US-China Education Review A, ISSN 2161-623X
January 2013, Vol. 3, No. 1, 19-26

The UMR Conception Cycle of Vocational School Students in Solving Linear Equation

Shao-Ying Li
National Taitung College, Taitung, Taiwan

Shian Leon
National Kaohsiung Normal University, Kaohsiung, Taiwan

The authors designed instruments from theories and literatures. Data were collected throughout remedial teaching processes and interviewed with vocational school students. By SOLO (structure of the observed learning outcome) taxonomy, the authors made the UMR (unistructural-multistructural-relational sequence) conception cycle of the formative and development of linear equation about the learners under equality axiom.

Keywords: SOLO (structure of the observed learning outcome), UMR (unistructural-multistructural-relational sequence) conception cycle, linear equation, vocational school students, remedial teaching program, concept circle

Introduction

Linear equation is the key point to learn from arithmetic to equation, it is also the beginning to learn abstract algebra thinking for students. Students usually have learning difficulties about letter and operation of linear equation. Küchemann's (1981) descriptions categories of interpretations of the letters could be as follows (Hart, Kerslake, Brown, Ruddock, Küchemann, & McCartney, 1981, p. 104):

> A. Letter evaluated: The category applies to responses where the letter is signed a numerical value from the outset.
> B. Letter not used: Here the children ignore the letter, or at best acknowledge its existence but without giving it a meaning.
> C. Letter used as an object: The letter is regarded as shorthand for an object or an object in its own right.
> D. Letter used as a specific unknown: Children regard a letter as a specific but unknown number, and can operate upon it directly.
> E. Letter used as a generalized number: The letter is seen as representing, or at least as being able to take, several values rather than just one.
> F. Letter used as a variable: the letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values.

According to these categories, Küchemann (1981) identified four levels of understanding of generalized arithmetic and pointed out what obstacles students encountered: Level 1, using the letters as object by evaluating the letter or by not using the letters at all; Level 2, these are more complex than level 1, though the letters still only have to be evaluated or used as object; Level 3, children can use letters as specific unknowns, though only when the item-structure is simple; and Level 4, children can cope with items that require specific unknowns and which have a complex structure.

In addition, Foster (1994) found that students had different degree of difficulty when answering such questions that could be considered as the early form of algebra learning: (1) $4 + 3 = \square$; (2) $4 + \square = 7$;

Shao-Ying Li, Master, lecturer, National Taitung College.
Sheng Leon, Ph.D., professor, National Kaohsiung Normal University.

and (3) □ ＋ 4 = 7.

From these, Foster (1994) found that the third equation was the most difficult for students, because students cannot use the strategies of arithmetic to solve it. If use letter to instead of □ in the same questions, some students will consider that they are different. In recent years, Lima and Tall (2008), Freitas (2002), and Vlassis's (2002) researches showed that 15-16 years old high school students' performance was not well on linear equations, and that will affect their fellow learning about application of equation. In Taiwan, there are many high school students who do not understand letter or how to get the solution of linear equation that makes them down in mathematics. According to the above, the authors want to explore the students' conceptions and find out the UMR (unistructural-multistructural-relational sequence) conception cycle by SOLO (structure of the observed learning outcome) taxonomy in solving linear equation during remedial teaching. The authors think that it is necessary to understand what kind of difficulties and conception students have when learning linear equation, so that they can help them to overcome it.

Literature Review

Tall and Thomas (2001) distinguished three levels of algebra:

(1) Evaluation algebra: the evaluation of algebraic expressions, such as 4×A1 + 3 as in spreadsheets or in the initial stages of learning algebra. Tall and Thomas (1991) used BASIC program to express "A+3", and let students produce meaning through input "A". The experimental group is more superior than control group in concept problem-solving. By computer sheet, students could strengthen their conception of operating letter and find the same results between different equations;

(2) Manipulation algebra: where algebraic expressions are manipulated to solve equations;

(3) Axiomatic algebra: where algebraic systems, such as vector spaces or systems of linear equations are handled by definition and formal proof.

It can be found that evaluation algebra is the easiest to learn and axiomatic algebra is the most difficult to learn to students. Thurston (1990) thought different arithmetic methods to calculate "3 + 4", including "count-all", "count-both", "count-on", "count-on from larger", "derived fact", "know-fact" etc.. Gray and Tall (1994) accorded with Thurston's conception, and suggested "procept" which indicates that the symbol acting was the pivot of processes and concepts. Therefore, the produce and operation of procept are the abilities of operating symbols.

In successful learners with algebra, they have some characteristics: (1) Crowley (2000) found that those who continued to be successful "had readily accessible links to alternative procedures and checking mechanisms" and "had tight links between graphic and symbolic representations"; and (2) when asking students to draw maps of their developmental conceptual structures, the higher achievers revealed concept maps which grew organically from previous maps whilst the low achievers tended to draw each successive concept map anew without connecting ideas coherently (McGowen & Tall, 1999).

Learning of algebra by using a collection of procedures may help students to pass exams in algebra, but it may not prepare them for future developments. In practice, students give their own cognitive meanings to algebraic operations (MacGregor & Stacey, 1993). Therefore, many students fail to give meanings that agree with standard mathematical meanings. They used short-term strategies that can (seem to) help at one stage, but fail in subsequent learning. For instance, the subject is still widely introduced by a technique that is called "fruit salad algebra" in which letters stand for objects, such as 3a + 2b being interpreted as 3 apples plus 2 bananas.

This can give short-term success, such as adding 3a + 2b to 4a + 3b gets 7a + 5b by imagining apples and bananas were put together. Such an image soon outlives its usefulness when expressions, such as 3ab are used. Are they three apples and bananas? It certainly is not three apples times bananas. So, how to understand letters and use them correctly is important in learning linear function.

In contrast, Kieran (1981) gave evidence that the equal symbol is often seen as a "do something symbol" rather than a sign to represent equivalence between the two sides of an equation. Such as "2 + 3 = 5" means "add 2 to 3 gets 5" and an equation, such as $4x - 1 = 7$, is seen as an operation to find a number which multiplied by 4 and 1 is subtracted, gives 7. Lima and Tall (2008) used three linear equations to 68 15−16 year-old students, as shown in Table 1. Their performance was not well. Question 1 and 2 were adapted from Freitas (2002) and Vlassis's (2002) researches, question 3 was designed by Lima and her colleague. $5t - 3 = 8$ could be "undone" by arithmetical reasoning, $3x - 1 = 3 + x$ had the unknown on both sides of the equation, and 2m = 4m was suggested by one of the teachers and caused great difficulty among his students. This was consistent with Filloy and Rojano's (1989) findings. The arithmetical notion does not apply to an equation of the form $Ax + B = Cx + D$; its resolution involves operations drawn from outside the domain of arithmetic—that is, operations on the unknown. Only 16 students out of 68 solved both equations 1 and 2 correctly.

Table 1

Students' Responses on Three Linear Equations (Lima & Tall, 2008)

Equation	$5t - 3 = 8$	$3x - 1 = 3 + x$	$2m = 4m$
Successful	25	25	7
Blank	16	11	27
Other solutions	27	32	34

Note. Total students: 68.

Vlassis (2002) used equations with the unknown on both sides, and showed that the balance model was a helpful metaphor for almost his students in giving meaning to the equals sign as equality between the two sides of the equation. However, it failed to be meaningful for many students in more general situations involving subtraction and negative numbers. Other difficulties like Freitas (2002) found that procedures related to phrases, such as "change side, change sign" which were also called Viète model (Filloy & Rojano, 1989) were usually meaningless to students and often resulted in mistakes. Collis (1972) thought 7 + 4 is procept and $7 + x$ is lack of closure. Many students remain process-oriented (Thomas, 1994), and think primarily in terms of mathematical processes and procedures, causing them to view equations in terms of the results of substitution into an expression (Kota & Thomas, 1998). Western people reading habit from left to right also makes confuse on operation signs and parentheses (Thomas & Tall, 2001).

From the above, there are some common problems when students learning linear equation: They do not understand the meaning of the symbols, they cannot calculate with unknown, or only using "undo" method to linear equation, etc..

Research Method

Content

Linear equation usually includes:

(1) Elements of equation (meaning of sign, letter represents number, and equation simplized);

(2) Solve linear equation (meaning of linear equation and solution of linear equation);

(3) Problem-solving by linear equation.

In this study, the authors focus on (1) and (2), (3) is excluded.

Purpose

(1) The authors used some questions which designed by researchers to survey and got responses from 131 vocational school students' responses;

(2) Exploring formation and developing conceptions when solving linear equation during remedial teaching.

Research Design

(1) Quasi-experimental design and interview;

(2) Sample: 131 first grade students in vocational school (16−17 years old);

(3) Researchers (one mathematics teacher, who is also mathematics education doctoral student, one mathematics education professor).

Intervention

This is a remedial teaching program. It includes the processes as follows:

(1) Analyze what is the pivot of students through pre-test. The test includes 20 items as shown in Appendix Table 1. These items were designed by mathematics teachers and mathematics education doctoral students. The reasons why the authors use these items as follow: (a) According to Lima and Tall's (2006) and Filloy and Rojano's (1989) researches, three items (items 5, 6, and 8) come from Lima and Tall's (2006) research. Filloy and Rojano (1989) considered that when students solve equations of the types: $\mathrm{Ax} \pm \mathrm{B} = \mathrm{Cx}$ and $\mathrm{Ax} \pm \mathrm{B} = \mathrm{Cx} \pm \mathrm{D}$, it is not sufficient to invert the indicated operations. It is necessary to operate on what is represented. Once they had mastered the use of the models for the type $\mathrm{Ax} + \mathrm{B} = \mathrm{Cx}$, they were given complex types increasingly: $\mathrm{Ax} + \mathrm{B} = \mathrm{Cx} + \mathrm{D}$, $\mathrm{Ax} - \mathrm{B} = \mathrm{Cx} + \mathrm{D}$, $\mathrm{Ax} - \mathrm{B} = \mathrm{Cx} - \mathrm{D}$, etc.. So, the authors designed the test as Appendix Table 1; (b) Every four items were grouped to test students' operation which Lima and Tall (2008) indicated "Change side, change sign", but do not know why. And the minus operation, also included the coefficient of x is 1 that always makes students confused by literature; and (c) Preliminary test shown as in Appendix Table 2;

(2) The kernel ideas of remedial teaching by theories and preliminary and pre-test are: (a) The letter is evaluated (Collis, 1975). Basically, letter in the symbol of linear equation is represented by a number (a solution); (b) The meaning of representation of symbol; (c) The meaning of equal sign (=); and (d) Equality axiom (operating on both sides of the equation in the same operation);

Data and Analysis

(1) The teaching video;

(2) Observation record by teacher;

(3) Student's portfolio;

(4) Interview;

(5) The data are analyzed and presented by SOLO taxonomy (Biggs & Collins, 1982) as shown in Figure 1 which is the UMR conception cycle of $(72 \div 36) \times 9 = (72 \times 9) \div (x \times 9)$ (Pegg & Tall, 2010, pp. 177-179).

According to the SOLO taxonomy (Biggs & Collins, 1982), the authors may category into a single procedure as U (uni-structural), several distinct procedures having the same effect as M (multi-structural), and the realization that they are essentially the same process as R (relational). The encapsulation of a process into an object is then extended abstract, producing an entity (a procept) which can be used as the beginning of a higher-level cycle of procedure—multi-procedure—process—procept;

(6) When analyzing the data, researchers must discuss to reach a consensus about the UMR conception cycle.

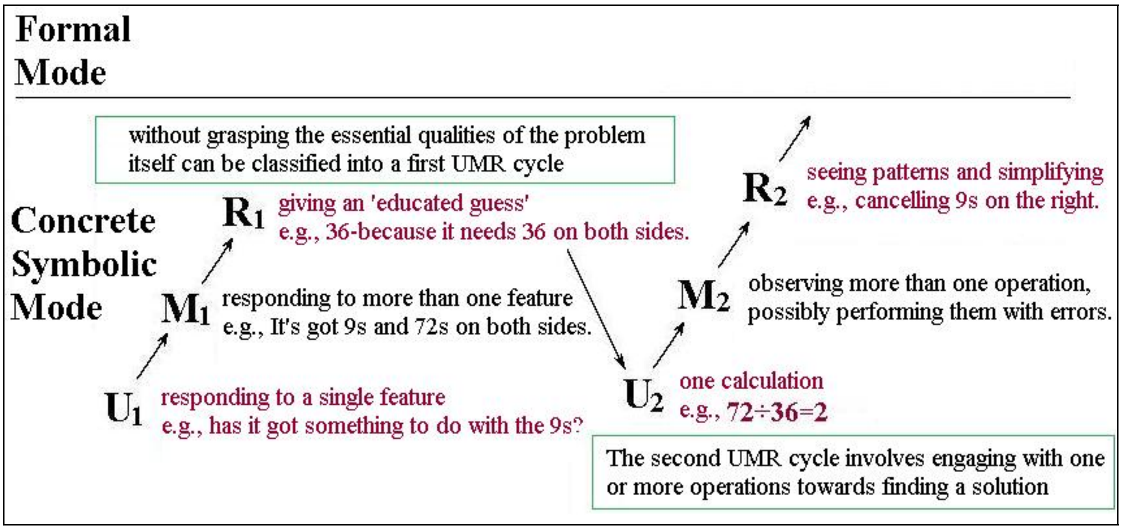

Figure 1. An example of SOLO model (Revised from Pegg & Tall, 2010).

Processes

Preliminary→pre-test→remedial teaching (six hours)→post-test:

(1) Analysis 37 second grade vocational students' features of solving linear equations from preliminary and 94 first grade vocational students' features from pre-test;

(2) 10 volunteers (first grade students) participated this remedial teaching after pre-test;

(3) Analysis students' abilities by pre-test, intervention, and interview after remedial teaching;

(4) Teaching and data collected, including students' portfolio recordings by teacher and post-test.

Results

In this study, data analysis is based on the data of the 10 students participating in remedial teaching, and accompanied by other students' answer from preliminary. Researchers from the content of the dialogue with each student to find out the structure, based on the SOLO classification model, resulting in UMR conception cycle which practice again in this remedial teaching to confirm the cycle of the UMR conception:

(1) The percentage of wrong answer for each question in preliminary test which all students participated average is about 20% (see Appendix Tables 2 and 3). The first and second grade students have no significant difference;

(2) Students' pre-test and post-test results shown in Table 2. The research data collected from 10 students. It is not easy for students to attend remedial teaching at the Saturday morning;

(3) Student A must be traced back to only unilateral unknown to start learning. Therefore, during this time,

he/she could not yet solve the linear equations which have unknowns in both sides. Question 7 does not appear before the process of this study, but the students might have done analogous one before;

(4) The authors can find four teaching features which students should have during the remedial teaching, shown as in Figure 2 (e.g., $3x+1=3+2x$).

(5) After the pre-test, the authors interviewed Students about question 8, students' responses as follows:

(A) substitute m = 0, It is OK, so m = 0;

(B) 2m − 4m = 4m − 4m, -2m = 0, $\frac{-2m}{-2}=\frac{0}{-2}$, m = 0;

(C) Substitute any number is not OK, so consider 0;

(D) 2m = 4m, (eliminate m) but 2 ≠ 4, so m = 0;

(E) if m < 0, substitute it, it is not OK; if m > 0, substitute it, it is not OK, so consider m = 0;

(F) 2m = 4m, 2m − 4m＝0, -2m = 0, m = 0;

(G) I do not know.

Table 2

Students' Pre-test and Post-test Results

Student	A	B	C	D	E	F	G	H	I	J
Pre-test	0	1	3	0	20	20	20	20	20	20
Post-test	0	19	19	19	20	20	19	20	20	20
Incorrect	blank	No.7	No.7	No.7			No.7			

Note. Numbers in the cells are showed as the number of right answers.

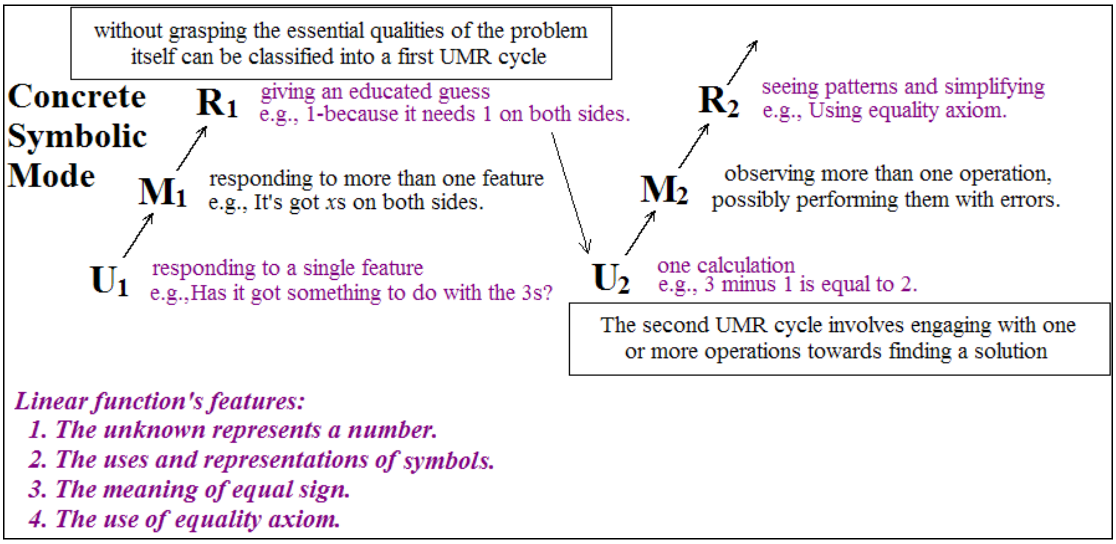

Figure 2. Students' UMR conception cycle in solving linear equation.

Discuss and Suggestion

In this study, the authors use UMR cycle and the concept of SOLO taxonomy to analyze the formation and development of students in the concept of solving linear equation. In the structure of the UMR, the researchers think that students must "awake" their own solution steps of the equation, manipulation, and judgment. Behind these operations, "cognition" must produce to monitor the existence of calculations, but do not know how to interpret. For example, in the pre-test, the student suddenly "stuck" like Lima's research by question 8 can be found. When students see the manipulation, such as 2 = 4 or -2m = 0 (pre-test question 8), they do not know how to do. In addition, some students could also get m = 0, if they can substitute m = 0 into the linear equation.

When student E−J use equality axiom, they do not know the reason why it is correct, but just know "change side, change sign". If they can know the reason why they are doing, the consolidation of the concept will be well. The students should overcome the four features as Figure 2 showed.

Suitable interventions from a teacher at the point of transition may be crucial for students learning algebra for the first time. The UMR conception cycle in this study could be helpful when teaching students, but teachers should know that what students know and what they do not know, in order to provide a better learning environment.

References

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning: The SOLO taxonomy*. New York, N. Y.: Academic Press.

Collis, K. F. (1972). *A study of the relationship between formal thinking and combinations of operations*. Newcastle, Australia: University of Newcastle.

Collis, K. F. (1975). *The development of formal reasoning*. Newcastle, Australia: University of Newcastle.

Crowley, L. R. F. (2000). Cognitive structures in college algebra (Unpublished doctoral dissertation, University of Warwick, England).

Filloy, E., & Rojano, T. (1989). Solving equations, the transition from arithmetic to algebra. *For the Learning of Mathematics: An International Journal of Mathematics Education, 9*(2), 19-25.

Foster, R. (1994). Counting on success in simple arithmetic tasks. Proceedings of *The 18th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 360-367). Lisbon, Portugal.

Freitas, M. A. de. (2002). First-degree equation: Methods of analysis and resolution of errors in high school (Master's thesis, PUC-SP, Sao Paulo).

Gray, E., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A proceptual view of simple arithmetic. *The Journal for Research in Mathematics Education, 26*(2), 115-141.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317-326.

Kota, S., & Thomas, M. O. J. (1998). Students' arithmetic preference: Effect on problem-solving ability. *Hiroshima Journal of Mathematics Education, 6*, 33-47.

Küchemann. D. E. (1981). Algebra. In K. M. Hart, M. L. Brown, D. E. Küchemann, D. Keslake, G. Ruddock, & M. McCartney (Eds.), *Children's understanding of mathematics* (Vol. 11-16, pp. 102-119). London: John Murray.

Lima, R. N., & Tall, D. O. (2006). The concept of equations: What have students met before? Proceedings of *The 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 233-241), Prague, Czech Republic.

Lima, R. N., & Tall, D. O. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics, 67*(1), 3-18.

Macgregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education, 24*(3), 217-232.

McGowen, M. A., & Tall, D. O. (1999). Concept maps and schematic diagrams as devices for documenting the growth of mathematical knowledge. In O. Zaslavsky (Ed.), Proceedings of *The 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 281-288), Haifa, Israel.

Pegg, J., & Tall, D. (2010). The fundamental cycle of concept construction underlying various theoretical framework. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education-seeking new frontiers* (pp. 173-192). Springer-Verlag Berlin Heidelberg.

Tall, D. O., & Thomas, M. O. J. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics, 22*(2), 125-147.

Tall, D. O., & Thomas, M. O. J. (2001). The long-term cognitive development of symbolic algebra. *International Congress of Mathematical Instruction (ICMI) Working Group Proceedings—The Future of the Teaching and Learning of Algebra* (Vol. 2, pp. 590-597), Melbourne.

Thomas, M. O. J. (1994). A process-oriented preference in the writing of algebraic equations. Proceedings of *the 17th Mathematics Education Research Group of Australasia Conference* (pp. 599-606), Lismore, Australia.
Thurston, W. P. (1990). Mathematical education. *Notices of the American Mathematical Society, 37*(7), 844-850.
Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics, 49*, 341-359.

Appendix

Table 1

(1) Pre-test Find x

(1) $3x + 1 = 3 + 2x$	(2) $x + 2 = 1 + 2x$	(3) $4x + 3 = 5 + x$	(4) $3x + 1 = 9 + 7x$
(5) $5t - 3 = 8$	(6) $3x - 1 = 3 + x$	(7) $x + 1 = 2x - 2$	(8) $2m = 4m$
(9) $1 - 3x = 5 + 2x$	(10) $2 - x = 1 + 2x$	(11) $3 - 4x = 9 + x$	(12) $2 - 3x = 5 + 7x$
(13) $3x + 1 = 2 - 2x$	(14) $x + 4 = 1 - 2x$	(15) $4x + 3 = 5 - 2x$	(16) $3x + 1 = 8 - 6x$
(17) $3x - 1 = 3 - 2x$	(18) $x - 2 = 1 - 2x$	(19) $4x - 3 = 5 - x$	(20) $3x - 1 = 9 - 7x$

(2) Post-test Find x

(1) $2x + 1 = 3 + 3x$	(2) $2x + 2 = 1 + x$	(3) $x + 3 = 5 + 4x$	(4) $7x + 1 = 9 + 3x$
(5) $8 = 4 - 5t$	(6) $x - 1 = 3 + 3x$	(7) $x + 1 = x - 2$	(8) $4x = 2x$
(9) $5 - 2x = 1 + 3x$	(10) $1 - 2x = 2 + x$	(11) $9 - x = 3 + 4x$	(12) $5 - 7x = 2 + 3x$
(13) $2x + 2 = 1 - 3x$	(14) $2x + 1 = 4 - x$	(15) $2x + 5 = 3 - 4x$	(16) $6x + 8 = 1 - 3x$
(17) $x - 2 = 6 - 4x$	(18) $x - 3 = 2 - x$	(19) $4x - 3 = 2 - x$	(20) $3x - 5 = 9 - 7x$

Table 2

Preliminary 37 11th-grade Students' Responses (Correct: 17 Persons, Blank 4 Persons)

Item	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Successful	33	32	31	30	26	31	29	23	28	31	30	29	31	31	28	29	32	29	29	30
Incorrect (%)	11	14	16	19	30	16	22	38	24	16	19	22	16	16	24	22	14	22	22	19

Note. The average incorrect rate: 20.1%, incorrect (%) is calculated by $\left(1-\frac{\text{number of correct}}{37}\right)\times 100\%$

Table 3

Pre-test 94 Students' Responses (Correct: 43 Persons, Blank: 11 Persons)

Item	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Successful	82	77	76	74	80	79	77	67	71	76	68	74	76	69	73	78	79	75	80	81
Incorrect (%)	13	18	19	21	15	16	18	29	25	19	28	21	19	27	22	17	16	20	15	14

Note. The average incorrect rate: 20%, incorrect % is calculated by $\left(1-\frac{\text{number of correct}}{94}\right)\times 100\%$