US-China Education Review A 5 (2012) 483-496
Earlier title: US-China Education Review, ISSN 1548-6613

Grasp of Consciousness and Performance in Mathematics Making Explicit the Ways of Thinking in Solving Cartesian Product Problems

Maria Tereza Carneiro Soares,
Maria Lucia Faria Moro
Universidade Federal do Paraná, Curitiba-PR, Brasil

Alina Galvão Spinillo
Universidade Federal de Pernambuco,
Recife-PE, Brasil

This study examines the relationship between the grasp of consciousness of the reasoning process in Grades 5 and 8 pupils from a public and a private school, and their performance in mathematical problems of Cartesian product. Forty-two participants aged from 10 to 16 solved four problems in writing and explained their solution procedures by answering the question: "How did I think to solve this problem?". The qualitative analysis identified three response categories as indicators of the participants' grasp of consciousness regarding their ways of reasoning involved in finding the solutions, which can be explained by the reflecting abstractions model. The statistical analysis showed those categories as associated with the performance: progressively more refined justifications came along with mathematically correct solutions to the problems. Differences in performance and the use of justifications for the solutions by participants of both types of school corroborate that result. A teaching process directed towards students' reflection and comprehension regarding specific schemes and conceptual mathematical relations may be at the root of these differences. Thus, teachers' interventions that encourage students to think about their own thinking are recommended.

Keywords: grasp of consciousness, Cartesian product problems, explicit reasoning by writing, mathematical education, elementary school

Introduction

This study deals with the relationship between the grasp of consciousness of the reasoning process involved in the solution of mathematical problems of Cartesian product and performance in solving those problems.

It comes from a wider research project concerning the ways of solving Cartesian product problems of Grades 5 and 8 Brazilian elementary school students from different schools located in different regions. However, it analyzes only a small part of the data obtained in the project in order to examine the hypothesis that there is a relationship between the performance in solving the type of mathematical problem and the levels of students' grasp of consciousness of the ways of reasoning they adopt in terms of the specific actions and

Maria Tereza Carneiro Soares, Ph.D., Programa de Pós-Graduação em Educação, Universidade Federal do Paraná.
Maria Lucia Faria Moro, Ph.D., Programa de Pós-Graduação em Educação, Universidade Federal do Paraná.
Alina Galvão Spinillo, Ph.D., Programa de Pós-Graduação em Psicologia Cognitiva, CFCH, Universidade Federal de Pernambuco.

relations involved in the solutions.

Since the analysed data came from students who enrolled at a public and at a private school from one specific area, this study made this as a secondary aim which is to verify how the supposed relationship comes about, considering the different school contexts of the participants.

This article is based on the premise that teacher interventions are crucial to guide students to think about their own thought processes, so that the knowledge they are focusing on is truly understood and transformed. In the specific field of mathematics teaching, this proposition is strongly presented when a meaningful and high qualified learning of mathematical concepts is supported in order to achieve a continuous and progressive development of the mathematical reasoning throughout the different school levels (Anderson, Olson, & Wrobel, 2001; Brito, 2006; Householter & Schrock, 1997; Kjos & Long, 1994; Klein, 1994; Spinillo, 1999).

Considering this framework, in this paper, we adopt Jean Piaget proposals in relation to the grasp of consciousness (Piaget, 1974; 1978). According to the genetic epistemology, the grasp of consciousness, as an aspect of the cognitive functioning ruled by the equilibration, is a process by which the subject becomes aware of their actions (schemes) and/or of the relations between their actions in the interaction with the object. Therefore, it is one of the dimensions that transforms these schemes in a concept or in a system of concepts (Piaget, 1974). According to this perspective, the action is understood as a “know-how” with a certain degree of autonomy. However, either in the case of a precocious and immediate success of a scheme or in the case of failure (the success being achieved beforehand), its conceptualization occurs by the grasps of consciousness “a posterior”, as explained by Piaget (1978), according to a movement from the peripheral regions of the adaptation to the object, in the interaction subject-object, towards the central regions of this interaction, that one of the internal coordination of the actions. So, progression in the construction of the knowledge would be tightly linked to “being aware” (to become conscious) of schemes and relations pertaining to a concept, which results in conceptualization in alternated and progressive cycles of “action-conceptualization” (Piaget, 1974; 1978).

According to this position, the claim that the grasp of consciousness is a process which imposes on researchers the need to identify and describes the different steps of its manifestation along the conceptual comprehension of a specific object of knowledge. In this sense, to analyse its presence definitely implies the identification of the schemes and relations that are typical of the comprehension of that particular object. Either focusing on a sequence of actions of the same subject (to search therein the transformation signs of those individual schemes along a task), or on the actions expressed by several subjects at a specific moment of a task (to capture the moment of the individual sequences by cross sectional approach), the analysis should always have an object for the schemes and relations among schemes belonging to the concept. Thus, in this study, the focus will be on the schemes and the relations among them, concerning the mathematical comprehension of the Cartesian product.

Literature Review

Another author who took a stand on this subject is Vergnaud (1990; 1996), stressed, in the construction of knowledge, the importance of the dialectic relationship between knowledge in action and theoretical knowledge, from the perspective that the conceptualization of reality happens via action, with a time lag between the planes of action and that of theory. This author sees as fundamental, within this dynamic, the complex process of making the knowledge explicit (“theorization”) by the person, transforming “action knowledge” into theoretical knowledge. In this transformation, actions are made up of one part automatism and one part conscious decision, which is inherent to cognitive functioning. From this perspective, it becomes possible to understand, along with Vergnaud (1996), how action is possible in those domains where a person's theorization is poor or inexistent, and how action can feed off theory and vice-versa.

Based on the genetic epistemology, Vergnaud (1990) conceived the scheme as a functional totality in the invariant organization of the actions for a specific category of situations. From these proposition results, the idea of conceptual field as a set of problems or a class of problems that to be solved requires concepts, procedures and symbolic representations closely connected to one another. A problem is any situation which, to be solved, brings to the subjects (in the school or elsewhere), the need to discover and to explore relations, to elaborate hypothesis and verify them.

In the case of mathematical knowledge, this process of relations elaboration is precocious, making sense when taking part of larger and more complex structures along subsequent evolutionary moments (Vergnaud, 1990; 1991). Composed by relational dynamic schemes, these relations attest the subjects' inferential and deductive activity as essential organizers of the knowledge.

Allal and Saada-Robert (1992) also revisited the genetic epistemology concepts of the grasp of consciousness and regulation, arguing that they allow us to better understand meta-cognition, a construct strongly present in a contemporary cognitive psychology concerned with a greater efficiency of school learning (Cardelle-Elawar, 1995; Goldberg, 1999; Jou & Sperb, 2006; Ribeiro, 2003; Wilson, 1998, 2001; Wilson & Clark, 2002).

In answer to this purpose, Allal and Saada-Robert (1992) proposed a close relationship between degrees of explanation of the regulations (implicit, explainable, explicit and instrumental zed) and levels of the grasp of consciousness: First, because regulations act on the reflexive overcoming of structures, but allow, especially, the opening of these structures; Second, because regulations make the subjects think about their own thoughts (metacognition), a double mechanism of construction, not only by guaranteeing the development of control operations, but also by guiding the construction of explicit forms of representation based on implicit forms. Therefore, for these authors, there are cognitive regulations (which deal with the structural and conceptual aspects of development) and meta-cognitive ones (of a functional order, according to various degrees of consciousness, used by the subject to manage procedures in the learning process).

Excluding, (obviously), the investigations of Piaget himself (Piaget, 1974; 1978), literature concerning the grasp of consciousness process in conceptual development and in learning, although restricted, points to some interesting results.

For example, in the research on the relationships between arithmetical representations and the grasp of consciousness at the cognitive construction, Ferreira and Lautert (2003) identified five moments of the grasp of consciousness in a child (six years four months old), while solving a problem of division by partition, from the complete absence of awareness of the totality of the elements to the representation of the remainder. For these authors, these different degrees of the grasp of consciousness were provoked by the intervention of the researcher and by the referent statements. These although important, did not lead to completing conceptualization and the interventions provoked re-adaptations, these were not sufficient for the child to understand the inter-relationships between the whole and the other terms of division.

Moro (2005) obtained results with elementary students in tasks related to multiplicative structures which cyclically alternated practical execution with oral interpretations and the production of written notes about the process, accompanied by explanation of the meaning of the written notes. The use of this alternated procedures is based on the high probability of activating a "being aware" of the "know-how", so it can be progressively conceptualized via the explanation of actions and of the relations between actions at the oral level, and later, at the level of written representation and its meaning.

English (1993) pointed out the abilities of 7- to 12- year-old children to monitor their actions, identify and correct their mistakes and recognize the structure of combinatory reasoning problems. The author showed that, in older children, there is a relationship between the grasp of consciousness of the underlying structure of the problems and the strategic changes needed in order to implement more effective solutions.

In other field studies conducted in relation to cognitive functioning of elderly, Santos, Rosseti, and Ortega (2006) analyzed the grasp of consciousness process of elderly people and adolescents which are related to their successes and failures along different sessions of playing a rule game (Quoridor). From the identification and the description of the different levels of the grasp of consciousness founded, the authors discussed numerous similarities between the levels in the elderly and the adolescents and they particularly underline how the fact of inducing the participants to think about their successes and failures, at each moment of the task, was productive to their performance and their mastery of the game.

It should be noted that, in the varied references reported above, there is the presence of some way to encourage the participants to think about what they had done, said and written. This fact calls our attention to how much, therefore, in a classroom learning situation, it must be taken into account the form of the intervention (that is, of didactical character) to activate the grasp of consciousness process.

Despite not always employing the same style of intervention to induce students' reflections about their own thinking (in this particular situation, their solutions to mathematics problems), there are a great variety of studies where, frequently, this issue is approached via the analysis of data which comes from students' requests for explanations to justify their ways of solving problems. This fact happens even in studies which is not supported by the genetic epistemology, a field where the elaboration and use of the Piagetian clinical-critical method (Bringuier, 1978; Domahidy-Dami & Leite, 1987; Spinillo, 2003) strongly mark the procedures destined to capture the students' own explanations with a didactical or a non-didactical aim.

So, from the perspective of the theories of information processing, Brito (2006) reminded us the contributions of techniques of asking students to "thinking aloud" in tasks designed to pave the way from mastering simpler problems to mastering more complex ones, among a set of competences and abilities which are indispensable to problem-solving. Along the same theoretical line, Lochhead (2000) defended the use of the "think back" technique as a key aspect of thinking aloud in a problem-solving situation with a partner. This technique, considered applicable in any area, results in reflective thought, that is, the ability to listen and follow one's own thought processes.

Wilson (1998) also recommended "thinking aloud" in a multiple methodological approach (observation, clinic interview, video and audio recordings) to promote and stimulate cognitive and meta-cognitive manifestations in the field of classroom performance when solving unfamiliar mathematical problem.

Using an intervention based on writing activities in the solution and creations of mathematical problems, Card (1998) found that most of elementary school students gained a meaningful improvement in solving problems with even better articulation between their processes of thinking mathematically and the processes of describing their problem-solving strategies.

Research Context

As already defined above, according to the genetic epistemology, in this paper, it is studied the participants' grasp of consciousness process of the specific schemes and relations involved in the mathematical comprehension of Cartesian product, in order to verify the existence of a relationship between this process and the performance of elementary school students in solving problems, concerning this kind of mathematical content.

The focus on problems of Cartesian product is linked to the interest in studying the reasoning process by the students in solving problems that are supposed to be complex to them. According to our interpretation, while solving complex problems, the students probably encounter the need to reflect on their ways of thinking more effectively than they would, while solving simple problems (whose solutions they already master and may be already automatic).

Therefore, among other possibilities, mathematical problems of Cartesian product have been considered as a challenge to Grades 5 to 8 elementary school students, the participants of the larger research project. According to Vergnaud (1983; 1985), within the framework of the theory of conceptual fields, such problems have their solutions based on establishing multiplicative relations in a combinatory nature. There are problems that involve a ternary relation between measures (of multiplication or of division), when one of them is the product of other two on the numerical and dimensional planes, and which is generalized by more than two measures. Its form of representation in the Cartesian plane covers the double correspondence involved (not a simple correspondence, as is the case with the structure of isomorphism of measures).

In the larger scenario where the preset study is located, it should be noted that the participants were asked to express their ways of thinking in writing, that is, the reasoning process activated by them when carrying out their specific solutions. Not only the sample size of the main research project guided that option for making possible the data collected from a high number of participants, but especially, it was seen as necessary to collect written data referred to subjects' "becoming aware" of their own actions and relations involved in the expressed solutions, because writing can be characterized as a recursive action on the part of the solver, of those who perform a task. The judicious literature revision provided by Escorcia (2007) spoke about this issue and attested that writing about one's own thoughts was a powerful means to develop reflection. Also, relevant to this question is the influence of Vygotsky's (1991) proposal that woriting, interpreted as an internal language, is a particular function of language. It takes for granted that the monologue occurring represents a superior psychic function on account of the cognitive effort and because of the control of the required procedures (Schneuwly, 1985).

It is supposed, therefore, that when challenged to write the explanations for their ways of reasoning when looking for their solutions, the subjects not only think about their own thinking, but above all else, constrain themselves to a reflection about it at a new stage, that is one of written representations. This representation expresses meaningfully the actions and relations systems taking place therein.

Thus, in order to achieve the aims mentioned above, and based on the adopted theoretical perspective, this study will seek answers to the following questions: Would the content of the written explanations expressed by the participants show signs of grasping consciousness about their own actions and relations involved in solving the proposed problems of Cartesian product? Would this process of grasping consciousness be associated with better performance in the sense that the participants who performed well would have a greater capacity to make by writing their means of thinking and solving the proposed explicit problem? Or are these instances unrelated? Are there differences between the results obtained from the public school and the private school participants we studied?

Method

Participants

Forty-two 10 to 16 years old students attending Grades 5 to 8 elementary school took part in this study. Twenty-one of them were from a public school and 21 were from a private school both located at a Brazilian metropolitan area. They were randomly sampled from a database of 175 students whose parents and school authorities agreed to their participation in a larger project designed to evaluate solutions to Cartesian mathematical problems. The public school students were from a low-income class and the private school students were middle class.

Procedures for Data Collection

Participants were asked to solve, in writing, four problems of Cartesian product and to explain their way of solving each problem. They needed to answer the question: "How did I think to solve this problem?". The problems were presented in the same order, written on a sheet of paper and read aloud by the examiner. The application was done collectively in a classroom. Here are the four problems:

Problem 1: Tatiana goes to a masquerade wearing a wig and glasses. In a store, she found 42 types of wigs and 26 different types of glasses. How many ways could she dress up by using one pair of glasses and one wig at a time?

Problem 2: In an ice cream parlour, there are six flavours of ice cream, three types of topping and two types of cones. How many different ways can ice cream be served, given that all ice cream served comes with a cone and a topping?

Problem 3: A soccer player can make 15 different combinations of socks and soccer shoes. He has three types of soccer shoes. How many types of socks does he have?

Problem 4: A snack bar can serve 8,184 different types of snack by combining sandwiches, ice cream cones and juice. To make up the snacks, the snack bar offers 31 types of sandwiches, 24 flavours of ice cream and different flavours of juice. How many flavours of juice does this snack bar offer?

These problems contain: two (Problems 1 and 3) or three (Problems 2 and 4) variables; variables with high values (Problems 1 and 4) or low values (Problems 2 and 3) can be solved by using multiplication (Problems 1 and 2) or division (Problems 3 and 4).

Analysis Procedures

Firstly, participants' problem-solving performance was analysed, that is, the frequencies and percentages of correct responses to each problem.

Secondly, a qualitative analysis of the written responses to the question "How did I think to solve this problem?" was made for the purpose of identifying categories of responses which would indicate signs of grasp of consciousness on the part of the participants in relation to the solutions found.

It was considered that the demand contained in that question implied to explicitly justify some reasons to adopting a determined procedure to solve the problem, that is, a solution procedure in terms of the actions and relations involved in it.

After repeated approaches to response categorization, the one adopted was carried out by three independent judges, and final classification was determined by agreement between two of the three judges. There were no cases of disagreement among all three judges in relation to the same justification along the different obtained categories.

For the purpose of looking for signals of the relationships under examination, statistical tests were carried out for comparisons between the obtained categories and their relationships with performance to each problem, according to the different types of participants' schooling.

Results

In sequence, the qualitative analysis of the participants' written responses to the request of making explicit their means to solve the problems is shown.

The categories identified can, thus, be described and exemplified:

Category I: Justifications have undefined character, grouping vague or confused responses, without any signal of the mathematical calculation procedures employed in the solution. It also includes cases of lack of justification (left blank). Examples: "Now I do not know how to say it" (Problem 3); "It came out of my head" (Problem 2); "I thought that it was almost a psychology test" (Problem 1);

Category II: Justifications have a descriptive character, grouping responses which are limited to describing (Pressiat, 1996) the mathematical operation used, including not only the appropriated calculations procedures (multiplications and/or divisions) to achieve the correct answers, but also the non-appropriated ones. Example: "I just thought of the LCM (Least Common Multiple)" (Problem 2); "I thought that 3 times 5 are 15, so it must be 5" (Problem 3);

Category III: Justifications have an explicative character. There are responses which involve true explanations about the calculation procedures used to obtain the solutions, including the mention of the quantities referents present in the statement of the problem, even though those procedures not always were the more appropriated to achieve the correct answer. Two types of explanation were identified:

Category IIIA: When explicative explanations have a particularized mark, grouping responses where a reason is formulated by means of connecting words, but this is limited to the evocation of the calculation procedure used in the specific solution to that problem. Example: "I first added the types of sandwich and the types of ice cream, and then I took the sum of the types of sandwiches and ice cream and subtracted it from 8,184, which came out to 8,129" (Problem 4);

Category IIIB: When explicative explanations have a generalized mark, grouping responses where the indicated reason has as its expression organized in connected premises with a conclusion which is a necessary result in terms of laws (generalization), as reflecting the process used in solving problems similar to the one that is being solved at the moment. Examples: "I figured that one wig had 26 combinations and two wigs had 52. So you have to multiply the number of wigs by the number of glasses, and the result was 1,092 combinations" (Problem 1); "He can match each soccer shoe with x socks, making 15 combinations. To find out how much x is, you just have to do the opposite operation" (Problem 3).

Intending to exam the hypothesis that there is a relationship between performance and level of the grasp of consciousness of the ways of solving the problems, the first step to test this possibility would be to know, in each group of participants (two different schoolings), which problem was the easiest (see Table 1).

Table 1

Number (out of 21) and Percentage of Correct Responses (in Parenthesis) in Each Problem According to Types of School

School	Problem 1 (%)	Problem 2 (%)	Problem 3 (%)	Problem 4 (%)
Public school	3 (14.3)	4 (19.0)	6 (28.6)	0 (0.0)
Private school	4 (19.0)	16 (76.2)	17 (80.9)	9 (42.8)

Students from the public school had very low percentages of correct responses, showing difficulty in solving each one of the four problems, especially Problem 4, which was systematically answered incorrectly. On the other hand, students from the private school had higher percentages of correct responses. Summing up all the correct answers (all the four problems), the former group totalized 15.5% of correct answers, while the latter one performed 54.8%, correctly.

For not rejecting the hypothesis, it was expected that: (1) In the public school, Category III justifications should be more frequent in Problem 3, the easiest (28.6% of correct responses), than the other categories; and (2) In the private school, Category III justifications should be more frequent in Problem 2 (76.2% of correct responses) and in Problem 3 (80.9% of correct responses), which were the easiest ones, than the other two categories.

According to Table 2, the hypothesis was rejected for the students from the public school, who tended to provide Category I justifications for all problems, independent of the performance they had. However, the hypothesis was not rejected for the students from the private school, since there is a relationship between the categories of justification and performance. These students tended to provide Category III justifications for the problems they answered correctly.

Table 2

Percentage of Justification Categories per Problem for Each Type of School

Category	Problem 1 (n = 21)	Problem 2 (n = 21)	Problem 3 (n = 21)	Problem 4 (n = 21)
Public school				
I	62.0	66.7	66.7	57.1
II	19.0	23.8	28.6	28.6
III	19.0	9.5	4.7	14.3
Private school				
I	38.1	23.8	23.8	33.3
II	9.5	19.0	23.8	38.1
III	52.4	57.2	52.4	28.6

Notes. Category I—No justification or undefined justification; Category II—Descriptive justification; Category III—Explicative justification.

Table 3 shows the distribution of each category by types of school.

Table 3

Mean Number of Categories as a Function of the School (Maximum: 4)

Category	Public school	Private school
I	2.60	1.86
II	1.79	1.86
III	1.62	2.29

Notes. Category I—No justification or undefined justification; Category II—Descriptive justification; Category III—Explicative justification.

The Friedman Test, applied separately into data of each school, showed significant differences between the justification categories only for public school students (X_r^2 = 13.743; df = 2; p = 0.001), whose justifications were concentrated on Category I, while for private school students, there is no greater incidence in any specific category. This result indicated that public school students demonstrated difficulties in providing explanations about how they had solved the problems. On the other hand, the private school students, besides Category I, provided descriptive (Category II) and explicative (Category III) justifications.

Comparisons among categories according to the types of school, by means of Mann-Whitney U test, revealed that Category I justifications were significantly more frequent among public school students than among the private school ones ($U = 112.0$; $p = 0.005$); whereas Category III justifications were more frequent among private school students than among the public school ones ($U = 86.0$; $p = 0.000$). No difference between students as function of the type of school was found in relation to Category II justifications. It seems that the public school students tend not to offer justifications, while the private school students tend to provide explanations.

It is worthwhile to explore the differences between schools in relation to the two types of explanations identified as Category III justifications: particularized and generalized explanations (see Table 4).

Table 4

Percentage of the Two Types of Explanation of Category III by Types of School

Category	Public school (n = 10)	Private school (n = 40)
IIIA	70.0	32.5
IIIB	30.0	67.5

Notes. Category IIIA—Particularized explanation; Category IIIB—Generalized explanation.

As it can be seen in Table 4, most explanations given by the public school students are of a specific nature (Category IIIA). Among the private school students, most explanations expressed general characteristics (Category IIIB). This goes beyond the proposed problem, applicable to other problems, similar to the one in question. Evidently, caution is needed in relation to this data, since due to the low frequency in the cells, it was not possible to apply any appropriate statistical tests which would allow us to generalize the results.

Table 5 shows the relation between the justification categories and the correctness of the responses to the problems.

Table 5

Percentage of Correct and Incorrect Responses in Each Category

Category	Correct response (n = 59)	Incorrect response (n = 109)
I	16.9	62.3
II	28.8	21.2
III	54.3	16.5

Notes. Category I—No justification or undefined justification; Category II—Descriptive justification; Category III—Explicative justification.

Here, we see that the greatest percentage of correct responses is associated with Category III, while the greatest percentage of incorrect responses is associated with Category I. It seems, therefore, that students who can correctly solve the problems are also those who can, in writing, explain the procedures implemented and the reasons guiding the forms of solution used.

Discussion

Dealing with the hypothesis under examination in this study, firstly in sequence, there is the discussion above the results related to the idea that the justifications to the question “How did you think to solve this problem?”, as it was categorized, show signs of grasping consciousness of the students' ways of reasoning when solving the problems of Cartesian product, and that there would be a relationship between this phenomenon and the performance on solving the same problems. Secondly, there is the discussion of the derived result of the comparisons between students from the public and from private school concerning the relationship on focus.

Two sets of results in this study support the statement that there is a relationship between the grasp of consciousness and participants' performance on solving the mathematical problems of Cartesian product: an association between justifications classified as Category I and the incorrect response, and also an association between justifications classified as Category III and the correct response.

So, what do the justification categories identified and described in this study say about signs of students' grasping of consciousness of ways of reasoning present in the solution to the problems?

First of all, we focus on the cases of undefined justification which also include those of absence of justification (Category I). What does this mean from a psychological point of view? According to Piaget (1974; 1978), there are degrees of consciousness a subject can reach about his own action; thus, the absence of a justification or a vague justification means that there is no realization of their schemes of solution (even the ones that could exist). This fact would characterize a moment of non-adaptation between those schemes and the cognitive demands involved in the action of explaining their ways of thinking and solving the problem.

The fact that there is an association between the absence of the grasp of consciousness and incorrect performance shows us the improbability that the participants could elaborate a proper justification, because they did not dispose of the coordinated solving schemes the consciousness of which they might grasp. Maybe they did not even understand what the text of the problems was suggesting about the mathematical relations involved. Or, in the likely absence of knowledge in action, there would be nothing for them to make explicit, so that action and its conceptualization could feed off each other (Vergnaud, 1990; 1996).

Within this line of thought, one may ask: What do the justifications classified as Category II (descriptive) and Category III (explicative) mean? According to our analysis, the content of these justifications would signal both a grasp of consciousness of the calculation procedures employed and then described (Category II), as well as a grasp of consciousness of the reasons that guided the use of these procedures, making up the explicative justifications (Category III).

It seems difficult to deny that these two categories represent two levels of the process of the grasp of consciousness of the ways of solving the problems. Thus, as the subjects are justifying their solutions, at least describing the sums used (Category II), they understand that the calculating actions used and now described are responsible for the solutions found (Piaget, 1974). However, there is still nothing involving the expression of some reason that justifies the choice of the operations used, which allows us to suppose that, even if the students possibly master some algorithmic procedures for solving the problem (even coming up with the correct solution), they are still not aware of the reasons why they opted to use that procedure. According to the data, descriptive justifications can be associated with correct or incorrect performance. When associated with correct performance, it is possible that the students already have a sufficient degree of understanding of the problem to generate the correct response, but have difficulties in explaining the reasons which guided the procedures adopted, although they can describe them.

However, another level of the grasp of consciousness is represented by Category III justifications. The explanation of the reasons for using certain procedures indicates that students inferred that the solution was the result of the specific schemes and relations of the concepts in question, at an explanatory level. This explanation can be a result of a grasp of consciousness about their ways of thinking, which results in their ways of solving problems. Therefore, it is at this level of the grasp of consciousness that it becomes possible to make a deductive interpretation about one's own thinking concerning the procedures used in the solution. This is true to the point that it was frequent for the participants to begin their explanations with expressions like "I thought that..." and used logical connectors like "because...". In these cases, there is an understanding of the mathematical reasons for carrying out the solution found, which makes them capable of externalizing these reasons in the form of an explanation. An important point to consider about this level of the grasp of consciousness is that it presents itself on two sub-levels, one of particularized explanation and the other of generalized explanation.

By restricting the explanations to a given solution geared specifically toward the problem to be solved, students are at the level of the grasp of consciousness implicated therein, which comes from reflecting abstractions (Piaget, 1977). There is reconstruction, and therefore conceptualization, at the level of the representation that was understood, in terms of coordination of actions (Piaget, 1974; 1978). However, despite of disposing of a structured comprehension of an operational order, concerning to the particular concepts and relations implied, they cannot obtain a grasp of consciousness of these structures, of the mechanisms in their composition, in order to take them as objects for reflection. For this reason, do they restrict them to the applicable "then and there" cases? (Piaget, 1974).

However, when they generalize their explanations, with propositions implicated in the form of rules for solution, the grasp of consciousness stretches into another level of the reflecting abstraction, becoming qualitatively more complex. This is the level in which the process of abstraction allows thoughtful reflection on their own thinking (in terms of reflexive abstraction, according to the Piagetian's perspective), which explains the reason why the subject can theorize. This is the reason why the explanations express general laws, extensible to every case where the solution applies. An example that brilliantly illustrates this level of abstraction is that when the students use algebra concepts in their explanations, such as using unknown numbers in their responses.

An important aspect to note in relation to these two levels of explicative justifications is what they represent in terms of understanding the mathematical concepts and relations in question. Because they are problems of a Cartesian product, the Category IIIA (particularized) justifications have an arithmetic slant, and the Category IIIB (generalized) justifications, an algebraic slant. This is a question which deserves an investigation in order to explain the forms of thinking in solving problems that involve combinatory reasoning.

Conclusions

The results indicate the existence of a relationship between grasp of consciousness, as defined in this study, and performance on solving the problems, supporting the idea that the refined justifications, from an explicative point of view, are accompanied by mathematically correct solutions to the problems.

On the other hand, it is necessary to be careful as to the interpretation of the complex relations between explanation and performance. An example of this complexity is the case in which the response is correct and the justification is vague or not supplied. Theorizing on this bit of empirical data, it is possible that this is an example of knowledge in action (or theorem-in-action) as postulated by Vergnaud (1990; 1997), which expresses the intuitive trajectory of the actions and operations carried out by individuals that, although it is present, is knowledge that cannot be verbalized. Another example of this complex relationship is the case of the descriptive justifications that do not appear to be clearly associated with a particular level of performance, since they accompany incorrect responses and as well as correct ones. However, when students are able to explain the reasons that guide their forms of solution, they associate this competency to an appropriate performance. They know and are able to explain this knowledge.

Explaining the bases of comprehension is a complex activity. This discussion could be extended to the level of a theoretical reflection about the relationship between implicit and explicit knowledge, has fascinated scholars of mathematical knowledge, such as Vergnaud (1996), and of linguistic knowledge, such as Gombert (1992) and of cognitive development, such as Piaget (1974) and Karmiloff-Smith (1995). According to Karmiloff-Smith (1995), there is a gradual process in which information implicit in the mind becomes knowledge that is explicit for the mind, in such a way that information becomes available both to consciousness as well as can be made explicit in language.

The written verbal explanation, required in the presentation of the students' justifications in this study, therefore, demand a high level of elaboration. In reality, it is also possible that the limitations of the participants in expressing themselves resulted from the fact that they were asked for explanations in writing.

Perhaps, if the demand for explanation had been inserted into a dialogue between the participant and the examiner, such as happens in situations of the clinical-critical style of interviews, the participants might have presented more explicative justifications. However, as previously mentioned, to write a justification allows a recursive action on the part of the one problem solver, bringing about, more than oral language, on activating the grasp of consciousness process.

Thus, the content of the explanations presented by the students in this study expressed their levels of grasping of consciousness of their way of thinking when solving the complex problems presented. The results sustained the idea that those levels are related to performance, hence correct responses were associated to explicative justifications while incorrect ones came together with vague and/or even absent explanations.

No evidence was found for the relationship between performance and the grasp of consciousness in the case of students from the public school, since these participants tended to give undefined justifications or did not give any justifications whatsoever, whether for the problems they solved correctly or the ones they solved incorrectly. For the students from the private school, however, that relationship was identified: Correct performance was associated with explicative justifications and incorrect performance was associated with undefined or inexistent justifications. It is important to remember that performance on solving the problems was better between the students from the private school than those from the public school.

One possible explanation for these results is that schools considered in this study differ in the way they conceived the process of teaching and learning mathematics: In the public school, it is likely that this teaching-learning process is less geared towards the proper reflection and comprehension concerning the schemes and the specific relations to the solutions of the problems, with the opposite occurring in the private school. In the latter, there is likely a process of mathematical understanding which is relatively more advanced, probably resulting from a mathematical education geared towards activating the learner's own elaboration of those schemes and relations, and a didactic approach which stimulates the student to give explanations about their procedures for solving mathematical problems.

It is worth noting that it is important to limit this explanation to the two schools considered in this study, since they are not absolutely representative of the universe of establishments they each belong to. As registered, although the data analysed comes from a sample of randomly chosen participants, we are dealing with a certain public school and a certain private school.

However, without neglecting the importance of the type of mathematical education in the explanation of the differences found between students from the two schools, we should point out another aspect that might have played a disadvantaged role on the results, especially for those enrolled on the public school: the writing of the justifications. It is possible that these students had only an elementary mastery of writing, and that this made the appearance of descriptive and explicative justifications difficult for them.

Furthermore, it is important to mention that the type of data analysed led only to the “then and there” expression of what each participant had to write as a justification for the solution adopted; therefore, we can say that only one moment, a momentous “state” of the process of grasping consciousness of those solutions procedures was captured. But we point out that, independently of this, there was room for the participants to manifest indicators relative to that process in the specific case on focus.

It is also worth to underline that, despite of being interesting, the results reported here require further investigation through future studies. Even so, they stress the relevance of the process of grasping consciousness of actions and relations activated in the subject-object interaction, when solving tasks towards the conceptual construction implied (Piaget, 1974; 1978).

They also provide evidence to renew the recommendations, already present in the literature (Ribeiro, 2003; Spinillo, 1999, 2003) that in education, and in this specific case, mathematical education, should be increased the value on forms of intervention in which the teachers activate the students’ capabilities of thinking about their own thoughts, a dimension extensively considered in the literature as meta-cognitive capacities. Thus, it seems productive to encourage students to reflect on their solutions and the reasons for using them, to test their hypotheses, to reformulate them, thus becoming aware of their ability to learn.

References

Allal, L., & Saada-Robert. (1992). La métacognition: Cadre conceptuel pour l'étude des régulations en situation scolaire. *Archives de Psychologie, 60*, 265-296.

Anderson, J. M., Olson, J. S., & Wrobel, M. L. (2001). Beyond computation: Improving mathematical problem-solving (Unpublished master thesis, Saint Xavier University-IRI). Chicago I. L.: Skylight. ERIC database. Retrieved from http://www.eric.ed.gov

Bringuier, J. C. (1978). *Conversando com Piaget* (M. J. Guedes, Trans.). Rio de Janeiro RJ: Difel. (Originally published in 1977)

Cardelle-Elawar, M. (1995). Effects of metacognitive instruction on low achievers in mathematics. *Teaching and Teacher Education, 11*(1), 81-95.

Card, R. A. (1998). The effects of writing in mathematics on second-grade student’s achievement and meta-cognition (Unpublished master dissertation, Saginaw Valley State University). Saginaw M. I. ERIC database. Retrieved from http://www.eric.ed.gov

de Brito, M. R. F. (2006). Alguns aspectos teóricos e conceituais da solução de problemas matemáticos. In M. R. F. de Brito (Org.), *Solução de problemas e a matemática escolar* (pp. 13-53). Campinas S. P.: Alínea Editora.

de Jou, G. I., & Sperb, T. M. (2006). A metacognição como estratégia reguladora da aprendizagem. *Psicologia Reflexão e Crítica, 19*(2), 177-185.

Domahidy-Dami, C., & Leite, L. B. (1987). As provas operatórias no exame das funções cognitivas. In L. B. Leite, & A. A. Medeiros (Eds.), *Piaget e a Escola de Genebra* (pp. 111-123). São Paulo SP: Cortez Editora.

English, L. (1993). *Development of children strategic and metastrategic knowledge in a novel mathematical domain*. Research Report. Queensland: Queensland University of Technology, Centre for Mathematics and Science Education. From ERIC database. Retrieved from http://www.eric.ed.gov

Escorcia, D. (2007). *Composantes métacognitivives et performance à l’écrit : une approche sociocognitive du travail étudiant* (Unpublished doctoral dissertation). Nanterre: Université de Paris X. Retrieved October 5, 2010, from http://www.halshs.archives-ouvertes.fr/docs/oo/17/09/24/PDF/Thèse_Escorcia.pdf

Ferreira, S. P. A., & Lautert, S. L. (2003). A tomada de consciência analisada a partir do conceito de divisão: um estudo de caso. *Psicologia: Reflexão e Crítica, 16*(3), 547-554.

Goldberg, P. (1999). *Increasing problem solving through the metacognitive skills of planning, monitoring and evaluating*. Research Report. Henry County, K. Y.: The Spencer Foundation. ERIC database. Retrieved from http://www. eric.ed.gov

Gombert, J. E. (1992). *Metalinguistic development*. Hemel Hempstead UK: Harvester Wheatsheaf.

Householter, I., & Schrock, G. (1997). Improving higher order thinking skills of students (Unpublished master's thesis, Saint Xavier University-IRI). Chicago, I. L. ERIC database. Retrieved from http://www.eric.ed.gov

Karmiloff-Smith, A. (1995). *Beyond modularity: A developmental perspective on cognitive science*. Cambridge, M. A.: MIT Press.

Kjos, R., & Long, K. (1994). Improving critical thinking skills and problem solving in fifth grade mathematics (Unpublished master's thesis, Saint Xavier University-IRI). Chicago I. L. ERIC database. Retrieved from http://www.eric.ed.gov

Klein, D. C. D. (1994). Transferring critical thinking skills across content areas: The effects of multiple content areas and summary generation on the analogical problem solving of regular and honors students. In *Annual Meeting of the American Educational Research Association*. New Orleans LA/Los Angeles C. A.: AERA/UCLA. ERIC database. Retrieved from http://www.eric.ed.gov

Lochhead, J. (2000). Sound thinking with thinkback. In Proceedings of *the Technological Education and National Development Conference "Crossroads of the New Millennium"* (pp. 1-13). Abu Dhabi: TENDC. ERIC database. Retrieved from http://www.eric.ed.gov

Moro, M. L. F. (2005). Estruturas multiplicativas e tomada de consciência: Repartir para dividir. *Psicologia: Teoria e Pesquisa, 2*(21), 217-226.

Piaget, J. (1974). *La prise de conscience*. Paris: PUF.

Piaget, J. (1977). *Recherches sur l'abstraction réfléchissante. 1/L'abstraction desrelations logico-arithmétiques*. Études d Épistemologie Génétique (vol. XXXIV). Paris: PUF.

Piaget, J. (1978). *Fazer e compreender* (C. L. P. de Leite, Trans.). São Paulo: EDUSP/Melhoramentos. (Originally published in 1974)

Pressiat, A. (1996). Passage de l'arithmétique à l'álgèbre ou modélisation algébrique? In G. Combier, J. C. Guillaume, & A. Pressiat (Eds.), *Les débuts de l'algèbre au college. Au pied de la lettre* (pp. 7-32). Paris: INRP.

Ribeiro, C. (2003). Metacognição: Um apoio ao processo de aprendizagem. *Psicologia: Reflexão e Crítica, 16*(1), 109-116.

Santos, C. C., Rossetti, C. B., & Ortega, A. C. (2006). O funcionamento cognitivo de idosos e de adolescentes num contexto de jogo de regras. *Estudos Interdisciplinares: Envelhecimento, 9*, 53-74.

Schneuwly, B. (1985). La construction sociale du langage écrit chez l'enfant. In B. Schneuwly, & J.-P. Bronckart (Eds.), *Vygotsky aujourd'hui* (pp.169- 201). Paris/Neuchâtel: Delachaux & Niestlé.

Spinillo, A. G. (1999). As relações entre aprendizagem e desenvolvimento discutidas a partir de pesquisas de intervenção. *Arquivos Brasileiros de Psicologia, 51*(1), 55-74.

Spinillo, A. G. (2003). Ensinando proporção a crianças: Alternativas pedagógicas em sala de aula. *Boletim GEPEM Grupo de Estudos e Pesquisas em Educação Matemática, 43*(3), 11-47.

Vergnaud, G. (1983) Multiplicative structures. In R. Lesh, & M. Landau (Eds.), *Acquisition of Mathematics: Concepts and processes* (pp. 127-174). London: Academic Press.

Vergnaud, G. (1985). *L'enfant, la mathématique et la réalité* (3rd ed.). Berne: Peter Lang.

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactiques des Mathématiques, 10*(23), 33-170.

Vergnaud, G. (1991). L'appropriation du concept de nombre: un processus de longue haleine. In J. Bideaud; C. Meljac, & J. P. Fischer (Eds.), *Les chémins du nombre* (pp. 271-282). Lille: Presses Universitaires de Lille.

Vergnaud, G. (1996). Au fond de l'action, la conceptualisation. In J. M. Barbier (Ed.), *Savoirs théoriques et savoirs d' action* (pp. 275-292). Paris: PUF.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes, & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 5-28). Hove: Psychology Press.

Vygotsky, L. S. (1991). *Thought and language* (A. Kozulin, Rev.). Cambridge M. A.: The MIT Press.

Wilson, J. (1998). The nature of metacognition: What do primary school problem solvers do? In *National AERA Conference*. Melbourne: AREA, University of Melbourne. ERIC database. Retrieved from http://www.eric.ed.gov

Wilson, J. (2001). Methodological issues of assessing metacognition: A new approach. In *Annual Meeting of the Australian Association for Research in Education*. Melbourne: AERA, University of Melbourne. ERIC database. Retrieved from http://www.eric.ed.gov

Wilson, J., & Clark, D. (2002). Monitoring mathematical metacognition. In *Annual Meeting of the American Educational Research Association*. New Orleans, L. A.: AERA. ERIC database. Retrieved from http://www.eric.ed.gov