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GRADUATE WORK IN MATHEMATICS IN UNIVERSITIES AND IN OTHER INSTITUTIONS OF LIKE GRADE IN THE UNITED STATES

INTERNATIONAL COMMISSION ON THE TEACHING OF MATHEMATICS

THE AMERICAN REPORT

COMMITTEE No. XII

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LETTER OF TRANSMITTAL.

DEPARTMENT OF THE INTERIOR,
BUREAU OF EDUCATION,
Washington, April 14, 1911.

SIR: At the Fourth International Congress of Mathematicians, held at Rome in April, 1908, an international commission was organized to make a general study of the progress of mathematical instruction in the various countries. The formal resolution adopted by the congress was as follows:

> The congress, recognizing the importance of a comparative examination of the methods and plans of study of the instruction in the mathematics in the secondary schools of the different nations, empowers Messrs. Klein, Greenhill, and Fehr to form an international commission to study these questions and present a general report to the next congress.

The committee of organization appointed members of the general commission, representing the various nations, and the following general officers were elected: President, Prof. F. Klein, Geh. Reg.-Rat., Göttingen; Vice President, Sir George Greenhill, F. R. S., London; Secretary, Prof. H. Fehr, Geneva. The commission soon found that it was necessary to extend the scope of the investigation to include the teaching of mathematics in higher institutions and elementary schools, thus covering the entire field of instruction in this branch of learning, and this investigation is now proceeding simultaneously in all of the leading countries of the world.

The commissioners appointed to represent the United States are Prof. David Eugene Smith, of Teachers College, Columbia University; Prof. William F. Osgood, of Harvard University; and Prof. J. W. A. Young, of the University of Chicago. This commission has been enabled to carry on their investigation by the financial support of several gentlemen who are interested in the subject and of a number of our leading institutions of learning. The Bureau of Education has assisted the commission by securing information from various institutions in the United States. Nearly 300 individuals who are engaged in giving instruction in mathematics have cooperated in preparing reports setting forth the present status of the subject in this country. I have the honor to transmit herewith the first of these reports, entitled "Graduate work in mathematics in universities and in other institutions of like grade in the United States," and to recommend its publication as a number of the Bulletin of the Bureau of Education.

Very respectfully,

ELMER ELLSWORTH BROWN,
Commissioner.

THE SECRETARY OF THE INTERIOR.

GRADUATE WORK IN MATHEMATICS IN UNIVERSITIES AND IN OTHER INSTITUTIONS OF LIKE GRADE IN THE UNITED STATES.

GENERAL REPORT.[1]

1. THE ESTABLISHMENT OF ADVANCED INSTRUCTION IN THE UNITED STATES.

Forty years ago the bachelor's degree granted on the completion of a four years' course of a general character marked not merely the close of a young man's liberal education, but also, except in the case of some lawyers, ministers, and physicians, the end of all academic instruction of any kind. In particular, apart from a few exceptional cases, no advanced instruction in mathematics was anywhere provided beyond the usually rather meager ingredients—hardly more than analytic geometry and a little calculus—of this college course, which consisted mainly of prescribed studies. As an external sign of this state of affairs we note that the master's degree, where it existed, was conferred for reasons having very little to do with study, while the doctor's degree was practically nonexistent.[2] The desire for higher education in America, which had been felt for many years by some of the leading minds of the country, had been able so far to achieve only momentary and sporadic success.

The most notable example of such a momentary success, so far as the study of mathematics is concerned, is to be found at Harvard during the fifties and early sixties, where, under the guidance of Benjamin Peirce, a band of young men devoted themselves successfully to the pursuit of higher mathematics.[3] A few of these have since attained world-wide fame, while others were influential in introducing advanced mathematical instruction into the United States 20 or 30 years later. Peirce's success in collecting at this time a fair number of competent students for graduate work seems to have been due primarily to the presence of the office of the American Nautical Almanac at Cambridge from 1849 to 1866, and, secondarily, to the found-

[1] No attempt is made in this report to summarize completely the reports of the three subcommittees or even to refer to all questions treated in them.

[2] Except at Yale University, where the degree of doctor of philosophy was established in 1860.

[3] Peirce was tutor or professor of mathematics at Harvard from 1831 till his death in 1880, but, except during the period here considered, it was only in the last ten years of his life that, under the influence of an expanding elective system, he again began to have an appreciable number of advanced students.

ing in 1847 of the Lawrence Scientific School,[1] which, in those early years, possessed, under the leadership of Louis Agassiz, Jeffries Wyman, Asa Gray, and others, some of the aspects of what is now known as a graduate school.

We notice, in passing, the contrast presented at this time, and for many years after, between the increasing supply of good astronomers in this country and the lack of men who, even by a stretch of the imagination, could be called mathematicians. It may fairly be said that the mathematical talent of the country was at this time diverted to astronomy.

Various circumstances united to bring a large measure of success in the establishment of graduate instruction in all fields, and in particular in mathematics, during the years 1870–1890. The great increase of wealth in the country brought with it endowments of many sorts which strengthened the older universities and established some important new seats of learning. Three things may be mentioned which, on this basis of material prosperity, did more than anything else to help forward the cause of graduate study in the critical period we are now considering.

1. *Study abroad.*—For many years an occasional American had gone abroad to complete his studies. Thus B. A. Gould, a pupil of Benjamin Peirce and a graduate of the class of 1844 at Harvard, who later became eminent as an astronomer, studied with Gauss in Göttingen and took his doctor's degree there in 1848. Similarly J. Willard Gibbs after taking his doctor's degree at Yale in 1863 spent three years (1866–1869) in Paris, Berlin, and Heidelberg, where he studied with Kirchhoff, Helmholtz, Weierstrass, and others. A few more cases of a similar sort might be recorded, but it was not until the end of the seventies or the beginning of the eighties that the stream of mathematical students from America to Europe (generally to Germany) became a steady one. This tendency to go to Germany for the closing years of study contributed probably more than anything else to build up sound standards of productive scholarship and of graduate teaching, without which all attempts to establish advanced instruction in this country must have remained abortive. Its success was in part due to the establishment and the wise administration of traveling fellowships, first at Harvard and then, to a much less degree, elsewhere. We shall return to this important matter of study abroad in a later section.

2. *The foundation of Johns Hopkins University.*—The magnificent bequest of Johns Hopkins of $3,500,000 for the foundation of a uni-

[1] In the same year the department of philosophy and the arts was organized at Yale with the purpose of furnishing "resident graduates and others with the opportunity of devoting themselves to special branches of study," these branches embracing "theology, law, medicine, and more particularly mathematical science and physical science and its applications." It was in this department that the doctor's degree was established in 1860, as noted above.

versity in Baltimore, and his wisdom in leaving his board of trustees a free hand in the organization of the institution resulted in the adoption, on President Gilman's initiative, of a plan whereby the ordinary undergraduate instruction was relegated to a subordinate position from the very start, so that the new university stood before the American public as the standard bearer of the higher education. This was of inestimable benefit in strengthening the hands of those members of the faculties of the older universities who had been struggling to establish and develop graduate instruction at their own institutions. The presence of the eminent English mathematician, Sylvester, as professor of mathematics during the first seven years of the Johns Hopkins University had also a marked effect in stimulating interest in advanced mathematical studies in America, though it is easy to overestimate his direct influence, as he was a poor teacher with an imperfect knowledge of mathematical literature. He possessed, however, an extraordinary personality, and had in remarkable degree the gift of imparting enthusiasm, a quality of no small value in pioneer days such as these were with us.

3. *The elective system.*—At the beginning of the period under consideration the lack of students qualified to undertake advanced work was most keenly felt and made any large success in the establishment of graduate instruction an impossibility. The adoption under the lead of President Eliot; first at Harvard and then to a greater or less extent throughout the country, of a far-reaching elective system in the four-years' undergraduate course furnished a possibility for the gradual extension of instruction in the special fields. Without entering on the question of the advantages and disadvantages of the elective system for the college itself, we may safely say that it provided a basis for advanced instruction without which any considerable development of such instruction, at least during the years of which we are now speaking, would hardly be conceivable.[1]

At the close of the period we are considering, when the idea of graduate instruction had already taken a firm hold on many of the stronger institutions of the country, the founding of Clark University exclusively for graduate study in mathematics, psychology, biology, physics, and chemistry gave a further impetus to specialization in advanced work, and the opening of the University of Chicago in 1892 may almost be said to mark an epoch in the development of graduate instruction in the West and Middle West; for, though that university had from the start an undergraduate department, it stood out,

[1] Cf., however, the closing remarks of sec. 2. What we desire to emphasize here is that an elective system so arranged as to allow some specialization in individual departments, not merely the choice between various *elementary* subjects, permitted a gradual development of more and more advanced instruction, the students being at first mainly undergraduates. *Such* a development could go on simultaneously at many places, while even a single attempt to duplicate the Johns Hopkins experiment would probably have quickly led to disastrous failure.

through the character of its faculty and the emphasis laid on research work, as a strong exponent of the graduate idea.

While in these universities, as well as at Johns Hopkins, advanced instruction was at once placed in a department by itself, in by far the larger number of institutions it developed very gradually within the old college, room being made for it by the elective system; and it was only slowly, even in the larger institutions, that small groups of graduate students began to collect. The somewhat unorganized condition, which was then the rule, is still to be found in the weaker institutions of the country and also in some colleges which in their chosen field of undergraduate work are strong, but which voluntarily renounce any substantial development of graduate instruction. The great universities, however, have all, since the year 1890, developed well-organized graduate schools frequented by the graduates of their own and other colleges. It may be added, to avoid possible misconception, that the graduate schools which sprang full-fledged into existence and those which developed slowly from the old college no longer form two distinct classes. Some of the strongest graduate schools in the country are now to be found among the last-named institutions.

In contrast with such countries as France, Italy, and, to a less extent, Germany, we note the complete lack of central control or organization in the United States. Many variations are hereby made possible which are, for a country like ours, almost a necessity; and competition, on the whole healthy, springs up between the different institutions.

In conclusion we note that of late years some technological schools (for instance the Massachusetts Institute of Technology) have undertaken a limited amount of graduate instruction in mathematics. As this instruction does not differ, except in the greater emphasis laid on applied mathematics, from that given at colleges and universities, and since the amount of such instruction at technological institutions is as yet very small in comparison to the whole amount of mathematical graduate instruction in the country, we have not thought it necessary to mention these technological schools specifically in what follows.

2. THE GRADUATE STUDENT OF MATHEMATICS AT THE PRESENT DAY.

Owing to the great variety of standards for the bachelor's degree in the different colleges of the country, the students of a single graduate school enter it with very diverse preparation. This is, however, not so disturbing as might be expected, owing to the fact that at every university in which a graduate school exists there is a collegiate or undergraduate department whose instruction is freely open to the graduate student who is in need of it. We may then say that not all work done by graduate students is graduate work. On the other

hand, the ambitious and capable senior in colleges allowing considerable freedom of election will frequently be doing work of a distinctly graduate character in the same classes with able graduates of colleges in good standing.

If we thus miss any sharp line of demarcation at the lower limit of the graduate school between graduates and undergraduates, we find a similar phenomenon at the upper limit where the graduate student often passes by almost imperceptible steps into the teacher. Indeed there are graduate schools, even among the better institutions of the country, the bulk of whose students are at the same time assistants or instructors. This, and the very high percentage of graduate students of mathematics the country over who are fellowship and scholarship holders are features of American education which, it is to be hoped, will gradually pass away.[1] They are closely related to the presence in graduate schools of large numbers of students of mathematics who have reached an age when their student days should be over. It can not be too strongly urged on all who give young men advice or who are influential, by awarding scholarships or otherwise, in shaping their careers that it is only in his first youth (not at the age of 30 or 35) that the foundation of real success can be laid by the student of mathematics.

A somewhat different class is formed by school-teachers in active service who are at the same time enrolled as graduate students of mathematics, but at any moment take necessarily only a small amount of work. The ambition of these teachers to improve their professional equipment is most laudable. When, however, as is sometimes the case, they form a considerable proportion of the enrollment of a graduate school, they may be a source of weakness to that school in spite of their earnestness of purpose.

The period spent by a student in graduate study varies from one to three, or even more years; and the amount of migration from one university to another does not seem to be large, although the great majority of students attend a graduate school at a different university from that at which their undergraduate years were spent.

We note also that in graduate work coeducation is the almost universal rule, not only in the great State and other western institions where coeducation forms an integral part of the scheme of education from top to bottom, but even in the most conservative institutions of the East, which do not admit women to their undergraduate departments. Apart from Princeton and the University of Virginia, where no women are admitted, it is only in women's colleges (Bryn Mawr, Vassar, etc.) and in some institutions for men which have held firmly

[1] Nothing in any way resembling this free award of financial aid is found necessary to induce strong men to attend schools of law or engineering. Cf. the closing lines of this section.

to the undergraduate idea, so that the amount of graduate work is very limited, that one sex alone will be found.

A striking and significant fact is that nearly half of all graduate students of mathematics come from small colleges. This is probably due to the fact that in such colleges students always have the opportunity to study the elements of mathematics and often something beyond the elements, while the inducements for them to turn away into other fields are slight in comparison to those offered at larger institutions where a richer elective system prevails. The tendency so strong in our day and country to regard the man of action as being of nobler clay than the man of thought and ideas, reinforced by the much greater financial prizes open to the former, whether he be lawyer, business man, or engineer, creates a situation where it is not easy to secure for mathematical study a due proportion of the strongest youth in our college communities.

3. THE ORGANIZATION OF ADVANCED MATHEMATICAL INSTRUCTION.[1]

The purpose of mathematical instruction should be fourfold:

I. To impart knowledge.

II. To develop power and individual initiative.

III. To lead the student to express adequately and clearly what he knows.

IV. To awaken the love of knowledge and to impart scholarly ideals.

The first of these aims, without attention to which the other three can not be obtained, has led to the great development of the lecture system which we find in all graduate schools, and to the use of the textbook and treatise either in connection with these lecture courses or independently of them, and of the original memoir the reading of which constitutes an art by no means easy to acquire, and which deserves special cultivation at the hands of the members of the teaching staff.

As means used under II may be mentioned: First, the solution of problems by students either in connection with the lecture courses or in special seminars or proseminars, and secondly, the writing of theses which may or may not be connected with the doctor's degree. This last is also the chief means employed under III, although the quiz (cf. subcommitte 3, section 4) is sometimes employed effectively for this purpose, and even the brief written problem is not without some value here.

Both II and III above should receive more attention than is now commonly given to them, while I is at present adequately treated, except, perhaps in the relative weakness of applied as distinguished from pure mathematics.

[1] Cf. also the report of subcommittee 1.

The aim indicated under IV depends for its attainment less on special methods of instruction and more on the personality of the instructor and his attitude towards science than do I, II, or III. To secure adequately the end in view, an instructor is needed who combines high *scientific* ideals with a commanding or sympathetic personality. Such men could do much to counteract the tendency noted in the closing lines of section 2, but, on the other hand, it is precisely this tendency which makes them difficult to secure. Their influence on undergraduate instruction should be no less valuable than in the graduate school.

We must come back once more to the lecture courses which everywhere form the backbone of graduate mathematical instruction. Such a course usually extends either through the whole academic year, that is from the end of September till early in June, or through the first or second half of this year.[1] The lectures, of somewhat less than an hour each, come usually three times (less frequently twice) a week. While much depends on the local traditions and the personality of the instructor, it may in a general way be said that these lectures have a far less formal character than is the case in European universities. Students will frequently interrupt the lecturer with a question, and short discussions between the instructor and one or more students will not infrequently take place, and at times the more formal quiz (cf. subcommittee 1, section 6, and subcommittee 3, section 4) finds its place here. Some tact and firmness are occasionally necessary to prevent the loquacious or thick-headed student from monopolizing the time of the class, but on the whole this frequent contact during the lecture between teacher and student is an admirable feature of American higher education, and counteracts, to a certain extent, some evils which usually accompany the lecture system. It is made possible by the smallness of the classes, an audience of 25 in a graduate course in mathematics being distinctly unusual.

The range of subjects covered by the courses offered in each graduate school is very great (cf. subcommittee 1, section 5). This is peculiarly the case in those institutions which have only recently begun a policy of expansion in their graduate work, where the first sign of such expansion often appears in an astounding increase in the number and range of courses offered, for only a small part of which there are students. Indeed, if students should present themselves, the capacity of the teaching force would be completely overtaxed. This is a state of affairs which no self-respecting institution should allow to continue, and there are signs that it is usually of only a temporary nature, since with a real strengthening of the mathematical department of such an institution this inflation tends to disappear. We

[1] Attention must also be called to work in the summer schools and summer quarters. Cf. subcommittee 1, section 4.

hasten to add that the stronger institutions, and many smaller institutions with a due sense of proportion, offer admirable selections of courses commensurate with their capacity and the needs of their students, courses which at each institution usually vary considerably from year to year. Even in the weaker institutions where a call for advanced instruction is hardly apparent, it may often be wise to encourage instructors to offer a course of a not wholly elementary character, as it will frequently be found to act as a tonic and, by keeping them in touch with the scientific side of their subject, enable them to make their elementary work more vital.

It was mentioned in section 2 that no sharp distinction between graduate and undergraduate work in mathematics can be made. Indeed it is hard to exclude entirely from graduate work anything above the first course in the calculus, now commonly taken in the second undergraduate year. The actual state of affairs is best expressed by regarding the group of courses just following this point, such as a second course in the calculus, the elements of determinants and of the theory of equations, projective geometry, a first course on differential equations, etc., as belonging both to graduate and to undergraduate instruction. From this latter point of view, however, these courses usually appeal only to the student of distinct mathematical ability and seriousness of purpose, whose presence in the course along with graduates does not very greatly affect the character of the course.

As the external signs of success for the graduate student we have the master's and the doctor's degrees. The first of these is commonly given for one year's graduate work done largely in one subject, such as mathematics or physics, and tested either by course examinations in which a higher standard is demanded than is accepted for undergraduates, or by a single examination covering the whole year's work. A thesis is also often required for the master's degree; but the work done on this thesis is not commonly of the nature of research work, and the degree is taken by considerable numbers of students most of whom never proceed further. This degree is given, and properly given, by a large number of institutions, many of which have only a very moderate strength in their graduate mathematical work. Under these conditions suggestions for a minimum standard for the degree are not out of place, and such suggestions will be found in the report of subcommittee 1, section 7.

The doctor's degree originally came to us from Germany, but has long been naturalized and is in all American institutions of good standing distinctly a research degree. In several of our stronger universities it has a standard at least as high as the best German standard. The requirements for the doctor's degree in universities which have been given to any extent during the last 10 years are tolerably uniform (cf. the report of subcommittee 2), but in this matter so much depends on the unwritten standards of individual professors or

departments that there still remains a great difference in the case with which the degree can be obtained at different institutions It is for this reason that the suggestion which is sometimes made that it would be well to attempt to formulate definite standards for the doctor's degree, to which the universities of the country should conform, seems to be of slight practical value.

In school and college work America adopts in one respect a very different standpoint from France and Germany, and this has a certain indirect influence on graduate work. We refer here to the fact that in the last-named countries a pupil will not be allowed to proceed from class to class, and, particularly, will not be allowed to pass the great educational landmarks (for instance graduation from the gymnasium in Germany) without conforming to a very exacting standard which a considerable percentage of each class fails to attain. In America, on the other hand, the teacher who tries to impede seriously the progress of any but the unusually lazy or stupid soon makes himself impossible. This is not the place to discuss the respective merits of these two points of view in the secondary school or even in the college; but when we come to the graduate student of mathematics it seems clear that the American attitude must be modified, and, as a matter of fact, in all the stronger institutions of the country a much greater ability and earnestness of purpose is demanded for passing examinations and securing degrees in the graduate school than would be allowed to pass muster in undergraduate work. Nevertheless, it is to be hoped that something more will be accomplished in this direction, and that, in particular, candidates for the doctor's degree will be made to feel that success for them at an institution of good standing is not a mere matter of time and patience. It is the more important to insist on this, since, as has just been said, the whole current of secondary and college education runs in another channel.

4. TEACHERS.

We must be concerned with this subject for two different reasons, first, because the great majority of graduate students of mathematics ultimately become teachers in secondary schools, colleges, or universities; and, secondly, because on the quality and efficiency of the teachers in the graduate school itself (professors, instructors, etc.) depends to such a large degree the quality of the school.

It is a favorable sign of the gradual elevation of the profession of secondary-school teacher that of late years many persons wishing to adopt this profession spend a year in study in a graduate school. It is true that this time is frequently not spent in the study of a single subject; but for the future teacher of mathematics (or of mathematics and some other subject) to have had a couple of graduate courses in mathematics, usually in the intermediate group referred to on page 14, is a very substantial gain over the conditions of 20 years

ago. It is to this class of students that the courses on the teaching of mathematics, which are now given at many colleges and universities, mainly appeal.

If we except this group who go into secondary-school teaching, and a second group who study mathematics as a tool for use in some other science, such as physics, it may be said with almost absolute precision that all other students of mathematics in graduate schools become instructors in mathematics in colleges or universities. The condition of 25 years ago, where college instructors in mathematics were taken from among the freshly graduated students of a college (usually the same college where they were to teach), has now become the exception instead of the rule; and where it still occurs, the appointment is usually a temporary one, both the instructor and the college expecting that, after a year or two of teaching, further graduate study will follow. The gain involved in this changed state of affairs, both in breadth of view and in real mastery of the subject, the teaching of which is to be the young man's life work, is so obvious as to require no further comment here. If the student can furthermore be given some comprehension of the fact that the science of mathematics is a living and growing one through contact with other students or instructors who are themselves contributing to this growth, and still more if he himself can take some part in the development of mathematical knowledge, his outlook on mathematics in particular and intellectual life in general will have been so broadened that he can hardly fail to become a better member of a college faculty than would otherwise have been the case.

After all this has been said, we must, however, admit that this question has also another side less pleasant to contemplate. What passes for original research, in this country more even than abroad, is often hardly a real contribution to mathematical progress at all, but merely a grinding out of results, which if they have only never been published before may be as unimportant and unattractive as you please; they form an "original contribution." One is tempted to answer, Yes, in the same sense as the brass button in the contribution box. We may feel certain that in the long run this will be the character of the research work done by students who have no real capacity or inclination for original work, but who are pushed into it by the increasing demand, on the part of certain heads of departments, for the doctor's degree as a necessary preliminary to college teaching. The pressure thus produced will surely, if persisted in, bring forth an increasing yearly crop of doctors—success can be obtained by almost anyone with a fair mathematical capacity and with sufficient industry and patience, either by going abroad or by going to one of the weaker American institutions with an ambition for giving the doctor's degree. It is doubtful if the time will ever come, certainly it will not come for a great many years, when all the members of the teaching staffs of the

large universities of the country, and the colleges of like rank, can be men with a real capacity for original investigation; the number of all such men in the country falls far short of (one might almost say that it is of a different order of magnitude from) the number of places to be filled.

The pseudo doctor, to whom reference was made above, is often narrowed rather than broadened by the bit of investigation which he has been set to do, and becomes thereby less effective as a teacher, investigation for him becoming a fetich for which he forgets all other ideals. Or, on the other hand, he may let all thought of original work drop out of his mind when once he has secured his degree. In either case the letters he places after his name ought not to go very far in recommending him for teaching positions. A broad and deep mathematical training should surely be demanded by all the institutions of the country which claim collegiate rank as a prerequisite for a permanent appointment on their teaching staff. They will naturally demand also some ability as a teacher. If in addition they can secure an investigator of a genuine sort, even though his caliber be slight, they should usually regard themselves as fortunate, though a few of the strongest institutions can and should set themselves a much higher standard. On the other hand, our stronger graduate schools should continue, as they are now doing, to encourage every capable student to try his hand at some piece of original investigation, but they should not hesitate, after a fair trial, to tell him, if that turns out to be the case, that he is not fitted for that kind of work.

No specific training for the profession of college or university instructor is commonly given in graduate schools apart from the training in mathematics (cf. subcommittee 3, section 5). The statement made in section 3 of the present report that the training in clear and adequate exposition which is given to graduate students of mathematics is frequently insufficient is of peculiar importance in relation to the future teacher. While it is probably not desirable to attempt to train the future college or university instructor in the art of teaching, the question whether more can not be done to lead graduate students of mathematics to express their ideas well both in spoken and in written form is worthy of serious consideration.

Let us turn now from the graduate students, who are to become college instructors, to the actual instructors and professors of mathematics in our colleges and universities. If we compare conditions at the present day with those existing 20 years ago, a very great increase in the standard of mathematical knowledge on the part of the teaching staff is evident. That the improvement here has not been even greater is due in large measure to the fact that the supply of well-trained graduate students falls far short of the demand. Weak appointments are also made from time to time, owing to ignorance on

the part of trustees or heads of departments of what really constitutes a mathematician, to the pernicious view that administrative ability may be allowed to take the place of mathematical ability, or to other like causes. Flagrant cases of this kind occasionally occur which make one blush for the good name of American universities, but such cases are now merely sporadic and one gains comfort by contemplating conditions in Germany only a hundred years ago. What is needed here, as in so many other places in American life, is a strengthening of *intelligent* idealism (we have more than enough misdirected idealism amongst us) based upon knowledge, and there seems every reason to hope that the great development of mathematics in this country during the last 20 years, evident chiefly in the growth and activity of the American Mathematical Society, will in an ever-increasing degree supply the intelligent and influential public opinion here needed. The shortage, above mentioned, in the supply of instructors in mathematics forms the most serious aspect of the situation.

For various further points: The excessive burdening of young instructors with drudgery, which still often occurs; inadequate salaries; the burdening of professors with administrative work; we refer to the report of subcommittee 3.

5. STUDY BY AMERICANS ABROAD.

No account of higher mathematical education in America would be complete without a reference to the part played by the study of Americans abroad. What an important factor this was in introducing advanced mathematical instruction and research into America has already been mentioned in section 1. In the early days the possibilities for advanced mathematical study in this country were very limited, so that it was natural that students able to do so should go abroad where they could find this opportunity in large measure. At the present day it may safely be said that at several of the stronger American graduate schools most American students find mathematical opportunities better suited to their needs than are to be found at any place abroad. Nevertheless, students still go abroad in apparently undiminished numbers to study mathematics,[1] and their decision

[1] It would be a matter of considerable interest to have statistics on the number of American students who go abroad each year to study mathematics and the length of time they stay. Such statistics would seem to be very difficult if not impossible to secure. As to the proportion of instructors of graduate courses in mathematics who have spent at least one year abroad, see the report of subcommittee 1, section 3. Far less important is the question of the number of doctor's degrees conferred on Americans abroad. Such information might be secured. We content ourselves with giving two such items, for which we are indebted to Dr. Dunham Jackson:

At Göttingen in the years 1889–1909, inclusive, 22 Americans received the degree in mathematics, while no degrees in mathematics had been conferred on Americans during the four previous years.

At Leipzig in the years 1885–1902, inclusive, 8 Americans received the degree in mathematics, while after this time Americans seem to have ceased taking the degree in mathematics there.

At present from 2 to 4 Americans take their degree in mathematics in Germany each year, as against an average of 16 or 17 in the United States.

to do this is frequently a wise one. Let us inquire how this can be the case.

There come first considerations of a nonmathematical character. It is desirable for everyone to become acquainted at first hand with other countries than his own, and this is doubly true for an American, for whom a period of residence in European countries is invaluable. It is true that the student often seems to have brought back from a year or two of residence abroad only a strengthening of his earlier national prejudices, since the mote in the neighbor's eye is so very easy to discern;[1] but if he is worth his salt, he brings with him a fund of impressions and experiences which, as time goes on, greatly enrich his life. For this reason alone study abroad is to be recommended even at some mathematical sacrifice. A second consideration is that the cost of living in Germany, to which country the great majority of students going abroad have always resorted, even after the great increase of the last few years, is still lower than in America, and in particular, the tuition fees are much less than in many of the larger American institutions, especially of the East. These facts largely counterbalance the expense of the trip across the ocean. Finally, it is to be remembered that a year or two of mathematical study in Germany, France, or Italy gives the student a reading and speaking knowledge of one of the great languages of modern thought, besides his own native English, such as can hardly be acquired in any other way.

When we come to mathematical considerations, the first question we must ask is whether getting a degree or learning mathematics is the prime object of the student going abroad. It is the former which, owing to circumstances mentioned in section 4, is too often uppermost in his mind. A student of this category had much better go abroad for his degree than to a second-rate American institution. Of course some care must be exercised by him in the choice of his university, or he must have good fortune in writing a thesis whose weak points are not evident on a superficial examination, but his task is, on the whole, not a difficult one, and he gets at least the advantage of a period of foreign residence.

For another class of men foreign study may be recommended without qualification, namely, for able students who have already had a substantial training in one of the better American graduate schools, or who have even taken the doctor's degree at such a school. Such men will naturally go either to one of the great mathematical centers like Paris or Göttingen, where they will have the opportunity to hear lectures by several of the leading mathematicians of the day, and, perhaps, to see some of them occasionally outside of the lec-

[1] There are also cases in which he takes so kindly to foreign conditions as to become out of touch with America. It is, however, rare that this state of affairs should survive his return more than a few months.

ture room; or they will select some mathematician of eminence in a particular field with whom they may hope to gain direct personal contact, and go to the university where he happens to be. Thus of late years a small but steady stream of American students has gone to Italy.

To the students just considered, and to some extent to their weaker comrades mentioned above, the period of residence at a great European mathematical center or of contact with an eminent mathematician at a less important European institution brings with it a realization of what high scientific ideals in mathematics are, and to what an extent they prevail abroad. Such ideals prevail also, it is true, at the strongest American institutions; but it is hard for the young American to appreciate their great diffusion in a ripened civilization until he has experienced it by personal contact.

REPORT OF SUBCOMMITTEE 1, ON COURSES OF INSTRUCTION AND THE MASTER'S DEGREE.

In preparing this report the committee has found the greatest difficulty in securing reliable information on the various topics considered. Nothing of the kind has been attempted since Cajori's report of 1890, "On the teaching and history of mathematics in the United States."[1] Valuable as this is, it covers only the period of the beginning of graduate study; and except for two brief statistical notes in the reports of the United States Commissioner of Education and scattered items of information in the Bulletin of the American Mathematical Society, there exists no published information on the development of graduate instruction in mathematics since 1890. The committee has, therefore, been compelled to gather its material almost entirely at first hand, relying mainly on personal correspondence and on statements contained in catalogues and other publications of American universities. It is needless to say that only a limited amount of information can be obtained in the former way, while as for the latter, it is too frequently the case that the facts desired are either very hard to get at, or are not given at all.

Over 30 colleges and universities in the United States offer graduate work in mathematics, but less than 15 have given a doctor's degree in that subject within the last five years. The following 24 offer courses of an advanced character and report three or more graduate students for the year 1908–9: Bryn Mawr, California, Chicago, Cincinnati, Clark, Colorado, Columbia, Cornell, Harvard, Illinois, Indiana, Iowa, Johns Hopkins, Michigan, Missouri, Nebraska, Northwestern, Pennsylvania, Princeton, Leland Stanford, Syracuse, Virginia,

[1] United States Bureau of Education, Circular of Information No. 3, 1890.

Wisconsin, Yale. To this somewhat arbitrary list the greater part of the statistics of this report refer, but even in the field thus limited it has been impossible to obtain entirely reliable information on all the points covered. Accordingly, though the numerical statements here given are believed to be in the main correct, it has seemed best not to give detailed statistics for any particular institution.

1. *Historical sketch.*—Genuine graduate work in mathematics may be said to date from the founding of Johns Hopkins University, and the assumption of his duties there by Sylvester, in the year 1876–77. It is true that courses of an advanced character had previously been given in various institutions, notably at Harvard, where Benjamin Peirce taught for nearly 50 years and exerted a profound influence on American mathematical development; but this earlier work was never of an exclusively graduate character, and was carried on with frequent interruptions due to the lack of properly prepared students.[1] Johns Hopkins was founded primarily as a graduate school and that department has always overshadowed the college. Its great success proved a powerful stimulus to advanced work along all lines in this country and led ultimately to the founding of separate graduate departments in most of the leading American universities.

The group of 8 or 10 whom Sylvester soon gathered about him, and their successors, carried something of their master's enthusiasm with them when they went out to other institutions, and in the course of a few years advanced instruction in mathematics was begun in a number of other colleges and universities. Almost as important were some of the indirect results of this influence; although the development at Harvard was independent of the Johns Hopkins movement, it is significant that this was also the period when graduate work in mathematics was there placed on a permanent basis.

The course of lectures delivered at Johns Hopkins by Cayley in 1882 had 14 regular attendants, and added materially to the force of this movement. After Sylvester's departure in 1883, his pupils and associates continued his influence. Cajori reports nearly a dozen institutions in which graduate instruction was offered in the decade 1880–1890. Among them he mentions specifically Harvard, Cornell, Princeton, Virginia, South Carolina, Texas, Michigan, and Wisconsin, as well as Johns Hopkins, and indirectly refers to the work at Yale.

After the departure of Sylvester and Cayley, it was natural that the more ambitious students should wish to continue their work under other mathematicians of the same order. Hence we find that by 1885 a number of Americans were studying in Europe, mainly in

[1] Exception may here be made of the work of Gibbs and Newton at Yale in applied mathematics. The former conducted strictly graduate courses yearly, beginning in 1871.

the German universities. It was no new thing for an American to receive a doctor's degree abroad, but up to this time such cases had been relatively few. From 1885 to 1893, however, American mathematical study continued more and more to derive its inspiration from the seminars of Leipzig, Berlin, and Göttingen. The period of English influence was replaced by one in which a German training was the greatest desideratum, and if a single man is to be designated as Sylvester's successor, it is probably correct to say that the majority of Americans who studied mathematics in European universities received their training at this time in the seminar of Klein. This period was marked by the institution of strong mathematical departments in the newly founded Clark University (1889) and the University of Chicago (1892), and culminated in the International Mathematical Congress held at Chicago in 1893 during the World's Fair, and Klein's Evanston lectures.

By 1893 Americans trained abroad had become an influential, and later a controlling element, and from that time American mathematics stood on its own feet. At present doctor's degrees in mathematics obtained each year at home and abroad are about in the ratio of 5 to 1. It is, however, true that many still go to the great universities of Europe, and it will be shown in a later section of this report that at least half of those giving graduate instruction in this country have studied abroad.

2. *Organization and personnel.*—At present practically all of the leading American universities have organized their graduate work into a separate school or college. With two or three exceptions, however, there is no division of the teaching force into graduate and undergraduate faculties. In particular, the same department of mathematics gives both advanced and elementary instruction, though not all its members may participate in the former. The number of those who give no undergraduate courses is very small. The work of the department is usually directed by one of the professors, designated as the head or chairman.

An examination of publications of the 24 institutions mentioned in section 1 shows that the total mathematical teaching force, not including student assistants, was 202 in the year 1908–9. Of these, 136 offered advanced courses, but probably a considerably smaller number actually gave such work, since it will appear in section 4 that the total number of graduate students specializing in mathematics at these institutions was less than 250. It thus seems that only about half the total teaching force at these institutions had a part in graduate instruction. It should be added, however, that in a number of cases large technological departments necessitate a great amount of elementary mathematical work and increase the number of in-

structors who must devote all their time to such courses. In only two or three institutions was advanced work given by every member of the department.

A further inquiry as to the amount of advanced work given by the 136 mentioned in the preceding paragraph shows that 45 undertook courses announced as primarily for graduates amounting to more than three hours per week. In only five universities were there four or more men offering more graduate work than this, while in 10 there was either only one, or else no member of the department offered more. It thus appears that the greater part of advanced instruction is in the hands of about one-fifth of the total teaching force. Naturally these are the older men, as a rule, but it is also evident that a considerable number of others give a limited amount of graduate work. In many institutions each of the younger instructors offers a single graduate course.

In the past the complaint has frequently been heard that American professors are overburdened. That this is still the case even with men giving a considerable amount of advanced work is evident from the fact that 9 of the 45 just mentioned have a teaching schedule of from 14 to 16 hours a week. However, there has undoubtedly been an improvement of late in this respect, and the average for all the 45 is about 11 hours per week—a figure that still leaves much to be desired.

As to the preparation of those announcing graduate courses, it appears that nearly five-sixths have obtained the doctor's degree, and probably half of the remainder have had an equivalent training. The instructional force would thus seem to be well equipped. More than half have received training in European universities; a partial list of those who have spent at least one year abroad shows 50 who have studied in Germany, 14 in England, 5 in France, and 1 in Italy.

There is a marked tendency in some of the older universities to recruit their teaching staff from their own students. Even when a period of residence elsewhere has intervened, such a practice has some disadvantages, but it also tends to give a permanent character to the work of the department.

3. *The graduate students.*—It may be said generally that the one prerequisite for status as a graduate student in an American university is the possession of a bachelor's degree. In many cases it is announced that this degree must have been given by an institution of equal rank, but as a matter of fact this stipulation is almost never enforced. The result of this is that there is no uniformity of preparation among those classed as graduate students, and frequently such men are taking work as elementary as the first course in the calculus. There are usually, however, more definite requirements to be fulfilled for admission to candidacy for a master's or a doctor's degree, but some

institutions make no distinction of this sort and none require all graduate students to qualify as candidates for those degrees. In view of the wide range of attainment which presents itself among those classed as graduate students, statistics on this subject have only a relative value.

Another difficulty that arises is in the estimate of the work of the summer schools which have sprung up in the last 15 years in connection with most of the larger institutions. Many of these have an attendance of from 500 to 1,500, a considerable proportion being teachers in secondary schools. The instruction is given by regular members of the university, or others specially engaged. The brief duration of these sessions, usually six weeks, and the fact that this is vacation work, very often bring about a serious lowering of standards. Usually only a small amount of advanced work is offered,[1] and the published lists of students give no clue as to the number taking mathematical courses. For this reason we shall here attempt to give no statistics as to graduate work in summer schools beyond the statement that 10 institutions out of the 24 appear to give such courses, and that at least in two or three western institutions the enrollment in these classes is larger than the registration in regular advanced work.

The total number of graduate students in attendance on mathematical courses for more than one term or quarter of the year 1908–9 was probably about 400. The 24 institutions especially investigated report 295, but many indicate only the department in which major work is carried, so that a considerable number specializing in physics and astronomy are not included in this figure. It is easier to obtain a fairly reliable estimate of how many have mathematics as their primary subject. The institutions mentioned report 225, not including summer students and those enrolled for only one term or quarter. Possibly 25 more should be added for universities outside this list, making a total of 250, a number less than twice that of instructors offering advanced courses. It may be interesting to note that 40 of the above-mentioned 225 were women. Only two of the leading universities now restrict their graduate attendance to men.

In this connection it would be highly desirable to present some statistics on the growth of graduate work in mathematics and its relation to the growth of graduate study in general. Unfortunately, little is to be found bearing on this point. In the report of the United States Commissioner of Education for the year 1896–97 (vol. 2, p. 1649) the number of mathematical students in 24 institutions is given as 247

[1] The summer quarter at the University of Chicago differs considerably from the summer session in other institutions in that it is of equal duration with each of the other three quarters, and is in other respects the same except in the character of the attendance. It is, however, divided into halves, and many students attend for six weeks only. It may be noted that the number of graduate students in mathematics enrolled in this one session is three times the average registration for any other quarter.

out of a total graduate enrollment of 3,204, but this probably includes summer registration and students specializing in other subjects. Since that time the number of graduate students in all departments has more than doubled, and mathematics would seem to have very nearly kept pace with the general development. In 1904 the same authority (vol. 2, p. 1424) gives figures for 10 institutions, showing 184 graduate students specializing in mathematics, inclusive of about 80 in summer sessions. This total has been increased by nearly 40 per cent in the last five years. It appears, however, that most of this recent growth has been confined to three or four institutions. Columbia reports an increase of over 300 per cent, and there has also been noteworthy development at the University of Illinois, large sums having been appropriated for the graduate school within the last two or three years.

Another indication of the increase of graduate students is afforded by a statement in the Bulletin of the American Mathematical Society (vol. 16, No. 2, p. 95), in which the number receiving the doctorate in mathematics from American universities in each of the last 11 years is given as 13, 11, 18, 8, 7, 14, 21, 11, 13, 22, 15. It may be noted that the average for the first three years is nearly six-sevenths that of the last three. Since regulations affecting the doctor's degree are not generally more stringent than they were 10 years ago, and no larger percentage of Americans are now studying abroad, these figures lead one to inquire whether advanced work is really increasing as fast as the enrollment would indicate. It is possible that special courses for teachers have largely swelled totals without materially adding to the amount of research. Another cause for the small and apparently decreasing percentage of doctorates may be the great demand for teachers of mathematics, due largely to the recent growth of technological work. This pressure has probably led many to discontinue their studies earlier than they would otherwise have done. Some light is shed on this point by the fact that in the year 1908–9 about 60 took the master's degree with mathematics as a major subject. Probably most of these had that degree only in view and terminated their graduate work as soon as it was obtained.

The bulk of advanced work is done by students receiving financial aid or on salary as subordinate members of the teaching staff. To be more specific, out of the total of 225 in the institutions mentioned, 57 received fellowships and scholarships, paying from $100 to $800 per year, and 61 were assistants or instructors. We may safely assume that a considerable proportion of the remaining 107 were secondary teachers or others taking but one course. There would thus seem to be relatively few giving the major part of their time to study without remuneration, and in fact only four institutions of the 24 have as many as six students of this class. The large number of assistants and instruc-

tors in graduate classes gives a hint of what is often a most unfortunate state of affairs. These are men who should be devoting all their time to graduate study, but they are usually so loaded down with class work as to be unable to do justice to the courses they attempt to carry, and so poorly remunerated that they can not save enough to provide for the additional year or two of research work necessary for the doctorate.

An examination of the migration of graduate students of mathematics develops some very interesting facts. In the institutions mentioned 70 per cent received their bachelor's degree at another college or university, and about 50 per cent received their undergraduate training at other places not on that list. One may gather that there is no great disposition among students in universities maintaining advanced work to go elsewhere for further study after receiving the bachelor's degree, but that there is a large flow from the small colleges to the graduate schools. This movement is a source both of strength and weakness to the latter; of strength from the constant accession of fresh and in the main good material; of weakness in that the preparation of many of these men is very defective. Only 25 per cent of the total enrollment had previously taken graduate work elsewhere, from which it appears that there is relatively little migration after graduate status has been attained.

4. Courses of instruction.—Before considering specifically the nature and extent of the courses offered, we will briefly indicate the limitation of their field. Only two institutions have separate departments for applied as distinguished from pure mathematics. Another assigns such work to the department of astronomy, but it is more usual to include courses of this kind under the head of physics. Frequently the field of this latter department has no clear demarcation from that of mathematics; in general, courses in mechanics and mathematical physics are offered by both departments, with some resulting duplication. This report will cover only the work specifically listed as mathematics.

Courses beyond the first year in calculus are nearly always announced under two headings, introductory and advanced, often styled, "for undergraduates and graduates," and "primarily for graduates," respectively. Although, as indicated by their title, courses of the former group are attended largely by undergraduates, a considerable proportion of the graduate enrollment is also to be found here, so that they properly form part of the field of the present report.

The American university is apt to be distinctly liberal in its announcements; thus we find institutions with relatively small equipment offering from 15 to 25 courses in the two groups, and, in fact, the average for the 24 mentioned is about 20. It should be added that in many cases it is stated that these courses are offered in alternate

years, but too frequently the lists are padded and no indication is to be found as to exactly what work was done in a given year. Something less than half the courses announced seem to have been conducted in 1908–9, about one-third of these being of the first group. We may note that the number of courses was thus about the same as that of graduate students, so that the average enrollment per class must have been very small.

The figures that have just been given clearly indicate the ambition of American universities to become centers of graduate study, and raise the question whether in many cases too much is not attempted on too little capital. It would seem that a concentration of advanced work in a dozen or so institutions would be a distinct gain, and that economy in the number of courses might in many cases be advantageously practiced. Often a course is given, not because it is needed, but in order that certain members of the department may participate in graduate instruction. There are, however, several institutions among the 24 mentioned which wisely limit their work to a fairly complete list of courses in the first group, and to but four or five each year of the more advanced sort. In such cases an adequate training for the master's degree is aimed at, but students who wish to go further are urged to continue their work at one of the larger centers of graduate instruction.

Naturally there is much diversity in the classification of subjects under the two groups; thus we might expect Harvard, which encourages an unusual amount of specialization in the undergraduate program, to include in the first group such courses as quaternions and vector analysis, infinite series, special topics in higher algebra, and theory of numbers. Differential geometry is similarly classed at Yale, Michigan, and Wisconsin, as well as one or two other institutions, while about the same number place in this group a first course on the theory of functions.

As already indicated, much of the work in applied mathematics is left to the department of physics. This is particularly true of introductory courses. Only 9 institutions of 23 whose practice was ascertainable announced under the head of mathematics a first course in analytic mechanics. Eight offered a second course in mechanics under the first group, but in only five cases was this given. It is, perhaps, more to be expected that celestial mechanics should be offered as a mathematical subject in but two institutions.

There is substantial agreement in classing the following subjects under the first group: Advanced calculus, advanced analytic or projective geometry, solid analytic geometry, determinants and the theory of equations, and an introduction to differential equations. The first two are almost always year courses, while the others are more often of half that length. The extent to which these may be

considered standard subjects is indicated by the fact that but one institution on the list fails to offer a course in advanced calculus, and in only one other instance was such a course not given, if we accept the evidence of catalogues; and practically the same is true of the second-named subject. Three failed to offer the third and the same number the fourth as separate courses in the group, though we may suppose the subjects to have been included elsewhere, while the last was everywhere listed, either by itself, or in a few cases as the second half of the course in advanced calculus; and in only a few additional cases were these courses not actually given.

Though there is thus seen to be a considerable uniformity as to introductory subjects, it is interesting to note some of the variations from the more usual lists. Five courses in the history of mathematics are announced, three in probabilities, one in finite differences, six in statistics and insurance, and one in the theory of envelopes. Harvard and Iowa combine algebra with projective geometry, and as noted above, in some cases the first course in differential equations is given as a part of advanced calculus. There would thus seem to be evidence of a tendency to consolidate the introductory work into as compact a group as possible, and this tendency is probably on the increase.

The establishment of special honor courses similar to those of Oxford and Cambridge has been much discussed of late, and has been recommended in recent reports of several university presidents, but so far Princeton alone has adopted such a system. This movement, though of great interest, will not be further described here, as it at present concerns undergraduates only.

In the consideration of the group of advanced courses it will be useful for some purposes to introduce what may appear to be an arbitrary distinction in classing institutions; those which gave more than six courses of this kind will be placed together, while the others will be regarded as constituting another division. As a matter of fact, however, this corresponds to a real difference in the character of the work undertaken; the 13 universities of the former class all have candidates for the doctor's degree among their students, while this is true of almost none of the others. As might be expected, a majority of the institutions giving more than six advanced courses are to be found among the older universities of the East.

As to subject matter, we may note that nearly everywhere by far more emphasis is placed upon analysis than upon algebra, geometry, or applied mathematics. In but one institution of the first division are more courses given in any one of the last three subjects than in analysis, and the same is true of the class enrollment. As to the relative attention paid to the other branches, this varies widely, depending chiefly on the special interests of members of the department.

Only two institutions of the first division leave all work in applied mathematics to other departments, and but two fail to offer courses in both geometry and algebra. We may conclude that there is generally a conscious effort to give a balanced curriculum, with especial emphasis on such subjects as the teaching staff is best prepared to offer. With institutions of the second division an even distribution of each year's work is, of course, impossible. Most of them, however, vary the courses given from year to year so as to include as wide a field as possible. In a few cases this seems to be only a matter of intention, and the courses actually conducted are confined to the subjects most interesting to the two or three men who have charge of graduate instruction.

All the universities of the first division include in their program from two to seven research courses or seminars, while none of the others has more than one. As this matter lies in the field of another committee, it will not be further discussed in this report.

Both in institutions of larger and of smaller equipment the practice of varying the program from year to year prevails. This is obviously a necessity where the enrollment is small. A number of the universities of the second division make it a rule never to repeat an advanced course in successive years, and this seems also to be true of at least three of the first group. In only two or three institutions are there more than four courses repeated year after year, and even in these cases the subject matter is seldom exactly the same. By far the most usual method of announcing courses consists in grouping them in pairs, sometimes from the same field but often widely differing in nature, offered by the same instructor in alternate years. Several institutions announce all of their work in this way. In a few cases more extended sequences are offered in which the cycle consists of more than two years. The effort thus shown to give variety to the work of the teaching staff and to cover as broad a field as possible is in the main most commendable, but it sometimes leads to the giving of advanced courses by men who may be qualified to do work in certain lines but are obviously unprepared for what they undertake.

Although there is a marked tendency not to devote whole courses to special topics, and one may note a certain conservatism in choice of subjects, most institutions offer at least one course in some comparatively narrow field, frequently one of recent development. Thus we find integral equations (given by three institutions) as well as point sets, divergent series, functional equations, modern theories in geometry, general analysis, differential invariants, linear systems of algebraic curves, imaginaries in geometry, symmetric functions, elasticity, and capillarity. Advanced courses in insurance, probability, and statistics appear in three announcements In two instances

the influence of Sylvester probably accounts for such subjects as the logic of mathematics and principles of mathematics.

Although this may encroach somewhat on the field of another committee, we may at least mention here the courses offered specifically on the teaching of mathematics. As might be expected, these differ widely in nature. We shall not here enter into their character, but merely report that all but seven institutions of the list offer work of this kind.

This section should not be concluded without some reference to the courses of lectures delivered by distinguished foreign mathematicians on various occasions within the last 30 years. It is true that comparatively few graduate students heard them, but the influence of Cayley's course has already been indicated, and the visits of Klein in 1893 and 1896 were probably no less important. Of late a number of such courses have been given, and the exchange system may possibly lead to further developments along this line.

5. *Methods of instruction.*—Practically all the work, both introductory and advanced, considered in the preceding section is given by means of lectures of a little less than an hour in length. Sometimes seminars or research courses meet for two consecutive hours. By far the most usual program consists of three lectures a week, but three or four institutions prefer two-hour courses; very few classes meet but once a week, or oftener than three times.

A questionnaire on methods of instruction was sent to 18 of the 24 institutions previously mentioned and replies were received from the following: Bryn Mawr, California, Chicago, Columbia, Cornell, Harvard, Illinois, Michigan, Missouri, Northwestern, Pennsylvania, Princeton, Wisconsin, and Yale. It appears that on some points there is a considerable diversity of practice, even among instructors in the same institution, for freedom of instruction is, fortunately, almost everywhere carefully preserved. However, in most cases the answers received stated that the department used substantially uniform methods.

The first question concerned the assigning of courses to the members of the department, and was put in the form, Are standard courses (the theory of functions, for example) given by the same or by different instructors each year? Only two answers indicated that the latter alternative was the rule. Two more, however, stated that a change in this direction was probable, while four indicated some variation in practice. The custom of permanently assigning certain fields to certain men thus seems to be distinctly in favor. In spite of the advantages of such a system it is undoubtedly true that in many cases it is carried too far, so that a subject comes to be considered a particular instructor's preserve, not to be trespassed upon.

An inquiry into the matter of examinations reveals two widely differing conceptions of graduate work. On the one hand, the practice of Harvard, for instance, in requiring searching written examinations at the end of each semester indicates a general system of careful supervision, while Columbia, Princeton, Chicago, Bryn Mawr, and California, which, it seems, ordinarily give no course examinations, represent the opposite extreme, where the student may take or leave the intellectual repast spread before him. Two or three other institutions allow oral examinations, and in some cases there is a difference of rigidity in the conduct of introductory and advanced courses. It appears that Yale, Harvard, Wisconsin, Northwestern, and Missouri represent the maximum of strictness in requiring written semester examinations. It is hard to determine any tendencies in this matter.

Another question that has a similar bearing concerns the giving of grades indicating excellence of work, a custom which prevails in undergraduate courses in American universities. The answers received show that the opposite practice is most usual in graduate work, but Harvard, Wisconsin, Chicago, Illinois, and Northwestern report that such grades are given.

To the query, Are problems assigned for report from time to time? an almost universal affirmative was returned, so that it would seem incorrect to conclude that anywhere is the student expected to be merely a hearer of lectures. It appears that in some courses in Columbia and Chicago this may have been the case, but the universal rule elsewhere is to require reports of some kind on assigned work. On the other hand, stipulated reading in connection with courses is almost never insisted on. Lists of authorities and references are assigned, but the responsibility for consulting these is thrown upon the student. Only three replies indicated an occasional exception to this rule, while but one or two stated that reports are required on such outside reading.

As already said, the work is given almost entirely in lectures, though only five replies indicated that textbooks were never used. However, it would seem that they are followed in only a few introductory courses, and that even here they are used rather as reference books. Perhaps the most usual exception is in the first course in differential equations. The replies received were unanimous in stating that there were no courses in which a textbook was used exclusively, no lectures being given.

It is customary to devote some hours or parts of hours to reports of students and discussion of lectures. However, at Harvard reports are given in seminars only, while at Princeton and Northwestern they are very infrequent. Yale, California, and Cornell do not give class

time to discussion of lectures, though the last named appoints regular hours for individual consultation. At Wisconsin oral quizzes on lectures are a feature. The most usual practice everywhere is to allow a few minutes of each hour for discussion, questions being frequently asked in interruption of a lecture. The instructor's procedure is, as a rule, quite informal. In some cases an hour is reserved once a week or twice a month for discussion and reports.

In addition to the regular lecture and seminar courses, some institutions maintain courses for reading and research, in which the student (usually there is but one) meets the instructor at intervals for report and consultation, but no stated lectures are given. Often, but not always, this is done in connection with the writing of a thesis. Harvard announced eight such courses last year, Cornell two, Michigan two, and Missouri one. In the last two cases only were written examinations given, but in all but two courses problems were assigned for report.

Work in absentia has decreased almost to the vanishing point, Chicago alone conducting a few graduate courses by correspondence.

A number of the larger institutions are well supplied with models, calculating machines, and other mathematical apparatus, but others are more meagerly equipped. In several cases the making of models has had a place in graduate instruction.

A special feature at Yale has been the writing up of lecture notes by students and placing them for reference in the seminar room. This practice has been followed elsewhere for certain courses, but does not seem to be the rule in any other institution.

In the main it would appear that little change in methods is contemplated, certainly no general movement is on foot. Two or three institutions have inaugurated considerable departures from past usage, but this is in each case an individual matter due to changes in the personnel of the department.

6. *Requirements for the master's degree.*—The master's degree "in course" is given by practically all American colleges and universities as representing one full year of graduate study.[1] The title of master of arts or of science is also given as an honorary distinction under entirely different conditions, but we will here consider only the degree "in course." This is sought especially by students who either have but one year to devote to advanced work or are not sure of being able to continue for the doctorate, since its possession is often a decided advantage in obtaining a teaching position. Even in secondary work the master's degree is frequently insisted on, and this tendency is so marked that in the opinion of many it will before long be as much

[1] Johns Hopkins University seems to be alone in requiring two years. Bryn Mawr gives the degree to no students who have done their undergraduate work elsewhere, except as an accompaniment to the doctor's degree.

in demand there as the doctor's degree now is for the candidate for a college position.

Excluding varieties of nomenclature corresponding to technical or professional courses, four master's degrees are given, namely, master of arts, master of science, master of philosophy, and master of letters; but of the last two the former is conferred by but 3 institutions of the 24 previously referred to, and the latter by only 1. Nearly half give the master's degree in arts only. The distinction is, except in a few instances, an artificial one in that it does not indicate the candidate's field of advanced work, but merely shows which bachelor's degree he has received. It is noteworthy that in 11 institutions of the 24 this distinction has been abandoned, and even where the degree in science is preserved it is often possible for any who prefer it to receive the arts degree, provided their work is not mainly professional or technical.

As noted on page 25, about 60 master's degrees were conferred on students specializing in mathematics in the year 1908–9, a number but little less than one-fourth the total graduate enrollment in that subject. Of the 24 institutions especially considered, only 5 or 6 failed to confer the degree in that year. Two institutions, Harvard and Columbia, gave the maximum number, 8 each.

The universal condition for admission to candidacy is the possession of a bachelor's degree from an institution of equivalent standing. Although there is some latitude in the interpretation of this rule, it is but fair to say that the student admitted with inferior preparation is usually required to make up his deficiencies, taking a longer period than the minimum of one year for obtaining the degree. Applications for candidacy accompanied by a program of studies must usually be filed early in the first semester and must be approved either by a committee of the faculty or by the dean of the graduate school. In case a thesis is required, the subject must be submitted soon after admission to candidacy. The number of courses which must be carried varies from three to five, but in the latter case an allowance is made for the thesis. Some of this work may be done in summer school, and in a number of institutions the degree may be obtained by attendance on three or four such sessions, with the usual requirements as to examination and thesis. Correspondence study or work in absentia has nearly disappeared so far as mathematics is concerned, though some catalogues still announce that part or all of the requirements may be thus met by students who have previously been in attendance. In a few cases where five three-hour courses must be carried a deduction is made for advanced work taken by the student as an undergraduate, but in excess of the requirements for the bachelor's degree. A combination with professional

work is sometimes possible; in the case of mathematics this must be confined chiefly to teaching and engineering.

In 1904 a committee, after two years' consideration and careful inquiry, reported to the Chicago section of the American Mathematical Society on the requirements for the master's degree, with especial reference to conditions in western institutions. We will here use their recommendations, which, they state, correspond with the requirements at several leading American universities, as a basis for discussing present conditions. Their findings will, however, be only briefly summarized, since the details are easily accessible.[1]

The first questions raised are: What should be the minimum acceptable time devoted to undergraduate mathematics and how should it be distributed? The committee agrees that this should be the equivalent of five three-hour courses carried continuously throughout the four years of the undergraduate course and distributed about as follows: In preparation for the calculus (algebra, trigonometry, and analytic geometry), five year hours; calculus, including differential equations and mechanics, six hours; advanced algebra and geometry, four hours. It will be noted that this list includes seven year hours in courses classed as "introductory" in section 5. Courses in applied mathematics are not specifically recommended, though it is urged that undergraduate instruction maintain a close contact with the physical sciences. Except that four undergraduate courses are more usually taken as a minimum, and that the exact distribution here shown is not always insisted on, it may fairly be said that the stronger institutions at present comply very closely with the above conditions.

The next section of the report reads:

"Upon the basis of the preparation indicated above, it is the opinion of the committee that the work leading to the master's degree should include at least 270 hours,[2] selected from the following groups of subjects. The selection should in any case include subjects from at least two of the three groups.

I. GEOMETRY.
- Projective geometry.
- Modern analytic geometry, algebraic curves and surfaces. Application of calculus to twisted curves and surfaces. (Differential geometry.)
- Solid analytic geometry.
- Descriptive geometry (Darstellende Geometrie.)

II. ANALYSIS.
- Theory of equations.
- Advanced calculus.
- Theory of functions.
- Differential equations.
- Theory of numbers.
- Invariants.

[1] Published in the Bulletin of the American Mathematical Society, 2d ser., Vol. X., No. 8 (1904), pp. 380-385.

[2] The term "hour" here refers to a single lecture hour, so that the above is equivalent to two and one-half three-hour year courses.

III. APPLIED MATHEMATICS.
Analytic mechanics.
Mathematical astronomy.
Mathematical physics.
Mathematical theory of probability.

This arrangement provides that the candidate, in case he enters upon his graduate work with the minimum mathematical preparation indicated above, and hence having a broader general training in his undergraduate course, should be required to take at least two-thirds of his year's graduate work in pure and applied mathematics. On the other hand, as it may be assumed that the student entering upon his graduate work with a larger mathematical credit than this has had less general training in his undergraduate course, it is the opinion of the committee that he should be permitted to take as much as one-half of his work in departments other than mathematical, providing, however, that the minimum requirement in mathematics indicated above is also fulfilled.

Present practice tends to reduce somewhat the minimum here recommended, so that in a majority of the larger universities two courses in mathematics are accepted, though as a matter of fact more are usually taken. Four institutions require a candidate to present three subjects and two others require work in at least one additional department. The majority would seem to permit the student to take all his work in mathematics if he so chooses, but with one or two exceptions there seems to be no institution where this is distinctly encouraged. No general rule obtains as to the distribution of the candidate's work; this is nearly always an individual matter decided by the instructor or committee in charge. It is probable, however, that the recommendation presented above represents nearly the average of practice.

An interesting variation from ordinary requirements is presented by the University of Michigan. In addition to Plan A, which calls for the usual amount of specialization, a scheme styled Plan B allows a wider choice of studies. Here the 13 hours to be taken may be distributed as is seen fit among three departments, and other requirements would seem to be less rigid, but the year's work will not usually count as one of the two or three necessary for the doctor's degree.

It is apparently nowhere stipulated, except at Bryn Mawr, where unusual conditions obtain (see footnote, p. 32), that the candidate be able to read a foreign language. The report under consideration recommends a knowledge of German and considers French also desirable. As a matter of fact, the requirements for the bachelor's degree are such that nearly every graduate has had at least one year's work in one of these languages, but seldom enough to make reading easy. Nevertheless, in most cases some reading of foreign mathematical texts is urged, though rarely required.

The committee quoted believe that all candidates should present a thesis, but that this need not embody original research. It appears that 13 institutions of the 24 adhere strictly to this rule, while in at

least 5 more the requirement is optional with the instructor in charge or the head of the department. In 3 or 4 others a thesis is never presented for the master's degree.

The last topic considered is that of examinations. The report mentioned recommends either a final test covering all the graduate courses taken by the student or the requirement that a high grade be obtained in term examinations in each course. The former alternative is the more usual one; in fact, it obtains at 18 of the 24 institutions considered, while in several of the others either procedure is possible. At Harvard one of the two highest of the four passing grades in each course is the sole requirement. The final examination may be either oral or written; practice seems to be fairly evenly divided in this matter, and if oral, it is usually conducted by a committee of instructors from the departments involved.

There are smaller and less reputable institutions in which the requirements are lower, and in fact some where all requirements are merely nominal. The present report has aimed only to present the state of affairs in the institutions which are really prepared to confer the degree.

REPORT OF SUBCOMMITTEE 2, ON PREPARATION FOR RESEARCH AND THE DOCTOR'S DEGREE.

The requirements for the degree of doctor of philosophy in mathematics are not exactly defined in all the institutions.[1] As a matter of fact, the policy is in force in some of them of definitely advising aspirants for the doctorate to complete their graduate studies at other universities in which graduate faculties in mathematics have long been strong and active. It is, however, clear that this condition is but temporary. The increase in the number of strong men in mathematics is resulting in added strength in an increasing number of institutions. It is, therefore, certain that the number of universities in which graduate work will be carried to a definite conclusion will be greater in the immediate future than at present.

On the other hand, in those universities in which, within the past 10 years, three or more doctorates have been awarded there is a degree of uniformity in the requirements which probably indicates and establishes a standard for the United States, the outgrowth of conditions peculiar to the educational system of this country. This standard may be described by detailing the course and career of a successful candidate in some one of these institutions.

[1] The institutions mentioned in the following pages are Brown, Bryn Mawr, California, Chicago, Cincinnati, Clark, Columbia, Cornell, Harvard, Illinois, Indiana, Johns Hopkins, Kansas, Michigan, Minnesota, Missouri, Nebraska, Northwestern Pennsylvania, Princeton, Stanford, Syracuse, Virginia, Wisconsin, and Yale. The data for Brown and California were taken from catalogues and are necessarily incomplete.

For admission to registration as a candidate for the doctorate, successful completion of an undergraduate course terminating with a bachelor's degree is required. While all undergraduate degrees can not at the present time be regarded as equivalent in educational units, greater uniformity in the future will result from the standardizing influence in this direction of the Carnegie Foundation for the Advancement of Teaching.

In his undergraduate years the candidate has pursued courses in mathematics including the calculus, and in some instances mechanics, projective geometry, advanced calculus, and theory of functions also. In any case the courses mentioned are regarded as introductory, and are required before advanced study begins. A reading knowledge of French and German is prescribed. The course of advanced study upon which the candidate now enters extends usually over three years. This time may, however, be spent in residence at one or more universities, with the exception of the final third year. During this year the candidate must reside at the university from which he desires his academic degree. The instruction during the period mentioned is for the most part in lectures, combined with work in the seminary, prepared and delivered by the candidate and his fellow students under the critical eyes of their professors. There may be no formal examinations during this period. The responsibility of mastering his courses rests upon the candidate. In addition to mathematics, one or two minor subjects are required, usually in applied mathematics. This portion of the requirement may, in some cases, be successfully met before the final year.

The mathematical club affords an opportunity for presentation of original papers and discussions of pedagogical and pertinent questions. In the mathematical seminary library are available the standard treatises for study and consultation in connection with the lectures.

The final test of the candidate's power in original investigation is the quality of his thesis. If this is satisfactory, and if he survives the ordeal of a final examination planned to determine the extent of his mathematical attainments, he must, as a final requirement, arrange for the publication of his thesis, in whole or in part, in a manner satisfactory to the department under which he has studied.

The concluding formalities accompanying the granting of the degree are simple—a small diploma fee and attendance (voluntary but customary) at the commencement exercises.

A summary of the existing conditions at the universities named above follows, the arrangement being topical for the sake of clearness.

1. CONDITIONS OF MATRICULATION.

The bachelor's degree accepted in matriculation is not specifically described at the following institutions, but all such degrees are considered equivalent: Cincinnati, Illinois, Indiana, Missouri, Stanford, Syracuse, Virginia, Yale.

"Graduates of this or other institutions in which the bachelor's degree is awarded for courses equivalent to those given at this institution" are accepted at Bryn Mawr, Chicago, Cornell, Kansas, Nebraska, Wisconsin.

A bachelor's degree from a college "in good standing" is required by Brown, California, Clark, Columbia, Johns Hopkins, Princeton, Pennsylvania.

Harvard University requires a bachelor's degree of "substantial character."

At Michigan the credentials of the candidates are examined by the graduate council.

It appears that at the majority of the institutions in question (16 out of 24) the mere attainment of the first degree is not deemed sufficient, but the value of the degree in educational units is decisive.

2. LANGUAGE REQUIREMENT.

All institutions prescribe a "good reading knowledge" of French and German. The practical point is that the candidate may have to consult treatises and memoirs in French, German, or Italian, and his linguistic knowledge must suffice for this purpose. In addition, Bryn Mawr specifies "some knowledge" of Latin as a requirement, and at Brown and Yale a "good knowledge" of Latin is required unless for some excellent reason the candidate is excused by the faculty.

3. TIME ELEMENT.

One year of continuous residence is required in all cases save by Bryn Mawr (two years), Chicago (one and one-half years), Columbia (two years).

Work of equal grade in other universities is generally credited for courses in the early graduate years.

It appears, as a rule, that three years of graduate study are necessary for securing the doctorate.

4. PRESCRIBED COURSES—NUMBER OF COURSES.

Rigidly prescribed graduate courses in mathematics do not, as a rule, exist. Specific exceptions are the following:

Courses totaling 20 hours for one year at Clark University. The titles of the courses are analytic geometry of higher plane curves, skew curves and surfaces, differential equations and calculus of

variations, real and complex function theory (including elliptic functions), theory of numbers, modern synthetic geometry, theory of substitutions, invariants, finite differences.

Undergraduate courses amounting to 8 hours for a year at Yale University. The subjects are projective geometry, advanced calculus, mechanics, function theory.

The courses just mentioned are doubtless required at all institutions in which degrees have been awarded, without specific mention of this fact.

The amount of work required is also not generally defined. There are some exceptions. At Yale a candidate must have successfully followed graduate courses amounting to 18 hours for a year. At Columbia a minimum of 15 hours is required; at Virginia, 9 hours of lectures each year; at Pennsylvania, 24 year hours.

The essential point is, of course, that the candidate must prepare himself for the final examination. To make a comparison of the exacting nature of this work in various institutions is difficult and perhaps impossible. The same course (in name) might well be quite vigorously conducted at one university, and at another partake of an easy character. On the other hand, the breadth of study required can easily be determined by taking into consideration minor subjects (see the next section).

5. MINORS.

The requirement as to minor subjects at Harvard University is quite definite. From the following four divisions of mathematics, (1) geometry, (2) arithmetic and algebra, (3) the differential and integral calculus and the theory of functions, (4) mechanics and mathematical physics, the candidate must designate one as his chosen field, his major. In each of two of the other three he must present a thesis on a topic assigned to him. "These minor theses are intended to test the capacity of the candidate to prepare, at short notice, brief expositions of subjects with which he has had previously little or no acquaintance. While no strictly original work is expected on these theses, the form of presentation must be original."

Minor subjects (one or two) are as a rule required, but not specified. Princeton offers an exception in specifying philosophy and a cognate subject. At Wisconsin one minor must be studied for two years, a second minor for one year.

At Yale the requirement of 18 year hours of graduate courses results in general in the pursuance of courses distributed among three of the main divisions in the Harvard scheme.

Mathematical physics is very generally recommended as a minor.

6. FINAL EXAMINATION.

Examinations in the courses are held by Columbia, Harvard, Illinois, Kansas, Missouri, Northwestern, Pennsylvania, Syracuse, Virginia, Wisconsin, Yale.

A formal, final, oral examination is required after acceptance of the thesis at the following universities: Chicago, Clark, Columbia, Harvard, Minnesota, Nebraska, Northwestern, Princeton, Wisconsin.

The final examination may be both oral and written at these institutions: Bryn Mawr, California, Cincinnati, Cornell, Illinois, Johns Hopkins, Kansas, Michigan, Missouri, Stanford, Syracuse, Wisconsin, Yale.

Yale offers an exception to the general rule of holding a final examination at the end of the candidate's course. At Yale the examination takes place at the end of the last year but one, or at the beginning of the last year. This determination of the candidate's attainments before work on the thesis is begun is intended as a check.

The final examination is public except at Clark, Cornell, Minnesota, Pennsylvania, and Yale.

Virginia requires no final examination; Pennsylvania, a written examination.

7. PUBLICATION OF THESIS.

Publication of the thesis, in whole or in part, is a requirement at all institutions (p. 36), except at Brown, Harvard, Michigan, Minnesota.

Special conditions imposed are: One hundred and fifty copies deposited in the college library (Bryn Mawr), 25 in the college library (Northwestern), 50 in the college library (Cornell), 100 in the college library (Clark, Chicago, Cincinnati, Illinois, Nebraska, Virginia, Wisconsin).

Printing of the entire thesis is required by Bryn Mawr, Chicago, Cincinnati, Indiana, Northwestern, Pennsylvania, Princeton, Syracuse, Virginia, Wisconsin.

Acceptance by a reputable journal of the whole or part suffices at Clark, Columbia, Cornell, Johns Hopkins, Indiana, Illinois, Michigan, Missouri, Nebraska, Pennsylvania, Yale.

At Harvard doctors' dissertations have "almost invariably been published."

8. VARIOUS.

A small diploma fee is usually required. This varies from $5 to $60 (Princeton).

The master's degree is in no case a prerequisite for the doctorate, but is recommended at Chicago, Cincinnati, Columbia, Illinois, Kansas, Minnesota, Missouri.

9. RESEARCH AND SPECIAL COURSES.

The candidate's ability for original investigation is, in general, stimulated through work in seminaries. Topics assigned to him for private research develop this faculty, and criticism and personal conference with his instructors further the same end. Seminary work consists in discussion and criticism of original papers presented by candidates. Reports on current literature are also called for—in particular, on contributions to those special lines in which students and instructors are peculiarly interested.

10. EQUIPMENT—SPECIAL FEATURES.

Collections of mathematical models are available in all institutions for study by the candidate. At Yale a special room is equipped with all necessary tools for the making by the students of simple models. With the development of graduate clubs at universities throughout the United States has come the establishment in each institution of a mathematical club. At Harvard the club is conducted entirely by the students, members of the faculty attending as guests. The organization at Yale differs from this only in the existence of faculty supervision through the secretary of the club, who is a full professor. This supervision is nominal. In general, it may be said that the mathematical club is an informal affair, professors and students meeting for extra-curriculum intercourse and for discussion of such papers and reports as may be presented by faculty and students without obligation so to do.

The library facilities are reported adequate for all lines of mathematical research, save at Indiana, Kansas, Michigan, Minnesota, and Syracuse. In addition to the resources of the university library there exists at many institutions a seminary library; in which are available current mathematical treatises and periodicals, duplicates of those stored in the main library.

The library of the American Mathematical Society is rapidly becoming for its members a most valuable possession. By exchange, files of all periodicals and proceedings are being completed, and these are made available to members wherever located, through the regular means of transportation.

11. DEGREES GRANTED.

Within the years 1900–1909 the doctorate in mathematics has been conferred upon approximately 140 candidates, the number at the various institutions being as indicated (in a few cases round numbers are given): Bryn Mawr 1, California 1, Chicago 30, Clark 9, Columbia 10, Cornell 13, Harvard 12, Illinois 1, Johns Hopkins 20, Kansas 1, Pennsylvania 8, Princeton 3, Stanford 1, Syracuse 1, Virginia 4, Wisconsin 4, Yale 21.

12. CONDITIONS IN THE FACULTIES.

At Clark, Chicago, and Johns Hopkins certain members of the mathematical faculties give graduate instruction only. The number of hours per week of such instruction is, respectively: Clark 7, Chicago 6–8, Johns Hopkins 4–7.

In general, graduate instruction is compensated for by a reduction in undergraduate hours. The following table makes clear the policy followed in the various institutions, the first number indicating the hours per week of undergraduate, the second of graduate instruction:

Bryn Mawr	9 and 3.
Chicago	6 and 4.
Columbia	6 and 6, or 9 and 3.
Cornell	11 and 4.
Harvard	4½ and 4½ and a reading course.
Illinois	10 and 3.
Indiana	5 and 4, or 5 and 6.
Kansas	7 and 5, or 10 and 3, or 15 and 0.
Michigan	12 and 3.
Minnesota	11 and 3.
Missouri	10 and 2, or 10 and 3.
Nebraska	10 and 3, or 15 and 0.
Northwestern	6 and 6, or 10 and 3.
Pennsylvania	10 and 3.
Princeton	6 and 3, or 9 and 3.
Stanford	10 and 0.
Syracuse	6 and 6.
Virginia	12 and 2.
Wisconsin	5 and 4, or 5 and 6.
Yale	10 and 3.

REPORT OF SUBCOMMITTEE 3, ON THE PREPARATION OF COLLEGE AND UNIVERSITY INSTRUCTORS IN MATHEMATICS.

In the preparation of the following report the committee has found little or no literature dealing specifically with the training of mathematical instructors for the higher institutions of learning in the United States. The subject is, indeed, one to which little explicit attention has been paid. The committee has therefore been obliged to beat out its own path. As, moreover, the usual preparation for the teaching of mathematics is inseparably connected with the history and organization of mathematical instruction in our country, it has been found desirable to treat the subject freely and discursively.

1. *Survey of past conditions.*—To obtain first an adequate comprehension of the conditions prevailing to-day in the teaching of mathematics, it is necessary to review briefly the past. The mathematical development in the United States during the last quarter of a century has been so unexpected and sudden, and so remarkable in its propor-

tions, that it can scarcely be termed an evolution, but merits rather to be called a revolution. Thirty years ago the contrast between the mathematical and the astronomical situation in this country was most striking. In astronomy we numbered some of the world leaders, but in mathematics there was, perhaps, not an American known abroad for investigational work. In 1876 the Johns Hopkins University was founded, and soon afterwards the English mathematician Sylvester came to this country and started a small group of investigators. Some time prior to this Benjamin Peirce had stood forth at Harvard as a unique figure, a solitary investigator in the field of mathematics. Possibly owing to his influence, there emanated from Harvard University about 1880 the policy of sending promising mathematical students abroad that they might reap the fruits of German scholarship. To a limited extent the example of Harvard was followed by other institutions. As our mathematical students returned from abroad, stamped like many another excellent product as "made in Germany," they brought with them a knowledge of the modern advance in mathematics and an appreciation of investigation. Then it was that the mathematicians of the country were wakened so as to perceive the infinitesimal development of their science in America and the boundless advance abroad. In this manner, and largely through the leadership of Johns Hopkins and Harvard Universities, there arose a small circle of investigators, fired with a determination to bring American conditions into some sort of correspondence with conditions abroad.[1]

Roughly speaking, the years 1880–1890 may be said to mark a new departure in the mathematical education of the United States. Prior to this time there had been fixity of aim and definiteness of character in the methods of instruction. The prevailing mode of instruction was the recitation system, interspersed with a few lectures. For study a textbook was used in which definite lessons were doled out daily, and upon these lessons the student was compelled to study and recite. Naturally he presented his work in the best form of which he was capable, so as to make the best impression possible on his teacher. The instructor in turn devoted his time to correcting the mistakes and deficiencies of the student. Thus from study of a definite text and correction by the teacher great stress was laid upon *form*, while little or no emphasis was put upon imagination and upon the student's initiative or power of selection. It was the teacher's province to select the food; it was the student's to digest the daily allowance.

The amount of mathematics taught at this time was extremely limited. Calculus was a junior or a senior elective, and its theory was

[1] For further information concerning the influences which made for progress, see Prof. Fiske's presidential address before the American Mathematical Society, Bulletin, vol. 11 (1904), p. 238.

taught from the purely formal side. Spherical and practical astronomy, Salmon's Conic Sections, and at most one or two other subjects (particularly quaternions) completed the college mathematical course.

For teaching, little knowledge of mathematics was demanded beyond the subject to be taught. After graduation the ablest students were usually called upon to teach without further study, but after the establishment of Johns Hopkins University a steadily increasing tendency was noticeable to demand first a year or two of graduate study. Yet the bright student still considered it preferable to try at once his hand at teaching, looking upon this as an experience which would fit him to profit by more advanced study later. If, on the other hand, he should fail as a teacher, no time would be lost in useless study before changing his vocation.

No special preparation for teaching was then required, nor indeed was any needed; for as a student the teacher had gone already through the machinery of the recitation mill, and this produced a certain perfection and uniformity of results. The polishing of the rough and projecting edges of the raw product through the friction of daily recitation, the demand for quick and ready expression by the student when on his feet, the insistence upon clearness, accuracy, and good English in the production of work (qualities which, it should be noticed parenthetically, were necessary to the professions of the law, ministry, teaching, into which the great majority of the students of that time passed)—all this constituted in itself an admirable preparation for teaching. Whatever may have been their deficiencies in exactness, extent, and depth of knowledge, the college mathematical professors of the time were commonly adepts in teaching the textbook material, and they knew how to present it so as to make it clear to the average man's comprehension. Like produces like. In consequence the young teacher, put in without special training to "sink or swim, live or die, survive or perish" according to his ability to control and guide the boisterous elements of his classes, generally fulfilled his mission satisfactorily.

The situation can be summed up by saying that the machinery was well adapted to its end and turned out its material after the model set. On the other hand, various other qualifications of supreme importance for the teacher were too often ignored or sorely neglected; for example, initiative in study, geometric and analytic imagination, rigor, and power of generalization. It is therefore not surprising that the mathematical training of that time failed completely to produce the true scholar or the great mathematician.

2. *Present day conditions.*—The conditions to-day with respect to teaching are in part a survival of those of the past, and in part the result of the ingrafting of new ideals and methods. The earlier methods, in so far as they were not indigenous, were an inheritance from

the English tutorial system. The changes have been due to contact with the mathematicians of the continent of Europe, and in particular with German mathematicians. As our students returned from study abroad and themselves became teachers, there sprang up a demand for up-to-date knowledge, for *thorough* scholarship and investigation. As a test and a certificate of some ability to meet these demands, the Ph. D. degree speedily became a very common prerequisite for desirable college appointments. So rapid has been the change in recent years, and so great the growth of the more prominent institutions of the country (especially the State institutions), that the supply of teachers and investigators has become inadequate and falls ever increasingly short of the demand.

While this condition exists in the more important institutions, the average small college west of the Alleghanies has been little touched by the awakening. Institutions of respectable standing with 800 or 1,000 students also exist which are substantially in the condition of the college of 25 years ago, and this, too, without reaching the degree of efficiency and excellence attained in the old-time eastern college. Between the two extremes lie a multitude of colleges in which the new influences are felt in varying degree.

In the better institutions the mathematical progress has been attended by a remodeling of the curriculum. Calculus has become to-day primarily a sophomore study.[1] In the last two years of the college course a considerable variety of electives has been introduced, extending as far as elementary courses in the theory of functions of the real and the complex variable. Graduate courses cover a range of topics fully as wide as in Germany, and perhaps even wider.

Speaking roughly, it may be said that the mode of teaching remains the same as of old up to the calculus. Here freer and more varied methods begin to prevail While in most institutions it is still found advantageous to teach calculus by the recitation method, yet a large part of the class hour is frequently spent by the teacher in discussion and explanation. The general view seems to be that about this time the student reaches a greater maturity of power, and the instruction he should receive requires therefore greater inspirational power and breadth of view on the part of the teacher. From this point on the recitation recedes into the background, being superseded by the lecture or lecture-and-quiz method. The subject matter thereafter is no longer closely prescribed by precedent, and the course of instruction is laid out largely in accordance with the taste, training, and ideas of the teacher.

Not only is the calculus a boundary line between two styles of teaching, but it also marks a sharp change in the problem confront-

[1] At Harvard it was so recognized as early as 1870. This is indicative of certain tendencies toward the elevation of mathematical instruction in America before the advent of foreign influence.

ing the teacher of mathematics, a change almost as great and important as that from the gymnasium to the university in the German educational system. The alteration in the educational problem at the end of the sophomore year is so well known and recognized that it has been proposed by some to relegate eventually the work of the first two years of the college course to the high school; by others, to separate the work of these two years into a junior college leading to an appropriate degree.

Without discussing the advantages and disadvantages of either change, we shall proceed to consider further the nature of the mathematical problem in our large colleges and universities before and after the calculus. It should be borne in mind that nothing can be said that will fit all of our universities and colleges, for their number is legion. Yet in the main the remarks below portray the situation correctly, for even in the academic department of Harvard University, where the lecture system is introduced from the beginning, it is accompanied by daily tests or problems in the elementary work.

In most of the leading institutions the mathematics of the freshman year has been made either elective or semi-elective in the college of liberal arts. Such, however, is its importance that classes in the elementary subjects of trigonometry, algebra, and analytic geometry are very large and create a peculiar problem of administration. In the engineering schools the same condition holds for the calculus also. A sample illustration may be taken from the University of Wisconsin. Here in the years 1908–9 and 1909–10 there were each semester in the College of Letters and Science a total of from 300 to 330 students to be given instruction in freshman mathematics. For this purpose they were divided into a dozen or more sections. This situation necessitates a large amount of work of a somewhat routine character. In the engineering school, in which mathematics is required, practically the same condition prevails. The adoption of the lecture system for these lower classes seems unwise because of its failure to meet the individual difficulties of the student and to give him the repetition and drill necessary at that stage of his development.

Even in these more elementary classes where the recitation remains, one notices a change in the character of the problem confronting the teacher, especially in the eastern section of the country.[1] College youths no longer form a carefully selected intellectual body designed solely for the professions. The increasing educational development of the country has touched all classes, even the foreign immigrant, and has increasingly brought to college young men preparing for business pursuits. The country has been so inundated by foreign immigration as to become the "melting pot" of the nations, and the high

[1] In the newer sections of the country pioneer or unsettled conditions have prevailed largely from the beginning.

school is the crucible. The pupils in the high schools, being in vast numbers the children of men and women who have learned the English language only after arriving in this country, mix foreign idioms with the grammar of the school. The classes are often very large (40 or 50 being not uncommon in a class), so that not much attention can be given to the individual; and the best students, in particular, suffer from receiving instruction appropriate to the mass. Moreover, the examination for admission to college has disappeared almost entirely and has been replaced by certification from the school.[1] For all these reasons the average student enters a university more poorly prepared than formerly, and brings with him less of the cultural influence of the select few in earlier days. Under these conditions the freshman instructor finds it impossible to maintain in his classes the former rigorous standards of good form, and of exposition in concise, lucid English. Consequently the recitation system no longer produces as satisfactory results as formerly, although it doubtless serves its purpose better than any substitute that has been suggested.

Subsequent to the calculus the number of pupils in mathematical classes is small. There remains chiefly the student preparing to teach in high schools and colleges, with a sprinkling of others who need mathematics for physics, engineering, or its other applications. The fundamental problem in teaching is henceforth to inculcate in the prospective teacher thorough scholarship and sound standards, which were necessarily somewhat neglected in the previous mass-instruction.

3. *The present preparation for teaching and its deficiencies.*—Owing to the changes above described the recitation system no longer affords the prospective teacher the same drill in presentation and the same feeling for form that it once did. Under the lecture system which succeeds it the emphasis is put primarily upon acquisition and upon extension of knowledge. But the mere hearing of lectures leaves the student passive and does nothing to tax actively his powers of presentation unless supplemented by other aids. (See section 4.) Furthermore, if the teacher himself is deficient in style and is so absorbed in acquisition and investigation that he slights the matter of presentation, the student is likely instinctively to copy his example and to remain content with crude and half intelligible exposition of his ideas. Thus it has happened not infrequently that a brigh. student has found nothing in his university training to correct the deficiencies of early years or his natural tendencies, and has gone out to his career of teaching only to meet with an initial

[1] Examinations are still required of all candidates for entrance at Harvard, Yale, and Princeton. Since the preparation of the report a modification of the method of admission at Harvard University has been announced.

disastrous failure, from which it was difficult and sometimes impossible to recover.

The introduction of the lecture method into our country has indeed been so rapid that an amalgamation of the old and new modes of teaching into a well-proportioned, coherent, and consistent educational system has been impossible. In consequence two dangers to-day beset the mathematician. On the one hand, there is extreme and unreasonable emphasis on the research side of activity at the expense of form and clearness and without due regard to values, any sort of investigation being prized even though of little or no intrinsic worth. This results in the elimination of any true interest in the actual teaching side of the profession. On the other hand, we see the teacher absorbed by personal interest in the student and by the pedagogical and administrative side of his work, to the neglect or exclusion of progress and true scholarship. Both deficiencies are found in our educational system. In a few institutions the first-mentioned evil may be found, while the latter is conspicuous in many.

It might be thought that the difference in the educational problems before and after the calculus would result in a corresponding differentiation of teachers into two classes, the emphasis being laid in the one class upon the imparting of knowledge and in the other upon its acquisition. In France, for example, a sharp difference is made in the preparation for teaching in the lycées and for teaching in the universities. For teaching in the lycées it is necessary to pass the Agrégation. This is perhaps the most searching examination to be found in any educational system of Europe. It includes, in addition to a test of mathematical attainments, an examination in powers of exposition. For the latter, reliance upon general education is not sufficient, even in a nation temperamentally predisposed to considerations of form and elegance, but in anticipation provision is made for special preparation and instruction. No such distinction has ever been made in our country, although in choosing men for professorships there is clearly an increasing tendency to distinguish, according to the nature of the work demanded, between those who are primarily educators or administrators and those who are scholars and investigators.

It continues to be the custom to train both these classes of men as teachers by setting them to teach freshmen, traveling upward by the rough and thorny road of experience.[1] Probably the drilling of large freshman classes affords the best training in teaching that can be given to the beginner. It possesses the further advantage that the younger instructors have more nearly the same interests as the

[1] The extent to which this is done is shown by the fact that out of 225 graduate students of mathematics in 24 universities, 61 were registered as assistants or instructors. In addition should be counted a certain proportion of 57 scholars and fellows.

lower-class students than do the older professors, and in consequence these students are likely to feel less hesitation in exposing their difficulties to the instructor than to the professor of greater reputation. Also the professor is saved thereby from routine teaching, which after many years of repetition is liable to lose its freshness and to become somewhat stereotyped, monotonous, and irksome; and he is left free for other university problems demanding his riper experience. In many institutions, however, it remains the practice for the professor to retain one elementary class (a freshman section, or more often the introductory class in calculus) for the purpose of supervision and of keeping in touch with the situation in the lower classes. The contact between professor and instructor when teaching different sections of the same class often affords the instructor most valuable training and help in teaching. It seems altogether probable that with the development of our mathematical education such work of supervisory instruction will be reserved increasingly for men of fine teaching capacity but without the taste and inventiveness for investigation.

The above system of freshman and sophomore instruction is, however, not without its serious and pernicious effects. Quite commonly 15 or more hours of this work are loaded upon the instructor, in addition to which he has numberless quizzes to conduct and "blue-books" or examination papers to correct, no distinction being made in the assignment of such work between the humdrum man and the scholar of talent and promise. But in many institutions the work is brightened by the assignment of a small class in some advanced subject. In a few instances (as at Chicago and Harvard) the proportion of "desirable electives" reaches a quarter or even a half of his total quota. Yet, in general, one result of the system of lower-class in uction has unquestionably been the draining of intellectual life out of many able men by an excess of routine work which leaves them inadequate time to carry on their study and investigation.

The handling of the huge and ever-increasing mass of students also necessitates a large amount of administrative work which is not connected with any particular branch of instruction. For this work the mathematical profession, in particular, has been looked to. By actual count there are to-day some twenty professors of mathematics who are either deans or similar administrative officers in their respective institutions. Among them are men of fine ability and scholarship, whose services are thereby lost to mathematical research and progress. Even when not occupying such a position, the mathematical teacher is supposed to be competent for administrative work and is liable to receive an extra large assignment of it. But the present situation, bad as it is, is better than the past, and gives hope of further betterment in the future. Probably also the seriousness

of the administrative problem has been responsible for another evil besides the absorption of so much of the scholastic ability of the country. It has helped to promote the pernicious view, still too largely "held by some people of whom one would expect better things, that a man can become a good college teacher by following about the same lines as the high-school teacher, learning a very little more of his subject than he has to teach, and developing important accessory qualifications, of which the one dearest to college presidents is his capacity for administrative work."

4. *Supplement to the lecture system.*—Diverse means are employed to correct the deficiencies of the lecture and recitation systems mentioned in the last section. Among the supplements to the lecture system the setting of problems, the seminar, and the quiz should be especially mentioned.

The first of these is used relatively but little. It was introduced at Harvard about 1880. Probably the bane of the American educational system as a whole, even to the present day, is the excessively great stress laid upon the acquisition and assimilation of material. Individual initiative in treating new questions is deserving of greater recognition and stimulus. Now, the aim of problem work is to encourage this. It is to be carefully distinguished from the merely illustrative example work in our lower classes and is more akin to the use of advanced problems at English universities. It prevents mere passivity on the part of the student and, if skillfully used, will stimulate the student's imagination and pass by imperceptible degrees into real original work. Its greater use and consideration in advanced mathematical instruction in our country can therefore be recommended. In exposition it is also not without very considerable value, since the student, especially when stimulated to it by his professor, may find a peculiar interest and satisfaction in formulating properly his own results. But it should be accompanied by other methods designed to develop more actively the power of presentation and to afford proper acquaintance with periodical literature and skill in its perusal.

The "quiz" is distinctly non-German in character and origin. Though native, it is somewhat akin to a species of French conference. In the quiz a review is made of the contents of a group of lectures recently given, and this is done by discussion between student and teacher, and by sharp cross-questioning on the part of the latter. Not infrequently new material is added in this manner to the lecture already given. The advantage of the quiz is not alone to the student. The teacher is enabled thereby to keep in touch with him; and from the democratic American point of view "keeping in touch" is a cardinal element of sound teaching. The indefinite spinning of lectures by the teacher, careless whether they are being

followed by the student, is foreign to our conception of education. Much could, of course, be said for the lecture system on the principle satirized in Clough's poem "The devil take the hindmost, O;" for, undoubtedly, by sifting the students and casting out the weak the best talent can be most rapidly developed. But one of the characteristic American aims is the development of an intelligent democracy rather than an intellectual aristocracy. It is greatly to be regretted that the quiz is not more frequently used, and its development more carefully studied. In the hands of the skillful teacher a quiz, say once in every four or five lectures, can be employed to instruct and aid simultaneously the weaker and the stronger pupils. Great insistence can be laid upon accuracy, clearness, conciseness, and upon thorough comprehension and expression of ideas; and the slipshod work, due so often to a mere hearing of lectures, can thereby be checked. While the quiz may impede rapidity of progress in a given subject, the loss is more than compensated by the hearty cooperation and the mutual understanding which it secures between teacher and student and by the added interest in the work.

The seminar and proseminar (i. e., a seminar not based upon research) are German methods adopted and adapted into the American system. In the seminar the student becomes the expositor, and oral reports are given on problems connected with his investigation or on his collateral reading. In the opinion of the committee the quiz and seminar together can be made an invaluable complement to the lecture system in the training of teachers.

Another very common aid is the mathematical club. This serves as a species of clearing house or a rallying point for the department, instructors and graduate students being brought together for discussion and report. Its advantages are very similar to those of the seminar. Here, perhaps more than anywhere else, the graduate student is anxious to show at his very best and is "tried out" to see what he can do. In some universities there are maintained two clubs of this character, one for advanced undergraduates and the other for graduates.

In certain institutions definite courses of reading are mapped out, while in other institutions the same end is sought by collateral reading assigned in connection with regular lecture courses. The bulk of this reading is in French or German. The purpose is alike to encourage wide reading, to give facility in reading foreign treatises, and to make the prospective teacher familiar with the literature of his subject and an adept in selecting material from widely different sources.

The usefulness of the M. A. and Ph. D. theses as preparation for teaching should not be overlooked. A doctor's thesis alone has proved to be a totally inadequate preparation in written exposition. For this reason some sort of a thesis should also be required for an

M. A. degree, unless some substitute is afforded, as, for example, in the additional "minor Ph. D. theses" at Harvard. The experience of editors of mathematical journals reveals on the part of the average young investigator lamentable lack of understanding as to what is clear, coherent, and readable mathematical English. Too often the editor is obliged to ask some one to assist the author by friendly criticism. Usually a remarkable improvement is noticed after the completion of three or four successive articles by the author. The responsibility for critical work of this character should not lie with the editor but with the university teachers. Nowhere can a student be taught clear exposition better than in written mathematical English, and in no department, perhaps, are the difficulties of exposition greater than in advanced mathematics. Hence in training for teaching, the use, and if possible the repeated use, of written exposition seems to the committee indispensable. The requirement of a written theme each semester in graduate courses in which no examinations are given is an expedient especially to be commended.

In many universities assistantships and fellowships are designedly used, not merely to supplement the educational force, but as an aid in the training of teachers. The assistant or fellow while continuing his graduate study is asked to teach a few hours per semester; thus theory and practice are developed simultaneously. In a few institutions, notably at Chicago and Columbia, there is an affiliated university high school at which selected students may also teach. In certain rare instances a peculiar cooperative scheme has been tried by the professor, who employs an advanced graduate student as his assistant in large classes. The latter is present at all class exercises, observing the methods of the teacher and taking his place in his absence. He is called upon for aid in correcting papers and to give individual assistance to the student. Thus at all stages of the class instruction he is in conference with the professor regarding the conduct of the work. This would appear to be true practical pedagogy, well worthy of further consideration and trial.

5. *Teachers' courses and teachers' colleges.*—In recent years the pedagogic side of teaching has been recognized at the large universities and not a few colleges by the introduction of the so-called teachers' courses. In a few universities courses of this character have been separated and organized into a teachers' college, notably at Columbia University, the University of Chicago, the University of Cincinnati. In these courses hints concerning teaching are given, attention is called to the essentials of good teaching and to the common dangers and pitfalls, and the pedagogically difficult parts of secondary mathematics are discussed. To assist the man who goes into college work it is the aim at Columbia University and the University of Chicago to "give a good knowledge of the history of mathe-

matics, of the curricula in different countries, and of the danger points which confront teachers in their early days of work." Special topics are sometimes assigned—for example, an historical study of the methods tried in the teaching of calculus.

Up to the present time the teachers' colleges have been patronized almost exclusively by elementary, secondary, and normal school teachers. They stand as a protest against the entire neglect of the pedagogical side of teaching. It is the apparent belief of these colleges that a great work can yet be done for college and university teachers, as well as for teachers in the lower grades. Not only is it said sharply that "training in the art of teaching is not even a minor requirement for a (higher) degree," but it is further urged that in mathematics above all subjects it is not enough for successful teaching that a man shall know the subject he teaches; he must have power of adapting it to the average man's comprehension. For this reason greater attention to pedagogy and to form is desirable in preparation for teaching, doubly so in a country where no fixed traditions prevail, and where the recent mathematical growth has been so rapid. Only in this way, it is asserted, can we save our schools and elementary college classes from the crude mistakes of the youthful teacher.

As yet, however, the influence of the teachers' colleges upon the preparation of college and university teachers has been indirect or felt purely as a protest. Special methods for the training of such teachers have not been there developed, and little, if anything, has been attempted in the way of actual courses for them. The whole question as to whether such courses should be given at all, and if so, what kind of courses should be given and where—whether in such colleges or in the graduate school—remains for determination. The whole field is a virgin one, and not to be developed hastily or inconsiderately. And, finally, it seems to be conceded that *dogmatic* instruction in the art and method of teaching would be injurious and disastrous in collegiate mathematics.

Little or no encroachment has in fact been made upon the graduate school by the professional educator. Against his view is urged the argument that there is every difference in the manner of preparing the elementary and the advanced teacher. In the elementary school the child moves in a world so different from the world of his teacher that a study of psychology and pedagogy helps to bridge the gap, but the student of 18 to 20 is sufficiently near his teacher in maturity that such aids can be dispensed with. The deficiencies of the American college teacher are rather on the side of an accurate, clear-cut knowledge; of a failure to distinguish between the husk and the kernel, and to insist without ceasing that the student shall search for the kernel. The prevalent opinion is that definite instruction in method, which

was the old-style pedagogy, crushes individuality, produces regrettable uniformity, stifles life and capacity for growth. Better an ultimate success obtained after partial failure and experiment and without suppression of individuality, than a dead-level success, which never rises above rule and precept. Hence in the graduate school the primary aim must ever be to build a sound mathematical foundation and to stimulate a discriminating interest and enthusiasm. The chief preparation is then to teach the student mathematics.

But after all this has been justly said, there remains with the committee the belief that the graduate instruction is arranged too exclusively with reference to research and with too little reference to the needs of the prospective teacher. A limited part of the time in the mathematical club and seminary might be devoted with profit to the more strictly professional side of the training and to a discussion of the broad and changing problems of mathematical education.

Furthermore, the tendency to undue specialization in some obscure corner of mathematics, with accompanying neglect of a broad mathematical interest and outlook, should somehow be counteracted. This might perhaps be done in part by carefully planned survey courses, such for instance as Klein's masterly lectures on projective geometry or "Höhere Geometrie." In particular these survey courses could be employed to bring to the student's attention such literature and lines of study as bear directly or indirectly upon the college subjects which he is most likely to teach. And above all, the need is for broad yet incisive introductory courses in the main and vital fields of mathematics.

6. *The amount of graduate study.*—Broadly speaking, the minimum requirement for teaching in college and university tends to become the equivalent of the master's degree. But in the better institutions it is recognized that a year or a year and a half of graduate training is by no means enough for a permanent college or university position, and the possession of the Ph. D. degree is therefore made a sine qua non. The situation thus created is in many ways unsatisfactory in regard to both degrees.

On the one hand, the demand for graduates with the doctor's degree has had increasingly the effect of making the Ph. D. degree a sort of college teacher's certificate, and has acted in a generous measure to cheapen the degree. Too many doctors have doubtless been turned out who have no actual interest in research and seek the degree solely for its commercial value. It has been the rule rather than the exception that the holder of the Ph. D. degree fails to go on to further research.

On the other hand, the situation with respect to the master's degree is even worse. It is in no way an adequate certification for college teaching. In many institutions it has been regarded as

essentially a degree for secondary teachers. In other institutions it is used with one class of students as a stepping-stone to a higher degree, with another class of students who are found not to possess the inventiveness and fertility of mind necessary for research it is both a recognition and discharge. It is indeed difficult to say for just what the M. A. degree does stand. The committee submits as a question worthy of consideration whether the M. A. degree might not be so administered or developed as to become a desirable teacher's degree in distinction from the Ph. D. degree as a certificate of research. Thus it could be used as a recognition alike for college teachers without the doctor's degree and for secondary teachers of decidedly superior capacity. Possibly in this way the Ph. D. degree could be relieved of some of the heavy load which it is now compelled to carry.

The present sharp differentiation of college mathematicians into two classes, the holders of the master's and of the doctor's degrees, is in many ways a most unfortunate one. Many men have stopped with the former who are capable of proceeding much further. The committee wishes to record its emphatic belief that every man should be encouraged to study just so far as his ability and taste may qualify him, without stopping at the line of either degree. Many a teacher without the kind of ability necessary for research can yet be encouraged to become a thorough scholar in some definite line, instead of looking to administrative office as his career.

More systematic encouragement for study after the attainment of the doctor's degree is also a desideratum. The introduction of the German Privatdocent system into this country is recognized to be impossible and totally foreign to the American spirit of financial independence after graduation. Yet some means should certainly be devised for recognizing better the more talented few and giving them opportunities for study commensurate with their ability. As before pointed out, the life and ambition of many young instructors have unquestionably been stifled by the load of 15 or more hours of classroom work, with the accompanying written exercises for correction. In the case of the ablest young instructors some financial provision for maintenance should be made with a reduced number of hours of work. The prevalent American practice is to relieve the instructor of his heavy hours of work only when he is about 40 years of age, or after he has proved himself a pronounced success by virtue of his ability or attainment. In opposition to this it can not be too much emphasized that the age of preparation for intellectual work of high order is from 20 to 30, or to 35 at the latest.

7. *The supply and appointment of teachers.*—The same general considerations which affect the supply and quality of teachers in other departments apply also to the teachers of mathematics. The serious feature in the situation to-day is a still increasing deficiency

in supply relatively to the demand. This necessitates also to a considerable degree a deficiency of quality, since places must be filled by such persons as can be obtained. The inadequacy of the supply is primarily a result of the enormous growth of the educational system of the country, but the deficiency in quality is due also to the rapid advance of the mathematical profession in standard and knowledge. On the absolute scale the number of good mathematical teachers in the United States has never been greater, yet never also has the shortage been greater. The expansion of the country and its increase in population has outstripped growth in education, and the outlook is that the situation will become even more acute in the immediate future. Not only must the prospective growth of the country, especially of the Middle and Far West, be taken into account, but also the awakening and development of our neighbors, China and South America. This expansion at home and abroad affords a great opportunity for the engineer. It attracts and doubtless will continue to attract a large number of the ablest mathematical students away from the teaching of abstract mathematics into the practice of the engineering and allied professions.

The attractions of these professions in themselves are enhanced by the remuneration. The financial rewards for the successful engineer are incomparably greater than those for the mathematician. Twenty-five years ago the standard salary for the full mathematical professor in our best colleges was $2,500, the variation to either side of this figure being, with few exceptions, slight. To-day the corresponding figure is $3,000, the variation ranging from $2,500 in small cities to $4,000 or $5,000 for a university situated in a large metropolis. For assistant professors $1,500 to $2,500 is the current compensation. For the successful engineer the compensation must be reckoned in five-place figures.

These salaries may very possibly appear adequate to the foreign professor not acquainted with American life and social conditions, but they are notoriously inadequate to the legitimate demands of life in this country. The high price of rents and cost of living, the cost of travel necessary in a country of large distances to attend conventions and to keep in touch with mathematicians at home and abroad, and many expenditures due to prevailing social conditions are recognized to press hard upon a college professor. If he has a family of three or four children, the financial pinch is likely to be felt daily and bitterly, and anxious thought must be spent upon the mode of spending one's income. Furthermore, the mathematician, by the very nature of his subject, is cut off from the usual modes of supplementing a professor's salary open to his colleagues, as for example by writing or by the delivery of public lectures, by legal and expert advice, by chemical or biological analysis, and so on. His only resource may be to write a textbook on one or more of the four ele-

mentary subjects in which there are large college classes. Hence we see issued year after year a large number of such books, the majority of which resemble one another like peas in a pod, and contribute little or nothing to mathematical progress. Rarely does the mathematician have both time and knowledge to write—for such fame and glory as there may be in it—the unremunerative advanced mathematical textbook or treatise so much needed in the English language.

With the obvious financial consideration there acts also another powerful force pulling men away from mathematical teaching. It is felt that the engineering professions give a chance to do "something worth while." In our rapidly developing material civilization the shallow view often prevails that the man of action stands on a higher plane than the man of ideas. The latter too is invaluable for highest development.

This sidetracking of genuine mathematical talent to engineering work is most seriously felt in applied rather than in pure mathematics. Precisely here, where the American mind might be expected to scintillate with flashes of genius, there is a real poverty of talent. While we boast our late lamented Gibbs—born like many a precious flower to blush unseen—it must be confessed that his career is indicative rather of the interest and talent which might exist in applied mathematics than of that which does exist. In support of this statement it suffices to point out how few university centers there are in the United States where a first-class training in applied mathematics can be obtained. The diversion of students to engineering is not solely responsible for this. It is in part a consequence of past influences when mathematics was pursued in our country as a branch of logic and a purely deductive science. A tendency to the purely formal side of mathematics may be noted from the days of Benjamin Peirce on. It is therefore not much to be wondered at (though much to be regretted) that American mathematical research has inclined too much to this formal side; for example, to multiple algebras, postulational and axiomatic foundations, theory of groups. Hence, there is need of great insistence to-day that applied mathematics shall be more abundantly pursued and for a longer period, and that men of sound mathematical training shall be brought into touch with vital physical problems.

The current method of filling vacant positions is the Socratic one of question and answer. The professor in charge of the department or some one of its divisions first makes necessary inquiries, writing particularly to the mathematical centers where the graduate students are most likely to congregate. When practicable, he supplements the information by personally meeting the candidate under most favorable consideration and thus gains first-hand impressions. If the position to be filled is a major and not a minor one, the candidate is

frequently invited to come to the university for a visit, to inspect and to be inspected. The considerations governing a choice of candidates are discussed with the president, or the president and dean, and usually the choice of the professor is accepted as a result of the conference. In some cases the selection must be approved by an academic council consisting of some or all of the full professors. Under normal circumstances the subsequent ratification by the board of regents or trustees is a matter of form, since the selection of instructors is properly deemed a prerogative of the president and board of instruction.

When the head of a department is to be chosen, a common practice is to appoint a committee of from three to five members, inclusive of the president or dean, and containing representatives of other departments than the one concerned. This committee then seeks for information in all possible ways.

Applications for a vacant position are often forwarded by those who would be glad to secure the appointment. To a large extent these applications are waste paper. Wider publication of vacancies has been advocated by some, but it is the almost universal experience of those who have the appointments to make that a too-wide advertisement results in a bothersome and useless hoard of applications from men, the number of whose submitted testimonials varies inversely as the square of their fitness. Information privately solicited is trusted more than the public testimonial. In the case of well-known men a hint of their availability is as effective as any formal application, but in the case of minor appointments information supplied by a really worthy applicant is sometimes a most important factor in the decision for his appointment.

To fill vacancies in small or less important colleges the teachers' agencies are often consulted, as well as by many normal and secondary schools. In such cases the standing of the college and the salary are not sufficient to attract the better trained men secured by the methods just described. These institutions rely particularly upon their recent and brightest graduates to fill the vacancies. To a great extent the practice of "inbreeding" by selection from the alumni of the institution has been current in our country, and this has been true even of some of the largest and most important of our institutions. When carried to excess the practice cramps and narrows the development of the college or university. It is increasingly felt that the best results are obtained by a generous infusion of instructors having other ideas and training than those characteristic of the institution itself.

It can not be said that mathematical proficiency always dictates the appointments. Not infrequently there are better men available. Yet this is no doubt less often the case than is realized by outsiders, since frequently appointments which are severely criticised have been

made only after better men have been sounded and have declined to be considered for the position. Other qualifications than mathematical proficiency must receive consideration, such as enthusiasm and the ability to stir mathematical interest in students, personality and genius to gather and keep coherent a group of promising instructors, ability to develop a department so as to make it useful and respected in the institution, and so on. Qualities such as these will doubtless weigh more in the future than in the past, while administrative ability will be less considered, owing to the increasing separation of business work. Doubtless in the past many sins of bad mathematical appointments can be laid to considerations of administrative ability.

Two qualities may be noticed as especially needed by the American teacher. The first is a broad, liberal culture. The pursuit of mathematics in itself is doubtless a narrowing one. There is perhaps no other science of which the development has been carried so far, which requires greater concentration and will power, and which by the abstract height of the qualities required tends more to separate one from daily life. A wide liberal culture therefore is eminently desirable for the establishment of that attractive personality so necessary for the best success in teaching.

The second quality, moral fiber and force, is demanded largely in all institutions, but especially in the small American college, where teacher and student come into particularly close personal relations. No just report concerning the selection and distribution of mathematical teachers can be given without some reference to this need. The denominational origin of most of the Eastern and older colleges accounts for the emphasis laid upon the possession of these qualities. While denominational lines have broken down and tend to disappear, and accordingly religious distinctions have been forced increasingly into the background in college and university education, the moral ideal nevertheless survives, receiving perhaps a greater consideration than in any other country except England. In the junior college—to designate thus the first two years of the college course—the student is still young and immature and in a formative period. Here the public claims that the student has as much a right to the care, time, and thought of the teacher as has investigation. It insists, even in the large State universities, that the youth shall be cheated neither out of its intellectual nor out of its moral birthright. The mathematical teacher with rare exceptions must begin his career either in the small college or in the junior college of the large university. He consequently must have the problems of the junior college at heart, and not neglect his students for his own investigation. Not a few talented men fail altogether because of such neglect or lack of interest, and thereby close their mathematical career.

In the graduate school the considerations of the last section fall mostly into abeyance since the student is now sufficiently mature to criticise rather than copy the deficiencies of his instructor. Intellectual leadership and vitality are here the crucial question. With increasing separation between the junior and senior colleges there comes an increasing difference in the kinds of teachers required.

The question "From what class of society are our mathematical teachers drawn?" is a complicated one, not to be answered by statistics. Have they come up through poverty and hardship, liking mathematics for its difficulties? Do they come from the great middle classes, or from well-to-do, cultured homes? Undoubtedly all classes contribute, but the chief source is probably the second, especially that section of the middle classes which is more poor than rich and the section which lies close to the cultured classes. To men from these homes the teacher's profession offers improved social conditions and increased opportunity. Its financial deprivations are borne with sufficient philosophy and complaisance, for the love of intellectual pursuits is the life of the teacher.

The large number of mathematical teachers coming from the very small college, where the curriculum is cramped and where mathematics is one of its chief components, should be especially noted. Probably these colleges furnish an abnormally large percentage of the mathematical students for the graduate school.

8. *Concluding remarks.*—The committee abstains from prophecy or the formulation of any definite and ideal program for the training of the advanced mathematical teacher. The situation regarding such training is still chaotic, and there is no method yet apparent of evolving order. The aim of this report has been rather to point out the causes, results, and deficiencies of the present method of training (or lack of method), and it will be enough if the committee shall have succeeded in arousing interest and stimulating thought upon the subject.

Certain general points may, however, be noted by way of summary or emphasis.

The introductory course in calculus, as already indicated, forms the beginning of the specialist's training to-day. Prior to the calculus the study of mathematics is pursued chiefly for its practical and disciplinary value, while thereafter it is pursued by the mathematician for its own sake.

The instruction of the mathematician and physicist for a longer time together in identical courses would undoubtedly have its advantages. Certainly the greater intermingling of pure and applied mathematics is desirable. The extent to which they have been separated is astonishing. The pure mathematician is prone to be either helpless or ignorant before the applications, while the applied

mathematician has rarely a good grip on modern mathematical principles. Not in a divorce from related subjects, but rather from the inspiration of contact with them, is the continued life and progress of that science to be sought which is the most perfect and coherent development of human thought.

The committee expresses the further wish that the early training of the mathematician on the purely mathematical side shall not be too narrow. The four great fields—analysis, mechanics, geometry, algebra (number fields, etc.)—should early be represented strongly in their main essentials and characteristics.

The matter of training in clear, coherent, and interesting presentation, both oral and written, should not be overlooked by the teacher, as is sometimes done, but he should be willing to give his time generously to their inculcation. A more abundant practice therein should be afforded the future teacher and investigator.

Probably greater cooperation between the experienced teacher and the beginner regarding the conduct of elementary college instruction is needed. This might be gained by more abundant discussion of the problems involved, by helpful suggestion, by mutual visitation of classes, and occasionally even by joint conduct of a class.

Finally, the amount of routine instruction, at least for the most promising of our younger college and university investigators, should be so regulated as to further rather than suppress their individual development. On the other hand, the man with a gift for teaching, but not for investigation, should be encouraged to obtain a broad and comprehensive knowledge of mathematics, and, obtaining this, should not be cashiered for failure to "produce" nor be spoiled as a teacher by being trimmed into a very mediocre investigator. The difference between the two kinds of gifts—the power to teach and the ability to extend the frontiers of our science—should be more clearly recognized and both should be amply rewarded. It is for a free and not a standardized development that we plead; and, above all, for greater freedom and leisure for the most able of our younger instructors, that they may achieve the best that is in them.

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