mi'kmaw K-6 schools over a nine-month period. Teachers, support staff, and elders were invited to participate. Ten after-school sessions were held in one school and twelve in the other school. In addition to our conversations, I also spent time working with teachers in their classrooms co-planning and co-teaching a lesson, or modelling a lesson. After-school conversations were recorded and transcribed. Classroom sessions were not recorded but field notes were kept and experiences from the classroom sessions were often discussed during our after-school sessions.

FINDINGS AND DISCUSSION

Through our conversations, four key areas of attention emerged as themes: 1) the need to learn from Mi'kmaw language; 2) the importance of attending to value differences between Mi'kmaw concepts of mathematics and school-based mathematics; 3) the importance of attending to ways of learning and knowing; and 4) the significance of making ethnomathematical connections for students. Within each of these categories, teachers identified conflicts that arise when worldviews collide and identified potential strategies to address these tensions. See the model below in Figure 1.

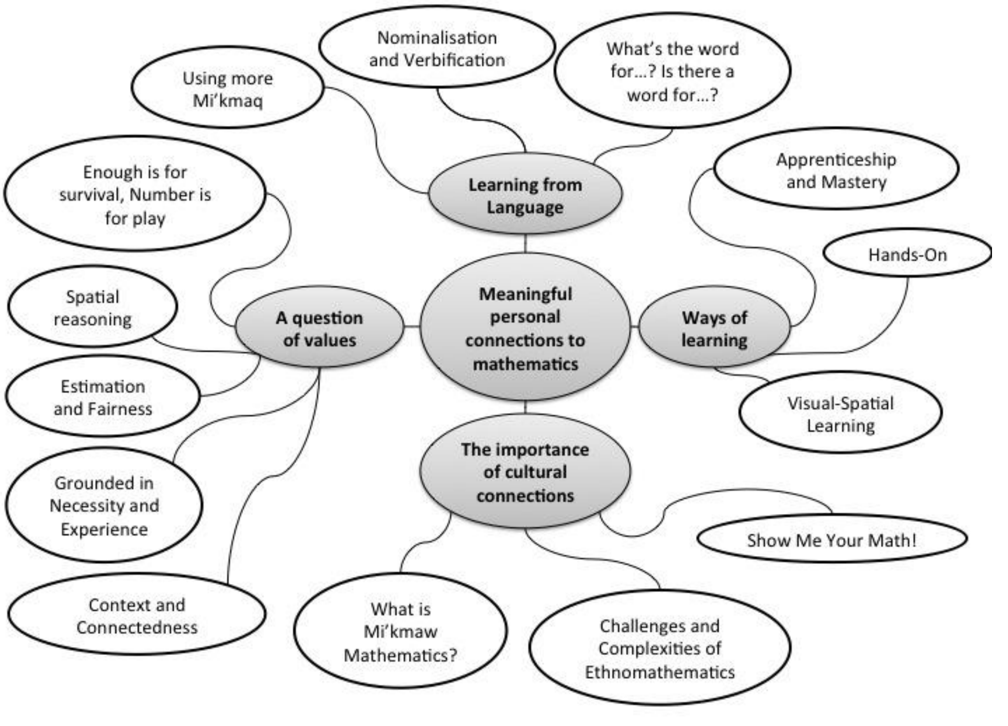

Figure 1

LEARNING FROM LANGUAGE

The need to learn from Mi'kmaw language was the most pronounced theme in the research. Thus I will highlight more of the findings from this part of the model and then briefly summarize the findings represented by the remaining parts of the model in sections below.

It has been argued that "a proper understanding of the link between language and mathematics may be the key to finally throwing off the shadow of imperialism and colonialisation that continues to haunt education for indigenous groups" (Barton, 2008, p. 9). This sentiment was supported by the participants in the study who felt strongly that language defines worldview and thus, by understanding Mi'kmaw language structures, teachers can gain greater insight into the ways of thinking of their students and be aware of potential tensions.

Conversations related to language focused on three main ideas. Firstly, there was a call to include more Mi'kmaw language in the mathematics classroom, with one group in particular stressing the importance of reclaiming mathematical words and supporting Mi'kmaq-speaking teachers to develop a lexicon of words that could be used in their classes. Participants cited examples of moments in their classroom teaching when they saw that a simple switch from English to Mi'kmaq resulted in increased comprehension. Elaine, an early elementary teacher commented on the way in which her students often do not understand what she means when she says, "How many?", but noted, "Say *'Tasikl* (how many – inanimate)?" and they get it."

Secondly, there emerged the notion that a great deal can be learned from studying the structure of the Mi'kmaw language even for non-speakers. In particular, this notion included a multi-layered discussion about what teachers, both speakers and non-speakers, can learn by asking questions such as "What is the word for…?" or "Is there a word for…?"

The idea of rooting to and learning from the home language has been advocated for in the literature on mathematics education in multilingual classrooms (Moschkovich, 2002; Setati, 2005). In my own teaching I often found it beneficial to learn Mi'kmaw words for mathematical concepts. Often, I would ask for words and occasionally I would discover that either the word did not exist or it was much more complex than I had anticipated. No matter what the result of my inquiry, I found the mere discussion had a profound effect on deepening my understanding of how my students might view mathematics.

Many of the research participants agreed that in order to better understand mathematical concepts our group needed to explore the Mi'kmaw ways in which we talked about these concepts. Richard proved to be a particularly helpful participant for this purpose as his knowledge of the language was often called upon. Many of our sessions would turn into mini language classes with Richard sharing his knowledge. Such an example arose when we were talking about fractions. Fractions are difficult for many children to learn. Yet through our conversations, we explored words that many children often hear at home when being asked to share treats with family members or friends. Richard explained, "*pukwe'* is part of something, but when you say *aqatiyik* that is half of it … now if a child understood Mi'kmaq very well, it'd be a lot easier for them to understand." Emily agreed, claiming "I was just thinking when you say *pukwe'*, that language is still used in the homes, *pukwe' iknumi kandiamul* ('give me a piece of candy'), you still hear that … it is used for everyday language." It was suggested that using these kinds of words might help students connect more with the often challenging concept of fractions.

In addition to knowing the Mi'kmaw words for concepts, it is also helpful to know when the concept does not have a direct translation. As Barton (2008) argued:

> *Different concepts are expressed in different languages, and some concepts are extremely difficult, some say impossible to translate between languages. The implication is that different quantitative, relational, and spatial concepts may also not be easily transformed into each other.* (p. 69)

If a mathematical concept does not have a direct Mi'kmaw translation, then it is likely that the concept is not part of the everyday language of the child. The result of this can mean that a concept that is thought to be quite simple in mainstream mathematics can in fact be quite complex for the child who is unfamiliar with the concept. This is an example of the type of taken-for-granted assumption of the school curriculum that may lead many Aboriginal students to have challenges with the subject matter or to disengage from it as has been discussed in the literature (see Cajete, 1994; Lipka, 1994; Nicol, Archibald, Kelleher, & Brown, 2006; Yamamura, Netser, & Qanatsiaq, 2003). An awareness of these potential conflicts may help teachers to mitigate them.

The word 'flat' is one example of a word that has no Mi'kmaw translation. I have asked on numerous occasions if there is a word for 'flat' and I have attempted to generate scenarios whereby we would need to use the word 'flat'. I asked about a flat tire but I was told that in Mi'kmaq we would say it was losing air. I asked about the bottom of a basket, suggesting it was flat, but I was told that it was the bottom; it had to be flat so that it does not roll around. Understanding that there is no word for 'flat' enabled us to think differently about how we describe a flat surface in mathematics.

An interesting connection to this notion occurred for me during a grade 3 lesson on prisms and pyramids. As we sat on carpet with students and asked them to say one thing about the prism that was being passed around, one young girl placed the prism on the floor and stated, "It can sit still!" Instantly I began to get excited by her answer. It made perfect sense that she would not talk about the flatness of the face but rather its usefulness. This connects directly to the relational way in which Mi'kmaw language is used and constructed. When I later recounted this story during an ad hoc session at the Canadian Mathematics Education Study Group Conference in Sherbrooke, Quebec (May 2008), Walter Whitely mentioned to me that the word 'polyhedron' actually is derived from the Greek word *hedron* which means 'seat', and 'polyhedron' means 'many seats' or 'many ways to sit'.

Some terms that are considered universal or commonplace in English may be more complex in Mi'kmaq as became evident on the day that Donna arrived for the research conversation, curious about the Mi'kmaw word for 'middle'. She and the speech language pathologist had encountered difficulty when working with a student on an assessment earlier in the week. The child had been asked to point to the object in the middle of a row to assess language processing skills. When the child could not perform this task, Donna wondered whether the child had difficulties processing the language or whether the child simply did not understand the word 'middle'. During our conversations in her classroom earlier in the day, she had talked with me about this matter and explained that once she had explained the term and provided the child with some experiences of 'middle', he was able to execute the task of pointing to the object in the middle quite easily. She said that she wanted to know if there was a word for middle in Mi'kmaq and she asked Richard this question when she arrived for our after school conversations that day.

Donna and I listened in on a long discussion between Richard and Elaine, another Mi'kmaw speaking teacher, about various ways you would describe middle things in various contexts, such as the middle shelf, the middle of the room, and so on, yet many of these words tended to be translated more as 'centre' or 'half way' rather than 'middle'. At the conclusion of this conversation Donna exclaimed with some degree of satisfaction, "Right, so it's a word you wouldn't really use."

When this type of language conversation happened, participants seemed to become aware of the different ways in which relationships are explained in different languages. This exposes the taken-for-granted assumptions embedded in school-based mathematics and allows teachers to begin to question this hegemony. English language has a way of talking about middle and uses this word in a variety of contexts; Mi'kmaq has several different ways of talking about the concept of middle but none of these words directly translate to the word 'middle'. Given this complexity, it makes sense that Mi'kmaw children may not find this concept as simple as the curriculum writers would assume it to be.

Thirdly, a closely related idea focused on investigating discourse patterns and the ways in which the Mi'kmaw language is structured. Most notably, a change in language-use patterns to reflect Mi'kmaw verb-based grammar structures, referred to as 'verbification', is exemplified as a strategy that holds promise for supporting Mi'kmaw students learning mathematics.

The Mi'kmaw language is a verb-based language. There is a sense of motion in the ways in which mathematical objects and ideas are expressed in Mi'kmaq. During one particular session in Wutank, Richard, a technology teacher and Mi'kmaw language expert shared with the group some ideas about the concept of 'straight'. He explained that the word *pekaq* means 'it goes straight'. There is a sense of motion embedded in the word. Similarly *pektaqtek* is a word to describe something that is straight such as a fence. He explained that there "is a sense of motion from here to the other end – *pektaqtek* [it goes straight]."

During the research, while working with a pre-service teacher in a third-grade classroom, numerous examples of verbification were heard in how the students explained their understanding of prisms and pyramids. Each group was given a prism or pyramid and asked to say why they felt it was a pyramid or a prism. One pair of students declared that they had a pyramid because it looked like a pyramid. When prompted to explain what they meant by that they said, "well it goes like this (gesture), into a triangle." This involved a hand gesture showing how the sides were merging. Another student also used a hand gesture to explain her declaration that her group had a prism "because it goes like this" and motioned her hands up and down in uniform fashion. A real challenge arose when it came time for the group with the triangular prism to report back. There was some debate about whether this should be a prism or a pyramid. "It kind of forms into a triangle," suggested one student, but this seemed to be not enough to commit to it being a pyramid. "What if we look at it like this?" I asked as I rotated the card on the board so that it now appeared to be standing on its triangular base. "Oh! It's a prism," a girl from the back offered, "because it goes like this," and she motioned again with her hands up and down in a uniform manner. This seemed to convince her classmates who offered supporting arguments such as: "Yeah, it's not coming to a point all around like the other ones." They all agreed that although it kind of looked like a pyramid in some ways, it was definitely a prism.

A QUESTION OF VALUES

The research conversations often turned to conflicting values that were apparent between school-based approaches to mathematics and Mi'kmaw ways of reasoning about mathematical questions. These value differences can provide teachers with insight that may enable them to anticipate points where two worldviews might bump up against each other and cause students to be conflicted and possibly disengage. These included a conflict between privileging numerical reasoning in mathematics curriculum over spatial reasoning more commonly used within the community and embedded in the language. Other Mi'kmaw approaches to mathematics identified included the common use of estimation, the value of playing with number, and the connection to necessity and intention. Many of these were noted as often being absent in school-based mathematics.

WAYS OF LEARNING

Our discussions in both research groups often turned to questions about children's preferred ways of learning and how they might influence the design of tasks for learning mathematics. It is important to avoid over-generalizations about aboriginal learning styles as "Aboriginal children [are] diverse learners. They do not have a single homogenous learning style" (Battiste, 2002, p. 16). There is as much diversity of learning styles within a Mi'kmaw class as there is in any class, so there cannot be a one-size-fits-all approach. That being said, some of the discussions focused on traditional apprenticeship models and mastery approaches to learning, as well as those related to visual-spatial styles of learning and hands-on learning. Other observations pointed to the role of gestures and embodied cognition. It was argued that understanding these different approaches to learning can provide teachers with additional strategies that can be employed in mathematics classrooms.

THE IMPORTANCE OF CULTURAL CONNECTIONS

The importance of making connections to the mathematical thinking that is, and has always been, evident in the Mi'kmaw community was seen as an important part of transforming mathematics education. As a group we explored some of the evidence of mathematical thinking that exists within the community's daily practices and recognized that there is far more to be done in this area. We also discussed some of things that have been done in both schools to strengthen the connection between school-based mathematics and community, cultural and everyday practices. We also discussed the challenges with such an ethnomathematical approach, including a risk of trivialization, and agreed that development of ethnomathematical resources must be developed in consultation with elders and community members to mitigate any potential issues.

CONCLUDING THOUGHTS

I have highlighted some of the key aspects of the model in this paper in an attempt to show the multiple aspects that need to be attended to if we are to decolonize mathematics education for Mi'kmaw students. Furthermore, in another Aboriginal context some of these issues may arise, and there may be other issues as well. What is essential is that conversations about these complexities need to happen with community members to discover potential complexities that may be working against students as they attempt to learn mathematics. It is important to question taken-for-granted assumptions and work together to find new ways forward in mathematics education for Aboriginal students. While many teachers may seek lessons that have cultural connections, this research has shown that such an approach misses much more significant issues. Asking questions about language, values, ways of knowing, and cultural connections is the first step and can lead to a more effective approach to decolonizing mathematics education for Aboriginal students.

ACKNOWLEDGEMENT

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Ad Hoc Sessions

Séances ad hoc

COMING TO KNOW MATHEMATICS: VIEWS OF TWO TEACHER MATHEMATICIANS

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Writing at the turn of the twentieth-century, in the essay, "Mathematical Discovery", Henri Poincaré (1914/2003) reflects:

> *One first fact should astonish us, or rather would astonish us if we were not too much accustomed to it. How does it happen that there are people who do not understand mathematics? If the science invokes only the rules of logic, those accepted by all well-formed minds, if its evidence is founded on principles that are common to all men, and that none but a madman would attempt to deny, how does it happen that there are so many people who are entirely impervious to it?* (pp. 46-47)

As students and teachers of mathematics, our greatest challenge is that of helping our students understand mathematics. So it seems to me that we have to go back to the beginning and ask: What does it take to understand mathematics? What does it take to 'know' the subject? Where are the challenges and the pitfalls?

I begin with the work of Leone Burton on mathematicians as enquirers. Burton, from a reading of the philosophical, pedagogical and feminist literature, identified four epistemological challenges to knowing mathematics: objectivity, homogeneity, impersonality and incoherence. She proposed a model for coming to know mathematics with five categories: 1) person- and cultural/social-relatedness, 2) aesthetics, 3) intuition and insight, 4) difference in approaches and styles of thinking, and 5) connectivities (Burton, 1995, 1999).

Burton (2004) tested her model in a study of 76 UK research mathematicians in the 1990s and found that the categories were 'remarkably robust'. I wondered whether, if asked about their trajectories and experiences in mathematics, the mathematicians I knew would speak of the same categories. In a pilot study with two mathematician colleagues, I found that only the first category relating to personal, social and cultural influences was prominent, with the subjects speaking of the other categories only when specifically asked. This finding has given the impetus for further work on our encounters with the subject of mathematics. More specifically, what are the factors that come into play in not only the mathematical experience but in *our* mathematical experiences?

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WHAT DOES 'BETTER' UNIVERSITY MATHEMATICS INSTRUCTION LOOK LIKE?

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A central motivation in mathematics education research is to understand forms of instruction which improve students' understanding of mathematics. There is a large body of research on different forms of elementary and secondary instruction and its impact on student learning, and while there have been decades of complaints about the traditional, lecture-based instruction most commonly used in university mathematics courses (Alsina, 2001; Bass, 2006; Kline, 1977; Seymour & Hewitt, 1997), there has been little formative research on what mathematics instruction at the university level might look like aside from a traditional lecture.

We propose a broad, overarching question: What does 'better' university mathematics teaching look like? Though little research exists, we have seen *recommendations* for improving instruction. These appear within the Scholarship of Teaching and Learning community (see ISSOTL) and many Teaching and Learning Centres within universities, for example the Carl Wieman Science Education Initiative and the Queen's University Society for Teaching and Learning. To further explore these recommendations, we ask two questions:

1. What variables can instructors manipulate when designing and teaching a university mathematics course?
2. Do we, as a community, know what specific actions help undergraduate students to learn mathematics better?

Our goal was a discussion with participants to develop a better understanding of the community's perspective on this issue. The skeletal outline that we drew up involved four elements: the *goals* of a university education, the *structure* of the environment in which we teach, *instructional design*, and the *actions* taken by the instructor while teaching.

In the end, we see that our inquiry was not sufficiently focused. In response to the first question, the consensus of the participants was that although we may be able to identify good teaching if we see it, there are too many variables to control in teaching to be able to construct a good teaching practice for all. The answers to the second question were very personal, and have all been seen in recommendations noted above.

Perhaps the most interesting element of the discussion was the presentation of goals: academic (focus on ideas, tasks, concepts); to transform our students; to engage and communicate with students. Thus, we might refine our inquiry by asking more pointed questions: "How would one measure instructional gains at the university level?" and "Does changing the structure of the course (in-class time, different tasks, assessment) affect student understanding more than the instructor's actions in the class?" Perhaps it would be wise to start with, "What does the instructor bring to the class that students can't get elsewhere?"

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EXPLORING VARIABILITY IN A DYNAMIC COMPUTER-BASED ENVIRONMENT

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Over the last two decades, research in post-secondary statistics education has focused on developing instructional materials and strategies that might support students' learning to become statistically literate, reasoning, statistically thinking, and in general, well-informed citizens. Central to these constructs of statistical literacy, reasoning, and thinking, is variability, which according to Alan Rossman, is the backbone of statistical thinking. Rossman (1996) argues that without variability, there would be no reason to study statistics. This *ad hoc* session focused on students' understanding and interpretion of formal measures of variability. For example, how dynamic technology might be used to model the relation between the mean and standard deviation for a given discrete dataset. The specific question was: "Given what we already know from research that students find the concept of variability quite challenging (see del Mas & Liu, 2005; Mathew & Clark, 2003; Garfield & Ben-Zvi, 2008), how might dynamic computer-based models support students' conceptual understanding of variability?"

Participants brainstormed on some of the available software that could be used to develop dynamic models for learning concepts in basic statistics. The list included *Wolfram Alpha*, *GeoGebra*, *The Geometer's Sketchpad* (GSP), and *Fathom*. One participant suggested checking on the Statistics Canada website, for resources that might help in the study, for example, getting some real data for students to use when learning the concepts. The ideas and suggestions from the session clearly represented the broad interest that participants had in computer technology and its applications. I was also privileged to receive some input from participants who knew about my presentation but were unable to attend. One conversation, in particular, was a discussion that I had after the *ad hoc* session with a CMESG participant from Brock University. It was a fruitful discussion in that it added more ideas and materials to my study. He shared some dynamic models they had developed at Brock University.

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VIRTUAL MATHEMATICS MARATHON: A MATHEMATICAL GAME FOR ALL CHILDREN

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Viktor Freiman, *Université de Moncton*

“*I am in grade 6 and I feel bored in my mathematical class. Thank you for posting interesting problems that I can solve every week and compete with other children!*”

This is a line from a student’s letter to the Virtual Mathematical Marathon team (VMM, http://www8.umoncton.ca/umcm-mmv/index.php). This quotation represents the intended audience of the new website well: junior high school students with mathematical ability and interest, especially students who would benefit from nurturing their talent, but lack challenge in their regular mathematics classrooms. As an open competition, it may also attract anyone who likes solving mathematical problems online. In addition, the site may serve mathematics teachers as a resource for classroom enrichment.

According to research, regular participation in extra-curricular mathematical activities is vital for proper development of mathematically inclined students (e.g. see Barbeau & Taylor, 2009; Krutetskii, 1976). These students need to meet other students with similar interests and compete with them in meeting mathematical challenges in a friendly learning environment. Mathematical clubs, Math League team games and individual contests are traditional ways of organizing and challenging mathematically talented youth. Net Generation-friendly, virtual competition may also help reach interested students in remote places, and connect children all around the world.

Virtual Math Marathon is a gradually developing bilingual (French and English) website. Its main activities are 15-week rounds with four new problems posted weekly. Students have a week to think about the problems on their own as well as to learn from analyzing suggested later solutions. The winners are the students who demonstrate not only good mathematical knowledge, intuition and problem solving skills, but also persistence and commitment to solve problems regularly over a three-month period.

The VMM team is composed of mathematicians and educators who enjoy working together helping mathematically inclined children to realize their full potential in the 21st century environment. Our international team includes university professors, programmers and students: Ed Barbeau, Mark Applebaum, Chadia Moghrabi, Natalia Vinogradova, Evgueni Vichnevetski, Elena Polotskaia, Ildiko Pelczer, Oumar Maiga, Adnen Barhoumi, Ian Payne, Dominic Manuel, and Karim Besbes. Support from the Canadian National Sciences and Engineering Research Council (Promoscience Grant), Canadian Mathematical Society, New Brunswick Innovation Foundation, Memorial University, as well as Université de Moncton is essential for project development.

The purpose of this note is to inform mathematics educators about the VMM and invite the contribution of ideas regarding its development and promotion among interested children. Contact us at viktor.freiman@umoncton.ca or mkondra@mun.ca.

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EARLY CHILDHOOD MATHEMATICS EDUCATION

Donna Kotsopoulos and Joanne Lee
Wilfrid Laurier University

Research has shown that young children from Asian countries are outperforming those from Western cultures beginning at age three, and differences are sustained and lasting (Miller, Kelly, & Zhou, 2000; Mullis, Martin, & Foy, 2008). Differences in mathematics achievement at these early ages have been found to be related to early learning environments and unrelated to parental education and socioeconomics (Duncan et al., 2007). It can be surmised that something very different must be happening in early childhood mathematics education in Asian countries. Recent large-scale studies of pre-school settings in the US found that young children spend only approximately 6% of their time in total on activities with either a primary or supplementary focus on mathematics (Barbarin et al., 2005). Even at home, families tend to focus more on literacy development and less on numeracy development (Cannon & Ginsburg, 2008). Low rates of engagement in mathematics in early childhood are not surprising. Studies have shown that (a) early childhood educators are underprepared in the area of early mathematical learning, and (b) appropriate professional development (early mathematical cognition and pedagogy) is scarce (Cross, Woods, & Schweingruber, 2009). More research in Canada is needed to explore the complex issues of early childhood mathematics education (ages 0 to 6). Currently, there is a dearth of education researchers addressing early mathematics learning. More research of varying types is urgently needed.

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THE CHALLENGES OF MATHEMATICS IN-SERVICE

Susan Oesterle
Douglas College

In the Fall of 2011, Douglas College is set to launch a new program targeted at elementary and middle school teachers who wish to improve their knowledge and skills for teaching mathematics and science: the Post-Baccalaureate Diploma in Mathematics & Science Teaching Program. As an instructor for the mathematics course within this program, I was interested in bringing together an *ad hoc* group for the sharing of information, wisdom, and resources related to supporting teachers in an in-service environment. As observed by Višňovská (2007), "[d]esigning effective professional development (PD) programs for mathematics teachers is a complex endeavour about which a lot remains to be learned (Borko, 2004)".

Particular challenges addressed in the discussion included:

- Teachers' mathematics content knowledge
- Teachers' attitudes towards mathematics and mathematics teaching
- Teachers' motivations for taking on an in-service program
- Diversity within the in-service group in terms of teaching context and grade-level focus
- Evaluation – graded assignments vs. mastery
- Selection of tasks that will be effective in building content knowledge and pedagogical content knowledge (PCK), while having a positive impact on affect.

Participants shared their experiences. Within the conversation there was much consensus, particularly around the benefits of taking advantage of the teachers' diversity and experiences, of making use of manipulatives to solidify teachers' conceptual understanding while building their PCK, of providing opportunities for reflection, and of building a learning community to encourage on-going support and professional development beyond the end of the course.

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Mathematics Gallery

Gallérie Mathématique

HIGH SCHOOL MATHEMATICS STUDENTS' TRAJECTORIES: TRACKING OR DIFFERENTIATING FOR SUCCESS?

Ralph T. Mason, *University of Manitoba*
P. Janelle McFeetors, *University of Alberta*

This poster presentation shared data from the longitudinal Trajectories Project, which followed high school students over a five year period, ending with their graduation in June 2010 (Mason & McFeetors, 2007). Ten cycles of interviews over the five years engaged students in describing their approaches to learning mathematics, their sense of purpose in relation to the mathematics they were studying, and the decisions they made that affected their success as high school mathematics students and learners. One major decision all students faced each year from grade 9 on was to select the mathematics course(s) they would take the next year, from a choice of four courses of significantly different academic difficulty.

In the United States, various theorists have suggested that *tracking* – the channeling of students into different mathematics programs based on their past performance – has resulted in substantially inequitable participation in the courses that lead to success in mathematically dependent post-secondary programs (El-Haj & Rubin, 2009; Moses & Cobb, 2001). In Canada, we tend to refer to students choosing different courses as *streaming*. Using data from the Trajectories Project, our poster displayed the pathways of students as they chose their mathematics courses together with statements of students explaining their course choices. A network diagram – where nodes represented mathematics courses selected and links represented the flow of students through courses – highlighted the tension between the nuances of the students' lived experiences and a compact visual representation.

The intention was to gather conference participants' perspectives on what characteristics and research pursuits would determine whether differentiated mathematics courses in Canadian high schools are legitimately meeting the different educational needs and capabilities of its students, or subjecting them to a sorting process which ranks them for exclusion from further study, perhaps replicating social class and cultural differences. Four questions were offered to guide CMESG participants' thinking:

1. Are students in the top streams *challenged to succeed*, or *taught to succeed*?
2. Are students *informed* to make good choices, or *slotted* by marks?
3. Does the range of streams offer different *content and experiences*, or only different *levels of difficulty*?
4. Do the lower streams offer *legitimate mathematics*, or only *fundamental arithmetic*?

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
  Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242

Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	ED 529557
Proceedings of the 2003 Annual Meeting	ED 529558
Proceedings of the 2004 Annual Meeting	ED 529563
Proceedings of the 2005 Annual Meeting	ED 529560
Proceedings of the 2006 Annual Meeting	ED 529562
Proceedings of the 2007 Annual Meeting	ED 529556
Proceedings of the 2008 Annual Meeting	ED 529561
Proceedings of the 2009 Annual Meeting	ED 529559
Proceedings of the 2010 Annual Meeting	ED 529564
Proceedings of the 2011 Annual Meeting	*submitted*

NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.