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ELEMENTARY SCHOOL MATHEMATICS

• NEW DIRECTIONS

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Foreword

INFORMATION on recent efforts to improve elementary school mathematics instruction is made accessible to States and local school systems in this bulletin. Numerous requests made to the Office of Education confirm the need for a single source in which factual descriptions of the major experimental projects and other improvement procedures are brought together for study.

No evaluation of individual projects is attempted. However, some guidelines for decision-making are suggested in the beginning and in the final sections of the bulletin. Each State or local school system will need to engage in careful study and thoughtful evaluation as decisions are made on the use of experimental programs in particular situations.

Appreciation is expressed to the following for their cooperation in reviewing a tentative draft of this bulletin: Directors of experimental projects; general curriculum consultants; mathematics consultants in State departments of education; mathematics educators; specialists of the elementary schools section, and specialists in mathematics, secondary schools section, U.S. Office of Education.

States and local school systems may wish to reproduce the entire bulletin or parts of it for their own use.

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Why the Increased Emphasis on Mathematics Today?

EDUCATORS ARE AWARE that changes are taking place in elementary school mathematics today. The purpose of this bulletin is to provide information for professional personnel responsible for elementary mathematics programs on (1) why changes are occurring; (2) the nature of the changes as seen from a study of selected experimental programs; (3) how changes are being accomplished; (4) the effect of changes on methods of teaching; and (5) implications for elementary mathematics instruction.

Teachers of elementary schools today find themselves in the midst of a general educational revolution with many measures underway to improve education generally. As a part of this reformation, elementary school mathematics is currently receiving considerable emphasis.

Reasons for Changes

One reason for change is found in the rapid advance of knowledge in mathematics which makes increasingly greater demands on an enlightened citizenry. Other reasons for change are (1) the need for more effective articulation from one grade to the next and from elementary to secondary school; (2) the recognition that the regular elementary mathematics program, limited mainly to emphasis upon the facts of the four fundamental processes, is somewhat barren of the fascinating, interesting phases of mathematics; (3) the need for better understanding of the structure of mathematics as essential to satisfactory and permanent grasp of subject matter; and (4) the need for improved mathematics programs for children of different abilities.

Historical Background

Historically, elementary school mathematics has gone through several changes which have influenced textbook writers and others responsible for the mathematics curriculum. Concern for the child as an individual and as a learner caused educators to question the grade placement and the usefulness of certain topics in elementary school arithmetic. As a result of research and experience certain topics were introduced later in the child's school life. Topics in arithmetic believed to have little use in daily life were omitted, and an effort was made to have children proceed in learning at rates more nearly commensurate with their abilities.

Elementary school mathematics also has been influenced by changing emphases in methods of teaching. As a reaction against a method based largely on repetition, some teachers experimented with incidental teaching where need was the paramount reason for learning. A study of results led to the conclusion that mathematics could not be learned well by either the drill or the incidental approach alone, nor even by the two combined. The heart of mathematics—its structure [1] and meaning—were neglected in both approaches.

For the past 20 to 25 years, much of the research, as well as some of the practice in the elementary classroom, has emphasized meaningful teaching, and in particular the "whys" of the processes as related to computational procedures. During this period, elementary school mathematics textbooks have reflected these emphases. Nevertheless in elementary schools today, the major method of teaching mathematics continues to be drill with rather fragmentary efforts in the direction of meaning.

Current Thinking

The realization that past achievements are not sufficient for today's world has brought together groups of professional people representing different backgrounds of experience and training to study educational problems and to suggest directions. An illustration of one such group effort was the Woods Hole Conference at Cape Cod where scientists, psychologists, educators, and others met to consider the goals and content of American education. The following statement about the problem of readiness for learning gives some indication of the thinking of this group. "We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any state of development." [2] A similar position was taken by the authors of the *Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics* in presenting a flow chart of 32 mathematical ideas [3] with suggestions for

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[1] Henry VanEngen. 20th Century Mathematics for the Elementary Grades. *The Arithmetic Teacher*, 6: 72 March 1959, defines structure "As the search for patterns . . . patterns which can be used to arrive at solutions to problems . . . ." An understanding of structure provides a method for solving "whole classes or groups of problems simultaneously. It is a powerful method which rests on abstractions and generalizations of a high order." For a more nearly complete explanation of the term "structure," *see* Jerome S. Bruner. *The Process of Education*. Cambridge, Mass.: Harvard University Press, 1961, p. 17-32.

[2] Jerome S. Bruner. *The Process of Education*. Cambridge, Mass.: Harvard University Press, 1961, p. 33.

[3] National Council of Teachers of Mathematics. *The Growth of Mathematical Ideas, Grades K through 12*. Twenty-Fourth Yearbook of the Council, 1959. The Council, 1201 Sixteenth Street, NW., Washington 25, D.C.: p. 480-498. See Appendix B.

ways that they might be developed from preschool through grade 12. Experimental groups throughout the country today are trying to identify the unifying ideas pervading all mathematics and to determine ways of beginning the development of these ideas in the elementary school.

The shift in terminology from "arithmetic" to "elementary school mathematics" implies a changing emphasis. The mathematics program for the elementary school child, formerly limited mainly to one branch of mathematics, is being broadened to include all branches which are appropriate for children within this age range.

Cooperative Experimentation

New experimental text materials are being written for elementary grades, tried out in the classroom, evaluated, and revised after use. If experimentation is to be successful, it must reflect the increasing mathematical knowledge of our times and, therefore, the greater need for structure, for pattern, and for relationships to bring meaning and understanding to this vast body of knowledge. Nor must we neglect to consider our knowledge of child development and our democratic ideal of helping each child learn all he is capable of learning.

Psychologists are assisting by turning their attention to the types of learning which go on in the classroom, and to the complexity of the classroom situations which produce general understanding of the structure of a subject. They are concerned with the relatedness of learning, with its value for later study, and with the process by which entirely new material becomes a part of learning that is familiar to the child. They characterize the child's learning of mathematics by several levels, each of which is pertinent to the development of fundamental mathematical ideas. There is (1) the level of the concrete or the world of things; (2) the level of internalized experiences in which the child imagines how experiences fit together or rejects trial and error; and (3) the use of abstract principles and generalizations. The degree of difference between levels depends upon the methods of teaching and upon the child's ability to comprehend. Many current educators are therefore led to conclude that certain topics can be introduced much earlier than was formerly believed.

Goals—Old and New

From the previous discussion it may appear that former goals for elementary school mathematics are being forsaken. This is not the case. The following frequently quoted purposes continue to be major goals for elementary school mathematics:

1. To develop concepts of quantity and of quantitative relationships; to develop the child's ability to think in quantitative situations.
2. To develop as high a level of skill in computation as is realistic in consideration of each child's potential.
3. To recognize those situations in daily living requiring mathematical solutions and the appropriate techniques for solving them.
4. To develop an understanding of our numeration system and to recognize the value of the base 10 in concept development as children work with processes.

To these the following goals, having some elements of newness, should be added:

1. To help each child understand the structure of mathematics, its laws and principles, its sequence and order, and the way in which mathematics as a system expands to meet new needs.
2. To help each child prepare for the next steps in mathematical learning which are appropriate for him in terms of his potential and his future educational requirements.

Some of the recent findings on learning and concept development have implications for the teaching of elemetary school mathematics. Studies in perceptual psychology indicate a relationship between the way a person acts and the way in which he perceives himself and his world. "If his perceptions are extensive, rich, and highly available when he needs them, then he will be likely to behave in effective, efficient, 'intelligent' ways."[4] A child perceives what seems appropriate to him in response to a need, in accord with his own goals and values. If threat is present, perception is blocked. From this point of view, intelligent behavior can be modified, changed, and shaped according to the quality of the person's perceptions. The task of the educator is to present challenging situations which the learner will approach with anticipation of success.

In the past much emphasis has been placed on what children know and on their misconceptions. Attention is now being directed to what children *can learn* about a given concept at different stages throughout the elementary grades and to the methods of presentation which are most fruitful. Evidence is accumulating to indicate that children can learn complex concepts and can begin learning them earlier than had been considered possible.[5] Crucial questions center around (1) the kinds of experiences which are appropriate to begin the develop-

[4] Arthur W. Combs. New Ideas About Personality Theory and Its Implications for Curriculum Development. *Learning More About Learning.* Washington, D.C.: Association for Supervision and Curriculum Development, 1201 Sixteenth Street, NW, February 1959. p. 13.

[5] O. L. Davis. Children Can Learn Complex Concepts. *Educational Leadership*, December 1959. p. 173-174.

ment of a concept; (2) how experiences may be planned and paced through the grades with appreciation for continuity and flexibility as to grade levels; and (3) variety of experiences for fostering and maintaining interest. Efforts which formerly went into finding ways of motivating repetitious drill are now being directed to ways of providing sequential practice in which succeeding exercises have elements of newness. Motivation, as well as depth and breadth of learning, is obtained through the selection of learning experiences.[6]

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[6] Ralph W. Tyler. Psychological Knowledge and Needed Curriculum Research. *Research Frontiers in the Study of Children's Learning*, Milwaukee, Wis.: School of Education, University of Wisconsin, May 1960. 55 p.

What Changes Are Taking Place?

Discovery and Intuitive Thinking

Some of the new mathematics content for the elementary grades has resulted from attempts to find out if younger children can successfully grasp simple selected geometric and algebraic ideas which in regular programs are met for the first time in the secondary school. Proponents of these innovations believe that through a discovery approach to learning children often grasp an idea intuitively long before they are ready for the detailed step-by-step analysis of the process. By an intuitive approach is meant a method which yields possible hunches or rapidly formulated ideas which will later be subjected to more formal analysis and proof. The method implies a freedom to make mistakes and to question. It makes use of what is known to arrive at a workable procedure as a starting place for solving a problem situation. Important aspects are the "critical question" and a low technical vocabulary. If the child can answer certain key questions, depth of understanding is assumed even though he cannot express his understanding in words.

Geometric and Algebraic Ideas

It is believed that the study of geometry can be expanded far beyond a meager knowledge of shapes, forms, and the computations required for finding areas and perimeters. It may include discovery of the principles underlying area and perimeter and the development of simple concepts with regard to points, lines, and planes in space. From their earliest school experiences with blocks and puzzles, children work informally with shapes and forms. While the lines, points, and planes are ideas, their representations in pictures or in the real environment of the child may be made concrete. In fact, it is possible that simple concepts of geometry are easier for the child to grasp than much of the abstract work with the operations of addition and subtraction which children have usually been expected to master during their first 2 or 3 years in school.

Evidence is accumulating from experimental centers to indicate that the mathematics program for children of grades 1–6 may be greatly enriched and broadened by including some simple algebraic ideas. The mathematical sentence in the form of an equation provides a way of getting at the nature of the operations of addition, subtraction, multiplication, and division. The equation lends itself readily to the use of letters or of frames, such as boxes, triangles, and ovals as placeholders. These are replaced by numerals to make the sentence true. For example, by using frames for the mathematical sentence $\square+1=\bigcirc$ children observe that the numeral which replaces the circle is always one more than the numeral which replaces the box and that there are an infinite number of replacements for both. Letters M and N ($M+1=N$) or X and Y ($X+1=Y$) may later be substituted for the frames to emphasize that any number may be used for M and that in this particular equation N will always be one more than M. They learn that replacements are made in such a way that the symbols on both sides of the equals sign represent different names for the same number. In other words, if the box is replaced by 3, the oval should be replaced by 4 to make the sentence true. The number sentence is $3+1=4$; $3+1$ and 4 name the same number. Children learn that sometimes a true sentence can result if one and only one numeral is selected. The only replacement for the box, in the equation $\square+3=7$ which will result in a true statement is 4. Sometimes several replacements are possible. The frames in the equation $7=\square+\bigcirc$ may be replaced by $3+4$, $4+3$, $1+6$, $6+1$, $2+5$, $5+2$, $0+7$, and $7+0$. When the child learns about fractions many other replacements are possible. Sometimes none of the numbers children know will work. Replacement for the frame in the equation $\square+4=2$ must wait for an understanding of negative numbers.

Operations and Properties of Numbers

Subtraction and division are presented as the "undoing" of addition and multiplication, respectively. Subtraction reverses the action associated with addition or brings the elements back to their original condition. In like manner division reverses the action associated with multiplication.

Children learn that zero is a number having special properties. It is a number that designates not any. When it is added to a number, the same number results. When it is subtracted from a number, the same number results: $3+0=3$ or $X+0=X$, $4-0=4$, or $X-0=X$. It is known mathematically as the identity element for addition, because when zero is added to a number, the sum is the number itself. They learn that *one* is the number in multiplication which serves a similar purpose as zero in addition: It does not change the product: $8\times1=8$, or $X\times1=X$.

Principles involved in the commutative, associative, and distributive laws of mathematics are a fundamental part of the newer experimental programs. Through the application and understanding of these principles, children are assisted in developing not only skill but also concepts of the nature of the operations, appreciation for the flexibility which is possible in mathematics, and understandings underlying the algorithms or forms of recording mathematics. The significance of these laws and properties of mathematics becomes clear to children who have opportunities to discover them and to observe what happens as they are used. Their application may be made simply and concretely in the early stages of learning and extended as advanced mathematics is experienced in later grades. The definitions and illustrations below serve as a reminder of the laws and of some of their applications to the mathematics of the elementary school child.

Commutative Property of Addition and Multiplication

(In all illustrations a, b, and c are rational numbers.)

The order in which two numbers are added does not affect the sum.	$3+2=2+3$; $\frac{1}{3}+\frac{1}{5}=\frac{1}{5}+\frac{1}{3}$ $a+b=b+a$
The order in which the numbers are used in multiplication does not affect the product.	$4\times3=3\times4$; $\frac{1}{3}\times\frac{1}{5}=\frac{1}{5}\times\frac{1}{3}$ $a\times b=b\times a$

Associative Property of Addition and Multiplication

Numbers may be regrouped for adding without changing the sum.	$3+(2+4)=(3+2)+4$ or $3+6=5+4$ $a+(b+c)=(a+b)+c$
Numbers may be regrouped for multiplying without changing the product.	$2\times(3\times4)=(2\times3)\times4$ or $2\times12=6\times4$ $a\times(b\times c)=(a\times b)\times c$

The associative property is also used in adding and in multiplying fractions:

$$\frac{1}{2}+(\frac{1}{3}+\frac{3}{4})=(\frac{1}{3}+\frac{1}{2})+\frac{3}{4}$$

$$\frac{1}{2}\times(\frac{1}{3}\times\frac{3}{4})=(\frac{1}{3}\times\frac{1}{2})\times\frac{3}{4}$$

Children in elementary school make use of this property when they regroup numbers to add and subtract.

Illustration 1:

$$\begin{array}{r} 24 \text{ or } 20+4 \\ +37 \text{ or } 30+7 \\ \hline 50+11 \text{ or } 60+1 \text{ or } 61 \end{array}$$

Illustration 2:

$$\begin{array}{r} 46 \text{ or } 30+16 \\ -28 \text{ or } 20+\ 8 \\ \hline 10+\ 8 \text{ or } 18 \end{array}$$

Illustration 3: $9+7=(9+1)+6$ or $10+6$

As answers to column addition examples are checked, both the commutative or order property and the associative or regrouping property are used: When adding down, the numbers are grouped $(3+4)+6$ and when adding up they are grouped $(6+4)+3$.

$$\begin{array}{r} 3 \\ 4 \\ +6 \\ \hline \end{array}$$

Distributive Property

The distributive property combines addition and multiplication.

$$2(3+4)=(2\times3)+(2\times4)$$
$$2\times7=6+8$$
$$a(b+c)=(a\times b)+(a\times c)$$

Illustration 1: $8\times36=8\times(30+6)$

$$=(8\times30)+(8\times6)$$
$$=240+48=288$$

Illustration 2: $36\times24=(36\times20)+(36\times4)$

$$=(30\times20)+(6\times20)+(30\times4)+(6\times4)$$
$$=600+120+120+24=864$$

The Closure Property

The term closure refers to a set and to an operation on the elements of that set.

Under the operation of addition, the sum of two counting numbers (1, 2, 3 . . .) is always another counting number or another number in the same set: $3+4=7$; $77+9=86$. The sums 7 and 86 are counting numbers.

Under the operation of multiplication the set of natural numbers is also closed because the product of two counting numbers is always another counting number: $3\times 4=12$; $9\times 12=108$.

Counting numbers, however, are not closed under the operation of division. For example, $5\div 2$ gives a different kind of number than a counting number. $2-7$ gives another kind of number. Fractions and negative numbers are created in order to have closure. In other words the counting numbers are not closed under the operations of subtraction and division.

Emphasis on the laws and principles of mathematics illustrates the trend of the newer mathematics to extend and expand familiar topics of standard programs in both depth and breadth. Although an understanding of the basic ideas of these laws and principles has been included in the content for pupils in some textbooks, the laws and principles as such have rarely been explained in teachers' manuals and guides and teachers have had little opportunity to realize the importance of these structural features of mathematics.

Extensions of Familiar Topics

Other extensions of familiar topics expand the study of our decimal numeration system to include an introduction to negative numbers and powers in the upper elementary grades. An introduction into bases other than 10 may provide interest and variety, but it is intended primarily to promote a deeper understanding of our decimal numeration system. The emphasis on relationships within and between processes is continued as children are encouraged to reason from known to unknown facts: If $5+5=10$, then $5+6=11$; if $10+6=16$, then $9+6=15$; if $10\times 6=60$, then $9\times 6=60-6$ or 54; if $3+9=12$, then $12-3=9$ and $12-9=3$; if $4\times 6=24$, then $24\div 6=4$; if $\frac{1}{4}\times\frac{2}{3}=\frac{1}{6}$, then $\frac{2}{3}\times\frac{1}{4}=\frac{1}{6}$, and $\frac{1}{6}\div\frac{1}{4}=\frac{2}{3}$, and $\frac{1}{6}\div\frac{2}{3}=\frac{1}{4}$, etc.

Teachers of the early grades are familiar with a study of "groups" as background for the arithmetic operations of addition, subtraction, multiplication, and division. Some of the newer programs are using the concepts of sets and the terminology of sets as a more mathematically accurate introduction to these operations. A "set" is a well-defined collection of objects which are not necessarily alike in any way; for example, a triangle, a square, and a circle; and a balloon, a cart, and a jump rope. In each example there are three things. The objects of one set can be matched one-to-one with the objects of the other. The triangle can be matched with the balloon, the square with the cart, and the circle with the jump rope to show that these are equivalent sets. Both have the same cardinal number. The numeral "3" names the number of objects for either set. The objects in two sets may be combined to make a third set. A set may be separated into two subsets. One set may be compared with another. A set may be used repeatedly or divided into subsets having the same number of elements. These procedures of identifying, combining, separating, and comparing when performed with numbers are roughly analogous to the operations of addition, subtraction, multiplication, and division with numbers.

Still other extensions and applications of familiar topics include (1) a greater emphasis on the understanding of large and small numbers to help children better meet today's needs; (2) different ways of arriving at the same answer; (3) more emphasis on estimation and mental arithmetic as essential for the numerical thinking demanded by our society today; and (4) variety in practice procedures required for maintenance of skills which will foster interest and provide new learning at the same time.

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How Is Mathematics Taught?

Guidance in Thinking and Reasoning

In all of the newer programs today there is greater emphasis on both types of reasoning—inductive, from the particular to the general, and deductive, from the general to the particular. Children are guided to conjecture from individual situations to what the universal situation might be and to test their conjectures to see if they work. They are also led to arrive at conclusions deductively by reasoning from basic principles.

The newer programs demand careful development and guidance from the teacher. It may be that well-conceived standard programs suffered a basic weakness because of the ease with which the content could be assigned for independent work or for homework by children. Teachers who have tried out the new materials are finding that developing concepts and helping children build a mathematical structure require continuous guidance. While children are not left to their own devices, there is an appreciation for the different ways of thinking, for the fact that there is more than one way of arriving at an answer, and for the gradual refinement to higher levels of performance. Experimentation and discovery are encouraged, and the attitude that learning mathematics can be a creative adventure is fostered. When thinking is valued, oral participation by children and observation of them at work become important. Thus mental arithmetic, or ways of solving problems without pencil and paper or with a minimum of recording, is an ability to be stressed. The relation of the unknown to the known, which becomes apparent as patterns are discovered, makes learning easier and more permanent. The emphasis on generalizations helps the child to fit new bodies of learning into the overall structure without upsetting past learning or endangering ideas to be developed in the future.

Content and Method

In some of the newer programs content and method have been so closely interwoven that a separation of the two is extremely difficult. In fact, method is so significant a part of the programs that the technique of demonstration by highly skilled teachers has been a major means of dispensing information on the projects. Demonstrations with classes of children have been held in many parts of the country. Principals, teachers, and supervisors observe and later participate in a discussion of the classwork and of the project in general.

Supplementary Aids

Television, teaching machines, and programed learning materials are now being utilized to some extent in elementary school mathematics. Television programs have been developed for the inservice education of teachers. Some TV programs have the dual purpose of classroom instruction for the child and inservice education for the teacher. Some experimentation in the areas of teaching machines and programed learning is underway. The individualization of instruction and immediate awareness of whether an answer is correct or incorrect appear to be the plus values most often attributed to the teaching machines.

Many school libraries are building valuable resources for able learners by increasing their supply of trade books dealing with mathematics. Annotated lists of these with suggested grade level designations appear periodically in *The Arithmetic Teacher*.[1]

Concrete and pictured materials have long been considered essential aids to making mathematics meaningful for the child. Materials are also used widely in the newer mathematics programs. There is perhaps a trend toward a greater use of the type of materials which emphasize structure, such as, the counting frames, the abacus, and rods or strips based on a unit length. In regular programs the use of materials often represents a stage or level of development of a new mathematical idea. In the newer programs materials are used to stimulate, to initiate, and to assist the child's thinking process. Hopefully they are used under the guidance of the teacher when and if they serve these purposes.

Classroom Organization

Whether the classroom is organized as self-contained, departmentalized, homogeneous, heterogeneous, or according to some combination of these, it is obvious that great variability in both skills and understandings still exists within a single classroom. In fact, when differences increase as they should, few teachers feel satisfied that they

[1] *The Arithmetic Teacher*, published monthly eight times a year, October through May, by the National Council of Teachers of Mathematics, 1201 Sixteenth Street NW., Washington 6, D.C.

are achieving the goal of meeting individual differences adequately. Not only are there differences among children, but variability also exists among the skills and abilities of any one child. Certainly the aim is not to reduce differences, but with good teaching-learning situations to create even greater ones. As schools accept differences and plan ways of meeting them, they will find some practices and procedures for study and evaluation. Some of these, such as team teaching [2] and departmentalization [3] are administrative plans which require attention to the total organization of the school. The decision to select any organizational plan should be made with recognition of the limitiations and strengths within the local situation, on the basis of research findings, and in the light of the educational philosophy of the school system.

Even when the organizational pattern of the school facilitates the teacher's task, teachers still find that some grouping within the class is necessary. Subgroups, which are often organized after the initial introduction of a new topic, bring children together for different kinds of instruction. The slower children move at a slower pace, the mathematics content is limited to the easier, less abstract phases of topics, and a greater use is made of concrete and other types of learning aids. A good program for the slow child provides more experience of every type that is appropriate for him, with sufficient variation to stimulate interest and thought. The more able children move at a faster pace and deal with more abstract material. These children are often able to visualize and imagine the concrete situation to the extent that there is little need for actual manipulation of materials. The able child is challenged with more difficult problems and examples. It is justifiable to include some examples not frequently met in real life situations merely because they bring completeness to the structural patterns of mathematics. A tendency to demand that more problems be solved or more examples be worked when the child already has mastery of a topic must be avoided. One guideline which teachers have found helpful is the challenge to solve the same problem or work the same example in several ways. In followup discussions the various methods may be evaluated for accuracy and efficiency as children share with each other the thrill of inventing ingenious methods of solution. These children are encouraged to be re-creators of mathematics, to make discoveries for themselves, not be just learners of mathematics. Able children should have opportunities to practice and to reach a high level of performance in mental arithmetic.

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[2] Stuart E. Dean, Team Teaching, *School Life*, Vol. 44, No. 1, September 1961. p. 5–8.

[3] *American Association for the Advancement of Science, Science Teaching Improvement Program.* Study on the Use of Special Teachers of Science and Mathematics in Grades 5 and 6. 15 p.

Teachers also find a need for still another type of mathematics subgroup designed to meet special learning needs of children. Special instruction for this group may provide children with the prerequisite skills for long division or for division of fractions. At times all members of this group may have different needs so that instruction becomes individualized.

A few schools have provided special classes and teachers with special qualifications for very able children in mathematics in grades 5 and 6. Classes meet two or three periods a week and cover topics not included in the regular program. Instruction highlights exploration, discovery, and creative approaches to topics.

Teachers in self-contained classroom situations use special projects and assignments drawing content from recreational mathematics, such as games and puzzles, from the history of mathematics, from real life applications, from new topics, and from extensions of topics being studied. Self-teaching instructional lessons or units which can be used with a minimum of guidance from the teacher have proved a fruitful source of material for able children.[4]

Relation of Mathematics to Other Subject Areas

The mathematics content which is structural is probably best learned within the system and sequence characterizing it. Illustrations are the numeration system and place value; the operations of addition, subtraction, multiplication, and division with whole numbers and fractions; and the generalizations, principles, and laws of mathematics. Most of the newer programs, which are generally mathematically oriented, develop mathematical concepts from physical experiences which are then abstracted into an ideal system. Applications are chosen primarily to clarify and show the use of the mathematical principle involved. The applications, uses, and needs for mathematics are often realized in relation to other subject areas or to life experiences. Naturally no sharp lines of demarcation exist between the two although the emphasis usually may be clearly defined in one or the other category.

Mathematics often assists the learner in developing social studies or science concepts. The concept of interdependence, for example, is clarified as a child becomes able to think of distance in terms of reduction in the time required to travel from one place to another as a result of improved methods of transportation. Quantitative con-

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[4] Fred J. Weaver and Cleo Fisher Brawley. Enriching the Elementary School Mathematics Program for More Capable Children, *Journal of Education*, Boston University School of Education, Vol. 142, No. 1, October 1959. 40 p.

cepts are essential to meaningful interpretation of map scales, comparisons of population statistics, and area measurements. On the other hand as a child deals with very large numbers in his study of astronomy, or the very small numbers used in making fine measurements, he begins to appreciate the contribution that concentrated study of the structure can make to his understanding of both mathematics and science.

Many teachers have participated in special projects designed to relate some aspects of science and mathematics. One such effort is an Elementary School Science Project developed at the University of California.[5] Teachers will find another useful resource in Sawyer's booklet *Math Patterns in Science*.[6]

Problem-solving and the ability to do critical thinking are not unique to mathematics. Social studies and science abound in opportunities for the exercise of these abilities. One of the challenges of education today is to determine the contribution different bodies of subject matter can make to such fundamental abilities as problem-solving and critical thinking.

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[5] D. C. Ispen, Phyllis Raabe, and S. P. Diliberto. Coordinates—An Introduction to the Use of Graphs and Equations To Describe Physical Behavior. University of California, Berkeley, Calif., 1960. (Tentative form)

[6] W. W. Sawyer. *Math Patterns in Science*. American Education Publications, Education Center, Columbus 16, Ohio, 1960. 31 p.

SELECTED EXPERIMENTAL PROJECTS

AS STATED IN THE FOREWORD, it is not the purpose of this bulletin to evaluate or recommend programs. Rather the purpose is to acquaint school personnel with the various projects and to provide factual information and pertinent guidelines which may be helpful in improving elementary school mathematics programs throughout the country.

Teachers of elementary grades who have not had the advantage of recent inservice workshops or courses in modern mathematics may find it helpful to refer to a glossary of mathematical terms as they read the descriptions and illustrations from experimental projects.

Recent experimental projects, which have not been available to school personnel generally during the 3- to 5-year period since they have been underway, have been selected for discussion. The many inquiries received by the Office of Education indicate a widespread need for specific information on the projects selected. A brief treatment of some other programs directed toward the improvement of instruction in elementary school mathematics is found in a later section of the bulletin.

The purpose of these programs is not necessarily to create a new and different program in elementary school mathematics. Rather, through these programs an attempt is made to test out certain probable ideas on a wide scale to see how workable they are. One purpose is to help all teachers learn to do better what some teachers have been doing well for a long time, to help children see the basic structure unifying the subject as they learn mathematics, thereby helping them to think clearly and reason correctly. While the experimental programs give new dimensions to the learning of elementary school mathematics, it is not intended that they entirely replace the standard or present program. Most of the experimental programs claim that their work is within the framework of the existing curriculum. New ideas can supplement and implement the developmental program which presents a solid and substantial base on which to build.

Descriptions of selected programs are limited to the simpler ideas developed by each project and to changes in method and content. Teachers and other interested school personnel are encouraged to consult project materials directly to learn about the systematic development of the materials and to sense the upper levels of accomplishment attained by children using project materials in experimental situations.

While each project has unique characteristics, all share common goals of adding new content, although not the same content; presenting new approaches to method; emphasizing the meanings and understandings inherent in the structure of mathematics; seeking to motivate and stimulate greater interest in mathematics; and exploring mathematical content appropriate for different levels of ability.

Some differences in approach and in method are apparent. There is a difference of opinion on the need for exactness and precision in mathematical vocabulary. While all are agreed that mathematics must be learned meaningfully, some rely more on inductive discovery (from specific to general) and others on deductive reasoning from examples (from general to specific). Of course both methods are often evident within a single program. Some are broad in scope having as an explicit goal the development of a complete mathematics curriculum. Others aim to develop materials for teaching certain topics or to produce supplementary materials.

The experimental projects have been concerned with the total problem of finding out more about how children learn as well as with improving the mathematics program.

Some of the materials developed by experimental projects are now available from commercial publishers. For specific information on materials available and on how they may be obtained, the reader may write to the director of the particular project.

School Mathematics Study Group Program[1]

General Description

Consideration of the mathematics programs at the elementary school level by the School Mathematics Study Group began with a conference on elementary school mathematics called in 1959 by Prof. E. G. Begle, director of the project. In attendance were university professors of mathematics, high school and elementary school teachers,

[1] Director: Prof. E. G. Begle, Assistant Director: George L. Roehr, School Mathematics Study Group, School of Education, Cedar Hall, Stanford University, Stanford, Calif.

supervisors and education specialists with specific interest in mathematics, psychologists, and representatives from scientific and governmental organizations having an interest in mathematics. From this conference came the recommendation that a critical study of elementary school mathematics curriculum be undertaken. Among the aspects of the total elementary mathematics program suggested for study were: (1) the grade placement of topics; (2) development of concepts and mathematical principles; (3) the possible introduction of new topics particularly from geometry; (4) topics for able learners; (5) training for teachers; (6) the relation of elementary school mathematics to future study of the subject; (7) methods and materials for effective classroom instruction; and (8) the application of findings on concept-formation from psychology and child development to the learning of mathematics.

In March of 1960, a detailed outline of a suggested program for grades 4, 5, and 6 was developed. For 8 weeks during the summer of 1960 a writing team composed of classroom teachers or supervisors, mathematicians, and mathematics educators worked together to prepare materials. At this time units comprising a complete course for grade 4 and sample units for grades 5 and 6 with accompanying teachers' manuals were prepared. The format of the units is the "write-in" text workbook type with explanation and instruction; space is allowed for pupil answers, for completion of exercises, or for carrying out activities. In this tentative form tryouts were held in 27 experimental centers over the country involving around 12,000 students and 150 fourth grade, 110 fifth grade, and 110 sixth-grade teachers. Each center was associated with a college or university which provided consultant service for the participating teachers. Regular meetings were held to build the mathematics background for units to be presented and to evaluate those which had been taught. In evaluating each unit teachers were requested to give reactions to the appropriateness of sequence and grade level of each unit, the difficulty of the material from the standpoint of concept development and reading and the amount of time devoted to teaching the unit, the use of supplementary materials or additional practice exercises, and the effectiveness of the teacher's commentary. On the basis of the experiences of the participating teachers and their evaluations, the units were revised and completed during the summer of 1961 and tried out again during 1961–62. Following a final revision of units currently available for examination and study, textbooks for general classroom use will be made available. It is the intention of

SMSG in the near future to extend its study and writing downward to grades 1, 2, and 3.

According to the committee, the proposed SMSG curriculum is not radically different from the regular present-day program in its organization. The present program is viewed as being based on social applications while the goal of the proposed changes of the SMSG is to provide a curriculum based on mathematics. There are some new topics and new approaches to old ideas. The SMSG committee believes that a healthy fusion of the old and the new will lead students to a better understanding of the basic concepts and structure of mathematics and will provide a firmer foundation for understanding and use of mathematics. The committee hopes that the materials they have prepared will serve as models and as a source of suggestions for textbook authors and as aids in preservice and inservice teacher education programs.

Another point of view held by the committee is that skill in computation and insight go hand in hand in mathematics, both being essential to proficiency. Emphasis is placed on precision of language of mathematics and on knowledge of and appreciation for a mathematical system which expands gradually to make more and more mathematical solutions possible.

In an effort which is national in scope, the School Mathematics Study Group has research and development in the teaching of school mathematics as its primary purpose. It expects to continue to develop courses, teaching materials, and teaching methods; to promote inservice education; and to carry out long-term evaluations of its mathematics programs.

Nature of Content

The content includes:

1. Extensions of all of the content now covered in elementary school methematics with more breadth and depth, placing greater emphasis on the laws and principles of mathematics.
2. Introduction of simple concepts throughout the elementary grades on an intuitive level by methods which are consistent with later approaches to the topics. New topics include intuitive geometry and simple algebraic ideas.
3. Emphasis on precision of mathematical language as an aid to logical thinking, and on symbols as a means of short-cutting language.

A listing of the units for grades 4, 5, and 6 followed by selected illustrations from some of them will provide information on the nature of the proposed changes:

Grade 4

- Concept of Sets
- Numeration
- Properties and Techniques of Addition and Subtraction, I
- Properties of Multiplication and Division
- Sets of Points
- Properties and Techniques of Addition and Subtraction, II
- Techniques of Multiplication and Division
- Recognition of Common Geometric Figures
- Linear Measurement
- Concept of Fractional Number

Grade 5

- Extending Systems of Numeration
- Factors and Primes
- Extending Multiplication and Division
- Congruence of Geometric Figures
- Addition and Subtraction of Fractional Numbers
- Measurement of Angles
- Area
- Ratio
- (Supplementary Review Exercises)

Grade 6

- Exponents
- Multiplication of Fractional Numbers
- Side-Angle Relationships of Triangles
- The Integers
- Coordinates
- Division of Fractional Numbers
- Volume
- Organizing and Describing Data
- (Supplementary Review Exercises)
- Sets and Circles

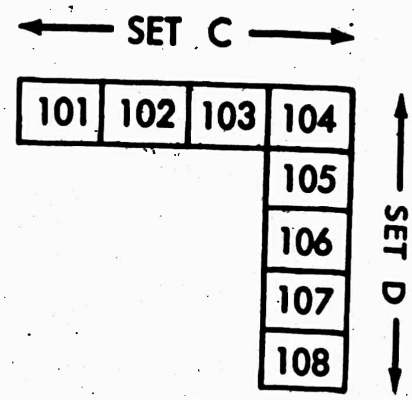
A more detailed description on selected units from grades 4, 5, and 6 follows:

Grade 4—Concept of Sets

The purpose of the unit on sets is to help children become better acquainted with the idea of *set* and to begin building a "set language" as aids to more effective learning of mathematics. The meanings of set, sets within sets, equal sets, empty set, union of sets, and intersection of sets are developed through discussion, classroom experiences and exercises similar to the following:

Illustration 1:

The picture at the right shows rooms in Jane's school. Rooms 101, 102, 103, and 104 have windows along the front of the building. Rooms 104, 105, 106, 107, and 108 have windows along a side of the building.



Set C={101, 102, 103, 104}. See figure.

Set D={104, 105, 106, 107, 108}.

We write: C∪D={101, 102, 103, 104, 105, 106, 107, 108}.

We read: the union of Set C and Set D is the set whose members are 101, 102, 103, 104, 105, 106, 107, 108.

Illustration 2:

Set R={10, 20, 30, 40, 50}

Set S={60, 70, 80, 90, 100}

Which one of the sets below is the union of Set R and Set S?

Set M={70, 90, 110, 130, 150}

Set N={100, 90, 80, 70, 60, 50, 40, 30, 20, 10}

Copy and finish: R∪S={N}

Illustration 3:

Set X is the set of numbers we use when we count by 5's, starting with 5 and ending with 30.

Set Y is the set of numbers we use when we count by 10's, starting with 10 and ending with 50.

Set X = {5, 10, 15, 20, 25, 30}
Set Y = {10, 20, 30, 40, 50}

The numbers that are members of both Set X and Y are 10, 20, and 30.

We write: $X \cap Y = \{10, 20, 30\}$.

We read: The intersection of Set X and Set Y is the Set 10, 20, 30.

Grade 4—Properties and Techniques of Addition and Subtraction

Addition and subtraction are explained as operations on numbers, or as a way of thinking about two numbers and getting a third number. Subtraction is presented as undoing addition or finding the missing addend. The properties of addition are emphasized.

Illustration 1: Write the correct numeral or word to complete this chart. The first exercise is done for you as a sample.

	Numbers operated on	Result	Operation used
1	5, 7	12	Addition
2	9, 3	____	Subtraction
3	10, 2	12	____________
4	10, 2	8	____________
5	10, 2	5	____________

Illustration 2: Statements like those in Box A are called mathematical sentences.

1. (a) Is it true that $6+4=10$?

   (b) Is it true that $9-3=6$?
   that $7+6 \neq 12$?
   that $10-10 \neq 10-9$?

A

$6+4=10$
$9-3=6$
$0+8=8$
$7+6 \neq 12$
$200-100 \neq 200$
$10-10 \neq 10-9$

Illustration 3: **Thinking About Addition Facts**

You show addition like this:

$9+5=14$

$$\begin{array}{r} 9 \\ +5 \\ \hline 14 \end{array}$$

The names of the parts in an addition sentence are:

9	+	5	=	14
addend		*addend*		*sum*

You read addition like this:

9 and 5 are 14, or 5 added to 9 is 14, or 9 plus 5 is 14.

Illustration 4: **Thinking About Subtraction Facts**

You show subtraction like this:

$9-5=4$

$$\begin{array}{r} 9 \\ -5 \\ \hline 4 \end{array}$$

The names of the parts in a subtraction sentence are:

9	–	5	=	4
sum		addend		addend

You read subtraction like this:

5 subtracted from 9 is 4, or
9 minus 5 equals 4.

Illustration 5: **Picturing Mathematical Sentences**

A number line can be used to picture mathematical sentences. On the number line below, two mathematical sentences are pictured:

$5+6=11$ and $11-6=5$

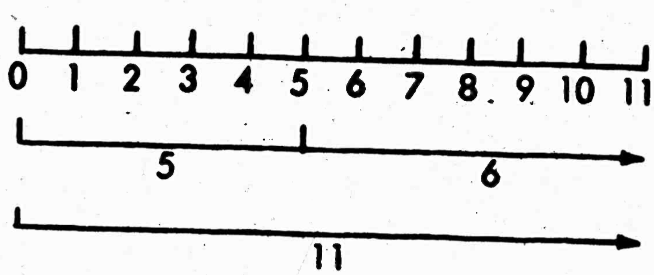


Illustration 6: **Addition will undo subtraction.** For example, if we start with 9, then subtract 2, and then add 2, the result is 9, the number we started with. Adding 2 undid subtracting 2. We can show this in these two ways:

$$9-2=7, \text{ so } 7+2=9,$$
$$\text{or } (9-2)+2=9.$$

Subtraction will undo addition. For example, if we start with 3, then add 5, and then subtract 5, the result is 3, the number we started with. Subtracting 5 undid adding 5. We can show this in these two ways:

$$3+5=8, \text{ so } 8-5=3,$$
$$\text{or } (3+5)-5=3$$

Illustration 7: **Addition is a Commutative Operation**

For example: $3+5=8$

$5+3=8$

The sum is the same even if the order of the addends is changed. So we can write: $3+5=5+3$

Illustration 8: **Addition is an Associative Operation**

Adding three numbers must be done in two steps. You may add 3, 4, and 5 in either of two ways.

$(3+4)+5=7+5=12$

$3+(4+5)=3+9=12$

The sum is the same even if we did group the addends differently. So, we can write

$$(3+4)+5=3+(4+5).$$

Grade 4—Properties of Multiplication and Division

The operations of multiplication and division are made intelligible to the child through an array or an orderly arrangement of objects in rows and columns. Multiplication is described as an operation on two factors to get a product. The application of the commutative and associative properties is made to multiplication. Division is explained as undoing multiplication or as the process of finding a missing factor. As a larger array is separated into two smaller arrays children are led to appreciate the contribution the distributive property can make to learning to multiply and divide.

Illustration 1:

An array is an orderly arrangement of objects in rows and columns.

The eggs in one-size egg carton form a 3 by 4 array. The crayons in one-size box form a 2 by 6 array.

Here is an array of 2 by 6:

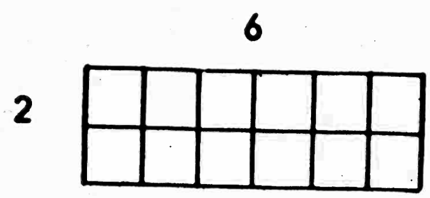


Illustration 2: **Multiplication**

We write a multiplication sentence like this:

$5 \times 4 = 20$

We read a multiplication sentence like this:

5 times 4 is 20.
5 times 4 equals 20.

The names of the parts of a multiplication sentence are:

$5 \times 4 = 20$

factor times factor equals product

When we operate on *two factors* and get a *product*, we *multiply*.

Illustration 3: Make a chart with 4 columns as shown below. Study and complete the chart.

Numbers operated on	Number which results	Operation used	Mathematical sentence showing operation used
4, 2	6	Addition	$4+2=6$
7, 4	3	Subtraction	$7-4=3$
4, 3	12	Multiplication	$4\times3=12$
45, 9	5	Division	$45\div9=5$
6, 4	24	----------------	-----------

Illustration 4: **Division**

Division is an operation of mathematics. It is one of the four basic operations.

An operation on numbers is a way of thinking about two numbers and getting one number. When we think of 6 and 2 and get 3, we are dividing. When we think of a number and one of its factors and get the unknown factor, we are dividing.

We can picture division with arrays. When we think of the number of columns there are in an array with 6 elements and 2 rows, we are *dividing*. When we think of the number of rows in an array with 12 objects in 3 columns, we are *dividing*.

Illustration 5: **Multiplication and Division**

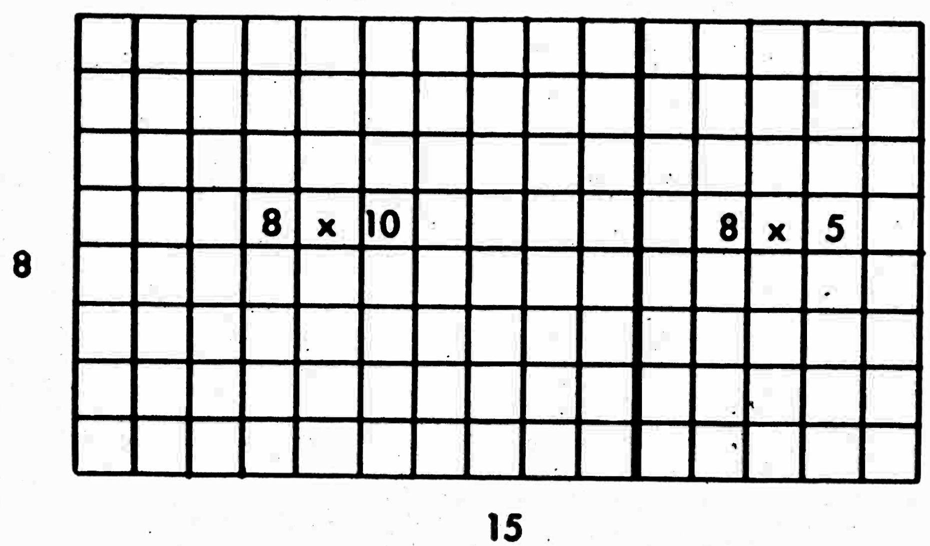
Multiplication will undo division. Think of 8, divide by 2, and then multiply by 2. The result is 8. The *multiplication* by 2 *undid* the *division* by 2.

$$(8 \div 2) \times 2 = 8$$

Division will undo multiplication. Think of 8, multiply by 2, and then divide by 2. The result is 8. The division by 2 undid the multiplication by 2.

$$(8 \times 2) \div 2 = 8$$

Illustration 6:



$$8 \times 15 = (8 \times 10) + (8 \times 5)$$

The heavy line shows a possible way to fold the array. The sentence below the picture shows the relation between the whole array and the two smaller arrays which the fold makes. The array can be folded in many other ways. Find six different ways of separating the array. Write the mathematical sentence for each separation.

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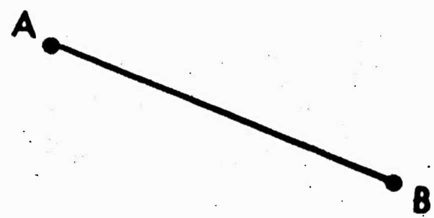
Grade 4—Sets of Points

Beginning concepts are developed for the geometric meaning of some of the following: point, space, line segment, line, the number of lines which can go through a given number of points, points and lines extended to form rays and angles, intersection of lines and planes and figures such as polygons and circles.

Illustration 1: **Space is the set of all points.**

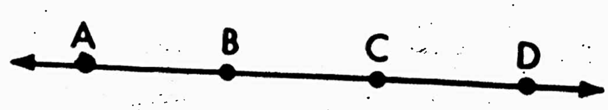
This means all exact locations everywhere. All the locations on the head of a pin, all the locations in your home, in your city and the sky above, in your country, in the world, and in the entire universe are points in space. Space as we now picture it is probably very different from the idea you had. Any object you can think of covers or occupies lots of points of space. For example, a ball, a block of wood, a room, a building, the earth are all occupying parts of space.

Illustration 2: On your paper draw a segment connecting the two dots as shown in the figure below. We shall represent a line segment in this way.



We name this "line segment AB." A short way to write "the line segment AB" is $\overline{AB}$. The line segment ends at points A and B. Therefore points A and B are called *endpoints*.

Illustration 3: Below is a picture of a line.



The arrows are used to show that it goes on and on in both directions without end. Only part of the line can be pictured on this page. We can call the line pictured, line AB.

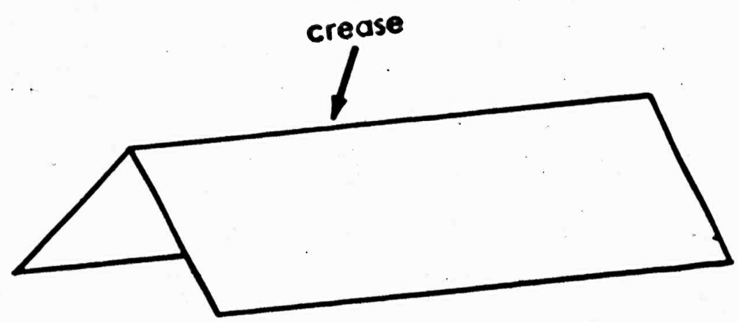
Both A and B name points on the line. C and D name other points on the line. We could also call this line, line CD, or line AC, or line AD.

Line AB is the same as line BA. What other names can this line have? Use just the points named.

Illustration 4: Put your finger at a point on the top of your desk. Are there many points which are not in the plane represented by the top of your desk?

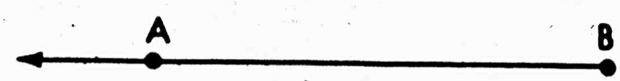
From now on we shall think of *a part of a plane* as a set of points in space. It is the kind of set suggested by all points on a flat table top, or on a wall, or on the floor. A piece of paper lying flat on your desk also suggests a part of a plane.

Illustration 5: Fold a piece of paper in half. We think of the crease as a line segment. Stand the folded paper on your desk so that the crease does not touch it.



Does this suggest parts of two planes which contain the line segment represented by the crease? If so, show them.

Illustration 6: A ray has one endpoint. The endpoint is named first. This is a picture of ray BA. What is its endpoint?

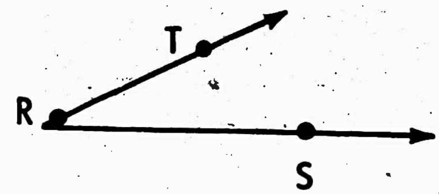


Ray BA is not the same as ray AB. Can you tell why? The endpoint of ray BA is B. What is the endpoint of ray AB?

We can say that a *ray* is the union of the endpoint and all points on a line in one direction from this point.

A ray is always part of a line. A set of rays is nicely represented by a beam of light from a flashlight. Each starts at the flashlight and extends in one direction without end.

Illustration 7:

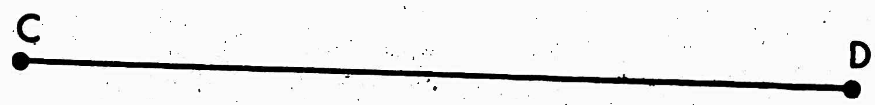


This drawing represents a new geometric figure called an *angle*. An *angle* is the union of two rays which have the same endpoint but are not on the same line.

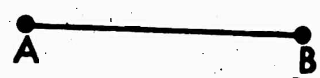
Grade 4—Linear Measurement

The unit on measurement is geometric in nature. Straight edge and compass are used to draw and compare segments. The gradual development from an intuitive understanding of differences in size to the selection of a standard unit and a suitable measuring scale is presented.

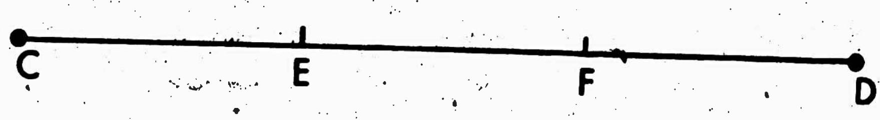
Illustration 1: We want to find the measure of $\overline{CD}$. We use our compass to help us measure a line segment.



Our unit of measure is $\overline{AB}$.

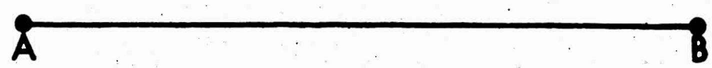


We lay off $\overline{AB}$ on $\overline{CD}$ three times.
We label the intersections E and F.
See the picture below.



We say the measure of $\overline{CD}$ is 3.
We write: m $\overline{CD}=3$.

Illustration 2: **Using your inch scale and your centimeter scale, find the measures of this segment in inches and in centimeters.**



m $\overline{AB}$ (in inches) = ______

m $\overline{AB}$ (in centimeters) = ______

Illustration 3:

1. Take a piece of wire, 15 inches long. Bend it to make a simple closed curve.

What is the length of the wire when it is bent in the shape of a closed curve?

2. Straighten out the wire, and bend it to form a different closed curve. What is the length of the new curve? What happens to the length of the wire when you change the shape of the simple closed curve?

Grade 5—Extending Systems of Numeration

Extensions of our base 10 numeration system include exploring large numbers by reading, writing, renaming in several ways, and expressing them in expanded notation form $(3{,}604=3\times1{,}000+6\times100+0\times10+4)$; exploring fractional numerals and decimal numerals through thousandths in a similar manner; providing experience with counting and place value in base five system of numeration and finally in any base.

Illustration 1: Since 300 means 3 hundred, we can write it as (3×100). 50 means 5 tens, which can be written as (5×10). 2 ones can be written as (2×1). Writing 352 as (3×100)+(5×10)+(2×1) is called *expanded notation.*

Look at the numerals in the chart below. Place values are written at the top of the chart. Use the chart to help you see how these numerals are written in expanded notation.

	1,000,000	100,000	10,000	1,000	100	10	1	
a				4	2	8	3	= (4 x 1,000) + (2 x 100) + (8 x 10) + (3 x 1)
b			2	3	5	8	4	= (2 x 10,000) + (3 x 1,000) + (5 x 100) + (8 x 10) + (4 x 1)
c		6	2	8	7	3	9	= (6 x 100,000) + (2 x 10,000) + (8 x 1,000) + (7 x 100) + (3 x 10) + (9 x 1)
d	7	9	4	3	2	1	5	= (7 x 1,000,000) + (9 x 100,000) + (4 x 10,000) + (3 x 1,000) + (2 x 100) + (1 x 10) + (5 x 1)

Illustration 2: Below are examples showing some of the ways a number can be named.

426,315=4 hundred thousands +2 ten thousands +6 thousands +3 hundreds +1 ten +5 ones

426,315=42 ten thousands +6 thousands +3 hundreds +1 ten +5 ones

426,315=426 thousands +3 hundreds +1 ten +5 ones

426,315=400,000+20,000+6,000+300+10+5

Illustration 3: We have learned to think about a decimal fraction like .73 as 73 hundredths. We also know that in .73, the 7 is in the tenths' place and the 3 is in the hundredths' place. This gives us another way to think about and name .73:

.73=7 tenths and 3 hundredths.

In the same way,

.49=__tenths and__hundredths.

We also can say,

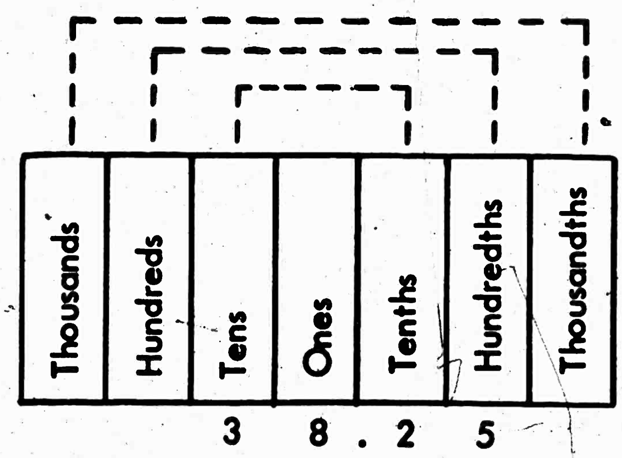
8 tenths and 2 hundredths=.82

In the same way,

3 tenths and 6 hundredths=__.

Illustration 4: We have been learning how to read and interpret decimal fractions such as .7 and .39 and .561. Many times we need to use numerals such as 38.25. These also are *decimal fractions*. We often call decimal fractions simply *decimals*.

The chart below will help us learn how to read and interpret other decimals.



We read 38.25 as: "thirty-eight *and* twenty-five hundredths." Notice that the word "and" is used between reading the digits to the left of the decimal point and reading the digits to the right of the decimal point.

Illustration 5:

Example A
XXXXX
XXXXX
XXX

Here is a picture of a group of 13 X's. In this group we have 2 sets of 5 and 3 ones. The numeral may be written 23_{five} (read "two fives and three ones" *or* "two three, base five.")

NOTE: The word "five" written to the right and slightly below the numeral shows that you are grouping in sets of five.

Illustration 6:

Number in base five system	Picture in base five system	Picture in base ten system	Number in base ten system
a. 22_{five}	XXXXX XX XXXXX	XXXXX XXXXX XX	12
b. 33_{five}	XXXXX XXXXX XXXXX XXX	XXXXX XXXXX XXXXX XXX	18

Study the chart above. What does the numeral 22_{five} tell us?

What does the numeral 12 tell us?

Are 12 and 22_{five} names for the same number?

Why are 33_{five} and 18 names for the same number?

Illustration 7: Separate the following amounts of money into quarters, nickels, and cents. Use the smallest number of coins.

How much money?	How many quarters?	How many nickels?	How many cents?	Base five notation
Examples: 14 cents	0	2	4	24_{five}
43 cents	1	3	3	133_{five}
(1) 23 cents				
(2) 26 cents				
(3) 29 cents				
(4) 33 cents				

Grade 5—Factors and Primes

The learnings emphasized in this unit are the ideas of different names for the same whole or fractional number, factoring into prime numbers, factoring as a way of finding different names for fractional numbers, the use of common denominators to compare the sizes of two fractional numbers, finding the greatest common factor and the least common multiple.

Illustration 1: **Write the following number as five products. Use only whole numbers as factors. Try five different ways.**

36 _____ _____ _____
_____ _____

Illustration 2: **A "factor tree" is a diagram which shows the factors of a given product. Take the number 24. Three product expressions can be given with two factors (each one greater than 1) as follows:**

$$24=2\times12$$

$$24=3\times8$$

$$24=4\times6$$

These factor trees would look like this.

$24=2\times12$	$24=3\times8$	$24=4\times6$
24 2×12	24 3×8	24 4×6

Illustration 3: **Every composite number is the product of smaller factors. If one of these factors is composite, then it also is the product of smaller factors. If we continue this, we must come to a product expression in which no factor is composite and every factor is a prime.**

$24=3\times8$	(8 is composite.)
$=3\times2\times4$	(4 is composite.)
$=3\times2\times2\times2$	(All are prime.)
$36=4\times9$	(4 and 9 are composite.)
$=2\times2\times9$	(9 is composite.)
$=2\times2\times3\times3$	(All are prime.)

This shows us that every number greater than 1 is either prime or is a product of primes.

Illustration 4: Suppose Set S is the set of all factors of 12 and Set R is the set of all factors of 18.

S={1, 2, 3, 4, 6, 12},

R={1, 2, 3, 6, 9, 18}.

Then the set of all factors of *both* 12 and 18 is

{1, 2, 3, 6}.

The members of this set are called the *common factors* of 12 and 18.

Illustration 5: If we know the greatest common factors of two numbers, we also know that all its factors will be factors of each of the two numbers. This set is the set of all factor common to both numbers. For example:

If we know that 8 is the greatest common factor of 24 and 32, then we know that the set of common factors of 24 and 32 will be the set of all factors of 8, or

{1, 2, 4, 8}.

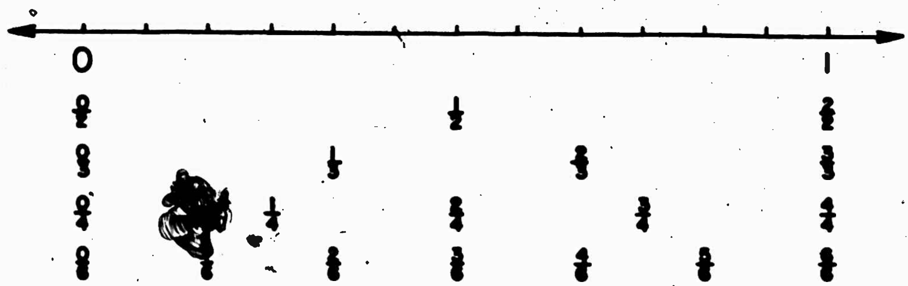
Illustration 6: A common multiple of 60 and 18 must have both 60 and 18 as factors. A number with both 60 and 18 as factors must be the product of *at least two 2's, at least two 3's,* and *at least one 5.* This means that 2×2×3×3×5 is the least common multiple of 60 and 18. We can now write:

$$2\times2\times3\times3\times5=(2\times2\times3\times5)\times3$$
$$=60\times3$$
$$2\times2\times3\times3\times5=(2\times3\times3)\times(2\times5)$$
$$=18\times10$$

Grade 5—Addition and Subtraction of Fractional Numbers

As the title suggests, the operations of addition and subtraction of fractional numbers are developed in this section. Naming fractional numbers in many ways leads into the skill of finding common denominators. The previously learned properties of addition are applied to fractional numbers.

Illustration 1: On the number line below, the fractions 2/4 and 3/6 label the same point. That point is also labeled 1/2. We may write 1/2=2/4=3/6.



The fractions 1/3 and 2/6 label the same point. We may write 1/3=2/6.

The fractions 2/3 and 4/6 label the same point. We may write 2/3=4/6.

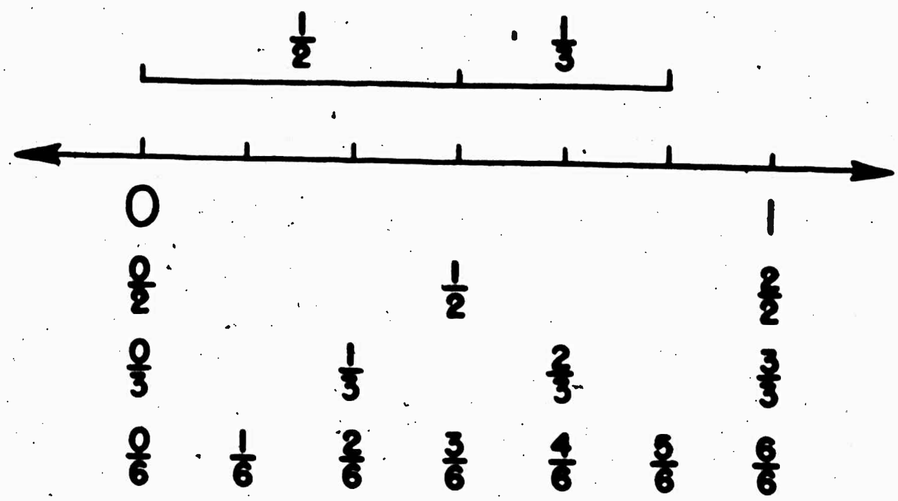
Illustration 2: The whole number named by the numeral above the bar in a fractional numeral is called the *numerator*. The whole number named by the numeral below the bar in a fractional numeral is called the *denominator*. By factoring the numerator and denominator of the fraction 4/8, we can write this fraction in the form $\frac{4\times1}{4\times2}$. The *common factor* of numerator and denominator in 4/8 is 4.

Illustration 3: Make a chart with these headings and fill in the blanks for each fraction given below:

Fraction	Factors in numerator	Factors in denominator	Factored expression (for the fraction) using greatest common factor	Fraction in lowest terms
Example: $\frac{6}{8}$	1, 2, 3, 6	1, 2, 4, 8	$\frac{2\times3}{2\times4}$	$\frac{3}{4}$
$\frac{6}{9}$	-------	-------	--------	---
$\frac{10}{15}$	-------	-------	--------	---

Illustration 4: If we multiply both numerator and denominator of the fraction $\frac{3}{4}$ by the same factor (excluding zero), we always obtain a fraction naming the same number as $\frac{3}{4}$ names. If this factor is 2, we obtain the fraction $\frac{6}{8}$. If this factor is 3, we obtain the fraction $\frac{9}{12}$. If this factor is 4, we obtain the fraction $\frac{12}{16}$. The fractions $\frac{6}{8}$, $\frac{9}{12}$, and $\frac{12}{16}$ each name the same number as $\frac{3}{4}$ names.

Illustration 5: To show $\frac{1}{2}+\frac{1}{3}$ on a number line, we need a suitable scale. A scale of sixths can show segments measuring $\frac{1}{2}$ and segments measuring $\frac{1}{3}$.



This suggests that $\frac{1}{2}+\frac{1}{3}$ can be written as

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}. \quad \text{Now, } \frac{3}{6}+\frac{2}{6}=\frac{3+2}{6}=\frac{5}{6}$$

Illustration 6: Set M = set of multiples of 8 = {8, 16, 24, 32, 40, 48, . . . }.

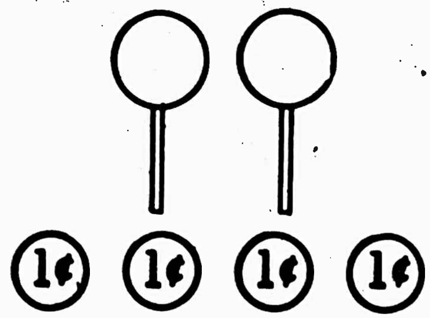
Set N = set of multiples of 6 = {6, 12, 18, 24, 30, 36, 42, 48, . . . }.

The intersection of the set of multiples of 8 and the set of multiples of 6 is the set of *common multiples* of 6 and 8. The smallest number in this set is the *least common multiple* of 6 and 8 or 24.

Grade 5—Ratio

The idea of a ratio relationship is first presented by means of sets of pictures. Mathematical sentences are written for described or pictured situations with the letter "n" used to indicate the unknown part of the ratio relationship.

Illustration 1: When Bill bought 2 pieces of candy for 4 pennies, we described this matching by writing 2:4. We read this, "two to four" or "two per four." We know it means "two pieces of candy for four pennies."



This same matching could be described by the symbol 1:2.

We could also use the symbol 3:6. This means 3 pieces of candy for 6 pennies.

Illustration 2: If there are 3 boys for 1 tent, draw a picture to show how many tents would be needed for 27 boys.

Here is a table which shows this information:

Tents	1	2	3	--	5	6	--	8	--	--
Boys	3	-	9	12	-	18	21	-	27	30

(a) For this same ratio, how many boys would sleep in 2 tents?

(b) Tell what numbers should be used to fill the spaces in the table. We can use these pairs from the table to write other names for this ratio. For example:

1:3=2:6=3:9=4:12=5:15=6:18=7:21=8:24 and so on.

Grade 6—Exponents

The exponent form is introduced as another way of naming numbers. Repeated factors and their relation to exponential notation ($4\times4\times4=4^3$) is studied. Treatment includes an extension of exponents to bases other than 10.

Illustration 1: If we were asked to write a number such as 3,000 showing the repeated multiplication of 10 we would write:

$$3{,}000=3\times10\times10\times10$$

30,000 would be written as:

$$30{,}000=3\times10\times10\times10\times10$$

How would you write 500; 8,000; 200,000; showing the factors of 10?

Illustration 2: Mathematicians frequently write the following products in this way:

$$(a)\ 10\times10\times10=10^3$$
$$(b)\ 10\times10\times10\times10\times10=10^5$$

In both of these examples 10 is the repeated factor. The small numeral 3 which is raised and to the right of the number 10, in (*a*), tells how many times the factor 10 is used. In (*b*), the factor 10 is used 5 times.

There is a short way for indicating how many times a number is to be used as a factor. This short way uses an *exponent* as in the following:

$5\times5\times5=5^3$

5^3 is read, "five to the third power."

3 is the *exponent*. 5 is called the *base*.

6^2 is read, "six to the second power."

$6^2=6\times6=36$

2^5 is called a *power*. In this example, it is the fifth power of 2.

$2^5=2\times2\times2\times2\times2=32$

2^5 is called the *exponent form* for 32.

Illustration 3: $5555=(5\times10\times10\times10)+(5\times10\times10)+(5\times10)+(5\times1)$
$=(5\times10^3)+(5\times10^2)+(5\times10^1)+(5\times1)$

The example above shows the numeral 5555 written in expanded form or in *expanded notation*. The last line shows the *exponent form*.

Illustration 4: Study the examples of multiplication of numbers in exponent form with the same base in the box below. Can you discover how the exponent of the product is obtained in each example?

(a) $2^3\times2^4=(2\times2\times2)\times(2\times2\times2\times2)=2^7$

(b) $3^2\times3^3=(3\times3)\times(3\times3\times3)=3^5$

In example (a) above, what is the base in each factor? What is the exponent in each factor? What is the exponent in the product? If the exponents 3 and 4 are added, what is their sum?

In example (b) the base in each factor is 3. What is the exponent in the product? What addition example gives you the exponent in the product?

Illustration 5: To find the product for 125×25 by using the exponent form, the factors must first be written with a common base. In the work in the box, what is the base for each factor? What is the product?

$125\times25=?$

$5^3\times5^2=5^{(3+2)}=5^5$

Illustration 6: Study the examples of division of numbers in exponent form with the same base, in the box below. Can you discover how the exponent of the quotient is obtained in each example?

(a) $3^3\div3^1=\frac{3\times3\times3}{3}=\frac{3}{3}\times3\times3=1\times3\times3=9=3^2$

or $=\frac{27}{3}=9=3^2$

(b) $5^3\div5^2=\frac{5\times5\times5}{5\times5}=\frac{5}{5}\times\frac{5}{5}\times5=1\times1\times5=5^1$

or $=\frac{125}{25}=5=5^1$

In example (a) above what is the exponent in the dividend?

In the divisor? In the quotient? What is the result when you subtract 1 from 3?

In example (a) we may think of 3^3 as 27. Then we divide 27 by 3 to obtain the quotient 9. Does 9 equal 3^2?

Grade 6—The Integers

The number system is extended to include the integers (whole numbers and their negatives): . . . $^{-}3$, $^{-}2$, $^{-}1$, 0, 1, 2, 3, ... The number line helps children visualize the operations of addition and subtraction with integers. The whole numbers count off units to the right from zero and the negative numbers count off units to the left from zero. Understanding is expanded as the application of the basic properties of addition is emphasized. Subtraction as the inverse of addition is stressed.

Illustration 1:

This is a number line showing a new kind of number. You will use these numbers now and later.

Find the dot numbered 0. As your eye moves to the right, the dots are numbered $^{+}1$, $^{+}2$, $^{+}3$, ... As your eye moves to the left from zero, the dots are numbered $^{-}1$, $^{-}2$, $^{-}3$, ... The numbers on the line are called *integers*. The set of integers is

$$(\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots).$$

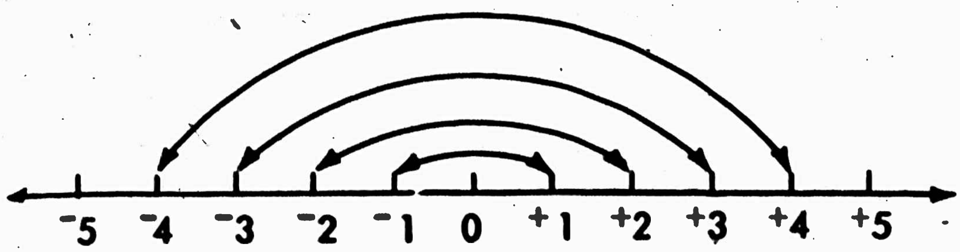
The integer "$^{+}2$" is read "positive 2." The integer "$^{-}7$" is read "negative 7."

Illustration 2: Look at the dot labeled $^{+}5$. Positive five is to the right of and greater than $^{+}4$, or 0, or $^{-}3$, or $^{-}8$. In fact, positive five is greater than any integer to its left. The mathematical sentences . . . $^{+}5 > {}^{+}4$; $^{+}5 > {}^{+}3$; . . . $^{+}5 > 0$; $^{+}5 > {}^{-}1$; . . . $^{+}5 > {}^{-}4$; . . . are ways of writing this.

Look again at the dot labeled $^{+}5$. Positive five is to the left of and less than $^{+}6$, or $^{+}7$, or $^{+}127$. In fact, positive five is less than any integer to its right. The mathematical sentences $^{+}5 < {}^{+}6$; $^{+}5 < {}^{+}7$; . . . $^{+}5 < {}^{+}19$; . . . $^{+}5 < {}^{+}127$; . . . are ways of writing this fact.

An integer is greater than any integer to its left; an integer is less than any integer to its right.

Illustration 3:



Look at the diagram above. Each integer is one of a pair. $^+4$ and $^-4$ form a pair. They are the same distance from zero. $^+4$ is the *opposite* of $^-4$; $^-4$ is the opposite of $^+4$. Zero is its own opposite.

Every integer has an opposite.

***Illustration 4:* The figure below shows two arrows with the same measure. One indicates a count of 6 in the positive direction; the other a count of 6 in the negative direction.**

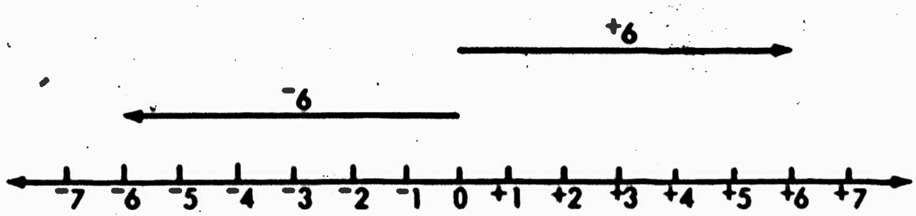
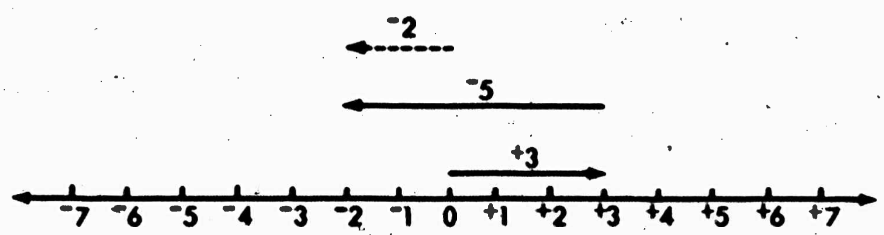


***Illustration 5:* Arrow diagrams may be used to rename integers. The diagram below renames $^+3+{}^-5$ as $^-2$. This may be shown by the mathematical sentence $^+3+{}^-5={}^-2$.**



The diagram is made by following these steps:

(1) Begin at zero and draw a solid arrow for the first addend ($^+3$).

(2) Begin at the head of the arrow for the first addend and draw a solid arrow for the second addend ($^-5$).

(3) Draw a "dotted" arrow from zero to the head of the arrow for the second addend. This arrow ($^-2$) renames $^+3+{}^-5$.

Grade 6—Coordinates

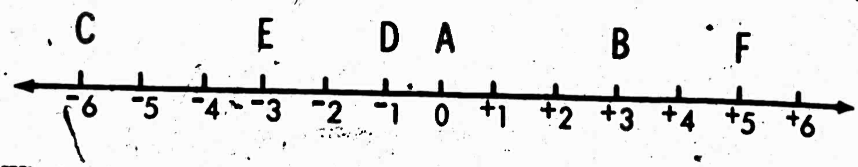
This unit presents the idea that the location of sets of points in a plane can be described by the use of reference lines and numbers

682481 O—63——4

which are called coordinate systems. A coordinate system is based on a one-to-one correspondence between a set of numbers and the set of points on a line.

Illustration 1: A number that tells both distance and direction of a point on a line from the 0-point is called the *coordinate* of the point.

On the number line below, what is the coordinate of B? of C? of D?



What point has the coordinate $^-3$? What point has coordinate $^+5$?

When you mark on a number line the points which have a certain set of numbers as coordinates we say you are drawing a graph of the set of numbers.

Illustration 2: The location of point P can thus be described not by one number, but by the pair of numbers ($^-1$, $^+2$). The numbers $^-1$ and $^+2$ are both coordinates of the point P.

The first number tells the number exactly below it on the horizontal number line. The second number tells the number exactly to the right of P on the vertical number line. The *order* in which the numbers are named is important, so ($^-1$, $^+2$) is called an *ordered* pair.

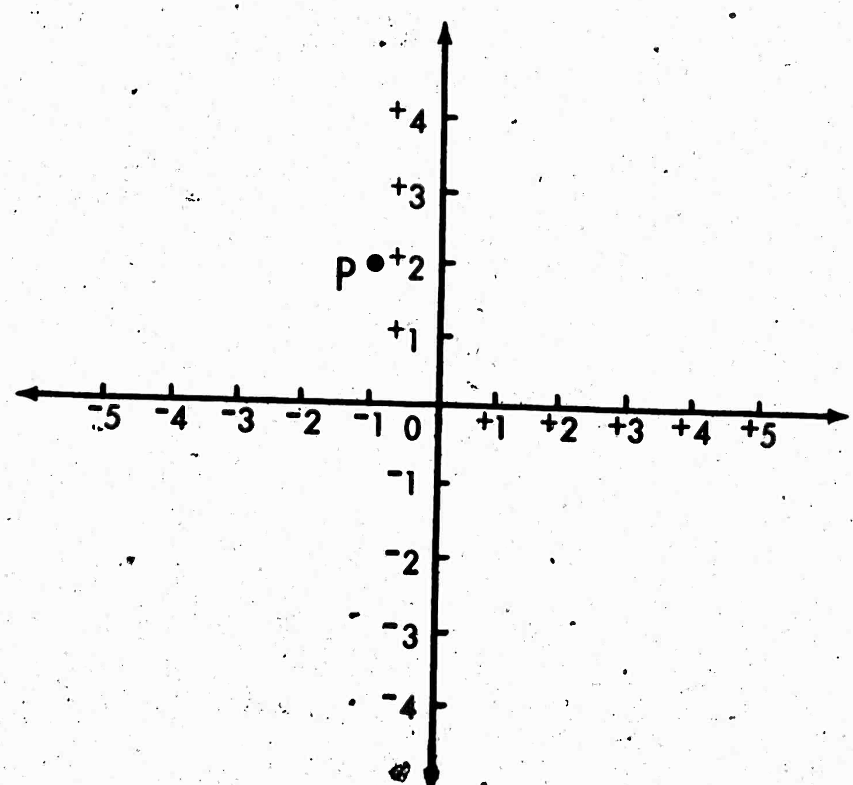


Illustration 3: Graph these ordered pairs. Label each with its letter and its coordinates.

H(⁺5, ⁻4)
J(⁻6, ⁻3)
K(0, ⁺6)
M(⁻2, 0)
R(⁻2, ⁺5)
S(⁺4, ⁺3)

When number lines are used in this way, we call each number line an *axis*. The horizontal number line is called the *x-axis* and the vertical number line is called the *y-axis*. The point of intersection of the x-axis and the y-axis is called the *origin*.

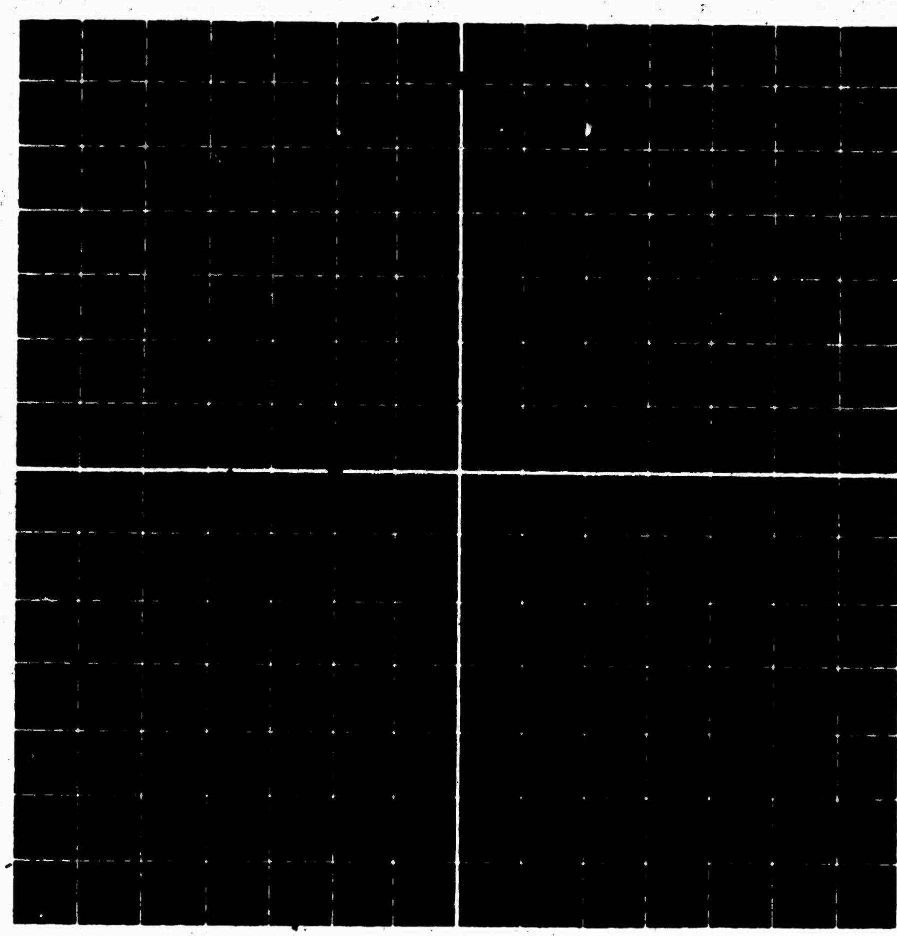


Illustration 4: You have been using two perpendicular number lines with the same zero point. We called these number lines the x-axis and the y-axis. These lines help you locate the point in a plane which is the graph of an ordered pair of numbers.

We say "ordered pair" because the order in which the two numbers are named is important. The point located by the pair $(^-3, ^+6)$ is a different point from the one located by the pair $(^+6, ^-3)$.

The first number in an ordered pair, which tells how far the point is to the right or left of the y-axis, is called the *x-coordinate*. The number which tells the distance of the point above or below the x-axis is called the *y-coordinate*.

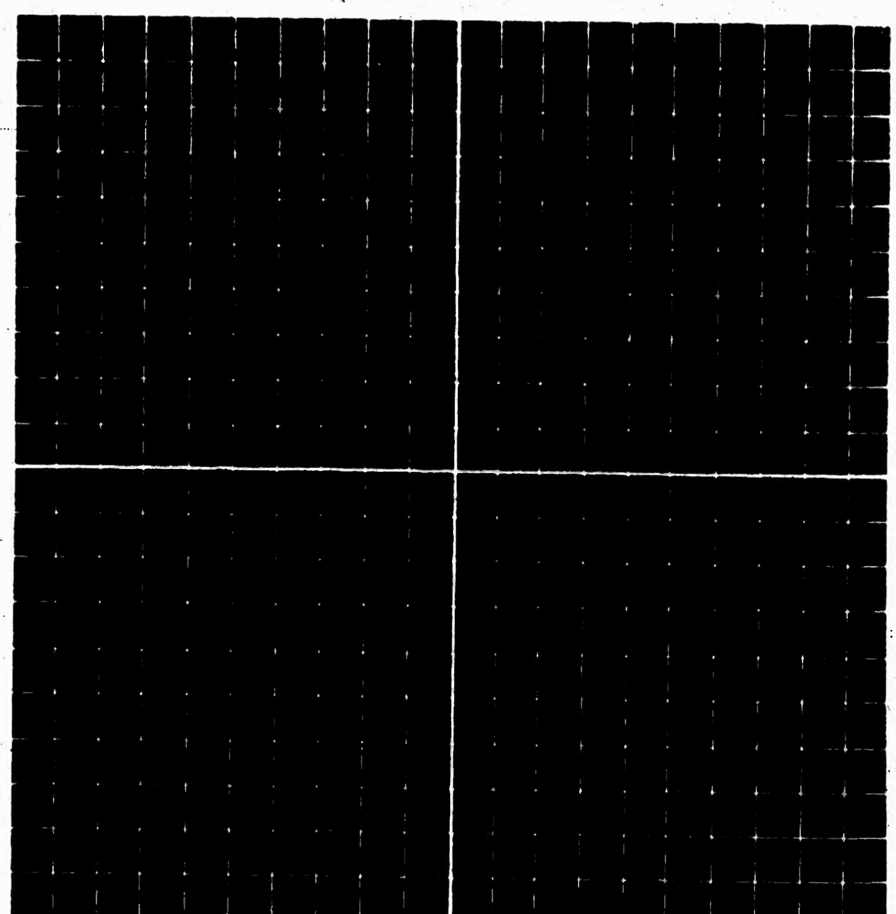
Illustration 5: You already know at least one meaning for "reflection." We think of a mirror or a pool of clear water as giving a reflection. Let us see what reflections are in geometry.

Graph this set of ordered pairs:

$$A(^+2, ^+7),\ B(^+9, ^+5),\ C(^+3, ^+4)$$

Draw segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. The union of these segments is a __?__.

Does your triangle ABC look like triangle ABC below? This drawing shows also triangle DFE which is a *reflection* of triangle ABC. Do you see why it is called a reflection?



Greater Cleveland Mathematics Program [1]

General Description

The Educational Research Council of Greater Cleveland, the parent organization of the Greater Cleveland Mathematics Program, is an independent, nonprofit organization which serves 25 school districts and 2 private schools in the greater Cleveland area. A major purpose of the Council is to develop the best possible curriculums, kindergarten through grade 12, and implement them immediately in the classrooms of the Council schools.

In March 1959 the advisory committee, consisting of the superintendents of schools from the participating school districts, requested that the Council make a concerted effort to improve the mathematics curriculum. As a result, the Greater Cleveland Mathematics Program was initiated.

The purpose of the project is to develop a new elementary and secondary curriculum that can be presented to all children in a logical and systematic way so that they will completely understand the basic mathematical concepts before they are taught the computational schemes or algorisms.

To accomplish its purpose effectively, the project makes extensive use of the discovery approach to learning. The project presents problem situations to students as if they had not been explored already by the great minds of the past and present. These situations are presented in such a manner that discovery has a good chance of taking place spontaneously; then students are led to the established symbolism. The logical structure of mathematics stimulates the imagination of children and leads them to appreciate mathematics as a dynamic and meaningful study by letting them experience the thrill of discovering or recreating some mathematics for themselves.

Personnel involved in the development and field testing of the materials include mathematicians, mathematics educators, consultants in mathematics, classroom teachers, elementary mathematics supervisors, and administrators.

During the first year of the project, teachers in the program were given foundation courses in mathematics by the project staff and

[1] Executive Director: Dr. George H. Baird, Greater Cleveland Research Council, 75 Public Square, Cleveland 11, Ohio.

received some materials to enrich their regular classroom program. The second year, teachers used the pupils' exercise sheet and teachers' guides which were prepared by writing teams of teachers and project staff members. These materials were revised during the spring and summer of the second and third years using feedback from approximately four hundred teachers per grade level.

The elementary phase of the program which began with the primary grades and progressed upward is now nearing completion.

A local television station has been used as a means of providing teacher training on mathematical content, teaching procedure, and the proper use of materials. Presently, field assistants from the project are assigned to grade levels K to 3 and 4 to 6 to assist the teachers in the classroom through classroom demonstrations, inservice meetings, and individual or grade-level conferences.

Enrichment topics at the fifth-and sixth-grade level extend the work on topics such as the following: products and factors; prime and composite numbers; geometry; sequences and series; simple informal proofs; and sentences, open sentences, and conditional statements.

Nature of the Content

The content of the GCMP draws heavily upon the principles of mathematics to help children learn the underlying structure of the material presented. In addition to the topics covered in the usual elementary school mathematics program, the GCMP contains units or exercises on number sequences; factors and multiples; prime and composite numbers; other numeration systems; an introduction to powers, roots, and negative numbers; physical geometry; and linear, area, and volume measurement. The concepts and language of sets are carried through the topics at all grade levels.

Beginning understandings of addition, subtraction, multiplication, and division are first developed by reference to sets of objects. Concurrently, equations are presented to describe the corresponding ideas with numbers.

Illustrations of the multiplication process are:

6×0 (six sets of zero)

3×1 (three sets of one)

4×2 (four sets of two)

In the equation $3 \times 6 = 18$, the factor 3 tells the number of sets being considered; the factor 6 tells the number in each set; the answer 18 tells how many elements altogether.

The equation form is used widely in all work with operations. The inverse relationship between addition of a number and subtraction of the same number and between multiplication of a number and division of the same number is stressed. Multiplication of whole numbers is also presented as repeated addition, and division as repeated subtraction.

The commutative, associative, and distributive properties are used to develop meaning for the processes of addition and multiplication. If children know the addition combinations through 10 and have an understanding of the associative property, they can find answers for addition combinations above 10:

$$7+8=7+(3+5)=(7+3)+5=10+5=15$$

The task is to think of the second addend as a sum of two addends the first of which will be added to 7 to yield 10. When the associative property is applied to multiplication, children learn to engage in thinking processes similar to the following:

$$60\times4=(10\times6)\times4=10\times(6\times4)=10\times24=240$$

After facts are explored and discovered, they are entered in addition and multiplication tables for later use in finding relationships.

Inequalities are considered to be as important as equations and are a part of the work at each grade level. The inequalities $2<7$ (read, "2 is less than 7") and $5+4>7$ (read, "5+4 is greater than 7") are true statements. The inequalities $3<3$ and $5<1+2$ are false statements.

Boxes are used as placeholders in equations such as the following: $3+\square=8$; $26+8=20+\square=$ _____. The use of letters as placeholders is begun in grade 4. Mathematics vocabulary is developed as needed.

Geometry units which begin at grade 4 are designed to develop understandings of such concepts as point, line, line segment, simple closed curve, plane, ray, and angle, through a study of representations in the child's environment or in pictures. The purpose is to build an awareness of geometric shapes and of their properties.

Materials which accompany the program include: (1) the counting-man for teaching counting and place value; (2) plastic numerals and operational signs; (3) a ten-frame for teaching combinations and ideas of place value for numbers 10–19; (4) a bead frame to teach the one-more concept and the structural properties; and (5) a flannel board.

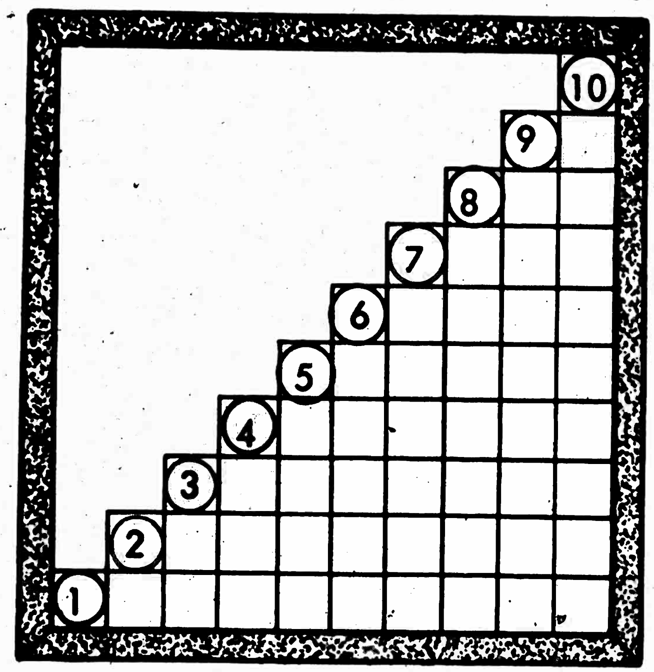
Procedures for Developing Concepts of Place Value

A concrete device, the countingman, is used to help children learn counting and the place value meaning of numerals. The digits 0 through 9 can be shown on one countingman; two countingmen are used to show the meaning of numerals 10 through 99; three countingmen are used to explain three-digit numerals, etc.

Each finger on the Ones man represents one set of one, each finger on the Tens man represents one set of 10, each finger on the Hundreds man represents one set of 100, etc. Later the countingmen are used to demonstrate the processes of carrying and borrowing, and higher decade addition and subtraction.

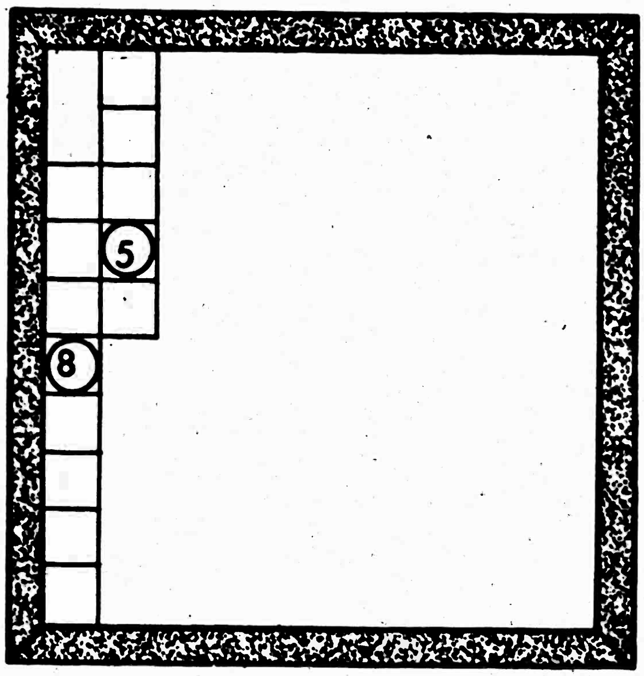
Procedures for Developing Understanding of Operations

1. The ten-frame is used in early grades for teaching combinations and ideas of place value for numbers through 19.



All combinations for 7 are found by placing the 7 strip at the left-hand side of the ten-frame and finding combinations of two strips which are the same length: 1+6; 2+5; 3+4 and the commuted forms: 6+1, 5+2, 4+3.

Addition combinations above 10 are found by thinking of the second addend as being itself a sum of two addends. For example, $8+5=8+(2+3)$; $8+2=10$; $10+3=13$; therefore $8+5=13$. The appropriate addend is determined by thinking $8+2=10$; $5=2+3$.



2. As cards containing combinations are shown to the class, children are requested to show other names for the same number with their plastic numerals and signs:

$\boxed{2+7}$ $3+6$; $1+8$; $4+5$

$\boxed{5+1}$ $2+4$; $3+3$; $1+5$; $4+2$

3. As cards with equations containing missing addends are exposed, children find the missing addend and read the complete sentence:

$7-\square=1$ $7-6=1$ or $1+\square=7$

$3+\square=8$ $3+5=8$ or $8-3=\square$

4. Children demonstrate an intuitive understanding of the associative and commutative properties as they describe the different ways of finding the sum of three addends:

$5+1+3=\square$; $5+1=6$; $6+3=9$

adding $1+3$ first; $5+4=9$

adding $5+3$ first; $8+1=9$

At first they may be instructed to find as many ways as they can without changing the order of the addends. This uses the associative property alone. They see other possibilities when they are allowed to change the order of addends and then associate, also. This uses both properties.

5. Plastic numerals and signs are arranged to show the process of adding three numbers:

14	$4+5=9;\ 9+3=12$
15	"2" goes in ones column;
+ 3	"10" goes in tens column;
32	$10+10=20;\ 20+10=30$

6. An addition table is used to tabulate known facts after the idea of addition as the union of sets has been developed.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

The commutative property is observed in the addition table as children see that $2+3=3+2$; $3+4=4+3$; etc. The "doubled" numbers $(1+1;\ 2+2;\ 3+3)$ are found on the diagonal from upper left to lower right corners of the table (2, 4, 6, and 8).

Adding a number and subtracting the same number are inverse operations:

$$3 + 4 = 7 \qquad 7 - 4 = \boxed{3}$$

(addend) + (addend) = (sum) (sum) − (known) = (unknown)

(addend) (addend)

This relationship is observed in the table. The sum and the known addend are located in the same column. The sum is found in the body of the table and the known addend at the top of the column. The unknown addend is found at the far left in the same row as the sum.

7. The understanding of two properties is essential for the process of adding 2- and 3-digit numbers to be meaningful for children: (1) the addends can be taken in any order without changing the sum (commutative property); (2) the numbers may be grouped to bring tens together and hundreds together (associative property).

$$
\begin{aligned}
583 &= 500+80+3\\
258 &= 200+50+8\\
\hline
&= 700+130+11\\
&= (700+100)+(30+10)+1\\
&= 800+40+1\\
&= 841
\end{aligned}
$$

A multiplication table is used to organize and tabulate combinations after they have been developed using sets and interpreted in terms of repeated addition.

As children construct and study a multiplication table they observe:

X	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

(a) The identity element for multiplication ($1 \times n = n$, where n is any number).

(b) The commutative property: $a \times b = b \times a$, where a and b are any numbers.

(c) The patterns of multiples for each factor found in the top row or in the left-hand column.

(d) The square of each factor is found along the diagonal from upper left to lower right-hand corner.

(e) That multiplying by a number and dividing by the same number are inverse operations.

$3 \times 4 = 12$
(factor) × (factor) = (product)

$12 \div 3 =$
(product) (known) (factor) (unknown) (factor)

A division combination is found by locating the product within the table, the known factor at the top in the same column as the product and the unknown factor at the far left in the same row as the product: $12 \div 4 = 3$.

8. Multiplication of whole numbers is also presented as repeated addition, and division as repeated subtraction:

$1+1+1+1=4$ $\quad$ $4\times 1=4$

$1+1+1+1+1=5$ $\quad$ $5\times 1=5$

$2+2+2+2=4\times 2=8$

Two is added 4 times; therefore
$4\times 2=8$

$$\begin{array}{rl} 8 & \\ -2 & \text{1st time} \\ \hline 6 & \\ -2 & \text{2d time} \\ \hline 4 & \\ -2 & \text{3d time} \\ \hline 2 & \\ -2 & \text{4th time} \\ \hline 0 & \end{array}$$

2 is subtracted from 8, then from 6, then from 4, then from 2; therefore, $8\div 2=4$

9. Children are guided to understand that the commutative and associative properties do not apply to subtraction and to division.

Subtraction is not commutative because $12-5\neq 5-12$. In fact, within the system of numbers they know, an answer for $5-12$ is not possible. Subtraction is not associative because $(12-5)-3\neq 12-(5-3)$.

Division is not commutative because $20\div 4\neq 4\div 20$. This operation is not associative because $(36\div 6)\div 2\neq 36\div(6\div 2)$.

10. Children use the distributive property to assist them in their thinking about multiplication facts and in analyzing multiplication examples.

If $5\times 9=45$, then $6\times 9=(5+1)\times 9=(5\times 9)+(1\times 9)=45+9=54$;
or $6\times 9=6\times(10-1)=(6\times 10)-(6\times 1)=60-6=54$

$8\times 24=(8\times 20)+(8\times 4)$
$=160+32=192.$

$247\times 89=(200+40+7)\times(80+9)$
$=200\times(80+9)+40\times(80+9)+7\times(80+9)$
$=16{,}000+1800+3200+360+560+63.$
$=21{,}983$

Procedures for Developing Concepts of Factors, Primes, and Composite Numbers

1. Numbers are expressed as products of pairs of factors:

$$
\begin{aligned}
36 &= 1\times 36\\
&= 2\times 18\\
&= 3\times 12\\
&= 4\times 9
\end{aligned}
$$

2. Numbers are expressed using the smallest possible whole-number factors starting with any known factor pairs:

36	36	36
3×12	6×6	4×9
$3\times 3\times 4$	$2\times 3\times 2\times 3$	$2\times 2\times 3\times 3$
$3\times 3\times 2\times 2$		

3. The greatest common factor (G.C.F.) for two numbers is found by factoring into primes and finding the combination of factors which is common to both numbers.

4. The least common multiple (L.C.M.) for two numbers is found by listing a set of multiples for each number in order and locating the smallest common element:

Find the L.C.M. for 12 and 18.
Multiples of 12 (12, 24, 36, 48, 60, . . .)
Multiples of 18 (18, 36, 54, 72, 90, . . .)
The smallest common element is 36.

Other methods of finding the greatest common factor and the least common multiple are developed.

University of Illinois Arithmetic Project[1]

General Description

Prof. David A. Page, director of the University of Illinois Arithmetic Project, and his staff have attempted to devise materials to help children view work in mathematics as a fascinating adventure. By exposure to interesting and different ways of approaching familiar tasks, children are encouraged to make their own mathematics discoveries, to develop mathematical insight, and to acquire an intuitive understanding of many mathematics ideas which have usually been initiated much later in the child's school life. No claim is made that the methods employed are entirely new. Rather an effort is made to give all children the opportunity to profit from the methods which some children and teachers have been using successfully all along.

In addition to working for better ways of presenting standard topics and attempting to adapt advanced topics for earlier use, the project team is inventing and developing new topics expressly for use in the early grades.

Although the materials prepared by the project are not recommended as a complete course of study, some topics are considered as a sequence for elementary classroom work and become the means whereby mathematical meanings for the processes of addition, subtraction, multiplication, and division with whole numbers and fractions are developed and practiced. They also serve as the vehicle through which algebraic and geometric ideas and other topics usually considered outside the province of elementary mathematics are introduced. Topics recommended for use as the need arises allow children to pursue mathematics beyond the usual limits of the elementary school.

Project materials are prepared with two major purposes in mind: (1) to assist the teacher in learning mathematics and in acquiring the background of understanding necessary for successful teaching, and (2) to develop types of exercises and activities in mathematics which appear appropriate for elementary children as a result of experimentation in the classroom. Professor Page urges teachers who wish to learn more mathematics to work all of the exercises recommended for children as they examine and study the material. The main ideas are

[1] Director: Prof. David A. Page, University of Illinois Arithmetic Project, 1207 W. Stoughton St., Urbana, Ill.

largely embedded in the exercises and often a sequence of exercises provides a hint at an important idea. He proposes that teachers learn mathematics just as children do—by wrestling with it. The exercises will in turn help teachers to present a variety of basic mathematical ideas to their students. No grade level is specified, for the exercises have been taught successfully in project classes to children of different abilities and grade levels from first grade on when suitable adaptations were made for pace of instruction and difficulty of material.

Work with experimental classes leads Professor Page to believe that considerably higher proficiency in computation can be obtained while at the same time a great deal of new, genuine mathematical content can be introduced. Exercises are therefore designed with a dual purpose: (1) to furnish much drill and practice on topics previously taught, and (2) to present new mathematical learning.

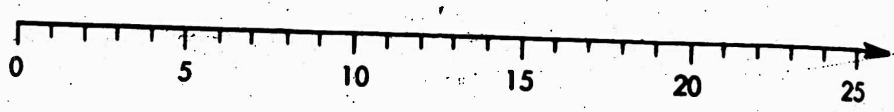
On the controversial question of terminology, the project team takes the position that ideas are more important than words and that an overemphasis on exact vocabulary may even interfere with concept development. For this reason a mixture of adult and classroom language is used to talk about the ideas to students. Few technical terms are used because the project teachers find that few are needed.

The project team has tried to develop activities which stimulate curiosity, exploration, experimentation, and discovery. Children are encouraged to estimate or approximate answers and to correct poor guesses. They are challenged to solve problems in different ways, to appreciate the many ways of arriving at an answer, to explain how they reason about a solution, and to respect the method requiring a high level of thought and reasoning. The material is characterized by less emphasis on computation and more on thinking and oral work. It is recognized that children learn best when they really want to find an answer and when the answer matters.

Nature of Content

Sample illustrations will provide some understanding of the methods used in developing the basic ideas stressed in the project materials. Early exercises include unique and varied ways of guiding children to learn mathematics through number line games. Exercises appropriate for younger children describe movement along a number line as jumps made by "crickets." For example, children are provided with the information that: (1) a cricket makes jumps along the number line; (2) a "plus" cricket starts at a given place and jumps to the right; (3) a "minus" cricket starts at a given place and jumps to the left. With this information and a number line to help them think, children solve problems similar to the following:

1. Illustrations of Number Line Games



(a) A $^{+}4$ cricket begins jumping at 2 and makes five jumps. Where is he? $(4 \times 5) + 2 = 22$.

(b) A $^{+}6$ cricket begins jumping at 5, after some jumps he lands on 29. How many jumps did he make?

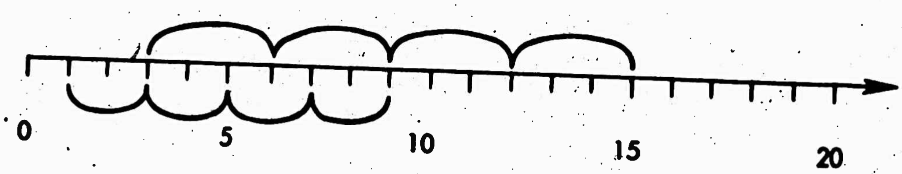
$$(6 \times \square) + 5 = 29$$
$$(4)$$

(c) A $^{-}6\frac{2}{3}$ cricket starts at 205 and makes 20 jumps. Is he above 100, exactly on 100 or below 100?

Solution: Below 100. (thinking: Just a $^{-}5$ cricket making 20 jumps from 205 would land at 105. A $^{-}6$ cricket would land 20 more units to the left, well below 100. A $^{-}6\frac{2}{3}$ cricket would land still farther left after 20 jumps.)

(d) Two crickets make trips. They make the same number of jumps. Tell how far apart they were before they started jumping and after they finished jumping.

Sample: Each cricket makes 4 jumps, the first cricket, a $^{+}3$ cricket, starts at 3. The second cricket, a $^{+}2$ cricket, starts at 1.



The crickets start 2 units apart and finish 6 units apart. (As well as working out each trip in detail, one could think: since one cricket jumps 2 units and the other one jumps 3 units, a jump with each puts them one unit farther apart if the faster cricket jumps ahead—and it does. Since at the start they are 2 units apart, after 4 jumps they are 4 units farther apart. They are 6 units apart after 4 jumps.)

2. Illustrations of the Use of Number Line Rules

Rule a: $\square \to 3\square - 5$[1] (Read—"Box goes to 3 times box minus 5")

Rule b: $\square \to \square - \square$ (Read—"Box goes to box minus box")

1. Start at 4 and make one jump with rule a.
2. Where must you start to land at 22 after making one jump with rule a?

[1] The reader is reminded that $3\square$ means 3 times the number in the box. Convention makes it possible to omit the multiplication sign.

3. Start at 2½ and make a jump with a.
4. Start at 3 and make a jump with b.
5. Start at 10 and make a jump with b.
6. Start at 10 and make a jump with a.
7. Starting at 10 which rule, a or b, takes you on the longer jump?
8. Starting at 1 which rule, a or b, takes you on the longer jump?
9. Start at 5. Make a jump with b. From where you land make a jump with a.
10. Start at 5. Make a jump with a. From where you land make a jump with b. Do you land in the same final place as you did in 9? (Example of noncommutativity.)

3. Illustrations of the Use of Frames

Many exercises developed by the project make use of frames, or hollow patterns, such as squares, triangles, ovals, circles, and rectangles, which are substituted for numerals in open number sentences. Frames play the same logical role as letters in algebra inviting children to fill the open space or to determine the missing number in a number sentence. The frame must be easy to make with the inside large enough to hold numerals: $\square+2=10$; $42=6\nabla$. Rules are set up for work with frames. Two rules which have been found useful by the project teachers are: If the same-shaped frame occurs more than once in an expression or sentence, the number used to fill one of the frames must also be used to fill all frames of the same shape in that expression. (This rule of "like" replacements holds only for frames of the same shape). If the frames are different in shape the different numbers may be used, although it is permissible to have the same number in frames of different shapes. The following illustrate the use of frames.

(a) *Instruction:* Fill the frames in each exercise to get 24.
$\square+\square+\square+\square=24$. The only possible solution with whole numbers for this number sentence is $6+6+6+6$.

If the number sentence is $\triangle\times\square=24$, any of the following solutions are possible:

3×8	6×4
4×6	8×3
2×12	12×2

If the number sentence is $\square+\triangle+\square+\triangle=24$, some of the possibilities are:

1+11+1+11	2+10+2+10
7+5+7+5	4+8+4+8
3+9+3+9	6+6+6+6

682481 O—63——5

(b) Use only the numbers 2, 3, and 4 for these frames. First substitute in such a way as to give the smaller result. Then find the largest possible result.

$\square + \square - \triangle$ Smallest $(2+2)-4$ (0)

$\square + \square - \triangle$ Largest $(4+4)-2$ (6)

(c) Find all the numbers which satisfy the equations:

$\square + \square = 15$	answer 7½
$1 = \diamond + \diamond + \diamond$	answer ⅓
$\bigcirc \times \bigcirc \times \bigcirc = 8$	answer 2
$5 + \diamond = 2 + \diamond + 3$	answer—every number
$\diamond \times \diamond = \diamond$	answer 1; 0

4. Illustrations of Equations

An equation is described as a sentence stating that two phrases are equal. Equations are solved by finding numbers that can be used for the frames to convert them into true statements. Children are encouraged to solve equations by informal intuitive methods.

a. Samples: $16 + \square + \square = 23$. Sixteen + what number + the same number $= 23$? (3½). $\triangle + 4 = 10$. What number $+4 = 10$? (6). $2\diamond + 14 = 20$. Two × what number $+14 = 20$? (3). $9\square + 24½ = 30½$. Nine × what number $+24½ = 30½$? (⅔).

b. Children are instructed to use the same number in all frames of the same shape. $\triangle\triangle\triangle = 64$. What number × the same number × the same number $= 64$? (4).

c. Some exercises with frames help children to discover that certain equations have no solution. $\triangle + \triangle + 3 = \triangle$. What number + the same number $+3 =$ the same number as was placed in the other triangles?

$$1 + 1 + 3 \neq 1$$
$$2 + 2 + 3 \neq 2, \text{ etc.}$$

d. Frame sentences with two different shapes may be filled with any pair of numbers that will fit. Numerals resulting in workable solutions may be recorded in an organized manner as discovered. The teacher may provide a partially completed chart for which children will find the missing number of the pair. For example, if the frame sentence is $\square + \triangle = 14$, children will ask themselves:

If the box is 5, what is the triangle? (9)
If the box is 3, what is the triangle? (11)
If the triangle is 8, what is the box? (6)
If the triangle is 7, what is the box? (7)

□	△
5	—
3	—
—	8
—	7

Some of the pairs of numbers that will fit, therefore are: 5, 9; 3, 11; 6, 8; 7, 7. Pairs which will not fit and would be rejected are 8, 7; 8, 8; 6, 9; etc.

5. Illustrations of the Use of Frames in Number Line Games

As children become more sophisticated, "cricket jumping" along a number line is changed to rules for jumping with frames.

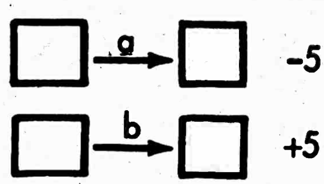
(a) A rule might be: $\Diamond \rightarrow 3 \times \Diamond + 5$. The rule is read "diamond goes to 3 × diamond +5. For example, if the starting place on the number line is 4, the rule says, "you start at 4 and go to $(3\times4)+5$ or to 17. If the starting place is 4, the solution is: $(3\times4)+5=17$. If the starting place is 5, the solution is: $(3\times5)+5=20$. If the starting place is 6, the solution is: $(3\times6)+5=23$, etc. If this particular rule is repeated three times using 4 in the diamond the stopping places on the number line are: 4; 17 or $(3\times4)+5$; and 56 or $(3\times17)+5$; and 173 or $(3\times56)+5$.

(b) Another rule might be: If children start at 7 and use the rule they land at 10. $(7+7+7-11=10)$. The distance from the starting point (7) to the landing point (10) is 3.
If they start at 8 they land at 13 and the distance between starting point (8) and the landing point (13) is 5. $(8+8+8-11=13)$.

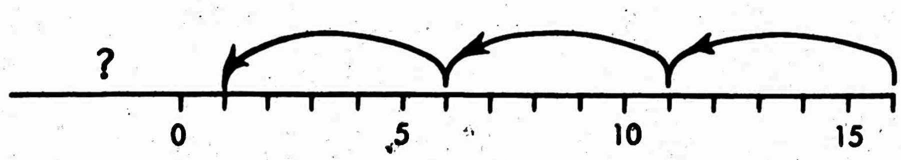
(c) Children may be instructed to devise a jump such that the distance between the starting place and landing place is zero. Children will determine starting place and landing place. $(5\frac{1}{2}+5\frac{1}{2}+5\frac{1}{2}-11=5\frac{1}{2})$.

6. Illustrations of the Use of Negative Numbers

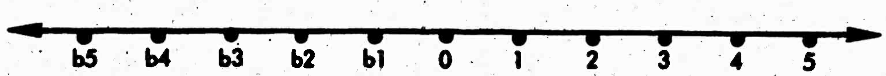
The project team finds that negative numbers come up spontaneously when a class considers what to do when they get close to zero. Consider the use of two number line rules such as:



Through previous experience children have discovered that if they start at 75 and make three moves with rule a followed by three moves with rule b they come back to 75. Now the question is, will this be true if they start any place—at 16, for example, and make four moves?



Three moves brings them to 1 and another move makes it necessary to go b4 (four below zero). Any other solution will not bring them back to 16 when rule b is applied. (b marks numbers below zero.)



Children discover that this extension of the number line to the left enables them to make as many jumps as they like in either direction.

Practice is provided in which a jumping rule, a starting place, and the number of jumps are given. The problem is to find the landing place:

a. Rule: ◇ → ◇ −2

Start at 9 and make ten moves.

b. Rule △ → △ $-3\frac{1}{2}$

Start at 15 and make seven moves.

c. Rule □ → □ −150

Start at 6200 and make three ~~moves~~.

d. Rule ✿ → ✿ +4

Start at $2\frac{1}{2}$ and make four moves.

e. Rule □ → □ +4

Start at b $2\frac{1}{2}$ and make two moves.

7. Illustrations from General Rules for Numbers

An understanding of the laws of mathematics is developed by giving a pattern of frames which are always true when numbers are substituted for them.

(a) The commutative laws for addition and for multiplication are presented by the frames: □ + △ = △ + □ and □△ = △□.

Generalizations are formed as substitutions are tried with whole numbers and fractions.

$3+4=4+3$; $\frac{5}{6}+\frac{2}{3}=\frac{2}{3}+\frac{5}{6}$

$2\times3=3\times2$; $\frac{1}{2}\times8\frac{1}{3}=8\frac{1}{3}\times\frac{1}{2}$, etc.

(b) The addition and multiplication laws for zero and the multiplication law for one are discovered by substituting for the frames:

0 + □ = □ $0+6=6$

$0+3\frac{1}{3}=3\frac{1}{3}$

$0\times\square=0$ $\quad$ $0\times 6=0$

$0\times\frac{3}{8}=0$

$1\times\square=\square$ $\quad$ $1\times 5=5$

$1\times 52.07=52.07$

(c) The distributive law of multiplication facilitates mental multiplication of some examples: $7\times 502=(7\times 500)+(7\times 2)$.

$$(9\times 4\tfrac{2}{3})+(9\times 5\tfrac{1}{3})=9\times(4\tfrac{2}{3}+5\tfrac{1}{3})=9\times 10=90.$$

The frames pattern for the distributive law may be stated as:

$$\square(\triangle+\diamond)=(\square\triangle)+(\square\diamond)$$
$$5(2+6)=(5\times 2)+(5\times 6)$$
$$\tfrac{1}{2}(10+6)=(\tfrac{1}{2}\times 10)+(\tfrac{1}{2}\times 6)$$

(d) The associative laws of addition and multiplication may be put into general form with frames as follows:

$$(\square+\triangle)+\diamond=\square+(\triangle+\diamond)$$
$$(\square\times\triangle)\times\diamond=\square\times(\triangle\times\diamond)$$
$$(8+5)+6=8+(5+6)$$
$$(3\times 5)\times 4=3\times(5\times 4)$$

As children work through exercises requiring the application of the laws of mathematics, they gradually develop short cuts and the ability to write equations with frames which will always give true statements when number substitutions are made.

8. Illustrations of the Use of Estimation

(a) Tell, if possible, without any computation, but by inspection which is the larger of the two numbers given:

$11\frac{1}{3}$, $5\frac{2}{3}$; $27\frac{1}{3}$, $26\frac{5}{6}$; $\frac{4}{5}$; $\frac{4}{7}$

(b) For the following exercises your only choices for answers are: $\frac{1}{100}$, $\frac{1}{10}$, $\frac{1}{2}$, 1, 2, 5, 10, 100.

For each exercise select from those above the answer which is closest. Use "inspection" and mental computation.

$20+\frac{1}{2}$; $20-\frac{1}{2}$; $\frac{1}{2}\times 20$;

$\frac{3}{8}+\frac{5}{8}$; $\frac{1}{2}+\frac{1}{8}$; $\frac{1}{2}+\frac{1}{10}$

9. Illustrations of the Use of Lattices

(a) Start a table or lattice of numbers on the board. After a couple of rows have students tell what number comes next or in any unfilled space.

"What number comes next?"

or

"What number goes here?"

20	.	.	.						
10	11	12	13	14	15	16	17	18	19
	1	2	3	4	5	6	7	8	9

As children study the chart guide them in establishing a code for using the chart:

5 ↑ means 15 (five is increased by 10)
20 ↓ means 10 (20 decreased by 10)
14→ means 15 (14 increased by 1)
16← means 15 (16 decreased by 1)

Use the code to provide arithmetic practice suitable for the age and ability of the students.

(1) 11 ↓ +12 ↓ +13 ↓ +14 ↓
(2) 10+17 ↓
(3) 13+4← +2→
(4) 14 ↑ +27 ↓
(5) 107 ↑
(6) 17 ↑ −7

(b) Use several arrows to stimulate children to think of fast methods of arriving at a solution.

(1) 4→ ↑ ← ↓ (4)
(2) 63→→→ (66)
(3) 44 ↑ ↓ ↓ ↓ ↑ ↑ ↓ (34)
(4) 17 ↑ ↑ ↑ ↑ ↑ ↓ ↓ (47)

Children are encouraged to share their methods of arriving at short cuts to the solution of exercises.

(c) Use a lattice to provide practice on multiplication facts.

21	22	.	.	.		
14	15	16	17	18	19	20
7	8	9	10	11	12	13
0	1	2	3	4	5	6

(1) 2 ↑ ↑ ↑ ↑ ↑ ↑ (44)
(2) 34 ↓ ↓ ↓ (13)

The student who knows 6×7=42 and 3×7=21 can respond more rapidly than one who must follow the chart step by step.

(d) Similar lattices and arrow patterns are developed for fractions.

15	15½	16	.	.	.				
10	10½	11	11½	12	12½	13	13½	14	14½
5	5½	6	6½	7	7½	8	8½	9	9½
0	½	1	1½	2	2½	3	3½	4	4½

(1) 6½ → → → (8)
(2) 25½ ↓ ↓ (15½)
(3) 3½ ↑ ↑ ↑ → → ← ← (18½)

Madison Project [1]

General Description

The experimental work now known as the Madison Project was begun in Madison Junior High School in Syracuse, N.Y., under the guidance of Prof. Robert B. Davis of Syracuse University. In general, groups of children who start working with project materials are followed throughout the remainder of their school program. For this reason the project's work has expanded to include grades 2 through 9. Madison Project maintains the original center at Syracuse University where the project was initiated. More recently a second center has been developed at Webster College, Missouri. A significant feature of the Webster College participation is their undergraduate teacher education program designed to prepare teachers in the use of Madison Project materials.

The three major purposes of the program are: (1) to promote greater interest in mathematics; (2) to stimulate children to think more creatively about mathematics; and (3) to provide a sounder background for future mathematics.

The author expresses some of the basic tenets of the Madison Project in the Teacher's Manual as follows:

> (1) *The absence of exposition.* In our use of the workbook, we almost never tell the students what to do, nor how to do it. Instead, we *ask questions.* The student learns by thinking through the questions himself (we call this the "Discovery Method,") or, in some cases, by imitating the behavior of the teacher or other students.
>
> (2) *Early introduction of concepts.* We try to get the student thinking about the basic concepts as early as possible. We avoid the use of extensive vocabulary. Names, and also calculations, tend to obscure the concepts, and it is the concepts that we want the student to think about.
>
> He can learn names and calculations later—after he has thought about the concepts in a creative way for some time. (By then he is ready for names and calculations, and they won't confuse him.)
>
> (3) *Conversations, not lectures.* We usually conduct our classes with a maximum of student participation. Every effort is made to get the students thinking and talking. We do not want them merely listening and accepting.

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[1] Director: Prof. Robert B. Davis, Syracuse University, Syracuse, N.Y., or Webster College, 470 East Lockwood, St. Louis, Mo.

(4) *Success experiences.* In our experience, nearly every student answer has some merit. We respond to every student answer as we believe scientists should—with respect. If it turns out to be right, it helps us with our work. But, even if it turns out to be wrong, it usually adds to our understanding—and so it, too, helps us along with our task. We try to avoid moral judgments. We prefer to say "Yes, that works" (if it turns out to) rather than saying "Good" or "Right"; if an answer turns out not to work, we usually say something like: "Well, that doesn't work; do you have any other suggestions?" We want the students to feel we are partners in an experience of intellectual discovery; we do not want them to feel that we are standing over them waiting to pass judgment on them.

(5) *The light touch.* The Madison Project introduces mathematical topics years earlier than most curriculums. This gives us the advantage of working without pressure. After all, our real purpose is to explore mathematics and to get students thinking on their own about the basic concepts of mathematics. We want them to enjoy it and to be motivated to learn more.

Nature of Content

The content of the Madison Project draws heavily upon intuitive algebraic and geometric ideas. For example, the purpose of the first lesson is to start students thinking about concepts of equation, open sentence, truth set, and inequality. The second lesson continues with these ideas and introduces tables and graphs. Forms such as squares and triangles are used as placeholders to be replaced with numerals so that the conditions of the problem will be met. Set language and symbolism provide an orderly arrangement for recording the results of thinking. Children are guided to discover patterns from the procedure of plotting on a graph pairs of numbers obtained from finding solutions for open sentences with two placeholders.

Each lesson reviews some of the earlier work and presents some new idea or concept. It is expected that the new idea will be mastered, not when first presented, but as it reappears in subsequent lessons.

The meaning of negative numbers and processes with them are introduced through considering such experiences as games won and lost or bills and checks brought by the postman.

The Madison material is intended for enrichment. In many places it is being used once a week along with the regular arithmetic program. The author has found that children in grades 4, 5, and 6 possess a remarkable ability to handle the abstract concepts presented in the Madison material, even surpassing students of grade 7 in enthusiasm, insight, creativity, and in the success they experience with the new ideas.

The Project has produced several 16mm sound films showing actual classroom lessons of children in grades 2 through 8. The films have been much more effective than descriptions in communicating methods of discovery, pupil participation, and the progression of mathematical ideas.

An important aspect of Madison Project activities is the maintenance of demonstration centers that may be visited by interested persons who wish to see the project in operation.

For the more mathematically oriented reader Professor Davis describes the nature of the content of Madison Project materials as:

> . . . an axiomatic approach to algebra, with the use of cartesian coordinates in graphing functions, with studies of various aspects of functions, with derivation of mensuration formulae in geometry, and with the notions of implication and contradiction. There is some relatively smaller emphasis on applications of mathematics to problems in the physical sciences. We make considerable use of matrix algebra, and, for example, use matrices as a basis for extending the real number system to the complex number system.

Examples are taken from early lessons on equations, sentences, inequalities, tables, and graphs.

Illustration 1: Children are asked to fill in the proper numeral in the squares in these open sentences:

$3+\square=5$ (2)

$2+(3\times\square)=14$ (4)

$2+(2\times\square)=11$ ($4\frac{1}{2}$)

The sentences above are referred to as open sentences because they have an unfilled placeholder. Until the placeholder is filled with a numeral it is not possible to tell if the sentence is true or false.

As children suggest numbers to try as replacements for the box in the third number sentence, they realize that 4 is too small and that 5 is too large:

$2+(2\times4)$ does not equal 11; it equals 10.

$2+(2\times5)$ does not equal 11; it equals 12.

These inequalities are expressed as: $2+(2+4)\neq11$ and $2+(2\times5)\neq11$. It is now clear that no whole number will fit. When fractions are included a replacement which will make the sentence true can be found. An orderly record of the numbers tried and rejected is kept to help children judge how close they are getting to the correct answer:

$1+(3\times\square)=5$	*Too small*	*Too large*
	1	3
Try 1: $1+(3\times1)\neq5$;		
		2
Try 3: $1+(3\times3)=10$; $10\neq5$		
		$1\frac{1}{2}$
Try 2: $1+(3\times2)=7$; $7\neq5$		
Try $1\frac{1}{2}$: $1+(3\times1\frac{1}{2})=5\frac{1}{2}$; $5\frac{1}{2}\neq5$		

What number larger than 1 and smaller than $1\frac{1}{2}$ will fit? Try $1\frac{1}{3}$: $1+(3\times1\frac{1}{3})=5$.

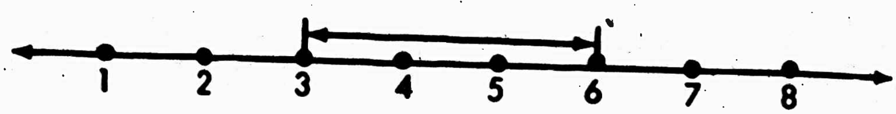
Illustration 2: When the meaning of the symbol $<$ (is less than) is taught, exercises with inequalities such as the following are introduced:

(a) $3 < \square$ — Three is less than what numbers? {4, 5, 6, 7, ...}

(b) $3 < \square < 5$ — 3 is less than some number which is less than 5; what is the number? {4}

(c) $2 < \square < 6$ — {3, 4, 5}

(d) $3 < (\square + 1) < 6$ — 3 is less than the box plus one, and the box plus one is less than 6.



The numerals 3 and 6 mark the boundaries within which the solution is found.

Illustration 3: Children are introduced to two variables as they try to find replacements for two placeholders in the same sentence. For example, the child is requested to find replacements for the box, or to find the truth set for the equation.[3]

For an equation such as: $\square + \triangle = 10$, the teacher may set up a table omitting the number for the $\square$ or the $\triangle$. The class helps to decide replacements.

Fractions are included as soon as the class discovers that 2/2, 3/3, and 4/4 equals one whole.

□	△
6	—
—	8
2½	—
—	3
—	9

[3] The Truth Set as used in Madison Project materials refers to the set of numbers (or, in the case of several placeholders, to the set of ordered pairs of numbers, etc.) which, when substituted for the placeholders in an open sentence produce a true statement.

Illustration 4: **When the truth set for an equation such as △= (2×□)+3 has been found children plot points representing pairs of numbers derived from the equation and discover the progression by 2 on the graph.**

□	△
0	3
1	5
2	7
3	9

If □ is 0; △ is 3
If □ is 1; △ is 5
If □ is 2; △ is 7
If □ is 3; △ is 9

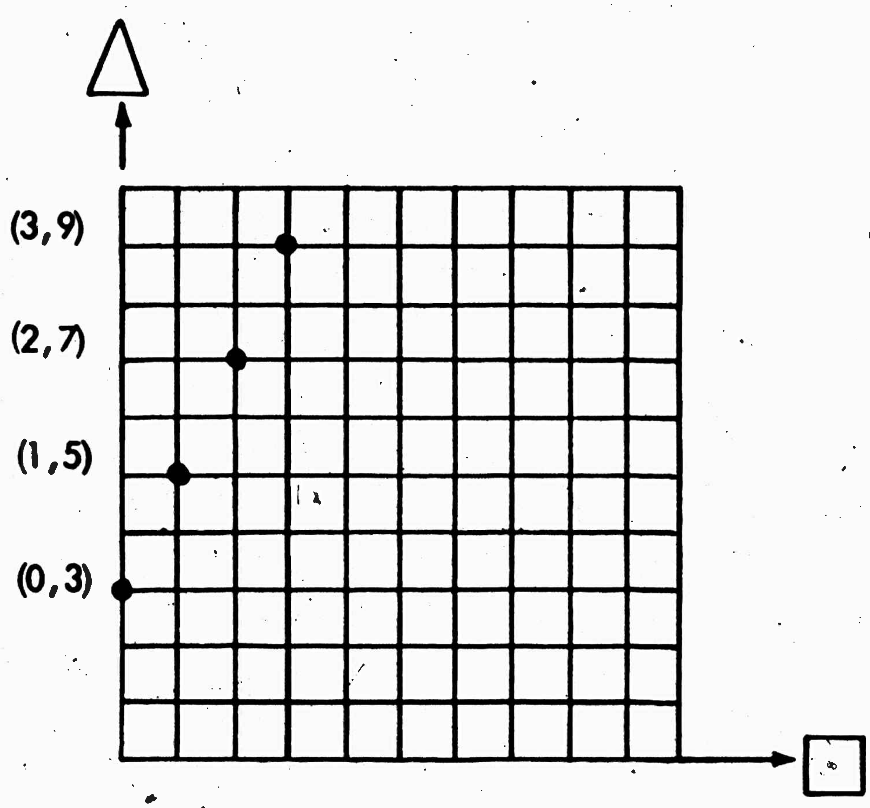
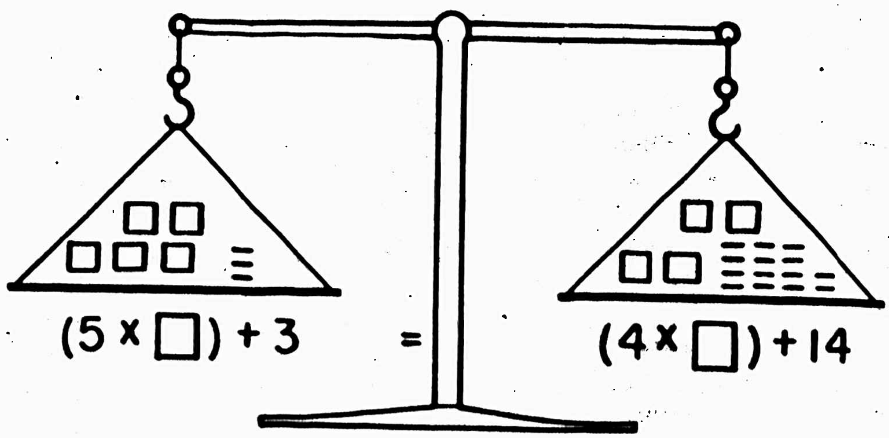


Illustration 5: Children are guided to discover the important generalization that if both sides of an equation are changed in exactly the same way the sides are still equal. A balance scale picture device is used to help children arrive at this conclusion as they find solutions for such equations as $(5\times\square)+3=(4\times\square)+14$.



The pictures represent scales with rolls of dimes. (Each roll contains the same number.) The problem is to simplify the situation keeping the scale in balance.

Take 2 rolls from each side: $(3\times\square)+3=(2\times\square)+14$.

Take 2 more rolls from each side: $(1\times\square)+3=14$.

Take 3 loose dimes from each side: $1\times\square=11$.

Prove by using 11 in the original equation and each time a change was made.

$(5\times11)+3=(4\times11)+14$

$55+3=44+14$ or $58=58$

$(1\times11)+3=14$ or $14=14$

$1\times11=11$ or $11=11$

Illustration 6: The following exercises show further work in finding truth sets for exercises with fractions.

(a) $\frac{\square}{3}+1=4$ what number divided by 3, plus 1 = 4? {9}

(b) $\frac{\square}{5}-4=0$ {20}

(c) $\frac{2}{3} \times \square = 12$ {4, 8, 16, 18}

Try 4: $\frac{8}{3} \neq 12$ (too small)

Try 8: $\frac{16}{3} \neq 12$ (too small)

Try 16: $\frac{32}{3} \neq 12$ (too small)

Try 18: $\frac{36}{3} = 12$ Truth set is {18}.

Illustration 7: The way in which children are guided to discover a mathematical truth may be illustrated by the following search for the correct method of checking:

$$(18-3) \div 2 = 7\frac{1}{2}$$

In order to check, the child knows that he does the opposite. If he assumes an incorrect order of steps in checking, the actual solution reveals his error.

$$\left(7\frac{1}{2}+3\right) \times 2 \neq 18$$

By further experimentation he finds that $(7\frac{1}{2} \times 2) + 3 = 18$, and draws the conclusion that the first thing he does to arrive at the original solution is the last thing he does when he uses inverse operations to check. The "undoing" is in reverse order to the "doing."

Illustration 8: Children experiment to find a pair of numerals to replace the $\triangle$ and the $\square$ so that two equations will be solved. For example, in the two sample sentences below, the replacement used for the triangle in the first sentence must also be used to replace the triangle in the second sentence. In like manner the same numeral must replace the square in both sentences.

$$\left\{\begin{array}{l} \triangle = 3 \times \square \\ \triangle = \square + 10 \end{array}\right\}$$

A child suggests 3 for the square.

$$\left\{\begin{array}{l} 9 = 3 \times 3 \\ 9 \neq 3 + 10 \end{array}\right\}$$

Other numerals are suggested either for the square or the triangle and solution sets are found:

$$\left\{\begin{array}{l} 15 = 3 \times 5 \\ 15 = 5 + 10 \end{array}\right\}$$

The solution set is {15, 5}

Illustration 9: Placeholder notation and the use of open sentences make it possible to use general statements about numbers. Through the use of this procedure identities are discovered. Identities are sentences which remain true regardless of the number replacement. Examples are:

$\square+6=6+\square$ $\square+0=\square$

$\square+\square+\square=3\times\square$

$2\times(\square+\triangle)=(2\times\square)+(2\times\triangle)$

Children try to find one exception or one case in which the number replacement fails to make the sentence true knowing that this proves the open sentence is not an identity.

Illustration 10: One of the techniques used for developing the meaning of negative numbers is a postman story:

Suppose a postman brings you a check for \$3. We can represent this as $^{+}3$. If he brings you a *bill* for \$2, we can represent that as $^{-}2$.

Suppose the postman brings you a check for \$5, and a bill for \$3. Are you richer or poorer? By how much? Can you make up a postman story for each problem?

$^{+}2+{}^{+}4=?$ $^{+}5+{}^{-}2=?$

$^{-}2+{}^{-}3=?$ $^{+}5+{}^{-}6=?$

$^{-}7+{}^{+}9=?$ $^{-}5+{}^{+}1=?$

$^{-}3+{}^{-}6=?$ $^{+}1+{}^{+}12=?$

$^{+}7+{}^{-}9=?$ $^{+}2+{}^{+}17=?$

Can you make up a story that will correspond to *subtraction?* How would you explain this?

$^{+}7-{}^{+}2=?$

Can you make up a postman story for each problem?

$^{+}9-{}^{+}2=?$ $^{+}15-{}^{+}5=?$

$^{+}8-{}^{+}2=?$ $^{-}5-{}^{-}1=?$

Illustration 11:

A truth set table provides number pairs for graphing.
The open sentence is derived from the graph.
The pattern is over 1, up 3;
the open sentence is $1 + (3 \times \square) = \Delta$

□	Δ
0	1
1	4
2	7
$^{-}1$	$^{-}2$
$^{-}2$	$^{-}5$

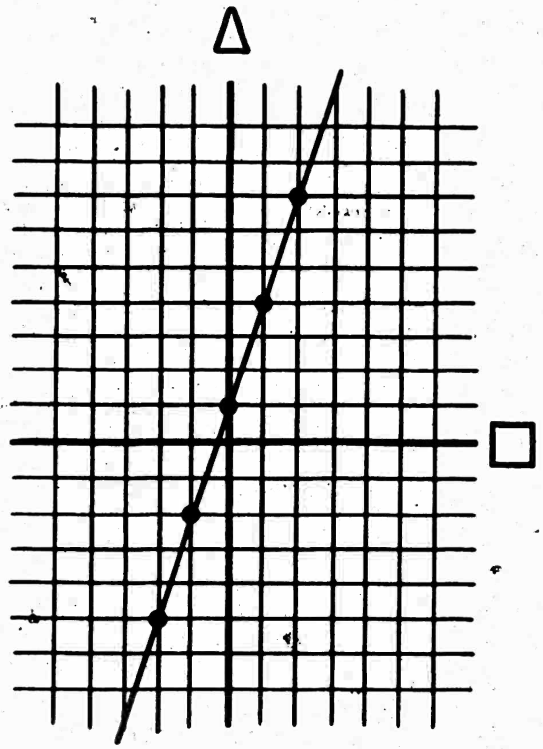


Illustration 12: Children learn to solve quadratic equations and enjoy the intellectual activity of discovering the two "secrets."

$$(\square \times \square) - (8 \times \square) + 15 = 0$$

The two numbers which will work in the equation are 3 and 5 because 3+5=8 and 3×5=15. In the process of discovering the two numbers, children experiment and try any numbers they think might work. If they find them by chance their task is to determine why they work so the next equations can be solved more efficiently.

Stanford Project [1]

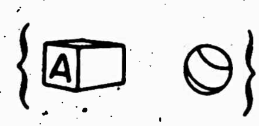
SETS AND NUMBERS

General Description

Prof. Patrick Suppes, author of Sets and Numbers, Books I and II, for the primary grades, believes that all mathematics can be developed from notions of sets and operations on sets. He states that his primary objective is to develop the elementary concepts and laws of arithmetic in a manner that is both pedagogically simple and mathematically exact. Professor Suppes finds the term "set" useful in helping children develop basic concepts. He believes that the child already has a workable idea of a set as a collection of or a family of things. This prior knowledge provides a simple and concrete means of introducing number operations which is easy to explain and understand. He holds the point of view that operations on sets are more concrete and comprehensible to young children than abstract operations on numbers. He holds further that numbers are easily understood and made meaningful when considered properties of sets. He conceives of numbers as properties of sets and the operation of addition of numbers as a general way of combining families of sets of things without paying any real attention to the things themselves. The introduction of a notation for sets permits consideration of the operations of arithemetic, such as addition and subtraction, without the symbolism of the Hindu-Arabic numerals in the early stages. Emphasis is placed on understanding through the development of a technical vocabulary and by a carefully constructed set notation for recording ideas. Counting is considered basic to addition. Other ideas given emphasis are the order of operations in a set, equality of sets, and zero as the number of things in an empty set.

Designed as a complete course in mathematics for grades 1 and 2, the texts are workbook type requiring the selection of the appropriate set or the insertion of an answer to complete a mathematical sentence.

Exercises are presented on three levels of abstraction. Naming objects within brackets constitutes the first step:



[1] Director: Prof. Patrick Suppes, Ventura Hall, Stanford University, Stanford, Calif.

In the second step objects within brackets continue to be presented. The N outside the brackets indicates that attention is now given to the number aspect of the set.

In the third step the Hindu-Arabic numeral for the set is introduced. The following illustrations serve to show the progression toward abstraction using the operation of addition:

The symbol "U" designates the union of the set containing the box and the ball with the set containing the drum, resulting in one set containing the three objects.

A set of two objects and a set of one object are equal to a set of three objects.

$2+1=3$
Two plus one equal three.

The results of experimental work begun in 1959–60 indicated that children experience very little difficulty with either the notation or the vocabulary. During the school year 1960–61, the program expanded to include 25 first grade classes representing a wide range of ability levels. The classes were taught by regular classroom teachers who had full responsibility for pacing the materials to the abilities of their children. Inservice work with teachers was provided by the project staff in an orientation meeting at the beginning of the school year followed by monthly meetings throughout the year.

An evaluation test of 74 items covering all types of problems in Book I yielded a mean score of 61.40 and a median of 65. An item analysis indicated that the least difficult items were those involving zero or the empty set, while the most difficult were the two-step type such as $2+\_\_=3+1$. Only in the instance of the two-step exercise, which proved to be more difficult for slower children, was there a difference in number of errors as related to abilities of the children.

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An evaluation comparing results for experimental and control groups showed the experimental group to be superior in items involving place value and writing numerals, and markedly superior in items involving arithmetical operations.

Projected plans include the preparation, tryout, and evaluation of materials for kindergarten and grades 1 through 6.

Nature of Content

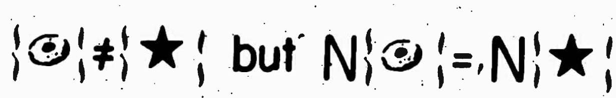
The following illustrations are typical of the concepts which are new or are presented by means of a different approach:

1. To describe a set, pictures of the objects in the set are placed in brackets:
   balloon and star

2. Order of objects is unimportant: balloon and star or star and balloon.

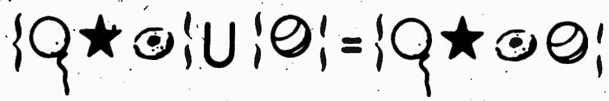
3. Zero as the number of things in an empty set: N{ }
4. Number of objects in a set:

Number as a property of a set:



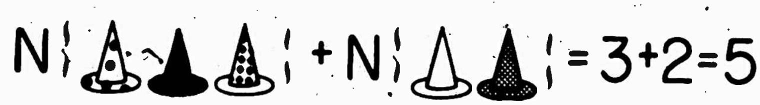
A cookie is not equal to a star, but the number of objects represented by the cookie and by the star is equal.

5. Addition as the union of sets:



A set containing a balloon, star, and cookie joined with a set containing a ball equals a set containing a balloon, a star, a cookie, and a ball.

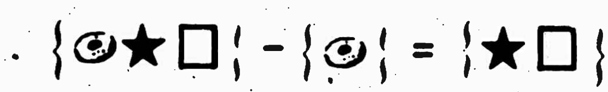
6. Notation for union of sets emphasizing the number of objects:



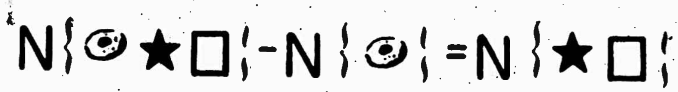
7. The missing members of sets:

How many objects are needed with a set of two to make a set of four objects?

8. Difference of sets:



The difference between the larger set and the smaller set is the set of the star and the box.



$$3-1=2$$

9. Equalities:

$$1+4=2+3$$
$$5=5$$

10. Combined processes:

$$4 - \_\_ = 3 = 2 + \_\_.$$
$$\_\_ + 2 = 4 - 2.$$

11. X and Y as unknown numbers:

$$4 - x = 1$$
$$x = \_\_$$

$x + y = 3$	$x - y = 3$
$y = 2$	$y = 1$
$x = \_\_$	$x = \_\_$

GEOMETRY FOR THE PRIMARY GRADES[2]

General Description

The geometry project had its informal beginning in the spring of 1958 when Profs. Patrick Suppes and Newton S. Hawley of Stanford University taught geometry in a first grade classroom 20 minutes per day for a 2-month period. Encouraged by the high level of comprehension and achievement by the class, the authors extended the project the following year. Worksheets were developed requiring the children to read instructions for themselves and enabling the regular teacher to conduct the lesson. Since the reading requirement necessitated skill in reading, these worksheets, organized into Books I and II, were recommended for students of all levels of ability in grades 2 and 3.

Among the goals they hope to accomplish through the project the authors list: (1) to stimulate reasoning, analytical and creative thinking; (2) to increase reading comprehension of the type needed to carry out instructions for executing accurate constructions and for interpreting them; (3) to introduce the child to a branch of mathematics other than arithmetic; (4) to verify their belief that an understanding of geometry will help children understand and analyze the physical world; (5) to introduce children to the important concept of precision; and (6) to provide another avenue through which stimulating and challenging experiences may be presented to able learners.

[2] Director: Prof. Newton S. Hawley, Department of Mathematics, Stanford University, Stanford, Calif.

Emphasis is placed on correct use of the pencil and on precision in the use of compass and straightedge in order that accurate ideas will result from constructions. The technical vocabulary necessary for reading and communicating geometric ideas is acquired as the need arises in construction work. Terms such as bisect, line segment, perpendicular, and pentagon are included in both the oral and reading vocabulary from the time that these ideas are first introduced.

While no previous knowledge of arithemetic is required for success in the geometry lessons, the authors stress the point that geometry is not a substitute for arithmetic which must also be taught as an essential branch of mathematics.

The Teacher's Manual provides information on the proper use of tools, gives general teaching suggestions, and includes facsimile pages from the pupil's text with answers, completed constructions, and specific suggestions to the teacher.

It is suggested that teachers give only minimum aid so that children will use their initiative to read and interpret the text. Minimum aid includes the following:

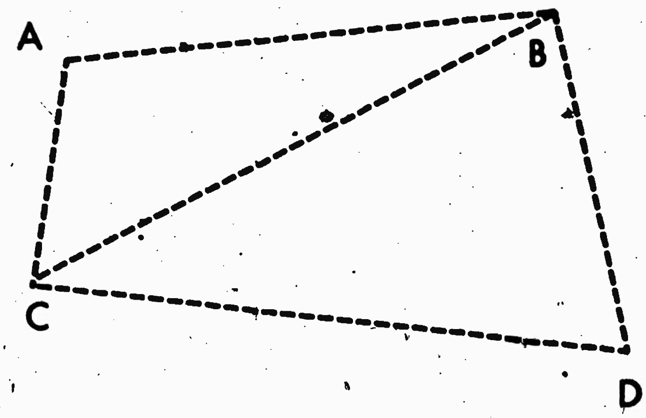
1. Instruct the children in the proper techniques for using the tools.
2. Insist on precision in the execution of the drawings.
3. Explain the meaning of the words which the children do not understand.
4. Give help to the class at large only when the majority is unable to proceed or unable to understand a lesson.
5. Give individual help to those children who need it. This would include correction of mistakes.

Nature of Content

Chapter one of Book I deals with lines. Children draw lines and line segments, draw lines through points, connect points, and draw lines determined by points to form triangles and quadrilaterals and locate points of intersection. They identify shapes, such as triangle, quadrilateral, pentagon, hexagon, and circle. They discover the maximum number of line segments they can draw to connect 2, 3, 4, or 5 points.

Problems are presented to be solved by drawing and by answering questions about the drawing.

Illustration 1: 1. Draw AB[3]

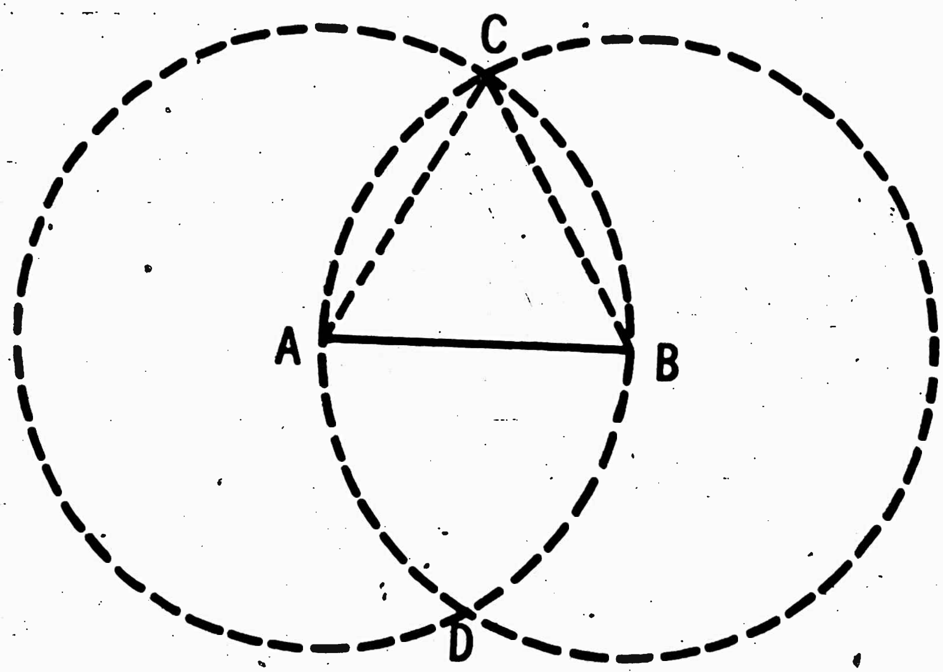


2. Draw CD
   Draw AC
   Draw BD
3. Connect C and B.
4. How many triangles do you see? 2

Chapter two of Book I on circles presents the following topics: inside and outside, center of circle, points of intersection of circles, radius of a circle, radii of equal lengths; labeling points, line segments, circles, and points of intersection; angles; vertex of an angle, angles of equal size; line segments longer than, equal to, or shorter than other line segments; equilateral triangles; arcs.

[3] Dotted lines denote children's drawings.

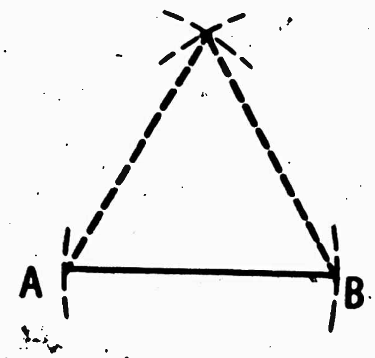
Illustration 1: 1. Draw the circles which have AB as a radius.



2. In how many points do these circles intersect? 2
3. Label these points C and D. (Put C on top.)
4. Draw AC, then draw BC.
5. Use arcs with radius equal to AB to determine:
   (a) Does AB equal AC? *Yes*
   (b) Does AB equal BC? *Yes*
   (c) Does AC equal BC? *Yes*
6. What kind of triangle is ABC? *Equilateral*

Chapter three of Book I deals with constructions. Children construct equilateral triangles and angles, bisect angles, compare angles, and construct squares.

Illustration 1: 1. Construct an equilateral triangle with base AB



2. Construct an equilateral triangle with base XY

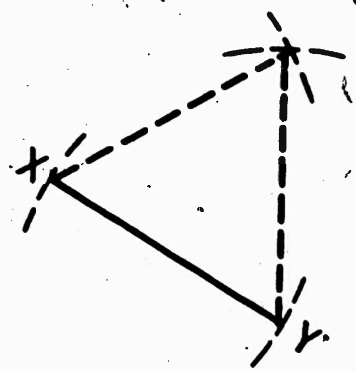
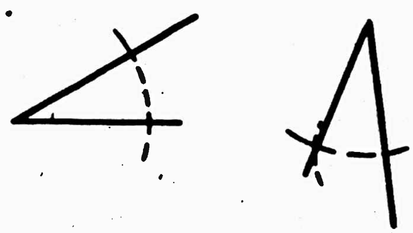
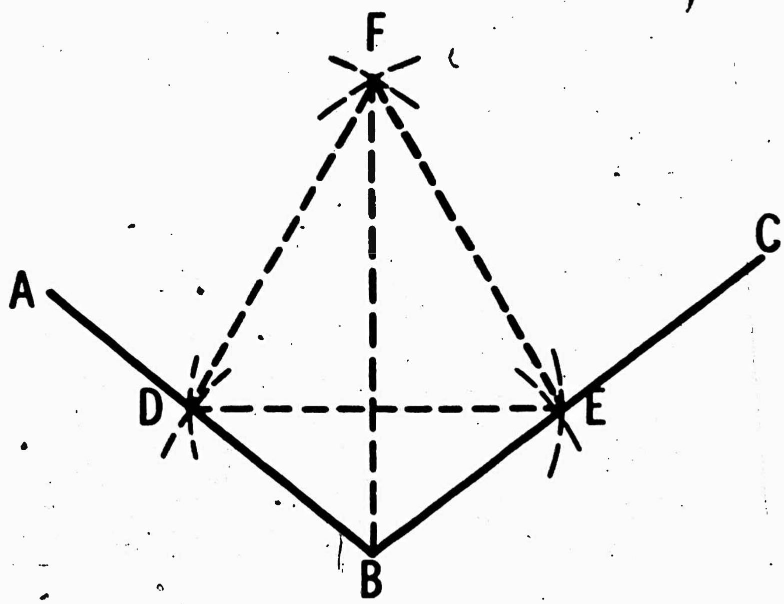


Illustration 2: Show the two angles are equal



Book II deals with the construction of perpendiculars and angles.

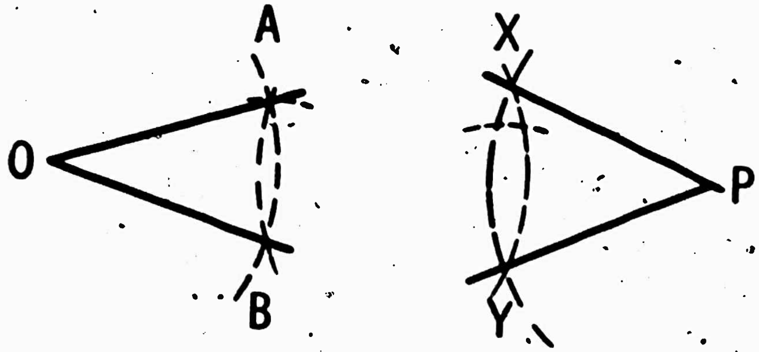
Illustration 1:



1. Draw arcs with equal radii and center B to intersect AB and BC.
2. Label as D and E these points of intersection.
3. Construct an equilateral triangle with base DE.
4. Label as F the vertex of the triangle.
5. Draw BF.
6. Does angle ABF equal angle CBF?

   Yes ______ No ______

Illustration 2: Problem: Compare angles.

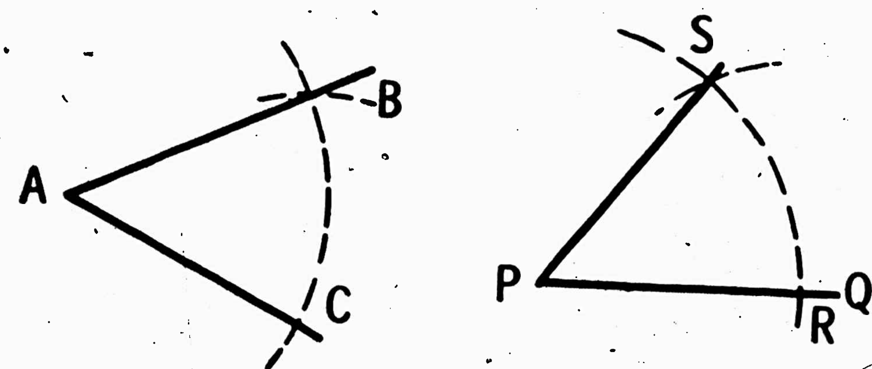


1. Draw arcs of equal radii centered at O and P.
2. Label the points of intersection A and B (about O) and X and Y (about P)

Rule: Compare AB and XY to compare the angles.

3. The angle at O is *smaller than* the angle at P.
   - (a) larger than
   - (b) smaller than
   - (c) equal to
   - (d) more than

Illustration 3: Problem: Reproduce a given angle.



1. Draw an arc with center P and radius equal to AC.
2. Label as R the point at which it intersects PQ.
3. Draw a second arc with center R and radius equal to CB.
4. Label as S the point at which the second arc intersects the first arc.
5. Draw PS.
6. Does angle BAC equal angle SPR?

Yes ______ No ______

Under the direction of Professor Hawley, the geometry project team will continue their experimentation.

MATHEMATICAL LOGIC [4]

The experimental project on logic for elementary grades is planned specifically for gifted students of grades 5 and 6. The project, under the direction of Prof. Patrick Suppes, at Stanford University, is designed to introduce the academically gifted elementary school child to modern mathematics and mathematical methods at a level which is rigorous but simple enough in presentation and context to permit relatively easy comprehension. The project director hopes to determine whether the kinds of deductive proof which characterize modern mathematics are within the capacities of able children of this age level, what difficulties they meet as they proceed with it, and the extent to which they are able to transfer skills of analysis and reasoning to other subject areas. Preliminary investigation from a pilot study continuing during 1960–61 leads Professor Suppes to believe that logic is not too abstract for elementary children and that, in fact, the ages of 10, 11, and 12 may be opportune for the introduction of abstract concepts. The participating students were able to attain a level of accomplishment equal to that of college students in the skills of deduction essential to mathematical reasoning when the material was paced more slowly and planned for the elementary school level.

In the expanded study conducted during the school year 1961–62, each class met for 30 to 40 minutes 3 days a week. Regular classroom teachers took an introductory course in mathematical logic in preparation for teaching one of the experimental classes. The project will continue during the year 1962–63.

[4] Director: Prof. Patrick Suppes, Ventura Hall, Stanford University, Stanford, Calif.

Minnesota Elementary Curriculum Project [1]

General Description

The Minnesota Elementary Curriculum Project is directed by Prof. Paul C. Rosenbloom, Minnesota School Mathematics Center, University of Minnesota. The aim of the Minnesota project is to produce a curriculum based on what children can learn, and then to prepare teachers to teach it. The materials in grades K–1 are based,

[1] Director: Prof. Paul C. Rosenbloom, Minnesota School Mathematics Center, University of Minnesota, Minneapolis, Minn.

in part, on the research of Muller-Willis and Rosenbloom on the development of mathematical concepts in young children. Other materials are based on small-scale experimentation by specially trained teachers.

As materials are written, tried out on a small scale, and revised on the basis of experience, they will be introduced into preservice education. Teacher education materials will be produced in conjunction with the new materials for children.

The main mathematical goal is to provide the children as early as possible with a geometrical model of the real number system. Children will be able to visualize concretely the relations between and operations on numbers. This model will apply to all the numbers they will meet in school. At first they will know only the names of the positive integers. As they go further in school they will not only learn names for more numbers, but they will also apply the processes they have already learned to the new numbers.

The geometrical model will enable even the slowest learner to work in full confidence that he can always get the right answer if he has enough time. The principal drill at first will be on the many concrete interpretations of the mathematical ideas in problems arising in the natural and social sciences. Only gradually will the pace be increased in order to stimulate pupils to remember the arithmetical facts.

Applications of mathematics are emphasized from the beginning. There is a strong emphasis on counting and measuring all sorts of things in the child's environment. The project aims at close coordination with the natural and social sciences. The material is intended to contribute also to the reading program, and connections with language, art, and music are brought in whenever possible.

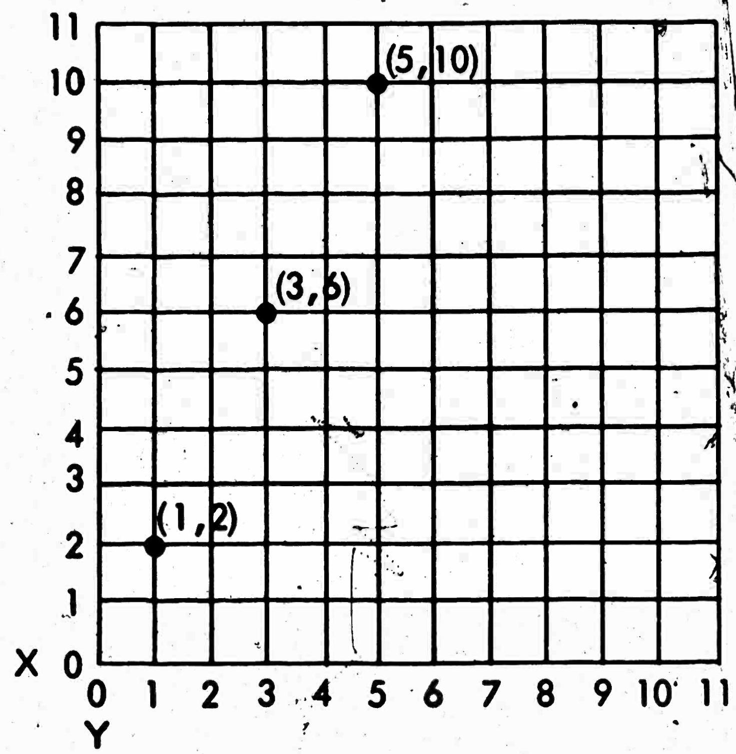
Materials for grades K–3 are being developed by a small writing team of elementary teachers, two editors, and an artist, under the direction of Professor Rosenbloom. They are being tried out in 10 schools during 1961–62, and will be used in about 20 schools during 1962–63.

An expansion to a science and mathematics curriculum project in grades K–9 is planned. Larger scale experimentation is planned, beginning in 1963, at centers established at teacher education institutions.

The point of view is that the mathematical structure is important for its own sake, but that a full appreciation requires a strong emphasis on the applications of mathematics.

Nature of the Content

The first important goal of the program in grades K–3 is to teach the use of a coordinate system. As soon as the pupil understands this, then whenever we wish to teach him how to graph the function [2] the child has a geometrical algorism for computing the operation.



[1] A function is a collection of ordered pairs (of numbers in these illustrations) usually having a definite relationship. For example, an illustration of a function consisting of three pairs such that the second element (number) is twice as large as the first is the following: (1,2) (3,6) (5,10)

Each first element is paired with only one second element. In other words each first element has a unique second element. In another illustration of ordered pairs the second element is five times the first element: (1,5) (3,15) (7,35) (10,50) (2½,12½) (2,10)

The collection of ordered pairs shown below does not qualify as a function because the same first element (2) is paired with two different second elements (8 and 12). (1,5) (2,8) (2,12)

In a function one set of numbers is paired with or mapped into another set. In the linear function shown below, for every number on the x axis there is a number on the y axis that is specified by the relationship between the pairs. This relationship is given by the equation $y=2x$.

The following outline indicates the sequence by which this goal is approached. Illustrative details are given only where the treatment is different from that of other programs. The time to be allotted to these topics is still under experimentation. The various concepts are worked into stories and games.

I. Sets
II. Correspondence
III. Number words and symbols to "ten"
IV. Common geometrical figures, closed and open curves
V. Measurement of length, number ray, addition of trips
VI. Number words and numerals to 99
VII. Area, volume, and mass
VIII. Ornamentation and symmetry
IX. Maps
X. Squareville (working with graph paper)
XI. Interpretations and applications
XII. Laws of arithmetic
XIII. Numeration
XIV. Technique of computation
XV. Applications

Most of units I–X have been taught in grades K–3.

Illustrative Details

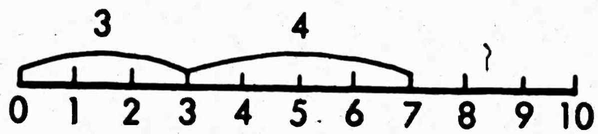
(a) Game: Place 16 objects on table. One child thinks of an object. Another tries to guess the object by asking questions which can be answered only by "Yes" or "No." The idea is to ask as few questions as possible.

Discussion: What was the yes-set for your first question? What was the no-set? How can you combine the information from your first two questions?

(b) Application of sets to classification of living things such as animals. What is a good way of arranging these animals in sets? Do you have a system? How are the animals in each set alike?

(c) Game of Nim[3] with two piles. What is the winning strategy?

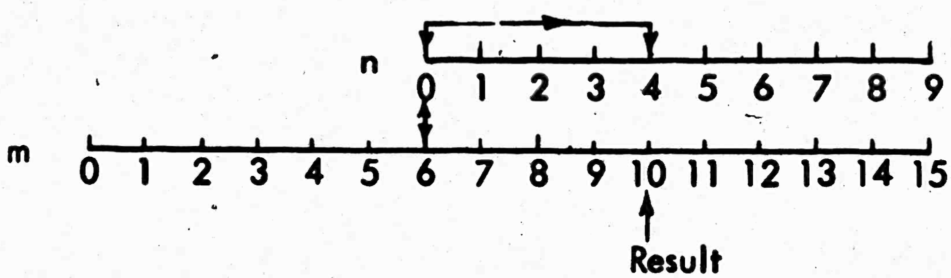
(d) Variations of the number line suggest the operations of addition and subtraction: To add 3 and 4, start at 0 on number line, lay off a distance of 3 spaces to the right, then lay off a distance of 4 spaces. Read 3+4=7 on the scale.



Two number lines manipulated by the child as a slide rule are used as shown in the diagrams:

To Add:

6+4=10



(a) Locate "6" on line *m*.

(b) Place line *n* as shown in sketch with the "0" above "6."

(c) Read over to "4" on line *n*.

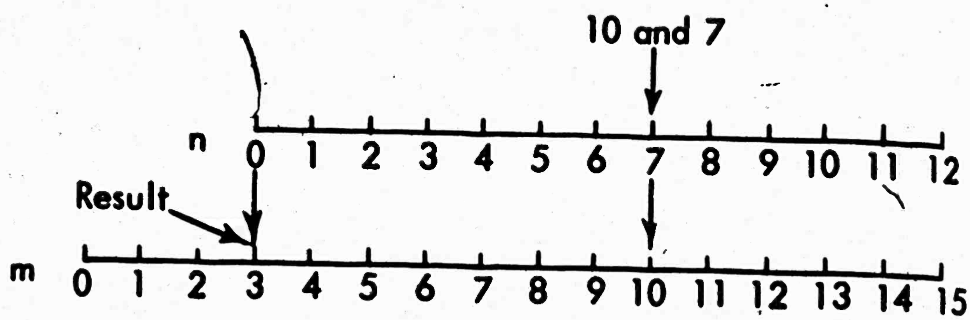
(d) Read the sum on line *m* below "4."

It is "10."

[3] Donovan A. Johnson and William H. Glenn. *Understanding Numeration Systems*, from the series "Exploring Mathematics on Your Own" St. Louis, Mo.: Webster Publishing Co., 1960. p. 41.

To Subtract:

10−7=3



(a) Locate "10" on the *m* line.

(b) Find "7" on the *n* line.

(c) Place the "7" above the "10."

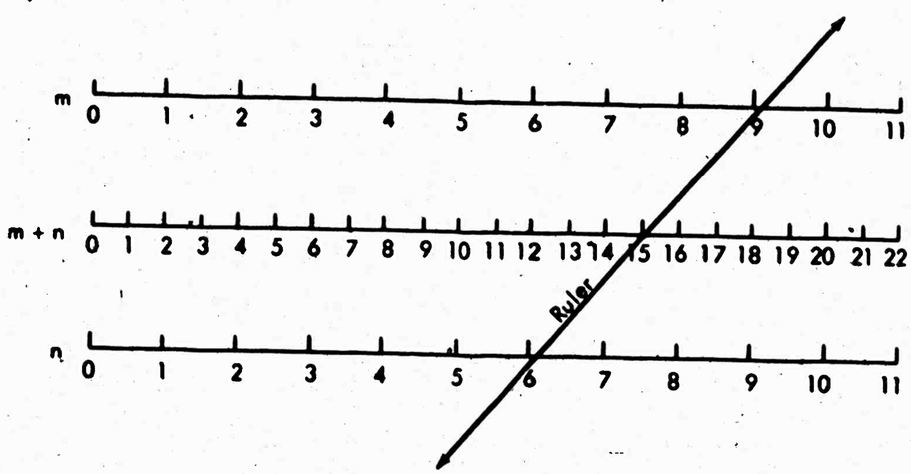
(d) Now read the point on *m* which falls below the "0" on *n*.

It is "3."

A graph arrangement of three lines (nomogram) shows the related values for the three numbers when the points are connected by a straight edge or ruler.

To Add:

9+6=15



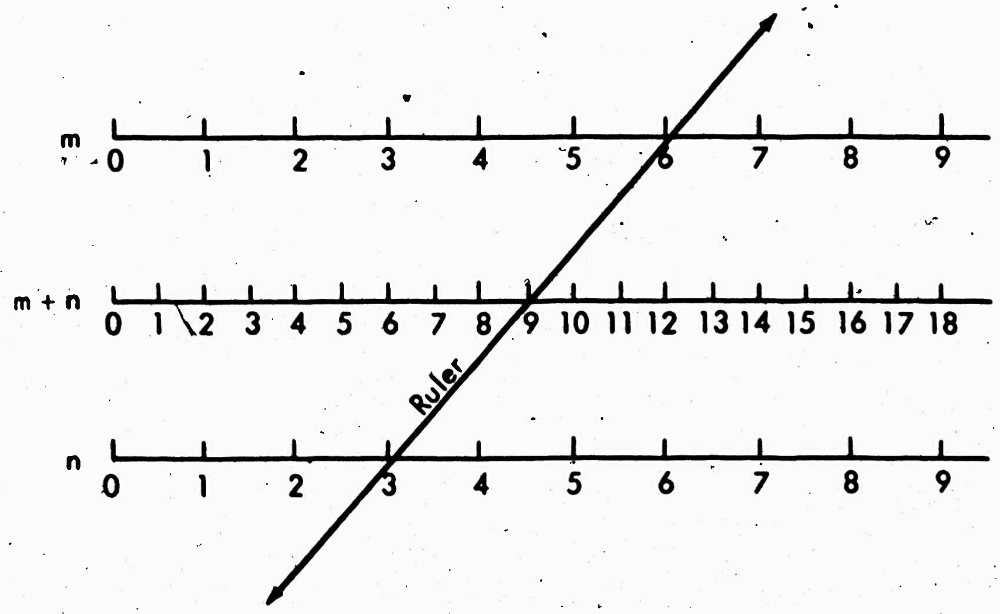
(a) Locate "9" on the *m* line and the "6" on the *n* line.

(b) Place a ruler so that the "9" and the "6" are in a line and visible.

(c) The ruler touches a point on the center line. Read this point. It is "15" and the sum for 9+6.

To Subtract:

$9-6=3$



(a) Begin with the center line marked $m+n$. Locate "9."

(b) Locate "6" on the m line.

(c) Place the ruler so it is in line with "9" and "6."

(d) Look at line n. Read the answer. It is "3."

(e) Clever, imaginative stories provide a medium for presenting mathematical ideas to young children in kindergarten and grades 1 and 2. Planned correlation of mathematics with language arts, science, social science, and music programs is achieved. For example, an original story for grade 2 helps children develop the concept that there are many ways to write the symbol for a number.

(f) Imagine you live in a country where the people have only 1- and 2-cent pieces. How many ways can you make a given amount of money into change? Make a table:

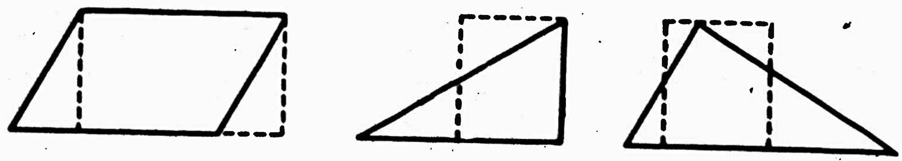
n	number of ways to make n cents.
1	
2	
3	
etc.	

682481 O—63——7

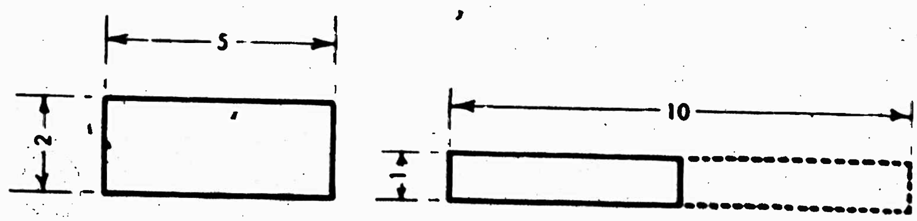
Is there a pattern? Predict the next number and check your answer. Can you predict how many ways you can make 20 cents into change?

Now do the same problem for a country where they have only pennies and 3-cent pieces, or where they have pennies, 2-cent pieces, and 3-cent pieces. Invent your own country with its own system of coinage, and solve the problem for your country.

(g) Jigsaw puzzles for transforming figures into rectangles:

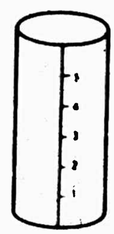


Can you transform these figures into rectangles?



If you transform the rectangle above into a 1 by 10 rectangle, you can measure the area by measuring the length.

(h) If you draw a number line on the side of a cylinder, you can measure volume by measuring length.



A spring balance also has a number line. You can measure mass, or weight, by reading off on the number line how much the spring stretches (Hooke's law).

(i) When a liquid is heated, it expands. If you put a colored liquid in a cylinder and draw a number line on the side, you can measure temperature by measuring length. This is how a thermometer works.

(j) Here is a map of a town called Squareville (graph paper). The streets run north and south, and the avenues run east and west. The streets are labeled 1st street, 2d street, etc., and similarly for the avenues. Locate 3d street and 4th avenue (the point (3, 4)). What is the address of this street corner? of that one?

Draw a highway from City Hall to 10th street and 10th avenue. What do you notice about the addresses along this highway? We call it "Equality Boulevard" because the street and avenue addresses are the same along this highway. We use "x" for the street address and "y" for the avenue address. Then we can write on the street sign simply "y=x."

Other Improvement Procedures

Support from Federal Government

The Federal Government provides financial assistance on a matching basis to State departments of education through the National Defense Education Act. These funds are used to purchase instructional materials and to provide supervisory service at the State level.

Among its many activities the National Science Foundation sponsors institutes for supervisors, principals, and classroom teachers. Institute offerings include foundation courses to strengthen mathematics background and opportunities for participants to learn about the new trends in elementary school mathematics and to plan improvement procedures for their localities. The fact that many applications must be turned down for each one accepted bears witness to the magnitude of the task of reaching all elementary school personnel, to their eagerness for information, and to the need for an expansion of offerings for elementary teachers.

Policies which govern the activities of the National Science Foundation and the Office of Education specify that funds may be used for research and development, for production, experimentation, and dissemination of information. Funds may not be used in any way to promote a particular curriculum or to interfere with local control.

Regional and State Efforts

Regional and State workshops are being held throughout the Nation in an effort to evaluate the present status of elementary school mathematics and to plan the logical next steps to be taken in improving the program. Curriculum committees made up of elementary school personnel and mathematics consultants are at work revising existing guides in elementary school mathematics or preparing new materials.

The State of California has held a series of Regional Conferences on Improving Mathematics Instruction in Elementary Schools. These Conferences were sponsored by the California State Department of Education under provisions of the National Defense Education Act.

Summary reports of these conferences are available.[1] The Advisory Committee on Mathematics to the State Curriculum Commission, as a part of its study of the mathematics curriculum, is attempting to identify strands or basic mathematical ideas which run from kindergarten through grade 8. The committee views the following strands as having particular significance for the elementary school level:

1. Number, structures, operation
2. Number systems and numeration
3. Geometry and measurement
4. Sets
5. The mathematical sentence
6. Logic
7. Graphs and functions
8. Applications of mathematics

In further work of the committee a sequential development of each of these basic ideas willbe recommended.[2]

LaGrange, Ill., has combined the resources of closed-circuit television and team teaching to help initiate School Mathematics Study Group materials into fourth and fifth grades. Teaching teams, including classroom teachers, a studio teacher, the mathematics consultant, and others, cooperatively plan and prepare instructional materials. The studio teacher telecasts part of the daily lesson. Each teacher prepares for the telecast and follows up with work in the classroom.

In 1958, the Texas State Board of Education appointed a commission on mathematics to study changes in mathematics and to make recommendation for improvement.[3]

In 1959, the production committee of the Illinois Curriculum Program published a bulletin, *Thinking in the Language of Mathematics*, the major purpose of which is to increase each student's ability to think with the ideas of mathematics. Six basic ideas are identified and learning experiences leading to their development are suggested.[4]

Georgia[5] is currently trying out a tentative course of study prepared under the direction of its State Department of Education. New Hampshire[6] is preparing a sourcebook for elementary curriculum development.

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[1] California State Department of Education. *Looking Ahead in Mathematics* (87 p.). and *Reports on Regional Conferences on Improving Mathematics Instruction in Elementary Schools* (85 p.). Sacramento, Calif., 1961

[2] George H. McMeen. First Report on the Work of the State Curriculum Commission's Advisory Committee on Mathematics.' *California Schools*, XXXII, No. 10 (October 1961).

[3] Texas Education Agency. *Texas Curriculum Studies*, Report No. 3. July 1959. 80 p.

[4] Illinois Curriculum Program. *Thinking in the Language of Mathematics*. Bulletin No. C-Two. 1959. 152 p.

[5] Georgia State Department of Education. *Mathematics for Georgia Schools*. Vol. 1. 1961. 303 p.

[6] New Hampshire State Department of Education. *Elementary Arithmetic Bulletin*. Concord, N.H. 1961. (Tentative)

The State of Illinois has developed a *Handbook for Elementary Mathematics Workshops* [7] for use in inservice education workshops throughout the State. Mathematics units on the following topics are covered: Numbers and Numerals, Numeration Systems, Sets, Operations of Addition and Multiplication, Operation of Subtraction, Operation of Division, Completion of Our Number System, and More on Number Structure. Subject-matter and explanatory material included in the handbook are intended to supplement lectures presented at workshop sessions and to stimulate discussion.

New York State has initiated the New York State Refresher Program which is centered in colleges and universities, with the State Education Department paying the student the amount of tuition for course work in elementary school mathematics. During the summers the State Education Department provides training for instructors who in turn offer courses in elementary school mathematics in different regions of the State. These courses carry local school system inservice credit.

Alabama, a pioneer State in the area of educational television, offers a weekly inservice program for teachers and two lessons per week for children in upper grades of the elementray school. A study guide for teachers is provided.

Contributions of Professional Organizations

The National Council of Teachers of Mathematics and the organization's official monthly publication for the elementary school teacher, *The Arithmetic Teacher*, give the professional elementary staff a continuous contact with new developments in elementary mathematics.[8] The national and regional meetings of the Council, the programs of its many affiliated groups, and its supplementary publications offer opportunities for stimulation and professional growth.

"Mathematics in the School" and "The New Mathematics" are titles of recent issues of *Educational Leadership* [9] and *AudioVisual Instruction*,[10] respectively. A reprint "A Mathematics Roundup" is also available from *Grade Teacher* magazine.[11] One section of *Updating Mathematics* [12] is devoted to topics of interest to the elementary school teacher.

[7] The Office of the Superintendent of Public Instruction. *Handbook for Elementary Mathematics Workshops*. Springfield, Ill., 1960. 128 p.

[8] National Council of Teachers of Mathematics, 1201 Sixteenth Street NW., Washington 6, D.C.

[9] *Educational Leadership*, March 1962. Association for Supervision and Curriculum Development, National Education Association, 1201 Sixteenth Street NW., Washington 6, D.C.

[10] *AudioVisual Instruction*, March 1962. Department of AudioVisual Instruction, National Education Association, 1201 Sixteenth Street NW., Washington 6, D.C.

[11] *Grade Teacher*, April 1962. Reprint Department, Leroy Ave., Darien, Conn.

[12] *Updating Mathematics*, Croft Educational Services, 100 Garfield Ave., New London, Conn.

Efforts of Local School Systems

Many local school systems throughout the country have participated in trying out experimental materials. Such participation is usually accompanied by seminars in which participants study the new content and acquire the necessary mathematical background for successful teaching.

Inservice programs at the local level include demonstrations and discussions of new methods and materials. These often serve as stimulus for a school staff to make a concentrated study of its elementary school mathematics program. Many local school systems, unable as yet to determine their direction with clarity, are producing leaflets, brochures, and booklets which serve a definite purpose at the time and which may later be incorporated into a guide or more permanent publication.

In the fall of 1961, the Department of Exact Sciences and the Television Teaching Program of the Detroit Public Schools offered a series of 16 half-hour arithmetic TV programs for teachers. The series was prepared by Dr. Phillip S. Jones, Professor of Mathematics and Education at the University of Michigan. The purpose of the series *Foundations of Arithmetic* was to assist teachers and administrators in developing or reinforcing their mathematical background. Lesson guides provided an outline of the presentation, references to supplementary readings, and exercises.

A television program for grades 4, 5, and 6, directed by Dr. Henry Van Engen, Professor of Mathematics and Education at the University of Wisconsin, has been viewed by more than 4,000 children in Madison, Milwaukee, and surrounding areas in Wisconsin during 1961–62. The programs were designed to offer teachers and children an opportunity to learn fundamental principles of mathematics together, to help teachers develop familiarity with new ideas, and to make learning mathematics a more enjoyable experience for children. Because of the vastness of our educational system, television offers a practical means of meeting these needs.

Twice weekly programs 15 minutes in length were followed up by regular classroom work during which time the ideas presented by TV were expanded and individual differences among children were met. Orientation sessions were held at 3-week intervals to give teachers a chance to evaluate past programs and accompanying materials, and to preview the work of forthcoming programs.

Cleveland Public Schools are experimenting with presenting arithmetic lessons by radio. Worksheets for children to use during the lesson are distributed by the Cleveland Board of Education. A teacher's guide provides information on the purpose of the lesson, steps to be taken before the broadcast, procedure while the broadcast is taking place, and followup work.

Some school systems are implementing their elementary school mathematics programs by the use of instructional materials designed to help children see relationships and patterns, and progress to abstract levels of thought in solving mathematical problems. One widely used type is a collection of rods based on a unit size such as a square inch or square centimeter. Other rods in the collection are multiples of the basic unit. In some variations the rods are grooved so that each unit is apparent; in others the differentiation in length is provided through the use of color. Another type presents variations of the abacus instructional materials designed to facilitate the teaching of the number system and place value, and to demonstrate a physical structure for the operations.

Contributions of Publishers

Publishing companies [13] are developing new programs in elementary school mathematics or are revising their regular textbook series to include the newer emphases. Some have made available supplementary materials, such as booklets on a variety of topics to enrich the regular program or to provide stimulating and challenging material for able learners. Materials for individualized work or independent study are also being offered.

Publishers often provide excellent service items which range from monographs or articles on elementary school mathematics to materials which can be used in the classroom.

[13] See Appendix C for list of publishers offering elementary school mathematics programs or materials.

Implications for Elementary School Personnel

The experimental projects described have demonstrated that the children who participated in the projects did learn the mathematics at the grade levels at which it was presented to them. These projects have served the Nation by adding some new dimensions and new directions to the elementary school mathematics program. They have helped teachers become aware of the need for background courses in mathematics and have stimulated them to request inservice courses or workshops. While the projects described are already having an observable effect, of particular significance is the long-range cumulative effect as materials are tried out on a broad scale by teachers from many sections of the country teaching children of varying abilities.

Questions of Concern

As school systems study the experimental programs for suggestions as to ways in which their own program may be improved, certain fundamental questions arise:

(1) Does the fact that content *can* be taught solve the problem as to whether it *should* be taught? Success in the experimental situation may or may not be sufficient evidence to justify inclusion of content in the mathematics program. Judgments must be made by local school systems on evidence of its value as consideration is given to:

(a) Present and future mathematical needs of children

(b) The value of the content from the standpoint of interest and motivation

(c) The extent to which it improves the acquisition of knowledges and skills which are deemed essential in the regular program

(d) Whether it offers methods and content of promise in promoting reasoning, thinking, and concept development.

(2) Does emphasis on a specified body of content represent the best expenditure of a child's time in terms of his mathematical growth and his general intellectual development at a given period of his school life? The answer to this question must be found as committees representing principals, supervisors, administrators, teachers, laymen, arithmetic specialists, mathematicians, and others concerned about the total growth of the elementary school child weigh the issues, resolve conflicts, and make decisions on action for the immediate

future with the understanding that continuous evaluation may indicate another direction at any time.

(3) Is the content appropriate for all children or only for the talented in mathematics? Certainly elementary schools should continue to seek better ways of differentiating the program for children of different abilities with content appropriate to their intellectual level. It may well be that some of the content from the experimental projects is more appropriate for the talented in mathematics than for all children. There is, however, some evidence that slower students profit significantly from these new approaches. In this connection it is important to remember that there may be a difference between the content which the adult views as difficult for children and the content which experience shows to be difficult. The adult is faced with the problem of changing old established patterns of thought and action. The greater flexibility of the child's mind and a smaller amount of accumulated experience make it less likely that blocking and serious interference with new learning will take place. Perhaps authors who prepare mathematics materials for children should reason with themselves thusly, "If it is worth while to get this idea across to children of this age and intellectual development, there must be a simple way to say it which will be clear now and which will not conflict with the precision which is desired for later work."

(4) Can continuous sequential development be provided once a new mathematical concept has been introduced? The introduction of new concepts, such as geometry in the primary grades, intuitive algebraic ideas, and factors and prime numbers requires organizational planning in advance, as well as careful diagnosis of students as they progress, to be assured that unprofitable repetition will be avoided and that gaps in learning will not occur. This also applies to such action as the decision to move into the work of the next grade level with either regular or new programs.

(5) Is it feasible to fuse a particular experimental program or selected ideas from several of them with the regular program?

Teachers learn about the experimental programs through summer study, inservice programs, and through their own reading. With encouragement from local leadership much experimentation is going on. Some of the projects can only be used widely as fused with regular programs because they are not designed as complete programs. Others are set up on a unit basis making it possible for the teacher to select units as supplementary work, as a substitute for a body of content in the regular program, or as a means of reinforcing learning in a given area. The fusion of a particular experimental program with a regular program, or the fusion of any two programs requires considerable organizational planning to insure desirable continuity.

(6) How can inservice opportunities be made available to teachers so that wide participation in program planning and implementation is possible? Personnel from the experimental projects and programs have been generous with their time. They have visited school systems upon request, given demonstrations with children, and held discussions with school personnel. They invite visitation to their experimental centers.

Colleges and universities are cooperating with local school systems to provide courses which include new trends in elementary mathematics and which will meet the raised certification requirements recently adopted by many State departments. By such means they are trying to narrow the gap between the increasing rate at which new mathematics is being created and applied and the mathematics the teacher acquired during his own schooling and preparation for teaching.

Scholarships are being made available in increasing numbers for the attendance of elementary school personnel at State and national institutes, workshops, and conferences to improve their understanding of mathematics. Local school systems are providing their own inservice programs with similar goals and purposes in mind. If the teacher has an excellent grasp of a concept, there is a good chance that a way can be found to present it to children of different ages at their level of understanding. Those responsible for inservice programs for elementary school teachers need to consider whether provision is being made for all teachers to learn the structure of mathematics to the extent that they can appreciate the importance of any part of the mathematics curriculum to the total program.

(7) What plans can be devised for continuous evaluation of the mathematics program as a part of the total school program?

Standardized tests will continue to be useful in measuring some aspects of any mathematics program. In particular, these tests measure certain types of computational skills which are included in the objectives of all mathematics programs. Teachers and school staffs will be able to measure subjectively such important objectives as the child's ability to apply mathematics to other situations and his attitudes toward and interest in mathematical experiences. Improved techniques of measurement will make it possible to evaluate more completely the results of the newer projects.

Conflicting Issues

School personnel are also concerned about certain influences and directions in the newer mathematics programs which appear to conflict with each other or to go in a direction counter to previously held beliefs. On the one hand there is the trend toward teaching mathematics so that large patterns or basic structures will be apparent to the learner. On the other hand the content of mathematics is being broken down into smaller and finer bits as material is adapted for teaching machines or for programed booklets. Are these necessarily in conflict? It depends on the purposes, the children concerned, on the timing of the two very different types of experiences, and on the quality of the content. The former emphasizes group interaction along with attention to individual thought processes and to new leads obtained from previous responses. The latter provides individualized learning with the opportunity for the child to know at any moment how well he is doing, a feat which is next to impossible in the regular classroom. There is the problem, however, of "keying children in" at appropriate portions of material so that their time will be profitably spent and not wasted on already well-learned material.

Another conflict seems to exist between the ideas formerly held that learning proceeds from the simple to the complex, or from concrete to abstract and the tendency to introduce precise mathematical language at an early age. Professor Bruner expresses the concern of many when he warns that enthusiasts must guard against ". . . the premature use of the language of mathematics and its end-product formalism, that makes it seem that mathematics is something new rather than something the child already knows."[1] In 1935, Dr. Brownell recognized and voiced a danger in teaching skills to children in exactly the same form in which they are used by the adult.[2] He warned against inferring methods of instruction from adult usage. Because the adult uses shortcuts in mathematics is no reason to assume that shortcuts represent the best techniques for teaching a process to children in the early stages of learning. Cannot the same question be raised with regard to the tendency to introduce adult language patterns of mathematics at an early age? Again long-range study is needed to determine if real understanding for these terms is maintained over a period of time so that they can serve a useful purpose in future learning of the child. Further experimentation and study of the language of mathematics may lead to the conclusion that the development of mathematical language follows the same developmental process of progressing from immature to mature levels as is accepted for other mathematical learnings. Other conflicts relate to the role of experience in learning. For example, mathematics by television is usually a sedentary type of learning. It is only as the

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[1] Jerome S. Bruner. On Learning Mathematics. *The Mathematics Teacher*, 53: 614, December 1960.

[2] W. A. Brownell. Psychological Consideration in the Learning and the Teaching of Arithmetic. *National Council of Teachers of Mathematics: Teaching of Arithmetic*. p. 1-31. National Council of Teachers of Mathematics, 1201 Sixteenth Street, NW., Washington, D.C., 1935. 289 p.

classroom teacher provides introductory and follow-up experiences involving interaction among children and between teacher and children that learning becomes an active process. This fact leads many educators to believe that one of the greatest values to be gained from television is inservice education for the teacher. Some television programs, such as the Washington County, Md., Project have been designed with the dual purpose of providing help for the teacher and the child at the same time. Arithmetic was considered one of the areas of neglect in the elementary school curriculum in Washington County when teaching by TV was undertaken. Results indicate that "after seven and a half months of teaching arithmetic to fifth-grade children, 18 classrooms reported a median advance corresponding to 17 months, with children of an IQ of 90 corresponding to 14 months."[3] Results on the Iowa Test of Basic Skills "obtained over several years' time showed striking progress in arithmetic learning in the television classes at all levels of ability in comparison with the period before television was used, as well as in relation to both non-television students and the standard national norms."[4]

In general, the emphasis in the newer projects and programs appears to be accompanied by a tendency to minimize the applications of mathematics. Granted that children need to learn some mathematics because of its contribution to structure, it is perhaps equally important that they recognize the direct application of mathematics to their daily lives. One major task of the teacher is to help them gain greater competence in meeting real life problems. It is doubtful that daily living affords a sufficient number of similar elements that this entire problem can be relegated to transfer.

The present controversy over whether experiences with mathematical content are social studies, science, or some other subject area should best be forgotten in the interest of developing a competent, elementary school child capable of dealing with mathematical situations wherever they occur. Professor Weaver sums up the situation, voicing both confidence and caution, when he says, "Let us do all that we reasonably can to improve the elementary school mathematics program along those lines where improvement may be needed. Let us make sure that we do not end up with a doubtful or even a poor bargain, gaining some things at the expense of others of acknowledged importance. Our safeguard is to be found in the planning and implementation of improvement efforts in full cognizance of *all* relevant instructional objectives."[5]

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[3] Henry H. Cassirer. Television Teaching Today. UNESCO, 1960. p. 36.

[4] *Teaching by Television.* A Report from the Ford Foundation and the Fund for the Advancement of Education, New York 22: 477 Madison Ave., 1961. p. 40.

[5] J. Fred Weaver. Basic Considerations in the Improvement of Elementary School Mathematics Programs. *The Arithmetic Teacher,* 7:270, October 1960.

Future Direction

The critical evaluation questions asked and the conflicting issues cited should not leave the impression that the status quo is adequate. Among several definite steps that any school system may take to initiate improvements are the following:

1. Examine its present elementary school mathematics program critically.
2. Provide sample sets of materials from experimental programs for staff study.
3. Encourage a few teacher-volunteers to try out one of the experimental programs and to evaluate the results.
4. Encourage teachers to try units or parts of experimental programs as supplements to the regular work in mathematics.
5. Encourage all teachers to begin to incorporate those aspects of the newer mathematics which are appropriate for the regular program.
6. Provide inservice opportunities for teachers to build background in mathematics and to improve their teaching methods.
7. Arrange for a demonstration of some aspect of the newer materials with children in the local school system.

APPENDIX A. Selected Bibliography

THE FOLLOWING TITLES of professional books, pamphlets, and magazines are selected because of the contribution they can make to the elementary classroom teacher who wishes to build a background of understanding for the changes in elementary school mathematics.

Analysis of Research in the Teaching of Mathematics. U.S. Department of Health, Education, and Welfare, Office of Education, Bulletin 1960 No. 8. 50 p.

BANKS, J. HOUSTON, *Learning and Teaching Arithmetic.* Boston: Allyn and Bacon, 1959. 405 p.

BELL, CLIFFORD; HAMMOND, CLELA D., and HERRERA, ROBERT R., *Fundamentals of Arithmetic for Teachers.* New York: John Wiley and Sons. 1962. 389 p.

BRUMFIEL, CHARLES F.; EICHOLZ, ROBERT E.; *and* SHANKS, MERRILL E., *Fundamental Concepts of Elementary Mathematics.* Reading, Mass.: Addison-Wesley Publishing Co., 1962. 340 p.

Educational Leadership. Association for Supervision and Curriculum Development, National Education Association. 1201 16th Street NW., Washington 6, D.C. March 1962.

Evaluation in Mathematics. Twenty-sixth Yearbook, National Council of Teachers of Mathematics, 1201 16th Street. NW., Washington 6, D.C., 1961. 216 p.

Frontiers in Mathematics Education. Department of Public Instruction, Lansing, Mich., Bulletin No. NDEA-310, 1961. 22 p.

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2. Invitation to Mathematics.
3. Understanding Numeration Systems.
4. Fun With Mathematics.
5. Number Patterns.
6. The World of Measurement.
7. Adventures in Graphing.

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APPENDIX B. A Flow Chart of Basic Mathematical Ideas[1]

MANY OF THE BASIC MATHEMATICAL CONCEPTS in this book are not confined to the classroom but occur throughout life. In some cases they are first met by the preschool child and then strengthened throughout his educational career. Some of these ideas, with examples of how they appear at different educational levels, are listed below. It is assumed that the teacher will think of many more instances where each of the concepts may be stressed. In fact, amplification of this chart might be a good teaching project for a series of working-discussion meetings of the mathematics staff of a school system. It is such spiral teaching that nurtures the full development of the idea within the pupil.

Idea 1. New numbers are invented by men and defined in terms of old numbers to do new jobs. Some of these jobs are to make the operations of division, subtraction, and extraction of roots always possible, to make linear and quadratic equations always solvable, or to represent geometric, physical, or other quantities such as diagonals of squares, temperatures "below zero," and losses (see chapter 2).

Grades K to 3. Children learn to use integers in counting concrete objects and in measuring. Seeing a collection of three balls, they not only identify the objects as balls but they also recognize threeness.

Grades 4 to 6. Children learn to use a pair of integers (a rational number) to describe a part of a unit. If □ represents one unit, $\frac{3}{4}$ is used to represent a portion of a unit. A pair of integers makes it possible to express the quotient of all division problems as a number. Thus $10 \div 3 = 10/3$, and $3x = 10$ now has a solution.

Idea 2. Each time a new number is invented the rules of operations with it must be defined. This is usually done to preserve the commutative, associative, and distributive principles of operation which applied to the natural numbers (see chapter 2).

Grades K to 3. Children learn that $3+4$ and $4+3$ give the same results; that $(2+3)+5$ is the same as $2+(3+5)$; that is, addition can be done in any order. In multiplication we may interchange the multiplier and the multiplicand; for example, $3 \times 4 = 4 \times 3$.

Grades 4 to 6. The associative and distributive laws are the essential ideas upon which are based "carrying" and "borrowing" in column addition and in subtraction. The commutative law is also used when one checks addition downward by adding upward.

The associative and commutative principles are extended to fractions. The commutative principle is used to show that $\frac{2}{3} \times 3 = \frac{6}{3}$. The problem can be written as $3 \times \frac{2}{3} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$ or $\frac{6}{3}$.

Illustrations of the use of the distributive law are in dealing with denominate numbers, 2×3 lb. 6 oz. = 6 lb. 12 oz.; in multiplying numbers written in decimal numeration $3 \times 24 = 3 \times (20+4) = 3 \times 20 + 3 \times 4$.

[1] Limited to material presented for elementary grades, p. 480–498. *The Growth of Mathematical Ideas K-12*, Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics, 1201 16th Street NW., Washington 6, D.C., 1959, 507 p.

Idea 3. Standard units of measurement are arbitrarily chosen, although the units may have arisen historically. Students should understand the nature and variety of standard units of measurement as well as the relation between units of the same measure. Pupils must have many experiences with physical examples of standard units to understand and appreciate them (see chapter 5).

Grades K to 3. Linear units: inch, foot, yard; liquid units (units of volume): cup, pint, quart, gallon; weight units: ounce, pound.

Some measurements of length should be made with inch rulers which contain no other units on the scales in order to emphasize the inch as a unit of measurement. Students can make their own units and use these to make measurement. From such experiences, pupils can see the need for standard units to communicate ideas. Pupils should be given opportunities to estimate and then check by measuring. They can be encouraged to compare parts of the body with standard linear units; then pupils can use hand span, thumb width, and arm length to make estimates.

Experimentation with measuring cups, pints, quarts, and gallons helps pupils see relationships between units.

Grades 4 to 6. Linear units: rod, mile; also such metric units as centimeter, meter; weight units: ton; area units: square inch, square foot, square yard, square centimeter, square meter, square rod, acre, square mile; volume units: cubic inch, cubic foot, cubic yard, cubic centimeter.

Give experience with measuring some quantity with different units to understand relationships between units. Pupils should be led to see which unit is more appropriate for measuring quantities of different sizes; e.g., miles to measure distance between two cities, inches to measure width of paper.

Classrooms should contain many standard units to give opportunity for pupils to become familiar with them. Estimates of measurements should be an integral part of the instructional program, e.g., stepping off the length of a room, estimating areas, and so on.

Idea 4. Estimates of results of computations are used as a check on computations and as a practical tool for everyday living. Such estimates may be done prior to the paper-pencil work or after it. The main purpose of estimates of results of computations is to help pupils deal with numbers sensibly (see chapter 5).

Grades K to 3. Estimating before computation: 31÷3. Result will be about 10.

$$\begin{array}{r} 10 \\ 3\overline{)31} \\ \underline{30} \\ 1 \end{array}$$

Estimating after computation: Does 10 seem reasonable?

Grades 4 to 6. Example: 76×84. My estimate is 80×80=6400 before computation. After computation: Does 6384 seem like a reasonable result?

Example:

$$\begin{array}{r} 3.72 \\ \underline{\times 4.6} \\ 2232 \\ \underline{1488} \\ 17112 \end{array}$$

The estimate should be in the interval 12 to 20. Where should the decimal point be placed in the result?

Idea 5. The ability to estimate is refined where pupils are able to locate two numbers such that the result is between them. Some pupils will be able to locate a small interval while other pupils will be able to give a large one (see chapter 5).

Grades K to 3. Exercises are needed to develop the pupils' ability to estimate such exercises as, $53+32$ is between 80 and 90.

Grades 4 to 6. On this grade level, the type of exercises might be: The sum of 53/16 and 45/12 is between 6 and 8. Some pupils will recognize that it is between 7 and 8. The product of 47×62 is between 2400 and 3500.

Idea 6. Number is an abstraction (see chapter 2).

Grades K to 3. A cardinal number such as 3 represents the common property of all groups with three elements, such as 3 dogs, 3 marbles, 3 pennies, and so on.

Grades 4 to 6. Fractions developed out of work with and talk of parts of things, as bushels of grain and pieces of candy, but they may be used to represent ratios and averages, and so on, i.e., in situations where no parts are involved. They may be treated logically as pure abstractions.

Idea 7. Mathematics is a kind of "if . . . then" reasoning in which the "then's" are supported essentially by definitions, assumptions, and theorems which have been proved previously (see chapter 4).

Grades K to 3. Two youngsters were checking their answers to the example: $9+7$. "Say, John, I figure that if $7+2=9$ and $7+7=14$, then $7+9=7+(7+2)=(7+7)+2=14+2=16$."

"That's the same answer I got," replied John, "but, I did mine a different way. I said if $7=1+6$, then $9+7=9+(1+6)=(9+1)+6=10+6=16$." Note the chains of logical reasoning in the way in which each of these students reached his conclusion.

Grades 4 to 6. "Hey, David, how many yards are there in a mile?" "Well, I don't know, but I think we can find out easily enough. If there are 5280 feet in one mile and there are three feet in each yard, then $5280\div3$ will tell us how many yards there are in one mile. If I have divided correctly, then there are 1760 yards in one mile."

Idea 8. Addition and multiplication are the basic operations in terms of which the inverse operations, subtraction and division, are defined (see chapter 2).

Grades K to 3. Subtraction is taught as the inverse of addition and thus the facts are taught together.

Grades 4 to 6. Multiplication-division facts are taught together. Checking subtraction by addition and division by multiplication are taught not merely as interesting devices but as an application of basic principles of inverse operations.

Idea 9. Numbers and their properties are the same, irrespective of the system of numeration; but our system of numeration with the base 10 and place value together with the associative, commutative, and distributive principles dictates the operation algorithms in arithmetic (see chapter 2).

Grades K to 3. Children learn that we group by 10. 10 ones make 1 ten. 10 tens make 1 hundred. We add and subtract tens and hundreds just like ones. The position in which we place a digit indicates whether it is ones, tens, or hundreds. The digit tells how many ones, tens, or hundreds. To facilitate operation we group to make a larger unit whenever possible in addition. And we regroup a larger unit to smaller units for purposes of subtraction.

Grades 4 to 6. The system of numeration is extended to thousands, to ten thousands, and so on, and later to parts of one. Place value is used to designate the denominator (10, 100, 1000, and so on) of the fractional part of the unit. We multiply and divide tens, hundreds, thousands, and so on. With common fractions we also build on an idea of grouping to make a unit of one, and separating one not only into 10 parts but also into halves, fourths, eighths, and so on.

Idea 10. As new numbers are invented, we must each time define what is meant by equality as well as by addition and multiplication. This leads to the fact that the same number may have many different names or representations (see chapter 2).

Grades K to 3. Each whole number may be the sum or difference of many pairs of integers; thus, $5=4+1=3+2=7-2=8-3$, and so on.

Grades 4 to 6. Fractions a/b and c/d may be defined to be equal if $ad=bc$. A logical proof can be developed later of the inductively derived fact that a fraction is equal to any new fraction derived by multiplying or dividing its numerator and denominator by the same number.

This leads to the many names for $\frac{1}{2}$, such as $\frac{2}{4}$, $\frac{5}{10}$, .5, and so on, as well as to the scheme for shewing that the integers correspond to certain special fractions (i.e., the integers are isomorphic to a subset of the rationals), thus $2=\frac{2}{1}=\frac{4}{2}$; $3=\frac{3}{1}=\frac{6}{2}$, and so on.

Idea 11. Numbers obtained by applying to quantities measuring instruments, such as rulers, scales, and protractors, represent approximations of the quantity. The numbers themselves are not approximate (see chapter 4).

Grades K to 3. Measurement to the nearest inch by using a ruler with inch marks only. The length of the paper may be nearer to 11 inches than 12 inches or 10 inches; students should see that one cannot measure exactly 11 inches. This leads to the idea that measurement is approximate.

Grades 4 to 6. Making measurements to the nearest foot, nearest inch, nearest half inch, and so on, to teach the idea of precision of measurement. Lead to the idea that precision of measurement is determined by the unit—the smaller the unit, the more precise is the measurement. But the recorded number still is an approximation of the quantity.

Begin the idea that, with each recorded measurement, the unit of measurement should be stated. When it is not stated, the unit must be inferred from the number. For example, a recorded measurement of 7.4 ft. probably indicates that the unit is one tenth of a foot.

Idea 12. Computation with numbers which represent approximations should be done with due consideration for that fact. The same computation procedures are followed for numbers arising from measurement, rounded numbers, and rational approximations for rational and irrational numbers (see chapter 5).

Grades K to 3. One child may make a flower bed 3 sticks long; another child may extend the flower bed 2 sticks. The entire bed is only about (not exactly) 5 sticks long.

Grades 4 to 6. The result of computations with numbers which are approximations should be rounded to the proper degree of precision. For example, the sum of 3.6 ft. + 4.73 ft. + 8.28 ft. should be rounded to 16.6 ft. Pupils probably are not ready to understand fully the general procedures for rounding results and will need to rely on directions from the teacher.

Idea 13. In algebra a variable is a symbol for which one substitutes the names of numbers (see chapter 3).

Grades K to 3. Elementary school children are familiar with the expressions 3+?=5 and 3+2=? A symbol has been used for which the name of a number may be substituted. In the sentence "The number of containers of milk for our class today is?" a symbol has been used for which a numeral may be substituted.

Grades 4 to 6. The following exercises from a sixth-grade arithmetic textbook indicate the occurrence of the concept of variable.

If $N \times 17 = 595$, tell what number N stands for. 4 bu. = ? pecks.

Idea 14. Variables in mathematics play a role similar to the role of pronouns in the English language (see chapter 8).

Grades K to 3. In place of the sentence, His score is always the same as her score, one could write J = M.

Grades 4 to 6. Instead of writing that the value of a fraction is not changed when its numerator and denominator are multiplied by the same number (except 0), the child may write:

$$\frac{a}{b} = \frac{ac}{bc} \; (c \neq 0).$$

Idea 15. A name of a thing is different from the thing (see chapter 8).

Grades K to 3. Names of children are on the blackboard, but the children are seated at their desks.

Grades 4 to 6. The symbol ¾ is larger than the symbol 3 but $\frac{3}{4} < 3$.

Idea 16. We may represent numbers geometrically showing some as lengths, some as points, and some as areas (see chapter 2).

Grades K to 3. By choosing a unit length and extending it by multiples of its own length, children learn sequence of counting numbers on a number line. The use of zero to mark an initial point on the number line introduces children to the idea of an enlarged set of numbers, the set of all integers.

Grades 4 to 6. Children learn to divide each of the segments of unit length into n equal parts, that is into ½'s, ⅓'s, ¼'s.

Idea 17. The language of mathematics has a grammar of its own just as the English language does (see chapter 8).

Grades K to 6. "2+3=5" is a complete mathematical sentence. Its structure is similar to that of the sentence, "The daughter of John and Mary is Susie."

Idea 18. A mathematical definition is a statement which gives us a set of symbols as a replacement for another set of symbols (see chapter 8).

Grades K to 3. Instead of saying three-tens, we say thirty.

Grades 4 to 6. 4×3 is defined as 3+3+3+3.

Idea 19. To indicate that two expressions name the same thing, we use the symbol "=" (see chapter 8).

Grades K to 3. To say that "2+1" and "3" are names of the same number, the child writes:

2+1=3.

Grades 4 to 6. To say that "½" and "50%" are names of the same number, the child writes:

½=50%.

Idea 20. You can see a numeral, but you cannot see a number (see chapter 8).

Grades K to 3. Mary wrote the numeral "3" which is bigger than the numeral "3" which Susie wrote, but they name the same number.

Grades 4 to 6. The Roman numeral "VII" and the Hindu-Arabic numeral "7" name the same number.

Idea 21. Recognition of the order relations for numbers is fundamental to estimates of computations (see chapter 5).

Grades K to 3. The early ideas of order involve only bigger and smaller related to physical size. As a child counts by ones, he begins to learn the order relation for integers. For example, 8 is more than 7, 6, 5, 4, 3, 2, or 1 and 8 is less than 9, 10, 11, 12, ... The idea is extended when he learns place value and sees that 40 is more than 30 and that 300 is less than 400. The order relation for fractions begins when children learn that $\frac{1}{2}$ is more than $\frac{1}{3}$ and that $\frac{1}{5}$ is less than $\frac{1}{4}$.

The order relation is used in estimations like these: Is $3+5$ more than or less than $3+3$? Is $8-6$ more or less than $8-4$? Is 3×4 more than or less than 3×5? Is $31\div3$ more than or less than 10?

Grades 4 to 6. As the concept of place value is extended to numbers with many digits, including decimals, the concept of order relation is also extended to examples like: 82,000 is less than 92,000; 1,000,000 is less than 10,000,000; 0.52 is less than .62. Ordering ideas as related to common fractions is extended. Children learn that $\frac{3}{5}$ is less than $\frac{3}{4}$, and generalize that, if the numerators of two fractions are the same, the one with the larger denominator denotes the smaller number. Likewise, pupils see that $\frac{7}{8}$ is more than $\frac{5}{8}$ and generalize that if the denominators of two fractions are the same the one with the larger numerator is the larger number.

Estimations such as these are appropriate: Is $310\div3$ more than or less than 100? Estimate the product 216×370. Estimate the sum $5\frac{9}{10}+4\frac{11}{12}$. Is $647-247$ more than or less than $842-247$?

Idea 22. In operating with new numbers, the former laws of addition, multiplication, order, and commutativity are usually preserved; but some of them may be changed in order to achieve some other desired advantages (see chapter 2).

Grades K to 3. Children learn to order numbers: 3 comes after 2 and before 4; 32 comes before 33 and after 31.

Grades 4 to 6. We order fractions but first we have to think of them as having the same common denominator. Then we order numerators as we do with the natural numbers. If all the numerators are the same, then we can order from smallest to largest by denominators, e.g., $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$.

$3\times\frac{3}{4}=\frac{3}{4}+\frac{3}{4}+\frac{3}{4}=\frac{9}{4}$, but the idea that multiplication is repeated addition, though it can still give us useful hints, is no longer logically sound when the multiplier is a fraction as with $\frac{2}{3}\times\frac{3}{4}$.

Idea 23. Sets of ordered pairs of numbers are an important concept in mathematics, and this concept is frequently used on all grade levels (see chapter 3).

Grades K to 3. When a child thinks one candy bar costs 5 cents, two candy bars cost 10 cents, and so on, he is thinking in terms of the ordered pairs (1, 5), (2, 10), and so on.

Grades 4 to 6. Fractions are sets of ordered pairs of numbers; for example, $\frac{2}{3}$ is the ordered pair (2, 3) and $\frac{4}{6}$ is the ordered pair (4, 6). In fact, both ordered pairs are the same number.

Idea 24. A conclusion is established by logical deductions from one or more statements called reasons (see chapter 4).

Grades K to 3. A child in a room of 32 pupils counts the number present on Friday to be 26. He then writes on the blackboard $32-26=6$ and says, "Six people are absent today." When another student questions the accuracy of his subtraction, the youngster replies, "Well, $26+6=32$." (This child is using a definition of subtraction intuitively.)

Grades 4 to 6. Mary adds a column of eight 9's and concludes that $8\times9=72$. She then declares, "If $8\times9=72$, then $9\times8=72$ and $72\div9=8$, and $72\div8=9$."

Idea 25. Induction doesn't "prove," it makes conclusions probable (see chapter 4).

Grades K to 3. Mary, after her first day at school said, "Mommie, I talked to every teacher at school. Do you know all teachers are women?" Her conclusion, by induction, was based upon a few cases. A larger number of cases would increase the probability of the validity of a conclusion.

Grades 4 to 6. John was developing skill in multiplying whole numbers and he remarked, "You always get an answer bigger than the other two numbers, except when one of the numbers is zero or one."

Idea 26. We often seek to establish the truth of a statement called a conclusion by arguing that it follows from one or more other statements called reasons that are only partial evidence of the truth of the conclusion (see chapter 4).

Grades K to 3. The pupils were discussing summer vacation plans. "How long does it take you to get to your grandmother's house?" one boy asked.

"About 3 hours. My grandmother lives on a farm."

"Gee, your grandmother must live three times as far away as mine! It only takes us one hour to go to Grandmother Miller's."

Grades 4 to 6. Using circle devices which they had made from pie plates, two fifth graders were attempting to find the answer to $\frac{1}{4}+\frac{2}{3}$. After considerable manipulation they concluded that the answer was less than one and that it probably was $\frac{11}{12}$.

Idea 27. To establish a conclusion, necessary inference may be used in the argument (see chapter 4).

Grades K to 3. Ann supported her statement that $7+8=15$ by this reasoning: "Well, 8 is the same as $7+1$, and 7 and 7 are 14 and 1 more makes 15. Do you see?"

Grades 4 to 6. George was adding $2\frac{3}{4}$ to $7\frac{1}{3}$. His reasoning might be outlined as follows: $2\frac{3}{4}+7\frac{1}{3}=(2+7)+(\frac{3}{4}+\frac{1}{3})=9+1\frac{1}{12}=10\frac{1}{12}$. In this same exercise Kathy proceeded in this manner: $2\frac{3}{4}+7\frac{1}{3}=2\frac{3}{4}+7+\frac{1}{4}+\frac{1}{12}=(2\frac{3}{4}+\frac{1}{4})+7+\frac{1}{12}=3+7+\frac{1}{12}=10\frac{1}{12}$.

Idea 28. To establish a conclusion, probable inference may be used in the argument (see chapters 4 and 6).

Grades K to 3. Two children are arguing over their answers to the problem $28¢+7¢=?$ One of the youngsters supports the answer 98¢ (adding the 7 and the 2) and the other one staunchly maintains that the answer is 35¢. Finally, after neither of them is able to convince the other, one declares, "Well, my daddy said that the answer is 35¢." (Reference to authorities can be very important in many arguments, though the authorities may at times be wrong.)

Grades 4 to 6. John had forgotten whether the divisor or the dividend was to be turned upside down when he was dividing common fractions. In finding the solution to $14\frac{5}{8} \div 3\frac{1}{4}$, John decided to work the problem both ways. His first attempt was to multiply $\frac{8}{117} \times \frac{13}{4}$ which he found to be $\frac{2}{9}$. Using his second method, he multiplied $\frac{117}{8} \times \frac{4}{13}$ and his answer was $4\frac{1}{2}$. It was then that he noticed that $14 \div 3 = 4+$ and $15 \div 3 = 5$. As a result of this estimation he decided that his first answer was incorrect, and $4\frac{1}{2}$ was the correct answer since he could find no mistakes in multiplication. From this experimental approach, John correctly inferred that in dividing fractions the dividend should be multiplied by the reciprocal of the divisor.

Idea 29. The inferences of statistics are inductions which are not certainties and must be given in terms of probability (see chapter 7).

Grades K to 3. If all the children in John's room are 8 years old, should he conclude that all the children in his grade are 8 years old?

Grades 4 to 6. If it rained each day last week, will it rain tomorrow?

Idea 30. Data from samples should be organized and summarized if they are to be analyzed for inferential purposes (see chapter 7).

Grades K to 3. If a first-grade pupil needs to estimate the number of containers of milk needed for the recess period, a study of the number used for the last few days will be helpful. Organizing figures in a numerical order often helps children to see relationships.

Grades 4 to 6. The importance of organization may be emphasized in constructing bar graphs from collected data.

Idea 31. Decisions about a whole population can be obtained most efficiently and economically by examining only a small part selected by a process based on the principles of probability (see chapter 7).

Grades K to 3. A child may count the students with blue eyes in one room and estimate the number in the whole school. A child may conclude whether peaches in a basket are ripe by feeling only one or two peaches.

Grades 4 to 6. A pupil may determine the average height of the boys in the room by measuring only a few. He might find those in the front seats an unsatisfactory sample.

Idea 32. Unless the sampling process makes it possible for each sample to have an equal chance of being drawn, inferences based on the sample may not be statistically valid (see chapter 7).

Grades K to 3. If it is desired to find the average weight of a second grade child, should we weigh only the fat children?

Grades 4 to 6. John was practicing on his most difficult division facts. Out of twenty of these facts, he correctly answered ten. Mary said John knew half of all the division facts. Was she correct?

APPENDIX C. Publishers of Commercial Arithmetic Textbooks

THIS PARTIAL LIST of publishers of commercial arithmetic textbooks is compiled from *Textbooks in Print 1962*, R. R. Bowker Co., 62 West 45th Street, New York. Other publishers known to have issued textbooks during the past year are also included. The list is provided for the convenience of the reader.

Addison-Wesley Publishing Co.
Reading, Mass.

Allyn and Bacon, Inc.
160 Tremont St.
Boston 11, Mass.

American Book Company
55 Fifth Ave.
New York 3, N.Y.

Arithmetic Clinic
4402 Stanford St.
Chevy Chase, Md.

Cuisenaire Co. of America, Inc.
246 East 46th St.
New York 17, N.Y.

Encyclopedia Brittanica Press
425 North Michigan Ave.
Chicago 1, Ill.

Ginn and Company
Statler Office Bldg.
Boston 17, Mass.

Grosset and Dunlap, Inc.
1107 Broadway
New York 10, N.Y.

E. M. Hale and Co.
1201 South Hastings Way
Eau Claire, Wis.

Harcourt, Brace, and World
750 Third Ave.
New York 17, N.Y.

D. C. Heath and Company
285 Columbus Ave.
Boston 16, Mass.

Holden Day, Inc.
728 Montgomery St.
San Francisco 11, Calif.

Holt, Rinehart, and Winston, Inc.
383 Madison Ave.
New York 17, N.Y.

Houghton Mifflin Company
2 Park St.
Boston 7, Mass.

Iroquois Publishing Company
1300 Alum Creek Drive
Columbus 16, Ohio

Laidlaw Bros.
Thatcher & Madison Sts.
River Forest, Ill.

J. B. Lippincott Company
E. Washington Square
Philadelphia 5, Pa.

Lyons & Carnahan
2500 Prairie Ave.
Chicago 16, Ill.

Macmillan Company
60 Fifth Ave.
New York 11, N.Y.

McCormick-Mathers Publishing Company
Box 2212
1440 E. English St.
Wichita 1, Kans.

Charles E. Merrill
1300 Alum Creek Dr.
Columbus 16, Ohio

New American Library of World Literature, Inc.
501 Madison Ave.
New York 22, N.Y.

Noble & Noble Publishers, Inc.
67 Irving Pl.
New York 3, N.Y.

Pitman Publishing Corp.
2 W. 45 St.
New York 36, N.Y.

Prentice-Hall, Inc.
Englewood Cliffs, N.J.

Rand McNally & Company
P. O. Box 7600
Chicago 80, Ill.

Row-Peterson & Company
2500 Crawford Ave.
Evanston, Ill.

Science Research Associates
259 East Erie St.
Chicago 11, Ill.

Scott, Foresman and Company
433 E. Erie St.
Chicago 11, Ill.

Charles Scribner's Sons
597 Fifth Ave.
New York 17, N.Y.

Silver Burdett Company
Morristown, N.J.

L. W. Singer Co., Inc.
249–259 W. Erie Blvd.
Syracuse 2, N.Y.

Steck Company
P.O. Box 16
Austin 61, Tex.

University Publishing Company
1126 Q St.
Lincoln 1, Nebr.

Vromans
367 South Pasadena Ave.
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