CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2010 ANNUAL MEETING / RENCONTRE ANNUELLE 2010

CMESG
GCEDM

Simon Fraser University
May 21 – May 25, 2010

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PROCEEDINGS OF THE 2010 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2010 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

34th Annual Meeting
Simon Fraser University
May 21 – May 25, 2010

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The organisational work for our 2010 annual meeting at Simon Fraser University enabled us to have another memorable meeting. Our local organisers, Peter Liljedahl, Rina Zazkis, Nathalie Sinclair, Sen Campbell, and Malgorzata Dubiel with exceptional help from Shiva Gol Tabaghi, Simin Chavoshi Jolfaee, Paulino Preciado, and Christian Bernèche, along with the rest of the SFU doctoral students, managed to do everything with utmost efficiency and pleasantness, and we thank them. We would also like to thank the Faculty of Education at SFU, the Vice President's office at SFU, PIMS, and MITACS for financial support for this conference. Finally, we extend our thanks to the guest speakers, working group leaders, topic session and ad hoc presenters, and all the participants for making the 2010 meeting a stimulating and worthwhile experience.

L'organisation locale de notre rencontre annuelle de 2010 à l'Université Simon Fraser nous a permis d'avoir une autre rencontre mémorable. Nos organisateurs locaux, Peter Liljedahl, Rina Zazkis, Nathalie Sinclair, Sen Campbell, et Malgorzata Dubiel, avec le support exceptionnel de Shiva Gol Tabaghi, Simin Chavoshi Jolfaee, Paulino Preciado, et Christian Bernèche, ainsi que les autres étudiants de doctorat à SFU, ont réussi à organiser la rencontre avec une grande efficacité et nous les en remercions. Nous aimerions aussi remercier la Faculté d'éducation à SFU, le bureau du vice-président à SFU, PIMS, et MITACS pour l'appui financier qu'ils ont fourni à la conférence. De plus, nous aimerions remercier les conférenciers invités, les animateurs de groupes de travail, les présentateurs de séances thématiques et d'ateliers ad hoc, ainsi que tous les participants pour avoir fait de la rencontre 2010 une expérience stimulante et mémorable.

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	9:00 – 10:30 Working Groups	9:00 – 10:30 Working Groups	9:00 – 10:30 Working Groups	9:45 – 10:45 Plenary 4 Dawson
	10:30 – 11:00 Coffee Break	10:30 – 11:00 Coffee Break	10:30 – 11:00 Coffee Break	10:45 – 11:15 Coffee Break
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	2:00 – 3:00 Plenary 1 – Q&A	2:45 – 3:15 New PhDs (3)	2:00 – 3:00 Plenary 2 – Q&A	
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4:00 – 5:00 Plenary 1 Byers	4:15 – 4:45 New PhDs (2)		4:15 – 4:45 Ad Hoc (3)	
5:00 – 6:00 Opening Session	5:00 – 5:30 Ad Hoc (1)	5:00 – 11:00 Dinner Cruise	5:00 – 6:00 AGM	
6:00 – 9:00 Reception	Dinner on Own		6:00 – 9:00 Dinner	

INTRODUCTION

Elaine Simmt – President, CMESG/GCEDM
University of Alberta

For more than thirty years mathematics educators (teachers, mathematicians, teacher educators, and educational researchers) have met annually in one of our Canadian public universities to engage in scholarly discussion and debates about mathematics education. Last year Simon Fraser University hosted the Canadian Mathematics Education Study Group. The vibrant group of SFU mathematics educators, Peter Liljedahl, Rina Zazkis, Nathalie Sinclair, Sen Campbell, and their SFU graduate students, Shiva Gol Tabaghi, Simin Chavoshi Jolfaee, Christian Bernèche, Paulino Preciado, Darien Allan, and Sean Chorney, hosted a wonderful meeting for some 120 delegates. Delegates had the privilege to learn about our history as an organisation from former SFU professor and CMESG Elder, Sandy Dawson. We also heard about the history of the International Congress on Mathematics Instruction from another of our CMESG Elders, Bernard Hodgson. As former Secretary General of ICMI he was able to call out the names of our community members who have provided service on our behalf to the international community of mathematics educators. Our gratitude goes out to Bernard and Sandy for their sharing. Without the stories of our history we would have no memory from which to connect our past, present and future.

The SFU conference is now another piece of our history and the document you are now reading serves as an important record of our work. In it is an account of the proceedings of our annual conference, as told by the people who took leadership roles in the scientific program. So, rather than try to provide a summary of the meeting, please indulge me as I share my recollections of the meeting with a top ten list.

I know that I have been to a good conference when:

10) *I am challenged to give up my certainty about mathematics and mathematics learning, and embrace ambiguity, paradox and creativity;*

9) *I am challenged to give up my preconceived notions that as a teacher I am an exclusive holder of mathematics knowledge, and to make space for parents and community to contribute to the mathematics education of their children and youth;*

8) *I see six interesting topics for working groups and I try to sneak into a different one after each break;*

7) *I only have to choose between two topic groups and no matter which one I choose it will be a great choice;*

6) *The new PhDs make the old PhDs look—well, old!*

5) *At the breakfast table I make space for a newcomer only to learn that he is a football player and isn't attending the conference—but he liked his math teacher;*

4) *The guitars and the accordion come out on the cruise ship;*

3) *There is no pizza place open on Burnaby Mountain late at night but there is someone with a car who can find some take-out;*

2) *The beer appears with the pizza;*

And the number one way I know that I have been to a good meeting is when—

1) *I bring home a new math problem.*

Thanks to all of you for yet another great meeting.

Plenary Lectures

Conférences plénières

AMBIGUITY AND MATHEMATICAL THINKING

William Byers
Concordia University

INTRODUCTION

Bonjour. Je suis heureux d'être ici avec vous cet après-midi et mes remerciements à Florence Glanfield et Walter Whitely de m'avoir invité.

I would like to dedicate this talk to my colleague of many years, David Wheeler, who started this organization and published my first paper on mathematics education many years ago called, if I recall correctly, "Dilemmas in the Teaching and Learning of Mathematics." My talk today will be consistent with the position I took in that paper and will also be in the Wheeler tradition of thinking about what he called "mathematizing," that is, thinking of math as process versus purely as content.

WHAT IS LIVING MATHEMATICS?

Les mathématiques constituent un domaine de connaissances dynamique, toujours en processus de changement et d'évolution non seulement dans le domaine de recherche mais aussi dans la façon qu'un étudiant comprend les maths. Comment trouver un moyen de parler des maths qui est en accords avec cette réalité dynamique ? C'est la question que je pose cet après-midi.

Mathematics is alive! It is dynamic, continually changing and evolving, i.e. it is process, a description that applies not only to mathematics as a whole, to the world of research, but also to the inner mathematical world of each and every student of mathematics. Learning is a dynamic use of the mind. Yet knowledge is static—we often think of mathematics as an objective set of facts and techniques. There is a tension here and inevitably this tension is reflected in the teaching and learning environment.

Our primary goal as teachers is to get students to think. What is thinking, why do we think? It is likely that we think in order to solve problems, or, to put it another way, thinking is a response to the problematic. Unfortunately, teaching often hides the problematic. If math is a set of techniques, if it is a set of facts, or even a set of theorems and proofs, if it just the right answer and not the wrong answer then the problematic is hidden from view. Suppressing the problematic results in a kind of rigidity that is the enemy of real thought and real learning and is the cause of the difficulties that many intelligent people have with mathematics. My talk will be about breaking down these rigidities that inevitably accumulate in our mathematical education and, in this way, opening the minds of students to the challenges that learning presents. Learning involves deconstruction and reconstruction. Leaving aside reconstruction for a moment, the key to this process of deconstruction will turn out to lie in those aspects of mathematics that are thrown out when mathematics is seen as focusing exclusively on the algorithmic, the precise, and the logically consistent, to the neglect of ideas, exploration, and understanding.

A NEW KIND OF PHILOSOPHY OF MATHEMATICS

My talk will be based on my 2007 book, *How Mathematicians Think: Using Ambiguity, Paradox, and Contradiction to Create Mathematics*. The title of the book wasn't my idea—Princeton's publicity department foisted it on me—but it does give you some idea of where I'm coming from. Maybe it would be useful for this group to replace the word "create" by the word "learn" or "understand." I've been influenced by people like Lakatos (1976) in his early mathematical years, Phil Davis and Reuben Hersh (1981; see also Hersh, 1997), Gian-Carlo Rota (1997), and many other mathematicians and philosophers who enjoyed standing back from time to time from the explicit content of mathematics and asking themselves global questions about the strange and wonderful world of mathematics—doing it, learning it, teaching it, and thinking about its intimate connections with the natural world.

I call what these people did the "philosophy of mathematics" but it is philosophy in a very naïve sense, namely, what we do when we, mathematicians and teachers, talk about our experience of doing mathematics. We have a great deal of experience with this kind of activity and the philosophy of math consists of drawing reasonable inferences from that experience.

AMBIGUITY

The idea behind my book is that we have an incomplete view (or myth) of what mathematics is. The reason why we subscribe to this myth speaks to the pervasive influence of mathematical formalism—in the belief, consciously or unconsciously held, that the logical structure of mathematics is definitive and absolute. Now this belief is not held as uncritically in the mathematics education community as it is for mathematicians, but nevertheless there is a discrepancy, in my opinion, between what we do when we research, teach and learn math, and what we say we do. This comes to the fore when we ask questions about process—how understanding comes about or how mathematics is created—but it is implicit in some of the informal language we use to describe mathematics, for example, what we mean by saying that some mathematical result is "deep" versus calling it "trivial."

The physicist Niels Bohr distinguished between two types of truth. An ordinary truth was one whose opposite is a falsehood; a profound truth is one whose opposite is also a profound truth. The Nobel Prize winner Frank Wilczek (2008) said, in this regard, that an ordinary mistake leads to a dead end whereas a profound mistake is one that leads to progress. We're looking in the direction of the deep, the profound truth, and even the profound mistake. They are not to be found within the formal structure of mathematics—there is something else going on. What is it, how can we talk about this missing X-factor that distinguishes the surface structure from the deep structure? In order to identify this factor I shall focus on something that you would think that we avoid like the plague in mathematics, namely ambiguity.

WHAT IS AMBIGUITY?

In the dictionary, ambiguity has two definitions: One is "obscure"; the other comes from the prefix "ambi" (or "two") as in "ambivalent" or "ambidextrous." We'll basically use the latter definition. To be precise, in a definition that comes from the writers Albert Low (1993) and Arthur Koestler (1964):

> *Ambiguity involves a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference.*

BINOCULAR VISION

Think about binocular vision. Each eye sees a given scene from a slightly different viewpoint; the brain receives these two inconsistent inputs and puts them together, unifies them. And when that unification happens something new appears, depth perception. This is a metaphor for how depth appears in math from situations of ambiguity, i.e. from unifying two different perspectives.

Situations of ambiguity are not static. There is a conflict in a situation of ambiguity that calls out for resolution. This resolution is what we call learning, understanding or creativity depending on the context.

GESTALT PICTURES

The writer Albert Low is the person who proposed this definition of ambiguity. He illustrates ambiguity by the Gestalt picture of the young woman/old lady. Notice that this picture can be interpreted in two entirely different ways but there is a conflict between the two interpretations in that you cannot see them both at the same time.

CREATIVITY AND JOKES

If you replace the word "ambiguity" in Low's (1993) definition with the word "creativity" then you get Arthur Koestler's (1964) definition of creativity. In his book, *The Act of Creation*, Koestler discusses many well-known instances of scientific breakthroughs from this point of view. Interestingly he begins the book with a discussion of jokes. For Koestler every joke is an example of ambiguity and therefore of creativity. There are two frames of reference that are in conflict and therefore produce a tension. This tension is resolved by laughter when you "get" the joke.

PARADIGM SHIFTS

It is interesting that the philosopher of science Thomas Kuhn (1962) used these Gestalt pictures to illustrate what he meant by paradigm shifts in science. The conflict that I pointed out, he calls "incommensurability."

AMBIGUITY IN MATH EDUCATION

In mathematics education David Tall and Eddie Gray (1994) have come up with the notion of "precept"; Anna Sfard (1994) has written about "reification"; Ed Dubinsky (1991) and his co-workers talk about "encapsulation"; and Lakoff and Núñez (2000) discuss metaphor in mathematics. *All* of the above ideas are variations of the notion of ambiguity.

In my view, ambiguity highlights an aspect of mathematics that contrasts with the purely logical. So the way we talk about mathematics will change if we accept ambiguity as an inevitable part of the mathematical landscape. From the point of view of pure logic, ambiguities are errors to be eliminated—but ambiguity cannot be eliminated. On the contrary, it is the feature of math that will give us an entry point into thinking about math as process and, in particular, into the process of understanding. It will also give us a way to think about the question about what makes some mathematics deep or trivial, about profound truths, and even interesting mistakes.

EXAMPLES

I will demonstrate more fully what I mean by ambiguity by showing how it appears at various levels of mathematics. Every stage in mathematical development consists of resolving the appropriate ambiguous situation—seeing the one idea that resolves the problematic situation faced by the student.

<u>Infinite Series</u>

I start with an old chestnut that many of you have thought and written about. Infinite decimals, and in particular, 1 = .999.... Students, even honours math students, notoriously have a lot of trouble with this. They want to say, "*.999... is not 1 but very close to 1*." How close? "*Infinitely close*." This just points out that even math students may not understand infinite decimals. What is going on here? .999… is usually not seen as an object, a number, but as a process. 1 is clearly an object. The equation says that a process is equal to a number. How can a process be identical to an object or, what is the same thing, how can a verb be equal to a noun? It seems like a category error. To "get it," to understand real numbers, you must realize that they can be thought of as both processes (an infinite sequence of approximations) and an object (the number that they are approximating). Remember that it is already a major intellectual accomplishment to think of an infinite collection (of rational numbers, here) as one thing. There are two points of view here that ostensibly seem to be in conflict but that are actually two ways of thinking about the same thing. Thus infinite decimals are ambiguous. Actually, we all know that the notation for infinite series is deliberately ambiguous. That's not a weakness but a *strength* of the notation. It's a strength because it makes the notation flexible and therefore the thinking that comes along with using that notation properly is also flexible. The strength of much mathematical notation resides in its ambiguity.

Arithmetic

The same ambiguity can be found in all of the operations of elementary arithmetic, which is one reason that kids have trouble with math in school. Other subjects do not use ambiguity in as basic a way that math does and therefore do not require the degree of mental flexibility that success in math requires. For example, -3 is both the number and the operation of subtracting 3. 2 + 3 is both the operation of adding and the sum. Bill Thurston (1990) has written about the great insight he had as a child that 126/37 (say) was a number and not just a problem in long division. He ran to his father with this brilliant insight, "*126/37 is a number*!" and his father just shrugged. "*Of course it's a number*." But Thurston had seen something important that was a significant step in his intellectual development. He had seen into the ambiguity of arithmetic. That's why arithmetic cannot just be memorized, you have to learn what is going on and learning often means seeing that it is ambiguous, that there is one idea that can be seen in two (or more) ways. Each step requires a leap, a discontinuous stepping up to a higher level, a higher point of view.

Algebra

The notion of "variable" is one of the great human inventions. It makes possible most of mathematics and science. But when you come to think about it, the proper use of variables, even in the simplest algebraic equations, involves the systematic use of ambiguity. What is this "x" that appears in "$3x + 2 = 8$"? How do we work with it? The variable stands for some unspecified number in the domain (which may only be defined implicitly). It stands for any number, for all numbers, but simultaneously it stands for some specific but unspecified number. Is it all or one, specific or general? At the beginning of the derivation the "x" stands for any number, at the end for the specific number 2. But at the beginning "x" is implicitly 2 and at the end it is also saying something about every number $x \neq 2$. Dealing with variables is a tricky, subtle affair. No wonder children have trouble with algebra. And the heart of the difficulty is ambiguity—that you have to think about this "x" in two ways simultaneously—as specific but unspecified.

Linear Algebra

Why in this regard is linear algebra, which, to my friends and I, was a relatively straightforward subject, so difficult for so many students? My answer is that linear algebra is full of ambiguities. To pick the most obvious, a matrix is both a rectangular collection of numbers and a function, a linear transformation. Often we even have separate courses, matrix algebra and linear algebra, which highlight this difference. Thus there are two totally different points of view. In the first, the rule for addition is clear but the rule for multiplication is obscure. It is only the second context that makes it clear why we multiply in the seemingly peculiar way that we do. Many of the important ideas in Linear Algebra, like rank, have these dual interpretations. So the student has this problem: when do you think of things in one way, when in the other? What is the *right* way to think of a matrix or anything else? The answer is that there is no right way, that the matrix is this ambiguous object—rectangular collection of numbers, transformation, row space, column space, etc. You have to see that behind all of these representations there is one idea with a whole bunch of interpretations or contexts, and you must be able to move easily from one to the other. Particularly ambiguous are things like the change of basis theorems where you have a linear transformation, a matrix, and two bases, and everything depends on everything else. It all seems very confusing!

ABSTRACTION IS AMBIGUITY

Matrices, functions which have a static (ordered pairs) definition and a dynamic (mapping) definition, and many other fundamental mathematical objects are ambiguous in this way: everything can be seen in more than one way. In fact, as the Harvard mathematician Barry Mazur (2008) pointed out in an article in the recent MAA collection, *Proof and Other Dilemmas*, the key thing in a mathematical situation is deciding what is the proper mathematical context within which to view a given object.

That is how abstraction works: in a function space, you are thinking of a function simultaneously as a point in an abstract space. Abstraction involves the ability to see things in multiple perspectives simultaneously; it involves ambiguity. Ambiguity is so ubiquitous in mathematics that almost everyone I mention this to has no trouble in coming up with some kind of ambiguity in their own fields of interest.

AMBIGUOUS THEOREMS

Now I want to move on to the theorems of math. "*Surely*," you would say, "*a theorem cannot be ambiguous*." But the essence of some theorems is precisely their ambiguity. I'll pick an easy example but you will all have your own. Think of the Fundamental Theorem of Calculus. Before the theorem there are two calculus subjects, often introduced in different courses. They have different histories and come from different kinds of problems. The fundamental theorem (FT) says that these two subjects are connected. However I would prefer to say that the FT is an insight into the fundamentally ambiguous nature of calculus, that there is one calculus with two frames of reference. Not only does the FT say that Calculus is ambiguous but it gives us a specific way of translating back and forth between the two worlds of calculus. It certainly does *not* say that there is an isomorphism between these two worlds. It's like translating from one language to another; a literal translation is bad; there are some things that you naturally can say better or at least differently in one language or the other. This is why it is good to speak two languages; it gives you a certain mental flexibility. The FT gives you important information; the situation is now richer and more useful, i.e. deeper. For example, you can prove the existence theorem for Ordinary Differential Equations by translating the ODE into an integral equation and then looking for the solution as a fixed point of an integral operator.

It was this kind of insight that seems to have been a key element in the proof of Fermat's Theorem. In the words of Barry Mazur about the (recently proved) Taniyama-Shimura-Weil conjecture,

> *It is as if you know one language and this Rosetta stone is going to give you an intense understanding of the other language. But the Taniyama-Shimura conjecture is a Rosetta stone with a certain magical power. The conjecture has the very pleasant property that simple intuitions in the modular world translate into very deep truth in the elliptic world, and conversely. What's more, very profound problems in the elliptic world can be solved sometimes by translating them into the modular world, and discovering that we have insights and tools in the modular world to treat the translated problem. Back in the elliptical world we would have been at a loss.* (As cited in Singh, 1997, p. 191)

This is for me a statement of the mathematical importance of ambiguity.

TAUTOLOGIES AS LOGICAL AND MATHEMATICAL AMBIGUITIES.

To write a computer program or a proof every detail seemingly has to be pinned down. This is the function of logical precision and is why I said earlier that logic seemingly prevents ambiguity from entering into the mathematical world. Ambiguity functions in the other direction through a structured or controlled imprecision. I'm not saying that this imprecision cannot be made precise but there is a tension here and I believe that we should be careful before we jump to conclusions about which tendency is the more basic and important. Actually even admitting that there are these two tendencies at work in math means that one has moved away from identifying math with its formal representation.

Even within logic, ambiguity has a way of sneaking in through the back door. I shall now argue that ambiguity is present in logic and in what I hope you agree is an obvious way. That is the reason why mathematics cannot be reduced to logic and explains why I claim that ambiguity is a more elementary notion than logic. The discussion revolves around the meaning and significance of tautology. The normal assumption is that tautology means identity, that two tautological statements are essentially identical. But, on the contrary, the statement "*P if and only if Q*" does not merely consist in giving the same information in two equivalent ways. It can do this when the statement is trivial like, "*a number is even iff it is divisible by two*." But more often, a tautological theorem is giving you important information. Take, for example, "*a real number is rational if and only if its decimal representation is eventually repeating.*" One side of this statement makes sense in the world of rational numbers, the other in the world of real numbers. The two sides evoke different contexts and, for this reason, it gives you important information. For example, it gives you an easy way to pick out the irrationals. There is non-trivial mathematical content here. It's not *just* a tautology. Such theorems for the continuity of real-valued functions of one variable (sequential continuity, open set continuity, and so on) generalize naturally into different worlds; so there is the metric space definition, the topological definition, and so on.

A large part of mathematics consists of elaborate tautologies. The great mathematician Henri Poincaré asked why, if mathematics consists merely of elaborate ways of saying "*P if and only if Q*", how are we to account for its power and effectiveness in describing the natural world? The answer to this seemingly perplexing question lies precisely in the observation that a tautology is ambiguous and can have non-trivial content. Remember that an ambiguity requires two frames of reference that differ from one another and are mediated by a single idea. In the statement "*P if and only if Q*" the *P* and *Q* are the two frames of reference. A rational number is a quotient of integers or it is a repeating decimal. On the surface of it these are very different ways of characterizing rational numbers yet the symbol or statement "*if and only if*" says that there is a unitary idea that is being expressed in these two different ways.

The previous paragraph shows that mathematics cannot be reduced to logic. If that were the case then logical equivalence would be the same as mathematical equivalence. But the value of the technique of "contrapositive proof," for example, is that it takes two statements that are logically identical yet are mathematically distinct. Thus you prove, "*the square of an integer is even implies the integer is even*" by means of the contrapositive, "*the square of an odd number is odd*". So you cannot say the two formulations are identical. Mathematical tautologies are often extremely valuable but their value lies precisely in their ambiguity. In my experience this is a point that some philosophers just do not understand. For them the logical level is the most basic. For us there is another, deeper level, the level of the mathematical content, the mathematical idea.

MATHEMATICAL IDEAS

Ideas are the currency of mathematics—we are in the business of teaching and learning ideas. An idea is an answer to the question, "*What is going on here?*" It reveals patterns or structure within the field that is under consideration. But ideas are slippery objects. An idea must be grasped; it cannot be memorized. And yet every mathematical situation revolves around some idea; every proof contains a central idea. When you grasp the idea, the rest is the mere filling in of details. Ideas are subtler and more basic than logic; in fact, *logic itself is a powerful idea.* Ideas are the ways by which our minds structure complex situations. But it's not productive to talk about ideas in terms of precision, in terms of right and wrong. Ideas are generative; they can be deep or shallow. They can come from anywhere; even a *paradox* can be turned into a fruitful mathematical idea as we see in the works of Cantor, Gödel, and Chaitin. As the Japanese mathematician Shimura said of his colleague Taniyama, "*he was gifted with the ability to make good mistakes*." What is a good mistake? It is one that contains a mathematical idea. A good mistake is worth pages and pages of formal reasoning, which is a good thing to remember when one is assessing the work of students. From the point of view of the ideas that support mathematics, we must learn to value those aspects of math that we mostly ignore or put down—the mistakes, ambiguities, contradictions, and paradoxes—all of which can be seen as opportunities to deepen our mathematical understanding.

THE VALUE OF THE PROBLEMATIC

The problematic has great value for learning and teaching. This is borne out by thinking about learning as overcoming a series of obstacles. Take "epistemological obstacles," those problematic situations that arose in the history of mathematics and are often recapitulated in the learning process of the individual student. As I said earlier, learning is a dynamic use of the mind. The essence of learning is that mental structures change and develop. For example, the psychologist John Kounios, who studies the neural basis of insight, defines creativity as the ability to restructure one's understanding of a situation in a non-obvious way. In the terms that I have been using, to develop a new frame of reference. Yet formal knowledge is static or we think of it as such. How do we get people to move from the static to the dynamic—how do we get people to think? My view is that we must present them with the problematic in a controlled manner. The role of good teaching is not, as so many seem to think, to hide the problematic or maintain that there is a royal road to learning that avoids difficulty. The virtue of the ambiguous, of paradox and contradiction, is that here the problematic is right out front. *Ambiguity is an opportunity to convert the static mind into a dynamic one.*

Logic emphasizes accuracy, precision and structure. Ambiguity, on the other hand, points to openness, flexibility, and creativity. But learning and teaching are essentially creative activities. Creativity does not follow some formula or algorithm. That is why you cannot learn for someone else. All you can do as a teacher is to set the stage. An ambiguous situation needs to be grasped. It does not exist outside of the mind of the learner or the teacher. The student is confronted by a situation of ambiguity—a challenge that they are called upon to overcome. But the teacher stands in front of the class as an expert who hides or otherwise does not acknowledge their limitations, their own unresolved ambiguities. If the teacher does not acknowledge their own unresolved ambiguities then they cannot really relate to the situation that the student finds themselves in. Of course we all know that every teaching situation is replete with ambiguities. Our success as teachers depends in large part in our response to the inevitable ambiguities of the teaching situation. That is why thinking of teaching as merely putting knowledge or techniques into the heads of students, or of writing down a formalized version of the subject, makes teaching easy but relatively meaningless.

We tend to identify success in mathematics with intelligence, whatever that is. But this discussion of ambiguity points to something else. Suppose you are teaching a young student to multiply and you introduce it as repeated addition. Suppose a student is an excellent adder and uses this skill in order to do her multiplication. Then she gets stuck when the multiplication problems get too hard for this strategy to be effective. At some point there are implicitly two or more ways to think of multiplication and the student must make a leap into a new way of thinking. Why can some students make this leap and others, who were perhaps more proficient at some preliminary stage, not do so? This is an interesting and complex question.

One factor that is at play in this situation and all situations of ambiguity is conflict between the two different frames of reference. Conflicts lead to **tension** and tension is often unpleasant. Maybe one of the predictors of scholastic success is the ability to manage tension. In a learning situation one is continually being put into situations of "not knowing" and you have to learn to navigate within such situations. One has to develop what could be called a certain stress tolerance. Can you stick with a situation that is unresolved or does it make you feel too nervous? Now I'm putting this in too negative a way because learning can be seen not as stress but as fun. In other words, one can learn to enjoy those unresolved situations. Of course this is unlikely if you are continually being put down for getting the wrong answer and are never given the opportunity to explore your own ideas. The mind is a wonderful instrument and there are all kinds of ways to understand a given mathematical situation.

CONCLUSIONS

In conclusion I reiterate, first of all, that mathematics is larger and deeper than its logical-formal presentation. Logic is a tool, not an end. Mathematics is essentially an ongoing process where our understanding at a given moment can always be deepened and altered in the future. Second, we must focus more of our attention on the problematic, which includes the ambiguous, contradictory, or paradoxical, and which, in the case of students' learning, includes mistakes and getting stuck at an inappropriate level. Let's remind ourselves that mathematics is basically a creative art form and should be communicated as such. Learning is an exercise in creativity.

Let me leave you with some questions:

1. A student of mine once commented about one of those vacuous proofs that you often encounter in math courses, "*I follow it but I don't understand it*." Is this distinction valid and is it related to the discussion of logic and ambiguity?
2. Is learning continuous or discontinuous? Can you teach understanding? If the answer to this is (at least in part), "*no, everyone must learn for themselves*," then what is the role of the teacher?
3. Is there indeed ambiguity in math? Can you identify particular ambiguities in the math that you teach?
4. How can students be taught to identify mathematical ideas? What is the relation between proof and ideas?
5. What is the value of the problematic—of ambiguities, mistakes, paradoxes, etc.—and how can they be utilized in the teaching situation?

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LEARNING FROM AND WITH PARENTS: RESOURCES FOR EQUITY IN MATHEMATICS EDUCATION

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This paper draws on almost 20 years of working and conducting research with parents, in particular with Mexican-origin parents in working-class communities in the Southwest of the U.S. Despite the specificity of the context, I argue that the lessons learned can be applied to other contexts to promote a more equitable approach to mathematics education. I start with my motivation for this line of work, that is, why work with parents? What took me there? I will then describe different approaches to engaging in conversations with parents about mathematics and mathematics education. I report on findings from our research on: 1) Parents' perceptions about the teaching and learning of mathematics; 2) Valorization of knowledge; 3) Language and mathematics (focusing on parents and children students whose first language is not the language of schooling); and 4) Parents-children interactions around mathematics. I close with some implications for teacher education.

FUNDS OF KNOWLEDGE FOR TEACHING PROJECT

In my early work in mathematics education I became intrigued by the notions around situated cognition (Brown, Collins, & Duguid, 1989) and the studies that documented out-of-school mathematics practices (Nunes, Schliemann, & Carraher, 1993). The notion of task relevance as playing a role in students' success in completing a task was one of the reasons why I became interested in this line of work. Shortly after my arrival to the University of Arizona I was introduced to the Funds of Knowledge for Teaching Project (González, Moll, & Amanti, 2005; Moll, 1992). A key concept in this project is a rejection of a deficit view towards low-income, non-dominant communities. (I use the term "non-dominant" based on Gutiérrez (2005) who writes, "*this term better addresses issues of power and power relations than do traditional terms*" (p. 3).) Instead of seeing these communities as lacking what is needed for success in school, the researchers (university-based and school teachers) in the Funds of Knowledge project focus on the resources, experiences and knowledge that are present in any community. As Moll, Amanti, Neff, and González (2005) write, "*we use the term **funds of knowledge** to refer to these historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and well-being*" (p. 72). In order to learn from the community, teachers visit the homes of some of their students. In these visits, teachers use ethnographic methods to learn from the families. It is important to understand that the purpose of these visits is to learn about the funds of knowledge that reside in any family. Teachers use very detailed questionnaires that range over the following themes: family structure; labour history; household activities; mathematical attitudes (this was added in a later version of the original Funds of Knowledge for Teaching project, as I became involved and worked on a project more specific to mathematics teaching and learning); parental attitude (parenting; money; religion; education; ethnic identity). Teacher-researchers who participated in the Funds of Knowledge for Teaching project have written about the impact that these household visits had on their teaching (and on their students). In particular, they stress the importance of developing relationships with students and their families; they also describe classroom applications in which parents (or other family members) shared their expertise in learning modules that the teachers designed based on their home visits (see for example, Amanti, 2005; Floyd Tenery, 2005; Sandoval-Taylor, 2005). One of the teachers in the later implementation of the project said, reflecting on the household visits, "*I guess realizing that the home is a real learning place, real learning environment, you know, I didn't think it was so much a learning environment as it is*."

This is a fundamental aspect of this approach: for all of us to see the homes of students, particularly non-dominant students, as "real learning places." Making connections between the household knowledge and mathematics teaching and learning in school is not a straightforward task. In Civil (2007) I address some of the tensions that I encountered in my work when trying to bridge funds of knowledge and school mathematics knowledge. Some of these tensions have to do with our values about what we count as mathematics. In González, Andrade, Civil, and Moll (2001), we also discuss the issue of transforming community knowledge into school mathematical knowledge. In particular we describe my experience trying to make sense of a seamstress' practice from a mathematical point of view. It was experiences such as this one, talking with women about their uses of mathematics, that led a colleague of mine and me to question what we saw as somewhat one-sided conversation in the home visits, in the sense that a main goal was for the teacher-researchers to learn from the families. We wanted to develop opportunities where the mothers (at the time we had been visiting with a group of mothers in a literacy project) could learn mathematics (per their request) but also, we were planning on learning about mathematics from them. This is how the mathematics workshops for parents started (Civil & Andrade, 2003).

ENGAGING WITH PARENTS IN MATHEMATICS

This early work with the group of mothers was key to the development of a much larger parental engagement project. Key to this early work and the work in Funds of Knowledge is the concept of "confianza" (trust). As Moll (2005) writes:

> *We found that a funds of knowledge approach through its emphasis on teachers engaging households as learners and thus forming what we call relationships of confianza with parents, may help create new options for parents, especially mothers … to shape their relationship to the school and the schooling process.* (p. 280)

As one of the mothers in the early work we did in mathematics wrote, "*for me the most important foundation was the confianza that each one offered me…. I can say that all that I now know and have learned has been accomplished by means of the confianza*."

To establish this relationship of trust takes time. It is part of the rapport building that accompanies the kind of ethnographic work that I do in my research. Thus, when we started a much larger project that had multiple levels of activities and many people involved, still it was very important to me to establish "confianza." This meant spending a lot of time in the field, attending mathematics workshops and "math for parents" courses, at times three to four evenings per week. This is part of my research approach. This does not mean that at every event I was there "collecting data." For me, it was important to help create a safe environment in which to engage with the parents in a dialogue about mathematics and its teaching and learning.

There were several goals for this larger project on parental engagement in mathematics:

- To engage parents as learners of mathematics.
- To familiarize parents with current mathematics education pedagogies.
- To develop an awareness of the mathematics instruction in their children's classrooms.
- To facilitate dialogue between parents and teachers that challenges power relations in schools.

This last goal (Civil & Bernier, 2006) has become particularly important to me and it is one of my current interests. Most of my work through this project, however, centered on the researchers-parents dialogue. It is through this dialogue that we were able to learn about parents' values and beliefs about the teaching and learning of mathematics (Civil, Planas, & Quintos, 2005; Civil & Quintos, 2009) and it is this dialogue that led us to the concept of "Tertulia" which I address later in this paper.

As we explain in Civil, Bratton, and Quintos (2005), our work in this large parental engagement project led us to a redefinition of parental involvement. Parental involvement is often characterized by physical presence of parents in the schools, leading to schools' (teachers', administrators', even other parents') deficit views of parents who "don't come to school," particularly in working-class, non-dominant communities (Civil & Andrade, 2003). Instead, our work is grounded on the literature on parental involvement from a critical perspective (Calabrese Barton, Drake, Pérez, St. Louis, & George, 2004; Delgado-Gaitán, 2001; Olivos, 2006; Valdés, 1996) and draws on the concept of cultural and social capital applied to parental involvement (Lareau, 2000; Lareau & Horvat, 1999). A key concept in our work is that of parents as intellectual resources (Civil & Andrade, 2003). By this concept we mean an interest in parents' views and understandings of mathematics and a desire to learn from them and build mathematics instruction on these adults' knowledge and experiences. We focus on the strengths and assets of the families and communities in order to change the focus from needs of the communities to the possibilities present within the communities (Guajardo & Guajardo, 2002). I concur with Valdés (1996) when she expresses her concern for any effort at parental involvement that "*is not based on sound knowledge about the characteristics of the families with which it is concerned*" (p. 31).

Throughout the project we looked at parents in four roles:

Parents as parents: Many parents joined the project for their children, to help them with their homework, or to show them that they were also engaged in learning. To be a role model for their children was a key motivation. As Bertha said:

> *We can be a model for our children. If we have opportunities to grow, if we have some kind of knowledge we can support our kids better and they can see, "oh my gosh, they're doing this for themselves and also for me" and they can feel stronger.*

Parents as learners: Parents enjoyed coming to the workshops and the Math for Parents courses to learn for themselves. Many of them commented that they liked the opportunity to be with other adults (many of them knew each other from the schools/community) and talk about mathematics. It offered them a change from their household activities (most of the participants were women). They enjoyed working in groups and looking at mathematics from a different approach from the one they had experienced in their own schooling. As one mother said, "*I went through my whole life being told how things were [in math] and not given any freedom to figure it out on my own.*"

Parents as facilitators: An innovative aspect of this project was to have teams of teachers, school administrators and parents facilitate 2-hour mathematics workshops for the community at large. While parents expressed being nervous at the idea of "teaching" mathematics to other parents, many of them shared with us that this aspect had been the most powerful. They felt that as parents they could understand where other parents may be coming from and thus relate to them perhaps more easily than teachers. This component also brought up power issues (Civil & Bernier, 2006), as this quote from one of the mothers, Marisol, points to:

> *It was hard in the beginning to work with the teachers. "They are the best." They don't give you the opportunity that you may know more or bring other ideas. Now we are more equal…. Now they rely on me, they check with me, they make you feel that you are important to them. One teacher once told me "you just hand out papers" and I was upset.*

Parents as leaders: Parents in this project had multiple opportunities to take on leadership roles, as facilitators of workshops and later on as mentors of teams of facilitators. But this project also aimed to promote a sense of leadership that would lead to action beyond their own children's education and more at the school (or district) level. Verónica captures this quite insightfully when she said,

> *How are you [to mothers in group] promoting this program to motivate parents for children's success in mathematics? What are we going to do? Because what I do is to come to help my child, mine. But that that doesn't mean the success of a district, of a school. No, not of the district nor the school.*

This comment took place as part of a series of sessions we continued to hold after the official part of the project had ended. A group of mothers wanted to continue meeting. We (the research team) were also interested in continuing the conversations and in fact we wanted to take a more critical approach. That was the origin of the Tertulias. During 2003-04 we had 16 Tertulias with 15 participants (14 women, 1 man).

The "Tertulias Matemáticas" (mathematical circles) combined exploring mathematics with discussion about the teaching and learning of mathematics within the schools in the parents' community. The discussions during the Tertulias were more critical than in past events in part because the group of mothers had been involved in the project for at least two years, and some for the whole duration (4 years). They knew each other well and they knew us. There was "confianza." These critical dialogues mostly centered on the question of "how can all students in this school district be successful in mathematics?" (Quintos, Bratton, & Civil, 2005). These dialogues served to uncover a series of themes that we have been studying since and that I address in the next section. The Tertulias were also an arena for leadership development. This became clear in the discussion about the district plans to continue the original parental engagement project, as the funding ended and it was up to the district to decide how to continue it. That continuation, however, took a different form from the original project as district administrators tended to make it into a project in which teachers facilitated workshops for parents and at most included a parent to translate into Spanish. Where did the idea of parents as facilitators go? The mothers who attended the Tertulias certainly voiced their views of what the project was supposed to be about:

> ***Jillian:*** *…the whole object of [this project] was for parents to come in and teach other parents, so they didn't feel so uncomfortable, intimidated….teachers can come in and teach because that's what they do, but when you have another parent come in teaching you…you can absorb a lot more.*
> ***Bertha:*** *The point is to be part of the school, be part of the community like parents…to me the main point was parent involvement…to me the point of [the*

> *project] was using the parents, using in the right way, using parents to teach other people.*

In particular when the mothers in the Tertulia saw how the district was planning to continue the project, they expressed their disappointment:

> ***Darla:*** *Isn't that kind of defeating the purpose behind [the project]? Because [it] was supposed to be parents teaching parents so…. Allowing teachers to take over what we worked so hard to set up, that's almost like slapping ourselves on the face; we put in a lot of time, a lot of practice.*

Some mothers came up with different action plans that they could implement at their sites (several of them either worked or volunteered at one of the schools). One of these mothers was instrumental in a later project I directed, as she had been organizing mathematics workshops at her school, which became one of the sites for our next iteration of working with parents and mathematics. From a point of view of both research and outreach, I consider the Tertulia format a very rich approach to establishing a two-way dialogue between parents and researchers. What remains to be explored is how to bring in teachers (and other school personnel). In the most recent project, CEMELA (Center for the Mathematics Education of Latinos/as), we had a couple of teachers particularly involved in the courses for parents and sometimes in the sessions that were modeled after the Tertulias, but we have not studied the teacher component in a systematic way. Through CEMELA we have conducted research with teachers (through Teacher Study Groups) and with parents (through Tertulias/courses for parents). This has given us insights into some of each other's views (teachers and parents) and these insights point to the need for stronger dialogue between teachers and parents to help clear up misperceptions and miscommunications. In the section that follows I focus on some key research findings from our work with parents, in particular with immigrant parents.

SOME RESEARCH FINDINGS

I have organized this section along four themes that we have explored throughout the different projects we have had with parents and mathematics.

PARENTS' PERCEPTIONS ABOUT THE TEACHING AND LEARNING OF MATHEMATICS

As we have written elsewhere (Civil, 2008b; Civil & Planas, 2010), immigrant parents generally think that the level of mathematics teaching is lower in the U.S. (receiving country) than in their country of origin (Mexico) and they often express a concern for the lack of emphasis on basics (particularly the learning of the multiplication tables). As I note in Civil (2008b) this perception is shared by immigrant parents in other parts of the world. I argue, along with other researchers (Hamann, Zúñiga, & Sánchez García, 2006; Macias, 1990) that this concept of the level being higher or lower is a complex issue that needs careful analysis. These authors, as well as the parents in our research, and the teachers in a Mexican school near the border with the U.S. (which we visited to gain a better understanding of the teaching of mathematics in a sending community) note that the curricula seem more demanding in Mexico, and that often students who have been schooled in the U.S. and transfer to a school in Mexico are placed in a lower grade than they would have been in the U.S. While parents may perceive that the level is higher in Mexico and perhaps more demanding, they also express their preference for the U.S. system as providing more options (in terms of course-taking, such as art, computer classes, music) and resources. Furthermore, it is important to point out that their perceptions may be a combination of what they see happening in Mexico right now (through their relatives, for example) and what they experienced as students. As Marina says:

> *My son who is in second grade, maybe it's a technique for learning how to multiply. For us, it was by singing. But we don't know how to reason, that is the difference. Maybe now they are using a way to make them think, without having to, like be just singing them, you know. And without knowing what they are saying.* [March, 2007]

Marina often brought up this notion of reasoning, of learning the why behind procedures, as something that her son was learning at school and that she did not learn when she went to school. She recalls memorizing things such as the times tables, without understanding how they worked.

In summary, Latino/a parents in our studies value education and are sometimes puzzled by what they see as lack of rigor, little homework, and not enough emphasis on the basics. Providing opportunities for them to explore mathematics as learners may lead them to reflect further on the pedagogical approaches that their children are experiencing, as is the case of Marina. These reflections bring to surface the values that we all have about what counts as mathematics and how it should be taught. This is the second theme in this section.

VALORIZATION OF KNOWLEDGE

Our work with parents often brings to the surface the fact that all of us (teachers, students, parents, researchers) have specific values about what we count as mathematics, what we see as appropriate teaching approaches, and so on. Abreu's writing about the notion of valorization, in particular as it applies to home and school mathematics (Abreu, 1995; Abreu & Cline, 2007), has been very influential in my thinking about parents' knowledge and school knowledge. Abreu's research in a sugarcane farming community in Brazil, as well as her later work with immigrant students in the U.K., shows that children, when talking about the school versus the home approach, tended to view the school approach as the "proper" way to do things. In my own work with preservice elementary teachers, I noted their preference for "school" approaches (even though they did not necessarily feel comfortable with those) over their more "informal" methods (which often made more sense than the procedures they had learned (Civil, 1990; 2002). This concept of valorization of knowledge is particularly important, I argue, when those implicated are non-dominant students and their parents. As Quintos, Bratton, and Civil (2005) write,

> *The knowledge that working class and minoritized parents possess is not given the same value as that which middle class parents possess and the ways that these parents are 'involved' in their children's schooling experience are defined according to the ways in which middle class parents participate in their children's schooling.* (p. 1184)

Civil and Quintos (2009) describe how conducting classroom visits with parents can be a powerful way to engage in a discussion about our different values. Through these visits parents share their views on what they would like teaching and learning mathematics to be like. For example after a visit to a 7th grade mathematics class (students ages 12-13) a group of mothers commented on the fact that the seating arrangement was in groups, thus with some students with their backs to the white board, a situation very different from when they went to school where they sat in rows facing the board. The researcher asked them what their thoughts were on this group sitting arrangement:

> ***Berenice:*** *Well, I think there is more distraction like that in, by being in a group all the time… rather than being individually. You're there by yourself attentive to what, to what the teacher is going to say, and to…*
> ***Dolores:*** *Or many, or many times, Jesús [researcher's name], if you're in a group, the other one is going, is going to copy the one who, who…*
> ***Berenice:*** *Yes*
> ***Dolores:*** *who is doing it right. So he is going to depend on the one on the side.*

> ***Berenice:*** *On the neighbour.*
> ***Dolores:*** *I think that individually they learn better. They work harder.*
> [Debriefing Classroom Visit; October 23, 2008]

Just from this brief excerpt we get a glimpse of how these two mothers may be viewing learning mathematics as primarily an individual endeavour. These views may clash with what the teacher is doing in her classroom. Elsewhere (Civil & Planas, 2010) we elaborate on this notion of valorization of knowledge with parents, in particular around the example of the representation of the algorithm for division in the U.S. and in Mexico. This topic comes up frequently in our discussions with parents who often view "their" method as being more efficient than the ones their children are learning in school. As several authors in Abreu, Bishop, and Presmeg (2002) point out, immigrant students (or children of immigrant parents) experience many transitions in school mathematics. In our studies we see that parents (like everybody else) have very definite views about the teaching and learning of mathematics. When these views are different from those of the school, which is often the case when navigating school systems from different countries, children are likely to be caught in the middle. An important implication for teacher education programs is the need for teachers to realize that there are other ways of doing mathematics that may be different from the ones they were taught. This does not mean, as a preservice elementary teacher wrote expressing concern: *"Are we expected to learn all these ways?"* I think that the key element here is to not only realize that there are different ways but also to develop the mathematical background to be able to make sense of them and to engage in discussions about the value we give to these different ways.

LANGUAGE AND MATHEMATICS

The third theme in our research relates to language and mathematics, specifically in the context of working with parents whose first language is not English (the language of instruction in their children's schools). In a survey of the teaching and learning of mathematics with immigrant students (Civil, 2008b) the issue of "language as a problem" is pervasive across many countries where the language of instruction is not the first language for students in the classroom. For policy-makers and teachers, children not being proficient with the language of instruction often becomes a key obstacle to their learning of mathematics. Instead of focusing on the resource in knowing multiple languages, schools often focus on a deficit in not knowing English (or the corresponding language of instruction) well enough. This thinking may lead to placement decisions that are based on language at the expense of students' knowledge of mathematics (and other subjects). As Valdés (2001) writes, "*students should not be allowed to fall behind in subject-matter areas (e.g., mathematics, science) while they are learning English*" (p. 153). How aware are immigrant parents of placement decisions? Civil (2008a; in press; Civil & Menéndez, in press) describes the case of Emilia, an immigrant mother who, in a first interview about three months after their arrival to the U.S., appeared to be satisfied with the fact that much of what her oldest son was learning in mathematics he had already seen in Mexico, because that way he could focus on learning the English language. It is interesting to note that in an interview 21 months later she wondered why her two boys who were in different grades would sometimes bring the same homework. As she said, "*it bothers me a bit because it leads me to believe that, as if the eighth is at the same level as the seventh, you know? One assumes that the eighth is at a higher level.*" (Emilia, Interview #3, December 2007).

Emilia's sons were caught in the implementation of a language policy in my local context that segregates English Language Learners (ELLs) for at least 4 hours a day to focus on the learning of English. In the specific case of this school during 2007-08 they tried an earlier version of the policy and ELLs were kept together for 5 or 6 of their classes. While the school's intention may have been to provide a supportive environment for ELLs, interviews with several of the students indicated that they would have rather been in the "regular" classes (Civil, in press; Civil & Menéndez, in press; Planas & Civil, in press). In fact, the mothers of some of them shared their children's perceptions about this arrangement in which ELLs were kept separate from the non-ELL students with us:

> ***Roxana:*** *He (her son) does say that he wants to go higher.* (Laughs.) *He is going for, he says "I want to get to my final goal," he says "I haven't reached it yet," he says "I am working to get there," he says. He says that [the ELL arrangement], it's like he's not very convinced of being there. He wants more.*
> ***Mila:*** *They are embarrassed. Larissa (her daughter) says, "Mom I am embarrassed to go to the [classes for the ELLs]." That's what she says.*
> [Interview with Mila and Roxana, April, 2008]

These issues of placement and being taught apart from the non-ELLs raise questions as to whether ELL students are receiving the appropriate mathematics instruction. But there is another aspect of the language and mathematics theme that directly connects to our work with parents. It has to do with how language policy affects parents' participation in the school. In U.S. schools parental involvement is often characterized by presence of parents in schools (e.g., to help out with field trips; to volunteer at the school, including assisting in classrooms) and by parents providing support for homework. After Proposition 203 was passed in 2000, bilingual education in Arizona became severely restricted. This affected parents in the communities where our work is located since they felt they could not participate as much in their children's schooling due to "the language issue" (Acosta-Iriqui, Civil, Díez-Palomar, Marshall, & Quintos, in press; Civil & Planas, 2010). As one mother told us:

> ***Verónica:*** *I liked it while they were in a bilingual program, I could be involved…. I even brought work home to take for the teacher the next day. I went with my son and because the teacher spoke Spanish, she gave me things to grade and other jobs like that. My son saw me there, I could listen to him, I watched him. By being there watching, I realized many things. And then when he went to second grade into English-only and with a teacher that only spoke English, then I didn't go, I didn't go.*

Cándida reflected on how she could help her children with homework when they were in the bilingual program but how that changed when they were put in English-only classrooms:

> ***Cándida:*** *I remember that they gave her homework that had English and had Spanish, and so I could help them a little more. But when it was only English, no. Then I felt really bad. I was frustrated because I couldn't explain it to them and I would have liked to explain it to them and I couldn't. I was frustrated.*
> [Interview February 2006]

Several parents in our studies have mentioned their frustration at not being able to help their children with the mathematics homework even though they have the mathematical knowledge. Their children try to translate into Spanish, but their academic Spanish is often not developed enough and thus this makes for a complex communication situation in which parents and children often end up frustrated. Although some of this is to be expected when the language of instruction is different from the primary language in the home, I want to underscore the affective impact of policies that directly or indirectly send a message that one's first language is worth less than the language of instruction. As Stritikus and Garcia (2005) write, "*the normative assumptions underlying Proposition 203 position the language and culture of students who are diverse in a subordinate and inferior role to English*" (p. 734).

In our local context, comments by the general public in media outlets often conflate the use of languages other than English (which in our case, most of the time means Spanish) with issues of immigration. As Wright (2005) writes in his analysis of Proposition 203:

> *There is also collateral damage as the result of the political spectacle surrounding Proposition 203. The issue was promoted as proimmigrant and supposedly only dealt with the narrow issue of the language of classroom instruction. However, it sparked widespread debate about immigration and immigrant communities as a whole, stirring up strong emotions about illegal immigrants and directed attacks on the Hispanic community in particular.* (p. 690)

In July 2010, The Arizona Educational Equity Project under the auspices of The Civil Rights Project at University of California, Los Angeles, published nine papers describing several studies undertaken to analyze the educational conditions of ELLs in Arizona with the implementation of the 4-hour block separation. In one of these papers, Gándara and Orfield (2010) focus on the consequences of segregation through a review of the literature as well as by looking at the current situation in Arizona. They ask, "*Is the four-hour Structured English Immersion block that is being implemented today in Arizona a return to the Mexican room?*" (p. 9). By the Mexican room the authors refer to the segregation of Mexican American students in the 1940s, a segregation that resulted in inferior education. Later on they write, "*As devastating to the educational outcomes as segregated schools are for minority and English learner students, perhaps even more pernicious is the internal segregations that goes on **within** schools*" (p. 10). As they write in the conclusion, "*[the segregated 4 hour block] is stigmatizing, marginalizing, and putting these students at high risk for school failure and drop out*" (p. 20).

My point is that we cannot ignore or pretend that these issues do not affect children's schooling, including their mathematics education. Ruiz (1984) writes about three orientations towards language—language as a problem, language as a right, and language as a resource. Language policies such as the ones in my context take the view of language as a problem, putting the burden on the families for not knowing the language of instruction. I wonder what kinds of experiences for children and their parents would be available if the language policies promoted a view of language as a right and language as a resource.

INTERACTIONS BETWEEN PARENTS AND CHILDREN ABOUT MATHEMATICS

This fourth and last theme brings the other three themes together. In some of the schools where we did the series of mathematics courses for parents, their children also attended the sessions. This allowed us to study the interactions between parents and their children as they engaged in doing mathematics together. Through these interactions we were able to see and hear parents' perceptions about teaching and learning mathematics, issues related to valorization of knowledge, and the role of language, since many of these interactions took place in Spanish as the dominant language, but English was also present as the language that children often used. I illustrate the nature of these interactions with two vignettes.

The first vignette involves a father (Sergio) and his seven-year-old daughter (Berta) and took place in Spanish. The focus on that evening's session was on explaining to the parents how the school was teaching addition and subtraction. The approaches were quite different from what the parents had studied in school, and different from what we could label as "traditional" algorithms. So, for example to add 23 + 46 + 7, the problem was presented horizontally and the children were expected to combine numbers in any way they wanted, yet using strategies such as groups of ten, friendly numbers, etc. For example, a child may do 7 + 3 = 10, then do 20 + 40 = 60, and so 60, 70, 76. While Berta is trying to do the addition following what she has been learning in the classroom, her father on his worksheet sets it up vertically and tells her "*so that you won't get confused*" and gives it to her to do it that way "*do it like this, look*" (pointing as his addition set up vertically). Berta then proceeds to add the ones column, "*sixteen*"; the father says, "*put the six here*" (in the ones place); she also writes the "1" (the 10 from the 16) in the tens column and completes the addition. Berta then moves on to the next one that was also presented horizontally in the worksheet (23 + 26 + 27 =) and she rewrites it right away vertically as her father had done with the previous one. She has some difficulties with this one and the father corrects her, though Berta asserts herself and says that she wants to do it, and in the end says, "*You don't have to tell me because I want to learn*."

The school where these sessions took place was using a "reform-based" curriculum for mathematics. One of the goals for having these workshops with the parents was to explain to them how and why mathematics is taught differently from what they may have experienced. But as we can see from the vignette I just presented, perceptions and values are very present. For this father, his vertical way of doing addition made sense and that is the way he chose to work with his daughter. Berta did not really understand the regrouping ("the carrying") in the traditional method. I am not saying that she had a better understanding with the alternative method. This would require further investigation. What I want to point out is that perceptions about how mathematics should be taught and valorization of knowledge play a role in these interactions (see Civil (2006) and Civil & Planas (2010) for more examples on this topic). Berta's father was quite explicit in his approach, telling Berta what to do. But I refrain from characterizing this interaction as direct, authoritarian, etc. without knowing more about the way this family interacts. Berta was quite assertive in telling her father not to do it for her and her father pulled back. We have much more data on Berta and her parents in these workshops, as well as an interaction around homework in their house (Civil & Planas, 2010). What I can say is that the interactions were quite spirited.

The second vignette involves a mother and her nine-year-old daughter (Alma). The theme of the workshop was the array model to explore multiplication. The participants had different-size arrays and on the grid side they had to write what multiplication sentence each array represented (the dimensions) and on the other side of the card/array they had to write the answer to the multiplication sentence. This was in preparation for a game they were going to play. Alma picked up a card that showed an 8-by-12 array:

> *1.- **Mother:** So, you are going to put eight times twelve and what other way you're going to do it?*
> *(Alma writes 8 × 12 on the grid part of the card)*
> *2.- **Mother:** ¿Qué otra manera puedes decir? [What other way can you say it?]*
> *(Alma writes 12 × 8.)*
> *3.- **Mother:** There you go. (Pause) OK, equal?*
> *4.- **Alma:** Should I put it on the back?*
> *(Alma is following the facilitator's instructions about putting the answer on the back, but her mother does not and asks her to write the answer to 8 times 12 on that side too.)*
> *5.- **Mother:** Sigue escribiendo, no, ¿Cuánto es ocho veces doce? [Keep writing, no; How much is eight times twelve?]*
> *(Alma does not say anything)*
> *6.- **Mother:** Ahora le ponemos a contar, ocho, diecises, veinticuatro; four times? [Now let's count, eight, sixteen, twenty-four.] OK, let's go easier.*
> *7.- **Alma:** English please.*
> *8.- **Mother:** English please, OK; these are twelves (with her pencil, goes over one row of 12 on the array); we are going to go easier. Ten times eight, how much is ten times eight?*
> *9.- **Alma:** Ten times eight?*
> *10.- **Mother:** Remember, it's eight and you add what? (Brief pause) A zero*
> *11.- **Alma:** Hmmm*

*12.- **Mother:** so, it's eighty plus eight?*
*13.- **Alma:** eighty plus eight is eighty-eight*
*14.- **Mother:** plus eight? (pause) How much is plus eight, eighty-eight plus eight*
*15.- **Alma:** eighty-eight plus eight…*
(pause)
*16.- **Mother:** Está nerviosa [She's nervous] (As if talking to the camera.)*
*17.- **Alma:** Eighty-eight plus eight*
(pause)
*18.- **Mother:** What about eighty plus sixteen?*
*19.- **Alma:** Eighty plus sixteen*
*20.- **Mother:** It's ninety what?*
(Alma is whispering)
*21.- **Mother:** Six, don't get nervous.*
(Mother checks that Alma puts 96 as the answer for 8 x 12 and asks her about 12 × 8; Alma puts 96 there too.)
*22.- **Mother:** Turn around (the card), put the answer there.*
(Alma writes 96; the mother asks her to take another card and to do the same thing they have just done.)

[Math for Parents – October 5, 2006]

This excerpt highlights different strategies that this mother is trying to use to help her daughter find 8 times 12. First she starts counting by eights (line 6), but realizes that this may be too hard. Then she breaks the problem as 8 × (10 + 2) (line 8), but the strategy is not made explicit. Then, when Alma doesn't seem to know what to do for 88 + 8, the mother breaks it differently as 80 plus 16 (line 18). Eventually the mother is the one who gives the answer. Throughout the interaction, it is not clear at all what her daughter is thinking since she remains mostly silent. This is not an isolated experience as most of the data that we have with this child shows her as hardly ever talking. Her mother is quite strict about what her daughter needs to do and how: telling her to write 8 × 12 and 12 × 8, guiding her throughout, asking her to write the answer on both sides of the card. Yet, later in the session when they are actually playing the game, we have an example of the mother learning alongside with her daughter and playfulness in how both interact during the game. The mother picks up a card that shows 36 as the product (and corresponds to a 9-by-4 array). Alma is to come up with a number sentence that matches the array. She is using a chart that shows her the different products and she says "*six times six*":

***Mother:** Six times … Noo… (mother looks puzzled and keeps looking at her card)*
(Alma looks at the card again and at her chart.)
***Alma:** You're lying.*
***Mother:** You know what? You are right…. But it's also another different answer for thirty-six, but you are right, six times six is thirty-six, you're right, but you know what, we have another answer for thirty-six, check it out.*
(Mother encourages her to look at the chart, shows her the 6 by 6 but tells her there is another answer; Alma comes up with 9 times 4; the mother asks her for the other way, 4 times 9; and it's Alma's turn.)
***Mother:** Oh please, don't get a big one for me.*
(Alma is laughing and ends up picking up a card representing 90)
***Mother:** You are bad girl… Ninety…*
***Alma:** It's easy (laughs); oh my gosh; why are you thinking? It's so easy*
***Mother:** Let's see nine times (pause) ten.*
(Alma puts the card back on the pile.)
***Mother:** I got it! Or ten times nine.*
***Alma:** You didn't have to think because you already knew it.*

In this excerpt we see the mother validating her daughter's answer (six times six) and encouraging her to look for other possibilities. She seems to be learning too, based on the puzzled look in her face and how she comments "*you are right, but it's also a different* *answer…*" When it is the mother's turn to determine the dimensions of the array, the daughter teases her and gives indication that she is aware that her mother is playing along and that she knew the answer right away. The tone is relaxed, teasing, and playful; it is different from the first episode earlier in the session. This mother shared with us on more than one occasion that she enjoyed coming to the workshops because she was also learning along with her daughter and she was learning how to explain things to her daughter.

As I mentioned earlier, the three themes (perceptions, valorization, and language) were often present in these interactions. I want to illustrate the language aspect. In the first dialogue excerpt, Monica (the mother) was mixing English and Spanish, but in line 7, her daughter says, "*English please*." As an isolated incident, we may not have much to comment on, but an important aspect behind CEMELA is its holistic approach in which we try to work with parents, teachers, and children. Thus we had other opportunities to interact with Alma and her mother. In an interview in November of that same year we learned that Monica tried to speak in Spanish to her daughter as often as possible since at school everything was in English. But she acknowledged that she also used English with her daughter:

> ***Monica:*** *The language is pure Spanish, and sometimes in English, it depends what I want to tell her because I want her to understand what I'm saying, I also try to tell her in English how I can, but in Spanish she gets frustrated.*

Alma shared with us that English was easier for her, but that when she went to Mexico to visit she spoke Spanish. Monica's English was quite good but still she did say that in explaining things to her daughter, sometimes it was hard to try to do it in English. Monica's case is one of several in which the parents are trying to ensure that their children learn/maintain their home language, although it may be a difficult task given that English is the dominant language in a big part of the children's world (see Menéndez, Civil, and Mariño (2009) for more on language and interactions).

Our analysis of the many interactions between children and their parents in the workshops reveal a wide range of styles. Some parents take over and end up doing the task themselves. Others guide their children through questions and hints, while some others engage in direct teaching. Finally there are those who learn together with their children. This wide range of interactions underscores the importance of not essentializing groups (in our case parents of Mexican origin) through stereotypical descriptions.

CLOSING THOUGHTS

I would like to close this paper with some thoughts on implications from this work with parents for teacher preparation and professional development. Much remains to be done in strengthening the dialogue between teachers and parents and children from non-dominant communities. In analyzing the teaching and learning of mathematics in schools that were highly effective in high-poverty communities, Kitchen (2007) highlights the building of relationships among teachers and between teachers and students as one of the key themes the research team found in common in the schools they studied. Our work in CEMELA stresses the importance of building relationships with the students and with their parents. Particularly in the case of immigrant families where the parents may have been schooled outside the U.S., their approaches to doing mathematics are likely to be different from what their children are learning. Their children are trying to navigate between two cultures (Suárez-Orozco & Suárez-Orozco, 2001); thus, it seems especially important that teachers and parents engage in authentic two-way dialogues (Civil, 2002) in which they all learn about each other's perceptions and expectations for the students' learning and teaching of mathematics. Regular series of workshops (short courses) such as the ones I have described may provide an avenue for the joint exploration of mathematics, hence learning from each other. The current efforts in teacher preparation to incorporate "mathematical knowledge for teaching" (Hill, Ball, & Schilling, 2008; Hill et al., 2008) need to include an awareness and disposition to work with mathematical approaches (content and pedagogy) that may be different from the ones teachers and preservice teachers know, particularly when these approaches come from non-dominant students. Hill, Ball, and Schilling (2008) define KCS ("Knowledge of Content and Students") as:

> *We propose to define KCS as content knowledge intertwined with knowledge of how students think about, know, or learn this particular content. KCS is used in tasks of teaching that involve attending to both the specific content and something particular about learners, for instance, how students typically learn to add fractions and the mistakes of misconceptions that commonly arise during this process.* (p. 375)

This definition does not include knowledge of students in the line that I advocate for in this paper, that is knowledge of students (and their families) from a socio-cultural perspective (e.g., knowledge of their funds of knowledge). An effort in this direction in teacher preparation is the work of Corey Drake and colleagues at six different universities through a recent NSF-funded project (TEACH MATH). This project aims to redesign mathematics elementary teacher preparation by focusing preservice teachers' learning on children's thinking about mathematics and on community knowledge (Bartell et al., in press, present some pilot work for this project). This work, as well as the work I have presented in this paper, have equity as their driving force.

The participation of teachers (and preservice teachers) in short courses with parents, home visits, classroom visits with parents followed up by debriefing (Civil & Quintos, 2009), are possible approaches to building relationships and establishing dialogue, and to see parents as intellectual resources, thus moving away from a deficit view that often characterizes parent-teacher interactions in schools, particularly in low-income, non-dominant communities. Teacher education efforts need to find ways to engage teachers and preservice teachers in meaningful work with parents and their children that centers on mathematics and allows them to understand the complexities of different perceptions and valorizations of knowledge, as well as the role that multiple languages play in children's learning of mathematics.

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COLLABORATION ET ÉCHANGES INTERNATIONAUX EN ÉDUCATION MATHÉMATIQUE DANS LE CADRE DE LA CIEM : REGARDS SELON UNE PERSPECTIVE CANADIENNE

ICMI AS A SPACE FOR INTERNATIONAL COLLABORATION AND EXCHANGE IN MATHEMATICS EDUCATION: SOME VIEWS FROM A CANADIAN PERSPECTIVE

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From 1999 to 2009, I had the utmost privilege of being the secretary-general of the International Commission on Mathematical Instruction (ICMI). This position has provided me with a unique context to both witness and participate in various actions aiming at fostering the development of mathematical education as considered from an international perspective. I wish to use the opportunity offered by this CMESG talk to reflect on these truly exciting years spent as a member of the ICMI Executive, and to share some insights and experiences gained from my involvement in the international community of mathematics education in the context of the ICMI programme of actions. It may be particularly timely to venture into such kind of reflections as the Commission has celebrated its centennial two years ago, a circumstance that offers a most useful framework for such considerations.

My presentation will start with some glimpses into the history of ICMI since its inception in 1908, a history that I shall aim at describing through some of its highlights. I will then examine with greater details some more recent episodes of ICMI life, especially episodes in which I was myself involved. My reflections will be largely inspired by the angle I take of a Canadian perspective on such matters, as I wish to stress here the rich contribution that the Canadian community has already brought to the mission of ICMI and comment on the roles that Canada might continue to play. I also wish to look at some of the main foci of the actions of ICMI, in particular over the last decade, and see how these can serve to shed light on a possible evolution within the Canadian landscape around matters related to mathematics education. These reflections will touch issues such as the responsibility of "rich" countries as regards the pressing needs in less affluent parts of the world, the perennial difficulty of the mutual understanding and respect between the communities of mathematicians and of mathematics educators, and the structural obstacles encountered in Canada as regards the setting up of a body representing the country in the framework of the ICMI sub-commissions.

INTRODUCTION

Pendant onze années consécutives, au cours de la période allant de 1999 à 2009, j'ai eu l'honneur et l'immense privilège d'occuper la fonction de secrétaire général de la Commission internationale de l'enseignement mathématique (CIEM, alias ICMI —

International Commission on Mathematical Instruction). Ce poste m'a fourni un contexte unique pour être à la fois un témoin et un acteur de nombreuses actions visant à soutenir le développement de l'enseignement et de l'apprentissage des mathématiques, considérés selon une perspective internationale. Je souhaite profiter de la tribune que m'offre cette conférence annuelle du Groupe canadien d'étude en didactique des mathématiques (GCEDM) pour jeter un regard sur ces années passionnantes que j'ai vécues au sein du Comité exécutif de la CIEM, et pour partager certaines perspectives et expériences qui se dégagent de mon implication dans la communauté internationale d'éducation mathématique via le programme d'actions de la CIEM. Il est sans doute particulièrement propice de s'aventurer maintenant sur un tel terrain de réflexions alors que la Commission vient de célébrer son centenaire il y a tout juste deux ans, un événement qui fournit une toile de fond des plus pertinentes pour mes propos.

Mon exposé commencera donc par un survol de l'histoire de la CIEM depuis sa création en 1908, histoire que je chercherai à baliser en faisant ressortir certains de ses temps forts. Je voudrai par la suite examiner plus en détails certains épisodes plus récents de la vie de la CIEM, en particulier des épisodes auxquels j'ai moi-même pris part. Mes remarques seront teintées en grande partie par le biais que je prends d'une perspective canadienne sur mon sujet. Je veux en effet mettre en lumière ici la riche contribution déjà apportée par la communauté canadienne à la mission de la CIEM, et réfléchir sur les rôles que le Canada peut continuer à y jouer dans le futur. Je compte également voir de quelle manière certaines des actions de la CIEM, notamment au cours de la dernière décennie, peuvent éclairer une possible évolution, au plan canadien, des tenants et aboutissants du champ de l'éducation mathématique. Ces réflexions porteront sur des questions telles la responsabilité des pays « riches » à l'égard des besoins pressants des régions moins favorisées dans le monde, l'éternelle difficulté d'une compréhension et d'un respect mutuels entre les communautés des mathématiciens et des didacticiens des mathématiques, et les obstacles de nature structurelle qui surgissent au Canada lorsqu'il est question de mettre en place un organisme susceptible de représenter le pays dans le cadre des sous-commissions de la CIEM.

Je suis d'autant plus reconnaissant de l'invitation qui m'a été faite de préparer cette présentation qu'elle me fournit une occasion en or de conjuguer deux pôles ayant occupé, et occupant toujours, une place centrale dans mon engagement comme mathématicien : le GCEDM d'une part, organisme qui a exercé une influence déterminante sur mon cheminement professionnel personnel et dont je me réjouis d'être l'un des membres fondateurs (je fais en effet partie des « vieux routiers » qui étaient là dès la rencontre tenue en 1977 à Kingston…); et la CIEM d'autre part, très présente dans ma vie professionnelle depuis trois décennies — mais surtout au cours des onze dernières années, alors que j'en étais le secrétaire général (période correspondant quasi au tiers de ma carrière!). Et le plaisir de me retrouver devant vous aujourd'hui est incontestablement décuplé à la vue de tant de visages nouveaux, et souvent jeunes, signe indubitable de la vitalité autant de notre champ professionnel que du GCEDM en tant qu'organisation.

COUPS D'OEIL SUR LE PASSÉ ET LE PRÉSENT DE LA CIEM

La Commission internationale de l'enseignement mathématique existe aujourd'hui, comme entité légale, à titre de commission de l'Union mathématique internationale (UMI / IMU — *International Mathematical Union*). En vertu de son Énoncé de mandat — voir clause 1, CIEM (2009) —, la mission de la CIEM, en tant que commission de l'UMI, se définit comme suit :

> *ICMI shall be charged with the conduct of the activities of IMU bearing on mathematical or scientific education, and shall take the initiative in inaugurating appropriate programmes designed to further the sound development of mathematical education at all levels, and to secure public appreciation of its importance. In the pursuit of this objective, ICMI shall cooperate, to the extent it considers desirable, with groups, international, regional, topical or otherwise, formed within or outside its own structure.*

Il est intéressant d'observer que c'est donc de son statut de commission en charge des questions d'éducation au sein d'un organisme de mathématiciens que la CIEM puise son existence formelle, tout en jouissant, il faut le souligner, d'une latitude considérable dans ce cadre pour ce qui est de ses actions. Il faut de plus noter que par le biais de l'UMI, la CIEM se retrouve au sein de la grande famille de l'ICSU (International Council for Science), ce qui, en vertu des statuts de ce dernier organisme, entraîne l'adhésion au principe fondamental de l'universalité de la science, basée sur la non-discrimination et l'équité.

Cette présence de la CIEM au sein de l'UMI n'est cependant pas sans parfois soulever des interrogations dans certains milieux quant à l'à-propos d'un tel rattachement. C'est là une question brûlante à laquelle le tout premier Comité exécutif de la CIEM auquel j'ai participé, de 1999 à 2002, a rapidement été confronté, notamment à la suite de certaines tensions avec l'UMI au cours des années 1990. Je ne souhaite pas me pencher dans le présent texte sur les événements ayant mené à de telles tensions, et renvoie le lecteur intéressé à Artigue (2008) ou Hodgson (2009). Je veux simplement mettre en relief ici la décision claire et unanime de cet Exécutif de contrer toute velléité de quitter l'UMI, en raison de la perte fondamentale qu'aurait représentée à nos yeux une telle séparation. Nous avons ainsi été amenés à réfléchir avec vigueur, et rigueur, à la nature de la CIEM et à ce que nous souhaitions qu'elle soit, en tant qu'organisme ayant pour mandat la promotion de l'éducation mathématique sur le plan international. Le contact régulier et la collaboration avec les mathématiciens, malgré certaines différences de culture entre le milieu des mathématiques et celui de la didactique des mathématiques, nous paraissaient y jouer un rôle essentiel. Notre action à cet égard s'est alors dirigée vers le renforcement et le développement des liens de la CIEM avec l'UMI. J'y reviens plus loin dans le présent texte.

Je souhaite faire maintenant un bref survol de l'histoire de la CIEM, tout en portant mon attention sur quelques aspects qui me paraissent particulièrement instructifs. Outre le texte Hodgson (2009) mentionné plus haut, on trouvera plus de renseignements sur l'histoire de la Commission dans l'article Howson (1984b) qui offre une synthèse des soixante-quinze premières années de la CIEM, dans certaines sections du livre Lehto (1998) traitant de l'histoire de l'UMI, ou encore dans les actes Menghini, Furinghetti, Giacardi, et Arzarello (2008) résultant du symposium organisé à l'occasion du centenaire de la CIEM.

LA CRÉATION DE LA CIEM ET SES PREMIÈRES ANNÉES

C'est lors du quatrième Congrès international des mathématiciens, tenu à Rome en 1908, que la Commission a vu le jour. La résolution créant la Commission faisait état de l'importance d'effectuer une étude comparative des méthodes et programmes d'enseignement des mathématiques à l'école secondaire et mandatait à cette fin un comité sous la présidence de l'éminent mathématicien allemand Felix Klein (1849-1925), avec mission de présenter un rapport au congrès suivant de 1912.

Les premières années d'existence de la CIEM furent ainsi marquées par une grande effervescence autour de comparaisons curriculaires, quelque 150 rapports ayant été publiés au moment du Congrès international des mathématiciens de 1912 et 50 autres étant en préparation, tel que le rapporte Lehto (1998, p. 14). Le mandat de la Commission fut alors renouvelé et malgré les difficultés liées à la Première Guerre mondiale, certaines activités de réflexion autour des programmes continuèrent au fil des ans. Lehto (1998, p. 14) indique qu'en 1920, 187 volumes contenant 310 rapports touchant 18 pays avaient été publiés.

Mais les tensions internationales résultant de la guerre, et leurs répercussions fort importantes sur les échanges entre scientifiques, eurent pour effet de ralentir considérablement les contacts internationaux. Même si certains travaux se déroulèrent au cours des années suivantes sous les auspices de la CIEM, celle-ci se dirigeait alors vers une phase de quasi stagnation — voir à ce sujet Schubring (2008). Ce ne sera finalement qu'après la Deuxième Guerre mondiale, dans un contexte où les milieux scientifiques voulaient échapper aux tensions majeures ayant fait suite au grand conflit mondial précédent, que la CIEM renaîtra en tant que commission éducative de l'UMI, nouvellement créée en 1952 — voir Lehto (1998, pp. 91 *sqq.*).

LA CIEM, DEUXIÈME MOUTURE

Dans son allocution Bass (2008) lors du symposium du centenaire de la Commission, l'ancien président de la CIEM Hyman Bass utilise l'expression « ère Klein » pour décrire la période de l'histoire de la CIEM allant de 1908 jusqu'à la Deuxième Guerre mondiale, l'influence du premier président de la Commission étant alors nettement prépondérante. On peut y voir comme principaux acteurs des mathématiciens qui, à l'instar de Klein, manifestent un intérêt pour les questions éducatives. Il s'agit d'une période d'une grande activité, principalement au cours des premières années. Mais, tel que mentionné plus haut, les difficultés ayant suivi la Première Guerre sont venues mettre un frein à cet élan. À noter que la plupart des pays impliqués à cette époque sont européens : tel est le cas, lors de la création de la CIEM en 1908, de 17 des 18 pays membres (« pays participants ») avec droit de vote — voir CIEM (1908); le Canada s'y trouvait, mais en tant que l'un des quinze « pays associés » (dont trois pays européens) sans droit de vote.

Lors de son redémarrage en 1952 au sein de la toute nouvelle UMI, la CIEM entreprend au tout début des travaux dans la lignée de ses actions de l'avant-guerre, les comparaisons curriculaires y ayant encore la part belle. Mais un renouveau vient s'instiller au fil des ans dans les actions de la Commission, reflétant ainsi les champs d'intérêts plus « modernes » se développant au sein de la communauté de ceux qu'on appellera plus tard les *didacticiens des mathématiques*. Éventuellement la CIEM deviendra elle-même un moteur de ces changements, notamment sous l'influence de Hans Freudenthal (1905-1990), président de la CIEM de 1967 à 1970. L'état des lieux évoluera alors de manière essentielle, si bien que Bass décrit cette période, ainsi que les années plutôt fastes qui suivirent, comme l' « ère Freudenthal ». Les activités de la CIEM connaissent alors un essor considérable.

C'est en effet durant la présidence de Hans Freudenthal, et à son initiative même, que se sont produits deux événements qui ont joué un rôle charnière dans le développement du champ de la didactique des mathématiques : le lancement de la revue *Educational Studies in Mathematics* en 1968 et la tenue en 1969 du premier Congrès international sur l'enseignement des mathématiques (*International Congress on Mathematical Education* — ICME).[1] La motivation sous-tendant de telles actions de Freudenthal peut sans doute se comprendre à la lecture d'une résolution adoptée lors du congrès ICME-1 (ESM, 1969) :

> *La pédagogie de la mathématique devient de plus en plus une science autonome avec ses problèmes propres de contenu mathématique et d'expérimentation. Cette science nouvelle doit trouver place dans les Départements de Mathématiques des*

[1] Le 7e Congrès international sur l'enseignement des mathématiques (ICME-7), il convient sans doute de le rappeler ici, a eu lieu à Québec en 1992 — voir Gaulin et al. (1994) et Robitaille et al. (1994).

> *Universités ou des Instituts de Recherche; ceux qui se qualifient dans cette discipline doivent pouvoir accéder à tous les grades universitaires.* (p. 285)

Ces actions de Freudenthal reflètent, et en même temps nourrissent, l'émergence d'un nouveau champ de recherche, avec ses acteurs souvent nouveaux eux aussi. Il n'est plus question ici de mathématiciens ayant un intérêt occasionnel pour les questions éducatives, mais bien de « chercheurs professionnels » s'intéressant aux questions d'enseignement et d'apprentissage des mathématiques : les didacticiens. Les considérations d'ordre psychologique ou concernant les théories de l'apprentissage occupent une place croissante, voire prépondérante. Conséquemment, les travaux s'éloignent des comparaisons de curricula nationaux, l'accent étant plutôt mis sur un contexte davantage centré sur la « personne », la démarche de celui ou celle qui apprend étant au cœur même du processus — voir Furinghetti (2008, pp. 49-50). Pour reprendre l'observation de Sfard (2008), cette évolution dans les thèmes de recherche signalait un passage depuis l'« ère du curriculum » vers l'« ère de l'apprenant » — pour éventuellement devenir, au cours des années récentes, une recherche inscrite dans l'« ère de l'enseignant ».

Autre élément qui vient distinguer la CIEM sous Freudenthal de celle de l'époque de Klein, une attention particulière est alors de plus en plus portée sur l'objectif de répandre l'action de la Commission en Asie, en Amérique latine et en Afrique, notamment en ce qui concerne la situation particulière des pays en développement. Notons au passage que cette attention envers les besoins des pays en développement a pris une place encore nettement plus marquée au cours des dernières années — j'y reviens un peu plus bas.

La période de la présidence de Freudenthal et les années qui ont suivi peuvent à certains égards être vues comme des années d'abondance. Non seulement y voit-on l'émergence d'une nouvelle discipline, la *didactique des mathématiques*[2] en tant que champ de recherche, tel que nous le connaissons aujourd'hui, mais de plus les activités de la CIEM s'amplifient, de même que les axes de recherche alors mis en évidence. Cette évolution n'est pas sans lien avec la montée de phénomènes pédagogiques forts, tel le mouvement des « maths modernes » (*new math*), de même qu'avec l'arrivée sur la scène didactique d'un nouveau joueur, la *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM) — voir Furinghetti, Menghini, Arzarello, et Giacardi (2008, p. 134). L'accent mis dans les travaux de la CIEAEM sur des aspects jusque-là laissés pour l'essentiel de côté, tels l'étudiant, le processus d'enseignement en tant que tel ou encore les interactions dans la salle de classe, a certainement contribué à la « renaissance » de la CIEM vers la fin des années 1960 décrite par Furinghetti et al. (2008).

C'est aussi dans la mouvance suivant la présidence de Freudenthal qu'est née l'une des composantes importantes de la structure de la CIEM, les *groupes d'étude affiliés*. Les premiers à voir le jour sont HPM (*International Study Group on the Relations between the History and Pedagogy of Mathematics*) et PME (*International Group for the Psychology of Mathematics Education*), tous deux lancés officiellement en 1976 sur la base de groupes de travail sur ces mêmes thèmes organisés lors du congrès ICME-2 (1972). La création de tels groupes d'intérêt est venue enrichir le cadre des travaux chapeautés par la Commission tout en manifestant la variété qui s'était alors développée au sein de la communauté de la CIEM dans les manières d'appréhender divers aspects de l'enseignement et de l'apprentissage des mathématiques.

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[2] En raison de spécificités linguistiques, cette nouvelle discipline est désignée en anglais par une expression au sens plutôt vaste, le vocable *mathematics education*, et ce malgré la suggestion faite très tôt par Hans-Georg Steiner d'utiliser, à l'instar de nombreuses langues, une expression telle *didactics of mathematics* — voir à ce sujet Furinghetti et al. (2008, p. 132) et Hodgson (2009, p. 94).

LA CIEM AUJOURD'HUI

Il peut être intéressant, pour bien saisir ce qu'est la Commission internationale de l'enseignement mathématique aujourd'hui, de reprendre les propos tenus en 1982 par Bent Christiansen, vice-président de la Commission 1975 à 1986, lors d'une rencontre entre les Exécutifs de l'UMI et de la CIEM — voir Lehto (1998) :

> *ICMI should not be seen as powerful leaders of the development in mathematics education. In fact, the Commission and its EC [Executive Committee] should not decide what are proper or relevant solutions to problems in our field. But there was urgent need for a structure under which interaction and exchange of views can be facilitated.* (p. 260)

La CIEM n'est donc pas là pour déterminer quelles seraient des solutions pertinentes ou préférables aux grands problèmes contemporains de l'éducation mathématique (des solutions qui auraient ainsi porté *de facto* une sorte de « sceau de validation CIEM », pour reprendre l'image un brin malicieuse utilisée par l'ancien président Jean-Pierre Kahane), mais plutôt pour proposer un cadre favorisant tant les interactions et échanges d'opinions que les analyses critiques. Lehto poursuit ses commentaires sur les propos de Christiansen en soulignant qu'il relève de la CIEM d'offrir un type de leadership et une structure en lien avec les besoins d'une communauté en croissance intéressée par les questions relatives à l'enseignement et l'apprentissage des mathématiques, le tout se déroulant, insiste Christiansen, sous les auspices de l'UMI.

Somme toute, la Commission peut être vue aujourd'hui comme un organisme visant à favoriser la réflexion, selon une perspective internationale, sur l'éducation mathématique prise comme un domaine tant de recherche que de pratique et comme offrant, à cette fin, un forum à tous ceux qui se sentent interpellés par une telle mission. Tel qu'évoqué plus haut, des changements substantiels se sont produits au cours des dernières décennies dans le champ d'intérêt sous-jacent aux travaux de la CIEM, notamment en lien avec la reconnaissance de la didactique des mathématiques en tant que discipline : il s'agit là d'une évolution de fond dont la Commission a été le témoin et qu'elle a accompagnée, voire encouragée ou même provoquée à l'occasion.

En vue de remplir ce rôle, la CIEM peut s'appuyer sur un cadre structurel riche et fécond. S'y retrouve d'une part l'UMI, organisme duquel relève formellement la Commission et avec lequel elle a su développer, notamment ces dernières années, des liens étroits et productifs. Par ailleurs, la CIEM peut aussi compter sur un réseau d'*organisations affiliées* regroupant tant des groupes d'étude (tels HPM et PME mentionnés plus haut) que des organismes multinationaux dont la mission touche l'enseignement des mathématiques. Les autres organisations présentement affiliées à la CIEM sont : IOWME, WFNMC et ICTMA en tant que groupes d'étude, et CIAEM, CIEAEM et ERME à titre d'organismes multinationaux (on trouvera la signification de ces sigles sur le site de la CIEM). Il s'agit là d'un réseau en évolution, ce qui reflète bien la richesse et la diversité du champ de la didactique des mathématiques aujourd'hui.

Je termine ce survol de la nature et de la mission de la CIEM en mentionnant, parmi ses activités courantes, trois catégories qui en constituent somme toute le cœur et sur lesquelles je voudrai revenir dans la deuxième partie de ce texte :

- les congrès internationaux sur l'enseignement des mathématiques (CIEM) qui se déroulent à tous les quatre ans (le prochain, ICME-12, se tiendra à Séoul en juillet 2012);

- les Études de la CIEM, lancées au milieu des années 80 — chaque Étude portant sur un thème précis (par exemple, la dernière Étude en cours, la 21e, concerne l'enseignement des mathématiques dans des contextes de diversité linguistique), s'appuyant sur une conférence où la participation est sur invitation seulement, et menant à la publication d'un livre présentant l'état des connaissances sur ce thème;
- les conférences régionales de la CIEM, visant à favoriser le développement de réseaux dans des régions données du globe (principalement en Afrique, en Amérique latine, en Asie du Sud-Est ou encore dans la Francophonie).

UNE PERSPECTIVE CANADIENNE SUR LA CIEM

Dans cette deuxième partie de mon exposé, je souhaite me pencher sur les liens qui relient la CIEM à la communauté canadienne. Je parlerai aussi bien de liens du passé que de divers mécanismes que mathématiciens et didacticiens canadiens ont présentement à leur disposition en vue de collaborer aux programmes de la Commission. Le Canada, je l'ai déjà mentionné plus haut, était présent dès les tout premiers débuts de la CIEM. Mais c'est en fait surtout depuis la reconstitution de la CIEM au sein de l'UMI, dans la deuxième moitié du XXe siècle, que la présence canadienne s'est véritablement fait sentir. J'en donnerai quelques exemples plus bas. Je voudrai aussi examiner comment certains phénomènes se déroulant au sein de la CIEM peuvent parfois s'avérer utiles en apportant un éclairage intéressant sur la scène canadienne. Je souhaite enfin évoquer au passage quelques pistes de réflexion en vue d'accroître la présence canadienne au plan international par le truchement de la CIEM.

Je souligne d'entrée de jeu que les liens entre la communauté canadienne et la CIEM sont déjà fort riches, et ce à plusieurs égards. J'en retiens un seul cas d'espèce pour l'heure. Plusieurs personnes ayant joué ou jouant des rôles majeurs à la CIEM ont eu l'occasion d'interagir avec les mathématiciens et didacticiens canadiens, notamment dans le cadre des activités du Groupe canadien d'étude en didactique des mathématiques. Ainsi les présidents de la CIEM que j'ai eu l'honneur et le plaisir de côtoyer au cours de mon mandat de secrétaire général ont tous deux été conférenciers invités lors des rencontres annuelles du GCEDM : Hyman Bass (président de 1999 à 2006) a donné une conférence plénière lors la rencontre de 2002 et Michèle Artigue (2007-2009), à celle de 1995. Il en est de même du président actuel, Bill Barton (2010-2012), conférencier plénier en 1999. Notons par ailleurs qu'Henry Pollak, conférence plénier à notre rencontre de 1985, a rempli deux mandats sur le Comité exécutif de la CIEM (1971-1974 et 1983-1986). Enfin Jean-Pierre Kahane (président de 1983 à 1990) est venu régulièrement au Québec, principalement dans le cadre d'activités mathématiques mais aussi à titre de conférencier principal lors des États généraux de l'enseignement des mathématiques tenus en 1990 ou encore dans le cadre du Forum canadien sur l'enseignement des mathématiques organisé par la Société mathématique du Canada (SMC) à Montréal en 2003.

PRÉSENCE CANADIENNE À LA CIEM

<u>Présence canadienne dans le Comité exécutif de la CIEM</u>

Il y a plusieurs contextes que l'on peut envisager quand il est question de la présence canadienne au sein de la CIEM. L'un d'eux est par exemple à titre de membre de son Comité exécutif. Il est intéressant de constater que trois collègues m'ont précédé sur cet Exécutif. Il y a eu tout d'abord Ralph L. Jeffery (1889-1975), mathématicien de l'Université Queen's qui a fait partie, de 1952 à 1954, du tout premier Comité mis en place quand la CIEM est devenue une commission de l'UMI, lors de la création de cette dernière en 1952. Par la suite, Stanley H. Erlwanger (1934-2003) de l'Université Concordia, que je me rappelle avoir côtoyé lors des premières rencontres annuelles du GCEDM, a été membre de l'Exécutif de 1979 à 1982. Enfin Anna Sierpinska, elle aussi de Concordia, a été sur le Comité exécutif de 1991 à 1998, dont les quatre dernières années à titre de vice-présidente. Si on combine le mandat d'Anna avec le mien, le Canada aura donc vu pendant 19 années consécutives un membre de sa communauté siéger sur le Comité exécutif de la CIEM, de 1991 à 2009.

Il est sans doute utile, à cet égard, de rappeler ici un changement structurel majeur survenu naguère à la CIEM — un changement qui, il convient d'insister, était tout à fait impensable il y a moins de dix ans. Depuis la création de l'Union mathématique internationale, l'élection de l'Exécutif de la CIEM était la responsabilité de l'Assemblée générale de l'UMI. Or cette Assemblée générale, lors de la réunion tenue en 2002 à Shanghai, avait mandaté le Comité exécutif de l'UMI afin de revoir la procédure d'élection de ses divers comités (parmi lesquels l'Exécutif de la CIEM) de manière à la rendre plus transparente et moins sujette aux conflits d'intérêts, notamment pour ce qui est du rôle joué par l'Exécutif sortant de l'UMI dans la préparation des listes de candidats. J'ai été largement impliqué dans les discussions — dans les faits, je crois qu'on pourrai parler de négociations — visant à définir des règles d'élection appropriées aux besoins et à la spécificité de la CIEM. Le lecteur intéressé à mieux comprendre cet épisode de la vie récente de la CIEM pourra consulter Hodgson (2009). Je me bornerai à dire ici que lors d'une rencontre avec le président de l'UMI, Sir John Ball, dans le cadre du congrès ICME-10 tenu en 2004 à Copenhague, non seulement le fait d'avoir une procédure spécifique à la CIEM (comité de nomination distinct, etc.) a aisément été accepté, mais aussi, poussant le raisonnement jusqu'à son terme, le transfert à l'Assemblée générale de la CIEM de la responsabilité de l'élection elle-même a finalement été vu comme une conclusion naturelle. Dénouement qui, tout juste quelques années auparavant, aurait été considéré comme totalement inespéré et inattendu! Bien sûr, cette entente avec l'Exécutif de l'UMI ne suffisait pas, car il appartenait à l'Assemblée générale de l'UMI de se prononcer *in fine* sur la nouvelle procédure. C'est lors de l'assemblée tenue en 2006 à Santiago de Compostela, à l'occasion du Congrès international des mathématiciens de Madrid, que ce vote historique a eu lieu. Il est clair que la stature en tant que mathématicien du président de la CIEM alors en poste, Hyman Bass, a joué un rôle essentiel dans le fait de convaincre l'assemblée des mathématiciens du bien-fondé du nouveau mode d'élection proposé.

Les conséquences de ce changement structurel sur la vie de la CIEM sont énormes. La donne se trouve ainsi sensiblement modifiée quand vient le temps de penser à des candidats pour l'Exécutif, notamment à la présidence de la Commission, la traditionnelle exigence plus ou moins tacite d'inviter un mathématicien de très haut calibre à occuper ce poste n'étant plus de rigueur — voir à ce sujet Lehto (1998, p. 65). (Il convient cependant de noter que dès l'élection de Michèle Artigue à la présidence de la CIEM lors de l'Assemblée générale de l'UMI de 2006, ce schème avait été mis en veilleuse, tant par le fait que les mérites professionnels de la candidate pressentie ne reposaient pas sur l'excellence d'une carrière à titre de mathématicienne « pure et dure », que pour ce qui était d'avoir finalement une femme à la présidence de la Commission.) J'invite donc la communauté canadienne à réfléchir, autant dans une perspective à court qu'à moyen terme, à la façon dont le Canada pourrait éventuellement envisager la proposition de candidatures aux divers postes de l'Exécutif à la lumière de ces changements récents et tout à fait fondamentaux.

Présence canadienne dans certaines activités de la CIEM : les congrès ICME

Il ne m'est pas possible ici de faire une analyse complète de tous les rôles joués par des Canadiens dans le cadre des activités de la CIEM. Je souhaite néanmoins en faire ressortir certaines grandes lignes. Ainsi, le tableau suivant donne le nombre de Canadiens ayant participé aux Congrès internationaux sur l'enseignement des mathématiques (ICMEs), sans doute l'activité la plus remarquable de la CIEM.[3] J'ai choisi arbitrairement de mettre la participation canadienne en comparaison directe avec celles de l'Australie et de la France.

ICME	Nombre total de participants	Canada	Australie	France
1	665	23	3	202
2	1384	52	17	125
3	1854	53	38	194
5	1786	40	814	56
6	2414	47	113	112
7	3407	543	182	142
8	3467	56	135	71
9	2012	17	64	24
10	2324	64	98	52

Il est clair d'une part, à l'examen de ces données, que la tenue du congrès dans un pays donné (ICME-1 en France, ICME-5 en Australie et ICME-7 au Canada) a une influence considérable sur la participation du pays hôte. Mais ce qui me semble par ailleurs intéressant de faire ressortir ici est le taux relativement constant de la participation canadienne au fil des congrès — de l'ordre de 2 à 3 % —, exception faite cependant d'ICME-9 au Japon en 2000 (la perception en bonne partie erronée d'un pays où tout coûte extrêmement cher a sans doute joué ici). De plus, si on se compare à la France, la présence canadienne aux ICMEs ne paraît pas trop mauvaise. Du côté de la France, le taux de participation est peut-être relié en partie à un facteur linguistique, l'anglais étant bien sûr la langue de travail des congrès ICME (et de la CIEM de façon générale). Peut-être aussi les sources de financement sont-elles en cause. Mais la comparaison avec l'Australie s'avère nettement moins avantageuse pour le Canada. Outre ICME-9 (certains y verraient peut-être en cause, malgré tout, une certaine « proximité » géographique), les cas d'ICME-6 (Budapest, 1988) et d'ICME-8 (Séville, 1996) sont particulièrement notables. Je ne connais pas de façon fine les infrastructures internes en Australie, mais il est frappant de voir que malgré certaines ressemblances générales entre le Canada et l'Australie en tant que nations, et aussi le fait que la population de cette dernière n'est que les deux tiers environ de la nôtre, les Aussies ont systématiquement une présence plus soutenue, et ce malgré un certain isolement géographique et les coûts de transport afférents.

Diverses raisons pourraient être avancées pour expliquer ce qui en est du Canada, mais à tout événement je soutiendrais pour ma part que l'augmentation de la participation canadienne aux ICMEs doit être vue comme un objectif hautement souhaitable. Il y a indéniablement à cet égard des obstacles liés à la recherche de soutien financier. Mais il me semble aussi y avoir des grands besoins du côté de ce que j'appellerais un soutien « moral », notamment en ce qui concerne la jeune génération. Il y a peut-être lieu de sensibiliser plusieurs membres de la communauté canadienne, entre autres parmi les jeunes, au fait que les congrès ICME sont

[3] Je tire ces données soit d'analyses comme Becker (1977) pour ICME-1, 2 et 3, soit de rapports tels Howson (1984a) et (1988) pour ICME-5 et 6, soit des Actes des congrès pour ICME-7, 8 et 9, ou encore du site Internet des congrès dans le cas d'ICME-10. Je ne possède malheureusement pas de données sur la participation pour les congrès ICME-4 et ICME-11. Autant que je sache, les données pour ICME-7 et ICME-8 incluent les personnes accompagnantes.

« pour eux » et peuvent les aider dans leur cheminement professionnel, voire personnel. Le GCEDM a certes joué, et peut encore jouer, un rôle important dans cette optique — tout comme le représentant canadien auprès de la CIEM. Mais c'est peut-être là un élément où la mise en place d'une sous-commission canadienne pour la CIEM pourrait avoir un impact indéniable (dans certains pays, la sous-commission joue en effet un rôle clé dans l'obtention de subventions des gouvernements en vue de couvrir des frais de déplacement). Je reviens un peu plus bas sur la création d'une telle sous-commission.

Toujours s'agissant des congrès ICME, il est intéressant d'observer que plusieurs Canadiens y ont occupé des rôles majeurs. Ainsi trois des nôtres ont été invités comme conférenciers pléniers : Maria Klawe à ICME-7, Anna Sierpinska à ICME-8 et Gila Hanna à ICME-10, cette dernière dans le cadre d'une entrevue plénière. De plus trois membres des « Survey Teams » lors d'ICME-11 venaient du Canada : Fernando Hitt, Eric Muller et Luis Radford. Je note aussi que dix-huit des conférences régulières au cours des cinq derniers ICMEs ont été données par des collègues canadiens :

- ICME-7 : John Clark, Michael Closs, Bernard Hodgson, Tom Kieren, Ronald Lancaster, Fernand Lemay;
- ICME-8 : Claude Gaulin, Carolyn Kieran;
- ICME-9 : Gila Hanna, Nathalie Sinclair et Peter Taylor conjointement, Walter Whiteley;
- ICME-10 : Nadine Bednarz, Jonathan Borwein, Eric Muller, David Pimm;
- ICME-11 : Carolyn Kieran, Louise Poirier, Christiane Rousseau.

Et je passe sous silence le fait que des nombreux responsables de « Topic Study Groups » et autres groupes similaires aux congrès ICME venaient du Canada.

Si ces résultats ne sont pas trop décevants, la question qui se pose est de savoir comment maintenir, voire éventuellement améliorer, cette situation. Il n'est peut-être pas indifférent à cet égard qu'à l'occasion un collègue canadien puisse se retrouver sur le Comité international du programme d'un ICME donné. Et on peut observer que cela s'est produit régulièrement dans le passé. Ainsi, outre le congrès ICME-7 pour lequel il était de tradition qu'une forte délégation canadienne fasse partie du Comité de programme — David Wheeler (1925-2000) à titre de président, et Eric Muller, Roberta Mura, David Robitaille et Anna Sierpinska comme membres —, on voit les collègues suivants à tour de rôle membres de ce comité : Claude Gaulin (ICME-3 et 4), Stanley Erlwanger (ICME-5), David Wheeler (ICME-5 et 6), Gila Hanna (ICME-9), Carolyn Kieran (ICME-11), Bernard Hodgson (ICME-8, puis ICME-9, 10, 11 et 12 *ex officio*). Bien sûr une telle participation aux travaux du Comité de programme se doit d'être totalement indépendante de toute attache nationale. Mais elle peut servir à s'assurer d'un certain respect, autant que faire se peut, pour des équilibres régionaux — et cela ne concerne évidemment pas que le Canada. Bref, il me semblerait important que la communauté canadienne cherche assez régulièrement à suggérer de fortes candidatures pour les Comités de programme des prochains congrès ICME. La nomination de ce Comité par l'Exécutif de la CIEM se fait habituellement au cours de l'année où se tient le congrès ICME précédent. (Ainsi, la composition du Comité de programme d'ICME-13 devrait être finalisée avant la fin de 2012.) Mais le plus important demeure le fait de répondre assidûment, via le représentant canadien à la CIEM, aux appels de suggestions de candidatures pour divers rôles sur le programme scientifique des congrès ICME. Il est important que de telles propositions soient bien étayées et visent divers équilibres, notamment en ce qui concerne la présence de chercheurs bien établis et la place que doit occuper la génération montante. Le GCEDM peut sans doute jouer un rôle important dans l'identification de candidats valables à cette fin.

Présence canadienne dans certaines activités de la CIEM : Études et autres activités

Il pourrait être pertinent d'examiner en détails la contribution canadienne à divers autres secteurs d'activités de la CIEM. Je me bornerai d'abord ici à quelques observations en vrac. Ainsi, on peut se réjouir de la contribution régulière de notre communauté francophone, et principalement québécoise, aux rencontres *Espace mathématique francophone* (EMF), un réseau régional de la CIEM axé sur un cadre linguistique — le français — plutôt que géographique. Depuis la première rencontre EMF organisée à Grenoble en 2000 par les cousins français à l'occasion de l'Année mathématique mondiale, ces rencontres se sont succédées à tous les trois ans (Tozeur 2003, Sherbrooke 2006, Dakar 2009, et bientôt Genève 2012), toujours avec une forte présence québécoise et canadienne. Je retiens aussi deux activités *ad hoc*. Tout d'abord, la rencontre (sur invitation) organisée en 2000 à Genève pour célébrer le centenaire de la revue *L'Enseignement Mathématique*, l'organe officiel de la CIEM depuis le début de la Commission — voir les actes Coray, Furinghetti, Gispert, Hodgson, et Schubring (2003). Parmi les 52 participants à cette rencontre, quatre étaient canadiens, dont deux des présentateurs principaux : Claude Gaulin et Gila Hanna. Dans le même esprit, on peut s'intéresser à la rencontre tout à fait extraordinaire tenue à Rome en 2008 au splendide Palazzo Corsini à l'occasion du centenaire de la CIEM — ce palais étant à la fois la résidence de l'Accademia Nazionale dei Lincei[4] et le lieu même de la naissance de la Commission un siècle plus tôt. Il est frappant que treize des 161 participants à ce symposium (encore une fois sur invitation) provenaient du Canada, trois d'entre eux jouant un rôle majeur sur le programme — voir les actes Menghini et al. (2008).

Mais c'est davantage sur les *Études* de la CIEM que je souhaite me pencher maintenant. Lancé au milieu des années 1980, le programme des Études (*ICMI Studies*) représente aujourd'hui un créneau fort important au sein des activités de la CIEM. Il ne saurait être question pour moi de rendre compte ici de façon détaillée de l'ensemble des Études et de leur ampleur. Le lecteur intéressé par une description générale de la « philosophie » des Études et de leur mode de fonctionnement, ainsi que par un survol des cinq premières Études, pourra consulter Hodgson (1991). Je veux seulement insister, selon la perspective du présent texte, sur le fait que nombre de Canadiens ont été impliqués à divers titres dans les vingt Études qui se sont déroulées entre 1985 et 2010, notamment par des textes acceptés pour présentation lors des conférences (près d'une centaine) ou encore comme membres des Comités de programme. Je souligne aussi que trois collègues ont agi à titre de responsables ou coresponsables de ces Études : Gila Hanna (Études 7 et 19), Anna Sierpinska (Étude 8) et Ed Barbeau (Étude 16).

Soit dit en passant, on peut peut-être s'étonner du fait que, malgré une implication d'une telle importance dans les Études de la CIEM, aucune conférence de ces Études ne se soit encore tenue au Canada. Ce serait pourtant là un service louable à offrir à la communauté internationale et qui demeure de taille raisonnable, tout en ayant au demeurant des retombées intéressantes au plan local, par exemple quant à la stimulation potentielle auprès d'étudiants de 2e et 3e cycles.

Toujours en lien avec les Études, je souhaite surtout livrer un témoignage personnel. La scène se déroule en 1984, lors de la 8e rencontre annuelle du GCEDM qui se tient à l'Université de Waterloo (j'étais alors relativement jeune en carrière…). À mon arrivée à l'Université, je suis apostrophé par David Wheeler — alors représentant canadien auprès de la CIEM — qui me lance avec son magnifique accent « British » : « *Bernard, have you heard of the forthcoming*

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[4] L'Accademia Nazionale dei Lincei, fondée en 1603, est la plus ancienne société scientifique au monde. Elle compte parmi ses tout premiers membres Galileo Galilei (alias Galilée en français).

ICMI Study on informatics and the teaching of mathematics? You ***must*** *submit a paper!* » Comment résister à une telle « invitation »?

Il en est résulté le texte Hodgson et al. (1985), écrit en collaboration avec trois collègues ontariens que j'ai eu le plaisir et le privilège de fréquenter fréquemment au fil des ans, Eric Muller, John Poland et Peter Taylor, et qui a été accepté pour présentation lors de la première Étude de la CIEM, tenue à Strasbourg en 1985. Un premier jet pour ce texte avait été ébauché sur les lieux mêmes de la rencontre de Waterloo, sans doute au retour d'une des légendaires « tournées de pizzas »…

Cet épisode reste encore fortement gravé dans ma mémoire. C'est moi qui avais eu la tâche de présenter notre texte à Strasbourg. Je me rappelle qu'il y était question entre autres de fractals, un sujet dont la popularité était alors en forte croissance. Lorsque j'ai réalisé, lors de ma présentation orale, que Benoit Mandelbrot en personne était assis sur la toute première rangée, j'ai senti ma gorge s'assécher… : comment « oser » parler de fractals devant leur « père »? Mais ce que je retiens surtout de cet épisode, par-delà l'excellent souvenir de ce bon David, est l'importance d'encourager, de stimuler — ou même davantage… — les jeunes à participer à des activités telles celles proposées par la CIEM. David a bien eu raison de me « pousser dans le dos » pour que je soumette un texte à l'Étude de la CIEM : sans son intervention, je n'aurais sans doute pas cru que ce cadre pouvait me convenir, que je pouvais vraiment y « appartenir ».

De façon générale, il relève au premier chef sans doute du directeur de thèse de sensibiliser ses étudiants à l'à-propos et à l'importance d'aborder les questions d'éducation mathématique selon une perspective internationale, et aussi de les convaincre — même lorsqu'ils sont devenus jeunes profs — de soumettre leurs travaux dans des contextes internationaux. Dans le cas de la CIEM, le représentant canadien peut également jouer un rôle important à cet égard, notamment auprès des collègues en début de carrière, en les incitant à l'aide d'une pression certes modérée, mais quand même soutenue (à la David…). Plus généralement, j'encouragerais le GCEDM à demeurer à l'affût des activités en marche de la CIEM et à s'assurer non seulement que les collègues canadiens en soient bien informés, mais même qu'ils se sentent fermement encouragés à soumettre des contributions. Les rencontres annuelles du GCEDM peuvent sans doute servir de creusets afin de faciliter le développement de collaborations à cet égard. Peut-être même un jour un organisme comme le GCEDM, ou encore une sous-commission canadienne pour la CIEM — si elle voit le jour — pourrait-il recueillir et administrer des fonds visant à faciliter la participation aux Études de la CIEM, tout comme pour les ICMEs.

<u>Présence canadienne… en puissance</u>

Mes prochains commentaires touchent un élément de la vie de la CIEM où la présence canadienne s'avère encore à venir. En 2001, le Comité exécutif de la CIEM a annoncé son intention de créer deux prix visant à reconnaître des contributions majeures à la recherche en didactique des mathématiques — voir Bass et Hodgson (2001). Ces prix portent les noms de deux présidents de premier plan de la CIEM. La *médaille Felix Klein*, nommée d'après son tout premier président (1908-1920), récompense l'œuvre d'une vie, tandis que la *médaille Hans Freudenthal*, du nom du huitième président de la CIEM (1967-1970), récompense un ensemble de travaux d'intérêt majeur sur un thème précis. Ces prix, qui reviennent à tous les deux ans, ont été décernés pour la première fois en 2003. En plus de récompenser les mérites de collègues et l'excellence de leurs travaux, ces prix visent aussi à mieux faire connaître le champ de la recherche en didactique des mathématiques.

À quand une médaille de la CIEM qui viendrait souligner les recherches d'un membre de notre communauté canadienne? Bien sûr cela risque fort d'advenir un de ces jours — on n'a qu'à observer la qualité des travaux réalisés chez nous. Pour cela, il faut cependant monter des dossiers de candidature. Ceux-ci peuvent émaner de l'initiative d'individus, mais il me semble que c'est là un contexte où la communauté dans son ensemble peut jouer un rôle. Dans un tel cas, il appartiendrait sans doute au GCEDM d'être l'instigateur d'une telle démarche. Sans en faire une action « politique » pour autant, j'aimerais donc inviter l'Exécutif du GCEDM à réfléchir sur des façons d'encourager et de coordonner la préparation de dossiers aux prix de la CIEM. Il ne s'agit évidemment pas de systématiquement inonder le Comité de sélection de propositions de candidats. Il s'agit plutôt selon moi de rester éveillé à ce qui se passe autour de nous et de ne pas se gêner pour mettre en évidence l'excellence exceptionnelle de membres de notre communauté.

COLLABORATIONS INTERNATIONALES À L'INTENTION DES PAYS EN DÉVELOPPEMENT

Dans les années qui ont suivi sa deuxième naissance en tant que commission de l'UMI, la CIEM a manifesté un souci plus marqué pour les besoins des pays en développement. Ainsi, certaines activités ont alors été organisées en Amérique latine, en Afrique et en Asie, souvent avec le soutien de l'UNESCO. C'est d'ailleurs dans un tel contexte qu'est née au début des années 1960 la CIAEM, à l'initiative du président de la CIEM à cette époque, le mathématicien américain Marshall H. Stone (1903-1989).

Les actions de la CIEM envers les pays en développement ont pris un tour nouveau au cours des années 1990 à la suite de la décision de son président Miguel de Guzmán (1936-2004) de créer un *programme de solidarité*. Une des volets importants de ce programme est la levée d'une « taxe de solidarité » de 10 % sur les frais d'inscription aux congrès ICME afin de faciliter la présence de participants de pays en développement. Lancée lors du congrès ICME-8 tenu à Séville en 1996, cette taxe a depuis permis d'offrir une aide financière partielle à plusieurs centaines de didacticiens, mathématiciens et enseignants des pays moins fortunés, améliorant ainsi de façon non négligeable la représentation de ces pays aux ICMEs. D'autres volets du programme de solidarité concernent des missions effectuées dans des pays en développement et soutenues par la CIEM, le soutien à des réseaux de conférences régionales, notamment en Afrique, ou encore la participation des pays en développement lors des Études de la CIEM. On peut aussi mentionner, au chapitre des actions envers les pays en développement, la collaboration de la CIEM à l'exposition *Pourquoi les mathématiques?*, organisée sous le parrainage de l'UNESCO. On estime — voir le rapport Hodgson (2008) — que depuis sa création en 2004, cette exposition a été vue par plus de 800 000 personnes, jeunes, parents, enseignants. Elle a visitée plus de 50 villes dans une vingtaine de pays, en majorité des pays en développement (notamment le Cambodge, le Laos, la Thaïlande, le Vietnam, l'Inde, le Pakistan, le Bénin, le Mozambique, la Namibie, le Sénégal, l'Argentine, le Brésil, le Chili ou le Paraguay).

L'objectif ultime de la CIEM en soutenant de telles actions est de favoriser l'enseignement et l'apprentissage des mathématiques dans les pays en développement en faisant en sorte que les communautés dans ces pays soient en lien direct et soutenu avec la communauté internationale représentée par la Commission. Mais c'est là, on s'en doute, un objectif colossal qui se heurte à de nombreux obstacles. Au premier chef, bien sûr, des obstacles financiers! Trouver les sommes nécessaires à de telles actions demeure un défi majeur et constant. Mais il y a plus. L'intégration fructueuse des collègues des pays en développement aux réseaux de la CIEM entraîne une nécessaire évolution depuis l'approche traditionnelle d'une action dirigée *envers* les pays en développement jusqu'à une vision davantage symétrique de collaboration et de solidarité. Il faut donc non seulement changer les modèles de type « Nord-Sud » pour des modèles « Sud-Sud », mais même rechercher des occasions d'échanges « Sud-Nord » en misant sur des expertises particulières qui se retrouvent maintenant dans des pays en développement ou en émergence. Le lecteur intéressé trouvera dans Artigue (2008) des commentaires plus poussés concernant les actions de la CIEM dans les pays en développement et la nécessité d'un tel changement de perspective.

Mais *quid* du Canada à ce chapitre? Notre pays a une très longue tradition de soutien aux pays en développement, tradition qui, il faut l'admettre, tend peut-être à s'effriter — par manque de volonté politique, entre autres. Cela dit, les membres de notre communauté doivent se convaincre, et convaincre également leur entourage, de la responsabilité que nous avons envers les collègues des pays en développement. Soutenir l'éducation mathématique dans ces régions demeure à coup sûr un moyen privilégié en vue de les aider à progresser et à atteindre leur plein potentiel. Il s'agit là, à n'en pas douter, d'un devoir et d'une obligation morale qui nous incombent sur le plan de l'équité et de la solidarité. Bien sûr nous offrons dans nos universités divers programmes visant à faciliter la présence d'étudiants étrangers. Mais il y a lieu de se questionner également sur la façon dont la communauté canadienne pourrait contribuer davantage à des programmes de coopération internationale, soit collectivement, soit sur la base d'une participation individuelle. Déjà certaines actions de ce type existent dans le domaine des mathématiques, par exemple dans le cadre des programmes du *Centre International de Mathématiques Pures et Appliquées* (CIMPA), créé en France il y a plus de trente ans à l'initiative de la communauté mathématique française — voir Jambu (2006) —; on pourrait peut-être déplorer à cet égard la présence apparemment faible de mathématiciens canadiens dans un tel contexte.

Les Exécutifs de la CIEM auxquels j'ai participé ont voulu pousser plus loin les actions envers les pays en développement, notamment du côté de l'Afrique. Cela a donné lieu à la mise en place de deux réseaux régionaux africains, l'un en Afrique francophone dans le cadre des rencontres *Espace mathématique francophone* (EMF), et l'autre en Afrique anglophone, le *Africa Regional Congress of ICMI on Mathematical Education* (AFRICME). Mais il faut aller nettement plus loin, et pas seulement en Afrique. De concert avec l'UMI et l'UNESCO, la CIEM développe présentement un projet d'actions ciblées dans des pays en développement, un peu sur le modèle des actions du CIMPA. Il s'agirait d'ateliers d'environ deux semaines axés sur le développement professionnel des enseignants. S'y grefferait l'objectif de favoriser la mise en place ou le renforcement de réseaux locaux entre mathématiciens, didacticiens et enseignants. Il s'agit là d'un programme ambitieux nécessitant des moyens importants, tant sur le plan financier qu'humain. Il est permis de croire que des collègues canadiens pourraient y apporter une contribution importante en offrant leur expertise pour de telles actions. Nos liens traditionnels dans le cadre de regroupements de pays anglophones ou francophones rendent-ils certaines collaborations plus naturelles? Il s'agit là de pistes intéressantes et importantes à explorer au sein de notre communauté.

COLLABORATIONS ENTRE MATHÉMATICIENS ET DIDACTICIENS

J'ai fait allusion au tout début de ce texte à certaines tensions qui ont pu être observées au cours des années 1990 entre la CIEM et l'UMI, l'organisme dont elle relève. De fait, de telles périodes de tensions se sont rencontrées à divers moments au cours de l'histoire de la CIEM, en alternance avec des périodes plus chaleureuses, et parfois d'autres périodes qu'on pourrait qualifier de douce indifférence. Le lecteur intéressé à connaître la toile de fond à de tels changements de climat pourra consulter Hodgson (2009). L'aspect que je souhaite retenir ici principalement, tel que mentionné plus haut, est la décision ferme de l'Exécutif de la CIEM, il y a une décennie, de renforcer et développer les liens avec l'UMI, et ce sur une base concrète et pratique.

Ces renforcements se sont manifestés par le biais d'échanges plus fréquents entre les principaux responsables des deux organismes, à savoir le président et secrétaire de l'UMI d'une part, et le président et le secrétaire général de la CIEM de l'autre, de même que par des rencontres face à face régulières entre eux ou avec d'autres représentants des comités exécutifs. De tels contacts sont essentiels afin de permettre aux deux organismes de mieux comprendre la spécificité et l'expertise de chacun, et ultimement de mieux se respecter. Mais c'est surtout par le biais d'actions communes que l'UMI et la CIEM en sont venues à une collaboration forte et fructueuse. Certaines de ces actions, comme je viens de le signaler, concernent les pays en développement. Mais je retiens comme autres exemples de collaboration l'étude faite par la CIEM, à la demande de l'UMI, sur les entrées dans les filières mathématiques à l'université (projet *« Pipeline »*), de même que le projet Klein, lancé conjointement par les deux organismes il y a un an. Prenant comme inspiration les célèbres leçons de *Mathématiques élémentaires d'un point de vue supérieur* de Felix Klein, ce dernier projet vise le développement de matériel permettant aux enseignants de mathématiques de mieux apprécier l'évolution dans le champ des mathématiques au cours du dernier siècle et d'établir des liens entre les mathématiques scolaires et les mathématiques en général.

Les défis auxquels fait face la CIEM dans ses liens avec l'UMI peuvent être vus selon deux plans différents. D'un point de vue institutionnel, tout d'abord, le maintien des relations entre les organismes doit aller au-delà des personnes occupant les principaux postes de direction dans les comités exécutifs et devenir une partie de la mémoire des institutions elles-mêmes. Il est donc souhaitable que des rencontres quasi statutaires existent de manière à assurer des contacts réguliers entre les responsables des organismes. De plus, la mise en place de projets communs, tels ceux que je viens d'évoquer, peut fournir un cadre des plus propices au renforcement des liens entre la CIEM et l'UMI.

Mais il y a aussi un défi majeur à relever sur le plan des individus, c'est-à-dire en ce qui concerne les membres des communautés de mathématiciens et de didacticiens desservies par l'UMI et la CIEM. Vu sous l'angle de la CIEM, un aspect de la question réside dans la capacité (ou non) de la Commission à faire en sorte que le mathématicien « typique » soit convaincu que les activités de la CIEM sont aussi pour lui, et non seulement pour les collègues — didacticiens ou enseignants — qui sont plongés au quotidien en vertu de leurs fonctions mêmes dans les débats éducatifs. Autrement dit, tout en prenant acte de la spécificité du champ de la didactique des mathématiques tel qu'il s'est défini au cours du dernier demi-siècle et tel que reflété dans les activités de la CIEM, comment attirer davantage de mathématiciens aux divers programmes d'activités mis en place par la Commission?

La raison pour laquelle je souhaitais soulever ici cette question des liens entre matheux et didacticiens dans le cadre de la CIEM est qu'il me semble y avoir un parallèle intéressant à faire avec la situation qui prévaut à l'intérieur du Canada. Bien sûr, à la différence du binôme CIEM/UMI, le GCEDM n'est pas un organisme relevant de la Société mathématique du Canada, ce qui fournit un cadre formel tout à fait différent. Mais la question des liens, tant institutionnels qu'individuels, entre mathématiciens et didacticiens ou enseignants me semble tout autant d'actualité sur la scène canadienne que dans les cercles de la CIEM et de l'UMI. À cet égard on ne peut qu'encourager les liens formels entre la GCEDM et la SMC, par exemple par des contacts réguliers (voire statutaires) entre l'Exécutif du GCEDM et celui de la SMC, ou encore avec les responsables de son Comité d'éducation, de même que par la présentation régulière de rapports par un représentant du GCEDM au Conseil d'administration de la SMC. Déjà la nomination de la représentation canadienne auprès de la CIEM résulte d'une action conjointe entre le GCEDM et la SMC, et on peut espérer qu'une telle collaboration puisse s'étendre à d'autres volets. Par ailleurs les succès récents de projets lancés conjointement par la CIEM et l'UMI suggèrent que la mise en place d'actions communes sous la responsabilité du GCEDM et de la SMC pourrait être elle aussi un excellent moyen de favoriser les rapprochements entre les deux organismes et les communautés qu'ils desservent. La SMC, ne l'oublions pas, demeure le lien officiel au Canada pour ce qui est de relations avec l'UMI, et par ricochet, sur le plan formel, avec la CIEM.

De façon encore plus brûlante se pose la question d'assurer que davantage de mathématiciens canadiens voient le GCEDM comme reliés à une partie importante de leurs responsabilités professionnelles et qu'ils ont tout à gagner à utiliser les forums de rencontre avec les didacticiens offerts par le GCEDM. C'est là, j'en conviens, un défi de taille, et je connais pas de recette miracle afin d'augmenter la participation des mathématiciens à des activités proprement éducatives. Mais tant pour la CIEM que pour le GCEDM, ce défi doit être relevé, car il y va de la vitalité de la communauté, prise dans son ensemble, de tous ceux qui sont concernés par l'enseignement et l'apprentissage des mathématiques, et d'une meilleure compréhension et d'un plus grand respect entre les collègues qui y interviennent. Peut-on envisager un jour où un jeune mathématicien canadien, encore dans le premier pan de sa carrière, pourrait de manière quasi naturelle se voir comme appartenant également à la communauté du GCEDM (et non seulement à celle de la SMC) et participer sur une base assez régulière à ses rencontres annuelles, comme certains ont su le faire au fil des ans depuis les débuts en 1977?

CRÉATION D'UNE SOUS-COMMISSION CANADIENNE POUR LA CIEM

« *Y a-t-il lieu de créer une sous-commission canadienne pour la CIEM?* » Telle est la question que je soulevais dans une intervention lors d'une table ronde organisée dans le cadre du Forum canadien sur l'enseignement des mathématiques tenu à Montréal en 2003, reprenant ainsi une interrogation formulée une année plus tôt par Eric Muller (2003), alors représentant canadien auprès de la CIEM, dans la conférence qu'il a prononcée à la rencontre du 25e anniversaire du GCEDM. Je visais dans mes commentaires au Forum à proposer quelques éléments de réflexion quant à des gestes pouvant permettre d'accroître les échanges et la collaboration sur la scène pancanadienne sur les questions d'enseignement et d'apprentissage des mathématiques, éventuellement par la création d'une sous-commission canadienne pour la CIEM.

L'Énoncé de mandat de la CIEM prévoit en effet la possibilité d'une double « composante CIEM » dans chacun des pays membres — voir clauses 6 et 7, CIEM (2009). En plus de l'obligation de nommer un représentant auprès de la CIEM, chacun des pays membres est invité à se doter d'une sous-commission, décrite selon les termes suivants :

> *Any Adhering Organization wishing to support or encourage the work of ICMI may create or recognize, in agreement with its Committee for Mathematics in the case of a Full or Associate Member of IMU, a Sub-Commission for ICMI to maintain liaison with ICMI in all matters pertinent to its affairs. The representative to ICMI should be a member of the said Sub-Commission, if created.*

L'expérience suggère que dans les pays où de telles sous-commissions ont été mises en place, les retombées se sont avérées importantes et fécondes, notamment quant à la qualité et la fréquence des relations entre mathématiciens et didacticiens. Non seulement une telle sous-commission est-elle utile en favorisant grandement les échanges entre la CIEM et le pays membre (ces échanges vont dans deux sens: information *dans* le pays à propos de la CIEM et de ses activités, et information *sur* le pays auprès de la CIEM), mais surtout dans la grande majorité des cas la sous-commission vient favoriser, voire « forcer », les contacts et échanges entre les divers groupes ou associations composant le paysage en mathématiques et en éducation mathématique à l'intérieur du pays. Et c'est là, selon moi, un des aspects majeurs pouvant justifier la création d'une sous-commission pour la CIEM au Canada.

Je sais d'expérience, pour avoir été mêlé à une tentative de création d'une sous-commission canadienne au milieu des années 1980, que de nombreux obstacles peuvent se dresser devant une telle initiative. Certains sont d'ordre bêtement financier, ce qui avait d'ailleurs été essentiellement le cas lors de ce premier essai — l'objectif visé à ce moment-là était de fournir un cadre légal à la préparation d'une candidature canadienne pour le congrès ICME-7 qui s'est tenu à Québec en 1992, et une structure formelle *ad hoc* avait finalement été mise en place en vue de l'organisation du congrès. Mais il est permis de croire que cet obstacle financier n'est sans doute pas le plus crucial, la question de la représentation des diverses composantes canadiennes touchant le champ de l'enseignement et de l'apprentissage des mathématiques semblant nettement plus problématique. S'il n'est pas trop difficile d'imaginer des mécanismes permettant la représentation des divers organismes au Canada regroupant didacticiens et mathématiciens de tous les domaines (mathématiques, mathématiques appliquées, statistique, etc.), le cas des enseignants demeure nettement plus épineux, compte tenu du nombre d'associations d'enseignants de mathématiques en cause dans les diverses provinces et de l'absence d'une fédération canadienne. Pour être fonctionnel, un organisme comme la sous-commission doit demeurer de taille raisonnable, et il n'est pas facile de voir comment y représenter l'ensemble des associations d'enseignants existant au Canada. Mais cet obstacle est loin d'être insurmontable, si tel est le souhait des principaux intéressés. Certaines discussions semblent se dérouler présentement au sein de la SCM en vue d'explorer la possibilité de mettre en place un cadre favorisant l'institution d'une fédération des enseignants de mathématiques au Canada. Si elle devait aboutir à la satisfaction des parties en cause, une telle démarche pourrait ouvrir la porte à la création d'une sous-commission canadienne pour la CIEM. Le GCEDM se doit, selon moi, d'être partie prenante à une telle initiative.

CONCLUSION

La Commission internationale de l'enseignement mathématique n'est évidemment pas le seul organisme œuvrant en éducation mathématique sur la scène internationale. Mais c'est sans contredit un « joueur » important, à la fois par son histoire qui est un reflet fort instructif de l'évolution qu'a connue le domaine qui nous réunit tous, par les liens qu'elle maintient entre la communauté des mathématiciens et celle des didacticiens, et par ses actions qui sont aussi bien nombreuses et variées que riches en retombées.

Participer aux activités de la CIEM est certes, sur le plan personnel, une expérience importante et précieuse qui peut grandement stimuler et enrichir la personne qui s'y investit. À cet égard, je ne saurais qu'encourager tous les membres du GCEDM à profiter le plus pleinement possible du forum qu'offre la CIEM en vue d'échanges, discussions et collaborations, dans une perspective internationale, sur les questions vives à propos de l'enseignement et de l'apprentissage des mathématiques aujourd'hui. Et le GCEDM peut clairement jouer un rôle fédérateur afin de stimuler et de soutenir la participation de nombreux membres de notre communauté aux activités de la CIEM, notamment ceux qui sont en début de carrière.

À l'inverse, je ne saurais trop insister sur le fait que la CIEM a vraiment besoin de la présence canadienne dans le cadre de ses activités — et mon propos ne relève pas de la flagornerie! Cela est bien sûr le cas de tous les pays membres de la Commission. Mais on n'insistera jamais assez selon moi sur un fait essentiel, à savoir que la CIEM ou son Comité exécutif n'existent pas « en soi », à des fins propres, comme il va sans dire, mais bien en vue de servir la communauté internationale. Le succès que la Commission peut rencontrer dans l'accomplissement de sa mission se mesure à l'aune de la résonnance que ses actions ont effectivement dans les pays membres, des répercussions locales qu'elles entraînent, des progrès qu'elles peuvent éventuellement faire naître ou stimuler, tant aux plans national ou régional qu'international. Il est à cet égard hautement souhaitable, voire capital, que le Canada joue pleinement, au sein de la grande mouvance internationale que chapeaute la CIEM, le rôle qu'appelle l'expertise indéniable de notre communauté.

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Working Groups

Groupes de travail

TEACHING MATHEMATICS TO SPECIAL NEEDS STUDENTS: WHO IS AT-RISK?

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INTRODUCTION

This report attempts to reflect the discussions, dilemmas and understandings of the working group focused on teaching mathematics to special needs students, what we called "students-at-risk". All materials for the working group were prepared in both French and English however the working group was conducted predominantly in English.

The issue of student learning challenges in mathematics, and the interventions to support students with special needs, is a sensitive and controversial one facing our education systems. Our range of understandings and perspectives with regard to learning difficulties in mathematics has a considerable influence on the decisions that educators make. And these decisions have tremendous impact on student learning and well-being. We felt this was a worthy topic of exploration at CMESG, where multiple perspectives and a wide range of collective experiences would offer interesting ways to consider and re-consider our understandings of "students-at-risk" in the mathematics classroom. A major strength of the group was our ability to contribute through open dialogue that did not constrict, but rather expanded our thinking.

GUIDING QUESTIONS

In our working group, we engaged in mathematics tasks and had open discussion guided by key questions raised in the group. Questions included: Who are these students and what kinds of perceptions do we have about these students? What factors need to be considered in the learning environment and how do student difficulties manifest themselves in the classroom? What are the challenges faced by the teacher with this student population? Might we revisit our conceptions of these students? Is it the teacher who has the difficulties teaching in ways that meet the needs of these students? How do we define success for students? In this report we outline the various directions taken during group discussions, reflecting the diversity of participants' backgrounds and perspectives.

EXPLORATIONS

EXPLORATION ONE : WHO ARE THESE STUDENTS WE ARE TALKING ABOUT?

To launch our thinking collectively about students struggling in mathematics, we engaged in a task where the goal was to determine the value of a bouquet of flowers (see Figure 1).

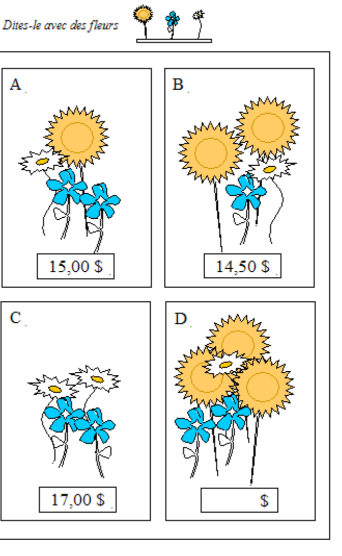

Figure 1. Initial algebra task to launch thinking about special needs students (Bélisle, 1999)

Participants were asked to solve the problem *without* using formal Algebra. We then shared our solutions and compared these to solution samples from field tests with special needs students, pre-service and high school teachers. Although participants in our working group sometimes struggled with finding an entry point to the task (without using formal algebra), students with special needs in the field trial all found viable entry points and showed nontrivial reasoning as well as productive behaviour patterns not usually associated with special needs student. Some students relied on the following observation: in each bouquet there is one blue flower and one white flower. They started with bouquet C in which the blue and white flowers appear twice – without the yellow flower. Others used the contents of bouquets A and B to find the price of bouquet D, while some students compared different bouquets' contents and their price (A and B; A and C) to find the relationship between the prices of two flowers, etc.

Exploration One led our group to reconsider what we mean by a "student-at-risk." Although the group in no way came to consensus, the following description gained momentum in the group:

> *A student is 'at-risk' of not having the opportunity to learn/of not having the opportunity to take charge of his or her learning/of not reaching his or her potential/of not being included in the learning/of not being part of the learning community/of not reaching the learning goals set by the teacher or school or curriculum, when the teacher is unaware of, or doesn't know how to help the student.*

We also considered the idea that a student who is "at-risk" may or may not have a formal identification, or be designated a "special needs" student.

EXPLORATION TWO : WHAT IS THE AT-RISK STUDENT FACING IN THE MATHEMATICS CLASSROOM?

In a second task on day two of our working group activity, we examined a problem presented to students and a transcript of student group work. The students had to calculate the extent of the melting of the Arctic icecap, using graphic representations of the forecasts of the melting published in local newspapers around that time. After helping them locate the Arctic on a globe, students were presented with the following figure explaining that the blue line represents the extent of the ice cap in 2003, and that the white surface is the projected extent of the icecap in 2020[1]. Students were then asked to calculate what the extent of the Arctic icecap will be in 2020 compared to 2003.

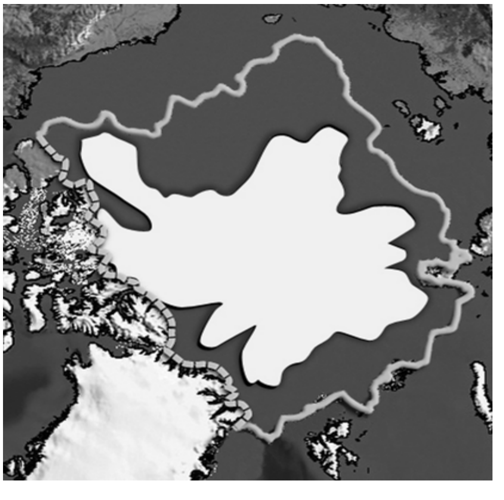

Figure 2. Representation of the icecap in 2003 and the projected icecap in 2020[2]

Researchers recorded group discussion about the task and generated a transcript of a sample group in which there was a student with special needs. Two moments when students are attempting to determine the area of the ice cap are excerpted and included here as an example:

[1] The representation has been slightly modified from the original, published in "La Presse", on November, 10th, 2004, in order to obtain a surface that will be easier to calculate for the children.

[2] Source: Arctic Climate Impact Assessment, 2004.

Moment 1

> During the first lesson in the sequence, Tristan imitates his teammates' strategy to subdivide the figure into triangles. However, he does not necessarily understand the purpose of this strategy:
>
> ***Jade**: But now we'll try to find some shapes. […] Look at this one, there is a triangle, it wasn't there before.*
> ***Tristan**: It was almost the only shape.*
> ***Jade**: Then there was a triangle. So…*
> ***Tristan**: Oh, there's a triangle!*
> ***Jade**: A shape… So afterwards we did like this and we divided it by 2. […]*
> ***Tristan**: I found another triangle. […] I have found 5! […] I have found 8! 8 triangles.*
> ***Juliette**: (Stops to find them and mark them.)*
> ***Tristan**: But it's because I don't have a pencil. (He looks around for a while, then he stops and talks again.) Hey guys, what are you doing?*

Moment 4

> When calculating the areas of the different surfaces (mostly by using a grid that is put over the image, followed by the counting of the number of squares that cover the surface to be counted), group members then need to come to a consensus within the group as to what surface would be reasonable. This constitutes a problem especially for Tristan. When he tries to calculate how many squares the area of the surface of 2020 measures, he counts 208 squares. Surprised by this result, Jade decided to go over Tristan's computation and comes up with a slightly different result:
>
> ***Tristan**: 208. […]*
> ***Jade**: There are 208 squares in this?*
> ***Tristan**: Yes, would you like to count it?*
> ***Jade**: Yes. Seriously, I would like to count it. […] (A few minutes later) Sorry, but there are 199 squares in it. […]*
> ***Tristan**: [I found] 208.*
> ***Mélodie**: You don't count well, Tristan. […]*
> ***Tristan**: (Directly talking to the camera) I screwed up, but we will start over tomorrow.*

The intimacy of looking in on the group of students via the transcript led to further discussion about the factors to consider when thinking about at-risk students in the mathematics classroom learning environment. Particularly in the example where Tristan did indeed have a viable solution of 208 squares, given the estimation strategies used for the task – and particularly, when considering the group dynamics at play. Our working group identified student factors, teacher factors and social factors to consider (see Table 1). This non-exhaustive list of factors highlights complex relationships and interactions of the learning/teaching process.

Student Factors	Teacher Factors	Social Factors
• Socio-economic • Respecting classroom norms • Cultural expectations and background • Specific versus general learning challenges • Organizational problems / Memory / specific learning disabilities • Perseverance, effort (high or low) • Affective (excitement, then dwindles over time, then resignation) • Task in grasp of student: Zone of proximal development in mathematics • Previous experience of the task • Smart (teacher needs to help the student reveal this and value student ideas) • Saving face	• Establishing, expanding and contracting the classroom norms (didactical contract) • Expectations of students • Teacher efficacy (belief that teacher has the ability to help all students learn mathematics) • Teacher effectiveness (range of strategies / repertoire, MKT) of the teacher • Pace of tasks • Design of tasks / nature of the tasks • Giving student-at-risk opportunity to be successful in front of peers	• Social isolation in group work (lack of cultural capital) • Eager to please or conform, but wants to do some math • Gives up authority to teacher and group when questioned (doesn't fight for his/her ideas) • Pace of the group surrounding the at-risk student *Group dynamics in the transcript:* • Group members question the mathematics (of counting), then question Tristan's ABILITY (to count) • Effort level of the at-risk student to stay with the group is high • Distractions are abundant • Gender relationships in group work (mixed gender group) • Size of the group (5 students, large group)

Table 1. Some factors to consider when thinking about students-at-risk in group-based math problem solving tasks

EXPLORATION THREE : HOW DO OUR PERCEPTIONS OF STUDENTS-AT-RISK INFLUENCE THE WAYS THAT WE TEACH MATHEMATICS TO THESE STUDENTS?

Our working group then examined results from a study (René de Cotret & Giroux, 2003) where the same teacher taught the same content to two different groups of students: one group had failed the course and was repeating the content while the second group was taking the course for the first time. The teacher taught the students differently in that the sequence was repeated consistently for the students-at-risk (teach the concept, practice the skill, take up the work), however the students in the "regular" class had more variety in the instructional sequence, including having students attempt to solve problems where the theory or concept had not yet been formally presented by the teacher, thus forcing those students to take responsibility for their learning (see Figure 3).

Comparaison de l'organisation d'une même leçon dans la classe d'élèves doubleurs (CD) et dans la classe d'élèves réguliers (CR)

<table>
<tr><th></th><th>Leçon 1</th><th>Leçon 2</th><th>Leçon 3</th></tr>
<tr><td>CD</td><td>|___T_1______/__D_1__|</td><td>|__C_1__/___T_2__/__D_2__|</td><td>|__C_2__/___T_3__/_D_3_____|</td></tr>
<tr><td>CR</td><td>|___T_1____/_$D_{1\text{-}2}$___|</td><td>|___C_1_/_T_2_/_C_2__/_$D_{2\text{-}3}$_|</td><td>|_C_2_/_T_3_/_C_3__/_$D_{3\text{-}4}$__|</td></tr>
<tr><td></td><td colspan="3">T_i : exposé de la théorie en lien avec les éléments de savoir i
D_i : tâches que les élèves ont à effectuer individuellement (devoirs)
C : correction en classe de ces tâches
En groupe Individuel</td></tr>
</table>

Figure 3. Comparison of two classes

One teacher, two classes: The class noted as CD in the left column included students with difficulties (who had failed the course once already and were repeating it). The class noted as CR in the left column consisted of students who were not experiencing difficulties in the course. T stands for “theory (or concept) introduced to students”, D stands for “practice tasks for students”, C stands for “corrections taken up in class”. The numbers with each letter stand for each of the concepts taught in class. Therefore, in the CD class, for a given concept, the lessons always have the same sequence: First, theory, then individual practice, then corrections in class. In the CR class on the other hand, there are some concepts, where individual practice precedes the presentation of the theory by the teacher (e.g. for concept 2).

Looking at these data led our working group to ask questions about why the teacher might have chosen to vary her pattern with the one group compared to the other. Why does the same teacher faced with the same content teach in a different way when working with special needs students? Some explanations/discussion points the working group came to were:

- Special needs students who were in a regular class before and failed need a different way of being taught. But is a rote transmission pattern of telling students how to use a procedure and practicing it helpful to the students?
- The “regular group” may be perceived by the teacher to have a greater tolerance of uncertainty, whereas the repeating group already has demonstrated uncertainty (by failing the course), so the teacher is reacting to that way of understanding the situation (teacher’s perception of students).
- The teacher has higher expectations of higher achieving students, and lower expectations of lower achieving students.
- Assumption of students who fail is that they have a cognitive deficit, instead of considering other factors.
- Question: How can students-at-risk build conceptual understanding if we are always giving them simpler tasks?

These observations and perspectives spurred our working group on to attempt to identify some of the key characteristics of high quality interventions. In small groups and then collectively, participants highlighted what they thought were some key characteristics of effective interventions. In doing so, we revealed to ourselves some of the principles of mathematics teaching that we believe underpin productive learning classrooms where students are supported in their mathematics learning.

Some characteristics of highly effective interventions:

- An ethic of care and respect – Establishment of a relationship of trust and caring between teachers and students.
- An environment where students receive feedback and have opportunities for one-to-one support as well as opportunities to (and an expectation that students) learn with and from peers.
- Access and entry points to the learning are available to students – tasks are generated or selected and refined to ensure that all students have access or entry points.
- Flexibility and careful selection of pedagogical strategies – teaching styles are adapted to the needs of the students.
- A belief that all students can do mathematics, and there are high expectations for students.
- An environment where teacher and student assumptions are acknowledged, challenged and revisited.
- A learning environment where actions of the teacher and students create windows for understanding – the teacher (and the class as a whole) tries to view a student's thinking from the perspective of the student to:
    - understand how the student is thinking, validate and give authority to student ideas;
    - enrich perspectives of mathematics thinking.
- A classroom that makes space for students and the teacher to increase or decrease the pace and/or modify the types of learning, tasks, strategies and representations that are used.
- There is a press for understanding and justifications, including taking up "mistakes" (with care).
- There is a focus on meaning-making and understanding of mathematics.

…In essence, highly effective interventions are embodied in... "good teaching."

Keeping the above characteristics in mind, our working group explored two different sample interventions.

EXPLORATION FOUR : MIGHT TECHNOLOGY-BASED LEARNING OFFER A POSSIBLE EFFECTIVE INTERVENTION FOR SOME?

The first intervention we explored was a set of online learning objects that focus on multiple representations of growing linear patterns (see Figure 4). Current research (Beatty, 2007; Beatty & Bruce, 2008; Bruce & Ross, 2009) is focused on the implementation of these algebra-based learning objects in two district school boards for Grade 7 and 8 students. The online lessons are intended to support mathematics learners who are struggling with linear functions. These lessons are combined with offline teacher lessons to use in an integrated fashion with the learning objects.

A lively discussion about the potential value and dangers of online learning was held, and successes so far with the combination of offline and online tasks for students that promote deep understanding of linear growing patterns were shared.

Study results indicate that achievement of the whole class is increasing as a result of use of CLIPS, including students who are normally considered at-risk in the mathematics classroom. The learning objects provide at-risk students with a different entry point to the learning that is predominantly visually-based.

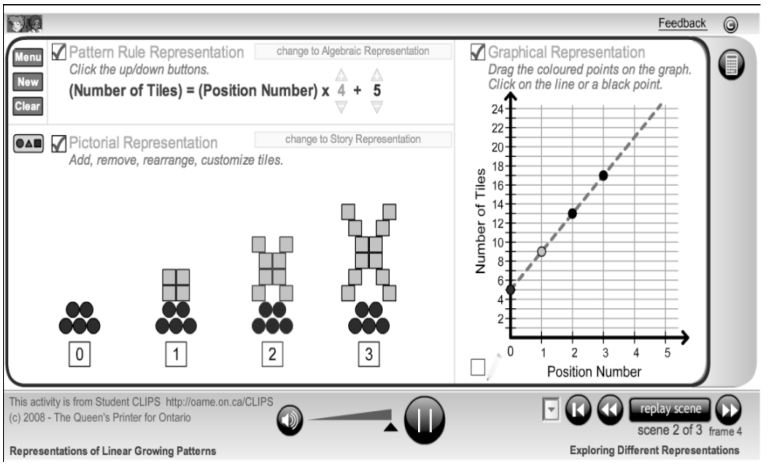

Figure 4. Online learning screen from the algebra lesson sequence known as CLIPS. See www.mathclips.ca

Interestingly, some teachers in the study resisted including their special needs and at-risk students in the online learning component of the study, thinking that it would be too difficult for them. After requesting that they reconsider, these teachers did use the online learning CLIPS with the whole class and were incredibly positive about the results in terms of level of engagement and achievement.

Working group participants considered whether learning objects might help a student like Tristan (in the Exploration Two transcripts) who would benefit from a non-threatening way of learning that could be undertaken *prior* to working in a group setting. The notion that Tristan might have a positive mastery experience that helped him prepare for group work was discussed in terms of overcoming affective damage of students like Tristan. We ended the online learning objects investigation thinking about the following question: What needs to be in place to positively change our perceptions and expectations of students-at-risk?

EXPLORATION FIVE : MIGHT A SPECIALIZED PROGRAM OFFER A POSSIBLE EFFECTIVE INTERVENTION FOR SOME AT-RISK STUDENTS? GINA'S STORY

The final discussion centred on a program for grade 10 students who had multiple challenges and were at significant risk of not completing courses, exams, and high school. One of our participants, Gina, told the story of her experience establishing and teaching in this program. The students in the program were a population that had failed courses 2 and 3 times. Students had hearing impairments, behaviour challenges, drug addictions, brain injuries, depression, learning disabilities, and organization deficits, to name a few. Everyone in the class is at risk in at least one way.

The program goal was to ensure that students' basic needs were addressed such as food, clothing, and positive interactions, and then this was followed by opportunities to learn mathematics. The ethos of care and respect was compelling. The course had high expectations for all students and there was a demand of an exam at the end of the program. The teacher built a supportive learning community with the students, and taking the lead from her students, began to shift her teaching practices. At the end of the program, every student passed the final exam where 62% had previously failed.

Our working group found this story compelling because it spoke so clearly to the characteristics of highly effective interventions we had earlier identified.

REFLECTION

At the end of the working group activities, discussions, and explorations, participants were asked to individually make a note about the ways in which their understanding of special needs students had been broadened or deepened through the working group discussions. To summarize, the following comments are drawn directly from some of the notes participants made:

1. In this group, I realized the broadness (and hence difficulty) of the “at-risk” label. Given this broadness of the notion, I realized that a learner experiencing “persistent” “not understanding” can be at risk if the appropriate help is not available for him (her). (I am interested in the phenomenon of “not understanding” in mathematics pedagogic situations). The point then is that there are different kinds of “at-risk” and ALL learners can be said to be “at-risk” in some way given the pedagogic environment in which they are located. The difficult work is identifying the potential / actual risk and finding appropriate “interventions” to help the learner.

   Finally, the teacher can also be said to be at-risk – at risk of having the learners not learn appropriately, or not receive the appropriate help that is needed for their specific situations.

2. It has only reaffirmed my conviction that “teachers matter most” (e.g. Gina’s story of her 31 “at-risk students”), and that our society would profit the most from building “good teaching” and “good teachers”. Just as our group had difficulty defining what “at-risk student” is (it seems to change with “who is the teacher”), so the “good intervention strategies” seem to be effective for every student when not applied in a “one size fits all” fashion (as too often happens), but responsive to both each individual student’s needs and the goals of instruction.

3. The key really is “good” teaching, which should arguably be more demanding for all students than it typically is. Our working definition of at-risk that places more responsibility on the teacher also has merit and could be a useful provocation in working with both preservice and inservice teachers.

4. Une des principales choses qui ressort du groupe de travail à mon sens est la difficulté d’établir une définition unanime et fonctionnelle de l’élève en difficulté, qui a été appelé «at-risk student» en anglais. Ramenant à la sempiternelle question : qu’est-ce qui est spécifique aux élèves en difficulté, la difficulté de cibler ou d’identifier les élèves en difficulté laisse croire qu’ils ne sont pas bien différents des autres, surtout sur le plan mathématique.

   En effet, la plupart des erreurs mathématiques rencontrées au cours des différentes études de cas pourraient être vécues par n’importe quel élève et non seulement par l’élève en difficulté.

   Une autre idée qui est ressortie de notre groupe de travail est le fait que plusieurs facteurs sont à prendre en compte pour considérer l’élève en difficulté, notamment les caractéristiques de l’élève, mais également les conditions d’enseignement (pédagogie, curriculum etc.), la nature de la tâche, les aspects sociaux et affectifs liés à l’apprentissage, etc. Ainsi, la difficulté rencontrée par l’élève ne lui est pas inhérente, elle découle d’une multitude de facteurs conjoncturellement situés qui créent les conditions d’apprentissage dans lesquelles l’élève peine à apprendre.

   Enfin, pour terminer sur une impression, il semble que la vision pédagogique ait prévalue sur la vision didactique dans l’appréhension des tenants et aboutissants de l’enseignement des mathématiques dans le groupe de travail. Ainsi, le regard gagnerait peut-être à être davantage porté sur les savoirs en jeu dans l'enseignement des mathématiques, plutôt qu'uniquement sur les conditions d'enseignement.

5. Il est intéressant de constater que certaines conduites d'élève en difficultés d'apprentissage ont été interprétées comme ne pouvant relever de cette population, ce qui montre plutôt, à mon avis, l'influence et la force des tâches qui sont proposées aux élèves en difficultés.

In conclusion, our working group was able to observe our own difficulties in ruling on a definition of the "at-risk student" but gained insights into what it means to work with struggling mathematics students and the prominent role that the teacher plays in knowing how to support these students in positive and productive ways. We believe that through discussions and explorations, participants developed a deeper and broader understanding of the multiple perspectives, factors, and subtleties involved in working with "students-at-risk".

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ATTENDING TO DATA ANALYSIS AND VISUALIZING DATA

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INTRODUCTION

Data analysis and the visualization of data have become an integral part of the curriculum in elementary and secondary schools as well as in pre-service teachers' courses. We must consider how to make these important and relevant topics meaningful to students as well as meeting the major challenge of ensuring that our future teachers have the confidence and knowledge to attend to this strand of the curriculum.

It is important that students and teachers develop an appreciation for the power and limitations of statistical inference and to develop the ability to recognize common pitfalls in the interpretation of data. How can we achieve this? What activities, experiments, simulations, and resources can we use and develop with students and pre-service teachers? In what ways can the technology that is readily available motivate and deepen understanding? How can we use existing indices and databases such as Statistics Canada's E-Stat, CANSIM and nationmaster.com to empower our students and pre-service teachers and help them make sense of our data-filled world?

We must further consider that the teaching of statistics generally takes place within mathematics or mathematics methods courses in the case of teacher training. How can we promote the synergy of these two disciplines, that of mathematics and that of data analysis, while fostering learning?

We devoted the first day to identifying skills and principles that are not inherently mathematical but important in understanding statistics. We identified the problem of inferring causality – particularly with non-experimental data – as a central and vital problem for statistical education and decided to spend the second day of the workshop exploring ways of structuring related concepts so they could be presented in the secondary curriculum. The third day was devoted mainly to the concept of correlation and its interpretation. We discussed ways of visualizing the properties of correlation using ellipses to approximate the shape of data in a scatterplot, including how to visualize correlation at a glance, and finally how to visualize whether a scatterplot reveals a statistically significant linear relationship at a glance.

Our three-day excursion into the didactics of statistics took us to many topics both planned and unanticipated. A condensed list follows:

1. Statistical understanding necessary for good citizenship and public leadership.
2. The problem of causality.
3. Mathematics and statistics: divergent or convergent modes of reasoning.
4. Using the latest available data and software for visualizing world demographic data: Hans Rosling and Gapminder’s goal of a “fact-based world view.”
5. Visualizing statistical concepts as well as data.

STATISTICS FOR GOOD CITIZENS

Statistical topics can be selected on the basis of the fact that they are mathematically tractable and would find a natural place in the mathematical development of the topic. We focused on identifying statistical topics for their social and conceptual importance.

Utts (2003), in an article entitled “What Educated Citizens Should Know About Statistics and Probability” lists seven topics:

1. When it can be concluded that a relationship is one of cause and effect, and when it cannot, including the difference between randomized experiments and observational studies.
2. The difference between statistical significance and practical importance, i.e. the strength of evidence versus the strength of an estimated effect, especially when using large sample sizes.
3. The difference between finding “no effect” or “no difference” and finding no statistically significant effect or difference, especially when using small sample sizes, i.e. the difference between evidence of absence versus absence of evidence.
4. Common sources of bias in surveys and experiments, such as poor wording of questions, volunteer response, and socially desirable answers.
5. The idea that coincidences and seemingly very improbable events are not uncommon because there are so many possibilities.
6. “Confusion of the inverse” in which a conditional probability in one direction is confused with the conditional probability in the other direction.
7. Understanding that variability is natural, and that “normal” is not the same as “average.”

The concepts listed by Utts (2003) are at the heart of the practical use and social relevance of statistics. They constitute the *raison-d'être* of the discipline in contrast with the building blocks of its mathematical structure.

A challenge in statistical education is to consider how some appreciation of these concepts can be stimulated in high school. An appropriate treatment of causality, for example, requires going far beyond "correlation is not causality," which leads to blanket uncritical scepticism towards most of the evidence on which personal and social decisions are based. It is crucial to understand the shortcomings of evidence, but it is equally important also to be able to make critical distinctions in the degree of evidence from different sources of information. We present a tentative approach for teaching "causality for good citizenship" below.

The K-12 GAISE (Guidelines for Assessment and Instruction on Statistics Education) report endorsed by the American Statistical Association (Franklin et al., 2005) suggests presenting a case-control study showing an association between smoking and lung cancer as an example of a study intended to elicit causal information from observational data.

STATISTICS AND MATHEMATICS: SHARED ROOTS, DIFFERENT DESTINIES

What skills or concepts have a different character from those of mathematics? How do we encourage teachers, whose own education has been oriented towards mathematics, to appreciate and teach statistical concepts? Statistics is a discipline that relies heavily on mathematics for its structure and development. But the application of statistics has a character that is almost antithetical to mathematics: uncertainty. In a sense, statistics is about forming the clearest insights where the clarity of mathematical thinking is not available. Fallacious thinking is much more widespread in statistics than in mathematics.

The discipline of statistics in universities emphasizes research into the formal mathematical structure of statistics which is rife with research questions. There is, generally, less emphasis on the meaning or application of statistics, issues that are more important for "citizenship" and the aspects of statistics that are most relevant for high schools. In a sense, it may be precisely those aspects of statistics that are least connected with mathematics that are most important in the context of high school education. This is a challenge since statistics is naturally closely associated with the mathematics curriculum and taught by teachers whose training is oriented towards mathematics. Fundamental statistics concepts may be better taught through stories than through formulas.

CAUSALITY FOR GOOD CITIZENSHIP

Most undergraduate textbooks in regression warn students to avoid causal interpretations with observational data admonishing that "*correlation is not causation*." Ironically most applications of regression would be of little interest if it were not for the tantalizing possibility that at least some of the coefficients have a causal interpretation.

Causality is arguably one of the most important statistical concepts, as well as one of the most difficult ones. A major contribution of R. A. Fisher in the 20th century is the principle of randomized controlled experiments. Fisher expressed the view that without random assignment to treatments there was no justification for causal inference. The broad acceptance of the need for randomized experiments is one of the great contributions of statistics to science. However, many of the most urgent and important issues facing humanity are of a causal nature but are not readily resolved with experimental data. The blanket prescription: "*correlation is not causation*" yields little guidance to help citizens and public decision makers assess the reasonableness of causal suggestions based on non-experimental, i.e. observational, data.

Filling this gap is an interesting educational challenge. Without diminishing the importance of experimentation, can we equip students with a critical approach to causal conclusions from observational data that finds a reasonable balance between uncritical acceptance and rigid scepticism? Many of the most important issues facing us today, appropriate policies in response to climate change for example, require causal inferences from data that are primarily observational. There is little opportunity for experimentation except in studying small aspects of the larger problem. There is no control planet and there is no time to do more than assess as intelligently as we can the data we have.

Although experimentation is the gold standard for causal inference, it is nevertheless true that initial decisions about which experiments to conduct must be based in part on the study of observational data. Moreover, many critical questions, even in areas such as pharmaceutical research where experimentation is well established, require the assessment of observational data. For example, the effect of compliance with treatment, in contrast with the effect of the intent to treat, requires the analysis of inherently observational data. The study of possible side-effects after a new pharmaceutical has been put on the market is also based on observational data.

Almost all inferences in economics are based on observational data. Thus, the intelligent assessment of causal claims from observational data is a vital skill for an educated public. There have been major advances in the past two decades in research on causal inference from observational data (Rosenbaum, 2010) but little of its content has percolated to the public.

A challenge for educators is to find a way of framing key concepts in causal inference in a way that allows them to be conveyed effectively without oversimplifying and contributing to the creation and entrenchment of fallacies.

Part of the problem in understanding issues of causality is to place them in context. Causality is not relevant when the only question is prediction. Causality is also not a crucial issue in the context of a randomized experiment where causal conclusions are not controversial. To help contextualize the problem we observe that most data sets belong to one of two types depending on the manner in which the levels of X, the potential cause, were determined:

1. experimental data where the levels of X are determined by an experimenter using random assignment, and
2. observational data where the levels of X were determined through processes not under the experimenter's control.

In parallel, there are two type of inference:

3. **causal inference**, in which we wish to infer what would happen to probabilities for a response, Y, if we were to actively change the levels of X, and
4. **predictive inference**, in which we merely wish to predict Y, given a level of X as it was passively observed.

Different contexts for statistical inference can be categorized with a 2-by-2 contingency table:

		Types of Inference	
		Causal	**Predictive**
Types of Data	**Experimental**	OK	problematic but rare
	Observational	most common and most problematic	OK

The common problematic and challenging context is that of causal inference with observational data. Freedman, Pisani, and Purves (1978) provide a revealing problem:

> *In 1964, the Public Health Service of the United States studied the effects of smoking on health in a sample of 42,000 households. For men and for women in each age group, they found that those who had never smoked were on average somewhat healthier than the current smokers, but the current smokers were on average much healthier than the former smokers.*
>
> *a) Why did they study men and women and the different age groups separately?*
>
> *b) The lesson seems to be that you shouldn't start smoking, but once you've started, don't stop. Comment.*

One could consider the ability to give a good response to this problem as a major goal of a good introductory statistics course. Students need to be able to identify the data as observational so that association between variables does not necessarily reflect a causal relationship. There are two related but distinct motivations to study men and women separately. Perhaps the relationship between smoking and health varies at different ages, an example of a possible interaction among gender, age, and smoking in their effect on health. More subtly but more crucially, even if the relationship between smoking and health is similar across genders and ages, studying men and women and different age groups separately allows the study to estimate the relationship between smoking and health, controlling for age and sex. The estimated relationship controls for the possible confounding effect of age and sex, obviating the possibility that an apparent relationship between smoking and health might just be the consequence of both smoking and health being related to age and sex.

Part (b) of the problem confronts the student with the possibility that even controlling for age and sex is not sufficient to isolate a causal relationship. Untutored, students will not distinguish between two types of explanations for the paradoxical fact that quitters are less healthy than continuing smokers. One type of explanation is that perhaps there are a disproportionate number of quitters who did so because of poor prior health: an explanation of the type "Y causes X." Or put differently, a third factor Z, pre-quitting health status, affects both the propensity to quit, X, and post-quitting health status, Y. Another type of explanation is that perhaps quitting causes many people to gain weight, or to develop other secondary pathologies such as high stress or depression, which in turn causes bad health. These explanations are fundamentally different. The former identifies possible confounding factors and the latter constitutes what can be called "mediating" factors. The identification of possible confounding factors offers alternatives to causality, while mediating factors offer an explanation that is consistent with causality. Mediating factors attempt to explain the otherwise paradoxical relationship between quitting and health.

An ideal response to this problem would involve the appreciation that both confounding factors and mediating factors may be at play. An analysis of observational data suggests a causal relationship only to the extent that the effect of plausible confounding factors has been controlled.

Students need a systematic approach that allows them to develop insights in the face of causal claims (or mere suggestions of causality) based on observational data. We suggest that students can be led, with the help of compelling examples, to understand that an association between two factors X and Y in observational data could indicate one of five possibilities:

1. perhaps X does cause Y;
2. perhaps it is really Y that causes X;
3. perhaps one or more other variable(s), Z, cause both X and Y, with a number of possibilities:
    a. some Zs can be measured accurately and controlled for in a statistical analysis;
    b. some Zs are known but difficult to measure accurately; or
    c. some Zs may be unknown and remain so until further scientific development;
4. perhaps the association arose by chance and other randomly selected samples would not display it;
5. perhaps the appearance of association is a consequence of selection: i.e. the sample was, through some process, intentional or not, selected to display this association although the association would not be present in the larger population from which the sample was obtained.

Association suggests causation only to the extent that we are comfortable in excluding alternative explanations 2 through to 5. Traditional statistical methods, Bayesian or frequentist, are aimed at dealing with 4. Alternative 5 can often be addressed by a review of data selection processes. The most problematic alternatives are, generally, 3b and 3c. The conclusion of causality from observational data rests on the hope that the effect of unknown Zs is too small to account for the observed effect.

Our typology of possible explanations for association leads to a natural explanation for the power of experimentation. With random assignment to levels of X, it can be seen that explanations 2 and 3 are excluded because levels of X are determined by a random process, neither by Y nor by any Z, except for the possibility that some Z, related to Y, may be randomly different for different levels of X. But this can only happen “by chance” which is alternative 4, which is quantifiable. Thus experiments leave only three possibilities: X causes Y, chance, or selection which is generally observable and can be excluded.

Understanding the causal interpretation of observational data is closely related to Simpson’s Paradox (Pearl, 1999), which refers to situations in which the direction of association between two variables, X and Y, is reversed when conditioning on a third variable Z. With observational data, the causal relationship is generally captured by a conditional association controlling for confounding factors. Simpson’s Paradox shows how failing to control for confounding factors can show an association whose direction is the opposite of the causal association. Pearl (2010) provides a recent overview of causal inference.

HYPOTHESIS TESTING: A FLAWED PARADIGM?

Almost all introductory courses in statistics present null hypothesis significance testing as a cornerstone of the statistical method. To make its seemingly convoluted logic easier to understand, an analogy is often drawn with proof by contradiction: *If A implies B and B is false*, *then A must be false*. The syllogism offered for hypothesis testing is: *If hypothesis A implies that B is unlikely* (where B represents outcomes that are equally or more consistent with a hypothesis A than the outcome observed), *and B occurs, then A is not likely*. In summary, we are invited to believe that if P(B|A) is small and B occurred, then we have evidence against A.

As plausible as the argument seems, it is seriously flawed, which becomes evident if there are other sources of information regarding the relative probabilities of A and B. An uncritical adherence to hypothesis testing as a way of assessing evidence against a hypothesis can lead to seriously wrong conclusions. Flagrant examples occur in medicine in the assessment of the results of medical tests (Gigenrenzer & Edwards, 2003) and in the assessment of legal evidence. For example Dawid (2001) reanalyzes the circumstances of the conviction of Sally Clark who was wrongly convicted in 1999 of murdering two of her children on the basis of evidence analogous to hypothesis testing. Although, using the logic of hypothesis testing, the P(evidence | innocence) would have been in the order of one in thousands (after correcting flaws in the original testimony that yielded 1 in 73 million), the more relevant P(innocence | evidence) would, Dawid argues, be in the order of 0.99. A video by Peter Donnelly (2005) presents the case in a way that is relatively easy to understand.

We identified three major problems with the hypothesis testing paradigm, each quite different in character. Their didactic consequences are not entirely clear. Although it is not a solution that addresses all problems, one conclusion is that it would be desirable to emphasize estimation and confidence intervals instead of hypothesis testing. This shift in emphasis from hypothesis testing to confidence intervals is increasingly being sanctioned in research (Wilkinson et al., 1999).

A fundamental problem with hypothesis testing is that it answers a question no one is really interested in asking. The question it answers is a proxy for the question that would be of real interest if it could be addressed. Hypothesis testing deals with the improbability of the data given the assumption that the hypothesis is true. What we would prefer to know is the probability of the hypothesis given the data. Alas we can only get that by "breaking the Bayesian egg," i.e. by formulating a prior probability for the hypothesis. To deal with the fact that we cannot get what we really want without paying a price we are reluctant to pay, we try to be content with a pale shadow of what we want. The fact that it is a pale shadow is rarely clearly articulated, so that people who learn hypothesis testing either misinterpret it as providing a probability of the null hypothesis, which leads to "Confusion of the Inverse," (item number 6 in the list by Utts (2003) above) or, those who understand hypothesis testing well enough to be aware of a problem feel a queasy disquiet when they think about it. Feeling that you do not understand hypothesis testing is a good sign that you at least understand something about it.

Bayesians feel that they solve the problem with hypothesis testing by adopting a Bayesian framework for inference. Staying within the frequentist framework that is still dominant in applied research (i.e. working without a prior probability), one could encourage a stronger emphasis on estimation with indications of probable error, such as with confidence intervals. Even within frequentist inference, one of the greatest flaws of hypothesis testing is that it conceals uncertainty; it wrongly creates the impression that a definite decision is justified.

In some applied areas such as psychology (Wilkinson et al., 1999), there has been a growing movement against hypothesis testing. Although manifestos against hypothesis testing have been published for a few decades, there has been little progress in abandoning hypothesis testing in practice. In many applied areas, hypothesis testing is used uncritically, sometimes with dire consequences as mentioned above.

Despite its flaws statistical hypothesis testing is likely to be quite durable. It creates the illusion of turning uncertainty into "decisions" without the necessity of taking prior information into account. Statisticians can take heart that the existence of such a thorny problem at the core of their discipline may be seen as a symptom of the importance of the questions the discipline attempts to deal with.

A possible practical didactic course of action on hypothesis testing is to avoid presenting it as a cornerstone of statistical methodology. It is appropriate to put much more emphasis on point estimation and estimation with an indication of error with confidence intervals, preparing students to understand a phrase they will encounter frequently: "*43% of responders in the survey supported candidate X. This kind of survey will be within 3% of the proportion in the population 19 times out of 20*." Although, at a deeper level, confidence intervals may be criticized by Bayesians for sharing flaws with hypothesis testing, at a more pragmatic level they are far superior to traditional hypothesis tests because they do not conceal uncertainty. Statistics is much less about eliminating uncertainty than it is about understanding and appreciating it.

NEW WAYS OF EXPERIENCING AND VISUALIZING STATISTICS

Teaching is a challenge. It is clear that statistics can not be presented entirely through its mathematical structures in high school. Statistics must somehow be experienced, presenting the educator with an interesting challenge. There have been significant developments in this direction in the last decade of which we briefly cite two. In connection with the development of a high school statistics curriculum in New Zealand, some very effective interactive graphical methods have been created to visualize sampling and uncertainty (Pfannkuch, 2008; Wild, Pfannkuch, & Regan, 2011).

A major recent development is associated with the Gapminder Foundation (http://www.gapminder.org) headed by Hans Rosling, a professor of international health at the Karolinska Institute. A goal of the foundation is to "*unveil the beauty of statistics for a fact-based world view*." Rosling has been influential in making global health and demographic data for recent years readily available for downloading and analysis. He has also developed effective graphical tools to visualize the data, especially dynamic changes over time. Presentations (Rosling, 2006; 2010) using this data and Gapminder's graphical tools are very compelling narratives about the directions of global health, stories that students at all levels can understand and appreciate, at least at a phenomenological level. The data provides a very rich opportunity for experiential learning on which to build deeper conceptual learning.

In statistics we generally think of graphics as methods to visualize data. The members of the workshop explored ways in which graphics can also be used to visualize statistical concepts such as correlation, regression to the mean, Simpson`s Paradox, consequence of measurement error, etc. (Monette, 1990).

CONCLUSIONS

Understanding statistical concepts becomes increasingly important in order to develop a critical appreciation of crucial social issues: a "data-based world view." Can the high school curriculum convey some of these statistical concepts? Or should it be limited to teaching the mechanics for computing the more elementary statistical techniques. We hope that, with the help of recent software and access to data, and through a greater awareness and understanding among teachers of fundamental statistical concepts, the teaching of statistics in high schools can find its ideal role.

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RECRUITMENT, ATTRITION, AND RETENTION IN POST-SECONDARY MATHEMATICS

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INTRODUCTION

The theme that our working group was assigned to discuss is very broad and far-reaching. Thinking about it, we published the following abstract:

> In the international context, universities in England are closing departments of Chemistry, and Physics – and leaving service teaching to other programs. Under financial pressure, universities in Canada are also starting to close programs with low enrolment. The key to these decisions seems to be enrolment and graduation rates. Mathematics programs are being judged by recruitment and retention numbers, as well as the quality of the support they provide through service courses. At the least, this impacts whether people are hired into mathematics departments. At the extreme, some programs within mathematics departments will be closed, or even whole departments might be closed (as has happened in some Ontario Colleges).
>
> There are a number of "causes" that have been suggested, some of which are observed on an international scale, some of which are local to the cultures of the region or the institutions, some of which connect to student motivation and the quality of student experiences. A number of responses have been suggested, including:
>
> - collaborations with other disciplines, such as education and interdisciplinary science programs;

- offering challenging mathematics in senior secondary education and undergraduate programs;
- refocusing on the balance between topics and processes in the objectives and courses of post-secondary mathematics programs.

To address these issues, we will draw from the research available, from the data and stories brought by participants from their institutions, and from broader recommendations from groups such as the Mathematical Association of America. We will also look at the image of mathematics, as well as the preparation for the mathematical sciences in the high schools, as they connect to recruitment. In some cases, the experience of mathematics in the first year of post-secondary education is very different from high school – does this contribute to attrition? We will investigate how other factors, such as quality of instruction or availability of adequate resources, influence students' decisions to stay (or not) in mathematics. Through the three days, we will consider what can be done to recruit and retain students with a strong interest in mathematics either as their major focus, or as part of their broader learning over several disciplines.

SETTING THE STAGE

In preparation for discussions at the conference, we produced a web page with references and suggested readings (listed at the end of this report).

We started the day by sharing our own personal perspectives and experiences: What attracted us to choose mathematics? Were there any defining moments? How did we decide that mathematics was going to be our career? Thinking of our own cases, we outlined a number of reasons why someone might (or might not) choose to pursue mathematics as undergraduate degree and later – possibly – as career.

To further our background, we read and discussed historical data on enrolment and graduation numbers, both Canadian (Fenwick-Sehl , Fioroni, & Lovric, 2009) and international (Holton, Muller, Oikkonen, Sanchez Valenzuela, & Zizhao, 2009). One important reason why we suggested the two papers was the search for evidence. It is amazing to what extent discussions on these issues are based on anecdotal evidence, personal (generalized) experiences, or no evidence at all.

Early into the session it became clear that the theme of recruitment and attrition is very broad, with a large spectrum of meanings, conceptions and misconceptions, and implications. Perhaps the best evidence of the complexity and broadness of the theme is the list of subtopics and questions that was generated:

1. Should each student get formal math education/instruction? Is formal math the only math that should be taught?
2. What do we mean by retention? Within a course? Within a program? Or broader? What constitutes success in the context of retention?
3. What is math in the 21st century? Should it be technology-dominated? Who is to decide about this? What are patterns of change in mathematics? How does mathematics relate to 21st century developments in other disciplines?
4. Assuming that we manage to keep *all* students that enrol initially in math programs, what would we do with them? What is our responsibility in terms of guiding students through the program, advising about careers, etc.?
5. What is an optimal number of mathematicians our society needs (and can support)? Are there reasons for concern, or is the present situation satisfactory?

6. Should math be made for all, or only for some? Does math in tertiary institutions have to be compulsory? Who gets to decide? Should we change our cherished views about this? What is the role of politics?
7. What are the ramifications of preparation of teachers (at all levels), their beliefs about math, and their prior experience of retention (for instance, in high school)?
8. How does the mathematics curriculum and pedagogy required to deliver the curriculum impact on the recruitment and retention?
9. How do service courses impact recruitment?
10. Importance of retention of math teachers.
11. Perception of attrition (at a personal level, in a classroom setting, in the design of a curriculum).
12. How to address adequately the transition into tertiary mathematics. Examine the role of the image of mathematics and mathematicians in transition.
13. Challenging the assumed language and metaphors (recruitment, attrition, retention). Issues of communication.
14. What kind of support systems are available (peer group, faculty, online, etc.)? For students? For teachers? For individuals or developing communities?
15. What is the impact of external policies?
16. How do external patterns of demographics impact on recruitment and attrition?
17. Can we "model" recruitment and retention? What are the appropriate measures and parameters?

It was not at all clear how one would collect evidence to address most issues raised in this list.

For example, what evidence would we look for regarding (13) above? We can look at how the terms are defined in the current studies, and whether there is a consistency (or lack of it) in how the terms are used. We might consider whether some studies draw policy recommendations from data that would shift if alternative definitions of "retention" and "success" were used. So this might be probing the theoretical framework used in the studies we find.

Moreover, we do not have precise definitions for the crucial terms "recruitment," "retention," nor "attrition." Without this, we might not start building any kind of framework within which we could study these phenomena using solid evidence.

Next, we discuss some emerging themes. Within each, we try to provide evidence, or suggest what kind of evidence might be desired.

MATH IN 21ST CENTURY

The emergence of biology as a major "user" of mathematics is perhaps the most recent of major factors that have shaped and reshaped mathematics and its applications. Advances and discoveries in biology (such as the human genome project, stem cell research, or cloning), and the impact on all life due to global climate changes, have contributed a great deal to a surge in interest in biological sciences. Advances in the area of bio-based fuels (and the general shift towards a bio-based economy) further contribute to the number of front-page news stories about biology and biological sciences, cementing, in public eyes, their present dominant position as *the* fields of scientific research. According to Statistics Canada, biology was the most popular field of study for doctoral students in 2004/2005. Of about 4000 students who earned their doctorates in that period, 21% were in biological sciences (CBC, 2008; King, 2008).

J. E. Cohen (2004) argues that it is mathematics that researchers in all areas of biological sciences are turning towards.

> *Mathematics broadly interpreted is a more general microscope. It can reveal otherwise invisible worlds in all kinds of data […] For example, computed tomography can reveal a cross-section of a human head from the density of X-ray beams without ever opening the head, by using the Radon transform to infer the densities of materials at each location within the head […] Today's biologists increasingly recognize that appropriate mathematics can help interpret any kind of data. In this sense, mathematics is biology's next microscope, only better.* (¶3)

Thus, what might work well in terms of attracting students into mathematics is the promotion of mathematics and statistics, or the mathematical sciences, as highly applied disciplines. Of course, we can also continue to benefit from continuing research in algebra or complex analysis, as well as logic and history of mathematics, but this will depend on continuing to attract students with these aspects of mathematics. However, given the breadth of the mathematical sciences, and the limited resources of some institutions, different institutions will specialize in distinct areas within mathematics and statistics and their applications, while offering a curriculum to our students that encourages students to be broad and flexible in their appreciation and skills in mathematics and statistics.

Although the idea of "selling" mathematics as an applied discipline seems to be fairly straightforward, it is not – it requires university mathematics departments to redefine their understanding and curriculum implementation of "applied mathematics." As an illustration: research in mathematics and biology that uses mathematics in a significant way and contributes to new knowledge in biology (but not visibly to new knowledge in mathematics) may not be recognized as a PhD thesis in mathematics.

As a second illustration, consider the connections between mathematics and mathematics education. Are these connections a form of applied mathematics? Some of the toughest and most interesting questions about mathematics concern the learning and teaching of mathematics, and these questions can be motivating to students, particularly given the significant fraction of undergraduate majors who are preparing for careers as teachers of mathematics. In what ways is research in mathematics education something that can be supported within a mathematics and statistics program, for example, so that it could be the core of a PhD thesis in mathematics?

The highest levels of education are not the only ones requiring changes. Appropriate education will have to be delivered throughout the whole educational system: "*Educating the next generation of scientists will require early emphasis on quantitative skills in primary and secondary schools and more opportunities for training in both biology and mathematics at undergraduate, graduate, and postdoctoral levels*" (Committee, 2003).

VIEWS AND BELIEFS ON MATHEMATICS AND SCIENCE

We could easily argue that, in our science-based society, mathematics indeed plays a very important role, and that it is not mathematics, nor its many applications and uses, that are on the decline. What seems to be on the decline is the interest in studying mathematics (and/or pursuing mathematics as a career option) among today's young people in the so-called Western world. Lack of motivation for (and, in some cases, negative experiences with) studying science and mathematics among elementary and high school students and their views of scientists and mathematicians, coupled with their parents' (sometimes quite unfavourable) opinions and beliefs about science and mathematics as careers, create a somewhat skewed image of mathematics and its role in society that might be quite difficult to modify.

As reported in USA Today, "*only half of children in grades 6 to 12 say that understanding sciences and having strong math skills are essential for them to succeed in life after high school*" (Feller, 2006). About 70% of parents polled stated that they believed their children are getting the right amount of science and mathematics. Although parents believe that, in principle, mathematics and science education are important (62% of parents said that it is crucial for most of today's students to learn high-level math, like advanced algebra and calculus), when it comes to their own children, they view it quite differently – only 32% said that their child's school should teach more math and science (Feller, 2006).

In 2005, as part of Einstein Year, Science Learning Centre in London, England, surveyed about 11,000 students aged 11-15 for their views on science and scientists. According to the survey, around 70% of students polled said they did not picture scientists as "*normal young and attractive men and women.*" And although they believe that science is important (around 80% agreed that scientists did "*very important work*" and 70% thought they worked "*creatively and imaginatively*"), very few students think that they will pursue science as a career (BBC, 2006).

Reasons that some students articulated, such as "*because you would constantly be depressed and tired and not have time for family*" or "*because they all wear big glasses and white coats and I am female,*" indicate that one of the serious issues that need to be dealt with in promotion of mathematics is the overall image of science and scientists.

A RECENT SURVEY IN CANADA.

The survey (Fenwick-Sehl, Fioroni, & Lovric, 2009) shows that recruitment and retention are not at the top of the agenda in mathematics departments across the country. However, we found that there are activities, organized at every university represented in our survey, that could be interpreted as efforts aimed at increasing numbers of mathematics students. The case of Brock University (Muller, Buteau, Klincsik, Perjési-Hámori, & Sárvári, 2009) shows that systematic, carefully designed, long-term efforts could be quite successful. Can we do better? Say, if we double the efforts at recruitment? The answer is not at all straightforward, as it depends on many complexities outside mathematics that characterize the social and cultural landscape in Canada (and in the whole world) in the first decade of the 21st century.

SHIFT IN PARADIGM THROUGH A SHIFT IN THE LANGUAGE?

Towards the end of the session, the discussion moved to a more philosophical standpoint. What if there was a shift in the language we use to talk about recruitment, retention and attrition? According to certain people, those terms seem to carry a negative connotation; mathematics can almost be compared to military conscription. Instead of *recruitment*, what if we talked about *invitation*, welcoming? Could *retention* be thought of as generating *engagement* and *well-being*? And what if the *attrition* of a mathematics student was simply seen as a stepping out to work on problems outside of mathematics – with a "*welcome to return*"?

Could students have the math that they understood as valuable to them? A lot of the aspects we tried to address during the session could be formulated in a more emancipated manner. In fact, this shift of the language cannot happen without a shift of paradigm. Despite obvious practical challenges, school administrators, legislators, and teachers could benefit from an exploration in this direction.

We tend to make mathematics courses depend on prerequisites that could be dispensed with. For example, a graduate student in biology needs to create a working model whose assumptions come from the literature, and whose output matches the data from his experiment. The learner has not done a modelling course, or a numerical analysis course, (but has done basic calculus, systems of linear equations, recursion, and a statistics course). What mathematics courses do we offer for such people who are based in other disciplines? Is the learner welcome to return?

In social science, it is assumed that someone with some "maturity" (random prior experience) can enrol in an upper level course, and learn/engage. Programs sometimes do this, for example in graduate mathematical finance programs which mix students with business degrees and students with mathematics degrees. Unfortunately, in the structure of the current mathematics curriculum there can be almost unrelated prerequisites, such as analysis for doing discrete models. Our experience is that this kind of situation negatively affects the way students approach mathematics and engage in mathematics studies.

How this issue is presented in undergraduate mathematics programs has an impact on future teachers and on their students preparing to enter post-secondary programs.

CONSEQUENCES OF THIS SHIFT

From this shift, we focused portions of our discussion on attracting students to continue with mathematics, and attracting people to return to mathematics. We discussed what attracted us to engage in mathematics and mathematics education, and connected these experiences to changes in pedagogy as well as content focus in university mathematics courses. We discussed classes in which students are active and are doing mathematics related to issues they cared about. We recognized this would cut across both traditional "service courses" and courses for majors. We considered the role of student self-efficacy and the beliefs of university instructors. When do students see themselves as "agents" who generate mathematical questions, rather than as learners following road maps in an alien landscape? Through examples, we considered what pedagogy and curriculum would help our students become empowered to use mathematics to address the problems that are meaningful to them, in their environment.

After discussion in subgroups, one group reported on some of these shared themes as follows: We discussed ideas for making the first year of the undergraduate program a more positive experience for mathematics students. These included: establishing learning communities (both in-person and online in nature, with self-selected or assigned groups to expedite the process and ensure full participation); clearly advertising tutoring services; assigning effective instructors to first-year courses within the department; advocating for smaller class sizes and student-instructor ratios for first-year classes; creating a more personalized curriculum that allows for student voice and choice (e.g., math projects, elective courses); placing a greater emphasis on technology and its applications in the math curriculum (e.g., modelling, spreadsheets, data analysis, simulations, computer algebra systems (CAS)); increasing communication between professors and their students (e.g., email chat/discussion forums, regular meetings with professors or TAs, more personalized assessment that provides direction in areas of improvement/focus); integrating curriculum so that it involves combinations of mathematics and other disciplines; celebrating mathematics projects/achievement in some form of public forum (e.g., math fairs/talks/newsletters/

websites); organizing/mandating co-op placements where mathematics students volunteer in the community to see first-hand the application of mathematics in the real world (or, video-taping guest speakers/professionals at the university and building a web-based collection of these talks); and, improving the tracking system of student achievement across courses to recognize academic issues/problems before they escalate.

This vision of students moving on from and returning to experiences labelled "mathematics," and being supported to add mathematical abilities when the need arises, suggests a much wider range of courses and pedagogies for our curriculum. Courses which mix graduate students in mathematics with graduate students in other disciplines are becoming more common: business and mathematics majors in mathematical finance, psychology and statistics majors in an applied statistics course, mathematics and computer science majors in computational geometry, mathematics graduate students and education graduate students in history and philosophy of mathematics. Flexible prerequisites, project work with students bringing different strengths to the problem solving, all come up as positive options.

We also discussed what preparation would help teachers of post-secondary mathematics to support such changes in courses, pedagogy, and program design. The references below by Bass (2003) and Chan (2003) are relevant here.

These are themes and possible changes that we recognized run through the development of mathematics and statistics education from K-12, and into graduate schools and the support for faculty members on through their careers. We also recognized the obstacles of finances and the connected issues of class size and time that make it hard to shift from these imagined possibilities into practice.

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CAN WE BE THANKFUL FOR MATHEMATICS?: MATHEMATICAL THINKING AND ABORIGINAL PEOPLES

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Cynthia Nicol
Randall Pyke
Gail Russell
Mary Stordy
Jennifer Thom

Onen sewathonsiyost kentsyohkwa
Now you-your-ears-good-make it-is-a-group
Now listen, this assembled group

kenh nikarihwesha
this it-is-a-matter-of-a-certain-size
for a little while

shonkwawi ne Shonkwaya'tison
he-to-us-has-given, he-our-bodies-formed
to what the Creator has given us

ne Ohenton Karihwatekwen
the before it-is-a-matter-comes
the words that come before all else

ne Kanonwaraton'tshera
the it-mind-remains-in-ness
the Giving of Thanks.

Can we be thankful for mathematics? This is the question we indirectly addressed in the Working Group on Indigenous peoples and mathematics education. As facilitators for the group we saw our role as inviting people into a conversation we have been having together (on and off) for a number of years. In the manner of all good conversations, over the course of our sessions, the focus shifted and flowed from one subject to another. It opened up spaces where we agree and others where we are still trying to come to consensus and understand each other; as always, much more was said in the spaces and relationships between the words we shared than in what was articulated aloud. Each of us took away ideas that we have continued to consider, to work with, to implement. How then to reproduce here a summary that does justice to this ongoing and always unfinished process? How can we represent the rich complexity that emerges when we take time to be together and consider things with good heart?

We have decided that all we can do, all we can ever do, is continue the conversation. As such, this paper is presented in three voices, each singular but part of much bigger whole. We hope that in mirroring the process we used in the workshop, we will invite more people to continue similar conversations, and to consider the relevant issues and complexities in their own contexts.

EDWARD

I have been thinking of the issues around the mathematical education of Indigenous people since I entered university, over twenty-five years ago. In that time I suppose I have learned something about the topic, although it doesn't feel that way. However, one thing I have learned for sure is the value of Indigenous methods in the study of Indigenous issues. As Indigenous people we learn, for example, the importance of opening and closing our gatherings with prayer, or to be more precise in the case of the Rotinonhsonni (Iroquois) tradition, the Opening Address, also known as the Thanksgiving Address.

(Eber Hampton, former president of First Nations University, once clarified the difference between a prayer and an address for me. He said he was at a gathering where an elder opened with prayer in Cree. At a break, one of the participants apologized to the elder for not understanding what he said. "That's OK," said the elder, "I wasn't talking to you anyway." That is a prayer, addressed to the Creator or other spiritual being. An address is addressed to the ordinary individuals gathered.)

Our tradition tells us that the Thanksgiving address was not always with us, but that at a time in the distant past, the people had forgotten the lessons the Creator had given us at the time of creation. We had taken to abusing the environment, to not living sustainably, and were suffering as a result. The Creator took pity on us, and sent a messenger known as The Fatherless Man to help us relearn the lessons of the Creator, to help us live once again in harmony and balance with creation. The oral tradition that The Fatherless Man brought us is the Opening Address.

The Opening Address tells us to be mindful of, or thankful for, all the good things in creation. It is structured in a way that facilitates memorization and oral transmission, as a linear narrative starting with Our Mother, the Earth, and going upwards, through the waters, the fish in the waters, the low plants, and so on, through the sun, the moon, the stars "on heaven," and then the various beings in heaven. Every good thing in creation has its place in the sequence from low to high, with one single exception: the people, who are mentioned first, that they may be at peace.

An interesting feature of the Opening Address is that its exact form is a personal matter: everyone who carries the Opening Address probably has a different version of it. The identities of the plants and animals and so on which are remembered, and the memorable features of each, vary from one reciter to another, depending on his [note to the reader: it is *his*, as the Opening Address is only delivered by men] state of knowledge. The changing nature of the Opening Address is a reflection, I believe, of the changing and imperfect nature of human knowledge, which is a result of our having forgotten the Original Instructions. We are still in a process of reacquiring the knowledge that was lost (or in some unfortunate cases, we are in the process of losing it again). From an academic point of view, the possibility of reacquiring knowledge is good news: it tells us that in our tradition, knowledge can be sought after and gained. Knowledge is not static in our tradition, as it is in some fundamentalist traditions. Not everything worth knowing can be found in our Creation Story, or in any story; we are all in the process of regaining the knowledge that was given to us at the time of creation and was then lost.

That point was first driven home to me when I heard a traditional Mohawk midwife named Katsi Cook speaking at a conference at the University of Toronto. She said that some think it's a tragedy when so much of our traditional knowledge is being lost. She, on the other hand, thought it was unfortunate but not a tragedy, as that knowledge is not necessarily lost forever. The knowledge we have of medicinal plants, for example, was not always with us, so it was acquired at some point, and if lost, can be acquired again. My interpretation of those comments is that it's unfortunate that we are so stupid that we keep losing things, but it's not a tragedy because no knowledge is ever lost forever.

Given that context, it is a natural question to ask where mathematics fits in this whole scheme, a question I have pondered since I began learning the Opening Address many years ago. (I consider it my prerogative as an Indigenous person to forget the exact numbers when convenient.) I thought it might make sense to put mathematics somewhere around the stars, as it is abstract and untouchable. It probably doesn't fit in heaven because that's where spiritual entities reside, and mathematical knowledge seems to me to be grounded in real-world affairs such as counting, measuring, locating, designing, playing, and explaining, although some might argue that mathematics is spiritual and otherworldly and "Platonic." Perhaps a case could be made for situating mathematics with the people, as it is a human activity, but I think biology would best go with the fish, plants, and animals, and chemistry with the earth, waters, breezes, and thunders, and astronomy with the sun, moon, and stars, and so on, so mathematics should go somewhere after the people but before the entities in heaven. Where, exactly, I'm not sure, but somewhere in there.

Which is why Dawn's simple and direct question was so surprising and enlightening to me: Can we (should we, must we) be thankful for mathematics? Perhaps mathematics really does not belong in there at all. Perhaps mathematics is not part of the Indigenous knowledge held and organized by the Opening Address. On the other hand, the Opening Address is supposed to keep us mindful of all the good things in creation. Mathematics is as incorporeal as the breezes, and as unreachable as the stars, but perhaps the problem is not the abstract nature of the field, but that it is not a *good* thing. I don't mean to suggest that it is a *bad* thing, but rather that it may be *no*thing – to us. Perhaps it is not really part of our Indigenous knowledge, the Original Instructions that the Creator gave us at the beginning of time. Perhaps the desire to fit mathematics into some Indigenous knowledge systems is nonsensical and ultimately in vain.

When I started on the path of thinking about Indigenous people and mathematics twenty-five years ago, it was with the arrogant assumption that the issues would be straightforward and easily solved; it's just that no one had tried hard enough before or had the right combination of knowledge. In that twenty-five years, I have learned one thing for certain (other than the value of Indigenous methods in Indigenous research), and that is humility. Every time I think I have an answer, it does not work, and I end up questioning my assumptions. From doing the same thing that everyone else is doing, only harder, to choosing "culturally appropriate" examples, to ethnomathematics, to Indigenous pedagogical methods, nothing has had the impact that I hoped it would have; nothing seems capable of turning the tide of poor achievement (at least, not on a national scale); nothing has engaged and seized the attention of Indigenous students in the way that I hoped it would.

Of course, it could be just my poor implementation of trying harder and choosing the right examples and methods that could be at fault. And indeed, there have been numerous successes in the field, some of which we are going to hear about in this working group. But overall, I feel that something is still missing, that there is some fundamental incompatibility that needs to be addressed before we can go all out with Indigenous mathematics and guarantee to our young people the opportunity to achieve the success that they deserve.

So I keep trying. Humility has taught me to remove myself from the picture, to the extent that I can, and to seek from other Indigenous people, and from Indigenous communities, to learn first what it is they wish to accomplish by studying mathematics, and second, what I might be able to do to help. Some, I find, want to go to university to learn to be doctors, lawyers, or engineers, in which case we can at least imagine that the "what" of mathematics education has been settled and we can concentrate on the how. But in other cases, people want to know how they can go back to the traditional ways of their ancestors, and in those cases I can't imagine what mathematics might even be helpful to them, as opposed to actively harmful.

I once met a man named Philip Wolverine who lived a traditional life, making with his own two hands everything that he needed. Philip told me that he had never turned a key in his long life. He had never driven a vehicle, operated heavy machinery, locked his house, or started a generator. When I offered him the opportunity to turn the key in the vehicle I was driving, he gave me a smile and gracefully declined the opportunity.

Why do some feel that mathematics is so important? I would argue that it's not intrinsically important; rather, it is important because we have built a world around us in which mathematics is important. Why we have done this is a reasonable question, one for which we should have an answer ready for Indigenous people who demand to know before they buy in to the project. What do we stand to gain and lose when we learn mathematics?

LISA

What messages do we need to hear? What are the complexities in mathematics education for Aboriginal children? How might we more fully come to appreciate these complexities and draw from them to inform our work? These are some of the questions we sought to explore as we gathered. While it is appropriate to reflect on why we teach mathematics, many Aboriginal communities are not asking this question because, quite simply put, they see it as essential that young Aboriginal children learn mathematics. As Canada's Aboriginal communities begin to re-establish their self-government and self-determination, they are confronted with the need to develop sustainable economies and manage natural resources while negotiating treaty rights and land claims within the context of a growing population and insufficient infrastructure. Aboriginal leaders are looking to the younger generations to acquire the knowledge and skills to address these challenges. Such capacity-building requires that young community members have education in related fields, especially in mathematics and science, but currently few Aboriginal students are choosing to pursue studies in these essential skill areas. This disengagement often begins as early as elementary school. There is a sense of urgency to examine the complexities related to this issue and our working group acknowledged that more awareness and a deeper understanding of the issues was definitely needed. I come to this conversation after over 10 years teaching mathematics in an Aboriginal community and another 5 years researching this topic in my academic career. I have continually struggled with way to negotiate the space between school-based mathematics and community knowledge and values. I have chosen here to focus on the *how*, as the *what*, in my research and work context, has been firmly established.

As part of our discussions, I shared a model from my own doctoral work (Lunney Borden, 2010) that outlines the themes that emerged during this research and highlights areas of potential tensions for Mi'kmaw learners with mathematics education. These themes were collaboratively developed with teachers in two Mi'kmaw schools over a period of one year through regular conversations regarding the challenges and complexities of teaching math in a Mi'kmaw context. Four key areas of attention emerged as themes: 1) the need to learn from Mi'kmaw language, 2) the importance of attending to value differences between Mi'kmaw concepts of mathematics and school-based mathematics, 3) the importance of attending to ways of learning and knowing, and 4) the significance of making ethnomathematical connections for students. Within each of these categories, teachers in the study identified conflicts that arise when worldviews collide and identified potential strategies to address these tensions (see Figure 1).

While the intent of this particular part of the session was to work through the entire model, our discussions surrounding the connection between language and mathematics took up the entire time allotted for this part. As such, I will briefly outline the findings for themes 2, 3, and 4 before engaging in a more thorough discussion of language and the questions that emerged during the working group session.

The importance of attending to conflicting values between school-based approaches to mathematics and Mi'kmaw ways of reasoning about mathematical questions were frequent points of conversation during the study. These value differences can provide teachers with insight that may enable them to anticipate points where two worldviews might bump up against each other and cause students to be conflicted and possibly disengage. These included a conflict between privileging numerical reasoning in mathematics curriculum over spatial reasoning more commonly used within the community and embedded in the language. Other Mi'kmaw approaches to mathematics identified included the common use of estimation, the value of playing with number, and the connection to necessity and intention. Many of these were noted as often being absent in school-based mathematics.

Similarly, during the research discussions, participants raised questions about children's preferred ways of learning and how they might influence the design of tasks for learning mathematics. It is important to avoid over-generalizations about Aboriginal learning styles as "*Aboriginal children [are] diverse learners. They do not have a single homogenous learning style*" (Battiste, 2002, p. 16). There is as much diversity of learning styles within a Mi'kmaw class as there is in any class, so there cannot be a one-size-fits-all approach. That being said, some of the discussions focused on traditional apprenticeship models and mastery approaches to learning, as well as those related to visual-spatial styles of learning and hands-on learning. Other observations pointed to the role of gestures and embodied cognition. It was argued that understanding these different approaches to learning can provide teachers with additional strategies that can be employed in mathematics classrooms.

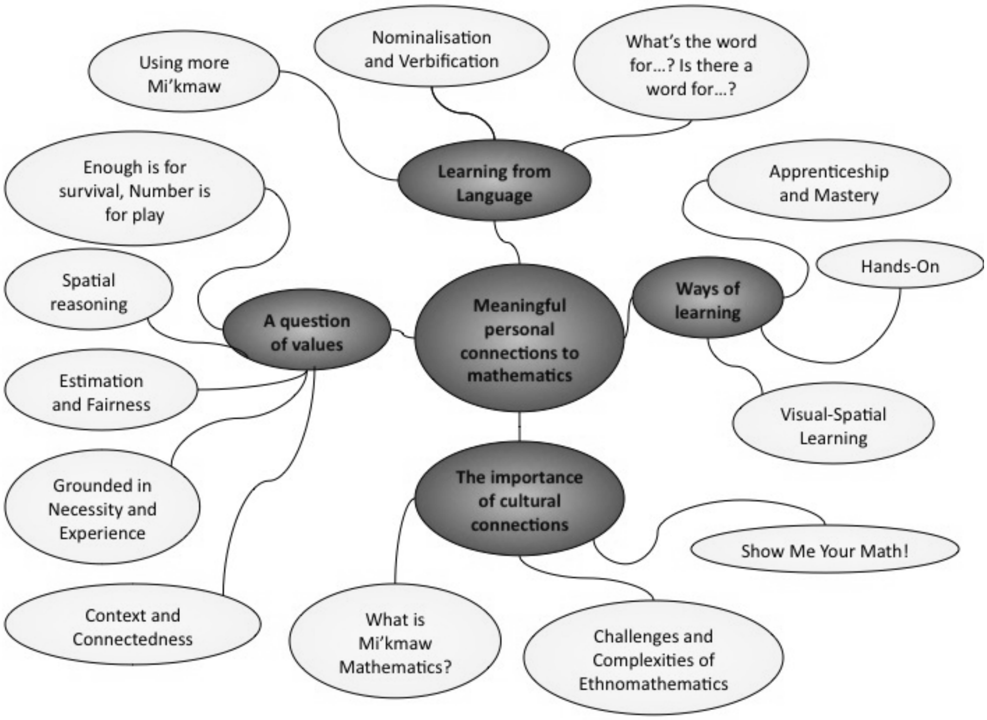

Figure 1. From Lunney Borden (2010)

The importance of making connections to the mathematical thinking that is, and has always been, evident in the Mi'kmaw community was seen as an important part of transforming mathematics education. The research group explored some of the evidence of mathematical thinking that exist within the Mi'kmaw community daily practices and recognized that there is far more to be done in this area. We also discussed some of the things that have been done in both schools to strengthen the connection between school-based mathematics and community cultural and everyday practices. As evidenced in the very popular Show Me Your Math Event (described later in this paper), mathematics may be used as a venue to reclaim what has been lost, to reconnect with traditional knowledge, and to enable students to see that their ancestors did use reasoning that is evident in modern day mathematics.

The need to learn from Mi'kmaw language was the most pronounced theme in the doctoral research. Barton (2008) has argued that "*a proper understanding of the link between language and mathematics may be the key to finally throwing off the shadow of imperialism and colonialisation that continues to haunt education for indigenous groups*" (p. 9). Participants in the study felt strongly that language defines worldview and thus, by understanding Mi'kmaw language structures, teachers can gain greater insight into the ways of thinking of their students and be aware of potential tensions.

Research conversations related to language focused on three main ideas. Firstly, there was a call to include more Mi'kmaw language in the mathematics classroom, with one group in particular stressing the importance of reclaiming mathematical words and supporting Mi'kmaw-speaking teachers to develop a lexicon of words that could be used in their classes. Secondly, there emerged the notion that a great deal can be learned from studying the structure of the Mi'kmaw language even for non-speakers. In particular, this notion included a multi-layered discussion about what teachers, both speakers and non-speakers, can learn by asking questions such as "*What is the word for…?*" or "*Is there a word for…?*" Thirdly, a closely related idea focused on investigating discourse patterns and the ways in which the Mi'kmaw language is structured. Most notably, a change in discourse patterns to reflect Mi'kmaw verb-based grammar structures, referred to as "verbification," is exemplified as a strategy that holds promise for supporting Mi'kmaw students learning mathematics.

As I shared the model with the working group, the majority of the conversation focused primarily on the role of language. With many people in the working group engaged in a variety of linguistic contexts, the connection between language and mathematics learning seemed to strike a chord and raised a number of key questions.

The merits of indigenous language instruction were noted, yet the complexities were also acknowledged. The majority of teachers, even those who speak the indigenous language, have been educated in a mainstream system and have learned mainstream mathematics. They may not know the words for mathematical concepts in their own language, and furthermore, the words may not exist.

Understanding how mathematical concepts are described, or not described, in an indigenous language can be very informative to mathematics teachers. I shared several stories from my own doctoral study, such as the non-existence of a direct translation for the term *flat* in Mi'kmaw. During my research study, I asked on numerous occasions if there is a word for *flat* and had attempted to generate scenarios whereby we would need to use the word *flat*. I asked about a flat tire but I was told that in Mi'kmaw we would say it was losing air or out of air. I asked about the bottom of a basket, suggesting it was flat, but I was told that it was the bottom; it had to be flat so that it does not roll around. It allows the basket to sit still. I asked about calm water but was told that the word used to describe calm water has embedded in it the potential to be rough, thus not flat. Understanding that there is no word for *flat* enabled the study participants to think differently about how we describe a flat surface in mathematics. This example highlights some of the taken-for-granted assumptions in mathematics education.

I also shared with the working group an interesting connection to the above notion that occurred for me during a grade 3 lesson on prisms and pyramids that I co-facilitated with a teacher participant in the study. As we sat on carpet with students and asked them to say one thing about the prism that was being passed around, one young girl placed the prism on the floor and stated "*It can sit still!*" Instantly I began to get excited by her answer. It made perfect sense that she would not talk about the flatness of the face but rather its usefulness. This connects directly to the relational way in which Mi'kmaw language is used and constructed. When I later recounted this story during an ad hoc session at the Canadian Mathematics Education Study Group Conference in Sherbrooke, Quebec (May 2008), Walter Whitely (personal communication) mentioned to me that the word polyhedron actually is derived from the Greek word *hedron* which means "seat," and polyhedron means many seats or many ways to sit. Thinking about how our Aboriginal students are speaking about, or not speaking about concepts informally, can help teachers to think about how best to approach some of these concepts.

I also shared with the working group another key idea about the structure of the Mi'kmaw language and its potential impact on mathematics learning. Mi'kmaw is a verb-based language. In Mi'kmaw, words for shapes and numbers act as verbs. Other indigenous languages including Maori share a similar grammatical structure (Barton, 2008). During one particular session in one of the two schools in the study, Richard, a technology teacher and Mi'kmaw language expert shared with the group some ideas about the concept of *straight*. He explained that the word *pekaq* means "*it goes straight*." There is a sense of motion embedded in the word. Similarly *paktaqtek* is a word to describe something that is straight such as a fence. He explained that here "*is a sense of motion from here to the other end* – pektaqtek *[it goes straight]*." Similarly, there are words like *kiniskwikiaq* that translates to "*it is forming into a point*" or "*it is coming to a point*." These exemplify the way verbs are inherent in Mi'kmaw descriptions of mathematical concepts.

The role of using verbs in mathematics teaching is something I had become curious about prior to beginning this project. I had noted in my own teaching, a transition from asking noun-based questions such as "*What is the slope?*" to asking verb-based questions such as "*How is the graph changing?*" I am certain that I did this quite unconsciously initially although I am also sure that I was listening to the way students were talking and tried to model my language with similar grammar structures. It was only upon reflection that I realized I was changing my discourse to be more verb-based than noun-based. I found in my own experience that students often understood better when I used more verbs and when we talked about how things were changing, moving, and so on.

Pimm and Wagner (2003) claim that a feature of written mathematical discourse is nominalisation – "*actions and processes being turned into nouns*" (p. 163). Mathematics as taught in most schools has a tendency toward noun phrases and turns even processes such as multiplication, addition, and square root into things (Schleppegrell, 2007). The dominance of English in school-based mathematics results in this objectifying tendency. "*We talk of mathematical objects because that is what the English language makes available for talking, but it is just a way of talking*" (Barton, 2008, p. 127). What would happen if we talked differently in mathematics? What would happen if we drew upon the grammar structures of Mi'kmaw (or other Aboriginal languages) instead of English? How might this change the experience of mathematics learning for Aboriginal students?

This examination of language and the role it plays in mathematics learning for Aboriginal students occupied a good portion of our working group time. Again, the complexity of the task emerged as we acknowledged that in order to address the needs of Aboriginal learners, teachers must come to truly understand the communities, the language, the various cultural and linguistic factors that influence students' ways of knowing. There is no simple solution. For many indigenous students, it is quite likely that the *how* of teaching/learning mathematics may be firmly rooted in their language structures.

EDWARD

I believe "nominalisation" in mathematics is the result of a deliberate attempt to remove time from mathematical discourse in Greek mathematics, probably in response to Plato's philosophical theories about eternal forms, and in response to Zeno's paradoxes of motion. So for example, the "kinetic" definition of an infinite set, "*for each element in the set there is a next element not already noted*" becomes the "static" definition, "*a set is infinite if it can be put into one-to-one correspondence with a proper subset of itself.*" With a little effort, one can see that the two definitions are equivalent. The kinetic definition, however, seems direct and personal, while the static definition seems oblique and impersonal.

Nominalisation may lead to tighter theories which are less vulnerable to paradox, but the process may not be helpful in mathematics education, particularly in Indigenous mathematics education, as Lisa has noted. Identifying nominalisation may however give us a way forward in Indigenous mathematics education: to "roll back" the mathematics of European peoples, back to a time when they could be considered "Indigenous." The meaning of polyhedron is an example of Indigenous, verb-like thinking, I feel. A systematic study of the Indigenous origins of Western mathematics may demonstrate that modern Indigenous and Western mathematics have common roots which can be exploited to give a pathway for Indigenous people into Western mathematics.

(A similar process may help English speakers learn Indigenous languages: rolling back English until it has regained features that seem strange in Indigenous languages. For example, going back to "you" and "thou" could help introduce the difference between second person plural and singular which is lacking in modern English but is present in most Indigenous languages.)

Another thought I have along similar lines is to mine the oft-maligned and probably under-appreciated mathematics of the Romans, who are sometimes considered to have been "too practical" to fuss too much with much of what we now call mathematics. The Romans seem to have done pretty well for themselves, and I can think of modern groups who are just as practical as the Romans were reputed to be.

DAWN

I come to this conversation from within the context of a university and 18 years of working with Aboriginal peoples and communities. As I have learned (and I have learned more than I imagined possible), my position has always been to find meeting places. Places where – for whatever reason – we can sit together, be together, and think together, even though we may not necessarily agree or reach the same conclusions. In the Native Access to Engineering Program at Concordia University, the meeting place opened up out of expressed desires of Aboriginal communities for improved math and science outcomes for their young people, and the engineering profession's desire to more adequately reflect Canadian demographics (hey, you have to start somewhere). Now as I work on my doctorate, I find myself wondering how meeting places open up, and where they are located. I am uncertain, as Ed says, whether mathematics is a meeting place in and of itself, but I think it plays a distinct role in other meeting places. The thing about meeting places is, once you been in them, once you have sat with the people and ideas there, you leave changed; everybody leaves changed.

One of my colleagues is fond of asking "*Why do we teach mathematics anyway?*" It is a question she uses to poke at people, to make them stop and think, to make them question assumptions about teaching and learning in mathematics with which they are – perhaps – a bit too comfortable. It is a good question to consider in relation to any subject area – why is it we teach anything? Why is it we choose to teach the things we teach? I recently asked this question to a group of preservice science teachers who were about to embark on their final teaching experience before graduation. It was met with stunned silence. And that, I told them, was okay. As beginning teachers caught between the tensions of surviving practica, completing courses, looking for work, and all of life's other stuff, their focus is resolutely on "What?" questions: "*What do I teach?*" "*What do I do?*" even, "*What do I wear?*" Sometimes all of us get caught up in "What?", but "Why?" is important, and in relationship to Aboriginal peoples and provincial/territorial curriculum requirements to integrate/infuse/include Indigenous perspectives across curricula for all students, "Why?" questions might begin to shift underlying philosophies of education for all of us.

So, why do we teach mathematics anyway? Our provincial/territorial governments tell us that mathematics is important. It is a key skill. It "*is essential for everyday living and in the workplace*" (Alberta Education, n.d.). It helps prepare people to become contributing members of the knowledge economy; "*students need a strong grounding in mathematics to* *meet the challenges of the 21st century and to be successful in their futures*" (Alberta Education, n.d.). None of that is wrong[1].

So, why do we teach mathematics anyway? Because, in some ways we know, we sense, our experience, our *sensus communis* (Gadamer, 1989) tells us, that, yes, people do use mathematics, they do think mathematically in a number of ways. It may not be the mathematics of school, but it is a mathematics of usefulness, practical knowledge, *phronesis* (Gadamer, 1989); as Bishop (1988) posits everyone practices mathematics through counting, measuring, locating, designing, playing, and explaining.

So why do we teach mathematics anyway? Because we want our children, our young people, to know what is possible, what they might become, the things they might choose to be. We want to support them in becoming. Using Aztec traditions as an illustrative example, Cajete (1994; 2001) says the primary impetus for education from Indigenous perspectives is to support young people in finding face, heart, and foundation; unique qualities of self, passion, vocation and the ability to express who they are in order to become complete people. It is a process that, like school curriculum, has key content: the ways of life and worldviews of the community in question. It is taught by example, occurs over a lifetime, and is embodied in community Elders. Learners (regardless of age) are supported in their engagement with the content, and are encouraged to what Maxine Greene (2004) would call *wide-awakeness*, a deep awareness of what is going on around them, what they themselves are doing, and how these things interact.

Interactions and relationships are key to this philosophy of education. In fact, one of Cajete's (1999) primary critiques of Western educational philosophy is its fragmentation of the world, which abstracts content from lived experience, people from nature, and learning from community; in a sense abstracting all relationship and subjectivity from experience and focusing solely on a (false) objective view[2].

Jardine (2010) offers a similar critique saying "*We [frequently] surround ourselves with things that don't go anywhere*." He sees the lack of connection and relationship in the classrooms where he conducts research, and in his undergraduate university classes where preservice elementary teachers do not understand or see addition and subtraction as inverse operations that are related to each other (let alone anything else). For these teachers, mathematics is not a field but a set of isolated facts and skills to be memorized.

Friesen and Jardine (2009) suggest these limits arise out of a school system still defined by the tenets of Taylorism and the cult of efficiency, and exacerbated by a current focus on high stakes standardized testing. But instead of becoming tied down by these limits, Jardine, Friesen and, long-term collaborator, Clifford (in various publishing combinations) choose to examine how disciplines become expansive and abundant (see Jardine, Friesen, & Clifford, (2006)) when conceived of as "*living fields of knowledge*" (Friesen & Jardine, 2009, p. 149); literally places that students journey through and come to know in their complexity. They are not alone in suggesting that what we teach in schools should be imbued with life.

Egan (2008) shares their position, saying that teachers focus on informational pieces of learning that lead to rote learning "*at the expense of living knowledge and imaginative engagement with it*" (p. 21). He espouses imaginative education which accesses somatic, mythic, romantic, philosophic, and ironic understandings. While he does not speak

[1] Although you do get the sense that there is a rather specific definition of success hidden in there somewhere.

[2] Not that there is not a place for this approach, but that it is overly privileged.

specifically of fields, his levels of understandings map onto each other, each providing a deeper initiation into the living world, from affective bodily senses through to deep exploration of relationship, contradiction, and meaning, and how they are related to and inform each other.

Like Egan, Cajete (1994; 2005) speaks of enlivening science learning through multiple foundations for education – the Mythic, the Visionary, the Artistic, the Environmental, the Affective, the Communal and the Spiritual – which reflect a more complex, relational being in the world. He makes clear connections to field. Describing the approach for his own doctoral work, Cajete (1994) writes of going to Tewa[3] Elders for advice about how to explore commonalities across different Indigenous nations' conceptions of the purpose of learning/ education. They suggested he heed the saying "*Pin peyé obe*", or "*Look to the mountain.*" Within the context of the mesas of New Mexico this advice makes perfect sense; there, the broad view required for true understanding of the context (or field) and connections (or relationships) in which you reside is only possible from high ground.

Without using terms such as *field* or *landscape*, teachers I have interviewed have expressed their understanding of mathematics in a manner very similar to those concepts. In describing how their students cannot grasp the meaning of basic operations or where the operations apply, it seems to me they are saying the students have no (or little) ability to draw on the field's "*living inheritance*" (Friesen & Jardine, 2009, p. 150) in an appropriate way. Yet, these teachers desperately want their students to be able to draw on this inheritance appropriately. As one of them said, "*They need to be using more of what I refer to as their math skills. Where its more the* ***intuitive math****, to try and figure out how to set something up and solve it*" [Emphasis added] (Personal communication, March 7, 2010). At first glance "intuitive math" may seem like an odd choice of words, but given the context of our conversation, this teacher was actually referring to the ability to draw on prior knowledge in the experienced, field(ed) ways described by Freisen and Jardine (2009) and Jardine, Friesen, and Clifford (2006).

The power of landscape, as a means for developing understanding (and wisdom) is also discussed by Basso (1996), "*familiar places are experienced as inherently meaningful, their significance and value being found to reside in the form and arrangement of their observable characteristics*" (p. 108). He describes Apache relationship to the physical landscape they inhabit, and how long-term experience and knowledge of the stories mapped onto physical space can be accessed in an almost *prescient* manner by Elders. In other words, Elders have cultivated their knowledge of the fields in which they live to such a degree that it is immediately accessible to them. Moreover, they remember what has occurred in such a way that they see new connections, new relationships – and hence how specific prior knowledge does or does not apply to them – as they emerge. Their wisdom is literally fielded; they walk through it in a physical, spiritual, emotional and mental sense.

As Basso (1996) underlines, the wisdom that resides with Apache Elders is only acquired after many, many years of mindful interaction with the land, stories and practices of the nation, but it begins at a very young age. And so, while there is a more spiritual sense of connection to the relationships expressed in Basso than most teachers would likely feel comfortable discussing with respect to schools in our current context, it is the cultivation of this sense of connection that Cajete (1994; 1999; 2001; 2005), Jardine, Friesen and Clifford (2006) and Egan (2008) espouse. It is only by inviting our students into this type of field, to experience for themselves the expansive relationships therein, by letting them find and map out the multiple paths that connect concepts, that we will provide them with the ability to

[3] Cajete is from the Tewa community in New Mexico.

come to know the landscape of a subject area. Allowing students to explore a field, a concept, an idea, lets them build maps of knowledge that at any point helps them to know where they are, where they are going, and the (likely) multiple paths for getting there.

I think this is why we teach mathematics, I think it is also why we can (much of the time) be thankful for mathematics as a field.

EDWARD

I think Dawn has pointed out another practical way to consider the issue of Indigenous people and mathematics, and that is to synthesize something new, to develop Indigenous mathematics and Indigenous mathematics education as a field. There is no reason why we have to limit ourselves to either traditional Indigenous ways of thinking or modern Western ways of thinking, but that we can develop a new third way which will satisfy the needs of modern Indigenous students better.

Developing such a third way is something none of us can do on our own, however, as the situation is complex and varied as one considers differences from one Indigenous community to another across the country. That emphasizes the need to periodically come together, to treat the issue as an ongoing conversation rather than a single problem to be solved. That in turn leads to the notion of professionalizing the discipline and developing its body of knowledge and experience. Many members of the working group made valuable contributions to the developing body of knowledge on Indigenous mathematics. The best that we can hope for is that the conversation will continue to be as fruitful long into the future.

CONCLUDING THOUGHTS

Our working group was a meeting place, we sat with each other and – over the course of our time together – engaged in challenging conversations that sometimes lead to moments of silence as the participants reflected on the enormous complexity of the task. Together we sat with the difficulties, and struggled to make sense of where to go from here and what might be the next steps. As facilitators we acknowledged that it is important for us to understand that there are no simple answers. We cannot simply rewrite some math questions using Aboriginal names or contexts and say that we have addressed the needs of Aboriginal students (or any of our students); there needs to be more substantive change. Yet, despite the challenges, doing nothing as an alternative would lead to nowhere at all.

We concluded with sharing some things that are happening in Aboriginal mathematics education across the country. Cynthia Nicol shared some of the work she has been doing with Aboriginal communities in BC. She explained how her research team is working with traditional stories from communities to make connections to mathematics learning. She shared a video developed by some students in Haida Gwaii that examined aspects of language and culture that had earned these students some accolades. She also shared that with another group of students in the city, this idea was far less popular as it did not connect to the experiences of Aboriginal youth growing up in Vancouver. For these students it was important to find a way to meaningfully connect mathematics with their experiences, which resulted in a service learning project for these students.

Dawn Wiseman shared some of the work she and Corinne Mount Pleasant-Jetté have undertaken in relation to Aboriginal mathematics and science education at the Native Access to Engineering Program. She explained how all of their initiatives – outreach, professional development, resources development, policy intervention – were based on valuing traditional ways of knowing, community ways of knowing and teachers' ways of knowing. Dawn shared examples of NAEP work that can be found at http://nativeaccess.com.

Lisa Lunney Borden shared examples of ethnomathematical work that has been done by Aboriginal students in Atlantic Canada as part of the Show Me Your Math (SMYM) Program that she and David Wagner have developed. Since 2006, over 1000 Aboriginal students have been involved in exploring the mathematics in their everyday contexts and presenting their findings in the form of Math Fair projects. Lisa shared some examples of student work that can be found on the website http://schools.fnhelp.com/math/showmeyourmath. She explained that participants have used this event to engage elders and youth in learning together, to reconnect with traditional knowledge, and to enable students to see the mathematical reasoning that is inherent in their community contexts.

Many participants also shared examples they knew of from their own experiences which highlighted some of the progress that is being made in transforming mathematics education for Aboriginal students across the country, but clearly such initiatives need to be more widespread.

While we may not have resolved any issues, we came together to hear what we needed to hear, and now, in knowing more we can move forward with this new knowledge and continue the conversation in new contexts. For, as we assembled, we heard new words, received new ideas, raised new questions, and now our work continues.

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BEAUTY IN APPLIED MATHEMATICS

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INTRODUCTION

It is notoriously difficult to say what we mean when we speak of beauty. Under the influence of Pythagoras and Plato, the concept has traditionally been associated with the harmony and mathematical regularity of the universe, and the experience of the beauty of mathematics was long considered to require a deliberate departure from immediate experience, human passions, and "the pitiful facts of nature" (see, for example, Bertrand Russell, 1925). In contrast to this strongly idealist view, it is now recognized that beauty, like aesthetics more broadly, has a context, and should be considered within that context. Nathalie Sinclair (2006) summarizes the definition of aesthetics proposed in Dewey's (1934) *Art as Experience* as follows:

> *Dewey claimed experiences, responses, and objects have an aesthetic quality when they provoke a pleasurable 'sense of fit' for the individual. Thus the aesthetic, for Dewey, pertains to decisions about pleasure as well as meaning, thereby operating on both affective and cognitive levels. Objects do not, of themselves, possess aesthetic qualities: they require a perceiver as well as a socio-historical context.* (pp. 88-89)

This is not to say that beauty cannot in fact catch us by surprise, as it does so often in mathematics or in landscapes. It does suggest, however, that there are various kinds of beauty, some of which depend on a great deal of context, personal taste and experience, and inter-personal negotiation.

In this working group we tried to identify and examine some of the beauty inherent in the practice of applied mathematics. At 25 participants, we were a large group with a rich array of expertise and backgrounds. During the introductory discussions it became clear that the range of reasons for joining the group included curiosity about aesthetics, about applied mathematics, about beauty in nature and in the arts, as well as an interest in new ideas for mathematics teaching.

For “pure” mathematicians it can be difficult at first to acknowledge that the much more fluid and interactive nature of mathematical application, and its socially constructed outcomes deserve to be thought of in terms of beauty. Certainly, as the working group discussions developed, the question “*Where is the beauty?*” was raised more than once. In the end, however, most of us agreed not only that there is beauty in the internal coherence of the mathematical model constructed to fit a problem, but also that there is beauty of a different sort in the fit itself, in the negotiations between problem solvers and in the tension between the “real” and the “ideal.”

To explore some of these manifestations of beauty in applied mathematics, we decided to look for beauty in the following areas respectively, more or less in chronological order:

- Mathematics in arts, crafts and techniques;
- Mathematics in engineering;
- Computer science as applied mathematics.

BEAUTY IN ARTS, CRAFTS AND TECHNIQUES

The importance of context and teamwork immediately became evident when we started the first day with a warm-up exercise in which everyone was given a short encoded French-language quotation to decipher, working in groups. The same handout also included a (longer) encoded English-language quotation for “homework.” The groups looked for patterns and relied on knowledge of the French language as context; thus they looked for the letter E, articles, and double letters. It is worth noting that teams who did not have a “context expert” who knew French went directly to the English exercise.

Following this warm-up exercise we divided into four groups. Each group was given an identical set of tiles, and asked to combine them in a way that the group considered beautiful (see Figures 1 and 2).

This resulted in a variety of designs and approaches. Some looked for symmetry; others favoured dynamism and even randomness or chaos. Some comments expressed during the activity pointed to the subjective aspect of beauty and the different emotional responses the same object can produce in different individuals. The activity was followed by a short presentation on these so-called Girih tiles: The physicist Peter Lu, was led by his knowledge of the mathematics and physics of quasi-crystals to conjecture that these tiles were used in the construction of the intricate line motifs seen in much ancient Islamic art. A video of Peter Lu’s discussion of his ideas is available at http://peterlu.org/content/decagonal-and-quasicrystalline-tilings-medieval-islamic-architecture). The activity done by the working group can be tried online at http://www.geopersia.com/.

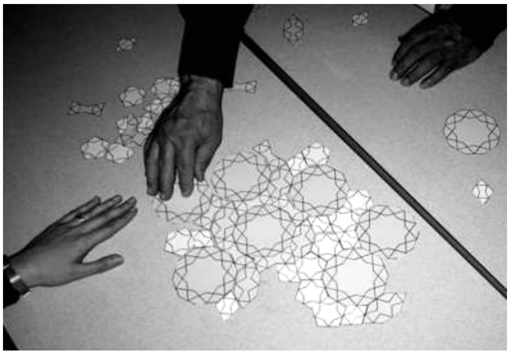


Figure 1. Collaborative design

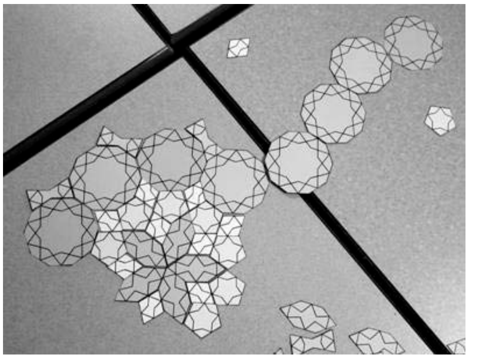


Figure 2. Static order and dynamism

As a second example of applied mathematics in arts, crafts and technique, David Lidstone presented a problem given to him by a luthier friend. The challenge was to calculate the positions of the additional frets on a guitar whose neck had been extended. The solution involved simple arithmetic, an understanding of the relationship between length and frequency of a vibrating string, and the twelfth root of 2. Some of the main challenges of the activity involved communication: understanding the problem as stated by the luthier and translating the solution into something that would be of use to him.

In our reflection on the first day's activities we noted the wide variety of Girih tile patterns produced by the groups. One group (Figure 1) worked out an agreement in which group members took turns placing one tile. As the construction evolved, the team extended this rule to allow the removal of a tile, and subsequently the switching of two tiles. This process illustrated quite eloquently the beauty that may emerge in the collaboration, negotiation, and use of iterative adjustments typical of the practice of applied mathematics. A second group (Figure 2) used the tessellating potential of the tiles to evoke a contrast between static order (often associated with pure mathematics) and dynamism and chaos (more commonly descriptive of applied mathematics). A third group was drawn to making organic patterns that emphasized the negative spaces produced by the tiles more than the way they fit together.

We discussed the relationship between art and craft, and felt that the boundary between these is often too sharply drawn. We also wondered whether in the case of the Girih tiles the craftsman's skill came first or whether there was a mathematician first who had worked out a Girih-tile technique, which then became a craftsman's skill handed down from generation to generation. Of course we did not have a definitive answer, though Peter Lu's suggestion is that the problem of producing the lines accurately and efficiently is best done by thinking of the patterns via Girih tiles. One of the participants (Nadia Nosrati) told us of family members in Iran who practice the craft to this day.

The discussion of the luthier's problem led to questions about well-tempered and equally-tempered scales, as well as Pythagoras' more ancient connection between simple fractions and harmony.

This led us to distinguish between mathematical techniques and mathematical ideas, with a reference to research indicating that conceptual understanding transfers more easily to new contexts than practical understanding. This idea-technique distinction pointed to the various discourses that can be used to render ideas and techniques, and the degree of sophistication in the mathematics invoked. France Caron drew our attention to the work of Chevallard, who describes mathematical practice of a given group (or *institution*) in terms of the *tasks* that characterise the practice within that group, the *techniques* that are applied to accomplish these tasks, the *technology* that corresponds to the reasoned discourse on these techniques, and the *theory* that supports this technology.

We wondered about the location of beauty in these various mathematical practices, and decided that it is (at least in part) located in the social aspects of working on a problem and that it is partly determined by the prevalent social aesthetic. It can also be found in the experience of convergence in an application of mathematics to a problem, either through a process of successive improvements of a model, or in an iterative mathematical process presented as a solution. There was some discussion about the tension between applied mathematics and pure mathematics – between those who want to explore the context and those who want to extract essences.

In looking at the beauty in social dynamics, we noted the different forms that it may take, depending on the interdisciplinary component of the work: in pure mathematics, collaboration typically emerges among a group of people who share a common understanding and want to play the same game; in applied mathematics, although people want to play a common game, their understanding is mainly complementary: not only do they need to find a common discourse that will support the dialog and collaboration, they also need to negotiate both the goal and the rules of the game.

The importance of such negotiation led us to ask whether there is more subjectiveness in applied mathematics. Nathalie Sinclair reminded us of a survey of mathematicians on what they considered to be the most significant theorems, theories, and findings. The surprise was not so much in the variety of the responses but in the reasons given by the mathematicians to justify their choices: they were usually related to personal experiences, and often had strong connections to particular periods or events in their own lives.

Another similarity between pure and applied mathematics was found in the necessity to state explicit assumptions: in applied mathematics, they determine the scope of application of the model considered and possibly its degree of sophistication; in pure mathematics they are central to the formulation and construction of proofs.

BEAUTY IN APPLICATIONS OF MATHEMATICS TO ENGINEERING

Using an article from a book on applications of mathematics (Banks, 2002) and a research paper (Weidman & Pinelis, 2004), we discussed recent attempts to reconstruct the process of designing and building the Eiffel tower. We considered the French government's 1886 call for proposals (Figure 3), and then worked in groups to think about the issues that might have arisen in the design process. We were surprised that the government imposed so few restrictions.

ART. 8. Les constructions principales seront entièrement établies en fer, avec remplissage en briques, maçonnerie, staff, etc.

ART. 9. Les concurrents devront étudier la possibilité d'élever sur le Champ de Mars une tour en fer à base carrée, de 125 mètres de côté à la base et de 300 mètres de hauteur.

Ils feront figurer cette tour sur le plan du Champ de Mars, et, s'ils le jugent convenable, ils pourront présenter en variante un autre plan sans ladite tour.

ART. 10. Les concurrents devront fournir obligatoirement :

1° Un plan général d'ensemble à l'échelle de $\frac{1}{5000}$;

2° Un plan d'ensemble du Champ de Mars à l'échelle de 1 millimètre pour mètre, avec la tour;

Figure 3. Tiré de l'arrêté du Ministre de Commerce et de l'Industrie, 1er Mai 1, 1886

Participants suggested that wind would be an issue, and that weight would be one as well. We wondered how Eiffel determined the precise angles at which the straight sections should meet to give such a smooth contour. Some tried to guess directly at the shape of the contour of the tower.

Figure 4. The Eiffel Tower

Eventually Leo Jonker presented two models, suggested relatively recently to account for the shape of the tower. Both models result in an exponential contour (which closely approximates the actual contour), though the underlying assumptions of the models are very different. However, the second model is based on historical evidence consisting of comments and writings of Eiffel himself, and appears to be the model Eiffel used. We learned that Eiffel modified the shape near the base, realizing that his model might not have taken into account all the eventualities that could occur, and that the assumptions behind the model were based on incomplete understanding of, for example, the effect of wind on a tower.

We discussed the beauty of the process as expressed by Eiffel, and conjectured what the experience might have been for the workers.

As for the beauty of the product itself, to many the shape of the tower nicely suggested a woman (one of its nicknames is "*La Dame de Fer*"), while to others the shape suggested the act of reaching up. Not everyone in the group thought the tower beautiful, however; and at the time it was built, most Parisians thought it was ugly.

Again we discussed the fact that art has a context, and that for the Eiffel tower this would include the many national monuments that line up with it, as well as the power of the French republic that such urban design wanted to emphasise.

Some suggested that there is elegance in the fact that the average density of the tower, taking account of all the empty space inside the structure, is about a tenth of the density of balsa wood: if it were possible to wrap it in air-tight plastic wrap it would float! Others marvelled at the way a number of straight line segments can create the illusion of a smooth curve, thereby illustrating the power of discretization in approximating the continuous. While some participants were less than thrilled by the perceived abstraction of the free-body diagram analysis that lies at the heart of Eiffel's calculations, others expressed delight at how apparently simple analysis determines such an elegant shape.

There are always several constraints that combine to shape the direction of a design project: the requirements expressed in the call for proposals, constraints that come from the situation in which the product will be deployed (e.g. wind), and those that you may impose on yourself for aesthetic considerations. The Eiffel tower is one of very few monuments that were named after the engineer who designed it – is this a testimony to the way the various constraints were met in this case?

BEAUTY IN COMPUTER MODELS OF CONTINUOUS PHENOMENA

Following this example from engineering, we turned to examples connected to numerical approximation and computer modelling. To start this off we studied the process that produces a straight line out of a set of points with equally spaced *x*-coordinates by keeping the two end-points fixed and iteratively applying the rule that the others adjust their positions to be half-way between their neighbours'. First we tried to embody the iterative process by having participants represent points; following this we demonstrated it using Excel (see Figure 5). The simple formula used in the spreadsheet led us to consider a "new" recurrence-based definition for the straight line, and connect it to other definitions of a straight line.

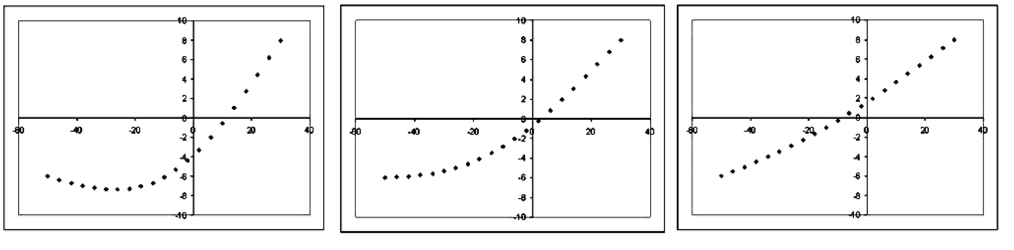
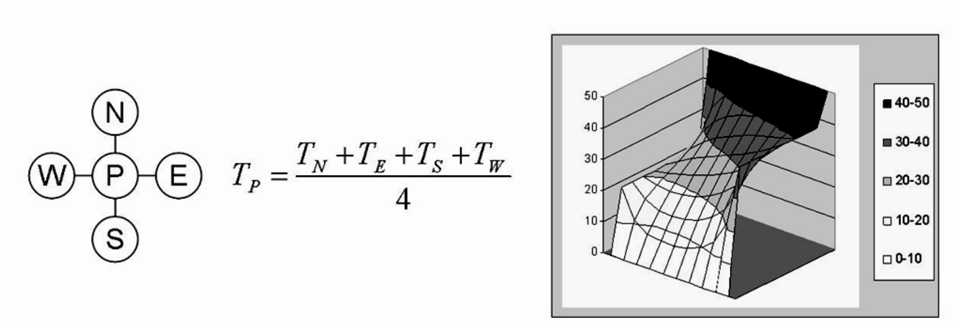


Figure 5. Iterative production of a straight line using Excel

We then generalized the process to a two-dimensional equally spaced grid by doing an analogous demonstration on Excel in which the middle of five contiguous points on the grid is repeatedly adjusted to the average of its four neighbours. This simple scheme was described as a discretization of Laplace's equation for heat equilibrium, and produced a smooth temperature surface every time the boundary temperatures were changed. Our exposure to these iterative processes led to a discussion of the value of introducing difference schemes into the high school curriculum.



$$T_P = \frac{T_N + T_E + T_S + T_W}{4}$$

Figure 6. Heat equilibrium illustrated using Excel

It was suggested not only that the beauty of the mathematics tends to come out in these changing images (which are produced by discrete processes) but also that this is often the only mathematization that matters. We combine such images when we think; and computers, in helping us combine images easily, allow a more dynamic understanding of our environment. It was suggested also that the images themselves can spark an interest in the underlying mathematics in students.

This led us to reflect on the relative importance of analytical and numerical mathematics in applications and the different qualities of beauty associated with the two. In this vein, we looked at computer animations brought by Sen Campbell. After looking at two different simulations of planetary motion, one done from the Copernican standpoint and the other one done from the Ptolemaic standpoint, we reflected on the way we judge the relative beauty and elegance of these two cosmologies, and asked whether the availability of computers affects the way we make that judgement. We wondered also how we would persuade someone else (especially students) to react to mathematical beauty the way we do, or whether it is even reasonable to expect that.

To further demonstrate the power of computer simulations as vehicles for understanding we looked at simulations of the gulf stream and of El Niño. We noted, though, that fascination with computer-generated images can also have its ugly side. Computer simulations can project a degree of authority that is false, appearing to relieve us from the need to understand their construction and unveil simplifications, omissions or distortions that went into that construction. To illustrate this, a story was told of a political battle over a landfill site whose proclaimed safety depended on a mathematical model constructed using open-source simulation software, and based on private data and undocumented assumptions made (consciously or not). We also heard about a paper on global warming that depended on key assumptions that were not warranted. Models are brittle in their sensitive dependence on underlying assumptions. This fact underlined the need to ensure students' ability to think critically about the assumptions underlying mathematical models, by teaching understanding of the kinds of processes that lead to their construction. For example, the assumptions made in a modelling process may reflect the tradition of ignoring outliers. As pointed out by Mircea Pitici, Taleb's (2007) Black Swan Theory suggests that outliers can collectively play a larger role than the regular and predictable events, and that we can easily be blinded by the comforting predictions that come from their dismissal.

Following this discussion of simulation, one of the participants (Olga Shipulina) told of her research into the spread of forest fires. Without discussing the details of the mathematics involved, she conveyed her strong affective reaction to the beauty of the differential equations involved, as well as the numerical methods needed to solve them, suggesting that the two contributed equally to the beauty of the model. Of particular interest to her was the fact that conservation equations equally apply to whatever volume is being considered, thereby allowing us to tackle a given situation at different scales.

Participants wondered how we can make understanding of these processes available to all students. Since context is particularly important in applied mathematics some stressed the role of shared, embodied, experience as part of a lesson. Another suggestion might be to bring into the classroom someone who is enthusiastic about a model, followed by someone who objects to it. However, the difficulty may not lie only in connecting mathematics with contexts, but may also point to the mathematical content taught. Today's presence and extensive use of numerical methods in applications may require that we revisit the way we articulate elementary courses in calculus: rather than focusing on traditional *techniques* which have a very limited scope of application, we should aim to develop deeper understanding of a theory of differential equations that stresses the control of solutions obtained by using contemporary methods and software.

FURTHER EXPLORATIONS

To round off our discussions we watched a short video clip describing computer graphics software for morphing a human face to show continuous transformations between various traits and emotions, and we learned that its appealing power and effect is made possible by rather simple iterative mathematics. We then divided into subgroups to go deeper into some of the issues that arose in previous discussions. After some negotiation, we decided on three groups, each discussing one of the following topics:

- Beauty, elegance, and aesthetics in mathematical tasks;
- Mathematics in the history of musical scales;
- Curriculum practice and the communication between practices.

Following an hour's discussion, we reassembled the groups for reporting and a general discussion.

BEAUTY IN TASKS

In its report, the group discussing the nature of beauty in tasks described a beautiful task, in applied mathematics, as one that possesses some or all of the following characteristics:

- It can be simply stated, but has interesting complexity underneath.
- It relates in some affective way to the one who takes it on.
- It can be entered into by the one given the task – you feel that you will be able to tackle it. In a classroom setting this means that the student must be able to trust that the teacher has judged this correctly.
- The description of the task is not too directive, so that it will not block the student's ability to enter into it – too much information stifles.
- It should evoke curiosity.
- It should invite creativity.
- It should involve some ambiguity.
- Its solution is likely to produce surprise and delight.
- It brings together a task and a structure, a situation and its model.

This led to discussion of the importance of developing a taste for a particular type of task. Developing this taste will generally take time and even persistence, and judgements of beauty will vary from one culture to another. The delight of a solution offered by someone who has worked through the process of constructing a model for an application, but whose delight in the beauty of a complex process can only be shared vicariously, can nevertheless encourage a taste for persistence in others.

It was added that applied mathematics brings together task and structure, object and abstract form, each with its own beauty, and does this through a process of reasoning and abstraction that is itself often beautiful and even elegant.

THE HISTORY OF MUSICAL SCALES

In the group that discussed the role of mathematics in the history of musical scales, one of the participants was an expert whereas the others were relatively unfamiliar with the subject. This led the group to remark on the challenge felt by the non-experts in attempting to follow the discussion of interval relationships in the various scales. The non-experts risked being intimidated by the complexity of the details. Imagining themselves as students they could see that in the classroom such complexity could be received by students as a form of violence. Since it was clear the complexity was inherent in the situation, this made us aware that it is sometimes necessary to persist before the beauty can be experienced, and that it is important for students to value that kind of persistence.

From a consideration of the importance of prior, embodied, musical experience for understanding discussions of musical scaling, this group shifted to a consideration of the analogous importance of prior experience in all teaching. Just as knowledge of music implies learning how to listen, so all learning involves some kind of learning how to see. In this connection it was noted that so many words, such as “brightness” and “colour” used to describe music, have a reference to sight. This was connected to a discussion of synesthesia, and led to the suggestion that in teaching, the emotional response evoked by gesture and eye contact can also be thought of as a type of synesthesia.

Viewing a YouTube video (http://www.youtube.com/watch?v=ne6tB2KiZuk) of Bobby McFerrin, who is able to get a general audience to chant in the pentatonic scale, and who can do this in a wide variety of cultural settings, suggested that some forms of embodiment may be independent of history or culture.

CURRICULUM PRACTICE AND COMMUNICATION BETWEEN PRACTICES

The third group studied curriculum practice and the communication between practices. This group noted how much context is problematic in applied mathematics, and that shared experience is necessary and must be allowed to evolve into owned experience. Teachers should ask themselves where to locate the agent of interest, and then open up space for owned experience to develop – for the development of an acquired taste, or a way of seeing. To achieve this, students should be exposed to a variety of problems, both big and small, seemingly patterned and apparently chaotic. This acknowledges the different kinds of beauty found in them. Sometimes unfamiliarity with the context can serve to enrich the learning experience for students. The group discussed examples of open-ended problems, and exercises for students that involved critiquing textbook problems and injecting the complexity they felt was appropriate. As another approach for making sure that applications are included in mathematics teaching, it was suggested that a course presentation make explicit the types of situations students should be able to address by the end of the course, and that it show a steadily increasing level of complexity of situations and tasks that students can tackle with new knowledge they have just developed.

Often the interest in a task is heightened if it is enculturated by means of a story, a physical experience, or the memory of a loving environment, as when a mother reads to a child who sits on her knee, or a problem is posed during a family meal. The affective response is necessary to support a cognitive response and should be seen as a way in. In this connection there was a reference to different kinds of rote learning, one that is superficial, but another that is akin to ritual, and can provide a way into mathematical thinking

SUMMARY

We spent the last half hour summarizing our discussions. This brought us back to the importance of an affective response as a way into a cognitive response, and the importance of play to allow students to find their own beauty. It was noted as well that in applied mathematics, there may be fewer “aha” moments than in “pure” mathematics. Yet, mathematical modelling allows us to get a better understanding of the world around us while minimizing the damage done or the costs incurred by our investigations. That alone should warrant a greater presence of modelling in the mathematics curriculum.

We asked ourselves once more: Where do we get our sense of beauty when we do applied mathematics? The answers included the different perspectives the process makes available to us; the connections we establish; and the process itself. Some brought forward the value of telling stories in which we share our delight. Others noted the aesthetic dimension of pedagogy itself (didactic engineering). Still others stressed the dynamic character of applied mathematics and enjoyed the fact that so often in applied mathematics there is a process of iteration that gradually converges to the “ideal” solution we sometimes imagine we are looking for when we study mathematics problems for their own sake.

One of the members of the group, Nadia, agreed to set up a discussion board where we can continue our discussions and eventually publish our report.

We ended our very enjoyable and informative series of discussions by formulating a few topics that might be suitable for future working groups:

- Follow up our discussions by thinking in more detail about modelling examples for use in high school, and for use in the training of teachers.
- Discuss the problem of constructing grade-specific modelling examples for both elementary and high school.

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NOTICING AND ENGAGING THE MATHEMATICIANS IN OUR CLASSROOMS

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Many characteristics describe the work of a mathematician. These characteristics just as readily apply to the work of "professional" mathematicians (e.g. people who "do math" as a career, researching and publishing in the field) as they do to "amateur" mathematicians (e.g. people who "do math" (without funding), be it students, teachers, or teacher educators). The focus of this working group was to explore different ways in which teachers, mathematics educators, and (professional) mathematicians come to appreciate themselves and their students as mathematicians.

Through engagement with mathematical tasks, our working group attempted to establish a sense of what it means to "be a mathematician." This developed from a shared vision of fundamental aspects of "doing math" that were exemplified in the tasks, discussions and experiences of our group members. Specific questions we designed to help shape our discussions included:

> *How is it that teachers/teacher educators/mathematicians come to notice and foster mathematical thinking in primary, secondary, and tertiary classrooms?*

This question is motivated by Wheeler's concern that "*the majority of teachers [do] not encourage their students to 'function like a mathematician'*" (Wheeler, 1982, p. 46).

> *How can we as teachers engage students as mathematicians and what types of tasks model what it is that mathematicians "do"?*

This question is motivated by a recognized disconnect between how students experience mathematics in the classroom and how professional mathematicians experience mathematics in research (e.g. Boaler, 2008; Lockhart, 2009).

INTRODUCTION

The emphasised text in the abstract above exemplifies the underlying themes and foci of our working group. While we continue to explore them (throughout our lives and careers), we present below a snapshot of the working group's engagement, thoughts and reflections that inspired and emerged. In what follows, each of us attempts to give voice to our personal and collective experiences, and we do so in a three-part discussion in which (i) Egan illustrates some of the considerations and intentions that went into selecting tasks that, for us, "model what it is that mathematicians do," (ii) Eva highlights the themes and events that emerged during the three days of the working group, and (iii) Ami reflects on corresponding experiences, *noticing* and *fostering*. Our intention is not merely to give a summary or overview of the experience of our working group, but rather to exemplify poignant issues connected to noticing and engaging that struck each of us in different, though related, ways. As such, each of the sections is written as a first-person narrative.

PINK PIG PONDERING

From the moment we received the invitation to help lead a working group on *noticing and engaging the mathematicians in our classrooms*, I knew I had the perfect opportunity to get (some) members of the CMESG/GCEDM community playing with and talking about little pink pigs, at least for a few days. For those of you not familiar with these little pink pigs – hereafter referred to as *mini pigs* – I'll take a minute to explain.

Mini pigs, for me, represent the modern day equivalent to astragali (i.e., ankle bones), which "*archaeologists have found...among the artefacts of many early civilizations*" (Bennett, 1998, p. 8) and, arguably, can be considered the ancestor of cubical dice. However, there are some fundamental differences between astragali and dice. For example, the six sides of astragali, unlike the six sides of a die, are not (necessarily) equally likely to occur. Further, and as another example, the six sides of the astragali are comprised of two different kinds of "sides" (four are long flat sides and two are shorter rounded sides), unlike a die whose six sides are all identical. My intentions were clear: I wanted the members of the working group to be working with and discussing astragali and not dice.

For me, the problem with using dice is a problem with what Taleb (2007) calls *platonicity*, which "*is our tendency to mistake the map for the territory*" (p. xxv). In other words, the problem with dice is a problem with mistaking the theoretical, perfectly true, cube where each side has an exactly equal chance of landing on any one of the six sides (i.e., the map) with the die in our hand (i.e., the territory). By introducing modern day astragali, and not dice to the working group, my intention was to engage in a variety of discussions on: classical, frequentist, and subjective interpretations of probability, sample space, equiprobability, elementary outcomes, events, and other topics, to establish a sense of what it means to "be a mathematician." What happened next, which admittedly, at first, caught us off guard, we now appreciate as a fundamental component to engaging the mathematicians in our classroom. While the following outlines specifics related to my mini pigs, similar discussions occurred for all the tasks we brought to the working group.

Here is the actual email communication that occurred after the initial posing of the problem to be used in our working group. (Note: the final version of the mini pigs problem is found in the appendix.)

> ***Egan:*** *do not accept any of Ami's rewording of the pig questions*
> ***Eva:*** *thanks for the comments. I accepted/rejected as suggested. I think we want to leave out some of the details from the problems because deciding what path to choose regarding such things, or even knowing the conventions, IS part of mathematical thinking,*
> ***Ami:*** *I agree with Eva's point, but do think the original problem is ill-posed (perhaps deliberately) and we could prepare to engage in discussion on when ill-posed questions are helpful in fostering math thinking (and why) and when they are not.*
> ***Eva:*** *P.S. the French translations of the pig questions are more akin to my revisions than the original phrasings and could probably be changed for consistency.*
> ***Egan:*** *Yes, the wording of the questions is deliberate. I don't, necessarily, agree with Ami's answers and, also, don't think if someone came up with the response they would be done with the task...two things I hope to happen during the working group. Nevertheless, how about the following changes...*
> ***Ami:*** *I like your wordings and don't think you've lost any of the power of the questions with them. For part c) do you mean that the pig will land both on snout and side at the same time? Or are you hoping they'll compare the probabilities of flipping snout or flipping side?*
> ***Eva****:* *the back and forth seems to have quieted down, so please see if the new, attached version is reflective of the feedback. (BTW I had to adjust the phrasing for 2 as it is a single pig being tossed 3 times so it doesn't make sense to say: "the probability of them landing...". Is that ok??)*
> ***Egan****:* *think it would sound better if they all said "what is the probability of landing." How about this....*
> ***Ami****:* *Egan, I'm still unclear about what you're asking in question 1c, and would like to go over it with you before we do the pig question in WG.*
> ***Egan:*** *Hi Ami and Eva: How about this....*
> ***Eva****:* *I can see that we want to avoid being too specific with the outcome description so as to avoid closing down avenues of discussion. Conversely, I think we want to respect certain norms so I would be ok with the curly brackets everywhere. Can you please come to an agreement by, say, Friday, so that I can make the copies?*
> ***Ami****:* *Egan: I agree, {snout, side} is great. Thanks for clarifying.*
> ***Egan:*** *Ok, but I am still looking for an acceptable alternative.*
> ***Egan:*** *Ok: How about this!*
> ***Eva:*** *Hi, I will go with this.*

As witnessed in the conversation presented above, a large portion of our preconference planning was spent formulating, via negotiation, the mini pigs problem and some of the others. Coming up with a precise formulation of the question was not a phenomenon restricted to the leaders of the working group. Members of the working group, who were orally presented with the mini pigs problem during the conference, also spent the majority of their time working on a precise formulation of the question. For me (whose group worked on this problem), the amount of time spent focusing on the problem was of note because, in many instances in probability, the question is the culprit. "*Outcomes are not uniquely determined from the description of an experiment, and must be agreed upon to avoid ambiguity*" (Weisstein, 2010). This agreement on the question, witnessed in our email communication above and also during the working group (WG) session, drew our attention to an interesting domain where we may be able to notice and foster mathematical thinking in our classroom.

Perhaps in a twist, to notice mathematical thinking in our classrooms, we may need to turn our attention from how our students answer a question to how they engage with the question asked. That being said, the resources we have available, e.g., textbooks, etc., present questions that, for the most part, are largely stripped of any purposeful ambiguity or issues surrounding the wording of the question. However, all is not lost. To engage our students as mathematicians and present them with tasks that model what it is that mathematicians do, i.e. negotiate the problem, we can provide them with tasks that encourage negotiation of the problem before we inevitably turn to our fascination with the solution. While probability is a natural place to find such questions (e.g., the Boy-Girl problem), other areas of mathematics are just as rich, as witnessed with the other questions presented in our working group.

INSTANCES OF THE DIGIT THREE

In the case of "Instances of the Digit Three," for example, although the formulation had also been revised before the conference, ambiguities came up during the WG and the meaning of the question was negotiated, as well.

The possibility of noticing "mathematical thinking" in ourselves or in others is mediated by a number of contextual and affective factors (Jacobs, Lamb, & Philipp, 2010), the first few of which, as discussed with respect to the mini pigs, are connected to the ways in which the initial problem is posed. For example, a problem can be very convergent in that the possible interpretations are few and almost equivalent. Conversely, a mathematical situation can be presented in a divergent enough way that it may not even include a specific question posed.

In the case of the work done on the problem known as "Instances of the Digit Three" (see appendix), I chose to present the problem in a format that was designed to be as specific as possible so as to reduce, wherever possible, possibilities of ambiguities that might generate interpretations of the problem other than the one that I intended. In the case in point, I provided the problem in written form, which gave the solvers a referable artefact, and included generous detail so as to converge on the meaning that was intended. At the same time, to make the problem more engaging, I added a "real life" context. These two factors can interfere with each other: the problem, as was the case in point, was interpreted by the participants through the lens of their previous experience with situations that were either similar to the provided context (is the "ground floor" Level 0 as it is in Europe, or Level 1 as it is in the US?), or mathematically analogous (a version of this problem has circulated previously that involved the sequential numbering of the pages of a book).

From the beginning of the participants' work on the problem, two methods of engaging with the ambiguities of the problem emerged. Some groups decided to work on "disambiguating" the problem as much as possible before initiating the solving phase. For example, the issue raised earlier about whether the ground floor counts as 0 or 1 was verified by me. Other assumptions and ambiguities were resolved internally, through agreement amongst the solvers of the particular group. In the second method, the solvers jumped in and began working on the problem, and negotiated the ambiguities as they arose. The latter method was justified by this group because, as was said, more ambiguities were almost certainly going to appear throughout the process, and so it was not feasible to "clean it all up" in advance.

Criteria for the negotiation of ambiguities fell into two main categories:

1. Efforts were made to keep the problem as "realistic" as possible, that is, ambiguities were resolved by selecting the situation that "could happen this way."

2. Decisions were made that involved accepting (possibly) unrealistic but pragmatic assumptions in order to position the problem at a comfortable location (accessible yet challenging), within the continuum of *trivial – easy – accessible/comfortable – challenging – perceived as impossible*.

During the negotiations that took place, reflections were made that incorporated three points of view. The speaker reflected on herself/himself as: (1) mathematician, (2) student, and (3) teacher. This distinction connects to an additional contextual factor that comes into play, involving the power dynamics and motivations of the participants.

In a conference working group, such as Working Group F of CMESG/GCEDM 2010, the participants were all present of their own volition. Further, they freely chose to participate in this particular set of activities. Although I, as the problem poser, was potentially seen as the owner of the problem, the other participants were not, therefore, under pressure to "perform" a resolution that I had designed and imposed on them. The consequence of this dynamic is that the solvers had the choice to engage with the problem as they saw fit, and to verify with myself, or not, whether their interpretation was as I had anticipated. This is decidedly different from the social context in classrooms, where power differentials abound.

As is typical in a problem solving situation involving a group, there were episodes during which some of the participants felt lost. Upon reflection, these conditions, which were sustained for various lengths of time, yielded descriptions such as:

- Being lost, but still following the language that the group is using
- Being lost completely (having a perspective that is so far off from the group's that it is unrelated)
- Being lost by oneself or with other people

A discussion of the way to get back into the flow, if possible or desirable, included active ways such as asking for explanations or clarifications, and more passive ways whereby things happened around the participant that allowed her/him to get back into the flow: it happened as s/he listened.

When the group was beginning to close in on the solution, comments were made that pertained to the "messiness" of the solution, and the fact that having well-organised data can improve the process of resolution. In response, other participants commented that sometimes it is easier to produce organised notes on a *post-hoc* basis, in retrospect. Indeed, the point was made that organising your thinking is a sign of "mathematical thinking" and that the organisation need not be linear, necessarily. Other signs included:

- Conjecturing
- Disproving
- Looking for patterns
- Looking for shortcuts
- Making *systematic* lists
- Looking for degrees of freedom (and reducing them)
- Thinking about the reverse problem (it was remarked, in fact, that a reversal of the wording of the problem would have made it cleaner, more accessible, perhaps even trivial)

At a later stage, participants also reflected on some of their problem solving experiences within the group. A turn of the table yielded what participants perceived their partners as having done that they could term "thinking like a mathematician." Examples of responses included:

- Pausing
- Verbalising
- Back tracking
- Never giving up
- Attending to details
- Writing down strategy
- Eliminating redundancies
- Finding easy starting points
- Finding a better way, *even in retrospect*
- Thinking of simpler problems and evaluating them
- Asking questions to clarify assumptions or to write down possibilities

A comment was also made concerning the fact that, in trying to ascertain what our partners do that is "mathematical," it can be difficult to tell which partner is responsible for what thought or idea.

REFLECTIONS ON NOTICING AND FOSTERING: A TEACHER, A TEACHER EDUCATOR, AND A MATHEMATICIAN WALK INTO A BAR…

At the very close of our working group (minutes after the end actually), an objection to one of our guiding questions arose; it was in regard to our distinction between teacher, teacher educator, and mathematician. To be clear, this distinction was made deliberately, and while some may have worried that it was divisive, we saw it as both relevant and important to noticing. The word "noticing" is used in a sense similar to (or at least our evolving understanding of) Mason's (2002) *Discipline of Noticing*.

Mason writes: "*the very heart and essence of noticing is being awake in the moment to possibilities*" (2002, p. 144), which stems from Gattegno's idea that only awareness is educable. For me, someone who is not so disciplined in noticing, being awake to possibilities is contingent on the focus of my attention in that moment. It is through this idea of focus of attention that there is relevance in the distinction between teacher, teacher educator, and mathematician. During the three days with our working group I noticed my attention focused on very different matters, depending for instance on which problem was being addressed or with whom I was working. What I noticed (in myself and others), how I interacted or responded, what choices were made, what was emphasised or avoided, and what and how I tried to foster, varied as my attention shifted between teaching, observing as a researcher, and mathematizing. That is, the distinct "roles" in which I found myself – teacher, mathematics education researcher (more so for me during the WG than "teacher educator"), and mathematician – carried with them different goals. Some of the goals were personal (what did I hope to learn from my colleagues) and some were communal or "public" (what did I hope my colleagues would get from the WG). My goals tended to be pretty broad, maybe a bit vague, and, despite varying with respect to the aforementioned roles, always, always, were the goals of a WG facilitator.

Part of being a WG facilitator, or at least one part of what I learned about being a facilitator for *this* WG, is that being able to move flexibly between one role and another is very important. Flexibility requires, in Mason's language, noticing a moment of choice, and then responding in an appropriate way. Mason (2002) writes: "*The real work of noticing is to draw the moment of awakening from the retrospective into the present, closer and closer to the* *point at which a choice can be made*" (p. 76). He is referring to a wide range of moments and choices, many of which are much more subtle than the ones I describe here. Nevertheless, the "roles in which I found myself" were made through choices in the moment, though they were largely defined by others – by their questions, their actions, by my interpretations of their actions and needs. In the following paragraphs, I share some of the moments I noticed and where my attention was focused at the time, as well as some of the associated choices made in relation to fostering.

Right off the bat, day one of our WG, I found myself in a role that many others can probably sympathise with: "math sales-person." I presented a problem (Cutting the Cube) that involved geometry, visualisation, imagination, and I was met with resistance. For a variety of reasons, not all of which were brought to light, my fellow group members really did not want to imagine suspending a cube and slicing it in half. I was left feeling worried, anxious, and even disappointed. Experiencing very much the same emotions many mathematics teachers feel when trying to convince, say, a group of teenagers that solving for x is a valuable way to spend their time. So, I reacted. That's probably the best way to describe it – I reacted to my colleagues' resistance by trying to "sell" the problem, by digging for any reason I could find that might convince them to give it a chance. Accordingly, my attention was at first focused on how to make the problem work for this group of people. As they got deeper into the solving, my attention shifted from their reluctance to their strategies, noticing what got them talking and questioning, and eventually (when my anxiety subsided) I noticed their math. *Their* math because each small subgroup did something different from the next, because they focused on different aspects of the problem than expected, because they did a variety of interesting things while I appreciated the math in their actions and in their work. I noticed an iPod being used as a ruler and later being abandoned for bits of paper with ruled markings. I noticed charts and diagrams, movement and gesture, furrowed brows and smiles; I thought "this is good" and chose to encourage people to share their thinking across the different subgroups. When someone did or said something interesting and unexpected, my attention shifted from facilitating the problem-solving to thinking about the math. While I can't speak for the specifics of what others attended to or noticed, I can say that what ended up being fostered in our group – including an exploration that went to a "surprising" level of depth and a discussion around multiple entry points and guidance – was a result of the focus of our attention, of what we were awake to notice.

It's appropriate to comment that these reflections are not about what I noticed and attended to, but what *was* fostered. While an individual may have intentions for fostering, what ultimately is cultivated is a result of the collective. This seems especially pertinent in a situation such as a CMESG working group where the collective includes a variety of perspectives and expertise. Thus, in my view, part of noticing (and fostering) is being aware of how to navigate your intentions for fostering within the dynamic of the group so that the activity you think is being encouraged is actually (or at least something close to) what is being encouraged.

Returning to noticing, another part of what influenced the focus of my attention was comfort and familiarity with the problem. Two of the problems addressed in the WG were relatively new to me (Cutting the Cube and Multi-facets), and so there were times where I really wanted to think about the math; sometimes it was to try and make sense of a new (to me) approach, other times it was to try and make connections between some tangential lines of reasoning and what I saw as the "heart" of the problem. With both problems, my colleagues ended up forming subgroups and working at their own pace and with their own strategies, as indicated above. With Multi-facets, there were some people who were more familiar with the math in question than others. As a result, some subgroups worked through the problem very quickly and others who explored more hesitantly, asking questions and struggling at times to visualize or represent the problem. My attention during this engagement was very much focused on the latter groups. The newness of this problem had consequences on how and what I could foster: for one, the math still had novelty and could capture my thinking; for another, there were unexpected interpretations of the problem, solving strategies, etc., for which I did not have handy responses. This was in contrast to my experience with the WG exploration of the Ping-Pong Ball Conundrum, which is very familiar, as I've used it previously in research. During this problem, the engagement was quite different: the group stayed as a whole, a timely and probing question asked by a group member instigated controversy and spurred discussion, and consensus on a final answer was not reached.

In this context, I found myself defaulting to a role as "education researcher" – noticing cues that triggered me to pose certain questions at certain times, not really with any intent to "lead" my colleagues to a solution, but rather with the intent to challenge and observe responses. In our group discussion of the problem solving, members reflected on the impact of some of my choices. They commented on their emotional responses to hearing that "there is an answer" though not being told what it is – some found it motivational, others found it intimidating. There was a sense that I "held some knowledge" but not a sense that I was thinking mathematically. A colleague observed that I never said "*hmmmm, I don't know*" during their exploration. Unlike facilitating the other two problems, with the Ping-Pong Ball Conundrum the responses, strategies, interpretations, and problematic issues were familiar and anticipated. What was fostered was a lively debate, an intuitive (rather than formal) address of the problem, and *reflecting-in-action* (Mason, 2002) of the problem-solvers as they compared and debated their points of view. What came to light was a tension between fostering mathematical thinking (or any activity) while not engaging in the same. There are questions about this tension that continue to attract my attention. It is something that needs more thought, but my instinct is that there is an important connection between how an individual may choose to foster mathematical thinking and how he or she perceives the role of "teacher," "teacher educator," or "mathematician" and what is being noticed when the focus of attention is not on mathematics.

CONCLUDING REMARKS

After this working group experience, what do we know for certain? That the first step to noticing, fostering, and engaging the mathematicians in our classrooms is to be given the opportunity to engage with mathematics, whether it be with mathematicians, mathematics educators, or mathematics teachers. In other words, to give an opportunity similar to the one we were given (and, looking back, are extremely thankful for).

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APPENDIX: THE PROBLEMS

CUTTING THE CUBE

Close your eyes and imagine a cube. Select one of the vertices of your cube, attach a string to it and hang your cube by that string. Now imagine a horizontal plane that will slice your cube exactly in half. What does the cross-section (the intersection of the plane and the cube) look like? How do you know?

EXTENSION

This next scenario will be hard to imagine, but we can try to extend some of the reasoning from before. Now, instead of a regular cube, "imagine" a 4-D hypercube – that is, a cube with four spatial dimensions. As before, we want to hang this hypercube by a vertex and slice it exactly in half with a 3-D hyperplane. We can't do much to visualize what this will be like, but we can still talk meaningfully about what the cross-section of the hypercube will look like. It will be a 3-D object and one that should have many of the same geometric (and algebraic) properties of the previous scenario. How can we figure out what this cross-section will look like? What shape will it be?

COUPONS UN CUBE

Fermez les yeux et imaginez un cube. Choisissez un de ses sommets et suspendez le cube par ce sommet. Imaginez qu'un plan horizontal coupe le cube en deux parties égales. De quoi a l'air la coupe, l'intersection entre le plan et le cube? Comment le savez-vous?

CONTINUATION

Ce nouveau scénario est difficile à visualiser, mais essayons de pousser le raisonnement précédent plus avant. Au lieu d'un cube ordinaire, prenons un hyper-cube 4-D, c'est-à-dire un cube à quatre dimensions spatiales. Comme précédemment, suspendons l'hyper-cube par un sommet et coupons le exactement en deux avec un hyper-plan 3-D. La visualisation de cette configuration est problématique mais il est tout de même possible de parler sensiblement de la forme de la coupe. Ce sera un objet 3-D, que nous pourrions tenir en main et qui devrait avoir des propriétés géométriques (et algébriques) communes à celui du scénario précédent. Comment pourrions-nous déterminer de quoi la coupe a l'air? Quelle forme a-t-elle?

PIGS

1. Two pigs are tossed.

a) Determine the possible outcomes.

b) Assign a probability to each outcome.

*Note: 1c), 2, and 3 can't be given out until 1a) and 1b) are completed by the group.

c) What is the probability of landing a snout and a side?

2. A pig is tossed three times. What is the

LES COCHONS

1. Deux cochons sont lancés.

a) Déterminez les résultats possible.

b) Assignez une probabilité à chaque résultat possible.

*Note : 1c), 2, et 3 ne doivent pas être posées avant que la réponse à 1a) et 1b) n'aie été établie.

c) Quelle est la probabilité qu'ils tombent un naseau et un côté?

probability of landing {2 sides, 1 back}? 3. Three pigs are tossed (all at once into a perfectly circular ring). What is the probability of landing {2 sides, 1 back}?	2. Un cochon est lancé trois fois. Quelle est la probabilité qu'il tombe {2 côtés, 1 dos}? 3. Trois cochons sont lancés (en même temps et dans un anneau parfaitement circulaire). Quelle est la probabilité qu'ils tombent {2 côtés, 1 dos}?
INSTANCES OF THE DIGIT 3 You are working for a big construction company that just finished constructing a very large office building that has 100 offices on each floor. The offices are numbered consecutively and it is your job to install the office numbers. The format of the numbers reflects the position of the office: for example, office 067 is the 68th office on the ground floor and office number 667 is six floors higher. The digits are sorted in boxes of 1000. Unfortunately, the 3's have to be back-ordered and there is only one box of 1000. Assuming you install the office numbers in order, what number will you be installing when you use the last digit '3' of the box? Could you reach the number in the previous question without having opened a second box of any other digit? EXTENSION Can you find the pattern that determines when the second box for each digit is needed?	**INSTANCES DU CHIFFRE 3** Vous travaillez pour une compagnie de construction qui est en train de terminer un édifice à bureaux énorme, avec 100 bureaux par étage. Les bureaux sont numérotés de façon consécutive et vous êtes responsables de l'installation de ces numéros. Leur format indique la position des bureaux : par exemple, le bureau 067 est le 68ème bureau sur le rez-de-chaussée et le bureau 667 est six étages plus haut. Les chiffres à clouer sur les portes sont rangés en ordre dans des boîtes de 1000. Par malheur, une seule boîte de « 3 » est arrivée et les autres sont en rupture de stock. Si vous procédez par ordre, quel numéro serez-vous en train de mettre en place quand vous utiliserez le dernier « 3 »? Pourriez-vous arriver à ce numéro sans ouvrir une des autres boîtes? CONTINUATION Pouvez-vous découvrir la régularité qui détermine quand la deuxième boîte d'un des chiffres est nécessaire?
THE PING-PONG BALL CONUNDRUM Imagine the following scenario… You have an infinite set of ping-pong balls numbered with the natural numbers and a very large barrel. You are about to embark on an experiment, which lasts 60 seconds. In 30 seconds, the task is to place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time (and working more and more quickly),	**L'AFFAIRE DE LA BALLE DE PING-PONG** Imaginez le scénario suivant… Vous avez un ensemble infini de balles de ping-pong numérotés avec les nombres naturels et un baril énorme. Vous êtes sur le point de commencer une expérience d'une durée de 60 secondes. Dans les premières 30 secondes, la tâche est de placer les premières dix balles de ping-pong dans le baril, et de retirer la balle numéro 1. Dans la moitié du temps restant, les prochaines dix balles sont placées dans le baril et la balle numéro 2 en est retirée. De

balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. After the experiment is over, at the end of the 60 seconds, how many ping-pong balls remain in the barrel? How do you know?	nouveau, dans la moitié du temps restant (et en travaillant de plus en plus vite), les balles 21 à 30 sont placées dans le baril et la balle numéro 3 en est retirée, et cetera. Quand l'expérience prend fin, après les 60 secondes, combine de balles se trouvent dans le baril? Comment le savez-vous?
BOYS AND GIRLS What is the probability that Anne Gull has two boys? *Anne Gull: I have two children.* *Matthew Maddux: Is the older one a boy?* *Anne Gull: Yes.* *Anne Gull: I have two children.* *Matthew Maddux: Is at least one a boy?* *Anne Gull: Yes.* *Anne Gull: I have two children.* *Matthew Maddux: Do you have a boy?* *Anne Gull: Yes. His name is Laurie.*	**DES GARÇONS ET DES FILLES** Quelle est la probabilité qu'Anne Gull a deux garçons? *Anne Gull : J'ai deux enfants.* *Matthew Maddux : Le plus vieux est-il un garçon?* *Anne Gull : Oui.* *Anne Gull : J'ai deux enfants.* *Matthew Maddux : Y a-t-il au moins un garçon?* *Anne Gull : Oui.* *Anne Gull : J'ai deux enfants.* *Matthew Maddux : As-tu un garçon?* *Anne Gull : Oui. Oui, il s'appelle Laurent.*
MULTI-FACETS Picture to yourself a length of rope, lying on a table in front of you. The cross section of the rope is a regular n-sided polygon. Slide the ends of the rope towards you so that it almost forms a circle. Now mentally grasp the ends of the rope in your hands. You are going to glue the ends of the rope together but before you do, twist your right wrist so that the polygonal end rotates through one n^{th} of a full revolution. Repeat the twisting a total of t times, so that your mental wrist has rotated through t n^{ths} of a full revolution. NOW glue the ends together, so that the polygonal ends match with edges glued to edges. When the mental glue has dried, start painting one facet (flat surface) of the rope and keep going until you find yourself painting over an already painted part. Begin again on another facet not yet painted, and	**MULTI-FACETTE** Imaginez un segment de corde, posé sur une table devant vous, dont la coupe transversale est un polygone régulier à n côtés. Vous glissez les extrémités de la corde l'une vers l'autre pour les connecter. Mentalement, attrapez les deux extrémités dans vos mains. Vous allez les coller ensemble, mais avant de le faire, vous allez tordre votre poignet droit pour que l'extrémité polygonale soit tournée de 1 n-ième de tour. Répétez cette torsion t-fois, afin que votre poignet imaginaire ait tourné de t n-ième de tour. À présent, collez les deux extrémités pour que les cotés des polygones s'alignent. Quand votre colle mentale aura séché, peignez une facette de la corde (une surface plane) et continuez jusqu'à ce que vous ayez peint toute la facette. Recommencez avec une nouvelle facette et une autre couleur. De combien de

use another colour. How many colours do you need?	couleurs aurez-vous besoin?
JEOPARDY On Friday March 16, 2007 history was made. For the first time ever there was a three-way tie (for first place) on Jeopardy! People involved with the show decided to contact you, yes you, about the odds, which you claim are....	**JEU TÉLÉVISÉ** Le vendredi 6 mars 2007, un moment historique a eu lieu. Pour la première fois, les trois joueurs furent ex aequo (pour la première place) dans un jeu télévisé. Les organisateurs ont décidé de vous demander, oui, vous, quelles sont les chances que cela se produise… Qu'en pensez-vous?
SU DOKU Can you create a completed Su Doku puzzle without an obvious pattern, starting with a blank grid? What are some useful strategies?	**SU DOKU** Pouvez-vous créer un puzzle Su Doku complet sans régularité apparente à partir d'une grille vide? Quelles seraient des stratégies utiles?
CHROMINO® Chromino® is a modified domino game whereby each tile is a rectangle which consists of three squares each coloured in one of five colours. Players alternate, laying down tiles to create a board. To lay down a new tile, it is required to touch the already-laid tile(s) in such a way that the new tile touches existing ones on at least two contact segments, and that the colours must match across the contacts. If every possible tile exists in the set, can you work out how they were placed to produce the provided image (see Figure 1)? EXTENSION If you had the choice of any tile in the set at each turn, what could be the most compact configuration you could construct?	**CHROMINO®** Chromino® est une version modifiée du jeu classique de domino. Dans cette version, chaque pièce est un rectangle constitué de trois carrés, chacun coloré d'une de cinq couleurs. À chaque tour, les joueurs placent une pièce sur la table. Pour pouvoir poser une pièce dans le jeu, celle-ci doit s'agencer à une pièce déjà posée de même couleur sur au moins deux des huit « arêtes de contact ». Si toutes les pièces possibles font partie du jeu, pouvez-vous déterminer où elles sont placées (figure 1)? CONTINUATION Si vous aviez le choix de poser n'importe quelle pièce à chaque tour, quelle serait la configuration la plus compacte possible?

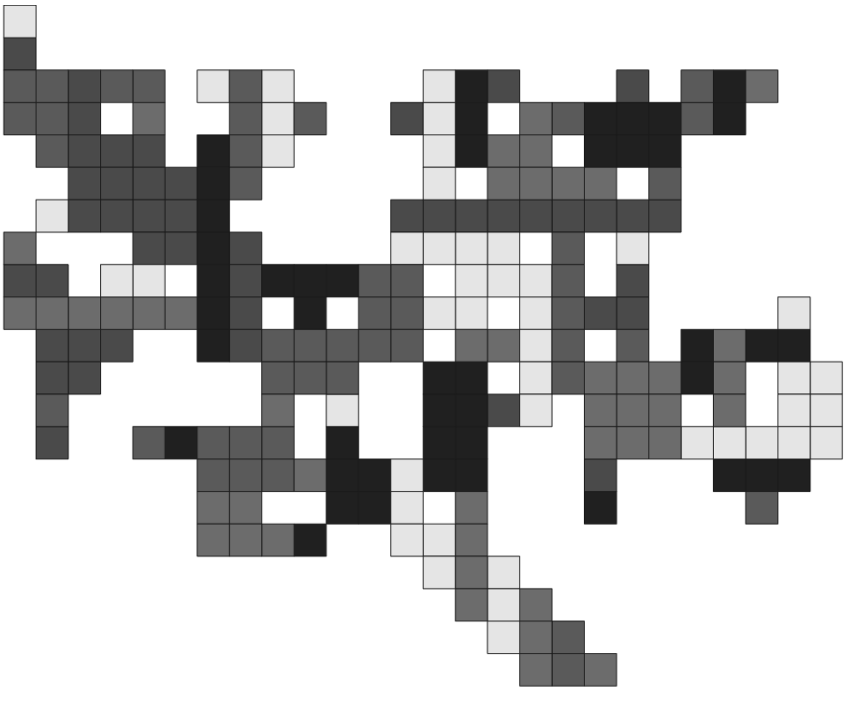

Figure 1. Chromino® configuration

Topic Sessions

Séances thématiques

ÉMOTION, RÉFLEXION ET ACTION: MATHÉMATIQUES ET ENSEIGNEMENT

Frédéric Gourdeau
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Lors de la rencontre du GCEDM de 2010 à Simon Fraser, j'ai eu le plaisir de présenter une séance portant sur la formation des enseignants de mathématiques du secondaire à l'Université Laval. J'ai voulu profiter de cette occasion pour poursuivre une conversation, amorcée pour moi en 1996 à Halifax; en acceptant que cette fois, c'était à moi de prendre la parole et contribuer de cette manière. Pour alimenter la conversation, j'ai voulu parler de ce que nous faisons à l'Université Laval mais aussi de mes réflexions personnelles, de mes idées et de mes questionnements. J'estime grandement les membres du GCEDM et je voulais donc partager pleinement avec eux. Dans ce texte, je serai assez fidèle à cette présentation en me permettant de présenter des idées et des réflexions, parfois plus ou moins clairement reliées, parfois plus ou moins bien articulées, dans l'espoir que cela contribue à nos discussions à venir.

L'ENSEIGNEMENT M'ENGAGE PERSONNELLEMENT

Enseigner est acte (inter-)personnel, qui dépend de qui on est, comme individu, et des personnes avec lesquelles on est. Il me semble donc devoir débuter par quelques mots sur moi.

En premier lieu, je suis un mathématicien profondément engagé en formation des enseignants de mathématiques au secondaire, à l'Université Laval depuis 1995. J'enseigne deux cours dédiés aux futurs enseignants; ceux-ci sont inscrits dans un baccalauréat en éducation de 4 ans dans lequel la formation disciplinaire figure au même titre que les cours de didactique, que les stages et que les autres cours en éducation. Ces étudiants suivent 14 cours de mathématiques et de statistique, dont 6 leur sont exclusivement dédiés. Ces cours, d'une durée d'une session chacun, comportent de 45 à 60 heures en classe, et sont suivis par 20 à 40 étudiants par session.

Je m'aventure maintenant dans une zone moins fréquentée des textes académiques en parlant de moi plus personnellement. Je le fais parce que j'estime que l'enseignement dépend profondément de qui nous sommes : de nos zones de confort, de nos forces et faiblesses, de nos valeurs. Ce qui me semble le plus pertinent ici a trait à mon engagement communautaire. J'ai fait de la radio communautaire, pendant deux ans, dans le cadre d'une émission intitulée *Québec sans frontières*. Je le faisais à titre de bénévole de Carrefour canadien international (CCI), une organisation avec laquelle j'avais fait du travail bénévole au Canada et en Sierra Leone avant mon doctorat. De plus, après mes études doctorales, et avant d'obtenir un emploi à l'Université Laval, j'ai été coordonnateur régional pour CCI pendant 4 ans, responsable de comités de bénévoles dans une vingtaine de villes au Québec et en Ontario. Je travaillais avec des bénévoles, tous adultes, qui apprenaient sur eux et le monde. Ils apprenaient à coopérer, tout en travaillant pour réaliser de la levée de fonds, des activités d'éducation au développement, et se préparaient à travailler outre-mer où à accueillir ici des gens d'autres pays. Ce travail a eu un impact majeur sur moi et il me ferait plaisir d'en parler davantage, mais je me restreindrai ici à deux apprentissages essentiels : l'importance du respect et de l'écoute, et l'extraordinaire potentiel du véritable travail en coopération.

Finalement, je suis père de trois enfants : Éloi (15 ans), Émile (17 ans) et Clara (19 ans). Cela influence tout ce que je fais, et qui je suis.

CAPACITÉS ET INCAPACITÉS

Je ne peux présenter un contexte théorique global et expliquer par la suite comment les cours de mathématiques que l'on donne à Laval se situent dans cette théorie. Je n'ai pas ces connaissances et ce n'est pas ainsi qu'ont été élaborés les cours. Je n'ai pas non plus de nouveaux résultats provenant de recherches en didactique des mathématiques. Et je ne prétends pas avoir pour vous des avancées révolutionnaires ou extraordinaires.

Cependant, je peux partager avec vous des idées, des réflexions, des activités; vous présenter mes réflexions, modestement et personnellement; et être ouvert au changement et à la poursuite de ma réflexion.

Comment vais-je réussir à partager tout cela? L'enseignement dépend de manière fondamentale de la mise en présence de personnes, dans un lieu et pour une période donnée. Être présent aux étudiants, réagir, lancer des défis, pousser plus loin, rassurer, clarifier, le tout en ayant une direction que l'on souhaite suivre, un objectif que l'on souhaite atteindre. Je vais essayer de donner des idées, des exemples, en espérant que se dégagera pour vous une image relativement claire.

LES ÉTUDIANTS

Ils les appellent les clients.

Quand j'écris *mes* étudiants… Comme *mon* fils, *mon* ami – un adjectif possessif qui marque la relation, et non pas la possession. (Quiconque croit posséder ses enfants s'expose à de vives désillusions. ☺)

MES ÉTUDIANTS…

- Ils sont des alliés. Ils sont *mes* meilleurs alliés. Ils sont *leurs propres* meilleurs alliés.
- Ils peuvent se remettre en question, se mettre au défi. Ils peuvent aussi défier le professeur, mais c'est surtout eux-mêmes qu'ils peuvent défier.
- Ils peuvent remettre en question la conception qu'ils ont de leurs cours à l'université.
- Ils peuvent aller plus loin que ce que l'évaluation permet de noter.
- Ils peuvent passer de «*Qu'est-ce que le prof veut?*» à «*Qu'est-ce que je veux?*». Je peux les y aider.
- Je me soucie d'eux, comme personnes. Je les aime – la langue française ne me donne pas d'autre mot pour décrire correctement ce que *I care about them* exprime pour moi en anglais.
- J'estime leur travail. Je me préoccupe des élèves auxquels ils enseigneront.

ILS DOIVENT REMETTRE EN QUESTION LEUR CONCEPTION DES MATHÉMATIQUES

Une conception qui peut englober plusieurs facettes.

- Comprendre ou chercher à comprendre des processus, des objets.
- Extraire des caractéristiques, des patrons. Représenter, trouver des structures.
- Communiquer à propos de ce que l'on comprend.
- Les mathématiques sont humaines, culturelles.
- L'aspect formel ou symbolique n'a pas besoin d'être présent pour qu'un énoncé soit mathématique – la forme n'est pas imposée.

On se doit d'aller au-delà des slogans qui simplifient à outrance, et notamment la description des mathématiques comme étant la science des patrons (en anglais, the science of patterns).

ILS DOIVENT RELEVER DES DÉFIS

Ils ont des défis personnels à relever. Plusieurs. C'est vrai… mais ils vont enseigner.

Ils pourraient mentir à leurs élèves. Acheter la paix. Être cool… oui, ils le pourront.

Ils doivent être responsables, se sentir responsables, non pas d'obtenir certains résultats sur le bulletin, mais bien de chercher à comprendre, de s'améliorer, de réfléchir et de remettre en question.

À PROPOS DES RÉSULTATS, DES NOTES

Habituellement, il y a des échecs dans les cours que je donne. Cela est toujours difficile mais n'est pas nécessairement négatif. Comprennent-ils pourquoi ils ont échoué? Ont-ils le sentiment que c'était juste, approprié?

Lorsqu'ils reprennent un cours et commencent à comprendre, que la brume se dissipe et qu'ils commencent à voir, alors leur confiance en eux-mêmes peut augmenter. J'ai souvent eu de tels témoignages. Passer un cours, le réussir de justesse, tout en sentant bien qu'on n'a pas vraiment compris : non, ce n'est pas une expérience positive. Se sentir incompétent en mathématiques peut être un problème.

ÉVALUATION

Je dois admettre que je ne suis pas parfaitement à l'aise avec l'évaluation. Il y a toujours une tension pour moi. D'une part, j'accepte que nous ayons une responsabilité d'évaluer et de certifier – un mot un peu fort – qu'un étudiant a démontré son apprentissage. D'autre part, j'aimerais que l'évaluation soit davantage un outil pour apprendre.

Il y a aussi une tension pour mes étudiants, bien sûr, mais c'est la tension entre étudier pour obtenir une note ou réussir un examen d'une part, et étudier pour apprendre et comprendre d'autre part, qui est problématique. Je tenais à reconnaître que ces tensions existent, à les énoncer. Ceci étant, l'évaluation ne constitue pas un des points centraux de cet exposé.

QUAND SUIS-JE UN BON ENSEIGNANT?

Si je considère mon enseignement, je peux me demander quand je crois faire du bon travail.

Lorsque je comprends tellement bien les mathématiques que je peux les oublier et me concentrer sur ce que je veux réussir à atteindre avec le groupe, avec les étudiants. Je peux me concentrer sur le fait qu'ils sont des enseignants en devenir, et que c'est leur capacité à devenir de meilleurs enseignants qui est cruciale. Je peux penser à leur amour des mathématiques, au plaisir qu'ils peuvent avoir à s'engager dans une activité mathématique. Je peux me soucier de la qualité de cet engagement. Je peux leur donner toute mon attention : j'en suis capable.

Je peux me soucier de la créativité, du plaisir et de la frustration; penser à la confiance en soi, au désir d'explorer, à la capacité de s'aventurer dans l'inconnu. Je peux porter attention à la communication. Je peux réfléchir à la culture. Et je peux faire tout cela tout en travaillant en mathématiques ou à propos de mathématiques. Des mathématiques qui sont liées au curriculum…

Et je me demande : de mes réflexions quant à mon enseignement, qu'est-ce qui s'applique aussi aux enseignants du secondaire?

MATHÉMATIQUES : UNE DESCRIPTION PARTIELLE L'ENTREPRISE MATHÉMATIQUE

Dans un des cours, nous discutons de la preuve et des preuves. Les notes de cours contiennent des exemples de preuves visuelles, de paradoxes, et présentent certains types de raisonnement utilisés dans les preuves (incluant les preuves visuelles et les exemples génériques). En classe, on discute à partir de paradoxes ou d'énoncés mathématiques relativement élémentaires afin de bien discerner le type de traitement logique que l'on fait de chacun. Ainsi, on essaie de bien distinguer ce qui découle de la définition d'un concept, et que l'on a donc pas à prouver, et ce qui requiert au contraire une preuve. On insiste sur la différence entre un argument heuristique et une preuve, ou encore entre une justification du bien-fondé d'une certaine définition et une preuve. Pour donner un exemple concret, on pourra aborder les règles de l'exponentiation en débutant avec le paradoxe $2 = 4^{\frac{1}{2}} = ((-2)^2)^{\frac{1}{2}} = (-2)^{2\times\frac{1}{2}} = -2^1 = -2$. Deux autres exemples sont la formule de volume de la sphère et celle de l'aire d'un rectangle – oui, vous avez bien lu, d'un rectangle.

QUELQUES THÈMES

- Les nombres et leurs représentations : plus précisément, les liens entre les propriétés des nombres et de leurs représentations. On considère ici les écritures en différentes bases des nombres naturels, rationnels et réels, et les liens entre les caractéristiques de cette écriture (telle que la longueur de la période ou de la pré-période) et les propriétés du nombre. Pour les nombres irrationnels, la représentation périodique de certains irrationnels sous forme de fraction simple continue est aussi abordée.
- L'infini et son étude. Des paradoxes, des surprises, des contradictions apparentes, et l'ingéniosité et l'imagination pour y faire face. Repousser les limites de notre imagination en discutant de cardinaux infinis.
- Aires et volumes, des concepts acceptés mais mal compris. Qu'est-ce que la longueur? L'aire? Le volume? Pourquoi est-ce que *ab* est l'aire du rectangle? Comment obtenir de manière élémentaire le volume d'une sphère ou d'un cône, ou l'aire d'une surface sphérique? On aborde aussi les liens avec la biologie, en utilisant *On Being the Right Size* (Haldane, 1928) et un texte que j'ai écrit pour Accromath, *King Kong et les fourmis* (Gourdeau, 2008).
- Les coniques, abordées en partie sous l'angle de la multiplicité des définitions. On réfléchit à ce que cela présente comme défi et à ce que cela peut apporter comme

avantage. On cherche à bien comprendre pourquoi on doit réconcilier ces définitions et comment s'y prendre. Les noms de Dandelin et Quételet sont évidemment au programme.
- Isométries, groupes de symétrie de figures du plan (rosaces et frises).

On travaille fréquemment avec des logiciels de géométrie dynamique et les étudiants acquièrent généralement une excellente maîtrise de ce type de logiciel, ce qui est facilité par une approche qui mise sur leur créativité; j'y reviens plus loin.

MATHÉMATIQUE ET CULTURE

Dans l'un des cours, les étudiants lisent des textes individuellement pour en discuter par la suite en classe au sein de petites équipes. Ces textes abordent différents aspects des mathématiques et de leur apprentissage, menant parfois à la rédaction d'un essai. Les auteurs de ces textes incluent : Philip Davis et Reuben Hersh, Ian Stewart, Bernard R. Hodgson, Jean-Marie De Koninck, Denis Guedj, Georges Ifrah, Louis Charbonneau, Louise Lafortune. Les sujets vont de l'histoire des mathématiques à des découvertes récentes en mathématiques, en passant par l'infini et par l'apprentissage des mathématiques à partir de situations réelles (ce qui est dénoncé dans le texte à ce sujet).

FAIRE DES MATHS

Dans toutes les situations décrites, on essaie d'amener les étudiants à *faire des maths* plutôt qu'à les apprendre, à découvrir et à s'investir personnellement. Comment fait-on cela? Voici quelques exemples.

EN CLASSE

On travaille parfois en petites équipes, parfois en groupe, ou encore on discute en classe à la suite du travail personnel fait avant le cours. Ce travail peut par exemple nous amener à justifier ou prouver un énoncé de différentes manières, selon les suggestions faites. Je vais généralement suivre les suggestions, écrire au tableau, puis demander si cela semble correct et complet. Parfois, je ne commenterai pas et les laisserai y penser pour y revenir au besoin par la suite. Je pourrai aussi compléter certaines parties, sans nécessairement écrire le tout complètement au tableau : je les inviterai à écrire une version personnelle, complète et correcte, dont nous pourrons discuter subséquemment. (Je n'en discuterai pas s'ils ne font pas le travail – j'amorce mes explications subséquentes à partir de leurs questions et de leur travail.) Je crois que ce genre de travail est utile pour eux puisqu'ils auront fréquemment à juger si une explication est valable, si un argument est correct, ce qui n'est pas une tâche facile.

Un autre type de travail consiste à trouver comment formuler mathématiquement les arguments présentés dans un texte tel que cela est fait dans le texte d'Haldane. Ils doivent alors eux-mêmes établir une notation appropriée.

Nous abordons à quelques reprises l'importance de l'utilisation d'une bonne notation en les amenant à choisir eux-mêmes la notation qu'ils utilisent. Ils n'ont sans doute pas eu l'occasion de le faire et cela leur permet de mieux comprendre le langage mathématique ainsi que les conventions et standards qui s'y rattachent (tel que l'utilisation de noms de variables différents pour différentes variables). Pour donner un exemple, lorsque l'on veut établir une formule pour calculer l'aire d'une sphère de manière élémentaire, on a besoin de gérer des passages à la limite en utilisant une notation appropriée – même si cette notation n'est pas forcément conventionnelle.

Comme enseignants, ils devront écrire et produire des documents mathématiques, ils devront parler de mathématiques : nos classes universitaires de mathématiques sont de très bons endroits pour essayer de le faire, et ainsi apprendre à mieux le faire. Plus largement, j'insiste sur l'importance d'apprendre à parler à propos de ce que l'on fait. Non seulement la verbalisation peut-elle nous aider à voir les choses, à les comprendre, mais nommer nous aide à réfléchir. Nommer un concept permet de le créer pour notre esprit; nommer crée une réalité, ce qui est plus que permettre de bien en parler.

RÉSOLUTION DE PROBLÈME

Les étudiants tiennent un journal de bord de résolution de problème, documentant et analysant leurs démarches de résolution selon les indications du livre L'Esprit mathématique (Mason et al). Au terme de ce travail personnel de 8 semaines, les étudiants remettent un manuscrit de 50 à 200 pages. Ce travail imposant leur permet d'approfondir leur conception des mathématiques en leur offrant la possibilité de travailler de manière créative en mathématiques et en travaillant sur certains problèmes bien plus que quelques minutes – parfois pour des heures, des jours, voire quelques semaines. Notons que pour ce travail, je les encourage à s'amuser, à réfléchir, à essayer des problèmes qui leur paraissent difficiles. De plus, les indications données incluent ce qui suit.

- Ne terminez pas certains problèmes et travaillez à fond sur d'autres.
- Si vous avez une idée au restaurant, au pub et travaillez sur un napperon, alors joignez-le à votre journal de bord.
- N'effacez pas et ne recopiez pas ce que vous avez fait pour que la présentation soit parfaite.

Le travail demande aussi une réflexion personnelle, guidée par des questions dont celles-ci.

- Quels genres de problèmes vous intéressent et pourquoi ?
- Quelles étapes sont plus faciles ou plus naturelles pour vous ?
- Pensez à vos futurs élèves : qu'y a-t-il de pertinent ici?

Ils travaillent par la suite en équipe de 3 ou 4. Pour cette partie, chacun contribue sa résolution d'un problème, choisi afin de bien représenter le processus de résolution de problème dans son ensemble, et une conclusion commune est rédigée.

Ce travail complexe permet d'aborder de multiples aspects que nous jugeons utiles pour des enseignants de mathématiques.

- Exemplifier, généraliser et prouver, ainsi que le rôle des conjectures.
- Intuition, induction, exemplification, représentation, dessin.
- Être perdu, avoir besoin de démarrer, manquer de confiance.
- Se tromper, un peu ou beaucoup. Faire des erreurs bêtes – mais le sont-elles vraiment? Faire des erreurs – sont-elles vraiment des erreurs? Le mot *erreur* est-il bien choisi?
- Commencer à comprendre, à voir. Voir simplement le complexe ne veut pas dire que le complexe est simple, ni qu'il est simple à voir. (Pourquoi est-ce que je n'ai pas pensé à ça avant?)
- Prendre son temps, prendre le temps de laisser notre esprit apprivoiser le tout.
- Communiquer aux autres, communiquer avec soi-même, justifier, prouver, se mettre au défi, être sceptique de ses propres raisonnements, clarifier sa pensée, comprendre et être compris.
- Communiquer par écrit. Est-ce que je sais lire? Lire des maths… Est-ce que je sais écrire? Écrire des maths… et être compris.

- Efficacité et clarté : alliés et ennemis à la fois.

CRÉATIVITÉ ET MOTIVATION

Les étudiants doivent créer une animation à l'aide d'un logiciel de géométrie dynamique. L'objectif principal poursuivi est de leur permettre de maîtriser suffisamment un logiciel de ce type pour pouvoir considérer son utilisation en enseignement lorsqu'ils le jugeront approprié. Cette animation peut porter sur un sujet réel ou imaginaire et doit être contrôlée à l'aide d'un curseur. Il y a des précisions techniques et des critères auxquels le travail doit répondre, mais je ne souhaite mentionner ici que deux aspects clés du travail demandé. Premièrement, ils peuvent être créatifs, créer une animation qui est belle sur le plan esthétique, et faire ce qu'ils souhaitent : un joueur de baseball frappant une balle, une annonce pour un film Ninja, un oiseau se déplaçant dans le ciel, un terrain de jeu en sont quelques exemples. En second lieu, ils disposent d'une grille de correction très précise qui leur permet de faire eux-mêmes l'évaluation de leur travail (à quelques points près) : cela semble leur donner des ailes et la majorité font bien plus que ce qui est évalué (et exigé). En plus, cette motivation (davantage intrinsèque) et la grande liberté de choix les amènent à comprendre beaucoup plus de choses que des travaux plus circonscrits et plus précis, comme ceux que je demandais auparavant.

CONCLUSION

Il y a tant à dire, tant à faire. Travailler davantage sur la pensée critique, sur le véritable travail coopératif. Et on pourrait faire tellement mieux, incluant davantage de collaboration entre les mathématiciens et les didacticiens des mathématiques dans le cadre de la formation initiale des enseignants : merci au GCEDM de permettre de tels échanges, de fournir un cadre dans lequel faire une partie de ce travail.

♦♦♦♦♦♦♦♦♦♦

DOING, FEELING, THINKING MATHEMATICS…IN TEACHER PREPARATION

At the 2010 CMESG meeting at Simon Fraser University, I had the privilege of being invited to share reflections, ideas and experiences with people who care about mathematics and who are serious about learning – their own learning as well as that of their students. I envisaged this session as part of an ongoing conversation, which started for me in 1996, in Halifax: for this part of the conversation, I was to do most of the talking; to contribute – this time – in this way. In this text, I will proceed as was done in the session, offering partly connected and partly disconnected ideas, in the hope that this might facilitate a discussion which is ongoing, and discussions to come.

TEACHING INVOLVES ME, AS A PERSON

Teaching is personal. It depends on who you are, on who you are with. So, who am I in this respect?

Well, first, I am a mathematician profoundly engaged in the mathematical education of prospective secondary school mathematics teachers (at Université Laval since 1995). I regularly teach two courses which are specific to prospective secondary school mathematics teachers: the students are enrolled in a four-year B.Ed. at the university. The program is a four-year integrated degree, with 14 courses in mathematics and statistics, 6 of which are specifically designed for them. These courses are each a regular one-semester course, which for us means between 45 and 60 hours in class. Depending on the year, there are between 20 and 40 students enrolled in a course. There are also three courses in *Didactique des mathématiques*, which are taught by colleagues in the Faculty of Education.

That was the safe part to talk about. I describe who I am without saying anything personal. However, much of what I do depends on who I am, on what I am comfortable with, or am good at, on what I believe in. So, here's a bit more. I have done some community radio: for two years, I was involved in a weekly show entitled *Québec sans frontières*. This was part of volunteer work with Canadian Crossroads International (CCI), an organization I had done volunteer work with (in Canada and in Sierra Leone) prior to my PhD. Between obtaining my PhD and getting a job at Laval, I worked for CCI as regional coordinator for 4 years. This was my full-time job. The work involved helping volunteers in twenty cities in Québec and Ontario as they were learning about themselves and the world, about cooperation, while they were busy doing development education activities, organising fundraising, preparing to go to work overseas, or to host Crossroaders from other parts of the world. This work profoundly impacted me and I could happily discuss it at length, but I will stress only two points: I learnt about respect, and about how much we can achieve when working cooperatively.

And, finally, I am a father of three, all teens: Éloi (15), Émile (17), and Clara (19). This also defines who I am.

ON WHAT I CAN AND CANNOT DO

I cannot present a complete coherent theory and then explain how everything we (at Laval) do in the mathematics courses fits into it. I do not have the knowledge.

I cannot present findings as one would do in a research paper in mathematics education.

And I do not claim to have amazing new results or amazing insight.

However, I can share ideas, thoughts, and activities; offer my reflections, honestly and modestly; be open to discussion; be open to change and reflection.

How will I do this? So much is about being with people, in a given space, for a given time – being physically present for the students, reacting, challenging, securing, clarifying, and coaxing in a wanted direction. I will try to present ideas, examples and reflections in the hope that this will enable you to gain some insight into the work we do at Laval.

ABOUT STUDENTS

They call them "clients," customers.

When I write *my* students…

Like *my* son, *my* friend – a "my" which doesn't mark possession but relationship. (Anyone who thinks they own their children is in trouble ☺.)

MY STUDENTS…

- They are allies. They are *my* best allies. They are *their own* best allies, collectively and individually.
- They can challenge themselves – and the lecturer – but themselves mostly.
- They can challenge the conception they have of their courses at university.
- They can go further than what the marking scheme will allow to mark.
- They need to go from "*What does the prof want?*" to "*What do I want?*" I can help them to do that.
- I care about them.
- I value their job. I care about the students they will be teaching for years to come.

THEY NEED TO CHALLENGE TO THEIR CONCEPTION OF MATH

What might this conception include?

- Knowing, getting to grips with something, which could be an object.
- Features. Pattern. Representation. Structures. A process. Humanity. Culture.
- Communicating about what we understand.
- A statement does not need to have symbols or look formal for it to be called mathematics.

We need to go beyond misleading popular statements, like the popular, *Mathematics is the science of patterns*; this is static, partial, so belittling.

However…

They also face personal challenges. Many. Yes, and they will teach.

They could be able to lie to their students. To buy peace. To be cool. Yes, they could.

They must be responsible, feel responsible – not for getting a given grade in a course, but for seeking to understand, to improve, to reflect.

ABOUT GRADES

Usually, some fail. This is difficult but doesn't need to be (mostly) negative. Do they understand why they failed, trust that it was appropriate?

When they retake a course and start to see the picture – a picture – which is not so blurry anymore, their self-confidence can be increased. I have had many testimonials about that. Getting a passing grade when you feel you have not understood is not a positive experience. Feeling incompetent at mathematics can be a problem.

EVALUATION

Evaluation is something I am not entirely happy with. There is a tension for me. On the one hand, I feel a responsibility to evaluate and certify – up to a point – that a student has demonstrated learning. On the other hand, I would like to focus more on evaluation as a tool to help learning.

There is also a tension for them, obviously, but the one which seems most problematic to me is the tension between studying to get a grade versus studying to understand and learn. I want to deliberately acknowledge that these tensions exist. That being said, evaluation is not one of the focal points of this talk.

ABOUT ME AS A TEACHER

I wonder: When do I feel that I am doing a good job?

When I understand the mathematics so well that I can forget about it and think about what I want to achieve in my course. I can focus on the fact that the course is aimed at enabling the students to become better teachers. I can think about their liking of mathematics, their engagement with it. I can think of different qualities that this engagement may have. I can focus on them: I am *able* to focus on them.

I can ponder about creativity, reflection, joy, frustration, feelings of self-confidence, desires to explore. I can think about communication. I can reflect on culture. And I can do all that while working on/in mathematics (or is it working mathematically?), mathematics which is linked to the curriculum.

And I wonder: How much of this applies to secondary school teachers?

MATHEMATICS: A PARTIAL DESCRIPTION

PROCESSES

Some of the courses we give at Laval explicitly include mathematical processes as part of the content. For instance, the first quarter of one course is centered on *proof and proving.* The lecture notes include examples of visual proofs, of paradoxes, and discuss various types of reasoning employed in mathematical proofs. In class, we have discussions starting with various paradoxes or simple mathematical statements to justify. We try to provide different types of explanation, depending on the statement to justify, in particular drawing attention to the distinction between the following: definition, proof, explanation, heuristics and tricks (for remembering). To give some context, one example might be the rules of exponentiation starting with a paradox like $2 = 4^{\frac{1}{2}} = ((-2)^2)^{\frac{1}{2}} = (-2)^{2\times\frac{1}{2}} = -2^1 = -2$. Another example is the formula for the volume of the sphere, or for the area of a rectangle (this is not a typo).

SOME TOPICS

- Numbers and their representations: more precisely, focusing on the links between properties of numbers and their representations. This includes looking at natural numbers, rational numbers and real numbers written in various bases, and the links between the properties of these representations (for instance, the length of the periodic and pre-periodic part) with properties of the number itself. For irrational numbers, we also consider the (sometimes periodic) representation as continued fractions.
- Infinity and the ingenuity needed to deal with it. Understanding various paradoxes about countable and uncountable sets, and exploring the existence of different infinite cardinalities.
- Area and volume: this is not as simple as many students initially imagine. What is length? What is area? Volume? Why is *ab* the area of a rectangle? How can we obtain, with elementary means, the volume and the surface area for a sphere, a cone? The role of shear (Cavalieri). Some links with biology, using *On Being the Right Size*

(Haldane, 1928) or a personal adaptation on King Kong and the ants, *King Kong et les fourmis* (Gourdeau, 2008).

- Conics. There are many definitions of the conics: Why should that be an issue? How can that be an advantage? Why do we need to reconcile these and how do we do it? The names of Dandelin and Quételet come to mind…
- Isometric transformation and similitudes, frieze patterns, symmetry groups of a 2D object.

Dynamic geometry is often used and students work with some software: they generally become very good at it, something which is encouraged by appealing to their creativity. I return to this below.

ON MATHEMATICS

As a part of one course, students read some texts which they then discuss in small teams in class. These texts are chosen to provide various ways of looking at mathematics and at the learning of mathematics. Sometimes writing an essay will be part of the work to be done by students in the course. Authors include: Davis and Hersh, Ian Stewart, Bernard R. Hodgson, Jean-Marie De Koninck, Denis Guedj, Georges Ifrah, Louis Charbonneau, and Louise Lafortune. Subjects include: history of mathematics; utility of mathematics; recent discoveries in mathematics; infinity; opposition to everyday or concrete mathematics (by one author); etc.

DOING MATHEMATICS

The work done in the classroom and the work done outside of the classroom share one aspect: as much as possible, the students should be *doing* mathematics and not only learning it. How is this achieved? Here are some examples.

IN THE CLASSROOM

Sometimes, we work as a group or in small teams, or discuss after some work done outside the classroom. This might involve justifying or proving statements in different ways, according to suggestions. I will tend to follow leads, write on the board and then ask what, if anything, is missing, or wrong. I might not comment on their suggestions, but invite them to think about it. I might also complete and explain some aspects, without necessarily clarifying everything in writing: I will then invite them to write a complete and correct version of a proof or justification. I believe it will be useful for them; I think they should be able to judge if a line of reasoning is correct. This, of course, is difficult.

In some cases, they might have to extract the maths from a text: for instance, with Haldane's paper. They will be asked to explain, using appropriate notation, what is written.

Using appropriate notation is something we revisit periodically. To develop the notation to write about a topic is not something students seem to have experienced much, yet it helps understand the mathematical language itself and standards we have in mathematics (for instance, using different variables for different values). This is done while working on area and volume, where they need to deal with limits in a correct way, using written notation which makes sense – even if it may not be conventional notation.

As teachers, they will need to produce written material dealing with mathematics, to talk about mathematics: our university classrooms can be a good place to explore and practice these skills. I stress the need to learn how to speak about what we do. Not only does it help us to see, but finding words helps us to think. It creates reality; it creates concepts; it does more than enable us to speak about them.

PROBLEM-SOLVING

Students have to do a problem-solving portfolio (which is referred to as *journal de bord*) based on their individual work with *Thinking Mathematically* (Mason, Burton, & Stacey, 1982). The work spans 8 weeks, and students hand in a manuscript which has between 50 and 200 pages.

This major piece of work offers an opportunity to work on the conception they have of mathematics. It offers the opportunity to be creative with mathematics, to be involved in solving a problem for more than a few minutes – and sometimes for many days or some weeks. Part of the instructions given for the work include…

- Have fun, work, think; try problems which may seem hard to you.
- Do not finish some problems, go crazy on some others.
- If you have an idea in a restaurant and work on a paper mat, don't hesitate to include it as part of your *journal de bord*.
- Do not erase or recopy so it's neat and tidy.

They are also asked to reflect on their work. Questions which can guide their reflection include the following. What types of problem attract you? Why is that so? What scares you? Think of your students to be: Is there anything that you think is relevant?

They then work as a team, each contributing the work they have done on one problem and which they feel is particularly representative of the problem-solving process, and write a joint conclusion to the work.

Here are some of the aspects which we look at in relation to this problem-solving work.

- Exemplifying – generalizing – proving – and conjectures as a mediator.
- Intuition, induction, exemplifying, drawing, representing.
- Being at a loss – need to start – feeling unsafe.
- Being wrong – so wrong. Making silly mistakes – are they? Making mistakes – is mistake a good word?
- Getting to grips. Seeing simply the complex doesn't mean the complex was simple. Or simple to see. (Why didn't I think of that?)
- Taking your time, letting ideas mature.
- Communicating, communicating to myself, justifying, proving, challenging my conceptions, clarifying my thoughts, understanding, being understood.
- Communicating in writing. Can I read? Read maths... Can I write? Write maths... and be understood.
- Efficiency and clarity: allies and foes.

CREATIVITY AND MOTIVATION

Using some dynamic geometry software, students are asked to create an animation which can be about anything, imaginary or real, and which can be controlled using a cursor. There are some technicalities which I don't want to go into, and my intent here is to mention what seem to be the key features of the work they do. Firstly, they can be creative, use nice illustrations and come up with their own animations: part of a baseball game, an ad for a new ninja movie, a bird flying in the sky, a playground, to give some examples. Secondly, they have a precise grid of evaluation and can pretty much determine their grade themselves (up to a few percent): this seems to free them immensely, and most do a lot more than what is asked for. Finally, with this motivation and freedom comes a lot more learning than with a more constrained piece of work. My main objective here is that they gain a sufficient mastery of this type of software to be able to properly consider using it in their teaching, when appropriate.

CONCLUSION

There is so much more to say, so much more to do. Working more on critical thinking, on real cooperative work. And we could do so much better, starting with improved collaboration between mathematicians and mathematics educators in the education of prospective teachers: thanks to CMESG for providing a space to do this.

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ON THE ORIGINS OF DYNAMIC NUMBER IN THE BREAKDOWN OF STRUCTURAL, METAPHORIC, AND HISTORIC CONCEPTIONS OF HUMAN MATHEMATICS

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INTRODUCTION

We have become curious about the specific tensions that arise when applying Dynamic Geometry paradigms of mathematical conceptualization and exploration to school arithmetic; to whole, natural, and integer numbers; and to the entire milieu of iteratively-constructed, discrete, monadic quantities that form the spine of curriculum from kindergarten up to—and well into—school algebra. At first blush, these conflicts may appear as inevitable and obvious outcomes of "the wrong tool for the job": why use Dynamic Geometry software in school arithmetic? But our own perspective is that they are novel incarnations of far broader conflicts in mathematics curriculum, history, and cognition. As such, our modest present technological task strikes us as having potentially more significant implications and outcomes than the more obvious "right tool for the job" context of Dynamic Geometry technology applied to school geometry curriculum.

In our Topic Group presentation, we brought together some of the diverse historical tinder inciting this perspective, and attempted to fan it with sparks of provocation into flame. Our motivation in this task is two-fold. On the one hand, some of the contemporary discourse on educational technology in mathematics is, frankly, dull; and consists of untheoretical, ahistorical recipe-making for which the end of the semester defines the horizon of possible relevance. We are attempting to dig a foundation for our own work here, deeper than one parameterized by version numbers and battery shelf-life. But any digging in the terrain of mathematical understanding rapidly also becomes an act of excavation, and so on the other hand we find ourselves discovering artifacts of earlier conflict between number and geometry, between mathematical technology and application, and between platonic stasis and vital dynamism. These moments inform not only the study of the past, where they contribute to what Netz (2009) calls *cognitive history*—a developmental history not of mathematical ideas but of mathematical ways of thinking—but also to the present and to the future. In the present, we find our own most simple and natural ideas about number—precisely those ideas that the school curriculum presents as well-polished gems—to be *received* ideas, to be evolved from and contingent on prior conceptual stratigraphy. For the future, this archaeology gives us

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opportunity, less to avoid new occurrences of such conflict or tension, than to benefit from the energies they unleash.

ORIGINS OF THE CURRICULUM

Our first point of provocation arises in charting the historical division of the numeric from the geometric in the mathematics curriculum. Our present curriculum descends—in only slightly modified form—from the medieval one, Boethius' quadrivium[2], which in turn draws on Proclus and through him the Pythagoreans as guiding authorities. Through this lineage we learn of a fourfold division among the mathematical subjects, conceptually arrangable as a two-by-two matrix. The dominant division here is between Arithmetic and Geometry, with the former, as Proclus writes, concerned with quantity and the latter with magnitude. In other words, if the first asks the question "how many?", the latter asks the question "how much?". The first looks to countable objects, the second to material extent—and so in its content division also reifies the familiar categorical opposition between the discrete and the continuous.

Quantity	*Arithmetic*	
Magnitude	*Geometry*	

Figure 1. Quantity versus magnitude in the quadrivium

The second axis of the quadrivial matrix is less familiar to the present, and perhaps therefore more exciting. Proclus describes it as identifying mathematical substance considered "*in regard to its character by itself, or in its relation to another quantity, magnitude as either stationary or in motion*" (1992, p. 30). Studying quantities such as 2 and 5 on their own is identified as Arithmetic, but the study of 2 in relation to 5, for the Greeks, is the study of Music, the third component of the quadrivium. By a similar argument, the mathematically *continuous* relationship of things to things forms the study of Astronomy. These fields' "mechanical" subjects are identified with mathematical epistemologies: the vibration of a partitioned string, and the succession of notes in a melody, reifies our understanding of ratio and interval, just as the motion of a planet among planets describes our conceptions of the curve and the locus.

Where a modern sensibility might relegate the technology of music and astronomy to—at best—applications of prior mathematical understanding, the Greek conception embraces the possibility that technology may *define* that understanding.

[2] The curriculum taught in the medieval universities, the *quadrivium*, consisted of four mathematical arts, as developed in this section. Their study followed initial work in the *trivium*, which was made up of grammar, dialectic, and rhetoric. (These are the *trivial* subjects, etymologically speaking.) The seven liberal arts together prepared one for the study of serious topics: philosophy and theology.

	"alone..."	"...in relation to others..."
Discrete	*Arithmetic*	*Music*
Continuous	*Geometry*	*Astronomy*

Figure 2. The complete quadrivium

TRACING THE OPPOSITIONS

While the structure we have shown in Figure 2 seems to offer tidy distinctions, they have not always lived well together.

THE HORIZONTAL DIVIDE : DISCRETE AND CONTINUOUS

We first examine the productive opposition between the discrete and the continuous in the history of mathematics. Early canonical examples can be found in Zeno's paradoxes, where the discrete interpretation of the journey from one place to another—as a countable number of ever smaller steps—clashes with the possibility of continuously moving over, or through, the same expanse.

The Cartesian marriage of Arithmetic and Geometry stands as another example of the productive opposition between the discrete versus continuous divisions of the quadrivium. In fact, Descartes' methodology for undertaking the more "general art" enables the continuous and undefinable magnitudes of lines and planes to be set up as proportions, as long as they have common measures (or units), so that these continuous magnitudes can, in fact, be understood as numbers. Thus, as Kline (1972) writes "*a place or linear figure represents no less and no differently a multitude of number than a continuous magnitude*" (p. 204). Descartes puts geometry in the service of arithmetic, now identifying "algebra" as symbolic logistic, with geometry "*interpreted by him for the first time as a symbolic science*" (p. 206, *italics in original*). In this opposition, geometry is sacrificed to the pursuit of algebra.

Preceding the work of Descartes, in the late 16th century, Simon Stevin proposed a different truce between the opposition by giving birth to a continuous conception of number. Stevin approached mathematics from a more practical perspective (drawing on his commercial, financial, and engineering experience), and—unlike Vieta, who preceded him (and Descartes who followed)—saw the Arabic digital and positional system as vastly superior to that of the Greeks. Stevin's main break with the ancient traditional conception of *arithmos* was to posit that the unit is, indeed, a number. For Stevin, denying the unit the status of number is akin to denying "*that a piece of bread is bread*" (Kline, 1972, p. 191). This conception of the unit allows Stevin to see zero as "*the true and natural beginning*" (p. 193), to which he assigns the symbol "0." The symbolic understanding of number thus endows it with a materiality "*compared to bread and water*," characterized by "*ever-continuing divisibility*" (p. 194). Number becomes assimilated to geometry formations. Indeed, Stevin writes "*the community and similarity for magnitude and number is so universal that it almost resembles identity*" (Kline, p. 194).

Stevin's take on number is interesting in part because of his own status in the field. As a practical man, his somewhat revolutionary ideas came from the margins of a canon that had long kept arithmetic and geometry separate. From our present position, it is difficult to decide whether Stevin was a founding father—a patriarch of modernity—in establishing a new foundation of continuous number—which, for example, enables the emergence of the number line as a model for the real numbers—or whether he was a revolutionary figure positing a sense of number still radical to our thinking half a millennium later. It is perhaps less rebellious to think of the continuum continuously, than as Stevin did, to have harboured continuous thoughts about the discretum.

THE VERTICAL DIVIDE : ALONE AND IN RELATION

In the opposition between the two columns of our matrix, the original conception of "alone" or "in combination" seems to shift to a very different distinction, that is, between the "pure" (arithmetic and geometry) and the "applied" (music and astronomy). In fact, jumping forward to more modern conceptions of the mathematical sciences, the right column disappears altogether and the quadrivium of the mathematical sciences becomes a bivium.

While some will not lament the disappearance of Music and Astronomy from the mathematics curriculum, it is surely worth probing the reasons behind the funnelling of the curriculum to the disciplines of the "alone." The pure-versus-applied distinction fails to capture an important dimension of the difference between the two columns. Both arithmetic and the geometric are fundamentally, for (most of) the ancients, about static objects. But the vibrating strings of Pythagorean music or, indeed, the temporally-dependent melody, and the motion of the heavenly bodies, both rely on temporality. Indeed, if we reconstitute the two columns as being the static-versus-dynamic, the jettisoning of Music and Astronomy takes on new meanings, which we will now explore in the context of the (historically dubious) nature of motion in mathematics. Our goal will be to link their banishment to the history of mathematical technologies, and to examine the bidirectional impact of mathematics and technologies on our emerging sense of what counts as mathematics in the curriculum—and what might count, in the future as it clearly has in the past, as mathematics itself.

ON TEMPORALITY AND MATHEMATICS

We have already mentioned bias toward the immobile in mathematics: Aristotle expressed this clearly in his Metaphysics. Plato's complaint about the discourse of ancient geometers (in the Republic) illustrates well the view that geometry should concern itself only with static objects:

> *[geometers'] language is most ludicrous, though they cannot help it, for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of squaring and adding and applying and the like, whereas in fact the real object of the entire study is pure knowledge.*

For Plato, knowledge itself is static, and so must be its objects.

In counterpoint to this static school, though, we also find mathematicians—such as Archimedes most notably—who eschew the distinction between mathematics and physics, and who readily use arguments involving mobility. For example, Archimedes describes the spiral as a point that moves with constant speed away from a fixed point along a line that also rotates with a constant speed around the fixed point. Or consider his method for finding the area of a section of a parabola by introducing a fulcrum through which the section can be balanced with a triangle. Or invoke his solution to the trisection of the angle, which involves a *neusis* construction in which a notched ruler slides into perfect fit atop another. If the history of technology symbiotically encodes the history of ideas, that the compass and straightedge emerge as the “fundamental” tools of plane geometry suggests Archimedes’ more temporal approaches lost out to the static, detemporalised ones of Plato.

Indeed, one can read the historical development of mathematics as an attempt to get rid of both motion and time, whether in the arithmetisation of analysis in the nineteenth century, or through the refusal to admit “mechanical” curves produced through moving objects as being geometry (Mancosu, 1996), or the attack on Cavalieri’s principle of infinitesimals (Palmieri, 2009), or Poncelet’s principle of continuity, or the constant criticism of Euclid’s (and also Cavalieri’s) method of superposition.

MATHEMATICAL TECHNOLOGIES

Let us return for a moment to the idea of the technologies that inform and co-create mathematics in an intertwined historical dance. We have mentioned already the vibrating strings of Pythagoras. But the compass and straightedge both require, when actually used to produce circles and lines, the creation of geometric objects over time, through certain movements of the geometer’s hand. Thus these technologies in their actual use, rather than in their historically subsequent vitiated form as metaphors and ideas, bear greater resemblance to Archimedes’ notched ruler in their fundamentally temporalized orientation to the production of mathematical knowledge.

Eventually, these technologies all get replaced by a single pervasive, and perhaps invasive one: paper. Once the compass has been used, the circle sits on the page as a detemporalised and depersonalised object. No one needs to perform the rotating action of the compass or the line of force of the straightedge—it “has been done” not only in the past tense but in the passive voice. Objects that exist on paper, as well as the words that describe them, can be handed from person to person and, as Rotman (2008) argues, take on the air of having always been true, having always existed. Thus the immanence of mathematics, as well as its purported permanence, both reflect and perhaps rely on the technological attributes of its media substrate.

In Stevin’s work, the introduction of the cipher by the Arabic mathematicians offers a new technology (symbolic notation) that enables him to see the unit, as well as zero, as a number. Indeed, it is Stevin’s use of the symbol “0” that allows him to understand number as material and, hence, as essentially continuous. But symbolic notation, as it develops in Descartes’ algebra, quickly became a technology of detemporalisation. One has only to consider the transition from parallel to perpendicular axes of the Cartesian coordinate system. While the former retain the behaviour of a function over time, the latter fixes that behaviour in time by showing all the possible positions of a function at once. Symbolically, this conventional technology of modern mathematics replaces time with the temporally-neutered parameter t, leaving an impression that not only has time ceased to exist, but with it, human agency over mathematical truth—which achieves transcendence only through active denial of that original agency.

Thus the pre-modern mathematical technologies—strings, compasses, rulers—can be seen as temporally dynamic in their orientation to performance. And modern mathematical technologies—like algebra, paper, and print (more than writing)—are anti-dynamic in their attempts to encapsulate, distill, abstract and detemporalize “time.” And what can be said of post-modern technologies? One striking characteristic of the digital age is how new technologies have arrived at temporalized—rather than atemporal—reifications, symbolizations, and manipulations of time itself. If we consider examples far from mathematics such as music and video recording, we see fundamentally new ways of working with time: not only can things be recorded so that temporal events can be captured with fidelity, but they can be played back again, at a later time of one's choosing, with full fidelity—or played faster, or slower, or randomly accessed. Or, moving ahead from these technologies of popular culture, we find the computer simulation, which enables precise experimentation over temporal events: temporal events can be manufactured and controlled, and re-played, in precise ways. Even the act of writing moves from the linear production of manuscript pages to the temporally bidirectional bricolage of cut and paste, undo and redo. Moving further towards the curriculum, we can see Dynamic Geometry environments as another post-modern technology that enables—and, indeed, insists upon—temporal studies of geometric behaviours.

We view powerful representations of manipulable time—representations that retain a temporal, dynamic quality—as signal characteristics of these new, "post-modern" technologies. Simulations that can be played over and over, or arbitrarily rewound; and interactive manipulations that can both stretch time and compress it newly permit the literal reinscription, reproduction, and transformation of time-based phenomena. They are thus literally, as well as metaphorically, dynamic.

MEDIUM AS MESSAGE

Looking across the relevant time periods, it becomes apparent how closely linked machines/tools are to mathematical ideas and activity. In the pre-modern era, the physical, dynamic tools of the compass, the notched ruler, *et al*., enabled mathematicians such as Hippias, Archimedes, and Nicomedes to construct a variety of curves. Much later on, in the modern paper age of technology, Descartes pressed his colleagues to decide which among the vast array of curves generated by the Ancient Greeks should be considered mathematical. He proposed that the ones involving a single motion (including the circle and the parabola) should be distinguished from those involving two (such as the quadratrix, the spiral), with the latter deemed "merely" mechanical. Descartes complained that the latter curves lacked precision but, as Netz (2009) points out, they also depend squarely on time in a way that single motions, which can be effectuated almost instantaneously, do not. Indeed, the single motion curves can easily be described, as Euclid does with the circle, for example, in a static way (*a circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another*). Descartes' mathematics is a function of his medium.

In comparing Euclid's definition to Hero's dynamic, mechanical, procedural approach (*a circle is the figure described when a straight line, always remaining in one plane, moves about one extremity as a fixed point until it returns to its first position*), we see how even in the pre-modern age, the drive to forget, and even bury, the tools that midwife mathematical objects, largely through detemporalisation, runs deep in mathematical history. This makes it difficult for the historian to excavate the material roots of mathematical objects, and hence, difficult to appreciate the extent to which technologies and mathematics are entwined. However, many scholars, from a range of disciplines, have insisted on a materially-framed historical approach to understanding knowledge of all kinds. McLuhan's famous dictum "*the medium is the message*" is most often thought of in terms of the relationship between different media and the information they carry, but mathematics is just a special case of this: mathematical ideas are not independent avatars capable of being communicated in any medium one wishes—they are borne out of particular technologies, which shape their very nature and, most importantly, mathematicians' ways of thinking about them. As Sinclair and Gol Tabaghi (2010) show, in describing mathematical objects, mathematicians, in both their gestures and their speech, draw extensively on the dynamic, material ways of thinking. Mathematics co-evolves with—and creates and is co-created by—its representational technologies.

Scholarship, such as that of Merlin Donald (1991) and Shaffer and Kaput (1999), emphasizes the role of representations in the development of human cognition. While the modern era of paper-based representations enabled the development of external symbolic representations, they argue that the new “virtual” culture (our post-modern age), which depends on the externalization of symbolic processing, will change the very nature of cognitive activity. Similar to those of McLuhan, such arguments depend on an epistemological assumption that knowing depends on the technologies (including the tools and representations) through and in which activity occurs. The widespread adoption and evolution of dynamic technologies implies the emergence of new dynamic mathematics. What are the implications for the mathematics curriculum?

A DIALETICAL VIEW (RETURN TO QUADRIVIUM)

We return to the quadrivium shown above, with a slight shift in the columns to indicate a more chronological depiction (time read left to right, since we are in paper!) of the four arts in relation to the technologies they presuppose (see Figure 3).

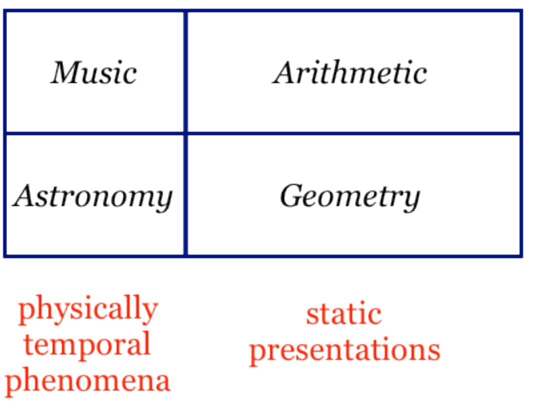

Figure 3. Revisiting the quadrivium

The presence of the post-modern age of technologies implies the addition of a third column to our matrix. Dynamic Geometry fits naturally into the column of the continuous subjects—with Astronomy and Geometry—in a progression from pre-modern, to modern, to post-modern technology. Instead of physically temporal (the compass) or statically antitemporal (the circle), Dynamic Geometry is *virtually* temporal (see Figure 4). This way of extending the quadrivium, which we have done through a historical study of technologies, leaves open the upper right cell, which should be concerned with the discrete, but also with virtually temporal re-presentation. We are structurally obliged to call this cell Dynamic Arithmetic or, perhaps more appropriately, to return to the goal we set ourselves in the title of this paper (On the Origins of *Dynamic Number* in the Breakdown of Structural, Metaphoric, and Historic Conceptions of Human Mathematics), Dynamic Number.

Conceptions of Dynamic Number already exist in familiar technologies that are not necessarily linked to the mathematics curriculum. As one ubiquitous example, many software programs enable users to change, in a number box, the value of a particular attribute that is simultaneously controlled by a slider or scrollbar. (Consider the line thickness setting in Photoshop.). These tools deploy numbers dynamically (in the sense that they can take on any value, sometimes in a given range)—and continuously (in the sense that they can be manipulated through continuously varying scrolling/sliding operations). Their actual number value at any instant is of less importance and than their possibility and promise of change, and (from the perspective of a manipulating agent) of their convergence toward desire.

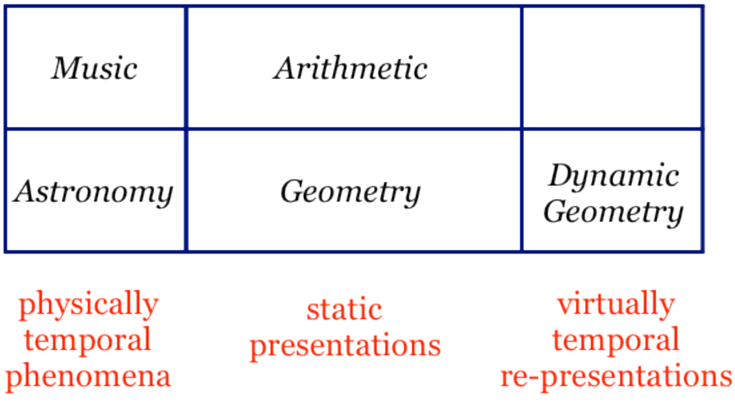

Figure 4. Technology-induced expansion of the curriculum

The spreadsheet amplifies this conceit to Stevinsonean dimensions, by replicating the single "dynamic number" across theoretically endless rows and columns of dynamic quantities. Actual numerical values cascade over these quantities as the effortless consequence of ever-changing input, and the very ephemerality, rather than the invariant essentiality, of specific values here, defines the larger "value" of the spreadsheet's mathematical model.

In our talk, we also explored several examples of dynamic number seen through Dynamic Geometry technology in the school curriculum. One such example was of the "dynamic number-line," which like its static counterpart, provides a geometric model of the real numbers that preserves order, scale and density. However, unlike its static counterpart, the dynamic number-line can be populated with points whose position—and value—can be moved. Where in Figure 5, the point happens to be located at 65.1, it can be continuously and dynamically dragged to 112.7—or any other value. Again, the actual value matters less, since the focus of attention becomes the motion of going right to increase value and left to decrease it. The draggability does not privilege any type of value, such as whole or integer, as static number lines often do; nor—as the third number line shows—does it fix itself to a certain scale.

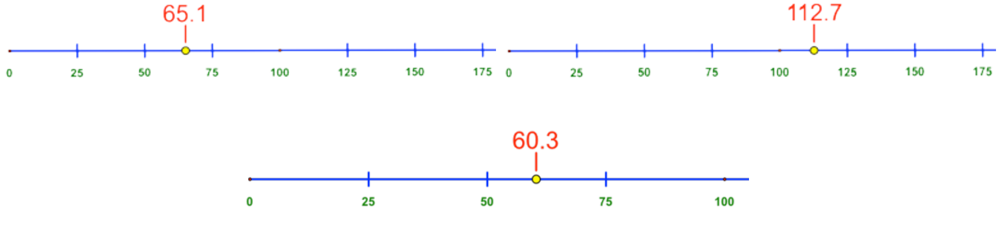

Figure 5. Points on the dynamic number-line

Restricting the domain to (0, 100) shows how the fundamentally relational idea of percentage (and proportion) can also be modelled by the number line. The "percentometre" tells us that the legs of a giraffe account for about 38% of its total height, and flipping the tool around, the head is about 1/3 of its neck (see Figure 6). The double motion of the scale and the point on the line, enables the part-whole numerical duplicity to turn into a single dynamic entity.

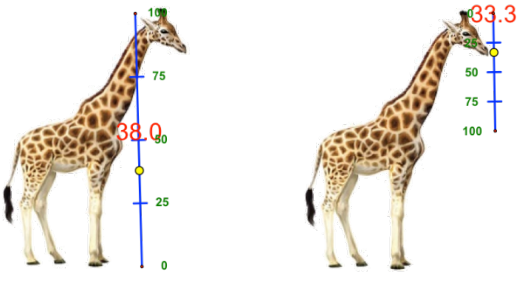

Figure 6. The “percentometre”

These examples of Dynamic Number may lead one to wonder whether, in fact, Dynamic Number is equivalent to Dynamic Geometry. Both draw on time and motion, which are the root metaphors of our embodied understanding of the mathematical continuum. At the same time, Dynamic Geometry is also a geometry of dynamics, and Gattegno tells us that an awareness of dynamics is at the heart of the study of algebra. And algebra’s expansion of arithmetic follows the same form as Dynamic Geometry’s expansion of “static” geometric relationships in its movement from instantiation to variation and from the possible to the general. In these senses, at least, Dynamic Geometry may after all be as epistemologically and technologically close to number as it has ever been to static geometry.

CONCLUSION

The idea of Dynamic Number, and the possibility of imagining a truly symmetric union between arithmetic and geometry, is clearly rooted in Stevin’s thinking, which, in a sense, gets derailed by the pervading modern technologies of paper and algebra. Nevertheless, we see this technology-driven examination of the quadrivium, which ends up at the Dynamic Geometry/Number finishing line, as the beginning of a mathematical and didactic voyage. Our school mathematics examples of Dynamic Number show how dynamic technologies can give rise to new mathematical conceptions, but that we are still swimming in the modern age understandings that shape our own historical understandings as presented in textbooks and curricula of the past (and present). When Dynamic Geometry software environments were first introduced, mathematicians balked: points were not supposed to move—this was not geometry! But the dynamic visualizations have taken hold, so that the idea of Dynamic Number is much easier to swallow. Still, much work needs to be done to investigate how it might be exploited and, perhaps more importantly, how it might affect current assumptions about schools’ curricular order of cognitive development.

While Shaffer and Kaput (1999) imagined a virtual culture in which mathematical processing could be achieved through digitally-sophisticated, post-paper representations, they imagined those representations as still *echt-symbolic*. Rotman (2008) instead points to the possibility of a post-alphabetic culture as a consequence of the continued development of digital technologies, which emphasise the visual, gestural, and haptic as primary modes of interpretation and communication. Does a Dynamic Number conception of denumeralized mathematics in some similar sense point to a post-symbolic, virtual, future culture? Certainly our own sense of the provocations we have examined here suggests alternate outcomes to the digital experiment than mere computational manservants shackled to the cause of curricular fidelity. The four mathematical arts of the Boethian curriculum have, in just under two millennia, been reduced to today’s Geometry and Arithmetic. We have imagined these arts’ post-modern and technological offspring as Dynamic Geometry and Dynamic Number. But if these two are in turn just halves of a single coin, then the quadrivium has produced a potential "univium" capable both of resisting inflationary rhetoric in the struggle between discrete and continuous, and of challenging the illusion of atemporal mathematics.

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New PhD Reports

Présentations de thèses de doctorat

ANALYSIS OF RESOURCES MOBILIZED BY A TEACHER AND RESEARCHER IN DESIGNING/EXPERIMENTING/REFLECTING UPON MODELLING SEQUENCES BASED ON ELEMENTARY COMBINATORICS AND AIMED AT INTRODUCING 7TH GRADERS TO MODELLING[1]

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OUR RATIONALE IN THIS STUDY

With hindsight, the main ideas that guided us throughout this research came from the work of the sociologist François Dubet (1994) and that of science education researcher Louis Martinand (1992). According to Dubet, we must acknowledge that there are many “gateways” between the world of “scholarly ideas” and that of “common sense ideas”, rather than opposing them. Thus, to advance knowledge, the task for researchers should be one of organizing an encounter between their perspectives on the actions/situations and that of practitioners. In a nutshell, there is a common ground between researchers and practitioners: which is *double relevance*. As for Martinand (1992), we borrowed from him the concepts of *practitioners' didactics* and *researchers' didactics*[2] which highlight experienced teachers as well as researchers contributing perspectives in supervising prospective teachers.

AT THE START: A DUAL PROBLEMATIC

As far as the teaching of elementary combinatorics is concerned, we perceived at the start of this research two shortcomings: one relates to standard teaching approaches, mostly characterized by a focus on models application, and the other pertains to the way teaching situations are developed (that is, so far it has been most of the time the business of researchers alone). Hence our endeavour is a) seeking an alternative approach focusing on model building (i.e. modelling) and b) taking into account teachers' perspectives on modelling. Thus, the main focus of this study was on teacher-researcher design, experiment and analysis of two instructional sequences based on elementary combinatorics and aimed at introducing

[1] Our translation. The initial dissertation's French title is : “*analyse des ressources mobilisées par enseignant et chercheur dans l'élaboration de scénarios d'enseignement en dénombrement visant le développement de la modélisation en secondaire 1*”. As this report gives only an overview, interested readers are invited to look at the book “As-tu vu les modèles? Conversation entre chercheur et enseignant” (Barry, 2010), which is the published version of the dissertation.
2 Actually, Martinand (1992) used the expression “critical and prospective didactics” (in French, *didactique critique et prospective*) instead of “researchers' didactics” we use here as to better capture his idea.

modelling. We documented and analyzed the collaborative process informed by collaborative research methodology (Bednarz, 2004). In what follows, we will present our research questions, theoretical framework, and research methodology, and share a few results of our study.

RESEARCH QUESTIONS

The main goal of this research was to analyze the contributions that we as a researcher and a teacher bring into the design, experiment and analysis of modelling sequences. The subsequent research questions pertained to these contributions in terms of:

- the tasks designed together;
- students' modelling processes;
- a teaching approach aimed at introducing modelling at 7th grade level.

THEORETICAL FRAMEWORK

Due to the collaborative character of our endeavour, our theoretical framework is an attempt to clarify concepts which can give insights into the contributing resources a teacher and a researcher (may) draw on/mobilize when working together. For the researcher it is obvious that his perspectives with regards to combinatorics and the modelling process could play a role in the collaboration, whereas the teacher may turn to what Martinand (1992) describes as practitioners' didactics. Other concepts have to be considered: the idea of theoretical and practical rationality (Weber, 1971; Desgagné, 1994), the notion of routines (Leinhardt, 1986), the concept of structuring resources (Lave, 1988). In what follows we present and discuss the researcher's perspective on modelling/models and what we termed (at the onset of this study) as the teacher's anticipated perspective during the collaborative process.

RESEARCHER'S PERSPECTIVE ON MODELLING/MODELS

Our modelling perspective is an integrative one as it draws on various streams such as applied modelling, theoretical modelling, educational modelling, and cognitive modelling, to give a few examples (see the survey of international perspectives on modelling in mathematics education, Kaiser & Sriraman, 2006). We see mathematical modelling as a cyclical process with two important and related stages: formulation and validation (Janvier, 1996), each being carried out through steps akin to those of many "models" of the modelling process (see for example Maaβ, 2005). Modelling is fundamentally concerned with creating and elaborating on models of situations (real, fictitious, etc.) by means of mathematical instruments. Moreover, in groups dealing with modelling tasks together, a modelling culture (Tanner & Jones, 1994) is shared whether the modellers are experts or novices. Considering students' modelling activities, following Gravemeijer's line of thinking, we focus on students' emergent models and on emergent modelling. The label "emergent" refers to the process by which models are first tied to a specific context/situation (a model-of) and later evolve by losing their dependency to a given situation (a model-for) (see Figure 1). This transition from model-of to model-for is related to the four levels of activity (activity in the task setting, referential activity, general activity, and formal mathematical reasoning) proposed by Gravemeijer (1999) and involves some sort of abstraction. As for emergent modelling, it can be defined as a learning process, preceding mathematical modelling, and which is better suited to the purpose of introducing lower secondary level students to the demanding process of mathematical modelling (Gravemeijer, 2007).

As for our perspective on combinatorics, we entered into the collaborative process having in mind a) existing conceptualizations of the notion of model and b) various aspects of combinatorial models underlying elementary combinatorial problems (see Figure 1). Broadly interpreted, mathematical models are more than equations or formulas and entail also iconic pictures, schemes, diagrams, symbols, operations, and so on (Van Den Heuvel-Panhuizen, 2003). For us, combinatorial models fall into two categories: external models (including figurative representations) and conceptual models which generally convey the fundamental structures underlying the various combinatorial tasks proposed to students (examples are: the three configurations of selection, distribution and partition, or some notorious combinatorial rules like the rule of sum or the pigeon-hole principle).

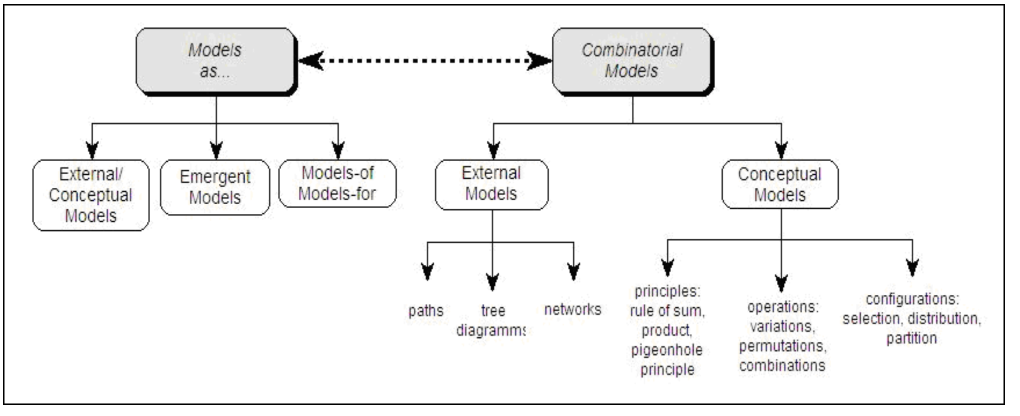

Figure 1. On models and combinatorial models

TEACHERS' ANTICIPATED PERSPECTIVE

Central here is the concept of *practitioners' didactics* which refers to what experienced teachers in the context of supervisions demonstrate or can make explicit as *knowledge of how to teach*. Seductive, this concept nevertheless has not been much developed in the literature, and so we tried to offer a first characterization for it. For that purpose Schon (1983), Leinhardt, Weldman, and Hammond (1987), Weber (1971) and to some extent Desgagné (1994) gave us valuable clues. Thus, anticipating the teacher's perspective during the collaborative process, we considered his practical knowledge (Schon, 1983) as a *situated* and *strategic* knowledge of how to teach in different settings (strategic as the goal of experienced teachers is always to succeed in engaging students into the learning process). Besides, as teachers have good and valid reasons to act as they act, practitioners' didactics should be construed as guided by a practical rationality (Weber, 1971) on which classroom routines (Leinhardt et al., 1987) of various kinds (support, management or exchange routines) draw.

TEACHERS' AND RESEARCHERS' CONTRIBUTIONS AS STRUCTURING RESOURCES

At the very beginning of our study, Lave's concept of *structuring resources* (Lave, 1988) had for us the potential of helping in further characterizing our respective contributions. Here again (as with the notion of practitioners' didactics), we were faced with a very seductive but elusive concept, that of structuring resources which needed to be better defined beyond Lave's vague indications as to the nature of such resources, (i.e. in our section dealing with results, we will show how in our emergent analysis of data we came to a more precise perspective on structuring resources). Going back to our earlier research questions, and informed by the concept of structuring resources, we tried in our study to document and analyze teachers' as well as researchers' respective contributions in terms of the *resources* (and ultimately the reference frames) they (might) mobilize in designing/experimenting/ reflecting upon modelling sequences aimed at introducing 7th graders to modelling.

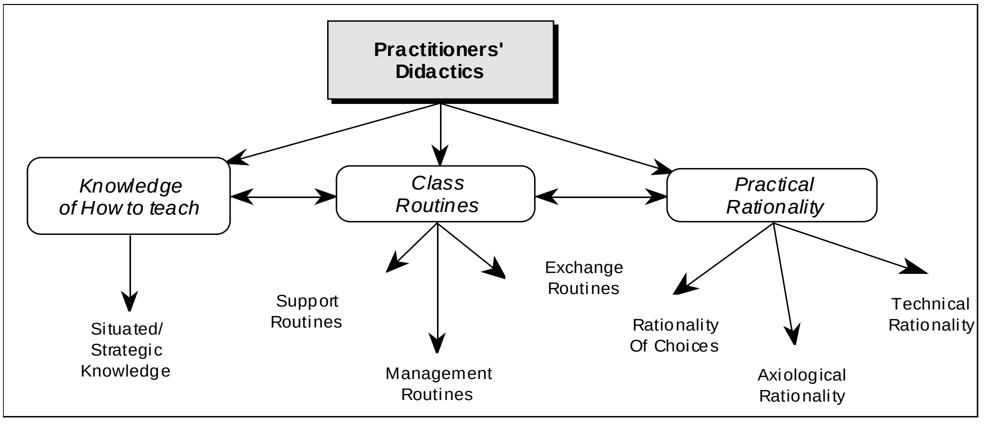

Figure 2. Key aspects of practitioners' didactics

RESEARCH METHODOLOGY

This study used a collaborative research methodology (Bednarz, 2004) whose focus is on doing research *with* practitioners (here a teacher) rather than doing research *on* practitioners.[3] Figure 3 sums up the three related stages of the research model developed in Quebec by Bednarz and Desgagné (see also Desgagné, Bednarz, Lebuis, Poirier, and Couture (2001) for further reading). With regards to co-situation, our concern with the modelling process had to be shared by any teacher with whom the research would be conducted. Fortunately, we found a teacher interested in developing his students' competency in "*solving situational problems related to mathematics*" as required by the curriculum in Quebec (MEQ, 2001), and for whom modelling appeared as an interesting avenue. Indeed, it is worth mentioning that the descriptive of that competency explicitly involves teaching students "*to **model** situations*" (MEQ, 2001). The experiment took place in two 7th grade classes in downtown Montreal with an average of 30 students per class.

At the co-operation stage, the research was structured around "reflexive meetings" and classroom experiments in which both the researcher and the teacher took part. The discussions focused on task design and classroom events analysis, and efforts were not only on analyzing/modifying the modelling tasks, but also in anticipating students' models, strategies and solutions. We also considered various ways of handling the activities. All researcher-teacher reflexive meetings were recorded and transcribed. Field notes on classroom events during the experimentation were also taken by the researcher, presented to the teacher (who commented on them), and used as a basis for discussions. Leading us into the co-production stage, this work helped us ensure that the knowledge produced by the research took a form useful to both teachers and researchers. And this includes the results of the further analysis of all these data, conducted using grounded theory (Strauss & Corbin, 1990). Important to highlight, we adopted a non-naïve perspective on grounded theory, where we accept the

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[3] To avoid any misunderstanding, we have to indicate that collaborative research methodology is not an attempt to dismiss valuable research on teachers, but is meant to bring researchers and practitioners closer and ultimately tackle the issue of the viability in teachers' practices of research theories/innovations.

existence of concepts close to our theoretical field, but which need to be reconstructed by the researcher to make sense of data. For such concepts Desgagné (1998) proposed the notion of available concepts («concepts disponibles» in French): in our analysis, Lave's (1988) "structuring resources" was one of them.

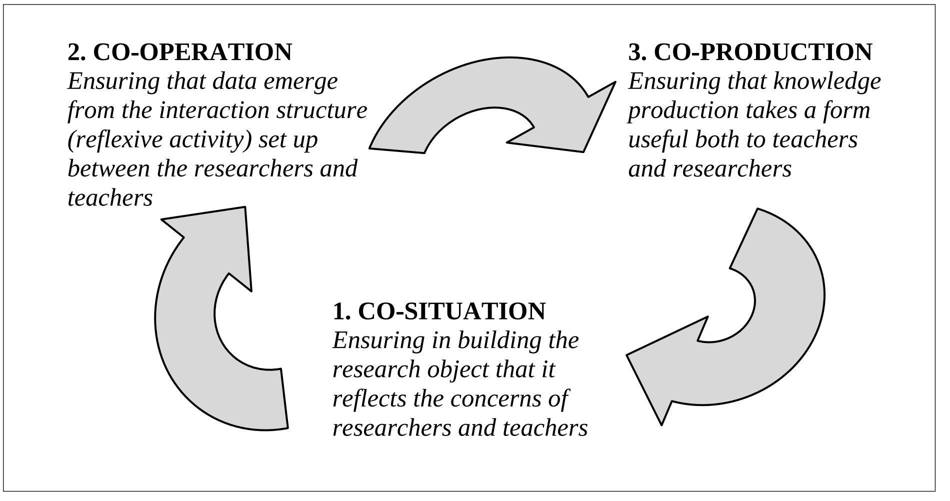

Figure 3. Stages in our model of collaborative research

RESULTS

Using grounded theory, our inductive analysis of the contributions of the teacher and the researcher led us to refine the concept of *structuring resources*. We identified various resources mobilized by the teacher and the researcher. These resources enable us to understand their respective perspectives on a) the combinatorial tasks at stake, b) students' modelling activities and c) a teaching approach aimed at introducing modelling. These resources are of two kinds: interpretative resources and action resources.

INTERPRETATIVE AND ACTION RESOURCES

Interpretative resources helped both teacher and researcher make sense, advance a reading, of issues addressed in this joint undertaking. As shown in Figure 4 these issues ranged from analyzing tasks, anticipating students' solving strategies, refining tasks, to planning the resulting sequences. In some cases the teacher's interpretative reading confirms, conflicts, or extends that of the researcher. Our study thus extends the notion of interpretative resources as envisioned in Experience Sociology (Dubet, 1994) which defines such resources especially in terms of argumentative and critical resources, enabling participants to position themselves with regards to theories proposed by researchers. In that perspective, interpretative resources are defined dichotomously in terms only of agreements and disagreements between actors and researchers. In between the two, agreeing or disagreeing, there is room for nuance.

As for action resources, we proposed this new concept to characterize data pertaining to resources other than interpretative resources, such as guidelines, ways of tackling given issues, proposals for concrete/practical task design or management, and so on. As with interpretative resources, action resources mobilized by the researcher and the teacher are of three sorts: they are either nested (teacher or researcher appropriates the other's resources), conflicting, or echoing one another (teacher or researcher develops, expands resources put forward by the other). Figure 5 identifies some of the action resources mobilized in the process of re-structuring tasks considered for the first modelling sequence built during the research.

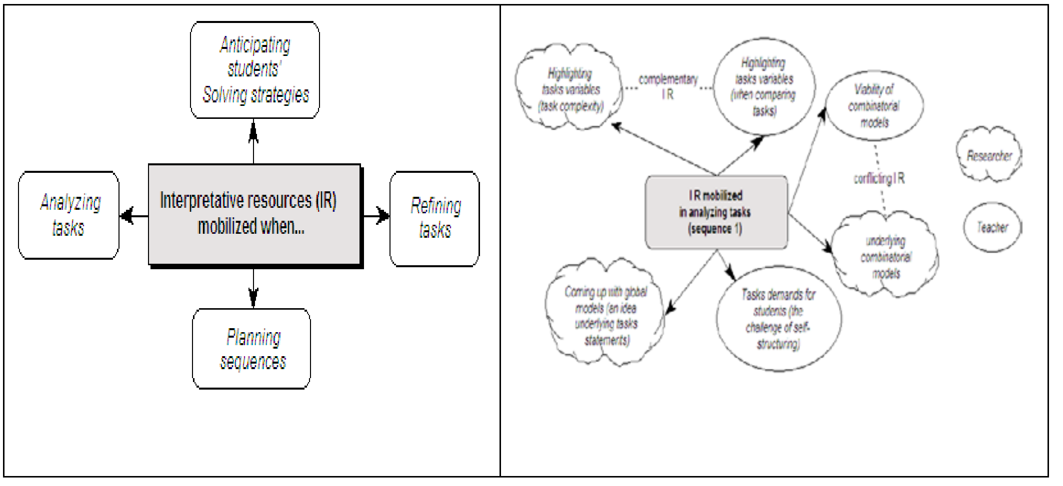

Figure 4. Examples of interpretative resources

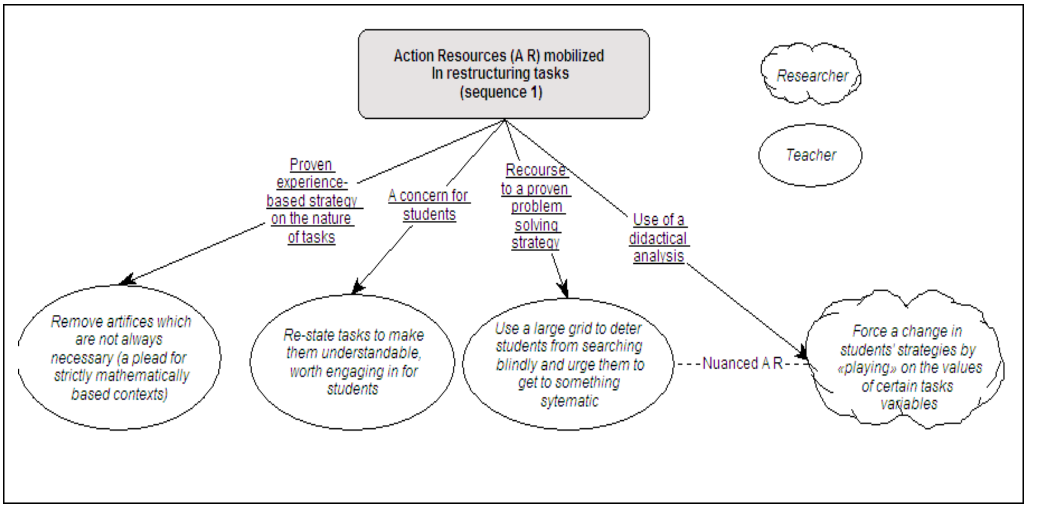

Figure 5. Examples of action resources

PRACTITIONERS' AND RESEARCHERS' DIDACTICS

Our research contributed in "fleshing out" the concepts of practitioners' didactics and researchers' didactics (Martinand, 1992) which remained unexploited, undeveloped. In the case of the teacher we were able to access several aspects of his didactics through a reference frame at work in his perspective on tasks, students and their evolution, and classroom experiments of the modelling sequences built together with the researcher. His practitioner didactics appeared shaped by different kinds of knowledge (knowledge of students, textbooks and curriculum), a mode of analysis characterized by a strong concern for students (their difficulties, weakness, interest, and possibilities), a concern for the installation and maintenance of classroom routines supporting problem solving activities, and an underlying rationale revealing various pursued purposes and principles (both didactical and pedagogical).

As for the researcher his didactics takes the form of a theoretical perspective, building upon his knowledge of combinatorics and the modelling process, a mode of analysis focused on the very nature of tasks (highlighting tasks variables as an approach in analyzing problems/situations often used by mathematics education researchers) and the modelling process, but also an underlying rationale revealing various pursued purposes and didactical principles.

CONCLUDING REMARKS

In conducting this research our perspective in modelling evolved. In that respect, our modelling perspective is an emergent one and happened to be a fallout of the collaborative process which challenged us to take into account the peculiar context of two 7th grade classrooms under the responsibility of a teacher who cared a lot about his students and never bothered compelling us to a down-to-earth approach in our common endeavour to introduce actual (and not epistemic) 7th graders to the often very demanding process of modelling. As mentioned earlier, we were faced in conducting this collaborative research with the daunting challenge of better characterizing two very seductive but elusive concepts: practitioners' didactics (Martinand, 1992) and structuring resources (Lave, 1988). In that respect, this study paves the way for research aimed at helping better understand expert teachers' situated practical knowledge and the various kinds of structuring resources they mobilize.

Finally, this study shows the richness of a dialogue between practitioners' didactics and researchers' didactics. We tried and succeeded in organizing an encounter between our perspective and that of a teacher as Dubet (1994) advocates. Practitioners' didactics and researchers' didactics can meet, in analysis as well as in action, and in ways that transcend conventional attributions in which researchers are considered only as designers and teachers only as executants. Last but not least, with respect to the introduction to modelling at lower secondary level, this research allowed us to highlight features worth considering. In upcoming papers we will focus on those features we consider as winning features.

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BEING (ALMOST) A MATHEMATICIAN: TEACHER IDENTITY FORMATION IN POST-SECONDARY MATHEMATICS

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The purpose of this research project was to uncover issues and difficulties that come into play as mathematics graduate students develop their views of their roles as post-secondary teachers of mathematics. Over a six-month period conversations were held with mathematics graduate students, exploring their experiences with and perspectives of mathematics teaching. Using hermeneutic inquiry and thematic analysis, the conversations were analysed and interpreted with attention to themes and experiences that had the potential to influence the graduate students' ideas about and approaches to teaching. Themes that are explored are: the structures of teaching assistant work, teacher versus professor, replication of not only teaching but also of identity, and resignation. Lave and Wenger's (1991) notion of legitimate peripheral participation is used as a framework to understand the mathematics graduate students' progression to becoming post-secondary teachers of mathematics.

INTRODUCTION

Mathematics departments are often one of the largest departments within institutions of higher education, providing prerequisite courses for students in diverse disciplines such as engineering, psychology, chemistry, business, medicine, physics, and education. Consequently, the teaching of mathematics at the university level is quite important in undergraduate education, and professors, instructors, and graduate teaching assistants in mathematics have a wide-reaching influence on the education of future researchers, teachers, and mathematicians (Golde & Walker, 2006). However, the format and style of post-secondary mathematics teaching has remained problematic for undergraduate success in mathematics and the sciences (Alsina, 2005; Kyle, 1997; National Science Foundation, 1996; Seymour & Hewitt, 1997).

Almost seventy-five percent of mathematics PhDs will become professors at post-secondary institutions dedicated to undergraduate education rather than research (Kirkman, Maxwell, & Rose, 2006), and so the development of teaching practices during graduate programs is essential in preparing mathematics graduate students for their possible future appointments. As a result, the preparation of the future mathematics professoriate has recently become a subject of investigation. In particular, the development of mathematics graduate students' teaching practices has become a focus of research for mathematicians and mathematics educators alike (e.g. Bass, 2006; Belnap, 2005; DeFranco & McGivney-Burelle, 2001; Speer, 2001).

The most recent research into mathematics graduate students' teaching has examined their classroom practices and possible connections between their practices and beliefs about teaching and learning. Researchers concluded that newly acquired positive attitudes and beliefs about teaching mathematics did not bring about hoped for changes to graduate students' teaching practices (Belnap, 2005; Speer, 2001). Although the mathematics graduate students in these studies developed a new vocabulary for discussing and describing teaching, these students also reported that they maintained a lecture-style form of instruction (Belnap, 2005). Other research has shown that enrolment in a course in pedagogy also did not produce expected changes to mathematics graduate students' teaching practices (DeFranco & McGivney-Burelle, 2001). In light of these conclusions, it appears that the experiences of mathematics graduate students and the development of their teaching practices are not yet understood.

The purpose of this research study was to understand what obstacles and issues might exist for mathematics graduate students that might prevent teacher preparation programs from taking root and being successful. Thus, this is a study of the experiences of graduate students in mathematics, who are in the process of becoming mathematicians and, most likely, future post-secondary teachers of mathematics. The goal of this study was to answer the questions: How do graduate students come to understand their roles as mathematics teaching assistants and possible future professors of mathematics? What experiences do graduate students in mathematics interpret as having meaning for whom and how they should be as mathematicians and professors of mathematics?

THEORETICAL FRAMEWORK

Lave and Wenger (1991) have offered the term *legitimate peripheral participation* in relation to a community of practice to name one central process by which novices gain knowledge and understanding about the practices of a community. This concept is described more fully as "*learners inevitably participate in communities of practitioners and that the mastery of knowledge and skill requires newcomers to move toward full participation in the sociocultural practices of the community*" (p. 29). As such, the concept of legitimate peripheral participation offers an interesting lens through which to interpret and understand what might be happening for the mathematics graduate students.

Lave and Wenger wrote "*Communities of practice have histories and developmental cycles, and reproduce themselves in such a way that the transformation of newcomers into old-timers becomes remarkably integral to the practice*" (p. 122). Further, they claimed:

> *Even in cases where a fixed doctrine is transmitted, the ability of the community of practice to reproduce itself through the training process derives not from the doctrine, but from the maintenance of certain modes of coparticipation in which it is embedded.* (p. 16)

Moreover, within the framework of legitimate peripheral participation exist issues of identity where Lave and Wenger describe how "*the development of identity is central to the careers of newcomers in communities of practice*" where "*learning and a sense of identity are inseparable*" (p. 115). Thus the notion of legitimate peripheral participation prompts an interesting question for this study: How are mathematics graduate students' identities as professors of mathematics formed as they learn about legitimate practices and become a part of the community of practice?

MODE OF INQUIRY

As "*hermeneutics […] holds out the promise of providing a deeper understanding of the educational process*" (Gallagher, 1992, p. 24), hermeneutic inquiry was chosen as the mode of inquiry for exploring the experiences that mathematics graduate students face in their programs. Hermeneutics helps to understand how we create and find meaning through experience and social engagement (Brown, 2001). Davis (2004) offered a description of hermeneutics as a mode of inquiry that asks, "*What is it that we believe? How did we come to think that way?*" (p. 206). Hermeneutic inquiry into mathematics graduate students' understandings, experiences, and ideas about teaching compelled a look at what is present in the structures of departments of mathematics that might cause these future professors of mathematics to adopt the teaching methods that persist as part of their role in maintaining "*certain modes of coparticipation*" rather than develop through their own interests in educating undergraduates about mathematics.

Because of its recognition of the interpretive work of data analysis, Braun and Clarke's (2006) six-stage process for thematic analysis was coupled with hermeneutic inquiry. Thematic analysis is flexible and "*has the potential to provide a rich and detailed, yet complex, account of data*" (p. 78). Further, the stages of thematic analysis are in accord with Laverty's (2003) description of a hermeneutic project where "*the multiple stages of interpretation allow patterns to emerge, the discussion of how interpretations arise from the data, and the interpretive process itself are seen as critical*" (p. 23).

RESEARCH STUDY

Graduate students in mathematics from an urban, doctorate-granting university were approached to be participants in this study. Six students agreed to participate. The group was fairly diverse in their backgrounds: three were master's students, three were doctoral students, ranging from a first semester master's student through a final year doctoral student; four were men, two were women; their ages ranged from 22 to 33 years; and there were four nationalities among them. While each of their paths to graduate study in mathematics was distinct, all but one of the participants expected to work in academia once they completed their degrees. During their graduate programs in mathematics, each of the participants had been assigned to teaching assistantship duties such as tutoring workshops where they helped students one-on-one with homework exercises, grading homework and exam papers, or leading one-hour sessions during which they presented mathematical topics similar to those in the affiliated lecture section of the course.

Carson (1986) and van Manen (1997) propose conversation as a mode of doing research within hermeneutic inquiry to explore and uncover one's own and others' interpretations and understandings of experience. In consideration of this, over a period of six months, a series of five audio-recorded conversations were conducted with the research participants. The first two meetings and the final meeting were conducted with each participant individually, where each meeting lasted approximately one hour. The third and fourth meetings were conducted with all participants present, each lasting just under three hours. A recursive process was used in which the topic of subsequent conversations was based upon themes from previous conversations. Throughout the project, the research participants had the opportunity to review the analyses in a collaborative effort to refine, augment, and improve the reporting of their experiences.

Each conversation was transcribed by the researcher, who listened for the topics of conversation and the language used by each of the research participants. Notes were made of the congruence among the research participants. Similarities were not limited to broad categories of their lives, such as how they each had to attend to their teaching assistantship duties or their graduate level course work. Opinions and perspectives about various aspects of their experiences also surfaced as important commonalities. These were grouped into themes using the guidelines of thematic analysis described by Braun and Clarke (2006) and van Manen (1997). The themes and the participants' comments within each theme were then assembled and analyzed using a hermeneutic, interpretive lens to understand what facets of their lives in graduate school were taken as having meaning for their identities as mathematicians and post-secondary teachers of mathematics.

FINDINGS

The participants in this study lacked a forum to discuss their views and explore different ideas for teaching, and they were left to find meaning in their experiences. They were left to creating understanding amongst themselves, relying solely on the reproduction of the teaching, the community, and the unitary identity they observed. They were limited in what they could do by the structures of the department and by the mentoring, or lack of it, from their faculty advisors. They had become resigned to a notion that there was only one way to teach mathematics and one way to be as a professor of mathematics. In their process of becoming mathematicians, they experienced a sense of reproducing an observed identity as post-secondary teachers of mathematics. These observations are explored in the themes below.

THE STRUCTURES OF TEACHING ASSISTANT WORK

The time and physical structures of the graduate students' work as teaching assistants were said to prevent them from being able to engage in meaningful experiences with undergraduates. In the tutoring centre, the number of students waiting for help and the hours spent helping students repeatedly with the same questions quickly diminished the graduate students' ability to provide meaningful learning experiences. The frustration and exhaustion within the tutoring centre was common among the graduate students. There was also a sense of disappointment of how things took place over time. In this regard, the graduate students weren't able to observe the undergraduate students' progress and understanding of concepts, and so the act of tutoring was felt as an unrewarding and tiring experience. One participant described how the lab situations became "*how fast can you turn them over*." Rather than being able to provide the undergraduate students with an in-depth learning experience, when there were many undergraduate students waiting for help, it became "*a lot faster to plug and chug*." Finally, another participant expressed the heavy load and exhaustion of the teaching assistant work: "*I just want to get the hell out of there*."

I AM PROFESSOR – HEAR ME NOT TEACH

The participants spoke adamantly about how university professors are not teachers. One participant said "*This is the first thing we need to get across is that professors and teachers are two completely different things*," while another participant remarked "*I never really saw them as teachers. I never saw them as teachers. I always knew there was a line between teachers and professors.*" With that perspective in mind, what did the participants learn from their professors about being teachers or teaching, especially when they looked to their professors as models for how they should be in the classroom? Lave and Wenger's (1991) work provides an interesting insight here. They stated, "*If masters don't teach, they embody practice at its fullest in the community of practice. […] Identities of mastery, in all their complications, are there to be assumed*" (p. 85). The participants' views of who they would become as professors became fixed to what they observed in the "masters" of their departments, and, as a result, to the idea that they would not be teachers.

REPLICATION OF IDENTITY AND PRACTICE

Similar to Lave and Wenger's (1991) idea that communities "*reproduce themselves*" (p. 121), the post-secondary teaching of mathematics, as viewed by the participants, appeared to be a practice of replication, a reproducing of others' teaching and the material in mathematics textbooks. The participants spoke of how they could rely solely on other professors' notes or on the mathematical material found in the textbooks, signifying the replication of material as sanctioned tradition in mathematics. Specifically, one participant spoke of the structure of all mathematics courses as "*definition, theory, example*," where replicating the fixed structures of mathematics texts and courses was a sufficient way of teaching mathematics. Another participant remarked, "*It's easy to keep teaching calculus like this. We've been doing it forever*," while another participant stated that for different sections of a course "*You pretty much do the same thing.*" The replication of practice in this study parallels what Seymour and Hewitt (1997) found, that "*teaching assistants had not received any instruction on how to teach effectively, were teaching in the same way that they themselves had been taught, and were, perforce, repeating the pedagogical errors of their professional mentors*" (p. 160).

The notion of a unitary identity for a professor of mathematics was also observed in the conversations. With regard to his own teaching, one participant spoke of how he could not work "*outside of a certain box*" in the department. Another participant stated, "*I don't really care who teaches calculus and who doesn't*," and so to him the person, and one's identity, did not seem to be of importance. This perspective came through for another participant who said, "*I don't know how much variety you can actually put in. How would professor A be different from professor B?*" In this statement was an interchangeability between professors, as though their identities might be so alike that it would not matter who was in the classroom. This resonates with Jardine's (2006) view that in mathematics there exists a "*mood of detached inevitability: anyone could be here in my place and things would proceed identically*" (p. 187).

RESIGNATION

The act of replication of mathematics teaching and the thought of taking on a particular identity in mathematics evolved into feelings of resignation among the participants. With regard to his current role as a graduate student one participant said, "*You can't have an opinion; you can't have anything except 'yeah, this is true.'*" Here it seemed that the participant was resigned to a passive position in his role as a mathematics graduate student. Further, when speaking about the possibilities for his future teaching practice and, in particular about the use of teacher-student discussions in mathematics, he said, "that's *never going to happen in math*," a statement that expressed a resigned view that there are not alternative possibilities for what can occur in mathematics classrooms.

Concerning his own observations of the ways in which the undergraduates were being taught by professors in the department, one participant remarked "*I might have the same complaints, but there is nothing I can do about it*," signalling a resignation to being unable to change the way mathematics courses are taught or structured. Even though he had acknowledged a previous interest in teaching and changing the way undergraduates are taught mathematics, another participant said "*I would not be able to change things even if I wanted to*." When this participant spoke of his future career as an academic, teaching was no longer of consequence to his success as a mathematician and future professor. In the final year of his doctoral program, this participant was an illustration of what Lave and Wenger (1991) refer to as the "*transformation of newcomers into old-timers*" (p. 121) and how "*an extended period of* *legitimate peripherality provides learners with opportunities to make the culture of practice theirs*" (p. 95). In this regard, it was clear that teaching was not a part of the culture of practice in the department.

IMPOSING LEGITIMATE PERIPHERAL PARTICIPATION ON THE POSSIBLE NEXT GENERATION

Finally, it was interesting to observe how the notion of legitimate peripheral participation made the participants' reactions to and interactions with undergraduates more comprehensible. In an interesting turn, it became the graduate students who had expectations of legitimate participation for undergraduates. The participants offered more in-depth learning experiences to undergraduate students who behaved in ways that were deemed sufficiently mathematical or displayed behaviour which demonstrated that the students treated mathematics as important. In contrast, students who did not exhibit such mathematical behaviour were not offered these opportunities. It appeared that the participants would only interact in meaningful ways with students who demonstrated legitimate behaviour, as though, along with the legitimate ways of becoming and being a mathematician, there were also legitimate ways of being an undergraduate student in mathematics.

DISCUSSION

The structures of the participants' programs, their teaching assistant work, and the suggestions that were put forth by the department either through direct communication or the lack of it seemed to point to a particular, sanctioned way of being and becoming a mathematician, a way of being which implied not only that teaching was unimportant, but also that it would be determined solely by what had been observed in texts and other professors' classrooms. Throughout the analysis of the recordings and transcripts, it became apparent that the participants began to make certain tasks more important than others through what they were and were not allowed to do as newcomers in the department. In this regard, it seemed that the participants were being primed for a particular way of being.

What became clear in this project is that the mathematics graduate students were on a path to becoming mathematicians, not post-secondary teachers of mathematics. They learned about the discipline of mathematics through their coursework and about mathematical research through undertaking their theses and dissertations. To be successful in their programs, they were compelled to become skilled in mathematics by mastering their coursework through earning high marks and undertaking a research project. In contrast to the mathematics they were required to become skilled in, the mathematics graduate students were not required to demonstrate competence in teaching, in how they interacted with students, how they presented material to a class, or how they assessed students' learning. There was no point in their programs where they were evaluated on their teaching, even though teaching had the potential to become a very important part of their future profession. The research work that they had to focus on, in a way, became the sole indicator of what would be expected of them in their careers.

Lave and Wenger (1991) remarked about identity: "*We have claimed that the development of identity is central to the careers of newcomers in communities of practice, and thus fundamental to the concept of legitimate peripheral participation*" (p. 115). Framed by the idea of legitimate peripheral participation, through their process of becoming mathematicians, the participants in this study seemed to move from a peripheral position to a slightly more central standing in the community as their identities became closer to that of a mathematician. This transition was not overt, nor was it explicitly stated anywhere. The participants did not report a public statement or even an acknowledgement that they had to abandon their own ideas about teaching, that they should no longer consider teaching important and, by maintaining "certain modes of coparticipation," they would move toward a more central position in the department. Rather, it seemed that the set-up, the structure of the department, the behaviours that were legitimate, and the progression to becoming a mathematician rendered it so.

One hope for this project was to find understanding as to why education programs for mathematics graduate students had failed to instil hoped for changes in future university teachers of mathematics (Belnap, 2005; DeFranco & McGivney-Burelle, 2001; Speer, 2001). It became clear in this study that the notion of teaching is problematic in university level mathematics. I suggest that before addressing the question of how mathematics graduate students might be prepared for teaching, an understanding of what it means to teach post-secondary mathematics is necessary. A question that I believe is important to explore and is posed to the departments of mathematics that are charged with educating mathematics graduate students: Why does the notion of teaching appear to be so disconcerting in post-secondary mathematics?

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SUBJECTIVE PROBABILITIES DERIVED FROM THE PERCEIVED RANDOMNESS OF SEQUENCES OF OUTCOMES

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Given the accessibility of the Internet, I have decided, for this article, to present the ABSTRACT, PREFACE, and INTRODUCTION of my dissertation. If, for some reason, you are interested in reading more about this research, you can freely access a full copy of my dissertation by searching for my name on the Simon Fraser University Library Webpage or simply entering: http://ir.lib.sfu.ca/handle/1892/10918. If, for some reason, you are interested in reading more about what I have been doing since my dissertation, please visit www.eganchernoff.com.

ABSTRACT

This research continues the longstanding tradition of taking an interdisciplinary approach to studies in probability education. Respondents are presented with sequences of heads and tails derived from flipping a fair coin five times, and asked to consider their chances of occurrence. A new iteration of the comparative likelihood task, which maintains the ratio of heads to tails in all of the sequences presented, provides unique insight into individuals' perceptions of randomness and associated probabilities. In order to develop the aforementioned insight, this research presents unconventional interpretations of the sample space – organized according to switches, longest run, and switches and longest run, which are all based upon individuals' verbal descriptions of the sample space – to help situate individuals' answers and justifications within conventional probability. In doing so, it will be shown that conventionally incorrect responses to the task are not, necessarily, devoid of correct probabilistic thinking.

The data for this research is based upon two interrelated questionnaires, denoted Study I and Study II. Answers and justifications from the 56 prospective elementary school mathematics teachers in Study I were employed to develop the second iteration of the questionnaire in Study II, which was given to 239 prospective mathematics teachers (comprised of 163 elementary school teachers and 76 secondary school teachers).

To accurately render the data compiled in Study II, an original theoretical framework, entitled the *meta-sample-space*, will be used with a new method, entitled *event-description-alignment*, to demonstrate, for the first time, that individuals' probabilities, derived from the perceived randomness of sequences of outcomes, are in accord with, or model, a *subjective-sample-space* partitioned according to said individuals' interpretation of the sequence of outcomes they are presented.

PREFACE

At the very core of my research is subjective probability. In fact, I see my research as the culmination of an intense interest with subjective probability. While certain mathematics education researchers claim, at present, that cognitive research on the subjective approach to probability does not exist, I disagree. In fact, I would argue that mathematics education literature and psychology literature are saturated with cognitive research on the subjective approach to probability. For example, I would argue that the normative, heuristic, and informal approaches to probability all fall under the category of cognitive research on subjective probability. In other words, because research has not been conducted specifically on the degree to which an individual believes in a particular proposition, does not mean there is an absence of cognitive research on subjective probability. Influencing the difference between my opinion and the opinion of others is how one defines subjective probability. Complicating matters, subjective probability means different things to different individuals. Alternatively stated, subjective probability is subjective. As such, and for contextual purposes, I present certain research that has influenced my interpretation of the present state of subjective probability in the field of mathematics education.

I contend the lack of a unified definition for subjective probability best captures and influences the current state of subjective probability in mathematics education. Further, the definitional issues that exist in mathematics education also exist in probability theory. In fact, the issues seen in mathematics education are, for the most part, derived from issues in probability theory. Gillies (2000) states:

> *The difficulty with this terminology is that the 'subjective' interpretations of probability include both the subjective theory of probability, which identifies probability with degree of belief, and the logical theory, which identifies probability with degree of rational belief. Thus, subjective is used both as a general classifier and for a specific theory. This is surely unsatisfactory.* (p. 19)

To address the difficulty presented, probabilistic philosophers have made further distinctions within subjective probability. For example, Hacking (2001) notes a distinction between "*personal*" probabilities and "*interpersonal*" (p. 32) probabilities. Unfortunately, the distinction between personal and interpersonal within subjective probability does not have a counterpart in mathematics education, yet. As such, and at present, subjective probability aligns with both the personal and interpersonal theories, and the lack of counterparts is representative of the current state of subjective probability measurement in mathematics education.

The state of subjective probability in mathematics education is also influenced by philosophical underpinnings. "*The research of psychologists Daniel Kahneman and Amos Tversky…has provided mathematics educators with a theoretical framework for researching learning in probability*" (Shaughnessy, 1992, p. 470). Kahneman and Tversky's (1972) use of subjective probability is as follows:

> *We use the term "subjective probability" to denote any estimate of the probability of an event, which is given by a subject, or inferred from his behavior. These estimates are not assumed to satisfy any axioms of consistency requirements. We use the term "objective probability" to denote values calculated, on the basis of stated assumptions, according to the laws of the probability calculus. It should be evident that this terminology is noncommittal with respect to any philosophical view of probability.* (p. 430)

Despite the noncommittal stance from (arguably and with all due respect to Piaget and Inhelder) the fathers of probability education and the polysemic nature of the term *subjective* *probability*, the current state of subjective probability in mathematics education is, I contend, aligned with the personal interpretation of subjective probability. In an attempt to demonstrate my assertion, definitions for subjective probability in mathematics education from significant pieces of literature are presented:

- "***subjective probability***, *describes probability* ***as*** *a degree of belief, based upon personal judgment and information about an outcome*" (Jones, Langrall, & Mooney, 2007, p. 913);
- "*In this subjective view, what is random for one person might be non-random for another…a degree of uncertainty specific for each person*" (Batanero, Henry, & Parzysz, 2005, p. 24);
- "*The basic assumption here is that subjects have their own probabilities which are derived from an implicit preference pattern between decisions*" (Borovcnik, Bentz, & Kapadia, 1991, p. 41);
- "*To a greater or lesser extent the probability is an expression of personal belief or perception*" (Hawkins & Kapadia, 1984, p. 349).

The examination of certain subjective probability definitions evidences an implicit(ly circular) definition of the term subjective. Subjective probability, when used in the field of mathematics education, is implicitly aligned with (1) the subjective theory of probability and not as a general classifier, and (2) the personal theory of subjective probability, not the interpersonal theory.

Given the implicit state of subjective probability in mathematics education described above, I contend there does exist a variety of cognitive research on the subjective approach to probability. In fact, I would argue that in my impending review of the literature, all research presented pertains to cognitive research on subjective probability. Further, I would also argue that the research I am about to present contributes to the existing cognitive research on the subjective approach to probability. Again, and as mentioned, other individuals may disagree with the statements I make here, but, and as I will soon show, who is considered "correct" is a matter of interpretation.

INTRODUCTION

"*The shooting of the hunters was dreadful*" (Paulos, 1980, p. 65). A number of interpretations coexist for the aforementioned statement. For example, if the hunters do not work on their aim they will never be able to hit their intended target, i.e., the shooting of the hunters was dreadful; in this particular interpretation of the statement the hunters' shooting ability is called into question. For another example, what did that cute, defenceless baby animal ever do to deserve being shot between the eyes, i.e., the shooting of the hunters was dreadful; in this second interpretation the shooting of an animal by the hunters is considered dreadful and the hunters' shooting ability is not under consideration. For yet another example, it should be understood that when people are walking around the woods carrying loaded weapons accidents are bound to happen, i.e., the shooting of the hunters was dreadful; in this third interpretation it is the hunters who are shot, not the animals, and (arguably) the hunters' shooting ability is not taken into consideration. There (co)exist at least three possible interpretations of the statement "*the shooting of the hunters was dreadful*"; at least three because the "*shooting*" discussed previously was conducted with some type of firearm and not a camera. As demonstrated, the statement is *multivalent* (i.e., has many distinct interpretations).

There is concurrency associated with the multiple interpretations of a multivalent statement. One method to describe the coexistence of interpretations is to declare the statement exists in a state of *superposition*, i.e., the statement represents all possible interpretations whether enumerable or not: poor aim, dead animals, dead hunters, and photographic shooting. Moreover, when the statement is in a state of superposition there is no one particular interpretation for the statement, because all interpretations exist at once. Limiting the statement to a single possibility, or to *collapse* the state of superposition, requires a particular interpretation to take place. Interpreting the statement "*the shooting of the hunters was dreadful*" collapses the state of superposition, i.e., coexistence of all possible interpretations, and limits the statement to one particular interpretation. For example, one individual's interpretation may result in a hunting accident interpretation, whereas another individual's interpretation may result in a marksman interpretation.

To determine which particular collapse of the superposition of interpretations has taken place, inferences can be made through the examination of comments made by individuals who have read the statement. Consider, for example, an individual who after reading the statement comments, "*I guess even if you wear a gaudy orange vest that does not mean you are immune from accidents*." One may reasonably infer that it is more likely that the individual has interpreted the statement in terms of a hunting accident interpretation, as opposed to any of the other available interpretations. Further, and as another example, the reading of an individual's comments such as, "*they need to spend more time practicing before they go out into the woods*" causes one to reasonably infer that it is more likely the individual interpreted the statement as a hunting accuracy interpretation, rather than any of the other available interpretations.

Examination of comments not only provides insight into the collapse of the superposition of interpretations for a particular individual, but also provides the opportunity for determination of whether an individual's interpretation matches the intended interpretation of an author. If a third party is the creator of the statement, or knows with certainty the intended interpretation of the author, they are then able to, by the reading of comments made by an individual, determine if the individual's interpretation aligns with the author's interpretation. Consider, for example, if it is known through some manner that the author who wrote the statement was in fact discussing hunting accidents. The individual who commented, "*they need to spend more time practicing before they go out into the woods*" has an interpretation of the statement that is less likely to align with the intended interpretation of the author. The individual who commented, "*I guess even if you wear a gaudy orange vest that does not mean you are immune from accidents*" is more likely to be aligned with the intended interpretation of the author.

The research method presented – consisting of: statement, multivalence, superposition, interpretation, collapse, comment, inferences, and intended interpretation – does not change throughout the remainder of the research; however, *what* is analysed via the research method does change. In general, the research method, as just detailed, can be described as a(n) exploration, examination, critique, creation, and testing of hypotheses generated via inference when individuals engage with multivalence. The most efficient way to describe the research method is to build on the already-defined terms of multivalent and multivalence to define: *multivalentology* as (a) the study of multivalence, or (b) the exploration, examination, critique, creation, and testing of (i.e., the study of) hypotheses generated via inference when individuals engage with multivalence; *multivalentological* as pertaining to multivalentology; and *multivalentologist* as a person who studies multivalence. Multivalentology is not restricted to statements. For example, the impending research is described by the author as a multivalentological disquisition on subjective probabilities involving the perceived randomness of sequences of outcomes.

An oft-used task found in psychology and mathematics education (an example is shown in Figure 1) will act as the medium for investigation. Despite a general structure (e.g., binomial experiment, probability of success equalling probability of failure, two or more sequences presented) associated with different variations of the task, there does not exist a common name for the task (e.g., HT-sequence problem, sequence task). As such, the author wishes to denote the task presented below (akin to different variations found in the literature) as the Comparative Likelihood Task, hereafter referred to as the CLT.

Which of the following sequences is the least likely to occur from flipping a fair coin five times? Justify your response:

a) TTTHT
b) THHTH
c) HTHTH
d) HHHTT
e) all four sequences are equally likely to occur

Figure 1. The comparative likelihood task: An example

The first chapter is dedicated to a review of the literature on the CLT. Recognizing the influential role psychologists have played in research on the CLT and probability education in general, Chapter One begins with a review of the seminal work on heuristics and biases by Amos Tversky and Daniel Kahneman. Next, literature in mathematics education is reviewed according to chronological periods of probability research in mathematics education.

In Chapter Two the author's novel framework based on the multivalence (i.e., characteristic of having many distinct interpretations) of CLT responses is presented. With the new *CLT Response Interpretation Framework (RIF)* in mind and embracing the notion that a literature review is, in essence, the explication of one particular or multiple interpretations, Chapter Two concludes with an interpretation of the research literature; achieved through the exploration, examination, and critique of hypotheses generated from research on the CLT found in Chapter One.

The last half of Chapter Three is comprised of the analysis of results from Study I, given to 56 prospective elementary school mathematics teachers. While results are first analyzed via the multivalence of the CLT Response Interpretation Framework (RIF), results are further analyzed through the author's second framework, the *CLT Task Interpretation Framework (TIF)*, based on the multivalence of the Comparative Likelihood Task, which is developed at the beginning of Chapter Three.

With the interpretive nature of the research in mind, Chapter Four presents the raw data from Study II, where 239 prospective mathematics teachers—comprised of 163 elementary school teachers and 76 secondary school teachers—are presented with an evolved version of the CLT implemented in Study I. However, it is not until Chapter Six—after the development of (1) the notion of a *subjective-sample-space*, (2) a framework known as the *meta-sample-space*, and (3) a description of the method, entitled *event-description-alignment*, in Chapter Five—that the results shown in Chapter Four are analysed.

While conclusions are also presented, Chapter Seven, the final chapter, is dedicated to a discussion on past, present, and future studies that investigate probabilities associated with the perceived randomness of sequences of outcomes. Finally, and as is customary, a research agenda for the CLT and the explicit statement of contributions to research in mathematics education are presented.

In general, the research described is derived from two main goals of the author. First, demonstrate the multivalence of elements of the CLT. Second, demonstrate that certain individual's answers to the Comparative Likelihood Task accord to a subjective organization of the sample space, which is based on their interpretations of sequences of outcomes. Throughout the research described above, it is shown that unconventional views of the sample space—organized according to constructs referred to in what follows as switches, longest run, and switches and longest run, which are all based upon individuals' verbal descriptions of the sample space—can help situate individuals' answers and justifications within conventional probability. In doing so, it is shown that normatively incorrect responses to the task are not, necessarily, devoid of correct probabilistic ways of thinking and, in fact, model particular partitions of the sample space. To aid in explanation the author proposes an original theoretical framework, entitled the *meta-sample-space*, which will be used with a new method, entitled *event-description-alignment*, to demonstrate, for the first time, that individuals' subjective probabilities involving perceptions of randomness for sequences of outcomes are in accord with, or model, a *subjective-sample-space*.

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LEARNING TO PLAY WITH MATHEMATICS ONLINE[1]

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As access to the Internet proliferates in schools across the country, possibilities for collaborative professional development of teachers abound. In this study, thirteen geographically dispersed participants, including ten teachers, a PhD mathematician, and two mathematics education specialists, came together to learn how to meaningfully connect with each other to play with mathematics. This paper hermeneutically describes and interprets our experiences with technology and with mathematics in four synchronous Elluminate™ sessions. Descriptions of encounters with technology and mathematics are rare. In our journey, we discovered how the nature of the mathematical task affected our interaction. Routine, procedural, calculation problems led to stifled, polite and disengaged involvement. Conversely, multifaceted, non-routine problems fostered mathematical spaces for discovery, collaboration, and broad connecting conversations.

As access to the Internet proliferates in schools across the country, possibilities for collaborative professional development of teachers abound. This hermeneutic study was about discovering how to connect geographically dispersed participants to play with mathematics as a meaningful experience of problem solving and discovery. In four synchronous collaborative online sessions, we discovered how the nature of the mathematics task affected our online interaction. Following a review of the literature that informs our understanding about mathematical play and online professional learning, I briefly describe my research approach. Lastly, I discuss the relevant findings.

WHY PLAY?

Mathematical play is often neglected in favour of the quest for learning skills and memorizing facts. To inform our understanding of what play is and why play might be meaningful to mathematics, I draw upon the notions of Huizinga (1950), Gadamer (1989) and Csikszentmihalyi (2000). Huizinga defines play as a voluntary activity, something the player chooses to do with "*fixed limits of time and place, according to rules freely accepted but absolutely binding, having its aim in itself and accompanied by a feeling of tension, joy and the consciousness that it is 'different' from 'ordinary life*'" (Huizinga, 1950, p. 28). Solving a mathematical problem with fixed rules and conventions is different from ordinary life. A

[1] This paper is based on a PhD dissertation that was supervised by Dr. Michele Jacobsen, Faculty of Education, University of Calgary. The author gratefully acknowledges the significant contribution of Dr. Jacobsen to this work.

struggle for understanding can be filled with a sense of tension. Joy follows the tension when the solution is discovered.

Beyond tension and joy, some problems cannot be ignored. Gadamer (1989) furthers this notion of play by freeing play from any subjective understandings. Play has its own life, its own spirit: "*Play itself contains its own, even sacred, seriousness…Play fulfils its purpose only if the player loses himself in the play*" (p. 102). There is no goal in play other than to play and the player himself gets played by the play (Gadamer, 1989). At times I have been captured by a mathematical problem and drawn into the play. Despite my attempts to ignore the problem, I am haunted as the problem creeps back into my conscious thoughts.

Davis, Sumara and Luce-Kaplar (2000) claim that creativity and play go hand in hand. Davis et al. and the *Concise Oxford Dictionary* (1964) suggest that play is often regarded as the "*opposite of work, and so it is often associated with distraction, purposelessness, and disorder. Play, that is, is generally regarded as what we do when serious responsibilities are fulfilled* (Davis et al., 2000, p. 146). The opposite of play is not work, but rather immobility or rigidity. A lack of play implies inertia. Arguably, the quest for learning mathematical skills and facts can lead to rigid pedantry.

Analogous to Gadamer's notion of play is Csikszentmihalyi's (2000) notion of *flow*. Csikszentmihalyi defines flow as the autotelic experience that captures your heart, mind and soul, an experience so desirable that one wishes to recapture it repeatedly (2000). As the artist, the athlete, the scientist or the mathematician gets absorbed into *flow*, the player gets absorbed in the play. Arguably, being in the flow is the same as Gadamer's (1989) notion of being caught up in the play. Captured in the play, I have spent days upon days caught in a circular struggle in problems where I could not find the solution; my absorption was fuelled with hope as new possibilities were tried. When I discovered the solution, I sought new problems to try.

Several researchers argue that play and/or *flow* are fundamental to learning mathematics (Davis et al., 2000; Friesen, Clifford, & Jardine, 2008; Heine, 1997). Mathematical play can lead to enjoying mathematics, awakening confidence, believing in reason, being able to justify, valuing the concepts and developing an appreciation towards mathematics.

ONLINE MATHEMATICS PROFESSIONAL LEARNING

The pedagogic format for this study was borrowed from a face-to-face professional learning program developed at the Galileo Educational Network (GENA) called *Lesson Study*. The program was loosely based on a Japanese form of professional learning where teachers collaboratively plan, implement and revise teaching lessons (see Fernandez & Yoshida, 2004; Stigler & Hiebert, 1999).

I was an active participant of Lesson Study for three years. During monthly sessions of Lesson Study, teachers gathered from across Calgary to work on rich mathematical problems with mathematicians and mathematics educators. The goal for Lesson Study was for teachers to learn practices that would cultivate imagination and creativity with mathematics. In this study, I transformed GENA's Lesson Study into an online format in order to broaden the geographic audience.

THE STUDY

The study was informed by the question "*how can mathematics be played online?*" To address this question, an interpretive hermeneutic approach was chosen to re-establish a research connection to original human experience. Hermeneutics is about finding practical knowledge in the everyday experience (Smith, 1999). Few studies focus on what teachers and students are *doing* with new technologies and how they are *adapting* to complex circumstances in educational practices (Friesen, 2009). This study attempts to address the gap. I chose hermeneutics to inform the study for several reasons: (1) hermeneutics is consistent with an emergent approach to designing online learning environments (Friesen, 2009); (2) rich, descriptive, context-dependent knowledge is valuable for understanding human learning processes (Flyvbjerg, 2001); (3) the fecundity of the individual case is a powerful interpretive tool for understanding pedagogy (Jardine, 2006); (4) exploration and discovery is necessary for learning and understanding in this study (van Manen, 1997); (5) hermeneutics permits a focus on mathematics and interactions with mathematics in accordance with Sierpinska (1994); and (6) hermeneutics situates the study in the lifeworld to facilitate understanding of our experiences with mathematics online.

STUDY PARTICIPANTS

Thirteen participants joined the study. Lily, a PhD mathematician from BC, assisted in developing each session. Lily[2] had a rare combination of characteristics necessary for the study: she is interested in K-12 education, values mathematical play, and is comfortable with technology. Sharon, a PhD mathematics educator, and Ella, a mentor for teachers, provided expertise on teaching mathematics. Ten teachers from around Southern Alberta were conveniently selected and then invited with a letter of introduction. Six teachers participated together at their school. The other five were from unique schools in different geographic locations.

INTERPRETATION

Hermeneutics is responsive to the situation at hand. There were several sources of data for this study. While the four Elluminate™ sessions were the primary data, conversational interviews at the beginning and the end of the study, asynchronous text discussions, emails, informal telephone and face-to-face conversations, and field notes supplemented the analysis. Each session was digitally recorded, preserving audio and visual images of each session. I listened to and observed each Elluminate™ session repeatedly to discover how mathematics and technology were experienced by the participants. The character of the phenomena took shape as I let myself relive the experience. As I wrote descriptions of the unfolding events, I returned to literature I had previously read, and then I repeated the process of listening, questioning, reading, and writing as per the hermeneutic process (Friesen, 2009; van Manen, 1997). When I became stuck, I consulted with experts. With this process of constant comparison, I returned to the hermeneutic process of writing and seeking understanding.

FINDINGS

In this next section, I describe our experiences as they unfolded chronologically and what we learned in our four Elluminate™ sessions. Due to the need for succinctness, the following chronologically summarizes, rather that narrates, our experiences. Each event has two

[2] Lily and Sharon chose to use their first names. All other participant names are pseudonyms.

components: (1) A pre-session online meeting where Lily and I tested the problems, and (2) the online sessions.

PRE-SESSION 1

Prior to our meeting, Lily emailed mathematical problems for our first session in a black and white PDF image. I reformatted, added colour images, and typed out the problems into PowerPoint slides that I could upload for the session. Once the slides were ready, Lily and I "met" on Elluminate™. Lily had never been in Elluminate™ or any other computer-mediated learning environment before. Lily stumbled for a short while as she learned how to depress the microphone button when she wanted to talk. However, within half-an-hour Lily felt comfortable enough with the technology to help moderate the session.

SESSION 1

Our first collective gathering was a frustrating breakdown of communication as we encountered firewalls, non-functioning microphones, and issues with usernames and passwords. A group of teachers in rural Alberta and one teacher in a Calgary school found that they could not access Elluminate™. With much deliberation through phone calls and emails, we concluded that firewalls were preventing the connection. Overcoming firewall issues were beyond the capabilities of the technical support assigned to help us: half of the invited participants could not connect.

Another participant had difficulty logging in. Her user ID and password would not work. We were also unable to alleviate her technical issues and she was unable to join the session. Of the six of us that could connect, three of our microphones were not working, including mine. In the heat of the moment, we were unable to find how to resolve the issues. Lily's microphone worked, leaving her as the only moderator.

The technical issues we encountered impeded access to Elluminate™ and our ability to converse. The playfulness in mathematics requires conversation: it is a dialectic between people and the mathematical play that is being invoked. When the conversation was interrupted then, most definitely, the play was interrupted and stopped. Yet still I saw evidence of tension, frustration, absorption, and to-and-fro conversations. However, the object of play during the session ended up being technology rather than mathematics.

In many years of experiencing online interaction, I had never encountered such overwhelming obstacles. I questioned the impact of technological issues with new instructors. What happens to new instructors who have this type of challenging experience the first time they attempt to teach in an online space? What is the likelihood they will want to continue? Will they believe in the possibilities for meaningful and engaged learning in online courses?

PREPARATIONS FOR SESSION 2

Shortly after Session 1, Zoe withdrew from the session. Considering our previous challenges from Session 1, I considered myself lucky that no one else withdrew too. Again, Lily emailed me problems for the session. Lily's choice of problems were word problems with an *Alice in Wonderland* theme. I added static images to the word problems to make the problems more visually appealing on the whiteboard. In our pre-session, we quickly looked at the problems and were satisfied with their functionality.

SESSION 2

For Session 2, firewalls were removed by upper level IT and we had all learned how to configure our audio microphone settings. Inadvertently, we were about to learn how the nature of the mathematical problem can adversely influence the interaction in the online environment.

During Session 2, the conversation among the participants was stifled. Lily repeatedly called upon people to invite them to participate. The interaction of Session 2 could be characterized as instructor-student with very little student-student interaction (Moore, 2007). Brice and Sherri were the only participants that appeared to be engaged in the session. In fact, a faded name accompanied by the word "away" reflected Samantha's absence for the entire session. Below in Figure , is a screen capture of the Elluminate™ classroom.

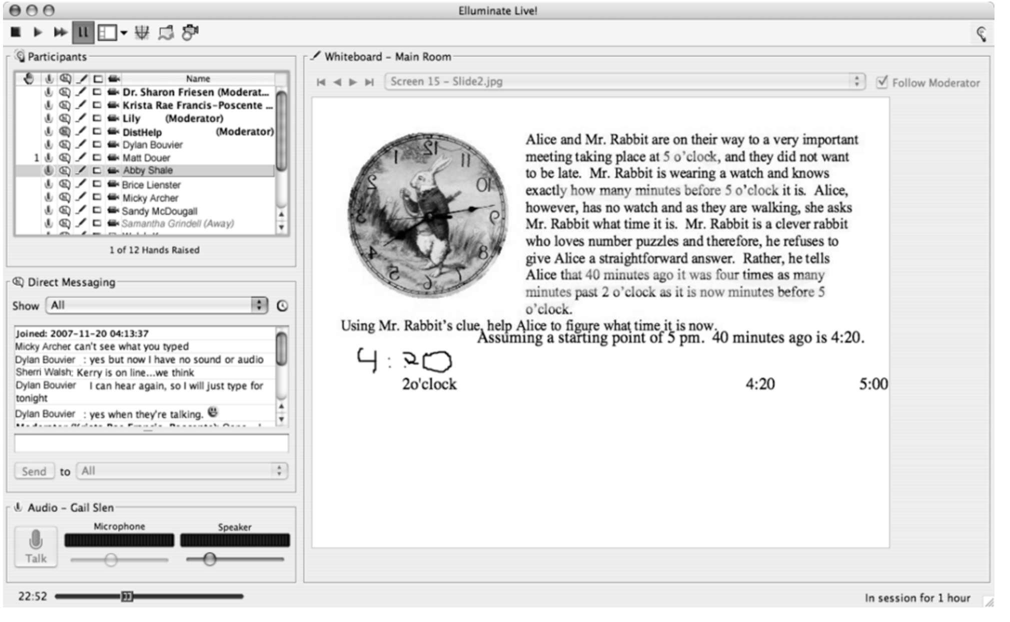

Figure 1. The Elluminate™ classroom

Upon reflection, we realized the wordy text occupied most of the available whiteboard space. Inadvertently, we had reduced the only space for whiteboard interaction (see Figure above). With the available tools, we had difficulty typing and drawing numbers on the whiteboard screen. Forming letters and numbers with a mouse was difficult. Consequently, the problem was unsuitable for the chosen media.

Fuchs et al. (2008) would characterize the Alice and Wonderland problem as a routine calculation problem requiring more linguistic deciphering than mathematics. There were no "aha" moments from the participants when the problem was solved, nor questions about the concepts. There was no tension, no absorption, and no spontaneous to-and-fro conversation. Besides being unsuitable for the media, the routine calculation problem did not invoke a sense of play.

PREPARATION FOR SESSION 3

For future sessions, we sought less routine calculation problems, with fewer words, and problems that could be supported with the use of manipulatives. I challenged Lily in Elluminate™ with a problem called Jumping Chips (Lewis, 2002). The problem required that we slide and jump chips to solve the problem. In response, Lily excitedly exclaimed she had some ideas for problems.

SESSION 3

Responsive to my suggestions, Lily sent problems for Session 3 with minimal text that required drawing on the whiteboard to demonstrate solutions (see Figure 2. below).

When participants logged in for Session 3, they were encouraged to practice drawing sticks on the title page. Within a few minutes, the page was messily marked up with black sticks. As the board began to get messy, Lily piped in, "*Could you please pick a colour and let me know who is drawing with that colour?*" Soon we had a colour coded key ascribing a name and colour identity to the sticks.

Lily's toothpick problems drew upon the Roman numeral system and algebraic reasoning. As Lily began to explain the problem, Brice jumped onto the board with an incorrect solution. Brice moved the vertical line of the plus sign to form IIII as the answer: IX – V = IIII. Lily used Brice's error as a chance for dialogue. Lily clarified, "*Roman numeral systems do not use four sticks to write a four.*"

A flurry of activity erupted. Micky rewrote the problem in red. Brice's orange answer disappeared in a blink of an eye. Anonymous green lines appeared. Maggy texted that she tried, but was on the wrong track. An orange answer suddenly appeared. Once again, Brice solved the problem. Lily encouraged Brice to justify his solution strategy.

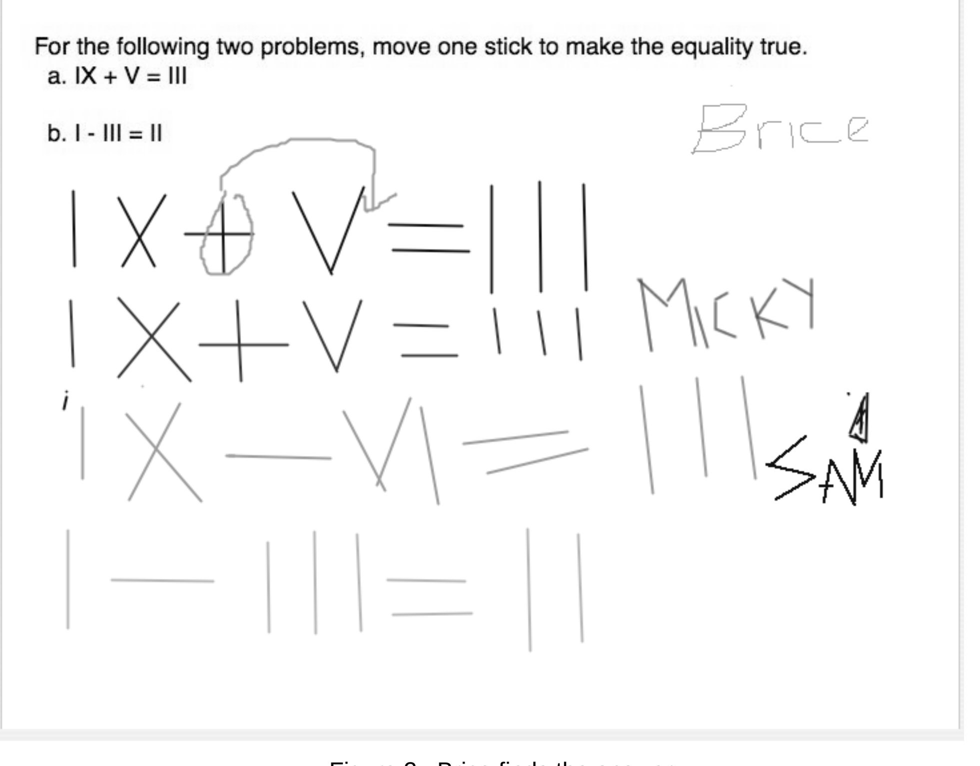

Figure 2. Brice finds the answer

Lily: Brice got that. Wow! That was really fast. How did you do that?

Brice: Once I figured out that we could just move the equation from addition to subtraction, I just looked at different combinations of numbers and operators.

Lily: OK, thank you Brice.

The polite stifled atmosphere dissipated into enthusiastic and unrestrained conversation. We talked about some of the emotional baggage teachers have surrounding mathematics: panic and being slow. While their subjective emotions about mathematics interfered with their playing, opening up to vulnerabilities relaxed the group. Lily and Sharon described the connections of mathematical creativity and competency. Everyone in the group contributed to the conversation and tried to solve the problem.

Session 3 was a turning point. With minimal text, the problem suited the whiteboard. By requiring participants to draw sticks for creating solutions, the problem encouraged interaction with the tools. By choosing unique colours, we created identities. The problem was non-routine and connected mathematical ideas. The participants felt comfortable making mistakes in problem solving and revealing their attitudes about mathematics. These conversations led us into deeper conversations about students and how to teach mathematics.

PREPARATIONS FOR SESSION 4

Lily brought Nim for our next session: a two person game that requires the removal of chips. In our test session, I had mistakenly thought that we could easily draw circles on the whiteboard. The task was too cumbersome and the formulated circles were too uneven. Recalling the ease of drawing lines, Lily suggested that I create a static image of the disks in rows. With a prepared template, uniformity and organization could be ensured. Also, a line through the disc provided an easy removal of chips. Following Lily's advice, I prepared various arrangements of circles.

SESSION 4

As soon as the first problem slide in Session 4 appeared, Brice and Sandy jumped in and played the game. After Brice won, Lily suggested that Abby and Gerald play together. Abby made the first move, crossing off the left-most dot.

Gerald crossed off the middle dot. Abby crossed off the second dot. Gerald crossed off the last two dots, won the game and exclaimed, "*It looks like I won.*" Lily replied, "*Gerald, could you take off your last two crosses. And Abby, could you remove your second stick. I would like to consider what was happening after the first two moves.*"

The board erased and Abby drew a line through the first circle again. Gerald drew his second line through the middle disc (see Figure below).

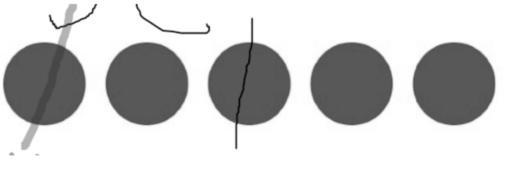

Figure 3. Session 4: Problem 1 – The first two moves again

Lily interjected, "*At this point can you see who can win? It is Abby's turn. Suppose we give Abby more time to think. Would it be possible for Abby to win from here?*" Abby crossed off the last dot in the row. Gerald was no longer able to win. Whichever dot he took, there would be one dot remaining. Lily probed, "*A change of Abby's second move meant she won. Why?*" Gerald articulated that if the first person took the middle chip and then copied the moves of the second player, the first person would always win. Lily restated Gerald's copying strategy as symmetry. For the remainder of the session, we explored if the strategy of symmetry worked with other scenarios. We played with six chips, seven chips, and then two-dimensional versions. Lily shuffled everyone up, ensuring that everyone played and that everyone was able to utilize symmetry as a strategy.

I was flabbergasted to find that our exploration of Nim was really an exploration of symmetry. I had only thought about symmetry as an object. Now, symmetry was a strategy for winning a game. All of a sudden, transformations and reflections had a new purpose. Whoever has the chance to keep the symmetry wins. Lily helped us discover how symmetry could be a strategy, a *process*, not just a *thing*. The order and beauty of symmetry was a path into the wilderness of mathematics.

DISCUSSION

This hermeneutic study sought to provide illustrations and insights into the interactive nature of playing with mathematics online. The key contribution of this study was the discovery of how the nature of the mathematical problem influences interactions both mathematically and online.

In Session 2, Lily provided us with a routine calculation problem that was hidden within a lengthy text: a word problem. In terms of mathematics and mathematical conversations, we found the conversation to be stifled and focussed on the solution procedures for solving the problem in question. One participant chose to ignore the problem and was "away" for the entire session. The conversations appeared to need Lily's coercion. Lily had to pointedly ask participants questions. In other words, no one jumped in. A procedural mathematical task did not elicit a sense of play. In terms of technology, the necessity of typing text and numbers for the solution was difficult given the tools we had. Perhaps a graphics tablet and pen would have made our ability to write on the whiteboard easier. However, the expense of graphics tablets prevented our ability to use them. A wordy procedural problem requiring text for the solution was an unsuitable problem for encouraging online collaborative problem solving.

With the complex, non-routine problems in Session 3 and 4, we made connections into the broader ideas of mathematics. In Session 3, two innocuous looking problems opened up and connected us to the concept of equivalence, the concept of number, and the history of our number system. Together we solved the problem and strengthened our mathematical understandings. Non-routine, complex problems helped us create mathematical spaces for discovery, collaboration and broad connecting conversations. Technologically, we learned to capitalize on our ability to draw lines with ease. Problems that required lines to demonstrate our solutions resulted in an explosion of drawing on the whiteboard. We engaged with each other and the problem using the whiteboard tools. Non-routine, complex problems that connect mathematical ideas and also require minimal drawing on the whiteboard were excellent problems for encouraging online collaborative and engaging problem-solving.

RECOMMENDATIONS

I have several recommendations before embarking on an online professional program. Firstly, anticipate technical difficulties and rely on the first session to trouble shoot and to introduce the participants. Have mathematical problems ready; however, hold off on the mathematical problem-solving until technical issues are taken care of. Plan time to play with the technology. Secondly, hold pre-sessions to test the problems in the environment before every session. In our pre-sessions, Lily and I made many discoveries together that informed our upcoming session. Our learning informed the upcoming sessions. Thirdly, the disciplinary expertise of the mathematician and the mathematics educator were invaluable to the emergent learning design process. The sharing of our understandings shaped our learning and the teachers' learning. Find and adapt problems that are suitable for the online environment. Drawing with a mouse and typing on the whiteboard was difficult. Problems that required line drawings for solutions worked the best. Lastly, of utmost importance is to find complex problems that draw upon multiple mathematical concepts. Routine calculation problems stifled communications and did not open up the frontiers of mathematics. Routine, procedural, calculation problems led to stifled, polite and disengaged participant involvement. Conversely, multifaceted, non-routine problems fostered mathematical spaces for discovery, collaboration, and broad connecting conversations.

FUTURE RESEARCH

Future research could move into other disciplines or continue to venture into mathematics education and professional learning. Questions we need to ponder include how different media affects the mathematical task and also how online professional development translates into teaching practice.

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STUDENTS' MODELS OF THE KNOWLEDGE TO BE LEARNED ABOUT LIMITS IN COLLEGE-LEVEL CALCULUS COURSES: THE INFLUENCE OF ROUTINE TASKS AND THE ROLE PLAYED BY INSTITUTIONAL NORMS

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This paper is a brief summary of my PhD dissertation. My research consisted in a study of instructors' and students' perceptions of the knowledge to be learned about limits of functions in a college-level Calculus course, taught in a North American college institution. These perceptions were modelled using a theoretical framework which combines elements of the Anthropological Theory of the Didactic, developed in mathematics education, with a framework for the study of institutions – the Institutional Analysis and Development Framework – developed in political science. While a model of the instructors' perceptions could be formulated mostly in mathematical terms, a model of the students' perceptions had to include an eclectic mixture of mathematical, social, cognitive, and didactic ***norms****. An analysis of the results shows that these students' perceptions have their source in the institutional emphasis on routine tasks and on the norms that regulate the institutional practices. Finally, I describe students' thinking about various tasks on limits from the perspective of Vygotsky's theory of concept development. Based on the collected data, I discuss the role of institutional practices on students' conceptual development.*

INTRODUCTION

My dissertation focuses on the teaching and learning of the concept of limit of functions in college-level Calculus courses. In the past, research in the didactic of mathematics which approached the same concept was influenced by psychology (especially Piaget and constructivism) and epistemology (especially Bachelard (1938) and his notion of epistemological obstacle). They related especially to the construction of the concept by students and the means of helping them to surmount the various cognitive and epistemological obstacles that had been identified. During the 90s, social and cultural perspectives started to draw more and more attention in the mathematics education community. Social interactions started to be seen as constitutive factors of teaching and learning phenomena. This shift, from the constructivist and individualist approach to the sociocultural approach of the development of mathematical concepts, led several authors to underline the influence of institutions in teaching and learning practices. My research question was based on the assumption that some "institutional" practices (where "institution" could refer to the various levels of institutionalization of teaching like, for example, school board, department of mathematics, a whole course, a section of this course, a subject of the course, a classroom, a final examination, etc.), in the form of definitions, properties, examples and exercises in textbooks and examinations, strongly influence what students learn about limits of functions at the college level. Barbé, Bosch, Espinoza and Gascón (2005) discussed the institutional constraints on the teachers' practices in the classroom in relation to the teaching of limits in Spanish high schools. My more general goal was to understand how these institutional constraints reach the students, independently of the personal mediation of a teacher. I initially considered this phenomenon from the perspective of the Anthropological Theory of the Didactic (ATD) (Chevallard, 1999). My intention was to describe the teaching and learning practices, in relation to the concept of limit of functions, in the considered institution, in terms of "mathematical praxeologies," i.e., in terms of organizations of mathematical tasks, the techniques to solve them and the discourses (technologies and theories) employed to produce and justify the techniques (Chevallard, 1999). But, while analyzing the data obtained in 28 interviews with college-level students, I realized that their implicit models of what they had to learn about limits of functions were very different from the mathematical praxeologies that I had identified in the institution. These models were not so "purely mathematical"; their structure was much more complex and eclectic.

In my research, I expected, of course, to see differences between the "scholar knowledge" about limits of functions and the "knowledge to be taught," and I knew that the "knowledge to be learned," which is also distinct from the "knowledge actually learned" by the students, is usually only a subset of the "knowledge to be taught." These distinctions are predicted already in the Theory of Didactic Transposition (Chevallard, 1985). I had the intention to explore the relations among these various types of knowing. The study of these relations by the means of the notion of praxeology, however, highlighted the difference between the institutional perspective of what must be learned and the students' perspectives of what must be learned. From an epistemological point of view, the "knowledge to be learned" can be a well defined object. From an anthropological point of view, its unity breaks up into at least two distinct, different praxeologies: students' and instructors' praxeologies. These differences are not only structural: while it seemed possible to classify the institutional praxeologies as "mathematical," students' praxeologies were of a heterogeneous nature, involving a mixture of mathematical, social, cognitive and didactic norms. In light of this awareness, the principal question of my research took the following form: *How do institutional practices influence students' perceptions of the knowledge to be learned about limits of functions at the college level?*

To tackle this question, I had to consider other associated questions; among them:

- What is an institution and when can we say that a practice is institutionalized?
- What is the institutional model of the knowledge to be learned about limits of functions at the college level?
- What are students' models of the knowledge to be learned about limits of functions? What is the relation of these models with the mathematical capacities of the students to deal with a task of the type "*find the limit of a given function*"?

To answer these questions, I considered a theoretical framework complementing ATD with a model developed in political sciences – the framework for Institutional Analysis and Development (IAD) (Ostrom, 2005) – and the theory of conceptual development of Vygotsky (Vygotsky, 1987). IAD defines what an *institution* is, in terms of the notions of *participant* and *position* of the participant in the institution, as well as *rules*, *norms* and *strategies* which constitute a system of mechanisms regulating participants' behaviour in the institution. The IAD/ATD combination is thoroughly discussed in Sierpinska, Bobos, and Knipping (2008) and Hardy (2009a). From this perspective, I could conceptualize the differences between the institutionalized models of the instructors of college-level Calculus courses and the spontaneous models of the students in these courses in relation with the knowledge to be learned about limits of functions. My analysis suggests that students' models are based on institutional norms and strategies rather than on mathematical rules, norms and strategies, and that this situation is not disputed by the instructors' models. Using Vygotsky's theory, I could characterize students' ways of thinking when dealing with tasks of the type "*find the limit of a given function*," in terms of levels of cognitive development. The Vygotskian perspective allowed me to explain how the social interactions – in this case, interactions of the students with the institution, negotiated by institutional praxeologies – can interfere with students' development of the concept of limit. For a detailed explanation of the complementation of the ATD/IAD framework with Vygotsky's theory of concept development, see Hardy (2009b).

Among the many questions and reflections triggered by the obtained results and their situation with respect to former research, I would like to stress two of them:

- If students' spontaneous models of the knowledge to be learned about limits don't correspond strictly to mathematical knowledge, to which other types of knowledge do they correspond?
- In terms of Vygotsky's theory of conceptual development, the interviewed students employed most of the time a way of thinking "by complexes" and not "by concepts" which is specific to scientific thinking, in general, and to mathematical thinking, in particular. Which would be the types of tasks which could promote students' use of the *conceptual* mode of thinking with respect to the tasks "*find the limit*"?

EDUCATIONAL CONTEXT

In the educational system studied in this research, "college" refers to an educational institution situated between high school and university. The high school curriculum in mathematics does not include Calculus. A first one-variable Calculus course is taught only at the college level, in academically oriented (as opposed to vocational oriented) programs leading to studying health sciences, engineering, mathematics, computer science, etc., at the university level. The majority of students enrolled in these courses are 17-18 years old. The course is usually a multi-section course, with the number of sections in large urban colleges often exceeding 15.

In the studied college, at the time of the research, there were 19 sections of the first Calculus course, taught by 14 different instructors, with 25-35 students enrolled in each section. The course in the college was run collectively by committees of instructors responsible for selecting an official textbook to be used in all sections, preparing the common "course outline," and writing the common final examination. All instructors teaching the course in a given semester would be automatically members of the ad-hoc "Final Examination Committee" for that semester. The course outline would be quite detailed, so that, in a given week, all sections would often be studying the same mathematical topic and working on the same homework assignments. Students from different sections usually study together, compare notes, and prepare for the final exam together, thus forming a "community of study," which has some control over what is going on in the individual sections. Students may, for example, inform their section instructor that another instructor is more (or less) advanced in the syllabus, or doing less (more) difficult problems.

As explained in the introduction, my main objective was to explain if and how the institution's perception of the knowledge to be learned influences the students' perceptions of this knowledge, independently of the personal mediation of a teacher. For this, I focused on the institutions "Final Examination Committee" and "community of study." Thus, I considered the instructors not as individuals participating in their classrooms, but as representatives of the institution's perceptions of the knowledge to be learned, and the students, not as participants in their classrooms – attending to the demands of a particular instructor – but as participants of a community that studies Calculus at the college-level and attends to the institutional expectations.

METHODOLOGY

My typology of tasks was based on a classification of the questions about limits of functions used in the studied college during several (six) years in its final examinations for the Calculus course. The analysis showed that the tasks appearing in the examinations always belonged to one of three types and therefore these three types of tasks were called "routine." The classification was guided by a generalization of the mathematical features of these tasks. I was interested in identifying the influence of the mathematical praxeologies developed around these tasks by the institution on the students' models of the knowledge to be learned about limits of functions. For this, I conceived a "task-based" interview (Goldin, 1997) made up of three phases. As the routine mathematical tasks appearing in the final examinations were all of the type "*find the limit of a given function*," the interview focused on this type of task. In the first phase, the students were invited to classify 20 limit expressions ($\lim_{x\to\cdots}[f(x)]$) according to a rule of their choice. In the second phase, the students were asked to find limits whose expressions superficially resembled the routine tasks but differed from them in the conceptual level. In the third phase, the students had still to find limits but this time the expressions of the limits did not resemble at all the routine tasks. All throughout the interview, the students were asked to think aloud. The script of the interview was guided by the principle of a comprehension as good as possible of the techniques and reasons the students were using to justify each step.

RESULTS

(I) I identified three types of routine tasks with their corresponding techniques. The presentation of these techniques in the textbooks and in the model solutions of the problems appearing in final examinations, written by the instructors and made available to the students, strongly emphasizes algebraic aspects (Hardy, 2009a).

(II) Instructors' spontaneous models of the knowledge to be learned about limits are restricted to the "know-how" block of a praxeologic organization, that is, with the system of the tasks and techniques. In other words, the instructors, as participants of the institution "Final Examination Committee", don't expect the students to reproduce explanatory discourses (Hardy, 2009a; 2010).

(III) The classification task in the interviews (phase one) aimed at understanding what reference framework is used by the students when they are confronted with limit expressions. At least 68% (19 out of 28) students proposed a classification which evokes the way in which mathematical concepts are presented in high school and college-level Algebra textbooks; the almost complete absence of references specific to mathematical analysis was remarkable (Hardy, 2009b).

(IV) In the second phase of the interview, while dealing with problems that strongly resemble the routine tasks, the students based their approach on the institutional norms: "*we do this because it is what we usually do (we were told to do) in these circumstances*". The justifications referred to the authority of the institution; the students tried to abide by the institutional norms and almost never referred to mathematical theory (Hardy, 2009a; 2010).

(V) In the third phase of the interview, the behaviour of the students changed completely. When dealing with problems essentially different from the routine tasks, the students based their approach on mathematical strategies and rules. Moreover, they tried to justify their reasoning in mathematical terms. *The students behaved mathematically, rather than behaving normally* (Hardy, 2010).

DISCUSSION AND CONCLUSIONS

(I) Students' behaviour in the classification task suggests that their concepts had become closely related to their didactic organization. Moreover, because of the institutional emphasis on the algebraic aspects, the students could have mistakenly interpreted the techniques as purely algebraic. Thus, for example, the students would consider the features "rational function," "radical," and "none of the latter," as their guiding features to choose a technique, instead of, for example, to consider types of indeterminations. In fact, the routine tasks are such that their interpretation and that of the corresponding techniques from an algebraic perspective is likely to produce correct solutions. The students can conclude from this that the algebraic approach always gives the good answer, and become thus unable to tackle the problems for which the algebraic approach fails. This conjecture, in connection with the students' behaviour in the first phase of the interview, was confirmed by their behaviour in the second phase, where the students were rather obsessed with finding an algebraic technique to solve the problems and, when this approach failed, they were unable to approach the problem in any other way (Hardy, 2009a).

(II) The absence of a theoretical (mathematical) block in the instructors' models of the knowledge to be learned deprived many students (26 of the 28 interviewed students; 93%) of means for developing mathematical justifications at the conceptual level. The students did not have any mathematical theoretical resource to reflect on their own behaviour and to justify it. Consequently, their explanatory discourses about why they would employ a technique rather than another made allusion to social validations of the techniques, i.e., to the institutional uses and not to the mathematical validity of a technique in a given situation. The analysis of students' behaviour in the second phase of interview confirmed this interpretation. This can be taken as an indication that instructors' models of the knowledge to be learned do not dispute the mode of thinking "by complexes" of the students in relation with the type of task "*find the limit*." Therefore, institutional practices do not fulfil the role of *pulling* students' cognitive development beyond their immediate individual capabilities – that is, they fall short of awakening "*a whole series of functions that are in a stage of maturation lying in the zone of proximal development*" (Vygotsky, 1987, p. 212; Hardy, 2009b).

(III) The "normative" character of instructors' models of the knowledge to be learned underlines an implicit institutional discourse of the type: "*this technique is used to solve this problem because it is the way in which things are usually done here*," instead of, for example, "*this technique is used to solve this problem because it is one of the mathematical strategies to find the answer and for this particular characteristic of the problem, it is an efficient strategy, better than….*" This can have the effect that the students learn how to behave *normally* rather than how to behave *mathematically*.

(IV) Selden, Selden, Hauk and Mason (1999) wonder why a student who has the necessary knowledge to tackle a given problem is unable to recover it [from his/her memory] and, if a student recovers the knowledge, how he/she does it? My research implies that, in the case of tasks of the type "*find the limit*," the knowledge to approach at least some tasks essentially non-routine (like the ones students were asked to solve in the third phase of the interview) seems to be on the "surface," recoverable at the slightest prompt. In Vygotsky's terms, this knowledge belongs to students' zone of proximal development. My thesis suggests that the reasons for which the students cannot recover this knowledge by their own means could be in a combination of the way in which the exercises are presented to the students, the study practices of the students, the routinization of problems in textbooks (Lithner, 2004), and the institutional norms controlling the tasks in the final examinations.

(V) The main difference between routine tasks and non-routine tasks is given by the categorization itself: problems that are practiced all the time, and problems that are not. Of course, the nature of the tasks is important – it is discussed in the thesis why it's in the nature of the tasks routinized by the studied institution that these cannot help students in the formation of a conceptual system to deal with tasks of the type "*find the limit*." It would not suffice, however, to transform the non-routine tasks into routine tasks and train students in solving them. What might help is a change in some institutional educative habits. For example, by creating situations where students are given a chance to engage in creative, critical mathematical thinking (the type of thought that students have used when dealing with the tasks proposed in the third phase of the interview).

(VI) As it was mentioned above, students' spontaneous models of the knowledge to be learned about limits of functions do not follow exclusively mathematical rules, norms and strategies but are strongly based on cognitive, didactic and social norms. The fact that these models do not correspond (exclusively) to the mathematical knowledge does not mean that it is not a certain kind of knowledge. The question of the type of knowledge that the students establish instead of that accepted by the community of mathematicians, in relation to a particular concept, was formulated by several authors. My thesis suggests that this knowledge is strongly related to a "*school survival knowledge*" (Sierpinska, 2000, p. 245); a type of knowledge necessary to succeed as an institutional subject.

(VII) In my thesis, I tried to stress the institutional relativity of the *knowledges* when they are considered from an anthropological point of view. However, even this relativity is not subtle enough. The concept of praxeology does not distinguish between the practices regulated by rules, and the practices regulated by norms. Consequently, although the concept of praxeology can be used to establish an epistemological model of knowledge, it may not be sufficiently sharp as a tool to describe the differences between the spontaneous models of the knowledge that the participants of the institutions can have. I found it useful to complement the ATD framework with the IAD framework to see and describe more clearly these differences.

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MAKING CONNECTIONS: NETWORK THEORY, EMBODIED MATHEMATICS, AND MATHEMATICAL UNDERSTANDING

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> *Networks are in the news and will likely remain there. To understand our world, we need to start thinking in these term.* (Buchanan, 2002, p. 22).

In the late 1990s, researchers began to develop the field of network theory as a means to explore the structural dynamics of the networks underlying complex systems in which coherent and seemingly purposive wholes emerge out of the apparently independent actions of individual elements. Network theorists focused not on characteristics of entities in the forms themselves, but rather on their interrelationships. By viewing the elements of a system as *nodes* in a network and their interactions as *links* among nodes, the system of entities and their connections can be portrayed by a graph. Using this technique, patterns not previously seen in complex phenomena emerged and simple, yet comprehensive, laws that describe network structure and evolution were formulated (e.g., Barabási & Albert, 1999). Scientists in many disciplines, ranging from physics to sociology, have found these principles invaluable in explaining how and why complex systems behave as they do.

Although the use of network theory in analyzing complex systems is rapidly expanding, it has not been applied to the field of mathematics education. It would seem desirable that educators be aware of these powerful and comprehensive methods of analysis. If techniques developed for exploring and understanding complex behaviour in other disciplines are applied to education, teachers may be provided with a different way of thinking, perhaps helping them to answer questions about complex educational systems that have proved intractable to date.

In my work, I argue that network theory offers a useful frame for talking about the mathematical understanding of individuals. I suggest that a network structure for subjective mathematics may be found in cognitive mechanisms presented in the theory of embodied mathematics, as put forth by Lakoff and Núñez (2000). They conceive of mathematics as being extended from a rather limited set of inborn skills to an ever-growing collection of *conceptual domains*. These are connected by *conceptual metaphors*, which carry inferential structure from one domain to another. Examination of the topology and dynamics of this web of ideas, which I refer to as the *metaphoric network of mathematics*, supports what might be seen as the beginning of a scientific explanation for mathematical understanding and learning. Inevitably, this work has implications for many areas in mathematics education.

In this paper, I present key points in my argument that a network model provides an appropriate and fruitful way to explore and interpret mathematical understanding. Following this, I briefly describe research carried out as part of my work with the intent of substantiating the thesis outlined here. In this study, I explored a small part of the proposed metaphoric network of mathematics, focusing on the concept of EXPONENTIATION.[1]

THE METAPHORIC NETWORK OF MATHEMATICS

CONCEPTUAL DOMAINS AS NODES

I propose to take, as the nodes of the metaphoric network of mathematics, conceptual domains like the ROTATION, ARITHMETIC, or SET. Each domain is a "*coherent organization of experience*" (Kövecses, 2002, p. 4) with many interconnected elements forming a complex of sensory experiences, language, and related concepts. Thus, a conceptual domain is a subnetwork of the larger network that forms the cognitive system (Lamb, 1999). This structure is dynamic, changing as an individual learns from new experiences, and differs from person to person.

While most work in this area has been done in other fields, I believe that the principles apply to mathematical domains. For example, the conceptual domain of CIRCLE contains many nodes representing a person's knowledge of and experiences with circles (see Figure 1). It is not possible to provide a complete map, for any concept may contain thousands of nodes. Moreover, each of these vertices, like the visual node labelled "seeing a pie" in Figure 1, is itself a network structure (Lamb, 1999). It includes the many different optical features (color, size, topping) that might be involved in a person "seeing a pie." Thus, the metaphoric network of mathematics displays the pattern of organization, forms nested within forms, that is typical of complex systems (Davis & Simmt, 2006; see Figure 2).

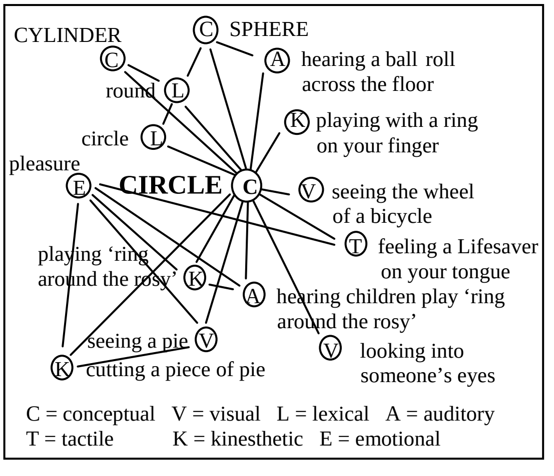

Figure 1. Part of the conceptual domain CIRCLE following Lamb's (1999) example of CAT

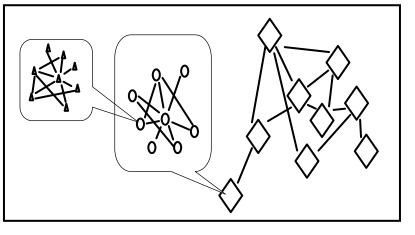

Figure 2. A network for "seeing a pie" nested within the subnetwork that is the CIRCLE domain, which is itself nested within the metaphoric network of mathematics

1 Throughout this work, the convention of identifying cognitive mechanisms using small capitals is used.

CONCEPTUAL METAPHORS AS LINKS

I suggest that the links of the metaphoric network of mathematics are conceptual metaphors. Particular features from one domain (the *source*) are mapped onto corresponding aspects of another domain (the *target*) by these cognitive mechanisms. Some source domains provide a framework for a variety of targets (Kövecses, 2002). For example, consider the many metaphors (some of which are listed in Figure 3) that can be based on the conceptual domain of SET. These metaphors all project significant characteristics from the source of SET onto various targets, thus developing common inferential structures in disparate domains. Just as source domains may be metaphorically linked to more than one target, some targets are connected to a variety of sources. For such conceptual domains, a single source does not possess enough structure to support all features of the concept (Lakoff & Johnson, 1999). For example, Lakoff and Núñez (2000) state that four grounding metaphors are needed to fully capture the many characteristics of ARITHMETIC (see Figure 4). Together, a collection of conceptual metaphors from distinct sources can construct a coherent understanding of their common target domain (Lakoff & Johnson, 1980).

AN ORDERED PAIR IS A SET.
A FUNCTION IS A SET.
A LOGICAL PROPOSITION IS A SET.
A NUMBER IS A SET.
A GRAPH IS A SET.
A LINE IS A SET.

Figure 3. Metaphors with a source domain of SET

ARITHMETIC IS OBJECT COLLECTION
ARITHMETIC IS OBJECT CONSTRUCTION
THE MEASURING STICK METAPHOR (measuring with segments)
ARITHMETIC IS MOTION ALONG A PATH

Figure 4. Grounding metaphors for the target domain ARITHMETIC

HOW THE METAPHORIC NETWORK OF MATHEMATICS EVOLVES

An individual's mathematical understanding is not static, but develops as he or she continues to learn. Metaphoric mappings bring new conceptual domains into being as they project inferential structure from one context onto another (Sfard, 1997). To illustrate, geometric meaning is given to complex numbers when *i* is metaphorically linked to rotation by 90° (Lakoff & Núñez, 2000). Entailments of metaphors can also lead to the development of new concepts. For example, the MEASURING STICK metaphor portrays numbers as physical segments (Lakoff & Núñez, 2000). Stretching this, any segment – like the hypotenuse of a right-angle triangle – can be considered a number. This gives meaning to a previously unknown domain, the irrational numbers. Nodes can also be added to the metaphoric network of mathematics through conceptual blends. These cognitive mechanisms project inferential structure from two unrelated sources onto a new blended domain. For example, the UNIT CIRCLE possesses characteristics of both a circle in the EUCLIDEAN PLANE and the CARTESIAN PLANE with its axes and coordinates (Lakoff & Núñez, 2000).

Although it is clear that the network model for mathematical understanding grows, connections do not develop in an evenly distributed manner. A disproportionately large number of new links involve nodes that are already highly connected, a property called *preferential attachment* (Barabási, 2003). A number of factors contribute to a node's ability to attract connections. Domains added to the network early in its development have more time to acquire links. Thus, sensori-motor domains encountered in early childhood, like BALANCE or ROTATION, are used as source domains for many mathematical ideas. Concepts that have a greater degree of "fitness" also tend to have more connections than other nodes. Certain domains, like SET (see Figure 3) are repeatedly employed as sources because of the power and utility of their particular inferential structures. Moreover, people tend to rely on sources with which they are already familiar.

Thus, the metaphoric network of mathematics exhibits both growth and preferential attachment. These two properties have been identified as necessary and sufficient conditions for a network to display a *scale-free* topology (Barabási & Albert, 1999). This pattern of organization is the "*common blueprint … [that governs] the structure and evolution of all the complex networks that surround us*" (Barabási, 2003, p. 6).

THE DYNAMIC BEHAVIOUR OF A SCALE-FREE NETWORK

In scale-free networks, clusters are formed within which every vertex is connected to a hub; these are in turn linked to more central nodes, and so on (see Figure 5). In this type of structure, a few nodes possessing very many connections coexist with numerous vertices that have only a small number of links; there is no intrinsic *scale*, or typical number of connections per node, in the network (Barabási, 2003). This distribution of connectivity plays a highly significant role in determining the dynamics of a scale-free network.

Highly connected vertices have a major effect on relationships within the network. For example, hubs create paths with only a few links between any two nodes in the system and, consequently, chains of interactions can spread quickly throughout the web. Hubs also ensure that scale-free networks are generally very robust. Since the majority of nodes have only a few links, a significant number can be removed from the system with little or no effect. However, the weakening of a key vertex may reverberate throughout the network; nodes directly connected to the hub fail first, nodes linked to these fall next, and so on. While such a *cascading failure* can go unnoticed for a long time, the collapse of one highly connected hub may eventually cause a large part of the network to disintegrate and become fragmented (Barabási, 2003; Watts, 2002; see Figure 6).

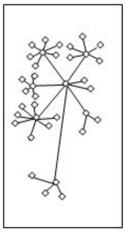

Figure 5. A simple network displaying a scale-free topology

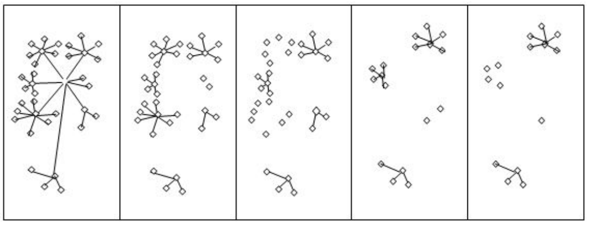

Figure 6. The effect of a cascading failure on a scale-free network

IMPLICATIONS FOR THE LEARNING AND TEACHING OF MATHEMATICS

To illustrate how the metaphoric network of mathematics can be affected by its scale-free topology and consequent dynamic behaviours, consider some of the many topics that can be based on ROTATION. Figure 7 displays just some of the domains that are jeopardized if understanding of this concept breaks down. Other mathematical ideas linked to these conceptual domains might collapse in turn. As the failure of this important source domain ripples throughout the metaphoric structure, an individual's understanding of mathematics may be severely compromised.

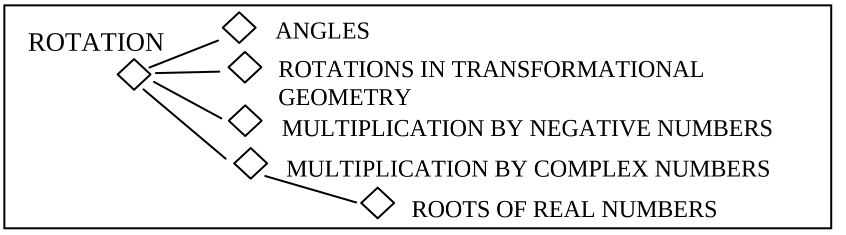

Figure 7. Some concepts metaphorically linked to ROTATION

There is some intrinsic credibility in the idea of cascading failures in subjective mathematics. As an educator, I believe I have witnessed this in the classroom. For example, the concept of EXPONENTIATION tends to be constrained by "definitions" of repeated multiplication. This interpretation works well for elementary arithmetic, but, when learners encounter negative or fractional exponents, they have trouble making sense of these new situations. Later, difficulties comprehending EXPONENTIATION lead to consequent problems working with connected domains such as POLYNOMIALS, QUADRATIC EQUATIONS, and LOGARITHMS. Experience in the classroom leads me to recognize situations where the catastrophic collapse of a student's understanding does occur.

To increase the robustness of the metaphoric network of mathematics, attention might be focused on a student's learning of major conceptual domains used in mathematics. However, little is known about which domains might be hubs in this network, and it is not clear that key nodes are the same for everyone. Moreover, even if a learner constructs stronger understanding of concepts that are seen as important, this will not eliminate the vulnerability that is characteristic of a scale-free network. Hubs would still be attractors of connections in the network of mathematics.

In order to improve the robustness of the network of metaphors, one must change its structure. Watts (2002) suggests that reducing the number of connections to a hub should lessen the likelihood of network failure: "*[e]ven in the event a hub did fail, fewer [nodes] would be affected, causing the system as a whole to suffer less*" (p. 193). However, it is not likely that a teacher would refuse to employ conceptual domains that are commonly used to provide coherent structure for other mathematical concepts, nor would this be responsible.

Another approach is required. Increasing the number of connections among conceptual domains would have the desired effect of reducing the network's dependence on its hubs. Adding even a few connections between clusters of nodes decreases the network's vulnerability (see Figure 8). The more distributed structure that results has sufficient redundancy to ensure that "*even if some nodes [go] down, alternative paths [maintain] the connections between the rest of the nodes*" (Barabási, 2003, p. 144). The collapse of one node is therefore less likely to cause the catastrophic fall of many other vertices and the network remains an interconnected whole. By promoting the establishment of more connections among mathematical concepts, teachers may be able to assist students to construct more robust understandings of mathematics. This tactic offers a more effective, and more acceptable, approach to preventing cascading failures in the metaphoric network of mathematics.

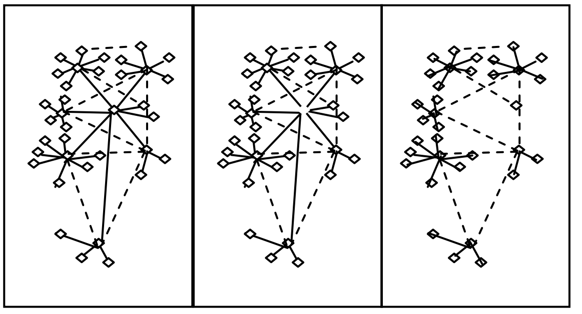

Figure 8. Increasing stability by adding links to a scale-free network

Therefore, I suggest that students be introduced to a variety of metaphors when learning about a new mathematical concept. This requires a shift in pedagogical thinking, for, in mathematics education, some metaphors are traditionally utilized to make sense of certain concepts (e.g., EQUATIONS ARE BALANCES). Such connections are strong because they are widely used and constantly reinforced. However, relying on a single link to provide structure for an idea is dangerous. Failure to comprehend the source it is based on can trigger a cascading failure in the metaphoric network, jeopardizing comprehension of many connected mathematical domains. If several metaphoric links are introduced to make sense of a concept, this is less likely to happen. Should a learner's comprehension of one source break down, he or she can rely instead on metaphors projecting inferential structure from other domains. The network of metaphors that constitutes a student's understanding of mathematics becomes more robust and less subject to the cascading failures and fragmentation that are characteristic of scale-free networks.

For this approach to succeed, teachers need to be supported. Professional development could help teachers to understand the important role metaphors play in students' learning. A reconceptualization of curriculum structures, highlighting multiple interpretations of concepts, would also be desirable. Before changes in these areas can be implemented, however, more needs to be known about which metaphors can provide important inferential structure for specific concepts. Systematic investigations that look for the many connections among mathematical ideas are badly needed. In an attempt to explore ways in which such research might be carried out and to substantiate ideas discussed here, I conducted a study for my dissertation that looked for links to the concept of EXPONENTIATION.

BEGINNING TO EXPLORE

This project consisted of three components: a review of the literature, interviews with mathematics educators and mathematicians, and a concept study. In each phase, I looked for representations that could be used to develop understanding of EXPONENTIATION – gestures, images, analogies, metaphors, models, activities, and applications. The third part of this work, a collaborative exploration by teachers, proved to be the most productive and exciting part of my research.

Numerous conceptualizations (too many to detail here) arose throughout the study, showing EXPONENTIATION to be far more than the most commonly found designation as *repeated multiplication*. These representations reflected conceptual domains to which EXPONENTIATION could be metaphorically linked. Not unexpectedly, source domains for the four grounding metaphors of ARITHMETIC (see Figure 4) were connected to EXPONENTIATION. Moreover, other domains that give meaning to the concept emerged. Figure 9 is an attempt to portray some of these findings, showing a small and almost certainly incomplete part of the metaphoric network of mathematics.

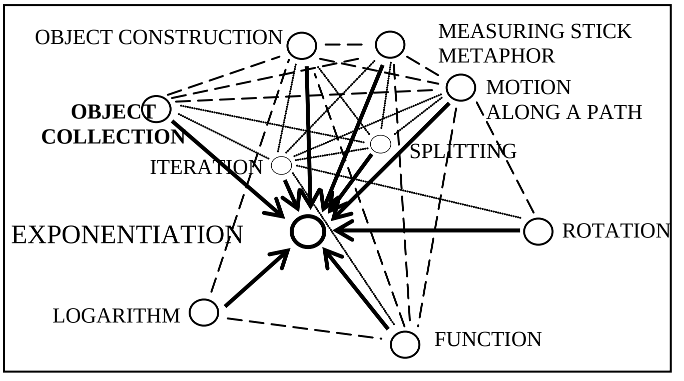

Figure 9. Part of the metaphoric network of mathematics involving EXPONENTIATION

It also seemed significant that most participants expressed positive views about the idea of using multiple metaphors to make sense of mathematical concepts. In a discussion of representations brought forth in the concept study, one participant enthused, "*These activities are often seen as enrichment – why are they not core activities? … If only we could talk about every topic in math in this way!*" Such comments encouraged me to think that teachers might be amenable to adopting the pedagogical approach suggested in this work.

REFLECTIONS

In my dissertation, I posit that it is important to consider mathematical understanding as a network structure. I further suggest that the theory of embodied mathematics offers a possible structure for a network of mathematical knowledge. While exploration of this model leads to valuable insights into the complex system that is subjective mathematics, I must acknowledge that different network structures may also prove effective in describing the cognitive dynamics that are an intrinsic part of the learning of mathematics. Moreover, I cannot expect the network model described here to provide explanations for everything that might be considered comprehension of mathematics, for "*developing an understanding of a complete system is a much harder task than simply looking at the underlying network*" (Newman, Barabási, & Watts, 2006, p. 415). Despite these qualifications, I hope that my speculations may be only a beginning for an important complex conversation in education.

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RECOVERING MATHEMATICS: AN ONTOLOGICAL TURN TOWARD COMING TO UNDERSTAND THE TEACHING OF MATHEMATICS WITH CHILDREN

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Memorial University

This research is an interpretive study into the teaching of mathematics. Drawing from Gadamer's (1989) ontological hermeneutics, this work examines lived experience through narrative pedagogic events to explore the idea of recovering mathematics as a living human enterprise for children and teachers in schools. To summarize a phenomenological hermeneutic study is extremely difficult if not impossible. The words – and the form of the words – are carefully chosen. As van Manen (1990) states, "*pedagogical writing requires a responsive reading….the reader must be prepared to be attentive to what is said in and through the words*" (pp. 130-131). In a synopsis of a dissertation, the meaning of the form and the writing is lost. For the purpose of disseminating my research I offer you a brief description and explanation of my dissertation.

Recovering Mathematics: An Ontological Turn Toward Coming to Understand the Teaching of Mathematics with Children begins with a narrative about a conversation I had with a grade one child in the hallway of an elementary school:

OUT IN THE HALLWAY

> "*I didn't know it was going to be so hard*," the young child said, standing outside of the classroom. Her arms were crossed, her brow furrowed and her bottom lip jutted out. Her dark eyes met mine. Then she looked away.
>
> "*What?*" I asked, not sure where we were headed.
>
> "*School. If I had of known it was going to be like this, I never would have come out.*" I bent down and squatted beside her, carefully adjusting my skirt in the process. I placed my hand gently on her back and tried to meet her eyes.
>
> "*Come out? You mean come to school?*"
>
> "*No! I never would have come out of my mom if I knew it was going to be so hard!*"

This brief conversation has proven to be the first major rupture to my pedagogic understanding of what it means to be standing in the living world of teaching children mathematics. In *Truth and Method*, Gadamer (1989) claims that the happening of events is essential for understanding. What my student, Isabel, said happened, and it shook me to my core. "*If I had of known it would be this hard, I never would have come out*." Her words were out there. They were loose in the world. They jarred me. In two months, had I managed to make school so terrible that she wished she had never been born? I did not know I had settled into an easy doing of things, of disappearing into the doing of things in a school with children. So now what was I to do?

> This child had spoken to me. And I needed to listen, to really listen and pay attention to what she had to say. What was it that we were doing that felt so hard? Clearly my student, Isabel, did not feel she belonged in the place of school.
>
> As I watched the young girl walk slowly back to her desk to finish her work, I realized that I didn't recognize anything anymore. With our talk out in the hallway, Isabel had transferred her suspicion about schooling to me. I felt like a stranger in my own body. Had forgetfulness guided me away from the centre of my own being? But what had I forgotten? My intentions were to be a good grade one teacher. I only wanted to do things right. I wanted to fit in with my grade team. I wanted to belong. I lingered on the question: What had I forgotten?
>
> The very night of my talk with Isabel, I began to pour over the Program of Studies for Mathematics (1996) for grade one. There were three specific outcomes that related to using a calendar such as: "*Sequence events within one day and over several days; Compare the duration of activities; Name, in order, the days of the week and the seasons of the year*" (p. 30). My students were already doing that with ease. Besides, were there not many ways of addressing these outcomes other than through a morning routine that I directed? Our calendar routine took up over three hours of instructional time each week.

I needed to find a path that would invite me to dwell with these children, and not merely navigate a classroom toward an already determined destination of grade two. So, could I be the problem? Had I lost my imagination? Might that be it? Was I so paralyzed by doing the right thing at the right time as a teacher that I forgot to be all right with children?

Gadamer (1989) suggests that everyday life is constituted through forgetfulness. Human life is the process of levelling out and flattening everything. Looking back, I see that I had flattened the landscape of that grade one classroom into a reduced pile of laminated apples and months of the year. My way of doing school had to change.

In the opening chapter and the chapters that follow, I unpack and live in the spaces of the complexity of teaching mathematics to young children. My desire to understand the teaching of mathematics is two-fold; I want to come to understand better the complexities of teaching mathematics to children, and in doing so I want to lay bare my own awareness of the difficulty that seems to have been forgotten and remembered, revealed and concealed. Had I completely overlooked the complexities of teaching mathematics? In the forward to the second edition of *Truth and Method*, Gadamer (1989) writes, "*My real concern was and is philosophic: not what we do or what we ought to do, but what happens to us over and above our wanting and doing*" (p. xxviii). The writing of this dissertation has helped me come to understand what has happened and is happening to me as a teacher of mathematics.

This dissertation considers the relationship of mathematics to teaching in terms of the past and the present, the particular and the general, the philosophical and the practical, the part and the whole. It is an exploration into what might be possible when it comes to teaching mathematics to children when the world, which includes the living world of mathematics, is allowed entry. Jardine (1994) describes mathematics as not being held in a fixed state. He goes on to write that,

> *It is, so to speak, a way which must be taken up to be a living whole. There is thus a way to mathematics. Learning its ways means entering into these ways, making*

> *these ways give up their secrets – making these ways telling again, making them more generous and open and connected to the lives we are living out.* (p. 270)

What may be possible for the teacher of mathematics if it is thought of in this way? Possible for the child? Possible for mathematics? What might it mean for pre-service teachers to be ready to teach mathematics to children? What does it mean for any of us to be ready to take up mathematics with children?

As an ontology of understanding, hermeneutics "*avoids both the subjectivizing involved in making interpretation a psychological process, and an objectivizing which omits/denies the interpretive moment in the reader*" (Palmer, 1999, ¶22). My dissertation reveals the nature of the social context in which I am situated. More importantly, readers of my text are able to share my experiences. I am not looking to pass on information to the reader through my text; I hope it evokes in them a new way of understanding the teaching of mathematics to children and a new understanding of who they might be in the world. I hope I have created a dialogical text that creates meaning for the readers but that also contains a critical reflexivity about my own pedagogical actions. In doing this, I have attempted to avoid any sense of romanticism around teaching young children mathematics or working with student teachers as they work with young children.

The hermeneutic circle is ubiquitous in descriptions of hermeneutics. Like Heidegger, Gadamer took up Schleiermacher's notion of the interplay between the part and the whole in the process of interpretation (Smith, 1994). This "*back and forth movement between the particular and the general, is more popularly referred to as the 'hermeneutic circle'*" (Davis, 1996, p. 21). As I move from the specific to the general, my understanding of both becomes deepened and this affects all other understandings. Gadamer brings the interpretive consciousness into his articulation of the hermeneutic circle. The hermeneutic inquirer cannot be a detached observer, as Davis (1996) points out:

> *Rather, the interpreter recognizes his or her complicity in shaping the phenomenon, simultaneously affecting and affected by both the particular and the general, thus wholly embedded in the situation. In other words, the 'object' of the hermeneutic inquiry is a moving target.* (p. 22)

By writing about particular pedagogic moments of awareness with students and mathematics, I come to understand not only the particular moment, but I come to a different understanding of the nature of mathematics in general, of teaching in general, and of difficulties of living on the earth ethically with one another. By remaining open in this cyclical process, I am in a state of becoming.

In this dissertation I point to moments of my own pedagogic awareness that I hope cracks open for the reader the living nature of teaching children and in particular, teaching the living discipline of mathematics. I am, as Jardine (2006) describes, "*working such matters out*" (p.161) through my writing to come to a deeper understanding.

Recovering Mathematics is comprised of six chapters. The opening chapter takes up the difficulty and complexity of teaching mathematics to children and shares with the reader how I arrived at my topic of wonder. The second chapter explores the living nature of mathematics. In order to come to understand the character of mathematics as a living discipline I turn to the work of scholars who write about mathematics as human and dialogic, complete with aesthetic dimensions. Their writing unearths for me a way into mathematics such that I have come to understand it as a human living enterprise. In order to better disrupt the traditional view of school mathematics, I unpack my own mathematical experiences and prejudices and lay bare my own teaching of the discipline. The third chapter takes up ontological hermeneutics and explores how I came to my method, the dialogical nature of hermeneutics, memory, truth, understanding through conversation, play and pedagogy, conversation, and the relationship between mathematics and hermeneutics.

The largest section of the dissertation is comprised of the fourth chapter. It is in this section of the work where I point to particular moments of pedagogic awareness that I hope breaks open for the reader the living nature of teaching children and in particular, teaching the living discipline of mathematics. This fourth chapter, then, is divided in two sections; the first section describes experiences from a classroom of grade one learners I once taught, and, in retrospect, they have taught me more than I them, and the second section focuses on learner-teacher classroom experiences in grade two.

The fifth chapter turns to my work with pre-service teachers and explores my experiences with them to come to understand the complexity and difficulty of teaching children mathematics and dwelling in the living space of teaching. From my initial work with pre-service teachers which led me to my topic of coming to understand the teaching of mathematics to children, I have come to the awareness that *how* students have dwelled with mathematics has formed *who* they have become, as learners and prospective teachers of mathematics. Part of my task, then, is to help them to challenge the assumptions they hold about what mathematics is for them, who they perceive themselves to be as learners of mathematics, and who they might become as teachers of mathematics, as well as teachers of the other disciplines for which they will be responsible. I argue in this chapter that there is a particular version of mathematics that many pre-service teachers hold, and I name this as dwelling with a “cover version” of mathematics. In particular this chapter explores keeping the difficulty of teaching alive and resisting the lure of technique.

Hermeneutics leaves me with the ethical task of deciding how to properly proceed and this is the focus of the final chapter. Gadamer’s hermeneutics calls to me and his ontology of understanding advances my work because of what I recognize as a hermeneutics of possibility, of hope, of generosity, and of responsibility. Engaging in conversation, looking for what might be true and for what might be possible is indeed a generous offering. Gadamer (1989) describes the task of hermeneutics in this way, “*What man* [sic] *needs is not just the persistent posing of ultimate questions, but the sense of what is feasible, what is possible, what is correct, here and now*” (xxxviii). In writing this dissertation, it has never been my intention to come to firm solutions as to the best way to teach children. The purpose of this hermeneutic journey has been to allow me to examine my lived experience so as to make meaning of those experiences. While keeping the original difficulty alive (Caputo, 1987), I am working toward understanding the teaching of mathematics to children. David Jardine said recently at a research institute on hermeneutics, “*We tell our stories to find out what they mean.*” This hermeneutic inquiry has not just been a telling of stories, but more importantly, I believe these narratives have been *telling of something* – telling of possibilities with children, of life in classrooms, of the difficulty of teaching, of who we might become as teachers, of the nature of mathematics, of what it means to be an elementary pre-service teacher, and of complexities surrounding mathematics education. Taking up mathematics as a deeply human living enterprise is not easy. To teach in this way is to tread what Dunne (1993) calls *rough ground* in all its messiness and complexity. It is worth it, however, to help students explore and come to know the landscape of mathematics, rather than experience what Boaler (2008) describes as the distorted image of school mathematics.

People who have experienced the distorted image of mathematics in schools have travelled along a fixed rutted path (Bransford, Brown, & Cocking, 2000) and many become lost when they do encounter the mathematical landscape. Indeed, such a landscape is alien territory. The majority of teachers of elementary learners, including myself, were schooled along such a path. This poses a challenge for coming to understand our way around the terrain and to being open to the wonder, open to the possibilities that being in a classroom with children and mathematics can provide. *Recovering Mathematics* allows readers an opportunity to come to a greater understanding of what it might mean to dwell on this earth with children when the living world of mathematics is allowed entry.

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Ad Hoc Sessions

Séances ad hoc

WHO TEACHES THE TEACHERS? CHALLENGING OUR PRACTICES

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University of Calgary

The following problems or challenges for teacher educators were posed:

- "After they leave us," why don't they (some of them) maintain their philosophical stance/ personal epistemology, and practice "what we taught them"?
- How does our practice, as teacher educators, impact their ability to maintain a self-awareness and sustain a critical practice?

During the ad hoc session, I described my current research entitled "Exploring Mathematics Teachers' Beliefs and Practices Through Participatory Action Research" where I ask what are the implicit and explicit beliefs of teachers regarding mathematics, teaching, and learning, how exposure to new and empowering teaching paradigms might make a difference for teachers, and how the practices of the study itself might have made a difference for teachers' sense of self-efficacy or empowerment. The five phases of my research included focus group questions which challenged issues of power in schooling; examples of pedagogies including constructivism, ethnomathematics, and criticalmathematics; reflective journaling and exposure to journaling practices; and the research participants' experiences in "stepping out of their comfort zone" and teaching, and then sharing that experience with their colleagues.

The written ideas of one of the research participants were shared as he navigated the five research phases from the beginning to the end of the study. Conference members in the session were asked to comment on the meaning of the teacher's narrative. They were then asked to discuss the narrative from their own research perspective, or from any of the following research areas: personal epistemology, beliefs/practices, research/practice, particular teaching/learning theories, empowerment, self-efficacy, critical thought, reflective practice, or other.

The questions and conversations that emerged from the conference members in the session are listed here:

- What are the barriers that teacher educators face?
    - How much influence, if any, do we have on our students' future practice?
    - Is it our responsibility to help them sustain their practice?
    - Are students' goals different from the ones we intend for them?
    - Are the structures of our pre-service programs aligned with our goals?
- Why don't students internalize the deep learning that we intended?
- How do our practices model what we would like them to practice?
- How do we encourage and monitor both the students' voices and ours?

WHAT DOES MATHEMATICS EDUCATION HAVE TO DO WITH CLIMATE CHANGE?

Richard Barwell
University of Ottawa

Climate change is one of the most pressing issues of our time and is increasingly reported in the popular media. Despite scientific understanding and concern dating back several decades, global action has proved entirely inadequate: global warming continues at an increasing rate, as do greenhouse gas emissions. This issue concerns me as a citizen. How can I respond to this concern? How do you respond? How can we, as mathematics educators, respond? So far, mathematics educators have not turned their attention to climate change or even to environmental issues more generally. The purpose of this ad hoc session, therefore, was simply to draw some attention to the role of mathematics in understanding climate change and to suggest ways in which mathematics educators could respond.

As prompts for discussion I shared excerpts of various documents and websites, including:

- Inter-governmental Panel on Climate Change. (n.d.). *Climate change synthesis report.* Retrieved from http://www.ipcc.ch/publications_and_data/ar4/syr/en/contents.html
- Mathematical Sciences Research Institute, Berkeley, CA. (n.d.). *Mathematics of climate change: A new discipline for an uncertain century.* Retrieved from www.msri.org/calendar/attachments/workshops/.../MathClimate.pdf
- Statistics Canada. (2007-2008). *Human activity and the environment, 2007-2008, Section 1, Climate change in Canada*. Retrieved from http://www.statcan.gc.ca/pub/16-201-x/2007000/10542-eng.htm
- *Climate Change – Has the Earth been cooling?* (n.d.). [Video recording]. Retrieved from http://www.youtube.com/user/potholer54#p/c/A4F0994AFB057BB8/7/xvMmPtEt8dc

The discussion touched on a number of issues, including some of the mathematics involved (e.g. calculation of global mean temperature, differential equations) and the role of the interpretation or misinterpretation of graphs, charts and other mathematical representations in the popular reporting of climate change. It is apparent from the four sources mentioned above, for example, that a degree of mathematical literacy is essential to be able to understand the evidence for climate change or participate in discussions about what needs to be done to avoid serious environmental, social and economic consequences.

CONSTRUCTION DU SENS DES OBJETS MATHÉMATIQUES CHEZ LES ÉLÈVES D'UN COURS DE MISE À NIVEAU

Analia Bergé
Cégep de Rimouski, QC

L'objectif de cette présentation ad hoc a été de partager une étude réalisée au Cégep de Rimouski concernant l'apprentissage et l'enseignement des mathématiques à des élèves du cours de mise à niveau pour Mathématiques 436.

La clientèle de ce cours a des caractéristiques particulières : il s'agit en général d'élèves qui ont eu de la difficulté en mathématiques à l'école et qui manifestent un manque d'intérêt et de motivation à les apprendre.

Traditionnellement ce cours débute avec une pratique algébrique (factorisation, exposants et résolution d'équations) suivi de l'étude de fonctions linéaires, quadratiques et d'autres fonctions de référence. Une telle structure, qui déploie des aspects plutôt techniques jusqu'à presque la mi-session, rend difficile, à notre avis, la construction du sens des objets mathématiques de la part de ces élèves.

Nous avons fait l'hypothèse qu'il est possible de favoriser une construction du sens chez les élèves visant l'imbrication des aspects techniques et conceptuels en effectuant une organisation différent des contenus du cours :

1. l'introduction d'éléments algébriques dans leur rôle d'outils de résolution de situations-problèmes;
2. la réalisation d'un travail algébrique après l'introduction de fonctions, une fois que les besoins d'une maîtrise des techniques algébriques sont reconnus comme pertinents par les élèves.

Dans cette rencontre ad hoc**,** nous avons discuté sur quels sont les avantages et désavantages d'une telle organisation d'un point de vue didactique et d'un point de vue institutionnel; et de quelle façon cette organisation peut-elle favoriser une meilleure compréhension de la part de nos élèves.

THE STUDY OF RESILIENCE IN MATHEMATICS EDUCATION

Pavneet Braich
Brock University

The purpose of the ad hoc was to explore the concept of *resilience* within mathematics education, since it was central to the research project I was conducting in the Masters of Education program at Brock University. Resilience describes "*a set of qualities that foster a process of successful adaptation and transformation despite risk and adversity*" (Benard, 1995, p. 2).The exploration of resilience stemmed from the search for possible ways to alleviate stereotypes in the mathematics classroom, one of which is a dividing notion that one either can or cannot do mathematics.

> *Too often, society has accepted the stereotype that mathematics is for the few, not the many. The reality is that mathematics is deeply embedded in the modern workplace and in everyday life. It is time to dispel the myth that mathematics is for some and to demand mathematics success for all. (Ministry of Education, 2005, p. 9)*

This stereotype seems to perpetuate a notion that it is acceptable for mathematics learners to disassociate themselves from being mathematics students. This illustrates a change that needs to occur within mathematics education. The ad hoc helped in building a discussion about whether there is a need for the study of resilience in mathematics education, and how teachers can help build certain characteristics of resilience in students.

A lot of emotions are involved in a mathematics classroom, and teachers must be prepared to deal with students who are generally unhappy in a mathematics classroom. How does a mathematics teacher deal with this while teaching the curriculum, as well as motivating students? Educators discussed the importance of teaching students how to deal with struggles and being frustrated, since these types of emotions are experienced by mathematics students. It needs to be understood that overcoming struggle is part of a mathematics student's identity. Teachers should encourage an understanding that "instant gratification" is not necessary, and students should be told that it is not necessary to achieve a right answer, right away. Teachers must create a culture in which students are willing to learn, and see beauty in mathematics, as opposed to feeling that they are just completing a compulsory subject. It was mentioned by another educator that the study of resilience seems to build on the beauty of mathematics.

Since the development of resilience was being related to the development of success within mathematics students, the question of classifying a successful student arose. Discussion evolved around success being equated with feeling able to do mathematics and remaining persistent in the learning of mathematics, along with appreciating the study of mathematics. The concept of dialogue within a mathematics classroom was understood to be significant, so students learn to connect ideas and connect with mathematics through *accountable* talk.

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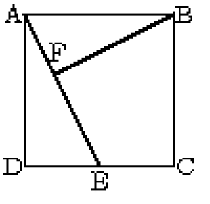
WHAT MY STUDENTS TAUGHT ME ABOUT JIM TOTTEN'S GEOMETRY PROBLEM IN FLM

Elias Brettler
York University

The goal of this session was to share the richness of possibility revealed by student responses to a geometry problem in FLM.

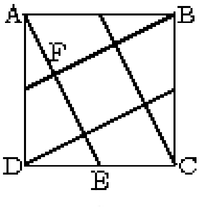
This problem from For the Learning of Mathematics was given to students in a first-year course, *Problems, Conjectures and Proofs*.

Given square ABCD with E the midpoint of the side CD. Join A to E and drop a perpendicular from B to AE at F. Prove that CF=CD.



The students were to look for two or more different ways to prove that CF=CD. An obvious solution is obtained by choosing coordinates making A(0, 10), B(10, 10), C(10, 0), D(0, 0). Then F has coordinates (2, 6) and the distance formula gives the length of CF as 10. Another proof based on measurement extends AE and BC to meet at G and uses the law of cosines in ΔCGF. The observation that the points E, F, B and C lie on a circle gives a third measurement proof based on the idea that chords which subtend equal arcs are equal in length. A proof based on classical construction ideas and the fact that an angle inscribed in a semi-circle is a right angle was given. Another used the idea of showing that C lies on the perpendicular bisector of FB. As the non-right angles which appear in the diagram are either equal or complementary, there are proofs which exploit this in order to use congruence or similarity. In one, the triangle ΔBCF is proved similar to the isosceles triangle ΔAEB.

Extend BF to meet side AD. The point of intersection is the midpoint of AD. Adding additional lines from the corners of the square to midpoints of opposite sides gives the following diagram.



Placing another copy below suggests two visually compelling proofs, one showing that CF and CD are diagonals of congruent rectangles and another showing that a parallelogram, two of whose sides are congruent to CF and CB, has perpendicular diagonals making it a rhombus.

This is an excellent problem for class use. It is simple to present and virtually any reasonable attack yields results. What remains is to identify what about this problem supports such variation.

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SOCIAL INTERACTION IN A VIRTUAL ENVIRONMENT: FOSTERING ONLINE COMMUNICATION AND COLLABORATION IN MATHEMATICS EDUCATION RESEARCH

Stephen R. Campbell
Simon Fraser University

Virtual environments such as “Second Life” <www.secondlife.com> are emerging as major cultural influences with significant opportunities and possibilities for mathematics education (Campbell, 2009; 2010). Second Life (SL) is a massively multi-user online social interaction virtual environment where individuals design and inhabit their own “avatars” or virtual bodies. In SL, individuals can socially and collaboratively interact in real time through their avatars with the avatars of others, via gestures and actions, and communicate through text messaging and voice-over-internet. In this session, we (the David Wheeler Institute for Research in Mathematics Education <www.educ.sfu.ca/research/wheeler> in collaboration with the ENL Group <www.engrammetron.net>) developed, introduced, and demonstrated an initiative to do just that: to use a virtual facility that we have developed in SL called “Wheeler Island” to foster world-wide communication and collaboration amongst mathematics education researchers.

We held this session simultaneously in the real world (middle top video inset), and in the virtual conference facility on Wheeler Island in SL (right video inset). Real world attendees were logged in as virtual world attendees. One attendee (middle lower video inset) attended the virtual session remotely using an ENL-based computer with eye-tracking monitor. He participated using voice over internet. Power point slides for this ad hoc presentation were controlled from the podium in the virtual conference facility. Both real and virtual sessions were recorded, and the acquired data were subsequently integrated and time synchronized, as per the figure above.

Beyond demonstrating that the Wheeler Institute can host virtual conferences and seminars on Wheeler Island in SL, we are making Wheeler Island accessible to math education faculty and graduate students to help foster new means for communication and collaboration in mathematics education research. Please contact the author at <sencael@sfu.ca> for more details.

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EDITING MATHEMATICS TEACHERS' JOURNALS IN CANADA: BRIDGING THE GAP BETWEEN RESEARCHERS AND TEACHERS

Sean Chorney, *Simon Fraser University*
Egan J. Chernoff, *University of Saskatchewan*
Peter Liljedahl, *Simon Fraser University*

Help! We're editing teacher journals in both BC and SK. What should be in it?

The interest for such an ad hoc gathering was to explore some guidelines for math teacher journals that would be useful and meaningful to the practicing teachers of our jurisdictions. A manifesto, if you will, to broaden both readership and relevance. A number of suggestions were offered and this summary will elaborate on some of the more dominant ideas that emerged in the discussion.

An appeal to and a drawing from post-secondary influence was a substantive point. It was thought that the inclusion of people from different institutions to work together both in content and organization would be a way in which to draw on different perspectives and to elicit articles from each respective group. It was suggested that one way to do this is to establish a peer review section. An editorial board could be established, made up of both teachers and post-secondary educators, to both evaluate and elicit articles from a practice/research paradigm. Articles of a more research-oriented framework could add a layer of sophistication, a characteristic assumed to be desired in a teacher journal. One member of the discussion maintained that the listing of references would be justification enough for such an academic paper, allowing teachers to pursue personal study, for example. Authors submitting to the journal would have a choice whether to have their paper be a part of the peer review process. University students registered in a Masters or PhD math education program could be required to participate in writing or be involved in peer editing. Keeping a relationship between current research and current practice was put forth as possibly the most relevant aspect of such a journal. The widening gulf between practice and research is a problem found in many contexts, and research-informed but practice-focused articles are a way to close this gulf.

Another major theme resulting from the discussion was the need for dialogue. Constant communication was thought to be essential in maintaining a teacher journal. Those seen to be part of a dialogue included: readership and editors, writers and editors, the various writers of current publications and the writers of past publications. Dialogue could be manifested in the form of editorial conversations as in FLM or in the form of a letter to the editors. It was recognized that a journal is not a static entity; it evolves in a way that may be initially unpredictable. Conversations are essential to elaboration and evolution of ideas. Ongoing discussion was seen as a necessary step in the publication process.

The following are pre-dialogue ideas but were considered as initial steps to communication. One thought was to have special issues focusing on a particular audience or a particular theme. In our discussion the example proposed was to have an exclusive elementary issue: an issue targeting elementary teachers, their needs, and their interests: to elicit articles from elementary experts and practicing elementary teachers. It was not seen as an extra issue but one of the regular publications devoted to the elementary contingent in our respective provinces. A dialogue between past and present was also addressed with the possibility of integration of past articles reprinted in current issues. Ideally the reprint could address a current theme or a current article seeking to provide answers: to consider how things have or have not changed in the field, and/or to reflect on past practices in light of current findings.

Another theme revolved around the technical organization of the journal itself. Ideas of structure and consistency were posited, such as an article moniker to identify a standard set of sections in each issue. Different categories could be established so as to appeal to different needs or interests. The editorial board could appeal to different groups to fulfil those needs. For example, “Teaching on the Edge” could be one of the categories fulfilling a personal look at teaching in a challenging setting where mathematics is secondary. Stories could be elicited from particular writers and would, very likely, appeal to a particular audience. Other categories would be resource websites, mathematical activities, book reviews, etc. and editors would appeal to particular writers for each of these categories.

The discussion was helpful and enlightening and the editors were thankful to those who joined in the conversation. In summary, our manifesto came down to this: ultimately the journal should evolve to offer both a resource and a tool to provide support to our teachers – a journal by the mathematics community for mathematics teachers.

REASONING DEMANDS IN MATHEMATICS TEXTBOOKS

Simon Karuku
University of Alberta

This presentation discussed the results of an analysis of the types of mathematical reasoning demanded in a10th-grade mathematics textbook.

The ability to reason mathematically is critical to students' success in mathematics (National Council of Teachers of Mathematics, 2000). Teaching mathematical reasoning is, however, a challenging task. To enhance their mathematical reasoning skills, students should be presented with opportunities to discover and explore new ideas, make and evaluate mathematical conjectures, develop and evaluate mathematical arguments and proofs, explore alternative solution strategies, justify results, as well as generalize and infer mathematical relationships. Mathematics textbooks can play a vital role in providing these opportunities to students.

Beginning September 2010, the Western and Northern Canadian Protocol (WNCP) will be adopting a new mathematics curriculum for schools in its jurisdiction. The 10th-graders will have three courses to choose from: Math 10C, Math 10-3, and Math 10-4. Math 10C is for students planning to attend colleges and universities (WNCP, 2008). My study explored how one of the approved textbooks for Math 10C (Van Bergeyk et al., 2010) promotes mathematical reasoning. A sample of 5% of the textbook pages was randomly chosen, and in each of the selected pages, all mathematical tasks were analyzed for reasoning demands using the framework of Lithner (2008).

Results showed that the dominant type of reasoning was imitative reasoning (characterized by memorization of solution or algorithm), which accounted for 57% of all the mathematical tasks in the sample. Creative reasoning (problem solving using flexible and novel methods that are based on the relevant mathematical properties of the concepts involved in the task) accounted for 23% of all the mathematical tasks in the sample.

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LAUGHING @ MATHEMATICS EDUCATION: EXPLORING POTENTIALS OF ONLINE MATHEMATICS COMICS (OMC) FOR SOCIALLY JUST PEDAGOGIES

Steven Khan
University of British Columbia

Mathematics and humour have an entangled history (Paulos, 1982). In the 80's and 90's printed comics such as the *The Far Side* and *Calvin & Hobbes* frequently poked fun at mathematics and mathematics education. Larson's *Hell's library* which is stocked only with word or story problems is commonly invoked in the field as in the opening pages of Reed (1999). In keeping with trends in popular culture such as the move to e-media for comics (Marvel's iPad comic store) and the visible commercial success of "geek" and comic (graphic novel) sub-cultures with mainstream audiences, math and science themed comics have established a successful presence online. Being relatively recent, the content of these OMCs has not yet been engaged with by the mathematics education community and I believe they provide a rich resource for pursuing discussions of social justice in mathematics education classrooms at all levels.

The questions that motivated this ad hoc session were: What types of mathematical consciousnesses and communities are being shaped by the world of online mathematics comics? Out of what types of consciousness and communities do such artefacts emerge? What are the implications and challenges for critical mathematics educators with interests in social justice and equity? Underlying these questions are assumptions about popular culture as a type of technology and the view that cultures shape and orient consciousness and influence communities.

I presented as "data" a non-random sampling of comics from *spikedmath*, *xkcd*, *abstruse goose* and *phdcomics* that were selected either for the mathematical content, their provocativeness, or because I thought they might be funny to a mathematics education audience. They were not meant to represent the complete spectrum of OMC's or to be representative of the individual websites. However in collecting and curating these images, themes emerged that included a fascination with the female body and adolescent sexuality. Indeed, one comic led to an interesting discussion of essentialist gender readings of mathematical ability. There was a brief conversation around the aesthetic qualities of the drawings and the composition of the comics. In future, a more detailed and comprehensive visual analysis of the comics themselves will have to be undertaken.

Open research questions include, "*Who is reading these comics? How do they interpret them?*" From a pedagogical standpoint, "*In what ways might these comics be used productively in the mathematics education classroom beyond providing a moment of levity?*" I suggest that while some of these comics are suitable for discussing mathematical ideas, their significance lies in opening a space for discussing mathematical values and critical issues such as gender, race, ability representations, stereotypes, and the hospitability of mathematics as a discipline. The arguments I presented were that popular culture artefacts such as OMCs ought to be seen as more than pleasant diversions from academic drudgery, that the subversiveness of some types of humour can serve to mask forms of continued discrimination, and that critically minded mathematics educators must perhaps engage in a double subversion

– a (de+re)-construction of such genres by deliberately selecting and engaging with the uncomfortable difficult knowledge that lies at the interface of mathematics, representations, and humour.

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THE RELOCATION PROPERTY

Kim Ledger-Langen
Spirit of Math Schools Inc.

In English, we read from left to right. We usually do mathematics in the same direction, but sometimes the calculations involved make this difficult. The relocation property helps to rearrange such calculations so that they are easier to do from left to right.

The relocation property combines the ideas of the commutative and associative laws and holds true for all four operations: addition, subtraction, multiplication and division. It is what people intuitively use to manipulate numbers in an algebraic equation.

EXPLANATION OF THE RELOCATION PROPERTY

Adding the numbers in a different order allows a person to calculate more easily what seems like a difficult question. The problem comes when you need to subtract. Consider the expression: 19 + 26 – 3 – 9 + 4 + 43. You cannot rearrange the numbers for easier calculation, because you have addition and subtraction and the commutative law does not hold for subtraction. Here is where the relocation property works. Consider each operation sign to be glued to the number that follows it. The relocation property states:

THE OPERATION SIGN GOES WITH THE NUMBER THAT FOLLOWS IT.

Now it is easy to rearrange the numbers in the example: 19 – 9 + 26 + 4 + 43 – 3 = 80.

The relocation property also works for multiplication and division. You simply remember that a multiplication or division sign must stay with the number following it when you do the rearranging. For example, for the expression 33 × 13 ÷ 2 ÷ 11 ÷ 13 × 10, one possible rearrangement that makes calculations easier is 33 ÷ 11, multiplied by 13 ÷ 13, multiplied by 10 ÷ 2. Therefore, 33 × 13 ÷ 2 ÷ 11÷ 13 × 10 = 33 ÷ 11 × 13 ÷ 13 × 10 ÷ 2 = 3 × 1 × 5 = 15.

IDENTITY ELEMENTS

Consider the expression: 5 ÷ 11 × 22. There are 2 signs, and they each go with the numbers that follow them, but what goes with the 5? Since the question involves only multiplication and division, the sign must be a multiplication or division sign. Putting a 1 × in front of the expression does not change the value of the expression (1 × 5 ÷ 11 × 22). Now the numbers can be rearranged for ease of calculation to: 22 ÷ 11 × 5 = 10. With use of prime factoring, and signed numbers, more complex expressions can then be calculated:

$$35 \div (-34) \div 13 \times (-22) \div 21 \times 9 \div (-15) \times 17 \times 26 \div 33 \times 6$$

$$= -26 \div 13 \times 17 \div 34 \times 9 \times 22 \div 33 \times 35 \div 21 \times 6 \div 15$$

$$= -4$$

Similarly, the identity element for addition can be used for relocation of addition and subtraction expressions, providing a 0 + before the first number in the expression.

In these examples, the addition and subtraction have been kept separate from multiplication and division. Combining the four operations can also be done, using the order of operations.

The relocation property is an important property that students use when they learn algebra. It is introduced long before the students begin algebra, so that when they learn algebra, relocation is second nature to them. Relocation is introduced in the Spirit of Math classroom in regrouping with positive and negative numbers in grade 1, and in grade 3 with multiplication and division. The complexity of the expressions increases as the students progress up the grades.

The benefits to the students are significant. Students have a greater appreciation and understanding of how numbers work together; algebra appears to be “common sense” when they learn it; students are naturally manipulating numbers in their heads and therefore arithmetic is done as easily as speaking a sentence. This property was developed by Charles Ledger in the 1980’s. He felt that there had to be one property that would work for all four operations.

BEYOND THE SUPERFICIAL: PROCEDURAL KNOWLEDGE IN UNIVERSITY MATHEMATICS

Wes Maciejewski, *Queen's University*
Ami Mamolo, *York University*

Research in conceptual knowledge is a current trend in mathematics education. In teaching, however, a balance between conceptual and procedural knowledge is struck, and, if one is more prevalent, it is often the procedural. This is especially evident in first-year university courses; a typical calculus textbook is largely composed of procedural questions asking for, e.g., the derivative or integral of provided functions. The long-standing distinction between procedural and conceptual knowledge (e.g. Hiebert & Lefevre, 1986) has attracted new attention from researchers aiming to refine these categories and definitions of knowledge. Star (2005) suggests that the description of procedural knowledge as rote learning lacking in complexity or richness, has inhibited research on procedural skill acquisition. The devil, Star argues, is in the definitions: conceptual knowledge is often viewed as "deep" while procedures are "superficial." Star challenges the existing definitions by positing the existence of "deep" procedural knowledge and "superficial" conceptual knowledge.

Our on-going research is motivated by Star's idea of deep procedural knowledge, which he describes as innovative, flexible, and deliberate in choice making to increase efficiency (Star, 2001). Given the emphasis on procedural knowledge in textbooks, and the accepted importance of conceptual knowledge, we wondered whether mathematics majors would demonstrate deep procedural knowledge when given the chance. Inspired by Star's (2001) description of deep procedural knowledge, we asked 21 third year math students to solve a typical derivative question taken from a first year calculus text in "as many ways as possible". Students had two weeks to solve the problem and had access to various resources (e.g. internet, texts, friends). While results are still preliminary, we note a few interesting points: (i) ~25% of students could not solve the problem at all; (ii) ~70% introduced irrelevant calculations / techniques; (iii) exactly one student solved the problem efficiently through a deliberate procedural choice. We are not yet in a position to draw conclusions, rather we find ourselves asking more questions. How is it that some students were unable to answer the question *at all*? What can we expect / hope for in the procedural knowledge of mathematics majors? We agree with Star that procedural knowledge and its development is a topic ripe for exploration, and we continue along this path.

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INQUIRY ACTIVITIES IN HIGH SCHOOL MATHEMATICS: ISSUES IN AUTHORING CURRICULAR PRODUCTS

Janelle McFeetors, *University of Alberta*
Ralph Mason, *University of Manitoba*
Elaine Simmt, *University of Alberta*

Teaching High School Mathematics through Inquiry is a design experiment research project (Cobb, Confrey, diSessa, Lehrer, & Shauble, 2003) made up of three researchers (the presenters) and eight mathematics teachers in Alberta and Manitoba. The team has been exploring what it means to incorporate inquiry-based learning activities in high school mathematics classes. The teachers designed and implemented activities to support rich mathematical learning. Data from students and teachers helped the team to develop notions about mathematical inquiry in classrooms and enabled the teachers to hone their activities for sharing with other teachers. In workshops for teachers, the dynamic processes of inquiry have been recreated with the teachers. However, our first attempts to capture the dynamics of inquiry lessons in static curriculum documents has brought forward questions of what to include (both in terms of mathematical content and pedagogical decisions the teachers made), how to structure the documents (the format), and how to open space for teacher-readers to shape their own inquiry activities.

In the presentation, we shared the curricular products constructed by the teachers for the purpose of discussing suggestions for implementation in other classrooms. Each teacher had chosen how to represent their inquiry activity differently: presentation slides containing activity sheets and teacher comments for a unit; a traditional unit plan with guiding thoughts and activity sheets; a Word document with teacher commentary added using the "comments" feature; and, an activity plan including intentions, content context, and possible questions. The materials can be found under the "T" link on the website www.thinkmathematically.ca.

In the ad hoc conversation, features within the examples were identified and explored by colleagues as producing curricular products which generate inquiry-based learning in other mathematics classrooms. While the form of the documents was not as important, the ways in which teachers wrote impacted the audience's reception and responsiveness to the activities. The use of first person pronouns communicated the experience of the teacher in a personal manner that allowed the audience to make sense of how to teach with the activities. To illustrate, where one teacher wrote "*The teacher may wish the students to use* …", another wrote, "*I now wanted to provide the students with an opportunity to* …". Modal verbs in the descriptions of teacher and student actions also impacted the way in which a reader could take up the teaching ideas. The use of "*should*" was seen as prescriptive, while the use of "*could*" or "*might*" opened up possibilities about which the teacher could decide. Again, a contrast was identified: "*I encourage students to* …" and "*Students should be asked to review their explanations*". In the ad hoc conversation, our colleagues helped us look at particular features of the documents which would enable the teachers to shape their writing so that other mathematics teachers might see the possibility of sponsoring rich mathematical learning. As researchers, we are encouraged to explore more deeply how Bakhtin's (1986) notion of dialogic texts could inform an understanding of curricular products as teachers engaging with teachers through text to improve mathematical learning.

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"I DON'T UNDERSTAND WHAT YOU DON'T UNDERSTAND!": A LOOK AT THE PHENOMENON OF "NOT UNDERSTANDING" IN MATHEMATICS PEDAGOGIC SITUATIONS

Ayopo Odunuga
University of Calgary

In this session, I discuss my ongoing doctoral work with a view to generate some discussion and hopefully get some useful suggestions from participants.

To start, I will give a brief overview of what the mathematics education literature has to say about "understanding" in mathematics. This is seen as important to a discussion of "not understanding" and what it can mean. The literature to be discussed will include Skemp's (1978) relational and instrumental (theoretical) characterization of understanding, Sfard's (1987) reification theory, Pirie and Kieran's (1989) recursive model of understanding, Tall's (1994) procept theory, Mowat and Davies' (2010) network theory of understanding, and Earnest's (2008) address through semiotics. The implication is highlighted that these theories (with the exception of Earnest, perhaps) consider understanding to be an epistemological, cognitive and mental phenomenon. Thus, it is easy from such a perspective to quickly assert what "not understanding" means as far as the learner is concerned, namely, the absence of links between nodes, or inability to cognitively tolerate ambiguous symbolism in processes and concepts, etc.

The contention however, is that the phenomenon of "not understanding" occurs on dual, qualitatively diverse levels. First, it occurs on an individual, cognitive level between the learner and the something that is not understood. This, I believe, is what the many theories and models of mathematical understanding refer to; a less than ideal state of affairs mentally that came about as a result of absence or inability. Secondly, this phenomenon also occurs on a social, inter-relational level that is occupied by the teacher and the learner. In other words, there is a "not understanding" between the learner and the teacher in relation to the thing that is not understood. Special attention is drawn to what I term "learner indicated not understanding" as opposed to "teacher identified not understanding". The latter, I note, inevitably lead to questions on student thinking which is typically a mental or cognitive issue. The former, on the other hand, since the initiative is from the learner, calls for interpretation on the part of the teacher.

To generate discussion and opportunity for questions and suggestions, attendees will be asked to discuss and respond to one of the questions in my survey that I am sending to research participants for my study. The question is: When and in what circumstances have you encountered students' expressions or phenomenon of "not understanding"? What do you think is not understood?

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GEOMETRIC REASONING: DEVELOPING HABITS OF MIND

Lydia Oladosu, *University of Calgary*
Olive Chapman, *University of Calgary*

Two of the issues raised in this session, which we address here, are what constitutes geometric *habits of mind* and how to develop it in secondary school students.

Mathematical habits of mind are productive ways of thinking that support the learning, and application of formal mathematics. Cuoco, Goldenberg, and Mark (1996) equate habits of mind with mathematical power. Specific to geometry, they (Cuoco, Goldenberg, & Mark, 2010) describe analytic and geometric habits of mind for high school mathematics as involving: reasoning by continuity; seeking geometric invariants; looking at extreme cases and passing to the limit; and modelling geometric phenomena with continuous functions. For Driscoll, DiMatteo, Nikula, and Egan (2007), geometric habits of mind involve: reasoning with relationships (i.e., actively looking for relationships such as parallelism, congruency, and similarities within and between geometric figures and thinking about how the relationships can help your understanding or problem solving); generalizing geometric ideas; investigating invariants (i.e., analyzing which aspect or attributes of a figure remain the same and which ones change when the figure is transformed in some ways through, e.g., translations, reflections, rotations, dilations, dissections, combinations, or controlled distortions); and sustaining reasoned exploration by trying different approaches and stepping back to reflect while solving a problem.

Students can develop habits of mind of learning geometry that do not reflect the above view of geometric habits of mind as a result of their experiences in learning. Such undesirable habits or ways of thinking could be attributed to their beliefs about geometry and how to learn it (e.g., a set of facts to be memorized) and geometric problem solving. Pedagogical approaches involving genuine problem solving, investigation, inquiry and argumentation, requiring students to explain and justify their thinking, and engaging students in geometrically important thinking could provide a meaningful basis to facilitate development of desirable habits of minds. Following is an excerpt from a transcript of a lesson of a high school mathematics teacher using an inquiry-based approach that is an example of this pedagogy. The topic is circles and lines from the coordinate and circle geometry portion of the curriculum. It is the opening lesson of this unit.

> *I began by having students write in their journals everything they knew about circles and lines. It may or may not be math related, but attempt to include math in some way. It can be in words, pictures or whatever. They then shared with their partner. We discussed briefly where circles and lines come from. The word circle comes from 'small ring' or 'persons surrounding a center of interest'. Its history goes back to the discovery of the wheel. The first theorems date back to 650 BC with the mathematician Thales. The word line has links to 'rope, row of letters' and 'rope, cord, string'. Linear functions go back to the discovery of the Cartesian plane and Rene Descartes. We then talk about how circles and lines exist in the world. I then send them on a journey around the school with their journal to find any examples of where circles and lines exist, together or separate, visible or behind the scene.*

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A PERFORMANCE ASSESSMENT APPROACH TO THE EVALUATION OF MATHEMATICS KNOWLEDGE FOR TEACHING

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The conceptualization of "teacher knowledge" was significantly elaborated by Shulman's (1986) pioneering work on the topic, in which he described domains of teacher knowledge beyond simply "knowledge of content" and "knowledge of pedagogy." His development of the concept of "pedagogical content knowledge" (PCK) helped researchers to understand that the specific knowledge held by teachers of a discipline was "qualitatively different" from the knowledge required in the learning of a discipline as a student or practitioner (Davis & Simmt, 2006). Following Shulman's work, Ball (1990) established a similar framework specific to the discipline of mathematics. The domain parallel to PCK in mathematics was termed "mathematics knowledge for teaching" (here referred to as "MKT"). Further work has been conducted following Ball's model in which additional sub-categories of MKT have been described (Ball, Thames, & Phelps, 2005), and multiple choice test items have been developed to assess MKT for teachers (Hill, Schilling, & Ball, 2004).

This ad hoc session presents a proposed doctoral research study in which a performance assessment approach is used to evaluate the application of mathematics knowledge for teaching by prospective math teachers. The study proposes the adaptation of a generic model of the Objective Structured Clinical Exam (OSCE: A type of performance assessment widely used in medical education) to the field of mathematics teacher education. In the adapted OSCE, prospective teachers cycle through multiple consecutive scenarios which present various teaching situations in which the prospective teachers are required to apply MKT in an interaction with an actor playing the role of a student. It is hoped that the study will demonstrate some possibilities for assessing the application of MKT beyond what is measurable on multiple-choice tests, as well as the general potential of using the adapted OSCE as a tool in the formative (and perhaps summative) assessment of student teachers in mathematics.

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TEACHERS' PROFESSIONAL DEVELOPMENT BY COLLABORATIVE DESIGN: WHY DOES IT WORK?

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The collaborative design among mathematics teachers and educators of teaching-learning artefacts, such as lesson plans, mathematical tasks, or assessment rubrics, has been used as a means for teachers' professional development—e.g. lesson study (Stigler & Hiebert, 1999), learning study (Marton & Tsui, 2004), communities of inquiry (Jaworski, 2009) and supported collaborative teachers' inquiry (Slavit, Nelson, & Kennedy, 2009). In order to understand and describe the type of interactions among participants in a team of collaborative design, I adopted a grounded theory approach analysing one particular case. Main categories and themes were developed. By interviewing participants of other cases of collaborative design, the themes were refined and expanded. The result is a set of four main themes that altogether describe the interactions of members in a team engaged in the design of an artefact: (1) *Anticipating* refers to both predictions about students' performance regarding the implementation of the artefact and the proposals for teachers' actions as responses to what they initially predicted; (2) *Achieving Goals* consists of the efforts to fulfil previously established goals for the artefact—proposing, or even withdrawing, mathematical tasks or activities for students is a part of the this category; (3) *Pursuing Coherence* includes discussions about the mathematical content and students' learning in a larger context—e.g. course unit, grade level, or post-secondary studies; (4) *Team Organization* describes the conversations of a team of collaborative designers regarding the organization of the team, such as the scheduling and division of labour. The first three themes are strongly interwoven and they often appear two or more at the same time.

Participants of this ad hoc discussed the relevance of these four themes as descriptions of interactions regarding teachers' professional development and complemented, from their own experience, some of the activities that might have been included in the themes. Particularly, the renegotiation of the artefact's goals as another type of interaction during collaborative design was discussed—this activity may be part of the achieving goals category.

Whereas attendees to this ad hoc agreed that the properties of the presented themes describe the type of interactions in their own experience on collaborative design, the debate on the selection and renegotiation of goals during the design of an artefact represents an opportunity for expanding and modifying the themes described above.

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REDUCING ABSTRACTION: THE CASE OF LOGARITHMS

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Simon Fraser University

In this ad hoc session, I presented a study which investigated the learners' mental process while coping with the abstraction level of the concept of logarithms. The theoretical framework used for this study is Reducing Abstraction (Hazzan, 1999). Reducing abstraction "*refers to the situations in which learners are unable to manipulate concepts presented in a given problem; therefore, they unconsciously reduce the level of abstraction of the concepts to make these concepts mentally accessible*" (Hazzan & Zazkis, 2005, p.101). Hazzan (1999) categorizes three abstraction levels: 1) Abstraction level as the quality of the relationships between the object of thought and the thinking person; 2) Abstraction level as reflection of the Process-Object duality; 3) Degree of complexity of mathematical concepts. However, because of the space limit, students reducing abstractions in only the first two levels have been discussed in this paper.

Analysis of the students' written work and interviews shows that as a way of coping with the complexity of the unfamiliar logarithmic function, some students find the rules given by the authorities (book or teacher) difficult and avoid using them while solving problems. Based on their previous experience of working with commutative properties of numbers such as $a \times b = b \times a$, some students develop their own faulty rules to work with logarithmic expressions. For example, $\log_3 4$ and $4\log 3$ are being treated as equivalent expressions by some of the students. Some others seemed to over-generalize the problem, as in $\log_a x + \log_a y = \log_a (x + y)$. One of the main reasons for such errors and misconceptions on students' parts can be attributed to their tendency to relate unfamiliar logarithmic equations to more familiar simple algebraic expressions and treat them with the commutative or distributive property. This can be interpreted as an act of reducing abstraction level (1) from Hazzan's (1999) perspective.

Furthermore, some students seemed to correctly evaluate $\log_3 9$ as evidenced in their written work, but their arguments (during interviews) for transferring the logarithm expression (using the formula) into a form so that they can use their calculator and get the answer shows that their conception of the logarithmic function is based on rules and memorized facts, but not on meaningful knowledge. They knew how to do it (process), but did not understand what it means (object). This tendency of focusing on the process aspect rather than the concept (object), according to Hazzan (1999), is reducing abstraction level (2).

Reducing abstraction as a theoretical framework has proved helpful in my attempt to understand the thought processes of learners while coping with unfamiliar (and complex) mathematical concepts. The results emphasize the importance of paying attention to the nature of students' understandings and possible misconceptions in designing instruction.

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IDENTITY OF A LEARNER IN AN UNDERGRADUATE MATHEMATICS PROGRAMME

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The purpose of this ad hoc was to help me narrow the questions to be explored in my research. The starting point of my research is a common observation that while many learners may be successful in mathematics, they may not perceive themselves as capable mathematics learners and may exist only on the margins of the practice. Solomon (2007) refers to this as a fragile identity. This phenomenon is not limited to the female population in mathematics, however females do appear to express such fragile identities more often, or at least more voluntarily (Solomon, 2007). Through this research, I would like to explore how undergraduate mathematics students identify themselves as being capable mathematics learners. Furthermore, I would like to examine whether differences exist in ways undergraduate mathematics male and female students identify themselves as being capable mathematics learners at the undergraduate level.

Identity is central to any socio-cultural learning. In mathematics, one's identity – how I am – is essential to their beliefs about themselves as capable mathematics learners and as potential mathematicians (Solomon, 2007). Learning mathematics may involve the continuous development of a student's identity as a capable mathematics learner. Researchers refer to identity as ways in which one defines him/herself and how others define them (Sfard & Prusak, 2005; Wenger, 1998). Identity includes one's perception based on their experiences with others as well as their aspirations (Black et al., 2010). As individuals progress through post-secondary, they develop a stronger sense of who they are as mathematics learners through their mathematics experiences such as: in lectures, classrooms, and seminars; in interactions with teachers and peers; and in relation to their anticipated future (Mendick, 2003; Sfard & Prusack, 2005; Anderson, 2007; Wenger, 1998).

In my research, the two main contributors to mathematical identity that I will be focusing on will be self-efficacy (self-perceived mathematics skill) and environmental factors (self-perceived mathematics skills based on others view and opinion of me). Assuming that identity, constituted of self-efficacy and environmental factors, is directly related to educational success and to setting personal goals, one might question whether it is also related to experiences of students in an undergraduate mathematics program. In which case, it may be necessary to investigate the role of mathematical self-efficacy and environmental factors of students in an undergraduate mathematics program as they are probable contributors to a learner's identity as a capable mathematics learner.

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RAPPORT AU SAVOIR ET SERVICE A.M.I.

Olivier Turcotte
Cégep de Jonquière

Depuis plusieurs années, je m’intéresse aux relations que les étudiants ont avec les mathématiques et les implications dans la salle de classe. L’attitude envers les mathématiques semble être un facteur important dans la réussite des étudiants. Lors de cette séance, j’ai commencé par décrire un service d’aide à la réussite particulier pour ensuite faire des liens avec la notion de rapport au savoir, tel que défini par Charlot (1997), afin de mieux caractériser les relations que les étudiants entretiennent avec les mathématiques.

Le service d’Aide Mathématique Individualisée (A.M.I.) existe depuis plus de 30 ans au Cégep de Jonquière. L’objectif principal est d’apporter un soutien individualisé aux élèves qui éprouvent des difficultés en mathématiques, peu importe la nature et l’origine de celles-ci. Le service prend la forme de rencontres en petits groupes d’étudiants (3 à 4) inscrits à un cours de première année du collégial animées par une enseignante ou un enseignant du département. Plusieurs caractéristiques semblent en faire un service unique :

- la motivation des étudiants semble être intrinsèque, c’est-à-dire qu’elle dépend des étudiants et des objectifs qu’ils se sont eux-mêmes fixés, puisqu’il n’y a pas de promesse de récompenses ni de menace de sanctions (aucune évaluation effectuée par l’intervenant);
- la disposition du local permet à l’intervenant de faire varier la distance didactique entre lui et l’apprenant, ce qui peut être favorable à la relation d’aide ainsi qu’à l’installation d’un climat de confiance propice à l’apprentissage des mathématiques;
- puisque l’intervenant n’a pas l’occasion de se préparer à toutes les questions, les étudiants sont alors en mesure de constater que l’intervenant n’a pas nécessairement réponse à toutes les questions et qu’il a droit lui aussi à l’errance lors de la résolution d’un problème, ce qui peut contredire certaines idées préétablies.

Comme il est difficile de mesurer l’effet d’un tel service sur la réussite des étudiants, mon projet de recherche vise plutôt à mieux comprendre les étudiants qui participent au service A.M.I. en analysant leurs rapports au savoir, à travers la perspective sociologique de Charlot. Ce dernier le définit comme étant « *l’ensemble (organisé) de relations qu’un sujet humain entretient avec tout ce qui relève de “l’apprendre” et du savoir* ».

Afin d’accéder à ces rapports au savoir, je compte utiliser des bilans de savoir (Charlot, Bautier, & Rochex, 1992) ainsi que des entrevues semi-dirigées afin de mettre en relation les dimensions épistémique (le rapport à « l’apprendre »), identitaire (rapport à soi) et sociale (rapport aux autres) du rapport au savoir des étudiants, selon une approche sociologique et didactique.

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USING MATHEMATICAL PROOF TO ENRICH CONCEPTUAL KNOWLEDGE

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Research shows (Selden & Selden, 2003) that students attempting to learn mathematics by reading mathematical proofs often focus on the local structure of proofs – whether or not it follows a familiar proof format or uses a seemingly valid logical progression – without considering the proof at a meta-conceptual level that might highlight the connections between the associated conceptions and promote change in conceptual knowledge in the sense of Hiebert and Lefevre (1986). While such change is generally quite slow (Vosniadou, 2003), it has been argued that instruction-induced conceptual change requires the promotion of meta-conceptual awareness of the concepts in question (Sinatra & Pintrich, 2003).

Studies at multiple academic levels support a belief that concept mapping, as laid out by Novak and Gowin (1984) can engender a view of mathematics as having conceptual structure, one buttressed by a socially validated body of knowledge upon which more formally expressed definitions and formulas rest (Afamasaga-Fuata'i, 2009). It was the juxtaposition of concept mapping with the need to promote meta-conceptual awareness that led me to propose a new method of reading/writing mathematical proofs: *proof mapping*. The basic idea is to take a received proof, such as one might find in a typical undergraduate mathematics text, and "explode" it using a modified process of concept mapping, reimagining the proof as a web that highlighted the connections between the concepts. While I feel this idea has great potential as a way for students to engage with mathematical proof, its precise format and the best methods to harness this idea as a pedagogical tool need considerable further exploration.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
  Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows :

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243

Proceedings of the 1994 Annual Meeting	ED 407242
Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	*submitted*
Proceedings of the 2003 Annual Meeting	*submitted*
Proceedings of the 2004 Annual Meeting	*submitted*
Proceedings of the 2005 Annual Meeting	*submitted*
Proceedings of the 2006 Annual Meeting	*submitted*
Proceedings of the 2007 Annual Meeting	*submitted*
Proceedings of the 2008 Annual Meeting	*submitted*
Proceedings of the 2009 Annual Meeting	*submitted*
Proceedings of the 2010 Annual Meeting	*submitted*

Note

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.