



CAL-PASS SAN DIEGO CENTRAL MATHEMATICS COUNCIL

The Concept of a Line

A study of its development through
the grade levels.

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This study originated from the realization that many community college students have significant difficulties with the concepts related to linear functions and their graphs even though these topics are included in the mathematics curriculum from about third grade onward. It is an attempt to identify possible gaps in the math standards in this area, as well as possible oversights in emphasis in classroom practice at various levels.

Linear functions and their graphs are important in their own right as models of practical situations. They are also used as regression lines in statistics and modeling, as simple first examples of various concepts in economics, and as a means of finding linear approximations.

Working with linear functions and graphs is also a critical part of meeting all three of the “essential areas of focus for all entering college students” in the 2010 Intersegmental Committee of the Academic Senates document. Note especially that the second of these states that “Emphasis should be placed on various representations of functions – using graphs, tables, variables, and words – and on the interplay among the graphical and other representations...” Students who are able to move seamlessly between a linear equation, a table of values, slope and intercepts and a graph will be prepared to do the same with other types of functions. Students who must rely exclusively on plotting points to produce a line will be ill-prepared to interpret the graphs of quadratic, exponential and other functions.

Several of the continuing difficulties that students have with linear functions and graphs overlap with their misconceptions about fractions. One of these is the unspoken assumption that the only numbers on the number lines are the integers. Therefore the act of connecting several points on the graph of a line is merely demonstrating that they all lie on the line, implying nothing about function values for non-integer inputs. Another misconception lies in seeing slope exclusively as a ratio of two numbers (rise over run) and not as a unit rate (a single number). These misconceptions can have consequences for future difficulties even with calculus.

In the following document we have quoted standards pulled from several sources including California State Standards (1999) (CSS), the California Common Core Standards for Mathematics (CCCSM), Statement on Competencies in Mathematics Expected of Entering College Students, and the National Common Core Standards for Mathematics (CCSM). In addition, the Cal-PASS deconstruction documents for Algebra I through Precalculus and the San Diego County Department of Education's backward mapping project for algebra supplied the example problems throughout the document.

For the grades 3 – 7, the examples are taken from the Backwards Mapping project from the San Diego County of Education Department. Those items marked AF are from the Algebra and Functions component. Those marked MG are from the Measurement and Geometry component. The first box indicates the actual state standard, and the second is a benchmark and the last box is an example created by the backwards mapping project participants.

For the Algebra 1, Algebra 2, Geometry, Mathematical Analysis and Linear Algebra standards, the examples are taken from the Cal-PASS deconstructed standards. Those standards with a * preceding them are referenced as key in the Statement on Competencies in Mathematics Expected of Entering College Students by the Intersegmental Committee of the Academic Senates of the California Community Colleges, the California State University and the University of California. (2010)

Notes refer to resources outside of the California State Standards, and comments refer to the opinions of the writers.

California Grade Level Standards

Grade 3:

Algebra and Functions (AF) 2.0 Students represent simple functional relationships:

2.1 Solve simple problems involving a functional relationship between two quantities (e.g., find the total cost of multiple items given the cost per unit).

2.2 Extend and recognize a linear pattern by its rules (e.g., the number of legs on a given number of horses may be calculated by counting by 4s or by multiplying the number of horses by 4).

Note: The Common Core Standards for Mathematics (CCSM) includes the ability to represent unit fractions on a number line. “Students ... use fractions to represent parts of a whole or distances on a number line that begins with zero.”

Grade 4:

AF 1.0 Students use and interpret variables, mathematical symbols, and properties to write and simplify expressions and sentences:

1.1 Use letters, boxes, or other symbols to stand for any number in simple expressions or equations (e.g., demonstrate an understanding and the use of the concept of a variable).

1.4 Use and interpret formulas (e.g., area = length \times width or $A = lw$) to answer questions about quantities and their relationships.

1.5 Understand that an equation such as $y = 3x + 5$ is a prescription for determining a second number when a first number is given.

| California State Standard | Benchmark | sample problem | | | | | | | | | | | | |
|---|--|---|-----------|------------|---|--|---|--|---|--|---|--|---|--|
| AF 1.5 Understand that an equation such as $y = 3x + 5$ is a prescription for determining a second number when a first number is given. CST (2) | AF 1.5 In an equation in two variables, find the value of the first variable when given the value of the second variable. | AF 1.5 Complete the table using the Rule: $y = 2x + 4$ <table border="1"><thead><tr><th>Input (x)</th><th>Output (y)</th></tr></thead><tbody><tr><td>0</td><td></td></tr><tr><td>1</td><td></td></tr><tr><td>2</td><td></td></tr><tr><td>3</td><td></td></tr><tr><td>4</td><td></td></tr></tbody></table> | Input (x) | Output (y) | 0 | | 1 | | 2 | | 3 | | 4 | |
| Input (x) | Output (y) | | | | | | | | | | | | | |
| 0 | | | | | | | | | | | | | | |
| 1 | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | |

Comment: This is the first time students see a number and a variable next to each other with no operation symbol. They need to understand that the implied operation is multiplication.

AF 2.0 Students know how to manipulate equations:

2.1 Know and understand that equals added to equals are equal.

2.2 Know and understand that equals multiplied by equals are equal.

Grade 4:

Measurement and Geometry (MG) 2.0 Students use two-dimensional coordinate grids to represent points and graph lines and simple figures:

2.1 Draw the points corresponding to linear relationships on graph paper (e.g., draw 10 points on the graph of the equation $y = 3x$ and connect them by using a straight line).

California State Standard

Benchmark

sample problem

| <p>MG 2.1* Draw the points corresponding to linear relationships on graph paper (e.g., draw 10 points on the graph of the equation $y = 3x$ and connect them by using a straight line).</p> <p>CST (2)</p> | <p>MG 2.1 Plot points on a graph that correspond to a linear relationship.</p> | <p>MG 2.1 Plot the points from the table. Connect the points to form a line.</p> <table border="1"><thead><tr><th>x</th><th>y</th></tr></thead><tbody><tr><td>0</td><td>3</td></tr><tr><td>1</td><td>5</td></tr><tr><td>2</td><td>7</td></tr><tr><td>3</td><td>9</td></tr><tr><td>4</td><td>11</td></tr></tbody></table> | x | y | 0 | 3 | 1 | 5 | 2 | 7 | 3 | 9 | 4 | 11 |
|--|---|---|---|---|---|---|---|---|---|---|---|---|---|----|
| x | y | | | | | | | | | | | | | |
| 0 | 3 | | | | | | | | | | | | | |
| 1 | 5 | | | | | | | | | | | | | |
| 2 | 7 | | | | | | | | | | | | | |
| 3 | 9 | | | | | | | | | | | | | |
| 4 | 11 | | | | | | | | | | | | | |

2.2 Understand that the length of a horizontal line segment equals the difference of the x -coordinates.

2.3 Understand that the length of a vertical line segment equals the difference of the y -coordinates.

MG 3.0 Students demonstrate an understanding of plane and solid geometric objects and use this knowledge to show relationships and solve problems:

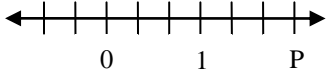
3.1 Identify lines that are parallel and perpendicular.

Number Sense (NS) 1.9 Identify on a number line the relative position of positive fractions, positive mixed numbers, and positive decimals to two decimal places.

California State Standard

Benchmark

sample problem

| | | |
|--|--|---|
| <p>NS 1.9 Identify on a number line the relative position of positive fractions, positive mixed numbers, and positive decimals to two decimal places.</p> | <p>NS 1.9 Plot decimals and mixed numbers on a number line.</p> | <p>NS 1.9 What value is represented by the point P on this number line?</p>  |
|--|--|---|

Note: The CCSM states that students “understand that the length of a number line (interval from 0 to 1) can be divided into parts of equal fractional length.”

Grade 5:

AF 1.0 Students use variables in simple expressions, compute the value of the expression for specific values of the variable, and plot and interpret the results:

1.1 Use information taken from a graph or equation to answer questions about a problem situation.

1.2 Use a letter to represent an unknown number; write and evaluate simple algebraic expressions in one variable by substitution.

1.3 Know and use the distributive property in equations and expressions with variables.

1.4 Identify and graph ordered integer pairs in the four quadrants of the coordinate plane.

1.5 Solve problems involving linear functions with integer values; write the equation; and graph the resulting ordered pairs of integers on a grid.

California State Standard

Benchmark

sample problem

| | | |
|---|---|---|
| NS 1.5 Identify and represent on a number line decimals fractions, mixed numbers and positive and negative integers. | NS 1.5 Graph decimals fractions, mixed numbers and positive and negative integers. | NS 1.5 Graph the following numbers on the number line: -5, 1, 5, $\frac{1}{5}$, $\frac{3}{4}$, $-4\frac{1}{4}$, 2.75, -3.5. |
|---|---|---|

Note: The CCSM includes the ability recognize that “equivalent fractions correspond to the same point on a number line.”

Comment: Does this standard mean no negative decimals or fractions even though the example includes them?

Grade 6:

AF 1.0 Students write verbal expressions and sentences as algebraic expressions and equations; they evaluate algebraic expressions, solve simple linear equations, and graph and interpret their results:

1.1 Write and solve one-step linear equations in one variable.

1.2 Write and evaluate an algebraic expression for a given situation, using up to three variables.

1.3 Apply algebraic order of operations and the commutative, associative, and distributive properties to evaluate expressions; and justify each step in the process.

1.4 Solve problems manually by using the correct order of operations or by using a scientific calculator.

Comment: Solving an equation is not directly related to the concept of a line, however for the student to understand what they are doing, a linear model is helpful.

| California State Standard | Benchmark | sample problem |
|--|---|--|
| AF 1.1 Write and solve one-step linear equations in one variable. | AF 1.1a Write one-step linear equations in one variable. | AF. 1.1 a Which algebraic equation best describes the total growth (T) in height of pine trees over a three-year period, if g equals the rate of growth in centimeters per year? A) $T = 3g$ B) $T = 3 + g$ C) $T = \frac{g}{3}$ D) $T = \frac{3}{g}$ |
| | AF 1.1b Solve one-step linear equations in one variable | AF 1.b Write an equation for the following problem and solve. The number of players on the football team increased by 3 equals 21. How many players were on the original team? |

Note: The CCSM states that students “understand that for a ratio a:b, the corresponding unit rate is a/b.” *This concept is basic to the understanding of the slope of a line as a unit rate.*

Grade 7:

***AF 1.0 Students express quantitative relationships by using algebraic terminology, expressions, equations, inequalities, and graphs:**

1.5 Represent quantitative relationships graphically and interpret the meaning of a specific part of a graph in the situation represented by the graph.

AF 3.0 Students graph and interpret linear and some nonlinear functions:

3.1 Graph functions of the form $y = nx^2$ and $y = nx^3$ and use in solving problems.

3.2 Plot the values from the volumes of three-dimensional shapes for various values of the edge lengths (e.g., cubes with varying edge lengths or a triangle prism with a fixed height and an equilateral triangle base of varying lengths).

3.3 Graph linear functions, noting that the vertical change (change in y -value) per unit of horizontal change (change in x -value) is always the same and know that the ratio (“rise over run”) is called the slope of a graph.

Comment: Students need to understand that slope refers to more than just the ‘rise over the run’. Other applications include unit rate and the relationship between the changes in the dependent and independent variables. Sometimes one is interested in considering a change in x prior to the change in y .

3.4 Plot the values of quantities whose ratios are always the same (e.g., cost to the number of an item, feet to inches, circumference to diameter of a circle). Fit a line to the plot and understand that the slope of the line equals the ratio of quantities.

Comment: The ratio of circumference to diameter is an irrational number.

Note: According to the CCSM, by grade 7 students should understand that “fractions and their opposites form a system of numbers called the rational numbers, represented by points on a number line.” In addition they “prove that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines.” It is in grade 7 that students “plot proportional relationships on a coordinate plane where each axis represents one of the two quantities involved, observe that the graph is a straight line through the origin, and find unit rates from a graph. Explain what a point (x, y) means in terms of the situation, with special attention to the points $(0,0)$ and $(1, r)$ where r is the unit rate.”

Note: The State of California has deemed that all students will take algebra in Grade 8, and hence the Algebra I standards are reflective of what is taught. The CCSM separates Grade 8 from the High School curriculum. For grade 8 the following concepts are included as core:

- “Students use linear equations ... to represent, analyze and solve a variety of problems. ... they understand that the slope (m) of a line is a constant rate of change.”

Grade 7:

- “Understand that the slope of a non-vertical line in the coordinate plane has the same value for any two distinct points used to compute it.” (Linear equations in two variables #3)
- “Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship; from two (x, y) values, including reading these from a table; or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.” (Functional relationships between quantities #6)
- “Understand that a straight line is a widely used model for exploring relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line...” (Statistics and Probability #3)
- “Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.” (Statistics and Probability #4)

Algebra I standards:

4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

Computational and Procedural Skills

1. Simplify the following expressions:

a. $-\frac{5}{2}x - 6 + \frac{1}{4}x - 5$

c. $-3(2x - 5)$

b. $5(3x - 4)$

d. $-2(3x - 1) - 2(5x - 4)$

***5.0** Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

1. Solve the following equations and inequalities:

a. $2x+3=15$

f. $-3x < 9$

b. $\frac{1}{4}x - 3 = -8$

g. $4x - 7 \leq 5x + 3$

c. $-4(m+6) = -36$

h. $5(x-1) > \frac{5}{4}x$

d. $\frac{x-8}{4} - 3 = -2$

i. $8 - 2y \geq 4 - y$

e. $5x + 2(1-x) = 2(2x-1)$

j. Solve $\frac{x-8}{4} - 3 = -2$ and justify each step.

k. The length of a rectangle is six less than twice the width. Its perimeter is 36 inches. Find the dimensions of the rectangle.

l. The sum of 32 and twice a number is at most 118. Determine all numbers with this property?

m. Jennifer's cell phone plan charges \$32.99 per month plus \$0.14 for each text message. Write a linear inequality and solve it to find how many text messages she can send per month and still keep her bill under \$50.

Comment: This last question was added because it was determined that a real world application was needed.

Algebra I standards:

6.0 Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

Computational and Procedural Skills

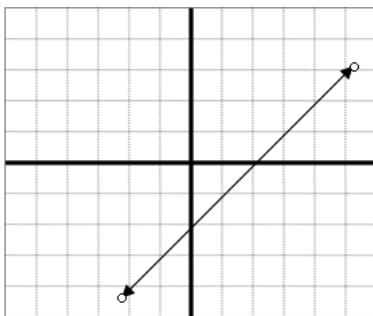
1. Find the “ x ” and “ y ”-intercepts for the line defined by the following equation:
 $2x + 3y = 9$
2. Use the “ x ” and “ y ”-intercepts to graph the line given by the equation above:
 $2x + 3y = 9$
3. Graph the following lines using the method of your choice. Identify and label the “ x ” and “ y ” intercepts for each graph if they exist:

| | |
|----------------------------|--------------------------------------|
| a) $3x - 5y = 10$ | d) $x = 3.5$ |
| b) $y = \frac{-2}{3}x + 4$ | e) $2x + 4y = 3$ |
| c) $y = 2$ | f) $\frac{1}{2}x - \frac{3}{4}y = 2$ |
4. Graph the solution set for the following inequalities:

| | | |
|------------------|-------------------------------|---|
| a) $2x - 3y < 6$ | b) $y \geq \frac{-3}{4}x + 2$ | c) $\frac{1}{2}x - \frac{2}{3}y \leq \frac{5}{6}$ |
|------------------|-------------------------------|---|

Conceptual Understanding

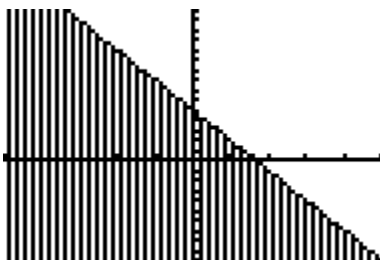
1. Sketch the graph of a line that has no x -intercept.
2. Identify the “ x ” and “ y ”- intercepts from the graph of the given line.



3. Can a line have more than one x -intercept? Explain your answer using a diagram.

Algebra I standards:

4. The solution to an inequality has been graphed correctly below. Insert the correct inequality symbol in the inequality below to match the graph of the solution. (Everything else about the inequality is correct – it just needs the correct symbol).



y $-3x + 5$

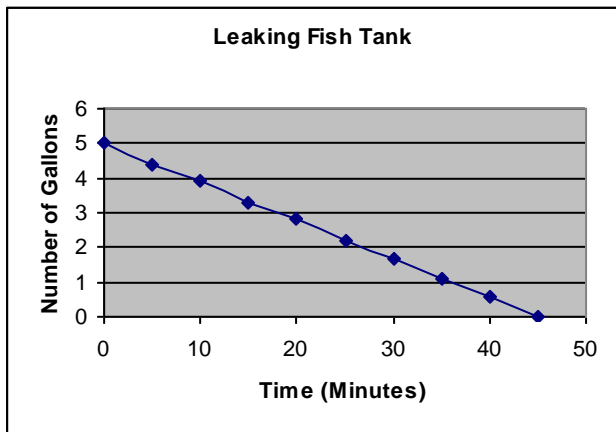
Insert correct symbol in

5. When is it advantageous to use the x and y-intercepts to graph the equation of a line? When would it perhaps be easier or better to use another graphing method? Give an example to illustrate your answers to both of these questions.

Problem Solving/Application

1. The graph displayed below is the graph of the following equation: $y = -\frac{1}{5}x + 5$, where “x” represents the amount of time that has passed since a 5 gallon fish tank sprung a leak, and “y” represents the number of gallons of water in the tank after the leak.
- What is the significance of the “x-intercept” in this situation? What information is given to us by this point?
 - What is the significance of the “y-intercept” in this situation? What information is given to us by this point?
 - What is the rate of leaking in gallons per minute?

Note: This question was added by the authors as an example of unit rate.



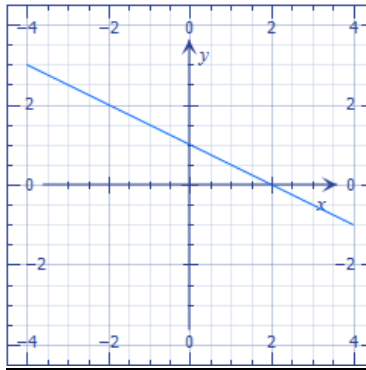
Algebra I standards:

2. The cost of a trash pickup service is given by the following formula:
 $y = 1.50x + 11$, where “x” represents the number of bags of trash the company picks up, and “y” represents the total cost to the customer for picking up the trash.
- What is the “y-intercept” for this equation?
 - What is the significance of the “y-intercept” in this situation? What does it tell us about this trash pickup service?
 - Draw a sketch of the graph which represents this trash pickup service.

7.0 Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.

Computational and Procedural Skills

- Given the equation of a line, $y = \frac{3}{2}x - 1$ determine whether or not the following points lie on the line: (4,7), (-6, -10) and (1/3, -1/2).
- Write the equation of the line with slope 2/3 and y-intercept of -2
- Write the equation of the line with a slope of 4 and passing through the point (-2, 5)
- Write the equation of the line passing through the points (-3, 7) and (4, -7)
- Write the equation of the line graphed below.



Conceptual Understanding

Consider the table of values below.

| Episode Number | Number of Survivors |
|-----------------------|----------------------------|
| 1 | 25 |
| 2 | 23 |
| 3 | 21 |

- Identify the input and output variables
- Determine whether or not the data could be linear.
- If the data is linear find and interpret the slope.

Algebra I standards:

Problem Solving/Application

A taxi driver charges a \$2.00 pick up fee plus \$3 for each mile traveled.

If you are interested in calculating the cost of your trip, what quantities would be of interest?

- d) Identify the input and output quantities.
- e) Make a table for inputs 0, 1, and 2.
- f) Identify and interpret the meaning of the slope.
- g) Identify and interpret the meaning of the vertical axis intercept.
- h) Write an equation describing the output as a function of the input.
- i) Use your equation to determine the cost of a 12-mile taxi ride.

***8.0** Students understand the concepts of parallel lines and perpendicular lines and how those slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

- a) **Recognize parallel lines as having the same slope and different y-intercepts graphically.**

Graph: $y = 2x + 4$ and $y - 2x = 1$ on the same rectangular coordinate system. Does it appear the two lines are parallel? Explain!

- b) **Recognize parallel lines as having the same slope and different y-intercepts algebraically.**

Algebraically, demonstrate the two lines whose equations are given by: $2y - 4x = 9$ and $y = 2x - 5$ are parallel.

- c) **Recognize perpendicular lines algebraically by intersecting at 90° angles.**

Graph: $y + x = 5$ and $y - x = 0$ on the same rectangular coordinate system. Does it appear the two lines are perpendicular? Explain!

- d) **Recognize perpendicular lines algebraically as having slopes whose product is -1.**

Comment: According to the CCSM, a geometric proof is included in geometry. Could an exercise that looks at it heuristically be included in the curriculum at this time? For example, comparing the graphs of a pair of perpendicular lines will show that one slope is positive and one negative and one has a steep ($> 45^\circ$) slope while the other is more shallow.

Algebraically, demonstrate the two lines whose equations are given by: $4y - 3x = 7$ and $3y + 4x = 15$ are perpendicular.

Algebra I standards:

- e) Understand the relationship between the slopes of parallel lines. See a above
- f) Determine the equation of a line parallel to a given line and passing through a point not on the line.
- g) Find an equation of a line, in slope-intercept form, through point $P(3,-2)$ that is parallel to $3y + 5x = 7$.
- h) Determine the equation of a line perpendicular to a given line and passing through a point not on the line.

Find an equation of a line, in slope-intercept form, through point $P(-12,4)$ that is perpendicular to $y - 6x = 3$.

***9.0** Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

Computational and Procedural Skills

- 3. Identify the intersection of two shaded areas created by the graphs of linear inequalities in two variables as the solution to the system:
- 4. Solve the system of inequalities $\begin{cases} x + y < 7 \\ 2x - 3y \geq 12 \end{cases}$ graphically.

Conceptual Understanding

- 1. Interpret the solution as representing the intersection of two lines or the same line:
Suppose a system of two linear equations in two variables has a solution, $(3,5)$. When graphing these two equations on the same Cartesian coordinate system, what would you expect the graph of the two lines to look like?

Suppose a system of two linear equations in two variables has infinite solutions in the form (x,y) . When graphing these two equations on the same Cartesian coordinate system, what would you expect the graph of the two lines to look like?

Solve the system: $\begin{cases} y - 3x = -3 \\ 3x + 2y = 12 \end{cases}$ by graphical method.

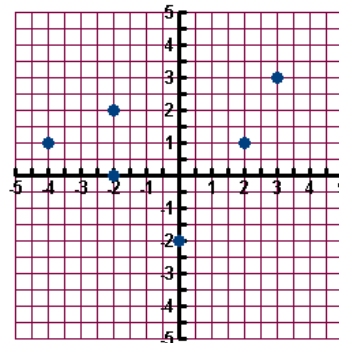
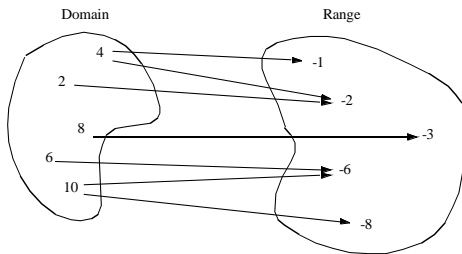
Algebra I standards:

Solve the system: $\begin{cases} y = 3x + 6 \\ -6x + 2y = 12 \end{cases}$ by graphical method.

***16.0** Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.

Conceptual Understanding

- Given the following sets of ordered pairs, give the domain and range of the relation; determine whether the relation *is* or *is not* a function:
 - $\{(-4,3), (-2,1), (0,5), (-2,-8)\}$
 - $\{(3,7), (1,4), (0,-2), (-1,-1), (-2,5)\}$
- Given the following figures, give the domain and range of the relation; determine whether the relation *is* or *is not* a function:



- Define *relation* and *function*. Compare the two definitions. How are they alike? How are they different?

***17.0** Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

Computational and Procedural Skills

- Complete each ordered pair so that it is a solution to $3x + y = 10$. Then identify the domain and range of the resulting set of ordered pairs.
 - $(1, ?)$
 - $(2, ?)$
 - $(?, 4)$
 - $(3, ?)$
 - $(0, ?)$

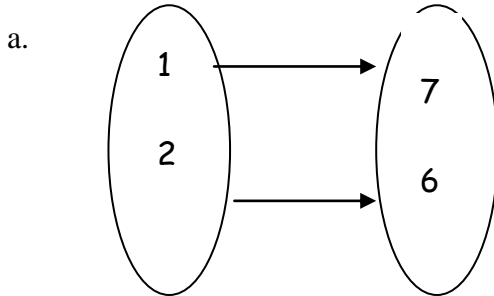
Algebra I standards:

Conceptual Understanding

1. State the domain and range of the relation:

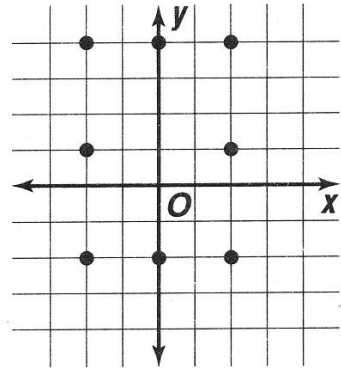
$$\{ (2,4), (2,5), (4,6), (7,2), (5,10), (8,4), (3,6) \}$$

2. Express the relation in each mapping, table, or graph as a set of ordered pairs and then state the domain and range of each:



| x | y |
|-----|-----|
| 3 | 4 |
| 5 | 6 |
| 7 | 8 |
| 7 | 6 |
| 5 | 4 |
| 3 | 3 |

c.



Problem Solving/Application

1. The table below shows the amount that a company charges for a bike rental. Identify the domain and range. Write a set of ordered pairs for the function. Assuming the cost of the bike rental is a linear function of the number of hours the bike was rented, find a formula for the function.

| | | | | | | |
|------------|----|----|----|----|----|----|
| Time (hrs) | 1 | 2 | 3 | 4 | 5 | 6 |
| Cost (\$) | 20 | 24 | 28 | 32 | 36 | 40 |

Algebra I standards:

2. The table below shows the per-minute rate for a cell phone. Identify the domain and range. Write a set of ordered pairs for the function.. Assuming the cost is a linear function of the number of minutes, find a formula for the function.

| | | | | | | |
|-----------|------|------|------|------|------|------|
| Minutes | 1 | 2 | 3 | 4 | 5 | 6 |
| Cost (\$) | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 |

3. The table below shows the distance that a car travels over time. Identify the domain and range. Write a set of ordered pairs for the function. Use the table to write an equation for this function (Comment: Assume the function is linear).

| | | | | | | |
|------------------|----|-----|-----|-----|-----|-----|
| Time (hrs) | 1 | 2 | 3 | 4 | 5 | 6 |
| Distance (miles) | 50 | 100 | 150 | 200 | 250 | 300 |

***18.0** Students determine whether a relation defined by a graph, a set of ordered pairs, or a symbolic expression is a function and justify the conclusion.

Computational and Procedural Skills

1. Determine whether or not the below relations are functions. Justify your answer.

a. $\{(4, 5), (-3, 6), (5, 6), (-2, 4)\}$

b. $\{(7, -3), (-3, 6), (7, -3), (-6, 5)\}$

c. $y = 3x - 1$

Conceptual Understanding

1. If you are looking at a graph, how do you determine whether or not it is the graph of a function?
2. Write a set consisting of three ordered pairs that is a relation, but not a function. Explain why the set of ordered pairs you wrote is not a function.
3. Draw a graph that is both a relation and a function. Write a sentence that states why the graph you drew represents both a relation and a function.

Geometry:

***7.0** Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.

Algebra II:

2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or with matrices.

Computational and Procedural Skills

1. Solve the system by graphing.

a.
$$\begin{cases} -3x + 2y = -6 \\ 2x - y = 4 \end{cases}$$

b.
$$\begin{cases} x \geq 0 \\ y \geq 0 \\ -x - y \geq -5 \\ -2x + 3y \geq -3 \end{cases}$$

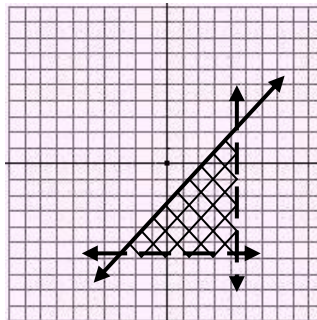
2. Solve the system by substitution:

a.
$$\begin{cases} -x + 2y = 11 \\ 3x - 2y = -13 \end{cases}$$

b.
$$\begin{cases} x - 3y = -2 \\ 3x + 9y = 9 \end{cases}$$

Conceptual Understanding

- a. If you graph two lines in the same coordinate plane, what are the possible outcomes?
- b. A system of linear equations may have infinitely many solutions. Explain how this is possible.
- c. Does every system of linear equations have a solution? Explain.
- d. After a solution of a system of linear equations is found, why should the solution be checked algebraically?
- e. If the solution exists, what is the solution of a system of linear inequalities?
- f. When is it advantageous to use the substitution method? The matrix method? Give an example to illustrate your answers to both parts of this question.
- g. Write the system of linear inequalities which describes the shaded region below (*Inside the triangle is shaded*):



Algebra II:

Problem Solving/Application

1. Your family receives basic television and two movie channels for \$32.30 a month. Your neighbor receives basic cable and four movie channels for \$43.30 a month. What is the monthly charge for just the basic cable? (Assume that the movie channels have the same monthly cost.) What is the monthly charge for one movie channel?
2. The senior class has a carnival to raise money for a senior trip. Student tickets are \$6 and adult tickets are \$11. Since 324 people were in attendance, the senior class raised \$2,359. How many of the people in attendance were adults?

24.0 Students solve problems involving functional concepts, such as composition, defining the inverse function and performing arithmetic operations on functions.

Computational and Procedural Skills

1. Let $f(x) = 3x + 2$ and $g(x) = 4x$; find $g(f(2))$:
2. What is the inverse of $f(x) = 4x - 6$?

Conceptual Understanding

1. Show that $f(x) = 4x + 9$ and $g(x) = \frac{x - 9}{4}$ are inverse functions.

Comment: An extension to this problem might be to ask, graphically what is the relationship between the two functions?

Problem Solving/Application

1. A department store is having a 20%-off-everything sale. You also have a \$10 coupon for any purchase.
 - a. Write the function M that represents the sale price of an item after the 20% discount, and a function K that represents the price of an item after the \$10 coupon:

Determine which is the best deal for you, discount then coupon, or coupon then discount, when buying an item costing \$25.

Algebra II:

25.0 Students use properties from number systems to justify steps in combining and simplifying functions.

Computational and Procedural Skills

1. What property of real numbers enables you to simplify $\frac{x^2 + 3x + 2}{x + 1}$ to $x + 2$ for all values of x not equal to -1 ?

2. What properties of real numbers enable you to simplify $\frac{x^2 + 3x - 18}{x - 3}$ to $x + 6$ for all values of x not equal to 3 ?

Comment: How does the graph of the original function differ from the graph of the simplified form?

Trigonometry

7.0 Students know that the tangent of the angle that a line makes with the x -axis is equal to the slope of the line.

Comment: Here is another example of a possibly irrational slope. Students will need more than 'rise over run' in order to interpret this.

Linear Algebra

6.0 Students demonstrate an understanding that linear systems are inconsistent (have no solutions), have exactly one solution, or have infinitely many solutions.

Computational and Procedural Skills

Determine if the following systems are consistent or inconsistent. Identify consistent systems as dependent or independent.

1.
$$\begin{cases} 2x - y = 4 \\ 4x - 2y = 8 \end{cases}$$

2.
$$\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$$

3.
$$\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$$

Conceptual Understanding

1. Given the following matrices, determine if the corresponding systems are consistent or inconsistent. Identify consistent systems as dependent or independent.

Where appropriate, give the solution.

a)
$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

c)
$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. If you solve a system of linear equations without using matrices, how do you know if the system is consistent or inconsistent? If consistent, how do you know if the system is dependent or independent.

8.0 Students interpret geometrically the solution sets of systems of equations. For example, the solution set of a single linear equation in two variables is interpreted as a line in the plane, and the solution set of a two-by-two system is interpreted as the intersection of a pair of lines in the plane.

Calculus:

4.0 Students demonstrate an understanding of the formal definition of the derivative of a function at a point and the notion of differentiability:

4.1 Students demonstrate an understanding of the derivative of a function as the slope of the tangent line to the graph of the function.

4.2 Students demonstrate an understanding of the interpretation of the derivative as an instantaneous rate of change. Students can use derivatives to solve a variety of problems from physics, chemistry, economics, and so forth that involve the rate of change of a function.

Additional Standards

From the National Common Core Mathematics Standards for High School, the following standards reflect the linear understanding expected beginning with the understanding that “linear functions with a constant term of zero, describe proportional relationships.

In addition, the following are from the linear, quadratic and exponential models standards:

- “Understand that a linear function defined by $f(x) = mx + b$ for some constants m and b , models a situation in which a quantity changes at a constant rate, m relative to another.” (1)¹
- “Understand that linear functions grow by equal differences over equal intervals...” (4)
- “Arithmetic sequences can be seen as linear functions.” (5)
- “Construct a function to describe a linear relationship between two quantities. Determine the rate of change and constant term of a linear function from a graph, a description of a relationship, or from two (x, y) values (including reading from a table).” (10)
- “Interpret the rate of change and constant term of a linear function or sequence in terms of the situation it models, and in terms of its graph or table of values.” (13)

The following are from the Statistics and Probability standards:

- “Functional models may be used to approximate data; if the data approximately linear, the relationship may be modeled with a regression line.” (Introduction)

The following are from Geometric Properties with Equations standards:

- “Understand that two lines with well-defined slopes are perpendicular if and only if the product of their slopes is -1 .” (1)
- “Use the slope criteria for parallel and perpendicular lines to solve geometric problems.” (7)

1 - The number in () identifies the corresponding standard in the CCSM document.

Mathematically Mature Concept of Line as envisioned by the authors

A line (or straight line) is the graph of the set of all solutions of a linear equation $ax + by = c$ or $y = mx + b$. (By ALL solutions, we mean that x and y can take on fractional and even irrational values unless a particular application requires only integer or rational values.)

A complete understanding of line includes the ability to move seamlessly between the equation, a table of values, a graph, and a verbal description. Graphing techniques should include making a table, finding the x and y intercepts, and using the slope and y intercept. Geometric terminology and descriptions of lines are a part of this understanding.

An understanding of slope includes the interpretation of positive and negative slopes, as well as zero and undefined slopes. The concept of slope should gradually expand from the formula and “rise over run” to include unit rates, rational and even irrational numbers. Unit rates are relevant in application and proportion problems. Rational and irrational slopes can arise in regression lines and derivatives.

A line, particularly in the form $y = mx + b$, is used in statistics and other applications to model a relationship if the data suggest that the relationship is linear. In addition it is used to approximate function values if the rate of change (slope, derivative) of the function is known.

A line can also be the boundary for the solution set of a linear inequality in two variables.

Lines are also important as asymptotes when describing the graph or behavior of functions.

For a complete development of the concept of a line, students should learn (not necessarily in this order) that:

- the integers can be represented as points at uniform intervals on a number line
- the absolute value of a number represents its distance from zero
- on the number line, the numbers increase as one moves from left to
- the values between integers can be represented on a number line (rational fractions first then irrational values)
- two points determine a line
- plotting input/output values yields a line given that a linear relation exists
- all lines are straight and continue infinitely in both directions
- the graph of a linear function results in a line
- the change in outputs for a change in inputs is constant for a linear function (slope, rate of change)
- the graph of a line represents all values for which the linear relationship holds and there are an infinite number of such values
- for some real world models a linear function only applies for specific values (ex. positive integers)
- the sign of the slope determines whether the linear function increases or decreases
- the graph of a line is read from left to right when determining whether the graph increases or decreases

- a slope of zero represents a horizontal line while an undefined slope represent a vertical line
- parallel and perpendicular lines can be defined both algebraically and geometrically
- both algebraic and geometric concepts of a line are complementary