Mevlana International Journal of Education (MIJE) Vol. 3(3)
Special Issue: Dynamic and Interactive Mathematics Learning Environment
pp. 36- 44, 01 July, 2013 Available online at http://mije.mevlana.edu.tr/
http://dx.doi.org/10.13054/mije.si.2013.04

# Is proving a visual act?

Vimolan Mudaly*
*Department of Mathematics, School of Education, University of KwaZulu-Natal, Durban, South Africa*

**Article history**

**Received:**
08 March 2012

**Received in revised form:**
-

**Accepted:**
06 May 2013

**Key words:**
Visualisation; proof; dynamic geometry;

This paper looks at the role of visualisation in the proving process. It considers the different functions of proof and then describes student responses when engaged in the process of discovering Viviani's Theorem. The findings show that learners can attain high levels of conviction when working in a dynamic geometry environment. In particular, students are able to grasp concepts easier when engaging with dynamic images, especially if these images create some cognitive conflict with their existing knowledge or ideas. Furthermore, the paper explores the student proving process from the context of proof as explanation. One of the important results of the paper shows that given the correct guidance, students may be able to proof simple mathematical results. Whilst this may be a small scale qualitative research, it still indicates to us that dynamic software can be used relatively effectively in mathematics classrooms, especially from the perspective of being able to engage visually with new mathematical concepts.

## Introduction

The idiom 'seeing is believing' is probably as old as Methuselah (oldest man to have lived – 969 years), but I've not seen Methuselah so I'm not sure whether he existed. Human nature is such, that for acceptance of the truth of anything, it is necessary to have clear, and sometimes, visible proof. A court is more likely to believe the words of a credible eyewitness than the musings of an interested other. Mathematics on the other hand is clinical and requires more than just a visual depiction of what seems to be true. The *Kanizsa Illusion* (Figure 1), is a useful example for the non-acceptance of a visual object as proof. The illusion is created that a square exists when in fact the square is an optical illusion created by the knowing mind.

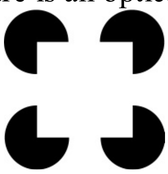

**Figure 1:** Kanizsa Illusion

The reason for the strict adherence to formalized proof in mathematics can be traced to the Greek mathematicians who saw the proof process as that of validation and certification (Davis & Hersh, 1984 : 249). This view of proof has since changed through the work of De Villiers (1986, 1991, 1992, 1995, 1996, 1997, 1998), Hanna (1996) and other researchers in the field. The earliest conception of proof served a pragmatic purpose and can be attributed to both the

* Correspondence: School of Education, University of KwaZulu-Natal, Phone: +27(0)312603682, Fax: +27(0)312603697, Cell: +27(0)829770577, Email: mudalyv@ukzn.ac.za

ancient Egyptians and the Greeks (Krantz, 2007: 2). They needed the mathematics when considering aspects related to building of structures that had utilitarian value. A mere description or a diagram would be sufficient to justify the mathematical validity of a statement or assumption. As the subject grew, mathematicians began to question the actual nature of mathematical proof. Thus the Greeks, Thales, Eudoxus and Theaetetus re-conceptualized mathemathematics by creating theorems and axioms as statements of facts that could be verified by using previously accepted mathematical statements or definitions. All of these led to Euclid's eventual formalization of mathematics in the form of Euclidean Geometry (Krantz, 2007: 3-4). Mathematics developed further throughout the middle ages and continued through the work of modern mathematicians. But the purpose has always been for verification and justification of mathematics statements. A more comprehensive and broader understanding of the proving process has been developed and this has certainly allowed for proving in mathematics to become much more acceptable to mathematicians and mathematics educators alike. It must be acknowledged though, that different standards and types of proof (Tall, 1995 : 28) exist at a formal level, and therefore different forms of proof might be appropriate in different contexts. Proofs vary in standard from the verbalising or the description of visual ideas in mathematics to the vast array of formal deductive systems (Tall, 1995: 29). Despite the method and context within which proof is done, proof has remained the main tool which mathematicians use for verifying, communicating, explaining, systematizing and discovering. Whatever our conception of the value and definition of proof we must acknowledge that "*mathematical proof provides a way of being absolutely sure. It is not the same as 'guessing', as 'checking a few cases', or 'giving a plausible-sounding, more or less intelligent explanation'.* ***Proof*** *is precise, it is exact, it is incontrovertible, it is objective; in short, it is* ***'bombproof'****"* (Gardiner,1995 : 10).

The value of a proof resides in the fact that proofs give mathematicians an assurance that a statement is true, if it has been proved using sound statements that were previously obtained and proved. Slomson (1996 : 12) stated that proofs "*give us the justification for the mathematical methods we use, and good proofs also help us to understand the mathematics".* Barbara Ball (1996:34) also conveyed the need for proof when she said that "*proof can bring understanding of why methods work and, consequently of how those methods might be adapted to cope with new and altered circumstances*" and "*we do them (pupils) a great disservice if we exclude it (proof) from the curriculum*".

Both Slomson and Ball reinforce the idea that proof is necessary but an important observation is that both acknowledge the fact that good proofs encourage understanding. Kitcher (1984: 181) argued that proofs not only increase understanding but creates new knowledge as well. But there are many reasons for proof. Proof for verification is concerned with the *truth* of a statement of proposition. According to Garnica 1996 : 257) "*proof is that which attests the veracity or authenticity, the guarantee, the evidence, the process of verification of the accuracy of operations and reasonings …".*

Proof as systematization is concerned with the logical organisation of propositions into a deductive system. In fact this aspect deals with the logical structure of the actual reasoning involved in the formal proof – the writing down of ideas in a logical sequence. The focus is on making logical connections between statements; statements that may already be known to be true or assumed to be true. Proof as discovery is supported by the fact that often whilst doing empirical work in mathematics, mathematicians stumble upon a new proof. This view was expressed by Peterson (1990 : 16) when he stated that discoveries are often not first made empirically, but can occur quite unintentionally during proof : *"Often, the obsession with proof is itself an important source of new ideas and mathematical methods. Efforts to prove* *that closed curves divide space into an outside and an inside led to the new mathematical field of algebraic topology, a central topic in modern mathematics. It's unlikely that any attack on a particular practical problem would have led to such novel abstract ideas".*

There are two lesser known functions of proof, namely, proof for communication and proof for self-realization. Proof as communication plays an important role in illustrating the social aspects of mathematicians. Tymoczko (1986 : 127), in an editorial, stated that *"verification of proofs is a public affair, an elaborate social process that proceeds by the canons and paradigms of a particular community of experts".* He further stated that the *"verification of proofs would involve such factors as the dissemination of results through a community".* Proof for self-realization serves an aesthetic function and is very important because it deals with exactly what the human mind feels satisfied with. Although most mathematicians know that a proof will benefit many others, the inner joy and personal satisfaction at discovering a proof is the main intrinsic motivating factor.

The final reason for proof and, the main thrust of this paper, is the one of explanation or illumination. Tiles (1991:7) stated that "by proof is meant a deductively valid, rationally compelling argument which shows *why* this *must* be so …" (my emphasis). This function of proof helps the individual make sense of a mathematical result and to satisfy the individual's curiosity as to *why* the statement or proposition is true. This function of proof has been neglected because proof has been seen as performing only the function of verification. Coe and Ruthven (1994 : 42) claimed that less emphasis has been placed on explanation because much writing about proof "has been from a philosophical rather than a pedagogical perspective". But Hanna (1996 : 16) stated that "with today's stress on 'meaningful' mathematics, teachers are being encouraged to focus on the explanation of mathematical concepts …".

Wittmann (1996 : 16) cited David Gale when he stated that "*the main goal of all science is to first observe and then to explain. In mathematics the explanation* ***is*** *the proof*"(bold added). Schoenfeld (1985 : 172) demonstrated this important function of proof quite succinctly when he stated that: *"'Prove it to me' comes to mean 'explain to me why it is true', and argumentation* (proof) *becomes a form of explanation, a means of conveying understanding".*

There are a few important questions that need to be raised here. Does this understanding and explanation actually arise out of only writing out a logical proof? If proof in mathematics is precise and formal, what then is the role of visualisation in conveying this understanding? This paper will attempt to answer these two questions.

## Research Methods and Methodology

This paper arises out of the results of research conducted in a school in Durban, South Africa. Framed within the interpretivist paradigm, the researcher collected qualitative data through one-on-one task based interviews. One-on-one task based interviews enabled the acquisition of rich data through careful probing. Seventeen Grade 9 learners were randomly selected from a group of 153 by their Computer Studies educator for the purposes of this research. They were duly informed of the objectives of the research and it was made explicit that they could opt out of the programme if they so wished. Furthermore, their parents also signed consent documents, allowing their children to participate. All of these learners were not exposed to the problem used in this interview before and were generally afraid of failure. The researcher met with them on several occasions prior to the interviews in order to set their minds at ease and to establish a rapport with them in order to obtain responses that were not accompanied by fear and anxiety.

### *Theoretical framework*

The theory that underpinned the analysis for this paper is an adaptation of the two models used in Mudaly (2012) and Mudaly and Rampersad (2010). This model (refer to Figure 1) describes the formation of knowledge or the transformation of existing knowledge through the active engagement of the learner with geometric diagrams. The learners' concrete experiences with diagrams, which they could see on the computer or even manipulate both physically and mentally, allowed them to make meaning of their task. The iterative process of seeing a diagram then adapting it, allowed for the learner to think both visually and analytically, through reflection and interaction. This leads to abstract conceptualization or the acquisition of new knowledge. This new or changed knowledge influences the acquisition of further new knowledge. The emphasis in this theory is the experiences of the learner and the way learners use their *a priori* knowledge to construct new information. This new knowledge is internalized and then used to the diagram that they are working with. Thus the internalized process leads to the externalisation of the new knowledge. This iteration may continue until the learner is able to resolve the problem set out in the task.

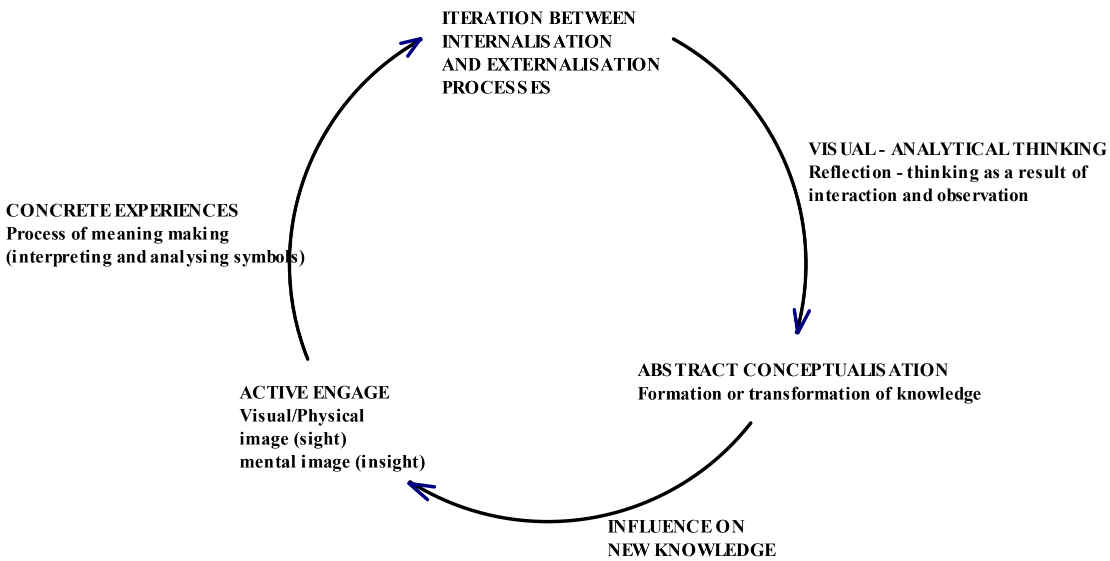


**Figure 2:** The process of acquiring new or transforming old knowledge (Adapted from Mudaly (2012) and, Mudaly and Rampersad (2010).

### *Data Analysis*

The question chosen for the interview was based on Viviani's Theorem and was extracted from the book *Rethinking Proof with Geometer's Sketchpad* (De Villiers, 2003). The question specifically handed to them was

> *Sarah a shipwreck survivor manages to swim to a desert island. As it happens, the island closely approximates the shape of an equilateral triangle. She soon discovers that the surfing is outstanding on all three of the island's coasts and crafts a surfboard from a fallen tree and surfs every day. Where should Sarah build her house so that the total sum of the distances from the house to all three beaches is a minimum? (She visits them with equal frequency.)*

The students were expected to read through the question and work on the computer using the software *Sketchpad*. Many of the main diagrams were provided in their skeletal forms, and learners were expected to manipulate these diagrams during the face-to-face interviews. For the purposes of this paper, the initial parts of the interview will not be discussed but it is necessary to highlight an important finding. The learners were asked to guess where they thought Sarah should build her house. The common response from all learners was that Sarah should build her house towards the centre of the equilateral island because *"...if you build anything in the centre then there is always a short distance around it"* (student response). This thinking stemmed from the idea of a circle with equal radii. But by the end of the interview, after they had worked through a series of sub-problems with dynamic diagrams using *Sketchpad* on the computer, all learners achieved a high level of conviction about the veracity of the solution to the problem that they worked with. This is not the focus of this paper and will therefore not be discussed. It suffices to say that though the learners were not inclined towards mathematical theory, they did want an explanation as to why the result was always true. Peter (pseudonym used) was asked whether there was any need for an explanation because he showed high levels of conviction.

Researcher : *Do you think that there is a need for an explanation ? Do you want to know why this is true ?*
Peter : *Yeah, there is a need for explanation.*
Researcher : *Why do you think there is a need ?*
Peter : *So we will be able to understand more clearly that diagram.*

This perhaps needs to be elucidated. The diagram that he speaks of really meant the *Sketchpad* diagrams that he actively moved around using the drag function. His experiences in the period prior to the question being asked had engaged him in the iterative process of drawing and moving a diagram and then reflecting on and adapting the diagram further. The fact that he could see the measurements alongside the diagram enabled him to make rapid hypotheses and these changed as he saw patterns emerging. Inadvertently, many of his original hypotheses were shown to be incorrect as he experimented with the software. The software enabled him to construct several triangles and manipulate them in a way that began to reinforce the result of Viviani's theorem. Although pencil and paper methods may have achieved the same goal, the time taken would have been much longer. So, in wanting to understand the diagram he really wanted to understand why the software produced the result. In essence, his high level of conviction created sufficient curiosity for him to find an explanation for the result.

This type of curiosity was displayed by many of the students interviewed. It all began with the students being given an equilateral triangle on the computer using *Sketchpad*, as stated in the question, and them determining the minimum sum of the distances from the house. At the outset the students had conjectured that the house should be in the 'centre' of the triangle. It was therefore very surprising for them when they manipulated the diagram by moving the point that represented Sarah's house around. In fact, through construction, dragging and measurement, they were able to discern that the house could be built anywhere within the triangle. Their active engagement with the diagram gave them instant feedback and in a sense created some cognitive conflict. It was noticed that all students began to smile when the observed result contradicted their initial expectation. Perhaps this implied that despite having a different result from that expected, it was still a pleasant experience. This part of their discovery took only a few minutes but the students achieved a high level of conviction. All the students were convinced of the result for the triangle that they had on the screen.

When asked whether the result will hold for any equilateral triangle they became unsure.

They were certain about the result for the triangle they worked with but they could not extrapolate the result to any triangle, so they were asked to make the triangle bigger or smaller by dragging any one vertex of the triangle. Their findings were the same and this second part of the discovery took far less time than the first part. They did not display too much of a surprise at this discovery and this can be attributed to the fact this new knowledge became subsumed into their prior knowledge. The diagrams that they were exposed to were dynamic ones and it afforded them the opportunity to engage with it in ways similar to replicating hundreds of similar diagrams. Whilst it must be acknowledged that pencil and paper methods could also have been used, but the time saved was be significant.

The activity that followed required the students to find a possible explanation for the result obtained. The diagrams that they worked with on the screen of the computer probably provided them with the necessary confidence but they did not find the task very easy. Nelson (2007: 1) stated that "a good visual proof is a picture or diagram that helps one see *why* a particular mathematical statement is true, and also to see *how* one might begin to prove it true". These students struggled for a while and only when they received a hint did they actually find a proof. They were advised to locate the different triangles and to then work with their areas. These hints may seem to be excessive but one must consider that these were grade 9 students and had not been exposed to proofs before. A typical example of a proof was given by a student.

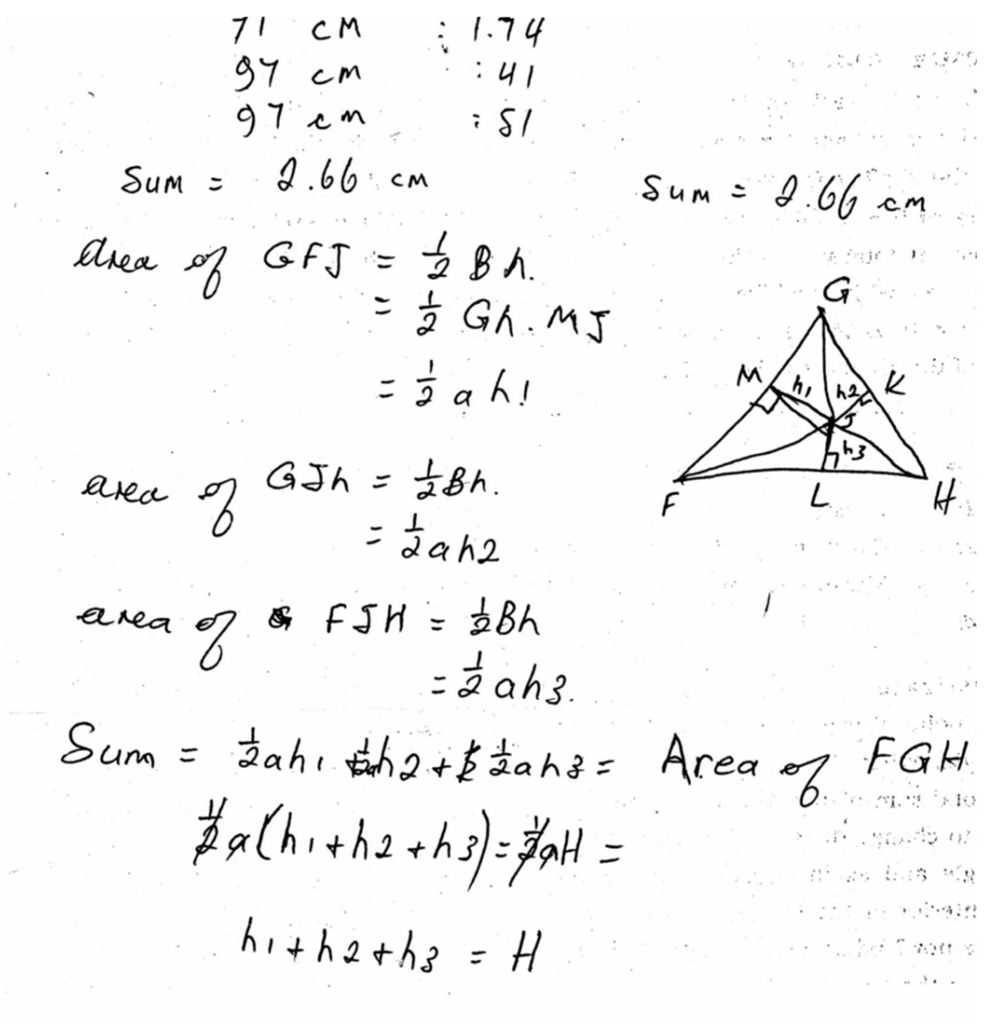


**Figure 3:** Student effort

From Figure 3, it should be noticed that the student started off by writing what was observed on the screen namely, the three different heights of the three different triangles. These heights represented the distances from the house to the beaches. At this point the student stopped and did not proceed any further. It seemed as if he had no idea as to how he should proceed any further. At this point he was advised to consider the areas of all the triangles in the diagram on the screen. He then began by drawing a triangle that may have looked similar to the one that he observed. He then connected F, G and H to J. There are a few points that can be hypothesized about the student's attempt at a proof. Firstly, it seems that he had a picture of the diagram in his mind because he uses symbols not indicated on his drawn diagram. It seems that he noted that all sides are equal and therefore each assigned the letter 'a' to each side. Also, he distinctly uses H for the original triangle and this too does not appear on his diagram. Secondly, after drawing his diagram there were more than just the three triangles available and it seems that he assumed that $h_1$ , $h_2$ and $h_3$ were heights of the triangle and therefore did not contribute towards creating new triangles. Thirdly, the student arrives at a final statement but does not explain what this statement actually means. There is no guarantee that he in fact actually understood the proof.

## Conclusion

This research has yielded some valuable results in terms of the teaching and learning of geometry theorems and problems. Given the fundamental importance of proof within mathematics as a discipline, dynamic exploration should become an essential part of the secondary school curriculum. Moreover, the teaching (and learning) approach used in the empirical research seemed to provide learners a greater, and more meaningful, understanding of the solution to the given problem. Although there was no real comparison, it can be assumed that pencil and paper constructions would have not been as effective due to time constraints and the learners' abilities to construct correctly. This study concentrated mainly on the introduction of proof to pupils as a means of explanation rather than as verification. As was observed the learners had a high level of conviction that the result was always true and this is significant in light of the poor attitude towards geometry in general. Their conviction did not waiver even though there were attempts to confuse them by asking them to change the size of the triangles. They may have had doubts at certain points but after within a few minutes of working again with *Sketchpad*, they reaffirmed the truth of their conjecture.

Proving the conjecture by themselves was not an easy task. Had they been students form a higher grade the result may have been different. Perhaps this experiment needs to be tried with students in higher grades who have actually already experienced some proving prior to the task. But in any case with some guidance it would have been possible to get them to complete a proof that would have provided a useful explanation for the result they obtained.

## References

Ball, B. (1996). How do I teach Proof ? *Mathematics Teaching, 155* pp. 31-33.

Coe, R. and Ruthven, K. (1984). Proof Practices and Constructs of Advanced Mathematical Students. *British Educational Research Journal*, 20(1) pp. 41-53.

Davis, P.J. and Hersh, R. (1984). *Proof in Mathematics : People. Problems. Results*. 1st edition. Editors : Cambell D. and Higgins, J.C. California: Wadsworth International.

De Villiers, M. (1986). The role and function of axiomatization. Stellenbosch: RUMEUS.

De Villiers, M. (1990). The Role and Function of Proof in Mathematics. *Pythagoras*, 24 pp. 17-23.

De Villiers, M. (1991). Pupils' needs for conviction and explanation within the context of geometry. *Pythagoras*, 26 pp. 18-27.
De Villiers, M. (1992). *Children's acceptance of Theorems in Geometry* – Draft for poster presentation at PME, August 1992.
De Villiers, M. (1995). Why Proof in Dynamic Geometry ? *Justifying and Proving in School Geometry*. London Institute of Education pp. 155-173.
De Villiers, M. (1996). Why Proof in Dynamic Geometry ? *Proceedings of Topic Group 8: Proofs and Proving: Why, When and How ?* Editors: Villiers M.D.and Furinghetti F. Centrahil: AMESA pp 24-42.
De Villiers, M. (1997). The Role and Function of Proof in Investigative, Computer-based Geometry : Some personal reflections, In King, J. and Schattschneider, D.(1997). *Geometry Turned On!* Washington:MAA.
De Villiers, M. (1998). An Alternative Approach to Proof in Dynamic Geometry in Lehrer, R. and Chazan, D. (1998). *Designing learning environments for developing understanding of geometry and space.* New York: Lawrence Erlbaum
De Villiers, M. (2003). *Rethinking Proof with Geometer's Sketchpad*. Key curriculum Press. 2nd edition. California.
Gardiner, T. (1995). The Proof of the Pudding is in the …? *Mathematics in school*, 24(3) pp. 10-11.
Garnica, A. (1996). Fascination for the technical, decline of the critical: a study on rigorous proof and the training of mathematics teachers. *Proceedings of Topic Group 8: Proofs and Proving: Why, When and How ?* Editors: Villiers,M.D.and Furinghetti,F. Centrahil: AMESA pp. 257-280.
Hanna, G. (1996). The Ongoing Value of Proof. *Proceedings of Topic Group 8: Proofs and Proving: Why, When and How ?* Editor: Villiers,M.D.and Furinghetti,F. Centrahil: AMESA pp. 1-12.
Kitcher, P. (1984). *The Nature of Mathematical Knowledge*. 1st Edition. New York: Oxford University Press.
Krantz, S. G. (2007). The History and Concept of Mathematical Proof. Downloaded from http://www.math.wustl.edu/~sk/eolss.pdf on the 23 April 2013.
Mudaly, V (2012). Diagrams in Mathematics: To draw or not to draw? Perspectives in Education, 30 (2).
Mudaly, V and Rampersad, R. (2010). The role of visualisation in learners' conceptual understanding of graphical functional relationships. African Journal of Research in Mathematics, Science and Technology Education. Volume 14 Issue 1. UNISA (36 – 48)
Nelson, R B (2007). A Sampler of Visual Proofs in First-year Calculus. CollegeBoard Advanced Placement Program. Lewis & Clark College. Portland, Oregon.
Peterson, I. (1990). *Islands of Truth – A mathematical mystery cruise*. 1st edition. New York: W.H. Freeman and Company.
Schoenfeld, A.H. (1985). *Mathematical Problem Solving*. 1st edition. Orlando: Academic Press Inc.
Slomson, A. (1996). Mathematical proof and its role in the classroom. *Mathematics Teaching, 155 pp.* 10-13.
Tall, D. (1995). Cognitive Development, Representations and Proof. *Justifying and Proving in School Mathematics*. London Institute of Education pp. 27-38.
Tiles, M. (1991). *Mathematics and the Image of Reason*. 1st edition. London: Routledge.
Tymoczko, T. (1986). The Four-colour Problem and its Philosophical significance in *New Directions in the Philophy of Mathematics*. 2nd edition. Editor : Tymoczko, T. Boston: Birkhauser pp. 243-265.

Wittmann, E. (1996). Operative Proofs in Primary Mathematics. *Proceedings of Topic Group 8: Proofs and Proving: Why, When and How ?* Editor: Villiers,M.D.and Furinghetti,F. Centrahil: AMESA pp. 15-21.