CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2004 ANNUAL MEETING

CMESG GCEDM

Université Laval
May 28 – June 1, 2004

EDITED BY:
Elaine Simmt, *University of Alberta*
Brent Davis, *University of Alberta*

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Proceedings of the 2004 Annual Meeting of the Canadian Mathematics Education Study Group / Groupe Canadien d'Étude en Didactique des Mathématiques

28th Annual Meeting
Université Laval
May 28 – June 1, 2004

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Remerciements

Le travail d'organisation de la rencontre annuelle de 2004 à l'Université Laval a été couronné d'un grand succès. Caroline Lajoie, coordonnatrice des rencontres annuelles au sein de l'exécutif du groupe, y est pour beaucoup. Sur le plan local, nous remercions Bernard R. Hodgson et Frédéric Gourdeau, qui ont su organiser la rencontre annuelle avec efficacité et entrain, Sylvie Drolet, qui a fourni un appui technique impeccable, ainsi que Jean-Lou De Carufel, Jean-François Maheux et Luc Tremblay, les trois étudiants qui ont aidé à ce que tout se déroule sans mauvaise surprise.

Les éditeurs voudraient remercier (en français !) Jérôme Proulx pour l'aide fournie à la vérification des documents écrits en français.

Schedule

	Vendredi/ Friday May 28 mai	Samedi/ Saturday May 29 mai	Dimanche/ Sunday May 30 mai	Lundi/ Monday May 31 mai	Mardi/ Tuesday June 01 juin
Matin/ Morning		**09h00–12h15** Groupes de travail / Working Groups	**09h00–12h15** Groupes de travail / Working Groups	**09h00–12h15** Groupes de travail / Working Groups	**9h00–9h30** New PhDs/Thèses **9h30–11h00** Table ronde/Panel **11h30–12h30** Séance de cloture/ Closing Session
Après-midi/ After-noon	**14h30–16h30** Registration/ Inscription **16h45–17h45** Cocktails GDM/CMESG	**12h15–13h45** Dîner / Lunch **13h45–14h30** Petits groupes/ Small Groups **14h30–15h30** Discussion de/of Margolinas **16h00–18h00** Groupes thématiques/ Topic Groups	**12h15–13h45** Dîner / Lunch **13h45–14h45** Conférence 2/ Plenary 2: Bouleau **14h45–15h15** Ad Hocs **15h30–** Excursion	**12h15–13h45** Dîner / Lunch **13h45–14h30** Petits groupes/ Small Groups **14h30–15h30** Discussion de/of Bouleau **16h00–16h30** Ad Hocs **16h30–18h00** AGA / AGM	
Soir/ Evening	**17h45–18h45** Souper/Supper **18h45–19h30** Opening **19h30–20h30** Conférence 1/ Plenary 1: Margolinas **20h30** Reception	**18h00–18h30** New PhDs/Thèses **18h30–20h00** BBQ		**18h00–** Excursion to/au Manoir Montmorency	

Introduction

Frédéric Gourdeau - Président, CMESG/GCEDM
Université Laval

La rencontre annuelle 2004 du GCEDM/CMESG a permis de poursuivre la tradition d'excellence établie au fil des ans. La richesse des échanges et le plaisir manifeste que nous avons eus à travailler ensemble ont fait de cette rencontre une grande réussite. Le contenu des Actes vous permettra d'en apprécier la valeur.

La rencontre 2004 avait une saveur particulière : tenue dans la ville de Québec, la présence francophone y était plus marquée qu'à l'habitude. Ainsi, avant même le début de la rencontre annuelle, les membres étaient conviés à des échanges avec le GDM (Groupe de Didactique des Mathématiques du Québec). En effet, l'exécutif avait réussi à organiser la rencontre annuelle en collaboration avec le GDM, ce qui a permis une plus grande participation des membres du GDM aux activités de GCEDM. La conférence de Jacques Bélair, donnée dans le cadre de la rencontre du GDM, ainsi que le conférence d'ouverture de la rencontre annuelle du GCEDM, donnée par Claire Margolinas, étaient ouvertes aux membres des deux groupes. Les échanges fructueux qui ont suivi, agrémentés d'un cocktail offert conjointement par la Faculté des Sciences de l'éducation et par la Faculté des sciences et de génie de l'Université Laval, sont un témoignage de l'intérêt d'une telle initiative. France Caron, pour le GDM, et Caroline Lajoie, pour le GCEDM, sont les principales responsables de cette belle réussite, et nous les en remercions.

Of course, the cultural impact of a meeting in Québec was not only visible in the larger participation by francophones at our meeting: it also manifested itself in the importance given to the evening meals which occupied us for many hours each day, and to the singing which accompanied two of those meals. Thanks to everyone for so playfully taking part in those activities.

En ce qui a trait au contenu des échanges que nous avons eus, je ne peux tous les commenter. Je peux certainement souligner les contributions de nos deux brillants conférenciers pléniers, Claire Margolinas et Nicolas Bouleau, ainsi que les réflexions de Claude Gaulin, Tom Kieren, Eric Muller et David Reid lors du panel du mardi matin. En ce qui a trait aux groupes de travail, qui caractérisent notre groupe, force est de constater que l'exécutif avait une fois de plus réussi à dénicher des thèmes répondant aux intérêts très diversifiés des membres du groupe. Cela n'est pas facile! Mathématiques, didactique, enseignement à tous les niveaux, formation des enseignantes et enseignants : les intérêts sont variés.

The reader unfamiliar with our group may not have noticed that of our two plenary speakers, one is a mathematician and one, a didactician. The diversity of our group is much larger than this may indicate and, wisely, the themes chosen for the working groups reflected this diversity of interest. Let me recall the titles: *Learner-generated examples as space for mathematical learning*; *Transition to university mathematics*; *Integrating applications and modelling in secondary and post secondary mathematics*; *Elementary teacher education – Defining the crucial experiences*; and *A critical look at the language and practice of mathematics education technology*.

Finalement, les séances thématiques et les présentations de thèses de doctorat récentes abordaient une variété assez spectaculaire de thèmes, parmi lesquels le choix a été pour moi et pour plusieurs assez douloureux. Je me refuse à les énumérer. Pour cela comme le reste, je vous invite à lire les Actes : ils sauront rendre justice à nos activités bien mieux que moi.

I do not want to end this introduction without expressing my most sincere thanks, on behalf of the members of the group, to the editors of these proceedings, Brent Davis and Elaine Simmt. For the fifth consecutive year, they have edited our proceedings and have, once more, done a superb job. *Chapeau !*

Plenary Lectures

Conférences plénières

La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe

Claire Margolinas
INRP, UMR ADEF, Marseille

Introduction

Mon exposé repose sur deux thèses, qui seront développées et argumentées tour à tour dans les deux parties du texte.

Thèse 1 : Le professeur « a » une situation

L'objectif que je me fixe dans la première partie, qui cherche à argumenter la première thèse, est de « déméler » un peu les différents éléments du travail du professeur.

J'emprunte à Dewey l'idée *d'avoir* une situation et non pas *d'être dans* une situation. Le changement du verbe permet de ne pas voir la situation comme étant *extérieure* au sujet, la relation sujet – situation est en effet plus complexe que cela : certaines connaissances du sujet font partie de la situation, certains éléments du milieu sont en quelque sorte incorporés – au sens littéral. Le verbe *avoir* transmet également l'idée que le sujet ne choisit pas d'entrer ou non dans une situation, il l'a, c'est un état et non pas une possibilité parmi d'autres.

J'appuyerai la première partie sur un bref exemple, celui de Marie-Paule, issu d'un système de protocole qui sera utilisé également dans la deuxième partie.

Thèse 2 : L'action du professeur en classe est rendue possible et contrainte par la situation qu'il a

L'intention de la deuxième partie est de montrer comment la situation du professeur permet de rendre compte du jeu du professeur entre ce qui est rendu possible et ce qui est contraint par la situation. Le professeur est trop souvent vu comme un personnage qui est en quelque sorte « tout puissant ». Les éducateurs pensent souvent qu'il suffit d'informer le professeur de tel ou tel aspect pour qu'il puisse agir, les journalistes taxent souvent les professeurs d'immobilisme… Dans les deux cas, la situation du professeur n'est souvent pas prise en compte, alors qu'elle renseigne sur ce que le professeur peut faire et aussi sur les ressources qui pourraient y être ajoutées pour que le professeur puisse faire autrement.

Cette deuxième partie ne comporte presque pas de nouvel élément théorique, elle est centrée sur l'exemple de Béatrice, qui nous permettra, sur un cas particulier, de faire des hypothèses sur les connaissances du professeur, leur stabilité, leur évolution ainsi que les possibilités d'échanges entre les professeurs.

I La situation du professeur

Développement théorique

Thèse 1 : Le professeur « a » une situation

Cette thèse est tellement générale qu'elle peut sembler inutile ! Elle nous est utile car (1) elle nous conduit à chercher comment modéliser une telle situation (2) elle nous engage à considérer la situation du professeur avec les outils de la théorie des situations de Brousseau (3) elle permet d'envisager que le professeur n'est libre de ses actions qu'à l'intérieur des possibles et des contraintes de la situation qu'il a.

Recherche 1 : Quelle est la nature de la situation qu'a le professeur ?

Lemme : Qu'est-ce qu'une situation ?

D'une façon très générale, la situation d'un sujet se caractérise par l'interaction avec un milieu, qui n'est pas tout l'environnement du sujet, mais seulement la partie (1) sur laquelle le sujet peut agir (2) de laquelle il peut recevoir des informations (3) qui peut rétroagir, c'est à dire renvoyer au sujet des informations non conformes à ses anticipations.

La connaissance se définit dans ce cadre par l'équilibre sujet – milieu. La connaissance est d'autant plus stable que l'interaction répétée avec le milieu permet de recueillir des informations correspondant aux anticipations du sujet, elle peut être déstabilisée quand le milieu rétroagit sur le sujet (figure 1).

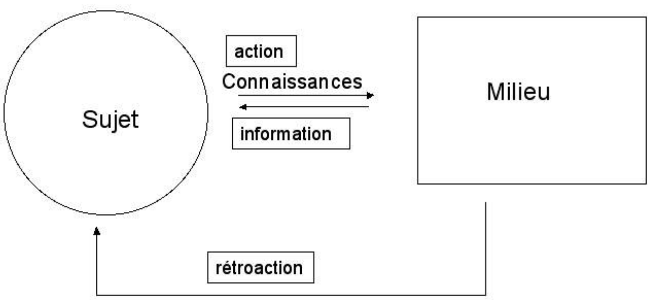

FIGURE 1 : Définition d'une connaissance

Transformation de la formulation de la recherche 1 Avec quel milieu le professeur interagit-il ?

Nous cherchons donc maintenant avec quel milieu le professeur interagit, en recherchant à chaque fois les trois caractéristiques de cette interaction (action, information, rétroaction).

Premier élément du milieu du professeur : le professeur interagit avec l'élève

L'interaction de l'élève avec les problèmes qui lui sont posés, que j'indique ici comme le milieu de l'élève (je ne développerai pas ce sujet ici), fait sans doute partie du milieu du professeur (figure 2).

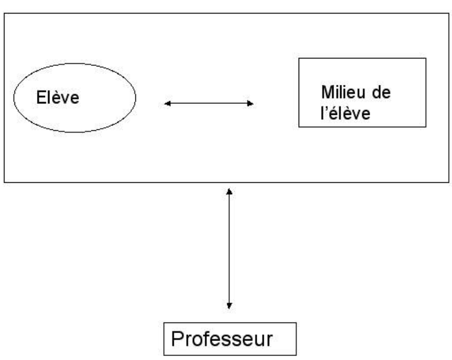

FIGURE 2 : Le professeur et la situation de l'élève

Le professeur agit sur ce milieu, puisqu'il peut par exemple modifier les problèmes qu'il donne à l'élève, il anticipe les informations qu'il peut recevoir, par exemple les réponses que l'élève peut apporter aux problèmes, et il reçoit des rétroactions, en particulier quand les problèmes qu'il pose ne donnent pas les résultats attendus.

Mais nous allons voir que ce n'est pas le seul élément de la situation du professeur, notamment parce que le professeur a pour intention (pour mission sociale) d'enseigner un savoir constitué.

> Lemme 2 : la situation du professeur n'est pas seulement la situation du professeur « en classe »

Pour considérer la situation du professeur, il faut changer de point de vue et « regarder autour » ou « en arrière ». Pendant longtemps, la situation du professeur a été considérée essentiellement dans sa dimension de gestion *en classe*, sans doute parce que la didactique est née dans le cadre de l'ingénierie. Quand c'est une équipe de recherche qui prend la responsabilité de la construction des séquences de classe, même si le professeur fait partie de cette équipe, le professeur n'est pas responsable, ou entièrement responsable, de la construction de la situation de classe, parce que l'équipe de recherche se substitue à lui. C'est alors seulement son « interprétation » en classe du scénario déjà écrit qui est visible… et souvent critiquée .

L'étude des classes ordinaires, dans lesquelles le professeur assume normalement l'ensemble de ses activités, engage à considérer aussi *le travail du professeur qui conduit à la situation de classe* et à envisager sa situation d'une façon beaucoup plus globale.

On peut dire que le chercheur change alors de point de vue dans la mesure où il regarde ce qui se passe *avant et après la situation de classe*, dans une part souvent « privée » du travail du professeur.

> Proposition : La situation de classe, dans laquelle le professeur interagit avec l'élève, est un premier niveau de l'activité professeur, je l'appelle le niveau 0 ou situation didactique (S0). À ce niveau, le milieu (M0) du professeur est l'activité de l'élève

> Transformation de la formulation de la recherche : quels sont les niveaux de l'activité du professeur et quels sont les milieux avec lesquels le professeurs interagit dans ces niveaux ?

Pour comprendre ce point de vue, plusieurs pistes sont possibles. Aujourd'hui, je partirai de la situation que nous avons décrite avant : la situation de classe, ou situation didactique. Il s'agit de la situation « de base » à laquelle j'attribue le numéro 0. Je vais considérer un autre niveau de la situation du professeur, qui englobe cette situation 0, que j'appelle situation de projet, à laquelle j'attribue le numéro 1 (figure 3).

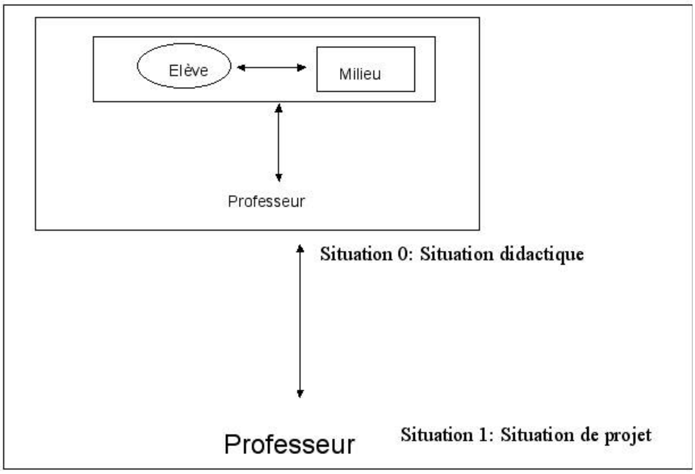

FIGURE 3 : Situations didactique et de projet

> Proposition : Si l'on considère l'interaction du professeur avec la situation didactique, on obtient un autre niveau du milieu du professeur, je l'appelle le niveau +1 ou situation de projet (S+1). À ce niveau, le milieu (M+1) du professeur est la situation didactique

Tout cela semble bien abstrait, peut-être. Voyons ce que signifie que le professeur interagit avec ce que sera ou ce qu'a été son interaction avec les élèves.

Quand le professeur prépare une leçon (quel que soit son style pédagogique), il imagine ce que pourra être sa gestion de la classe pour cette leçon et quelles pourront être les réactions des élèves. Il se souvient (s'il n'enseigne pas ce sujet pour la première fois) ce qu'on été les réactions des élèves dans une leçon similaire, soit par la forme, soit par le contenu.

Il réalise donc, hors classe, un travail qui implique un équilibre avec le milieu constitué par la situation didactique. Il a des connaissances qui lui permettent d'agir, d'anticiper sur cette situation, et aussi d'en interpréter des événements comme des rétroactions. La particularité de la situation du professeur ici est que la situation n'est pas immédiate: le milieu de la situation didactique n'est pas immédiatement disponible quand il planifie sa leçon et les rétroactions n'agissent pas directement sur la situation de projet puisqu'il n'enseigne pas souvent la même leçon rapidement.

La question du temps est cruciale pour comprendre la situation du professeur, surtout si l'on cherche à dépasser une description en terme « d'expérience ». Ce que l'on appelle souvent un professeur « expérimenté » est en fait un professeur qui a de « l'ancienneté ». On reconnaît par là que le temps joue un rôle dans la construction des connaissances du professeur, mais cette description ne nous dit rien de cette construction, ce qui n'est pas satisfaisant, ni d'un point de vue de chercheur, ni d'un point de vue de formateur.

Revenons maintenant à notre description de la situation du professeur. En nous « retournant », en changeant de point de vue, on a vu que la situation didactique, le niveau 0, ne pouvait suffire à décrire la situation du professeur et on a ajouté un niveau *surdidactique*, au dessus de cette situation.

Mais cela ne suffit pas, Selon le même principe récursif, on va considérer que la situation de projet de leçon est le milieu du niveau de la construction (+2), dans lequel le professeur construit l'ensemble du thème mathématique qui comprend la leçon du niveau +1 (figure 4).

> Proposition : Si l'on considère l'interaction du professeur avec la situation de projet, on obtient un autre niveau du milieu du professeur, je l'appelle le niveau +2 ou situation de construction (S+2). À ce niveau, le milieu (M+2) du professeur est la situation de projet

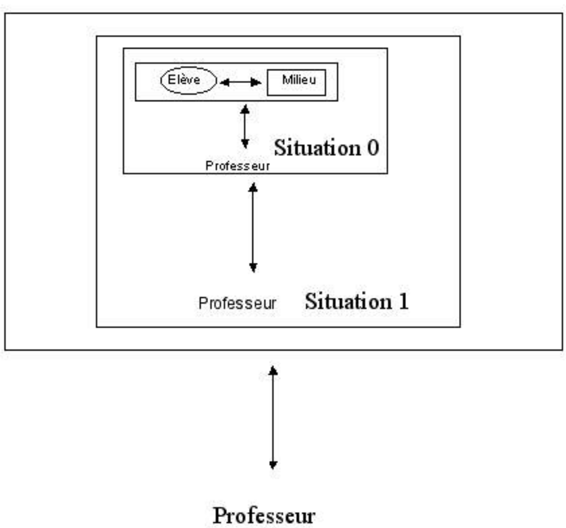

FIGURE 4 : Situations de construction

Quand il construit un thème mathématique (un chapitre, par exemple), le professeur doit imaginer les projets de leçons possibles. La plupart du temps, un professeur ayant déjà enseigné ce thème basera sa construction non seulement sur une articulation interne du thème, mais aussi (surtout?) à partir des projets de leçons disponibles, qu'il connaît déjà. On entend souvent les professeur décrire leur construction de thème en disant: « je fais d'abord cette leçon puis celle-ci », ce qui correspond bien à une interaction avec les projets disponibles. Un professeur qui n'a jamais enseigné pourra également considérer le manuel scolaire comme une source de projets possibles.

Considérer ces éléments comme un milieu nous amène à nous demander quelles sont les connaissances du professeur, ce que nous verrons dans un instant sur un exemple.

Je n'ajouterai qu'un dernier niveau surdidactique, selon le principe récursif précédent.

> Proposition : Si l'on considère l'interaction du professeur avec la situation de construction, on obtient un autre niveau du milieu du professeur, je l'appelle le niveau +3 ou situation noosphérienne (S+3), à ce niveau, le milieu (M+3) du professeur est la situation de construction

Pour ne pas alourdir le propos, je ne reporterai pas ici la figure (que tout lecteur peut construire pour lui-même) correspondant à ce dernier niveau, mais plutôt le tableau des niveaux (figure 5) que nous avons décrit jusqu'à présent, avant d'interroger et de décrire le niveau +3.

+3 Valeurs et conceptions sur l'enseignement/apprentissage

projet éducatif: valeurs éducatives, conceptions de l'apprentissage et de l'enseignement

+2 Construction du thème

construction didactique globale dans lequel s'inscrit la leçon : notions à étudier et apprentissages à réaliser

+1 Projet de leçon

projet didactique spécifique pour la leçon observée: objectifs, planification du travail

0 Situation didactique

réalisation de la leçon, interactions avec les élèves, prises de décision dans l'action

FIGURE 5 : Tableau des situations de 0 à +3

Nous retrouvons les trois niveaux dont nous avons parlé précédemment, la situation didactique, le projet de leçon, la construction du thème, et maintenant le niveau +3, situation noosphérienne.

J'ai décrit avant comment le niveau +1 interagissait (dans une temporalité décalée) avec le niveau 0, et que cette interaction était productrice et consommatrice de connaissances du professeur (le professeur agit avec des connaissances ; en retour les informations et les rétroactions du milieu renforcent ou modifient ces connaissances). De même, le niveau +2 interagit avec les niveau +1 puisque les projets sont en quelque sorte les « briques » qui permettent l'organisation possible d'un thème mathématique.

L'analyse que j'ai faite pour l'instant est une analyse qui remonte ainsi dans les niveaux, que j'appelle « analyse ascendante » (figure 6).

> Définition : L'analyse ascendante est celle qui part du niveau n et qui considère la situation S_{n+1} dans laquelle le professeur interagit avec le milieu $M_{n+1}=S_n$

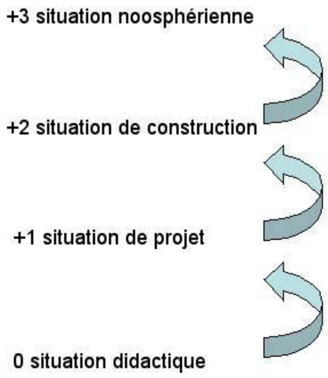

FIGURE 6 : Analyse ascendante

Si l'on prolonge l'analyse ascendante vers le niveau des valeurs et des conceptions de l'enseignement-apprentissage, que j'appelle aussi le niveau « noosphérien » ou « idéologique » (sans attacher de valeur négative à ce mot), ce point de vue nous permet de pointer que ce que le professeur a vécu comme situation didactique, ce qu'il a mis en œuvre comme projet, et les constructions qu'il a réalisé pour des thèmes mathématiques conditionnent en retour ses valeurs et ses conceptions de l'apprentissage, qui ne sont pas (ou pas seulement) de pures constructions intellectuelles indépendantes.

Mais l'on peut aussi inverser le mouvement et « descendre » du niveau +3 au niveau +2 et ainsi de suite, en réalisant une analyse « descendante » (figure 7).

> Définition : L'analyse descendante est celle qui part du niveau n et qui considère la situation S_{n-1} dans laquelle le professeur interagit avec le milieu $M_{n-1}=S_n$

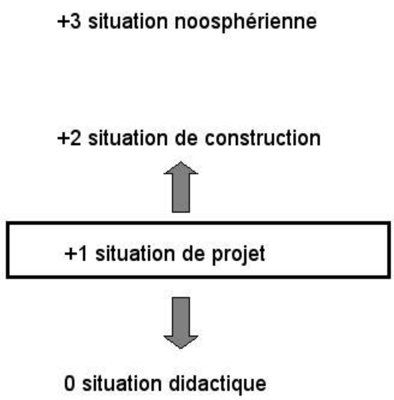

FIGURE 7 : Analyse descendante

Dans cette analyse, on va tout d'abord considérer la façon dont le professeur est inséré dans son « milieu » professionnel au sens social du terme et quelles sont les valeurs qu'il privilégie dans celles qui sont caractéristiques de cette profession, à une époque donnée, dans un lieu donné. Quand on examinera la façon dont il construit un thème mathématique, par exemple quand il choisit les documents sur lesquels il va s'appuyer, son interaction avec le milieu noosphérien conduit à considérer que certaines constructions sont plus légitimes que d'autres, voire même simplement à les privilégier, sans pouvoir s'exprimer à ce sujet. Le projet de leçon qu'il va construire est lui aussi conditionné par les choix opérés au niveau de la construction du thème et, enfin, la situation didactique qu'il peut vivre est elle-même très largement déterminée par les choix précédents. L'analyse descendante illustre sans doute de façon plus classique le point de vue du professeur.

Mais, à n'importe quel niveau, le professeur est en quelque sorte « tiraillé » par les interactions éventuellement contradictoire issues de niveaux « supérieurs » et « inférieurs » relatifs à ce niveau (figure 8). Ce sont donc les deux analyses et leurs éventuelles contradictions qui permettent de mieux comprendre la situation du professeur.

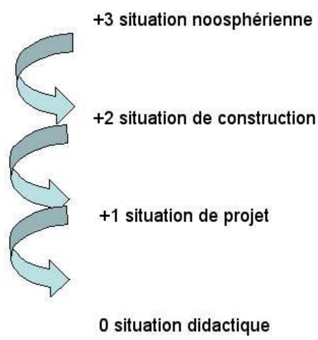

FIGURE 8 : Tension dans la situation du professeur

Par exemple, quand il planifie sa leçon, le professeur interagit à la fois avec ce qu'il croit possible comme réalisation en classe, en fonction de ses capacités à gérer une situation avec ses élèves, et ce qui serait cohérent de planifier en fonction de sa construction globale du thème ou de ses conceptions de l'enseignement-apprentissage.

L'action du professeur se réalise ainsi toujours sous une double contrainte.

Un exemple

Je vais maintenant développer un exemple au sujet de Marie-Paule, professeure « expérimentée », dont nous allons chercher à mieux comprendre les connaissances, au delà du terme non spécifique d'expérience.

L'observation dont je vais parler s'appuie sur un très vaste système de protocole. J'ai eu la chance de pouvoir observer le travail d'un groupe de quatre professeures, auquel appartient Marie-Paule, dont je vais parler maintenant, et Béatrice, dont je parlerai dans la deuxième partie de cet exposé.

Il s'agit d'un groupe de professeures qui ont décidé de travailler ensemble depuis le début de l'année scolaire pour préparer leurs cours dans une classe de 8ème grade (élèves de 14 ans environ), troisième niveau de l'enseignement secondaire français (classe de 4ème). Elles travaillent une heure par semaine en commun pour réaliser cet objectif. Marie-Paule est nettement le leader de ce groupe, même si institutionnellement il n'y a pas de hiérarchie dans celui-ci.

Elles ont accepté que je les observe : (1) Pendant l'heure de préparation des leçons d'une semaine, qui correspondait dans ce cas précis à un chapitre entier, j'ai pu recueillir les documents sur lesquels se basaient leur travail et faire un enregistrement audio : (2) Trois professeures ont accepté d'être filmées en classe pendant la première leçon du chapitre (Marie-Paule a été filmée sur les deux premières leçons) : (3) J'ai eu un entretien d'une heure, enregistré en audio, avec les trois professeures une semaine après la leçon. Par ailleurs, j'ai pu avoir accès à de nombreux documents utilisés par ces professeures (manuels scolaires, programmes officiels, etc.). Les documents recueillis et les entretiens d'une heure avaient pour but de renseigner d'une façon assez complète l'ensemble des niveaux de l'activité du professeur.

Marie-Paule est une professeure très reconnue dans l'institution scolaire, elle est assistante de l'inspecteur et formatrice à l'institut de formation et elle participe à des travaux de recherche-action en mathématiques depuis de nombreuses années à l'I.R.E.M.[1] de Clermont-Ferrand.

J'ai choisi de faire l'analyse *descendante* de la situation de Marie-Paule, et donc de commencer par caractériser le niveau +3 : celui de la situation noosphérienne, ou idéologique. Elle est donc très fortement liée aux valeurs institutionnelles de 1997 (année de l'observation) qui « mettent l'enfant au centre », pour lesquelles « le principe, c'est de leur faire découvrir par eux même ». Je considère cette phrase comme une connaissance, car elle caractérise l'équilibre de Marie-Paule avec les institutions auxquelles elle appartient (figure 9).

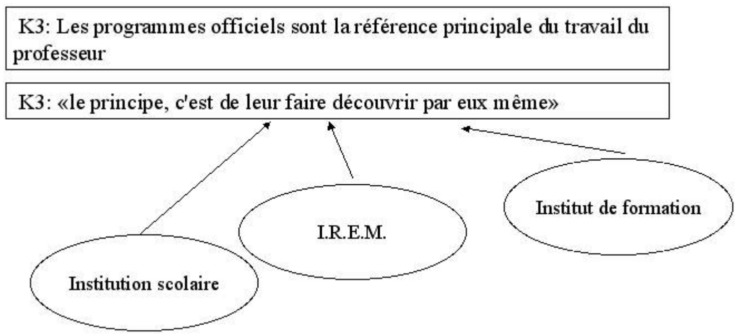

FIGURE 9 : Les connaissances de niveau +3

N'allez pas imaginer que ces connaissances pourraient caractériser tous les professeurs en France! De nombreuses observations montrent au contraire que les connaissances de ce niveau, même si elles sont fortement idéologiques, ne sont pas toujours façonnées uniquement par les instructions scolaires en vigueur : parfois le poids du passé scolaire est plus grand que celui des institutions actuelles, par exemple.

Si nous commençons à descendre, il nous faut donc décrire les ressources et les contraintes du niveau de construction du thème mathématique : niveau +2. La leçon étudiée se trouve dans un chapitre qui prend en charge une partie du programme. Étant donné la conformité de Marie-Paule à l'institution scolaire et au programme, le texte officiel est une partie essentielle du milieu de ce niveau. En voici quelques extraits (programme de 1985 en vigueur en 1997, un nouveau programme est en vigueur actuellement).

- Pour toutes les classes, les connaissances acquises antérieurement sont mobilisées et utilisées le plus souvent possible.
- Dans le plan, transformation de figures par translation ou rotation; translation et vecteur; polygones réguliers.
- Pour l'ensemble de cette rubrique, il s'agit d'un travail d'initiation; l'étude de ces notions sera poursuivie [dans la classe suivante].
- Les activités porteront d'abord sur un travail expérimental perrnettant d'obtenir un inventaire abondant de figures à partir desquelles se dégageront de façon progressive les propriétés conservées par translation ou rotation, propriétés qu'on exploitera dans des tracés.

Cet extrait de programme peut-être considéré comme la source principale des ressources (puisque le domaine est ainsi délimité) et des contraintes (puisqu'il faut se conformer au texte) dans la situation de Marie-Paule au niveau +2.

Plus près de la situation didactique, je considère maintenant la situation de projet (niveau +1), c'est-à-dire la situation dans laquelle Marie-Paule construit la première leçon du chapitre, qui est la leçon observée. Le problème est donc de chercher, au sujet de translation et rotation, un problème introductif permettant aux élèves d'utiliser leurs connaissances antérieures (symétrie axiale, symétrie centrale) et de s'initier aux propriétés de deux nouvelles transformation d'une façon expérimentale. L'environnement de Marie-Paule lui fournit la réponse (figure 10).

FIGURE 10 : Situation de projet (niveau +1)

Il s'agit d'une suite d'activités éditée par les I.R.E.M. qui répond à ces critères. Elle utilise ces problèmes depuis quelques années et elle en a modifié quelques variables, mais le problème de départ reste bien reconnaissable, ainsi que la forme de la gestion de classe, que l'on retrouvera dans le niveau suivant. Le texte du problème posé en 1997 est reproduit dans l'annexe 1.

La construction du projet de leçon de Marie-Paule s'appuie sur l'ensemble des activités et exercices qui sont prévus pour ce très court chapitre (traité en une semaine, au mois de juin), ainsi que sur la connaissance des réactions et notamment des difficultés des élèves les années précédentes et sur la connaissance des réactions des élèves de la classe à laquelle est destinée la leçon dans des activités similaires. Le groupe de travail influence également cette leçon, car même si Marie-Paule y est « leader », elle a pu y entendre d'autres opinions que la sienne (figure 11).

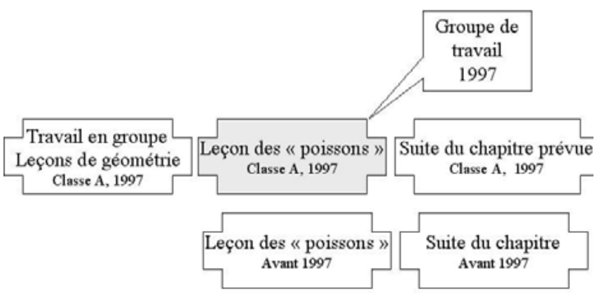

FIGURE 11 : Articulation temporelle du niveau +1

Le dernier niveau que nous allons étudier dans cette analyse descendante (nous verrons dans la suite que ce n'est pas le dernier niveau possible) est celui de la situation didactique : niveau 0. Marie-Paule maîtrise la plupart des éléments importants pour le travail dans la classe tel qu'elle l'a prévu, en fonction de la situation que nous avons étudiée.

(1) L'organisation de la séance prévoit un travail en groupes de quatre. Elle a des connaissances sur la perturbation que peut apporter le travail en groupe de quatre et la façon d'y remédier (placer les élèves dès le début du cours pour qu'ils n'aient plus qu'à se retourner).

(2) L'analyse du problème vu du point de vue de l'élève (que je ne peux pas faire ici par manque de temps) montre que l'usage des instruments géométriques est tout à fait crucial. En effet, on peut imaginer de répondre aux questions posées d'une façon assez peu formelle: « ça glisse » « ça tourne », ce qui n'est pas ce que souhaite ici la professeure. Marie-Paule insiste sur la nécessité de l'usage des instruments de géométrie, qui permet assez facilement de centrer les élèves sur la situation que souhaite installer la professeure.

(3) Elle est assez peu exigeante sur la qualité de la formulation des constructions des différentes transformations. Elle anticipe sur la leçon suivante, dans laquelle seules des connaissances de base sont nécessaires, conformément aux programmes.

La maîtrise qu'elle montre en classe relève de connaissances qui peuvent être décrites (au moins en grande partie). Certaines de ces connaissances, notamment ici celle très locales qui concernent l'usage des instruments, ou la nécessité ou non d'aboutir à une formulation précise, peuvent difficilement être acquises autrement que dans l'action répétée d'une même suite de leçons (d'une même construction, pour rester dans mon vocabulaire).

Marie-Paule prend sans doute des informations sur le travail des élèves, mais ces informations ne transparaissent pas dans son discours a posteriori, sans doute parce que son observation ne lui apporte rien qui contrarie véritablement son projet. Quand elle parle de cette séance, l'élève apparaît un peu comme un « élève générique », les élèves qui se détachent de son discours ne sont pas référés spécifiquement à cette séance là.

En conclusion de l'analyse de cet exemple, l'analyse de la situation de Marie-Paule montre une grande cohérence dans la « descente » du niveau idéologique vers le niveau didactique. L'expertise attestée de Marie-Paule dans son milieu de travail se révèle ici dans l'équilibre existant entre tous les niveaux.

Conclusion de la première partie

Le petit modèle exposé ainsi que l'exemple permettent d'envisager une cohérence dans le travail didactique du professeur, qui n'est pas toujours suffisamment mise en évidence.

Cette cohérence est très liée aux connaissances issues de la fréquentation de certaines institutions. Dans le cas de Marie-Paule, l'institution des I.R.E.M. joue un rôle très important, ce qui n'est bien évidemment pas le cas pour tous les professeurs.

Par ailleurs, le modèle permet de mieux comprendre que la situation du professeur est très complexe sur le plan temporel. Au cours même de la leçon, le professeur peut s'appuyer sur la connaissance de la même situation passée antérieurement pour anticiper les réponses des élèves, il peut aussi commencer à prévoir ce qu'il fera le lendemain à la lumière de ce qui est en train de se produire… Quand le professeur gère une leçon en classe, il n'est pas certain que ce qui se passe sur le moment soit l'élément le plus important. La situation du professeur ne peut en aucun cas être réduite à l'ici et maintenant de l'interaction de classe.

Ressources et contraintes de l'action du professeur en classe

Développement théorique

> Thèse 2 : L'action du professeur en classe est rendue possible et contrainte par la situation qu'il a

Nous allons nous centrer maintenant sur la situation *en classe* pour en comprendre les ressources et les contraintes. Tout d'abord, dans la situation de classe, le professeur rencontre l'élève : non plus l'élève générique qu'il peut imaginer pour bâtir son projet, ni l'élève passé, mais celui qui réagit ici et maintenant.

> Recherche : Quels sont les possibilités et les contraintes qui sont vues par le professeur dans la situation de classe ?

Le professeur qui interagit avec l'élève dans la situation de classe, quand il pose des problèmes aux élèves, se trouve non seulement en position de donner le problème et de discuter des solutions, mais aussi en position d'observer la situation de l'élève et de l'interpréter.

> Proposition : Le modèle de la situation du professeur doit être complété par un autre niveau du milieu du professeur. Je l'appelle le niveau -1 ou situation d'observation (S-1). A ce niveau, le milieu M-1 du professeur est la situation de l'élève qui résoud le problème posé.

Ce niveau n'est pas facile à nommer. Le terme « d'observation » que j'associe au niveau -1 a été parfois mal compris. Il ne signifie pas que le professeur n'agit pas dans ce niveau, mais qu'il n'agit pas *dans la part adidactique du travail de l'élève* c'est-à-dire qu'il n'agit pas pour influencer l'élève dans la résolution du problème dont il a fait la dévolution. Certaines actions du professeur, qui relèvent de ce niveau, ont au contraire pour but de renforcer l'élève

- +3 Valeurs et conceptions sur l'enseignement/apprentissage
  projet éducatif: valeurs éducatives, conceptions de l'apprentissage et de l'enseignement
- +2 Construction du thème
  construction didactique globale dans lequel s'inscrit la leçon : notions à étudier et apprentissages à réaliser
- +1 Projet de leçon
  projet didactique spécifique pour la leçon observée: objectifs, planification du travail
- 0 Situation didactique
  réalisation de la leçon, interactions avec les élèves, prises de décision dans l'action
- -1 Observation de l'activité des élèves
  perception de l'activité des élèves, régulation du travail délégué aux élèves

FIGURE 12 : Situations du professeur : tableau complet

dans sa recherche autonome. Par exemple, les actions du professeur comme : « donner la consigne, encourager les élèves, rappeller les règles de vie collective, peuvent relever de ce niveau dans la mesure où ces actions n'empiètent pas sur le travail de l'élève.

Sur le plan théorique, il est important de concevoir ce niveau parce qu'il nous montre que le professeur, en situation didactique (S0), est en tension entre ce qu'il perçoit de l'activité des élèves (S-1) et ce qu'il a prévu comme déroulement (S+1).

Il est important aussi car il permet de « raccorder » le modèle de la situation de l'élève (que je ne développerai pas ici, niveaux -3 à +1, voir Margolinas 2004) et celle du professeur. Dans le modèle de la situation de l'élève, le niveau S-1 correspond à la situation adidactique d'apprentissage (Brousseau 1990[2], avec une numérotation un peu différente, voir également Margolinas 1995). Cette dimension « d'observation » permet donc de rendre compte du travail du professeur pendant le processus de dévolution. L'observation dont il est question à ce niveau rend donc compte d'une activité essentielle du professeur qui, même s'il n'agit pas directement, collecte et interprète les informations qui lui sont nécessaires pour que la situation didactique soit viable.

Une étude de cas : Béatrice

Comme je l'ai annoncé en introduction, je vais maintenant centrer l'exposé sur l'exemple de Béatrice, jeune professeure qui enseigne pour la première année, à temps partiel (responsabilité d'une seule classe, de 8ème grade), le reste du temps étant consacré à sa formation (j'interviens en tant que formatrice dans l'institution de formation cette année là).

Béatrice fait partie du même groupe de travail que Marie-Paule. De plus, Marie-Paule est la conseillère pédagogique (tutrice) de Béatrice. Marie-Paule est également formatrice dans l'institut de formation où Béatrice est stagiaire. L'influence de Marie-Paule est notamment visible au niveau des connaissances de niveau +3, qui sont les mêmes chez les deux professeures, en particulier en ce qui concerne l'importance des programmes officiels.

Un problème dans l'interprétation du programme (niveau +2)

Béatrice a donc consulté très précisément les programmes officiels que nous avons déjà étudié dans l'exemple de Marie-Paule. Un problème se pose pour elle dans l'interprétation de la suite du programme (niveau +2). Voici la partie des compléments du programme qui mène à cette difficulté. La phrase problématique est en italique :

> 5. **Dans le plan, transformation de figures par translation ou rotation; translation et vecteur; polygones réguliers**
>
> Pour l'ensemble de cette rubrique, il s'agit d'un travail d'initiation; l'étude de ces notions sera poursuivie en Troisième.
>
> a) Comme en Sixième et Cinquième, les activités porteront d'abord sur un travail expérimental permettant d'obtenir un inventaire abondant de figures à partir desquelles se dégageront de façon progressive les propriétés conservées par translation ou rotation, propriétés qu'on exploitera dans des tracés.
>
> *La translation et la rotation n'ont à aucun moment à être présentées comme des applications du plan dans lui-même. Suivant les cas, elles apparaîtront dans leur action sur une figure, ou comme laissant invariante une figure*
>
> La translation sera reliée au parallélogramme.

Dans le groupe de travail, Béatrice et une autre enseignante (Danièle) ne comprennent pas quelle est la nature de l'interdiction très forte : « à aucun moment ». Elles interprètent cette phrase comme une interdiction de *parler* des transformations et même de *prononcer le mot transformation* (niveau 0). Marie-Paule, quand à elle, sait bien que ce n'est pas ce que veut dire cette phrase, parce qu'elle l'interprète par rapport aux programmes précédents (programmes de mathématiques modernes) dans lesquels l'introduction des transformation passait par la définition formelle de l'image d'un point, sans approche intuitive ni expérimentale. C'est par rapport à ce programme antérieur que la phrase prend son sens (figure 13).

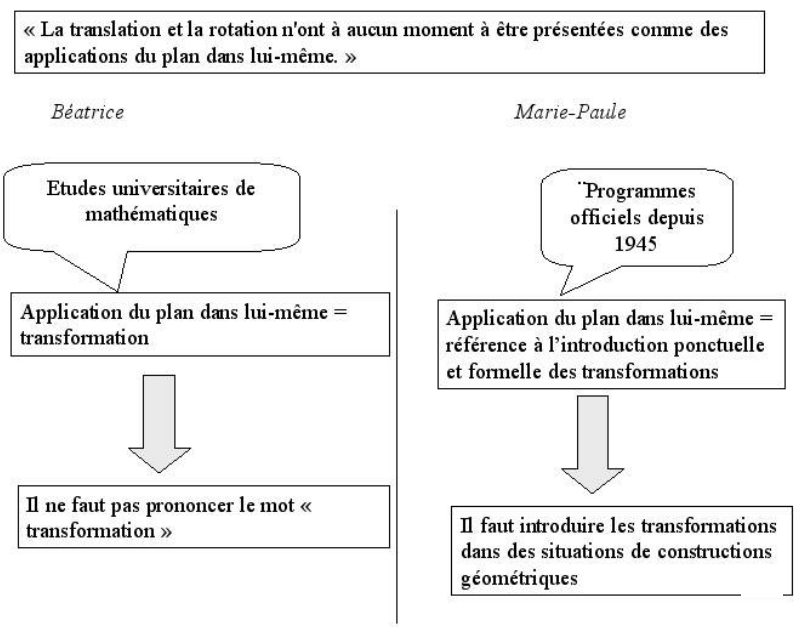

FIGURE 13 : Différences de connaissances entre Béatrice et Marie-Paule (niveau +2)

Interactions entre interprétation du programme (niveau +2), action (niveau 0) et interprétation des actions des élèves (niveau -1)

Quelles sont les conséquences observables de cette connaissance erronée de Béatrice ?

> Première conséquence en situation didactique (niveau 0, professeur) : Béatrice s'exprime difficilement dans la classe.

Voici des extraits de protocoles qui illustrent cette difficulté : à de nombreuses reprises (je ne donne ici que quelques uns de ces moments), Béatrice bafouille, elle hésite au moment où elle aurait pu prononcer le mot « transformation ». Elle remplace ce mot par « construction » et « situation ».

> 9h17 Béatrice (collectif, à voix haute): je vous ai dit qu'il y a une figure était l'image de l'autre par une certaine euh / à partir d'une certaine construction

> 9h27 alors je vais vous donner à chacun deux figures / vous donnez un nom à ces / à ces / situations

> 9h28 / donc vous pouvez tout écrire sur transparent / et donner un nom à la / à la situation /

Ce problème a également des conséquences sur la situation des élèves.

> Deuxième conséquence en situation didactique (niveau -1, élève) : Les élèves n'ont pas de mot pour parler des transformations

Les élèves n'ont pas eu d'indication sur la façon de nommer les objets mathématiques sur lesquels ils travaillaient. Puisqu'ils connaissaient déjà les symétries axiale et centrale, ils ont appelé « symétrie » toutes les transformations, en cherchant à trouver des qualificatifs pour les distinguer. Voici ce que répondent des élèves quand, une semaine après la leçon observée, ils répondent par écrit à la demande « d'expliquer pour un camarade absent ce qu'ils ont appris ».

> Adeline: Nous avons appris plusieurs sortes de symétries. La rotation, la translation et les vecteurs

> Vanessa: Il y a plusieurs sortes de symétries, j'en ai connu 2 de plus. Symétrie par translation et rotation

Malgré les définitions données dans le cours sur les translations et rotations, plusieurs élèves de la classe de Béatrice utilisent encore le mot « symétrie » pour « transformation ».

> Troisième conséquence (niveau -1, professeur) : Béatrice pense que les élèves prennent toutes les transformations pour des symétries

Béatrice sort de la séance très troublée, elle pense qu'elle n'a pas réussi à mener la séance comme elle aurait fallu. Une semaine après, dans un entretien long avec moi, elle s'exprime à ce sujet.

> Béatrice : / ils faisaient rien au début les miens / je comprends rien je me disais

> Béatrice : dans la mienne ça ne démarrait pas / et donc /[...] / donc ça démarrait pas parce que j'ai l'impression / l'explication que je me suis donnée c'est que la consigne était finalement pas très claire. / [...] et puis / y a quelqu'un qui a prononcé le mot symétrie / c'est Géraldine / et à partir de là tout le monde / tout le monde pensait que toutes les figures étaient euh / c'était des cas de symétrie

Béatrice attribue à une élève (Géraldine) cette idée de symétrie qui selon elle a ensuite gêné toute la séance en faisant dévier les élèves du problème posé.

Mon interprétation du problème de Béatrice correspond à une interaction entre les niveaux (figure 14). Quand elle a construit le thème et la séance, Béatrice a accordé une grande importance au mot « transformation » et au fait de ne pas le prononcer (analyse descendante du niveau +2 au niveau 0). Elle n'a pas anticipé la difficulté des élèves devant l'absence d'un terme pour décrire les « situations ». Pendant la séance de classe, elle observe les élèves et constate l'utilisation du mot « symétrie » (niveaux -1 et 0), ce qui remet en cause son projet de leçon, puisqu'elle croit que les élèves pensent avoir à faire uniquement à des symétries, au sens mathématique du terme. Les interactions entre les élèves conduisent à la diffusion de l'idée de Géraldine, ce qui est considéré comme catastrophique par Béatrice.

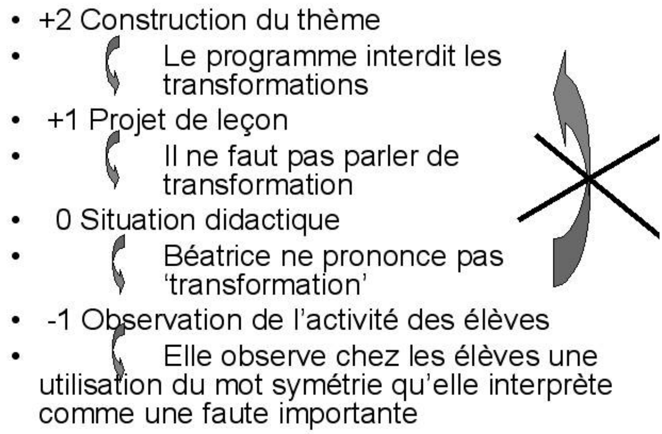

FIGURE 14 : Interactions effectives ou non entre les niveaux

Il serait intéressant de savoir ce qu’un collègue linguiste penserait des difficultés de Béatrice : son interprétation du programme la conduit à l’exclusion d’un *mot* ; l’intervention de Géraldine est considérée au niveau de l’introduction d’un *mot*. Pour Béatrice, le fait de nommer d’une façon ou d’une autre un objet mathématique est manifestement tout à fait essentiel, ce qui relève d’une conception proche du *nominalisme* en science.

Dans l’entretien qui suit, une semaine après, Béatrice ne fournit pas elle-même d’analyse de la situation qui lui permettrait de comprendre à la fois son erreur concernant le programme et la logique de l’usage du mot « symétrie » par les élèves. Elle n’envisage à aucun moment de s’interroger sure les programmes pour comprendre ce qui peut conduire les concepteurs à demander (comme elle le croit) de ne pas prononcer le mot « transformation ». Elle n’interroge pas non plus Marie-Paule sur cette question, alors que celle-ci connaît bien évidemment la réponse. Dans le groupe de travail, cette question n’est pas évoquée après la séance.

Il n’y a pas de rétroaction qui lui permette de modifier sa connaissance du milieu +2.

Un autre problème d’observation (niveau -1) et son évolution

Dans la séance, un autre problème se pose, qui relève directement d’une question d’observation et d’interprétation du travail des élèves.

Pour s’engager dans le problème et commencer à répondre aux questions posées, les élèves peuvent – voire doivent – tracer des liens entre les points homologues des figures dans les différentes situations.

En ce qui concerne les translations et les symétries, le tracé de *segments* entre les points homologues permet de conjecturer l’existence de certaines relations (parallélisme, orthogonalité, point de concours, équidistance). Dans le cas des rotations, le tracé de segments n’apporte pas d’information, c’est seulement le tracé d’arc de cercles qui est pertinent (figure 15).

Béatrice n’a pas prévu cette difficulté spécifique pour les rotations.

Plusieurs groupes, dont le groupe de Géraldine (figure 16), commencent par tracer des segments entre les points homologues, y compris pour les rotations. Dans le transparent du groupe de Géraldine, on peut voir à la fois des segments (en noir sur l’original) et des arcs de cercles (en vert) ; on peut supposer que ces éléments ont été tracés successivement. Sur ce même transparent, les élèves ont non seulement nommé correctement « rotation » la transformation, mais ont même su mesurer l’angle de cette rotation (61°). Les segments, mêmes s’ils sont présents (les élèves travaillent, à ma demande, avec des feutres indélébiles), n’ont donc pas empêché les élèves de répondre correctement à la question.

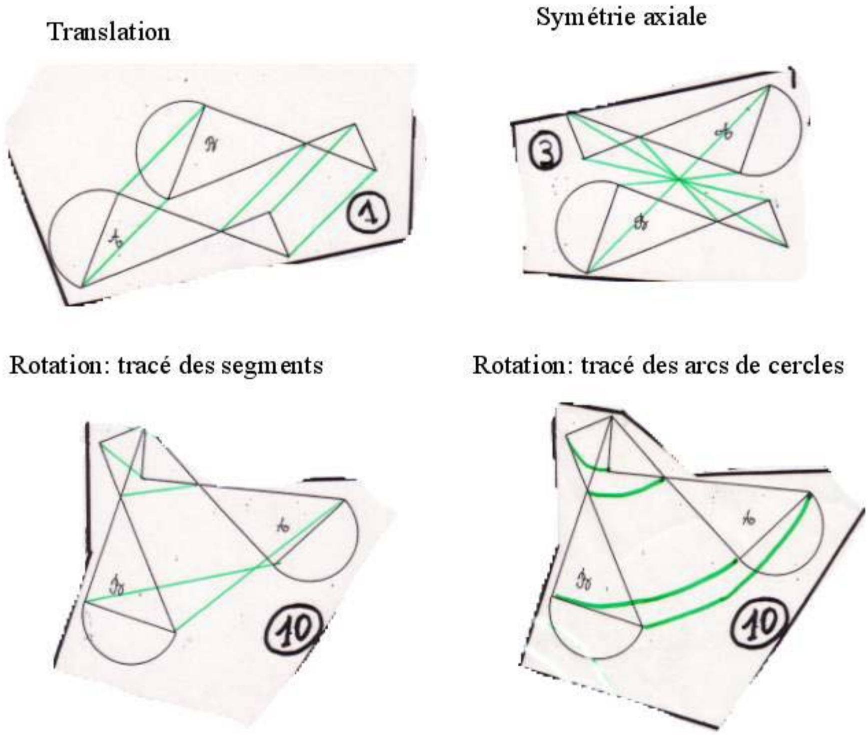

FIGURE 15 : Tracés des liens entre points homologues

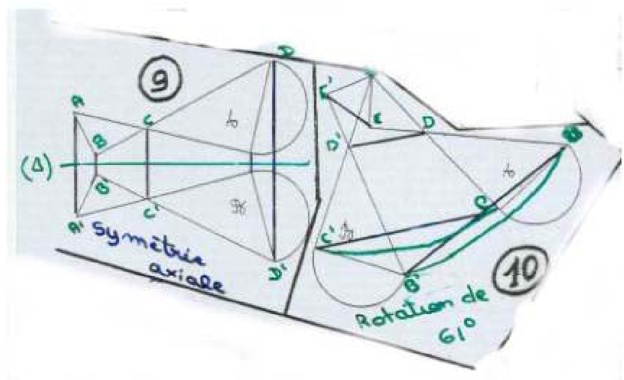

FIGURE 16 : Transparent du groupe de Géraldine (extrait)

Pour Béatrice, l'observation de segments tracés dans le cas des rotations va se cumuler avec le problème précédent.

À la 13ème minute du travail individuel (prévu pour durer 10 minutes, et qui en durera 20), elle intervient d'une façon assez agressive, car elle pense que les élèves ne « démarrent pas » :

> Béatrice (collectif, à voix haute): essayez d'être un peu malin / vous voyez qu'il y a une figure / je vous ai dit qu'il y a une figure était l'image de l'autre par une certaine euh / à partir d'une certaine construction / alors quand vous tracez une construction sur votre feuille / essayez d'être un peu malin / essayez qu'elle fasse ressortir la façon dont on passe d'une figure à l'autre / ça sert à rien de tracer des droites euh / si ça a pas de sens pour vous

Ces « droites » qui nont « pas de sens » jouent donc un rôle important dans son interprétation du blocage de la situation, elle dit une semaine après :

> Béatrice: / ils faisaient rien au début les miens / je comprends rien je me disais

Béatrice croit que les élèves ne parlent que de symétries et ne cherchent à construire que des symétriques. Au cours de l'entretien, confrontée au transparent du groupe de Géraldine (figure 16), Béatrice découvre qu'il y a une autre interprétation possible au travail des élèves :

> Béatrice: ah non parce qu'elle a fait des traits de construction / en fait / comme dans le / comme dans le / comme dans la / comme dans la translation / et y a que là où figurent des arcs de cercle / alors est-ce qu'elle a pas eu le temps de / les tracer ou / et qu'est-ce que ça veut dire aussi les segments qu'elle a tracés [...]

> Béatrice: on peut pas se poser la même question sur euh / puisque ici c'est des segments qu'il fallait tracer à chaque fois / d'ailleurs c'est bizarre / ça en enlève autant / pour les rotations / on trace pas des segments on trace des arcs de cercles et euh / j'avais pas pensé à ça justement

> Claire : et on peut pas tracer des segments justement / enfin ça

> Béatrice: oui on peut tracer un segment et son image mais on peut pas tracer / disons que quand on effectue la transformation les points qu'on écrit c'est pas des segments / et c'est pas des droites

On pourrait formuler cette connaissance de la manière suivante :

> C-1: pour caractériser les transformations, une stratégie de base peut consister à tracer des segments reliant les points homologues (cette stratégie n'est inadéquate que dans le cas de la rotation, où il faut alors tracer des arcs de cercles et non pas des segments)

Dans le cas de Béatrice, cette connaissance, envisagée seulement dans l'entretien mais pas formulée explicitement, n'est sans doute pas stable. Elle sera peut-être réactivée si Béatrice reprend un problème très proche l'année suivante avec une classe de même niveau.

On voit comment une interaction avec les niveaux supérieur pourrait s'engendrer et permettre de nouvelles connaissances, comme par exemple les suivantes :

> C+3 : Les stratégies des élèves peuvent être plus cohérentes que ce que je crois au départ.

> C+2 : Dans l'organisation du chapitre sur les transformations, la rotation pose un problème spécifique.

> C+1 : Il faut avoir le temps de traiter spécifiquement le cas de la rotation dans le problème des 'poissons'.

> C0: On peut reprendre l'idée des élèves de relier les points homologues et les orienter vers la production d'arc de cercle pour les rotations.

Néanmoins, dans la situation observée, on ne peut pas attester d'un tel impact sur les niveaux supérieurs, la connaissance nouvelle de Béatrice reste ainsi très locale, « confinée » au niveau -1. On peut remarquer que, dans le groupe de travail, aucun professeur n'évoque l'observation effective du travail des élèves et la difficulté particulière de la rotation.

Conclusion de la deuxième partie

Dans le cas que nous venons d'étudier, Béatrice, qui est une enseignante peu expérimentée, ressent une grande insatisfaction à l'issue de la séance observée. On pourrait donc s'attendre à des rétroactions des niveaux d'action (niveau 0) et d'observation (niveau -1) sur la planification de la séance (niveau +1) et plus généralement de son enseignement du thème des transformations voire même des mathématiques (niveaux +2 et +3).

En fait, ce que cette étude révèle, c'est que l'insatisfaction de Béatrice relève d'une observation assez fruste : le projet de leçon, tel qu'il a été conçu à l'avance, ne « passe » pas. Le professeur se voit ici comme un acteur qui interprète le projet et notamment l'avancée du processus d'institution. Ce que le professeur reconnaît ici, c'est que les conditions ne sont pas favorables à cette interprétation, mais les raisons de cette difficulté ne sont pas perçues.

L'observation du travail des élèves existe, mais elle est trop rapide (et donc superficielle) pour qu'une interprétation puisse avoir lieu, même après la leçon. Ce n'est que du fait du dispositif expérimental que Béatrice se trouvera face à une partie du travail effectif des élèves, ce qui lui permettra de commencer à construire de nouvelles connaissances.

D'une façon plus générale, il faut que noter que, dans le cours ordinaire de l'enseignement, la confrontation hors classe avec le travail de l'élève n'a lieu que sur la base de la mémoire des interactions, ou alors dans l'évaluation notée des copies. Cette rencontre avec le travail de l'élève a donc rarement lieu dans des conditions favorables à l'évolution des connaissances du professeur. La rétroaction de la situation d'observation (niveau -1) sur les niveaux supérieurs est donc soumise à des conditions de possibilité qui n'existent pas nécessairement dans l'exercice ordinaire du métier de professeur.

Quelques pistes pour une réflexion

Cette conférence avait pour but de donner des éléments concernant la situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe. J'espère y avoir réussi, en montrant que la situation du professeur doit s'envisager d'une façon globale et non temporelle et que les équilibres de cette situation doivent être pris en compte dans toute déclaration ou proposition concernant les pratiques du professeur. Parmi les conclusions et pistes possibles, j'en retiendrai ici deux qui me tiennent particulièrement à cœur.

Une hypothèse sur la stabilité des pratiques

On entend parfois des formateurs des éducateurs voire des chercheurs se plaindre d'une trop grande stabilité des pratiques des professeurs, parfois qualifiée « d'immobilisme » des enseignants. Le modèle que j'ai développé ici permet de montrer, pour une part de ces pratiques, la *fonctionnalité* d'une certaine stabilité.

Certaines connaissances, qui sont utiles au professeur dans l'action (niveau +1) ou dans l'observation des actions des élèves (niveau -1) sont en effet, pour certaines, d'un caractère très local et ne peuvent s'acquérir que dans l'action. Dans le cas de Béatrice, il s'agit ici des connaissances lui permettant de mieux comprendre les réactions des élèves, tant dans leur emploi du mot symétrie que dans celui de leur tracés. Ces connaissances, qui ne sont pas ici vraiment acquises par Béatrice mais à peine effleurées, sont très coûteuses : il faudrait que le professeur rencontre plusieurs fois la même situation ou le même type de situation mathématique pour que des connaissances puissent prendre sens pour le professeur et se stabiliser.

Le professeur ne vit de nouveau une même situation mathématique qu'au bout d'un temps long (parfois une année, parfois plus), le principe d'économie le conduit à conserver les « mêmes » problèmes, en les modifiant « au compte-goutte ». Béatrice, interrogée sur ce qu'elle ferait l'an prochain si elle avait une classe de même niveau, propose des modification de la *même* situation. Le groupe de travail utilise, avec des modifications assez subtiles, le *même* problème depuis plusieurs années.

L'observation du groupe de travail montre la difficulté d'un travail fin sur le problème posé, en 1997, le groupe a passé environ 45 minutes sur une « petite » reformulation de la consigne. Ce qui peut sembler stable, voire trop stable pour un observateur extérieur, est en fait une réalité qui bouge, mais dont le coût des modifications est très grand.

Ceci est sans doute renforcé par le fait que, à l'heure actuelle, les problèmes qui sont utilisés dans les classes par les professeurs sont peu partagés. Par exemple, il n'existe non pas un seul mais de nombreux manuels, il n'y a pas vraiment de problèmes « types » par lesquels tous les élèves « passent » (comme les anciens « problèmes de robinet »). La diversité des problèmes posés rend sans doute difficile la communication entre les professeurs concernant les connaissances locales.

On voit donc comment il peut être difficile en pratique, même si l'on est convaincu du bien fondé de certaines nouvelles injonctions générales (niveau +3), de faire « redescendre » ces nouvelles convictions sous forme de nouvelles progressions (niveau +2), de nouvelles séances (niveau +1) et surtout de nouvelles actions (niveau 0) associées à une prise en compte adéquate des actions des élèves (niveau -1).

Quelques questions pour la formation

Le problème qui se pose au formateur est celui de l'efficacité de son action concernant l'évolution des connaissances des professeurs dont il a la charge d'améliorer la formation mathématique et didactique. L'exemple de Béatrice indique en tout cas que les connaissances pertinentes n'évoluent pas facilement de façon spontanée, c'est à dire dans la situation ordinaire du professeur. Une intervention extérieure est donc décisive, ne serait-ce que pour révéler au professeur les actions effectives des élèves dans la classe. L'intervention à laquelle je fais allusion ici est donc celle d'un formateur qui a directement accès par son observation à certains éléments de la pratique d'un professeur.

J'ai eu l'expérience d'une structure dans laquelle ce type d'observation directe est possible : en France dans les Instituts de Formation des Maîtres, les stagiaires sont en formation par alternance et les formateurs interviennent à la fois en centre et par des observations sur le terrain.

Cette possibilité d'une interaction forte entre des interventions structurées dans l'institut de formation et des interventions sur le terrain me semble très importante. Il est parfois possible, dans ces conditions, de permettre une vraie rétroaction des niveaux inférieurs (0, -1) sur les niveaux supérieurs (+1, +2, parfois même +3), parce que le levier de l'action le permet. On voit parfois des étudiants-professeurs « découvrir », parce que la situation le permet, la pertinence d'éléments de formation plusieurs fois présentés sans succès.

D'une façon plus générale, l'action de formation est limitée par la pratique de base du professeur, chèrement acquise, et son utilité risque d'être très faible si elle ne s'appuie pas concrètement sur cette pratique.

Le problème est donc pour le formateur, non pas de partir de son propre point de vue et de chercher à le faire entendre au professeur, mais de partir du point de vue et des pratiques du professeur pour comprendre quels éléments sont déjà en cours de déstabilisation, et donc susceptibles d'évoluer. Plus facile à dire qu'à faire…

Notes

1. Institut de Recherche sur l'Enseignement des Mathématiques, où se retrouvent enseignants du second degré et universitaires pour élaborer des documents et des formations pour les professeurs de mathématiques.
2. BROUSSEAU Guy, 1990, Le contrat didactique: le milieu, *Recherches en Didactique des Mathématiques*, vol 9 n°3 pp. 309-336, ed. La Pensée Sauvage, Grenoble. On trouvera une partie de cet article également dans l'ouvrage BROUSSEAU Guy: 1997, *Theory of didactical situations in mathematics*, Balacheff et al.(ed.), Kluwer Academic Publishers: Netherlands (version anglaise) ou BROUSSEAU Guy, 1998, *Théorie des situations didactiques*, 395p, ed. La Pensée Sauvage, Grenoble (version française).

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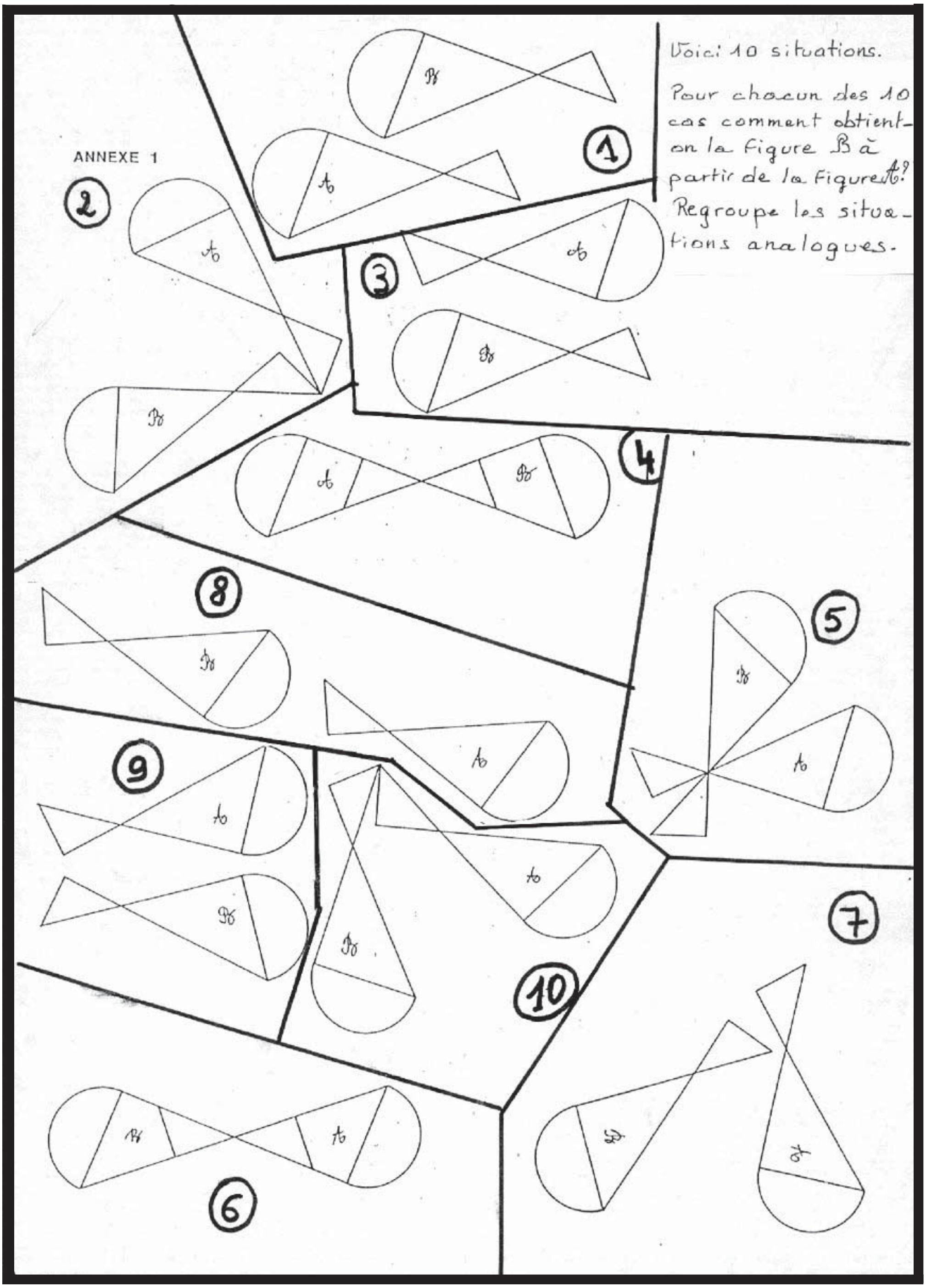
Annexxe 1 – La fiche du problème proposé en classe – Les « poissons »

Voici 10 situations. Pour chacun des 10 cas, comment obtient-on la figure B à partir de la figure A. Regroupe les situations analogues.



La personnalité d'Evariste Galois : le contexte psychologique d'un goût prononcé pour les mathématiques abstraites*

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Evariste Galois's personality:
The psychological context of an unusual fondness for abstract mathematics

Galois's name is today associated with group theory, of which he is considered a founder. By means of theoretical concepts he solved the general question of how to express the roots of an algebraic equation by means of auxiliary equations. This is what is known as the Galois Theory.

Nevertheless, a good twenty years after Galois's death, Alexandre Dumas, whose Mémoires *bear faithful witness to his time, was still completely ignorant of the mathematical activity of Galois, and presented him as a passionate, activist republican.*

His childhood, his behaviour with classmates and professors, his aptitude for abstract ideas and natural intimacy with higher mathematics, make Galois an interesting case both for a better understanding of mathematical research and also for the identification of some non-universal psychological features specifically associated with abstract conceptual ability.

Galois was anxious and uncompromising by nature, but he also possessed an exceptional vitality. Despite the numerous defeats encountered during his short life, he always found sufficient energy to undertake and see his projects through.

During this exposition we will meet some fundamental ideas of Lacan on the relationship between paranoia and creativity. It will be an opportunity to improve our knowledge of an author who is not always easy to approach due to his sophisticated use of the French language.

The plan of the lecture will follow three natural parts:

- *the story of the life, full of events, of this young man;*
- *shortly some insights in his mathematical ideas;*
- *and at last a discussion on psychology of discovery in mathematics.*

I

Né le 25 octobre 1811 à Bourg-la-Reine, Evariste Galois est mort le 31 mai 1832 à l'âge de vingt ans et sept mois des suites d'un duel. Ses travaux personnels furent conservés par son ami Auguste Chevalier et font l'objet d'une édition moderne[1] – il s'agit essentiellement de mathématiques et de considérations générales sur la science. Malgré une étude biographique fort documentée de P. Dupuy en 1896 qui est la source principale[2], on sait peu de choses sur son enfance. Son père, royaliste libéral, élu maire de Bourg-la-Reine durant les cent jours (1815), s'est maintenu à cette fonction malgré l'opposition déclarée du curé. Evariste est éduqué par sa mère jusqu'à l'âge de douze ans. Imprégnée de culture classique et de stoïcisme romain, elle lui fait traduire Cicéron et Sénèque. Plusieurs témoignages la décrivent comme une femme intelligente, chrétienne mais sans bigoterie, « réduisant presque la religion au

* Cet article est une version remaniée de l'étude «Une inquiétude créatrice» de mon livre *La règle, le compas et le divan*, Seuil 2002.

rôle d'enveloppe des principes de la morale ». Ces années d'enfance furent certainement les plus heureuses. Le cadre familial se compose d'une sœur aînée, dont la mère s'occupe vraisemblablement moins, et d'un frère plus jeune de trois ans. La complicité avec sa préceptrice maternelle est le registre des valeurs qui entourent sa vie intellectuelle. Pour le jeune garçon, sa mère est l'interprète des événements historiques récents de la Révolution, de l'Empire et de la Restauration sur fond de difficultés politiques du père.

Evariste quitte cette ambiance en 1823 pour l'internat au lycée Louis-le-Grand. Il est irrégulier dans son travail et sa conduite : un de ses bulletins mentionne « cet élève, qui travaille bien la généralité de ses devoirs, et quelques-uns avec ardeur et goût, se rebute facilement quand la matière ne lui plaît pas, et alors il néglige son devoir. [...] Jamais il ne sait mal une leçon : ou il ne l'a pas apprise du tout ou il la sait bien. Quant à ses qualités personnelles, elles sont bien difficiles à définir. Il n'est pas méchant mais frondeur, singulier, bavard, aime à contrarier et à taquiner ses camarades ». Cela le mène jusqu'à la classe de première où il subit son premier échec, fort humiliant : après un trimestre passé en classe de rhétorique, il est rétrogradé en seconde parmi des enfants qu'il ne connaît pas. Dès lors il adopte un comportement plus autonome vis-à-vis de la scolarité, il suit en parallèle la classe de mathématiques, ce qui était possible à l'époque, lit la *Géométrie* de Legendre, ouvrage remarquable par le jeu des idées qui semble avoir eu un rôle déterminant dans la motivation du jeune homme, il lit aussi Lagrange et d'autres auteurs récents. L'un de ses maîtres écrit « la fureur des mathématiques le domine. Je pense qu'il vaudrait mieux pour lui que ses parents consentent à ce qu'il ne s'occupe que de cette étude : il perd son temps ici et n'y fait que tourmenter ses maîtres et se faire accabler de punitions ». On sait par son ami Chevalier qu'il crut dès cette époque avoir résolu l'équation du cinquième degré (erreur commise par Abel également). L'année qui suit son redoublement, s'étant préparé seul, il se présente à Polytechnique, ce qu'il n'aurait dû envisager qu'après une année de mathématiques élémentaires et une de spéciales. Il échoue. Deuxième revers, prévisible étant donné son état d'impréparation, mais ressenti comme une forte déception tant cette école, aux relents de bonapartisme, avait de prestige à ses yeux.

Admis en classe de spéciales en sautant celle d'élémentaires, il a enfin un professeur qui reconnaît son talent : Monsieur Richard. Celui-ci note sur son bulletin « cet élève a une supériorité marquée sur tous ses condisciples » puis « cet élève ne travaille qu'aux parties supérieures des mathématiques ».Ainsi encouragé, Galois publie son premier mémoire (sur les fractions continues) et fait sa première communication à l'Académie des Sciences, concernant, d'après Auguste Chevalier, des résultats de la plus haute importance sur la théorie des équations.

Ce fut un nouveau déboire, d'une nature particulière. Que se passa-t-il vraiment ? L'historien Dupuy relate la chose ainsi : « Cauchy se chargea de présenter à l'Académie des Sciences un extrait de la théorie conçue par le jeune collégien ; il l'oublia ; l'extrait fut perdu pour son auteur qui le réclama inutilement au secrétariat de l'Académie ; il avait été égaré. Le peu d'attention donné par l'Institut au premier travail soumis à son jugement par Galois commença pour lui des douleurs qui, jusqu'à sa mort, devaient se succéder de plus en plus vives ». S'il est des oublis qui ont une signification inconsciente, c'est bien celui-là. Cauchy travaillait intensément sur ces questions et avait montré une force d'investigation impressionnante à ses contemporains. Qu'un garçon de dix-sept ans pût prétendre s'emparer de ce sujet était presque une insolence. Quels furent leurs échanges, en ont-ils discuté, la communication ne se fit-elle que par écrit ? On ne sait pas. Il convient de noter que les usages de la vie scientifique n'étaient pas les mêmes qu'aujourd'hui. Il était normal pour un jeune homme, comme Balzac nous le montre si bien, de rendre des visites. Cauchy lui-même avait été ainsi introduit dans le monde savant grâce aux relations de son père qui côtoyait Lagrange, Monge et Laplace par sa fonction de secrétaire-archiviste du Sénat. Les notables avaient des protégés dont ils pouvaient faire connaître les dons et les travaux. Ce système n'a jamais laissé passer à la notoriété des production franchement médiocres, mais il était très reproducteur de l'ordre social et choque aujourd'hui quelque peu nos habitudes républicaines. C'était déjà le cas à l'époque. Galois était républicain dans l'âme. Cette négligence de la part de Cauchy, royaliste, conservateur (légitimiste, il restera partisan de Charles X après la révolution de juillet 1830), anima le jeune homme d'un sentiment encore plus vif pour la République.

À l'été 1829 se produisit l'événement le plus grave, une quatrième déception suivie d'une catastrophe. Il se présente une seconde fois à Polytechnique, alors que dans le même temps une cabale est menée contre son père. Les milieux proches du clergé local avaient fait circuler des textes licencieux, faux, attribués au maire pour le discréditer. Evariste fut recalé à nouveau. Peu de temps après, son père se suicida, pris, d'après Dupuy, « du délire de la persécution ». Le cortège funèbre conduit par Evariste suscita une petite émeute de la population contre le curé.

Les sources de Dupuy sur les circonstances du suicide du père sont des renseignements fournis par des membres de la famille. On n'a pas d'autres faits relevant d'un tempérament paranoïaque, il semble que de nombreuses intrigues harcelaient réellement cet homme pour le faire craquer.

Vraisemblablement vers l'automne de cette même année 1829, il prend connaissance des résultats d'Abel sur la non résolubilité de l'équation du cinquième degré par radicaux et sur les fonctions elliptiques. C'est évidemment une désillusion qui l'obligera dans ses écrits ultérieurs à insister sur la plus grande portée de sa propre théorie. Un trait étonnant de la personnalité d'Evariste Galois est l'énergie qu'il parvient à mobiliser immédiatement après les échecs. Il réunit ses idées mathématiques tout en poursuivant ses études et présente l'ensemble de ses recherches à l'Académie pour le concours du grand prix de mathématiques en janvier 1830. Le manuscrit est remis au secrétaire perpétuel Joseph Fourier, l'inventeur de la théorie analytique de la chaleur et de l'analyse harmonique. Mais celui-ci meurt en avril avant de l'avoir examiné. On ne retrouva pas le manuscrit dans ses papiers.

Sans doute des contacts de politesse de Galois, directement auprès de Fourier ou par l'entremise d'un de ses professeurs, eussent permis de suivre l'instruction du dossier et limité les risques, mais on peut imaginer que le jeune républicain répugnait à ces comportements courtisans. Malgré cette vexation, il rédige à nouveau ces travaux et les publie en trois articles dans le *Bulletin des Sciences mathématiques* dirigé par A.-E. de Férussac.

Admis en février 1830 à l'École Normale, alors École Préparatoire, il n'est pas disposé pour autant à se consacrer enfin sereinement à ses recherches. Il s'insurge contre la personnalité opportuniste du directeur avant et après les Trois Glorieuses de juillet, et finit par se faire exclure de l'École en décembre. Les péripéties de ce nouvel échec sont révélatrices de son caractère. À la suite de lettres anonymes ou collectives contre le directeur et des réponses de celui-ci, Galois se met en position de porte-parole des élèves dans le conflit, sans s'assurer qu'il est réellement cautionné. Ses propos étant acerbes, il est effectivement désavoué et perd la partie. Après son exclusion, il publie encore dans la *Gazette des Écoles* une lettre ouverte à ses camarades dans laquelle il exprime, à leur place, où est le sens de leur honneur. « Il n'appartient ni à vous ni à moi de se prononcer définitivement sur le droit que s'est arrogé M. Guignault [le directeur]. Mais ce que vous ne devez pas souffrir, c'est qu'il vous charge de toute la responsabilité de mon exclusion; c'est qu'après les témoignages de confraternité que j'ai reçus de vous à mon départ, il ose déclarer que vous avez pris l'initiative pour amener mon exclusion ». La confraternité est perçue ici comme un lien inconditionnel, qu'il n'est même pas besoin de vérifier. En vérité, beaucoup d'élèves avaient été choqués de sa quérulence excessive et l'avaient dit.

Dix jours après son exclusion, prononcée le 3 janvier 1831, il ouvre un cours public hebdomadaire d'algèbre supérieure au 5 de la rue de la Sorbonne pour initier les étudiants aux mathématiques récentes. Annoncé par la *Gazette des Écoles*, la première séance s'ouvre avec une quarantaine d'auditeurs. On a l'impression que Galois prend appui sur ses infortunes pour repartir, tel Antée touchant le sol, à nouveau conforté dans sa révolte et son ambition. Le cours ne durera pas bien longtemps, sans doute trop difficile.

C'est à ce moment que Poisson, qui s'était intéressé à ses recherches et lui avait demandé de récrire le manuscrit perdu dans les affaires de Fourier, remit son rapport à l'Académie. Il déclara le texte *incompréhensible* (4 juillet 1831) et le travail ne fut donc pas approuvé. On lit dans les papiers personnels de Galois combien le rapport de Poisson fut ressenti plus profondément que d'autres écueils. Une reconnaissance méritée lui échappait. Dorénavant, il s'adressera volontiers dans les introductions de ses articles aux mathématiciens futurs. Pour une fois que son œuvre était examinée par un mathématicien compétent et *a priori* bienveillant, cette incompréhension tombait comme une sentence d'exclusion, non plus d'un établissement, mais de la science elle-même.

Quelques jours plus tard, il est arrêté pour port illégal d'uniforme, jugé et mis en prison. C'était l'aboutissement de divers démêlés avec la police pour agitations républicaines qui sont évoquées dans les mémoires d'Alexandre Dumas, au cours desquelles il avait été jugé une première fois pour insulte au roi et acquitté malgré un plaidoyer où il déniait lui-même l'éventualité de circonstances atténuantes[5]. En prison, d'après un compagnon de captivité, il souffre beaucoup par son jeune âge, il est physiquement et moralement torturé par des prisonniers qui le font boire en le manipulant par son sens de l'honneur.

Transféré à la maison de la santé du Sieur Faultrier en mars 1832, il se remet à ses recherches mathématiques et rédige quelques essais. À la suite du déroulement fâcheux d'une relation amoureuse avec Mlle Stéphanie D., il est provoqué en duel et meurt le 31 mai 1832, des suites de ses blessures après avoir confié à son ami Auguste Chevalier les étapes principales de sa théorie et avoir écrit deux lettres dans l'une desquelles il se plaint de mourir « victime d'une infâme coquette ».

Les circonstances de cette dernière et fatale mésaventure sont obscures. Ce n'est pas un mari trompé qui tira sur Galois, mais un républicain qui avait pris la position de défendre l'honneur d'une jeune femme. La question de la construction de l'échec se pose ici pleinement. Une relation amoureuse brève qui se noue et se dénoue est d'ordinaire une suite d'émotions joyeuses et de contrariétés qui ne sauraient aboutir à la mort que si elles sont conduites d'une certaine façon. Pour Galois, on en est réduit à des conjectures. Il se sentait membre du mouvement républicain et entendait se comporter de façon exemplaire à légard de ces critères. Dupuy écrit « rien n'était plus fréquent alors, que les duels chez les républicains, les patriotes : ils se piquaient de gentilhommerie en tout, aussi bien dans leur conduite privée que dans leur conduite publique et l'une des conséquences de cet oubli complet de soi-même, qui fait leur noblesse devant l'histoire, était la facilité avec laquelle, souvent pour de très légers motifs, ils se retrouvaient sur le terrain »[6]. Selon toute vraisemblance, Galois eut un élan amoureux vers Mlle Stéphanie D. et se comporta en conquérant. Mais rapidement il eut du mal à gérer cet amour qui ne se renforçait pas et voulut s'en défaire. Il s'y prit si maladroitement que la jeune femme se trouva offensée ou prétendit l'être et que deux patriotes se sont senti le devoir de défendre la cause de cet honneur. D'après Alexandre Dumas, celui qui tira sur Galois était Pécheux d'Herbinville, un de ses proches sur le plan politique[7]. Tout porte à croire que l'amour s'est tari sans être consommé et que Galois fut déçu de son propre comportement affectif. Il en déduisit immédiatement une loi supérieure et absolue : il n'était pas fait pour l'amourIl écrit quelques jours avant le duel « Comment se consoler d'avoir épuisé en un mois la plus belle source de bonheur qui soit dans l'homme, de l'avoir épuisée sans bonheur, sans espoir, sûr qu'on est de l'avoir mis à sec pour la vie ? » puis « C'est dans un misérable cancan que s'éteint ma vie. Oh! pourquoi mourir pour si peu de choses. [...] Je prends le ciel à témoin que c'est contraint et forcé que j'ai cédé à une provocation. [...] Mais mes adversaires m'avaient sommé sur l'honneur de ne prévenir aucun patriote. Votre tâche est bien simple [...] dire si je suis capable de mentir, de mentir même pour un si petit objet que celui dont il s'agissait ». C'est presque un suicide que de se mettre dans une configuration de contraintes morales où affronter la mort est la seule issue pour faire la preuve qu'on dit la vérité. Quelle fin dérisoire que d'être tué par des républicains au lieu de mourir pour la République !

Dix ans après sa mort, les papiers de Galois furent confiés par Auguste Chevalier à Joseph Liouville. En septembre 1843, celui-ci fit part de leur profondeur à l'Académie et trois ans plus tard publia dans le *Journal*, dont il était directeur, le plus important mémoire manuscrit de Galois. Ayant accompagné cette publication de l'avertissement qu'il allait en publier un commentaire ultérieur, tous les mathématiciens savaient que Liouville travaillait sur la théorie de Galois et attendaient cette synthèse pour investir ces nouvelles idées. Mais ce commentaire n'est jamais paru, acte manqué posthume si l'on peut dire, qui retarda notablement, d'après les historiens, la diffusion des idées de Galois[8].

Bien sûr, Galois eut aussi des succès. Il obtint le premier prix au concours général et publia son premier article en 1829 alors qu'il n'avait pas dix-huit ans, mais ce fut juste avant la mort de son père et son second échec à Polytechnique. Il fut admis à Normale mais en fut exclu peu après. Ses vrais succès sont posthumes. Son enterrement d'abord qui devint un grand événement républicain. Son œuvre ensuite qui émerveilla la postérité.

FIGURE 1

Galois was born on October 25th, 1811. He was educated by his mother until 12. He lived in the little town of Bourg-la-Reine, 10 km south of Paris where his father was mayor.

In 1823 he left Bourg-la-Reine to the lycée Louis-le-Grand in Paris as boarder.

FIGURE 2

The historical period is the Restauration. The kings have come back after Napoleon. In 1824, after the death of Louis XVIII, Charles X takes the throne, he is particularly conservative.

A school report of Evariste Galois at 15 in the lycée Louis-le-Grand.

"Never he knows badly a lesson, either he didn't learn it at all or he knows it well. About his own qualities, they are quite difficult to define. He isn't nasty but rebellious, singular, talkative, he likes to tease his classmates."

2° TRIMESTRE.

Note d'étude. — Conduite fort mauvaise, caractère peu ouvert. Il vise à l'originalité. Ses moyens sont distingués, mais il ne veut pas les employer à la Rhétorique. Il ne fait absolument rien pour la classe. C'est la fureur des Mathématiques qui le domine; aussi je pense qu'il vaudrait mieux pour lui que ses parents consentent à ce qu'il ne s'occupe que de cette étude: il perd son temps ici et n'y fait que tourmenter ses maîtres et se faire accabler de punitions. Il ne se montre pas dépourvu de sentiments religieux, sa santé paraît faible.

Rhétorique.

Note de M. Pierrot. — Travaille quelques devoirs. Du reste, causeur comme à l'ordinaire.

Note de M. Desforges. — Dissipé, causeur. A, je crois, pris à tâche de me fatiguer, et serait d'un fort mauvais exemple s'il avait quelque influence sur ses camarades.

Mathématiques préparatoires.

Note de M. Vernier. — Intelligence, progrès marqués. Pas assez de méthode.

3° TRIMESTRE.

Note d'étude. — Conduite mauvaise, caractère difficile à définir. Il vise à l'originalité. Ses moyens sont très distingués; il aurait pu très bien faire en Rhétorique s'il avait voulu travailler, mais, dominé par sa passion des Mathématiques, il a totalement négligé tout le reste. Aussi n'a-t-il fait aucun progrès. Je ne crois pas qu'il soit dépourvu de sentiments religieux. Sa tenue à la chapelle n'est pas toujours exempte de reproches. Sa santé est bonne.

Rhétorique.

Note de M. Pierrot. — S'est assez bien conduit, mais a peu travaillé : va mieux depuis quelques jours.

Note de M. Desforges. — Paraît affecter de faire autre chose que ce qu'il faudrait faire. C'est dans cette intention sans doute qu'il bavarde si souvent. Il proteste contre le silence.

FIGURE 3

"Very bad conduct... he does absolutely nothing for the class. He is dominated by the fury of mathematics... he looses his time here."

At seventeen the gifts of Galois are eventually recognised by his professor of mathematics Mr Richard who writes: "This pupil has a marked superiority on his classmates."

This pupil works exclusively on the highest parts of mathematics.

Galois obtained the first place at the "Concours Général" in 1829. He attempted to enter Polytechnique but failed twice.

He succeeded at Ecole Normale in February 1830, he is 18 and a half. Galois can't bear seeing the director of Ecole Normale to be Bonapartist under Napoleon, legitimist under Charles X, reformist under Louis-Philippe. He publishes open letters against the director. He is expulsed from Ecole Normale in January 1831.

The main difficulties of the young man were about his mathematical works : in June 1829 Cauchy looses the manuscript on the resolution of algebraic equations that Galois proposed to the Academy. The second manuscript given to Fourier is also lost after the death of Fourier in June 1830. In July 1831 Galois presented for the third time his ideas to the Academy. The paper is not lost but Poisson *considers it incomprehensible, so that it is not registered.*

Eventually, the ideas of Galois are known by the papers he wrote just before his mortal duel and the drafts found by Chevalier in his affairs.

II

Il arrive de temps à autre que les mathématiciens résolvent, sans que ce soit le but initial de leur travail, de très vieux problèmes. Les conjectures, contrairement à une idée reçue, ne sont pas des points de passage obligés de l'activité mathématique. En général, les théories fécondes produisent des résultats dans le champ qu'elles éclairent et des questions sans réponse accompagnent leur progrès comme des scories plus ou moins abondantes. Ces difficultés seront levées ultérieurement, peut-être par d'autres théories sur des objets différents. Au dix-neuvième siècle par exemple, les fonctions analytiques apportèrent de nombreux résultats en arithmétique sur la divisibilité et les nombres premiers. Plus récemment, la géométrie algébrique, fortement développée au vingtième siècle, fit tomber grâce aux travaux d'Andrew Wiles le grand théorème de Fermat resté 325 ans sans démonstration complète. Plus longtemps une question demeure irrésolue, plus les sentiers qui la rencontrent naturellement ont été battus et plus elle nécessite une pensée innovante. *A fortiori* lorsqu'elle n'a cessé de préoccuper de nombreux auteurs avec des réponses partielles suscitant la critique et l'émulation. Telle est la question de la résolution des équations algébriques d'où émergèrent les idées originales d'Evariste Galois.

La question qui préoccupait Galois, et grâce à laquelle son nom est resté à la postérité comme un des esprits les plus novateurs du dix-neuvième siècle, apparaît sous une forme restreinte dès l'antiquité : Telle figure géométrique peut-elle être construite *avec la règle et le compas* ? Les auteurs grecs et latins attribuent en général l'origine de cette exigence à Platon. Celui-ci, d'après Plutarque, récusa la solution apportée par Archytas et Eudoxe au problème de la duplication du cube — construire un cube dont le volume est double d'un cube donné — parce qu'ils avaient utilisé des procédés mécaniques et des instruments appelés mésolabes et corrompaient ainsi la géométrie[9]. Le fait que les *Éléments* d'Euclide s'attachent exclusivement, en matière de constructions, à celles qu'on peut réaliser avec la règle et le compas, c'est-à-dire au moyen de cercles et de droites, renforça durant le moyen âge et l'époque classique le prestige de cette interrogation. La peinture allégorique retint ces deux instruments comme symboles de la géométrie. Pour les polygones réguliers, on savait depuis l'antiquité que ceux de trois, quatre et cinq côtés étaient constructibles avec ces instruments ainsi que les hexagones, octogones, etc., qui s'en déduisent immédiatement. Mais on n'avait « rien ajouté à ces découvertes depuis deux mille ans », remarque Gauss lui-même, quand, à la toute fin du dix-huitième siècle, il montre la constructibilité du polygone de dix-sept côtés.

La résolution des équations algébriques par radicaux est une question de même nature : on demande si, avec des moyens imposés, addition, multiplication, division, racine carrée, racine cubique, etc., on peut exprimer les solutions des équations algébriques à partir de leurs coefficients. Les Grecs, et les Mésopotamiens déjà, savaient résoudre l'équation du deuxième degré à l'aide des formules qu'on enseigne aujourd'hui dans les lycées et qui font intervenir une racine carrée. Mais il faut attendre la Renaissance pour que la brillante école de géomètres italiens (del Ferro, Tartaglia, Cardan, etc.)[10] invente une nouveauté surprenante : l'équation du troisième degré peut également être résolue par radicaux avec des racines carrées et des racines cubiques. Quelques décennies plus tard, au milieu du seizième siècle, c'est au tour de l'équation du quatrième degré d'être résolue par Ferrari de façon similaire. Puis, plus rien. L'équation du cinquième degré résiste durant tout le dix-septième siècle. Au dix-huitième siècle, les plus grands mathématiciens travaillent sur le sujet. On essaie aussi de résoudre des équations algébriques de degré supérieur à cinq dont la forme n'est pas générale, ce qui est possible pour certaines d'entre elles sans qu'on sache très bien lesquelles. Progressivement, la situation se retourne. Avec les travaux de Vandermonde et de Lagrange, l'impossibilité de la résolution de l'équation générale du cinquième degré par radicaux devient l'hypothèse la plus vraisemblable : ces auteurs suggèrent que la difficulté vient d'une indiscernabilité des racines de l'équation avec les seuls outils d'équations auxiliaires solubles par radicaux. Si une racine satisfait une relation écrite avec ces outils, les autres la satisfont également, de sorte qu'on ne peut ainsi la caractériser donc non plus la calculer. En approfondissant ces travaux par l'étude des fonctions exprimables par radicaux, Gauss démontre la solubilité des équations de la division du cercle — les fonctions trigonométriques sinus, cosinus, etc., prises pour des arguments qui sont des fractions entières de 2p peuvent s'exprimer avec des radicaux[11] — et Abel parviendra en 1826 à démontrer l'impossibilité d'une solution par radicaux de l'équation générale du cinquième degré.

Aboutissement d'une longue histoire, ce résultat n'en fut pourtant pas le point final. La véritable compréhension de la question fut apportée quelques années plus tard par Galois selon des idées qui préfiguraient les mathématiques du vingtième siècle. Sa théorie permettait de comprendre clairement les phénomènes d'indiscernabilité, non seulement pour la résolution de l'équation générale du cinquième degré par radicaux, mais aussi pour d'autres équations algébriques de forme spécifiée, au moyen d'équations auxiliaires. Grâce à elle, on retrouvait les résultats de Gauss et d'Abel, et Galois déjà avait montré son application aux équations congruentes et aux fonctions elliptiques.

La personnalité très originale de Galois révèle certains traits typiques d'une forte créativité. Son œuvre est bien ténue en comparaison de celles d'un Gauss ou d'un Cauchy qui ont ouvert une multitude de voies nouvelles. Mais elle est particulière : prémonitoire par sa nature même. L'histoire des mathématiques dans la seconde moitié du dix-neuvième et du vingtième siècles abonde de travaux et de découvertes dans le langage des structures abstraites. On peut se demander ce qui se serait passé si les papiers de Galois avaient été perdus, ce qui faillit bien arriver. Raisonnablement, on peut penser que ses idées auraient été rencontrées si ce n'est par Liouville du moins par Jordan ou avec l'étude systématique des structures algébriques au vingtième siècle. La fascination qu'exerce Galois sur les mathématiciens d'aujourd'hui se situe précisément là. Il est un précurseur par une démarche authentiquement personnelle qui le rend attachant, il partage avec nous la familiarité des raisonnements abstraits et la conviction de leur puissance *une pensée contemporaine formulée sous la Monarchie de juillet.*

En quoi consiste précisément l'apport de Galois ? Il introduit le terme de groupe mais n'est pas l'inventeur de la notion. Elle est utilisée par Lagrange, Gauss et Cauchy, ce dernier propose des notations commodes pour les éléments des groupes de substitutions[12]. Les groupes utilisés à cette époque sont des groupes finis, c'est-à-dire des structures finies munies d'une opération interne analogue à la composition d'applications inversibles comme les substitutions ou les rotations. Depuis lors, les groupes ont pris une importance considérable dans de nombreux domaines des mathématiques et de la physique[13]. Les notions nouvelles introduites par Galois (en terminologie moderne : sous-groupe distingué, groupe simple, extension algébrique d'un corps, etc.) concernent les relations des groupes entre eux et leurs liens avec d'autres entités. Il a dégagé la structure sous-jacente du problème : correspondance entre les groupes et sous-groupes liés à une équation et les corps et sur-corps de nombres algébriques engendrés par ses racines. Il est l'inventeur d'une algèbre des groupes et des corps. Là réside son audace, penser les groupes eux-mêmes et les hiérarchies de sous-groupes comme on raisonnait précédemment sur les éléments des groupes. Ses contemporains n'ont pas compris ce langage, Siméon Denis Poisson, découvreur de formules profondes et pionnier du calcul des probabilités, avoue ne pas comprendre le mémoire de Galois qu'il doit présenter à l'Académie. Jean Dieudonné écrira : « lorsqu'on lit les quelques pages où Galois expose ses idées générales sur les mathématiques, on est frappé de l'allure étrangement moderne de sa pensée. Son insistance sur le caractère conceptuel des mathématiques, son aversion pour les lourds calculs masquant les idées directrices, son souci de grouper les problèmes selon leurs affinités profondes plutôt que leur aspect superficiel, tout cela nous est maintenant familier ; il est piquant que ses mémoires si concis soient pour nous bien plus clairs que les filandreux exposés que croyaient devoir en donner ses successeurs immédiats »[14]. L'apport de Galois est tout le contraire d'une astuce fortuite, il est parfaitement conscient du registre où se situe sa créativité. Il écrit lui-même qu'il fait l'analyse de l'analyse : « sauter à pieds joints sur les calculs ; grouper les opérations, les classer suivant leur difficulté et non suivant leurs formes ; telle est, suivant moi, la mission des géomètres futurs ; telle est la voie où je suis entré dans cet ouvrage [...] Ici on fait l'analyse de l'analyse, ici les calculs les plus élevés exécutés jusqu'à présent sont considérés comme des cas particuliers, qu'il a été utile, indispensable de traiter, mais qu'il serait funeste de ne pas abandonner pour des recherches plus larges »[15]. À l'époque de Galois, l'analyse est une des branches les plus abstraites des mathématiques. Elle comprend l'étude des fonctions, le calcul différentiel et intégral, les représentations des fonctions par des séries ou comme solutions d'équations différentielles, ce qui constitue l'analyse dite transcendante parce que cela va au-delà des calculs sur les fonctions algébriques (polynômes, fractions rationnelles et radicaux). « Les longs calculs algébriques ont d'abord été peu nécessaires au progrès des mathématiques, les théorèmes fort simples gagnaient à peine à être traduits dans la langue de l'analyse. Ce n'est guère que depuis Euler que cette langue plus brève est devenue indispensable à la nouvelle extension que ce grand géomètre à donnée à la science. » écrit Galois dans la préface (décembre 1831) de son mémoire. À partir d'Euler, en effet, les notations de l'analyse sont voisines de celles encore en usage aujourd'hui. Au dix-huitième siècle, les deux grandes figures d'Euler et de Lagrange ainsi que les Clairaut, Bernoulli, d'Alembert, MacLaurin, Laplace et Legendre perfectionnent ce langage dans le but de mener des calculs plus explicites en mécanique terrestre et céleste après les *Principia* de Newton[16]. À cette époque la notion de fonction est pensée comme expression analytique et non comme « application ». Les propriétés des fonctions analytiques et les limites exactes de cette classe ne seront dégagées qu'au siècle suivant. Il en résulte des moyens de démonstration formels très puissants, notamment par le procédé qu'une fonction définie localement par une formule est considérée comme ainsi complètement déterminée même en dehors du domaine connexe où la formule est développable en série entière convergente. La moisson de formules remarquables (fonction gamma et intégrales eulériennes, fonctions dites plus tard de Bessel, fonctions sphériques, fractions continues, etc.) s'accompagne d'une brume sur la validité des méthodes qui mènent parfois à des absurdités. Aussi la généreuse profusion de la « langue d'Euler » arrive-t-elle à saturation. Ceci pousse les mathématiciens de la génération suivante à aborder les questions générales, comme celle de la convergence des séries, avec plus de rigueur. Avec Gauss et Cauchy, ainsi que les autres contemporains de Galois (Fourier, Poisson, Monge, Abel), cette nouvelle exigence, au lieu de stériliser l'analyse, lui donnera des méthodes plus sûres et des fruits encore plus nombreux. « Depuis Euler les calculs sont devenus de plus en plus nécessaires, mais de plus en plus difficiles à mesure qu'ils s'appliquaient à des objets de science plus avancés, poursuit Galois. Dès le commencement de ce siècle, l'algorithme avait atteint un degré de complication tel que tout progrès était devenu impossible par ce moyen, sans l'élégance que les géomètres modernes ont su imprimer à leurs recherches, et au moyen de laquelle l'esprit saisit promptement et d'un seul coup un grand nombre d'opération. » Certaines méthodes de Bolzano à Prague ou de Cauchy à Paris sont si délicates et d'une rigueur si précautionneuse qu'elles suscitent des réactions. Certains (comme Laplace) s'empressent de vérifier leurs propres calculs, d'autres (comme Comte) croient pouvoir affirmer que la fécondité est hors de ces subtilités. Lorsque Galois présente ses travaux comme l'analyse de l'analyse, il prend donc une position extrême en faveur d'une abstraction encore plus grande. Il s'exprime ainsi :« Or je crois que les simplifications produites par l'élégance des calculs (simplifications intellectuelles s'entend, de matérielle il n'y en a pas) ont leurs limites ; je crois que le moment arrivera où les transformations algébriques prévues par les spéculations des analystes ne trouveront ni le temps ni la place de se produire; à tel point qu'il faudra se contenter de les avoir prévues. Je ne veux pas dire qu'il n'y a plus rien de nouveau pour l'analyse sans ce secours : mais je crois qu'un jour sans cela tout serait épuisé. » Il ne s'agit pas à proprement parler d'un métalangage. Ce serait anachronique de voir les choses ainsi. On usait volontiers à l'époque du terme de métaphysique pour évoquer les recherches sur les bonnes façons de présenter tel ou tel domaine des mathématiques[17]; puis le préfixe *méta* a pris un sens fort au début du vingtième siècle lors des travaux des logiciens sur l'axiomatisation et le programme méta-mathématique de Hilbert. À l'époque de Galois, il n'y a pas de langage mathématique formalisé et unifié, la théorie des nombres, les fonctions de variables complexes, l'algèbre linéaire, les déterminants et les séries de Fourier sont des constructions qui vont plus ou moins haut dans l'abstraction. En parlant de l'analyse de l'analyse, Galois situe ses idées encore plus haut. Évidemment, l'abstraction en mathématiques n'est pas féconde de façon assurée. Considérer comme Bolzano des fonctions très générales, façon quelconque d'associer à un nombre un autre nombre, ne fournit pas en soi de résultats nouveaux. En mathématiques, celui qui embrasse trop n'étreint plus grand chose et l'histoire des mathématiques est un processus d'abstraction très progressif. Le talent de Cauchy avec ses intégrales dans le plan complexe, celui de Galois avec la théorie des groupes ou plus tard celui de Hilbert avec les espaces fonctionnels munis d'un produit scalaire où la géométrie euclidienne s'applique à des points qui représentent des fonctions, résident dans la construction d'abstractions utiles qui fournissent des résultats hors d'atteinte par les approches anciennes et ouvrent de nouvelles méthodes. Ce niveau de généralité, utilisant la puissance de l'abstrait pour traiter de situation concrètes qui se développera considérablement au vingtième siècle avec les traités de Banach et de Bourbaki, est initié par Galois de façon consciente avec un jugement très sûr et étonnamment mature sur les mathématiques de son temps. Il semble porté spontanément, par un goût ardent, vers ces régions. D'où lui vient cette passion pour le dépassement conceptuel des méthodes éprouvées ?

Deux traits ressortent de la psychologie de Galois. D'abord, il ne se laisse pas aimer, quel que soit l'interlocuteur. Son ouverture affective n'est pas disponible. Galois aime ailleurs. Il écrit à son ami Auguste Chevalier le 25 mai 1832 : « uTme dis que ceux qui m'aiment doivent m'aider à aplanir les difficultés que m'offre le monde. Ceux qui m'aiment sont, tu le sais, bien rares. Cela veut dire, de ta part, que tu te crois, quant à toi, obligé à faire de ton mieux pour me convertir. Mais il est de mon devoir de te prévenir, comme je l'ai fait cent

FIGURE 4

fois, de la vanité de tes efforts ». Il fuit donc toute aide amicale, toute participation à la motivation de sa conduite. C'est qu'en effet, second trait corrélatif, il n'a besoin de personne pour vivifier son action, pour aiguiser par cette relation son appétit vital, il se réfère exclusivement à des liens sentimentaux sublimés, immuables, intangibles, en fonction desquels il interprète les événements du monde.

Sa production est incompréhensible aux yeux de ses contemporains. Rejeté de son vivant, phare aujourd'hui, il se situe en ce lieu précis où la nature de la connaissance scientifique nous provoque. Aussi voudrions-nous approfondir le cadre et la dynamique psychologique du chercheur en nous appuyant sur le cas de Galois, extrême et exceptionnel évidemment, mais si humain, si sincère et si prégnant qu'il est révélateur, comme limite, d'une situation dans laquelle bien des chercheurs se reconnaissent partiellement.

D'abord en ce qui concerne le rapport affectif entre le chercheur et la discipline, qui est complexe et ambigu, fait à la fois d'intimité complice et de compétition sociale avec des savants reconnus ou candidats à la notoriété. Ensuite et surtout pour tenter d'élucider l'état d'esprit qui est à la source de la connaissance neuve, c'est-à-dire le non-conformisme créateur qui, dans la science, ose poser une insatisfaction des idées actuelles et ressent comme un impératif de trouver des raisons nouvelles.

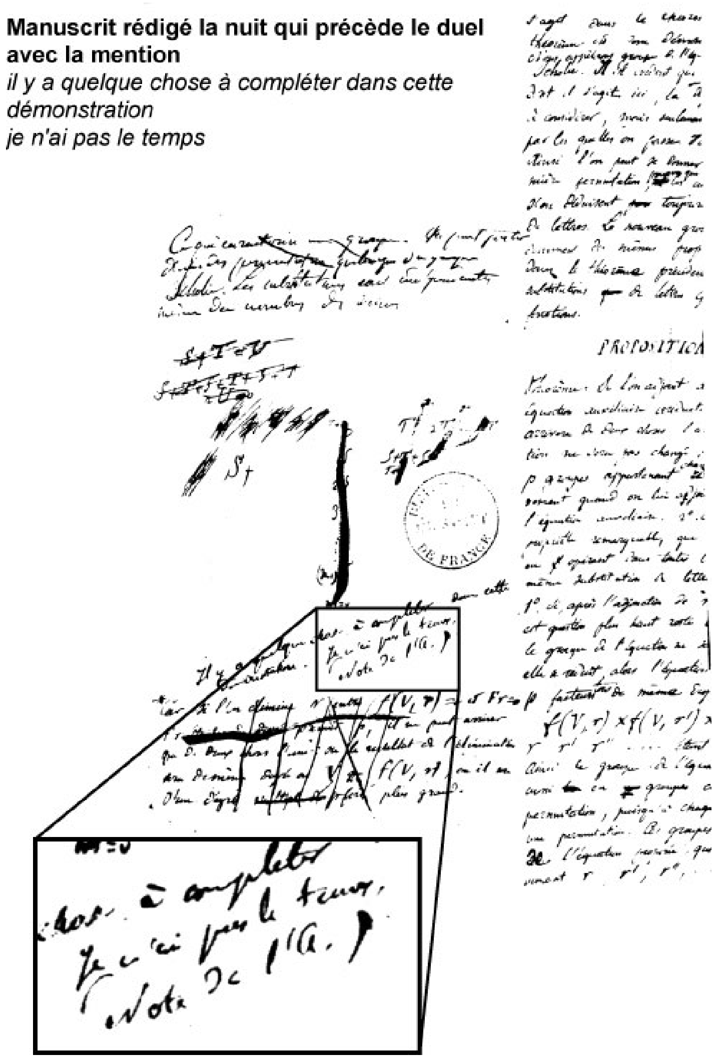


FIGURE 5 (Part of the text writen the night before the duel: "to be completed. I have not the time")

III

Le chercheur est institutionnellement en situation de double contrainte et ne sera reconnu que s'il brutalise un tant soit peu la discipline dont il souhaite recevoir des gratifications.

Dans le cas de Galois, cette échappée semble ne faire aucune difficulté. Il fait « l'analyse de l'analyse ». Mais son attitude est ambiguë dans ses rapports avec les représentants de la légitimité disciplinaire. Il semble vouloir et ne pas vouloir leur satisfecit, il ne fait pas ce qu'il faut pour l'obtenir et ne l'obtient pas. La double contrainte n'a pas prise sur lui parce qu'il s'est construit ses propres références morales que ses échecs ne font que conforter. Les seuls juges qu'il reconnaît, Lagrange et la postérité, sont inoffensifs. Le premier parce que mort, et les autres parce que projetés dans un avenir indéfini. Pourtant, l'ouverture imaginée par Galois n'est pas un échafaudage *ad hoc,* une perspective qui lui indiquerait une ligne de fuite. La théorie des groupes est induite d'une situation concrète ancienne, notoirement mal comprise, dont elle vient dénouer l'inextricable écheveau. C'est par des idées et un cheminement interprétatif que Galois procède, travail pour lequel il est particulièrement doué.

Cela pose donc la question très importante de la création interprétative en tant que source productive de science qu'il faut mener plus loin d'un point de vue psychologique. Dans l'œuvre de Freud, la démarche interprétative a, au contraire, une importance considérable, à trois niveaux.

Celui, tout d'abord, de l'interprétation des rêves dans la cure. « La technique que j'exposerai dans les pages qui suivent diffèrent de celle des Anciens par ce fait essentiel qu'elle charge du travail d'interprétation le rêveur lui-même »[18]. C'est l'énonciation par le sujet des signifiés qu'il découvre qui lui permet de s'en dégager et de remettre en mouvement des initiatives plus conscientes.

Le psychanalyste, quant à lui, tente de favoriser ce travail, par des relances et des questions. À ce second niveau, se constitue un savoir interprétatif des rêves, en évolution, auquel Freud a grandement contribué lui-même dans sa *Traumdeutung*, ses *Cinq psychanalyses*, etc. Cette connaissance s'appuie sur l'invention de concepts et de notions spécifiques (inconscient, libido, résistance, refoulement, narcissisme, Œdipe, etc.) qui perfectionnent, amendent ou dissocient des notions cliniques existantes et qui sont même, troisième niveau, structurées en une charpente métapsychologique que constituent les topiques, les types psychologiques, etc.

Dans ces deuxième et troisième niveaux, Freud emploie plus volontiers le terme d'hypothèse que celui d'interprétation pour désigner ce qui permet de comprendre les liens entre les faits affectifs vécus par le patient et ce qu'il rêve. Freud, qui est très attaché à présenter sa contribution aux savoirs des thérapeutes comme scientifique et qui vit à une époque où la science moderne des Pasteur, des Claude Bernard et des Ernst Mach, n'a pas encore été ébranlée par l'indéterminisme quantique, occulte, d'une façon somme toute assez étonnante, le fait que son apport à la science est largement dû à sa propre créativité interprétative, particulièrement talentueuse.

Vingt ans plus tard, Lacan présente le cas « Aimée » de façon bien différente. L'époque n'est plus la même, la physique a proposé le principe d'incertitude de Heisenberg, la logique a rencontré l'indécidable et l'art, après dada et le cubisme, s'alimente du rêve, de l'incongru et du fantasme en une sorte de parodie de la psychanalyse, le surréalisme. L'indéterminisme vient brouiller le discours épistémologique classique et l'interaction entre l'expérimentateur et son expérience redonne une place effective au sujet connaissant. Une des originalités de la thèse de Lacan est l'insistance avec laquelle il présente et défend le talent d'Aimée. Il cite le roman qu'elle a écrit, elle apparaît active, vivante. L'hypothèse selon laquelle ses dons auraient pu se développer, de façon purement artistique dans l'émotion d'une névrose à demi-maîtrisée, sans ces actes paroxystiques qui lui valurent d'être internée, est laissée ouverte[19]. Notons à ce propos que Galois fut arrêté pour avoir crié « À Louis-Philippe » en brandissant un couteau, lors d'une réunion publique. Il frôla donc également un internement de longue durée qui l'aurait anéanti.

Par la suite, dans ses *Écrits* (1966) et dans le *Séminaire,* Lacan forge une théorie plus élaborée qui dénonce tout déterminisme de la connaissance. Il critique la notion de vérité scientifique comme cause et plaide pour restaurer la place du *sujet*. Il ouvre une nouvelle image de la science comme paranoïa réussie, phrase à rapporter évidemment à l'intérêt qu'il porte à la vitalité du délire paranoïaque et à sa tendance à considérer avec bienveillance ce penchant très commun comme une composante créative de la personnalité[20]. Pour définir le savoir vrai, Lacan écarte l'absolu et l'universel du modernisme positiviste, au profit d'une acception plus proche de celle de la sociologie des sciences : une connaissance est délirante si elle est « l'expression, sous les formes du langage forgées pour les relations compréhensibles d'un groupe, de tendances concrètes dont l'insuffisant conformisme aux nécessités du groupe est méconnu par le sujet »[21]. Il situe ainsi la frontière au niveau de la lucidité du sujet quant au caractère hétérodoxe de son discours pour le groupe. La « connaissance paranoïaque », expression introduite par Lacan en 1955, est alors un témoignage, non pas désintéressé mais relevant de la dialectique de la jalousie[22] et introduisant une aptitude productrice de sens. Cela intéressera certains artistes et particulièrement Dali[23]. Alors qu'interprétation a essentiellement chez Freud le sens de déchiffrement, ce qui est conforme à une conception univoque de la science, Lacan en souligne le côté productif : « on peut dire que, contrairement aux rêves qui doivent être interprétés, le délire est par lui-même une activité interprétative de l'inconscient. Et c'est là le sens nouveau qui s'offre au terme de délire d'interprétation »[24]. Cette idée recevra toute sa force épistémologique durant le séminaire de l'École Normale : « La psychanalyse – je l'ai dit, je l'ai répété tout récemment – n'est pas une science. Elle n'a pas son statut de science et elle ne peut que l'attendre, l'espérer. Mais c'est un délire dont on attend qu'il porte science […] c'est un délire scientifique »[25] et il formule cet énoncé-clé, *le savoir, ça s'invente*, qui est à la fois un manifeste et une invitation à repenser l'épistémologie[26]. Cela veut dire que la connaissance se jardine toujours dans le terreau de l'inconscient et que la validité scientifique du savoir n'éclaire en rien la raison ou la déraison de l'auteur.

À cet égard, Galois est exactement à l'entre-deux d'un discours qui espère un satisfecit de ses contemporains par des textes qui leur restent néanmoins incompréhensibles, et d'une connaissance vraie recherchant consciemment l'attention plus approfondie que la postérité pourra lui accorder. Galois aurait fort bien pu se tromper dans ses mathématiques. Alors c'eût été l'échec sur toute la ligne, et son idéal ne se serait appuyé que sur un délire. Or ses démonstrations ne sont le plus souvent qu'esquissées et il eût suffit qu'un lemme fût erroné pour que tout s'écroulât. C'était tout à fait possible, surtout en ce qui concerne ces notions encore mal maîtrisées relatives aux nombres algébriques. Quinze ans après sa mort, une fort belle démonstration du grand théorème de Fermat fut publiée par Lamé (1847) utilisant les nombres complexes et les racines n-ièmes de l'unité. Si belle que certains ont pensé qu'elle était celle que Fermat avait en tête lorsqu'il écrivit dans la marge du traité de Diophante qu'il n'avait pas assez de place pour retranscrire sa preuve. La démonstration de Lamé utilisait comme évidente l'unicité de la décomposition en facteurs premiers dans les anneaux de nombres algébriques. Ce n'est que plus tard qu'on se rendit compte que cette propriété n'avait pas toujours lieu et qu'elle était fausse pour les racines n-ièmes de l'unité[27].

Qu'une pensée soit cohérente ne suffit pas à établir qu'elle est vraie. Sous l'angle de la créativité, le « déblocage » paranoïaque, pour reprendre les termes de Lacan, a cette particularité de faire bon ménage avec la logique. Celle-ci ne fait pas obstacle au délire. Elle est de son côté, d'autant plus efficace, comme toujours, que la sémantique est forte. La syntaxe est bonne servante du récit interprétatif. L'enquêteur qui échafaude une théorie sur des indices, articulant méticuleusement les déductions à partir des alibis, des empreintes, et des heures d'allers et venues, construisant grâce à ses dons psychologiques une interprétation des mobiles du ou des suspects, élabore un discours logique irréprochable, susceptible d'impressionner un amateur — tel le docteur Watson — mais ne saisit pour autant qu'une lecture des faits. La longue histoire des erreurs judiciaires montre combien une interprétation se renforce d'elle-même, en particulier par le terrible jeu des interrogations qu'elle suscite. Les coïncidences ne peuvent plus être fortuites dès lors qu'une théorie les explique. Aussi la créativité paranoïaque porte un plein questionnement sur la science : « I was led to consider the mechanism of paranoïac alienation of the ego as one of the precondition of human knowledge » écrit Lacan en 1951[28]. Plusieurs auteurs signalent un phénomène similaire durant la cure psychanalytique, la règle du tout dire induisant une sorte de paranoïa artificielle, au cours de laquelle le patient est parfois illuminé et dans l'angoisse d'une preuve à apporter. Une telle paranoïa artificielle se produit aussi dans le travail maïeutique de la recherche scientifique, particulièrement en mathématiques. La fréquence du penchant paranoïaque chez les grands mathématiciens, bien connue dans le milieu quoique difficile à valider, conforte cette hypothèse. Cependant, tout mathématicien est plongé de temps à autre dans l'angoisse de juguler ses illuminations. En mathématiques, l'illumination est en quête ou de preuve ou de réfutation. Ce travail vers la preuve et vers la réfutation, lorsqu'il échoue, laboure profondément la matière mathématique et donne à une autre illumination attendue une urgence tragique en même temps qu'une portée plus grande. La combinatoire des objets et des relations a noué un écheveau plus vaste, des enjeux plus considérables aussi. L'immensité des conséquences est source d'une profonde angoisse : évincer les pairs c'est tuer le père, se placer en position de meneur au péril d'un combat dont on n'a pas tous les éléments. Mais c'est aussi le plaisir de se mettre sur les rangs de cette confrontation constitutive de l'estime de soi. En préserver la perspective, la dessiner plus loin vers ses points de fuite, sans encore engager la lutte, telle est la folie naturelle et courante du chercheur. Toutes ces idées ni réfutées ni prouvées, interprétations de situations mathématiques sur le mode de l'éventuel, sont le délire de chaque mathématicien, vérité à laquelle il s'accroche, qu'il peut exposer avec passion à quelqu'un de confiance. C'est ce savoir qui lui donne l'énergie, la libido, de poursuivre sa vie de mathématicien, son investigation où peut-être les ingrédients d'une preuve se formeront à moins que d'autres illuminations surgissent avec leur cortège d'enjeux radieux et d'angoisses.

En matière d'interprétation dans la création scientifique ou artistique, la force tonitruante du terme « paranoïa » crée une difficulté de langage. Ce serait une erreur de pointer Galois comme paranoïaque. On peut dire que la science comme logomachie est paranoïaque, que tout scientifique a un penchant paranoïaque plus ou moins accusé, mais c'est bien parce que Galois est sain d'esprit et qu'il porte des jugements très sages, profonds et matures sur les mathématiques de son temps, que sa créativité nous provoque. Je voudrais au contraire saisir cette occasion pour estomper les contours de ce vocable grâce au fait que faire des mathématiques est une activité honorable et utile.

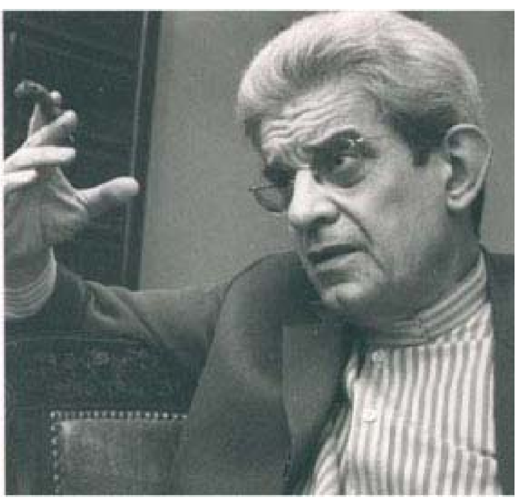
Comme plusieurs auteurs, je parlerai de paranoïa en tant que penchant universel du système psychique qui s'amplifie quand les projections positives vers les autres diminuent. Elle peut nous entraîner lorsque le moi, gestionnaire des relations avec le monde, se recroqueville, s'étiole, à la suite de circonstances douloureuses. Face à d'autres réactions dépressives, elle est une quête d'interprétation conceptuelle des souffrances affectives. Elle est aussi une fuite vers une vérité abstraite, les difficultés les plus fortuites s'interprétant à l'aune de la sphère des généralités érigées en système. Au demeurant, ce besoin d'interprétation est un moteur, et la paranoïa se trouve ainsi au cœur d'une réflexion sur la psychologie de l'invention.

Sérieux et Capgras, dans leur ouvrage de référence *Les folies raisonnantes, le délire d'interprétation*[29] constatent que « le mécanisme de l'interprétation délirante […] ne diffère en rien de ce que l'on observe à l'état normal ». Ils considèrent, avec Meynert, que « chez tout individu, l'idée délirante existe à l'état d'élément inconscient, réduite au silence par le fonctionnement normal des facultés ». Par ailleurs, lorsqu'elles se développent les pathologies paraphréniques ou paranoïaques[30] ont la particularité de ne pas altérer le fonctionnement lucide de la plupart des facultés de jugement. « On sait que l'éclosion d'un délire d'interprétation, même très actif, n'est nullement incompatible avec l'existence des plus brillantes qualités intellectuelles. Raison et délire marchent ici de pair, génie et folie peuvent même s'associer. »[31] Certains auteurs décèlent toutefois une différence de nature à laquelle Sérieux et Capgras dans leur souci de rationalité semblent souscrire : « Le mécanisme de la production des idées délirantes, dit Cotard, ne diffère pas foncièrement du mode habituel de formation des opinions erronées. Dans ces deux cas, la conviction pénètre non par l'entendement mais par le sentiment […] L'existence de convictions erronées, d'idées délirantes est donc la marque de la prépondérance du sentiment et de la faiblesse relative de l'intelligence. »[32]

Les mathématiques démentent nettement une telle assertion. Si la séparation entre intelligence et sentiments n'est pas possible en mathématiques, une telle césure paraît encore plus douteuse en tout autre domaine. Or, tout habitué des colloques et autres séminaires sait bien que les mathématiciens communiquent entre eux de manière allusive, se promènent à grandes enjambées parmi les notions et s'efforcent de transmettre leur motivation par des analogies incomplètes ou autres plaidoyers d'intérêt.

Le plus curieux, essentiel pour « faire des maths », c'est que la rigueur est ressentie même dans les développements informels. Orateur et auditeurs évoluent dans une région qui donne la sensation d'une harmonie particulière liée à des signifiés rigoureux. Un lien de nature affective engage la compréhension ; là, les mathématiques fonctionnent. En général, les auditeurs mettent à l'épreuve l'intuition partagée par des expériences de pensée rapides, des exemples simples. Dans cette région harmonieuse abonde parfois une création dénuée d'angoisse, émaillée de petites trouvailles. Les longues recherches sont autrement plus douloureuses. Est-ce une douleur de parturition ? Non, la souffrance est préalable, parce que rien ne vient. Elle est morale, faite d'espoirs inaboutis, de séduction frustrante. Combien de fois m'est-il arrivé de laisser l'ouvrage sur un indice prometteur pour pouvoir dormir. On ne peut trouver le repos que sur la voie d'une nouvelle interprétation avant qu'elle ne s'effondre. Je voudrais témoigner ici d'une expérience personnelle corroborée par d'autres mathématiciens : certaines phases de la recherche mathématique m'ont plongé dans un état d'angoisse et de fragilité affective d'une nature proche des récits de cas pathologiques de paranoïa ou de paraphrénie.

Si les mathématiciens parlent peu des ces périodes d'effondrement, c'est que les récits de trouvailles, comme tous les souvenirs, sont repérés par des chaînes d'événements heureux séparés de zones où l'on oublie même que tout s'est effacé. Le dialogue en profondeur ravive chez certains le souvenir de crises, alors transmuées positivement. Elles deviennent un climat d'une rare intensité, indice de grandes choses : durant ces périodes, on manie des idées qui eussent forcé le respect des esprits les plus pénétrants du passé, on a quitté la bimbeloterie, on est dans les coulisses de la scène où l'histoire s'écrit, le trac en est la preuve. Les mathématiques sont capables d'exercer une superbe fascination, une séduction hautaine, en comparaison de laquelle tout le reste est futile, qui opère comme une drogue et engendre une dépendance d'autant plus forte que les grandes idées qui nous ont été léguées par les maîtres des siècles écoulés montrent que certains de ces paradis ne sont pas artificiels.



The psychoanalyst Lacan, starting in his thesis from a case of paranoia, proposed a new conception of scientific knowledge based on the interpretative ability of the subject.

Lacan's patient was a women he calls Aimée. She had been institutionalized after trying to kill a famous actress at the end of a theatre performance.

The new conception of Lacan came from the observation that Aimée possessed an interpretative ability that gave her a real artistic creativity. She had begun a novel, and showed an interesting literary talent.

FIGURE 6

Lacan introduces the concept of "paranoiac knowledge» and emphasizes that «knowing is the result of inventing". He presents "science as a successful paranoia".

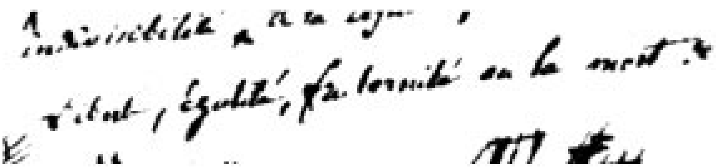


FIGURE 7

On a paper of Galois : «libertyequality, brotherhood, or the death»

Notes

1. R. Bourgne et J. P. Azra *Evariste Galois, Ecrits et Mémoires mathématiques*, Gauthier-Villars 1976.
2. P. Dupuy, «La vie d'Evariste Galois, *Ann. Sci. de l'Ecole Normale*, 3° série, Vol 13, p197-266, 1896.
3. P. Dupuy, op. cit.
4. Archives du Lycée Louis-le-Grand.
5. Dans Mes Mémoires 1830-1833 chapitre CCIV, Alexandre Dumas raconte l'affaire du banquet républicain auquel il participait lui-même et d'où il s'enfuit par une fenêtre assez piteusement, craignant les représailles au moment où Galois leva son couteau en criant À Louis-Philippe… ce que tout le monde entendit… s'il trahit ce qui resta inaudible dans le brouhaha. Puis, Dumas rapporte les dialogues du procès qui suivit où Galois aurait confirmé que « la marche du gouvernement peut faire supposer que Louis-Philippe trahira un jour, s'il n'a déjà trahi ». Et après le verdict d'acquittement, Dumas se demande si les jurés tenaient Galois pour fou ou s'ils étaient de son avis.
6. loc. cit. page 247.
7. Dumas parle cependant de ce Pécheux d'Herbinville comme d'une tête brûlée : « ce charmant jeune homme qui faisait des cartouches en papier de soie, nouées avec des faveurs roses ».
8. La première publication sur la théorie de Galois fut celle de l'Italien E. Betti « Sulla resoluzione delle equazioni algebriche » 1852, cf. J. Lützen Joseph Liouville 1809-1882, Master of pure and applied mathematics, Springer 1990.
9. Plutarque, *La vie des hommes illustres, Marcellus.*
10. Cf. A. Dahan-Dalmedico et J. Peiffer, Une histoire des Mathématiques, Le Seuil, 1986.
11. Par exemple cos (2π /17)=

$$-\frac{1}{16}+\frac{1}{16}\sqrt{17}+\frac{1}{16}\sqrt{34-2\sqrt{17}}+\frac{1}{8}\sqrt{17+3\sqrt{17}-\sqrt{(34-2\sqrt{17})}-2\sqrt{(34+2\sqrt{17})}}$$

12. Cf. J. Dieudonné, *Pour l'honneur de l'esprit humain*, Hachette, 1987, Chapitre V
13. Voir H. Weyl, *Symétrie et mathématiques modernes*, (1952), Flammarion, 1984.
14. J. Dieudonné introduction à l'ouvrage de R. Bourgne et J. P. Azra Evariste Galois, *Ecrits et mémoires mathématiques*, Gauthier-Villars, 1976.
15. Mémoire rédigé lors de son incarcération à la prison de Sainte-Pélagie.
16. Mentionnons également Pierre Varignon (1654-1722) et Sophie Germain (1776-1831), cf. A. Dahan-Dalmedico, *Mathématisations, Augustin Cauchy et l'Ecole française*, Ed. du Choix 1992.
17. Cf. par exemple, le fort intéressant ouvrage de Lazare Carnot *Réflexions sur la Métaphysique du calcul infinitésimal* 1813.
18. S. Freud, *L'interprétation des rêves*, (1898-1929), Presses Universitaires de France 1967.
19. Aimée avait été internée pour avoir tenté de poignarder une actrice célèbre à l'entrée des artistes d'un théâtre où elle se rendait pour jouer ce soir-là.
20. Lacan s'exprime comme s'il regrettait de n'être pas le premier inventeur de cette formule : « Pourtant si l'on aperçoit qu'une paranoïa réussie apparaîtrait aussi bien être la clôture de la science [...] on retrouve là la même impasse apparente [...] Peut-être le point actuel où en

est le drame de la psychanalyse, et la ruse qui s'y cache à se jouer de la ruse consciente des auteurs, sont-ils ici à prendre en considération car ce n'est pas moi qui ai introduit la formule de la paranoïa réussie. » Séance d'ouverture du séminaire 1965-66 « La science et la vérité », in *Ecrits II,* Le Seuil, 1971.

21. J. Lacan, *De la psychose paranoïaque dans ses relations avec la personnalité* (1932), Le Seuil 1975.
22. Cf. J. Allouch, *Marguerite ou l'Aimée de Lacan*, EPEL, 1990.
23. Cf. l'article «Interprétation paranoïaque-critique de l'image obsédante <L'angelus> de Millet» de Salvador Dali (*Le minotaure*, I, 1933) cité par J. Allouch *ibid.* p. 629, ainsi que l'article « Ducasse, Duchamp, Dali… » de R. Brossart, *Littoral* n°31-32, mars 1991.
24. Thèse, *op. cit*. p. 293.
25. *Séminaire* 1977, cité par J. Allouch *op. cit*. p 455.
26. J. Allouch place en exergue de son chapitre « Marguerite sachante » ce propos de Lacan « Il a fallu Freud pour que je me la pose vraiment [cette question] c'est : qu'est-ce que c'est que le savoir ? […] J'y ai été happé parce que la patiente de ma thèse, le cas Aimée, elle savait. Simplement, elle confirme ce dont vous comprendrez que j'en sois parti, elle inventait. Bien sûr ça ne suffit pas à assurer, à confirmer que le savoir ça s'invente, parce que, comme on dit, elle débloquait. Seulement, c'est comme ça que le soupçon m'en est venu. Naturellement je ne le savais pas. » *Op. cit*.
27. Voir W. et F. Ellison « Théorie des nombres » *in Abrégé d'histoire des mathématiques*, sous la dir. de J. Dieudonné, Hermann, 1978, tome I p. 191 *et seq.*
28. Cf. J. Allouch *ibid*. p. 603.
29. Alcan, Paris 1909; réédition, Laffitte reprints Marseille 1982.
30. La nosographie désigne sous le nom de paraphrénie des délires moins systématisés et plus changeants que ceux de la paranoïa. Au début du vingtième siècle, Sérieux et Capgras considèrent que ce qu'on appelle « délires systématisés » en France correspond à ce qui est groupé sous le nom de « paranoïa » à l'étranger.
31. *Op. cit*. p. 366.
32. *Op. cit*. p. 226.

Annexe

Préface du traité de Camille Jordan sur les Substitutions
et les équations algébriques, 1870.

Le problème de la résolution algébrique des équations est l'un des premiers qui se soient imposés aux recherches des géomètres. Dès les débuts de l'Algèbre moderne, plusieurs procédés ont été mis en avant pour résoudre les équations des quatre premiers degrés : mais ces diverses méthodes, isolées les unes des autres et fondées sur des artifices de calculs, constituaient des faits plutôt qu'une théorie, jusqu'au jour où Lagrange, les soumettant à une analyse approfondie, sut démêler le fondement commun sur lequel elles reposent et les ramener à une méthode véritablement analytique, et prenant son point de départ dans la théorie des substitutions.

L'impuissance de la méthode de Lagrange, pour les équations générales d'un degré supérieur au quatrième, donnait lieu de croire à l'impossibilité de les résoudre par radicaux. Abel démontra, en effet, cette proposition fondamentale; puis, recherchant quelles étaient les équations particulières susceptibles de ce genre de résolution, il obtint une classe d'équations remarquables qui portent son nom. Il poursuivait avec ardeur ce grand travail lorsque la mort vint le frapper; les fragments qui nous restent permettent de juger de l'importance de cet édifice inachevé.

Ces beaux résultats n'étaient pourtant que le prélude d'une plus grande découverte. Il était réservé à Galois d'asseoir la théorie des équations sur sa base définitive, en montrant qu'à chaque équation correspond un groupe de substitutions, dans lequel se reflètent ses caractères essentiels, et notamment ceux qui ont trait à sa résolution par d'autres équations auxiliaires. D'après ce principe, étant donnée une équation quelconque, il suffira de connaître une de ses propriétés caractéristiques pour déterminer son groupe, d'où l'on déduira réciproquement ses autres propriétés.

De ce point de vue élevé, le problème de la résolution par radicaux, qui naguère encore semblait former l'unique objet de la théorie, n'apparaît plus que comme le premier anneau d'une longue chaîne de questions relatives aux transformations des irrationnelles et à leur classification.

Galois, faisant à ce problème particulier l'application de ses méthodes générales, trouva sans difficulté la propriété caractéristique des groupes des équations résolubles par radicaux, la forme explicite de ces groupes pour les équations de degré premier, et deux théorèmes importants relatifs au cas des degrés composés. Mais, dans la précipitation de sa rédaction, il avait laissé sans démonstration suffisante plusieurs propositions fondamentales. Cette lacune ne tarda pas à être comblée par M. Betti, dans un Mémoire important, où la série complète de ces théorèmes de Galois a été pour la première fois rigoureusement établie.

L'étude de la division des fonctions transcendantes offrit à Galois une nouvelle et brillante application de sa méthode. Depuis longtemps Gauss avait démontré que les équations de la division du cercle étaient résolubles par radicaux; Abel avait établi le même résultat pour les équations de la division des fonctions elliptiques, en supposant la division des périodes effectuée; proposition que M. Hermite devait étendre aux fonctions abéliennes. Mais il restait à étudier les équations modulaires dont dépend la division des périodes. Galois déterminant son groupe, remarqua que celles de ces équations dont le degré est 6, 8 ou 12, peuvent s'abaisser d'un degré. M. Hermite, effectuant cette réduction, montra qu'il suffisait de résoudre des équations des quatre premiers degrés pour identifier la réduite obtenue dans le cas de la quintisection avec l'équation générale du cinquième degré, ce qui fournissait la solution de cette dernière par les fonctions elliptiques. M. Kronecker parvenait en même temps au même résultat par une méthode à peu près inverse, que M. Brioschi a reprise et développée dans quelques pages remarquables.

Une autre voie féconde de recherches a été ouverte aux analystes par les célèbres Mémoires de M. Hesse sur les points d'inflexion des courbes du troisième ordre. Les problèmes de la Géométrie analytique fournissent, en effet, une foule d'autres équations remarquables dont les propriétés, étudiées par les plus illustres géomètres, et principalement par MM. Cayley, Clebsch, Hesse, Kummer, Salmon, Steiner, sont aujourd'hui bien connues et permettent de leur appliquer sans difficulté les méthodes de Galois.

La théorie des substitutions, qui devient ainsi le fondement de toutes les questions relatives aux équations, n'est encore que peu avancée. Lagrange n'avait fait que l'effleurer; Cauchy l'a abordée à plusieurs reprises. MM. Bertrand, Brioschi, Hermite, Kronecker, J.-A. Serret, E. Mathieu s'en sont également occupés; mais, malgré l'importance de leurs travaux, la question était si vaste et si difficile, qu'elle reste encore presque entière. Trois notions fondamentales commencent cependant à se dégager : celle de la primitivité, qui se trouvait déjà indiquée dans les Ouvrages de Gauss et d'Abel; celle de la transitivité, qui appartient à Cauchy; enfin la distinction des groupes simples et composés. C'est encore à Galois qu'est due cette dernière notion, la plus importante des trois. [...]

Discussion

Ne croyez-vous pas que l'exemple de Galois risque de renforcer l'idée que les maths sont faites pour les surdoués et non pour l'élève de base ?

Il est incontestable que les mathématiques avancent par les surdoués, comme la musique. Mais par le cas de Galois j'ai surtout voulu montrer un trait universel de notre psychologie auquel est lié le plaisir de l'activité mathématique et qu'on peut sans doute cultiver chez chacun : une sensibilité interprétative qu'il ne faudrait pas étouffer ou refouler. Cela se relie, il me semble, à la question du symbolique. Certains enseignants pensent que c'est brutaliser l'élève que de lui faire désigner des entités par des lettres et de lui apprendre à manier ces lettres au lieu des objets qu'elles représentent et qu'il faut lui épargner ce vertige autant qu'on peut. Je pense au contraire que cette difficulté pédagogique doit être abordée en elle-même, les maths sont aussi l'apprentissage d'un langage, de divers langages, les exercices de modélisation sont ici un moyen extrêmement utile.

Dans la classe de mathématiques, il est d'usage quand un élève défend une affirmation, de ne pas attacher la thèse à la personne de l'élève, mais plutôt de considérer que tout le monde est en charge de la défendre ou de la contredire, dans le cadre de la recherche d'une vérité partagée. Dans votre présentation de l'histoire de Galois, on voit au contraire comment Evariste considère ses travaux comme éminemment personnels, comme partie essentielle de sa vie. Cela questionne alors le professeur vis-à-vis de l'élève qui propose une preuve : ne faut-il pas au contraire accepter une personnalisation forte de celle-ci, quitte à trouver d'autres moyens pour permettre à l'élève de l'abandonner en cas de réfutation ?

Il s'agit de la question de la gestion de l'énergie de motivation. Evariste Galois nous montre l'exemple extrême d'une confiance absolue dans ses idées, envers et contre tous, cette conviction lui donnant une fougue au delà du bienséant et du raisonnable, avec un courage et une détermination assez admirable. Dans une classe, la force d'une conviction lorsqu'elle apparaît chez tel ou tel élève, est un événement très positif sur lequel l'enseignement prend appui. Elle doit être à mon sens en permanence encouragée, et même félicitée. Les mathématiques présentées de façon apodictique et castratrice détruisent le goût du défi qui est aussi ancien que les mathématiques elles-mêmes. Mais il faut être bon joueur, ce qui nécessite une éducation chez l'enfant, un climat est à créer où la rigueur doit finalement gagner sans que les tentatives soient découragées. Lakatos rend bien ce climat dans *Proofs and refutations*, et ses idées sont plus pertinentes, à mon avis, dans le champ pédagogique que sur le plan épistémologique. La personnalisation peut être poussée pour autant que l'élève est en état de défendre discursivement son point de vue, d'argumenter par déduction et par des exemples dans un langage partagé que le professeur améliore. Lorsque la conviction utilise des outils symboliques (et c'est le cas de Galois qui croit à ses entités abstraites que Poisson ne comprend pas), le travail de rendre à la rigueur ses droits est particulièrement important et formateur.

En présentant les mathématiques comme le résultat de tentatives de preuves et de réfutations il manque à la pensée de Lakatos précisément l'importance de l'interprétatif, le goût et le talent de donner du sens à des situations opaques et *a priori* inextricables qui se trouvent ainsi éclairées. Les groupes n'ont pas été inventés par des tentatives de preuves et de réfutation, les nombres complexes non plus, la notion de dérivée non plus, ni celle de fonction analytique, etc.

Vous faites le lien entre ce talent et la paranoïa, pouvez-vous mieux définir celle-ci ?

Ce serait un vaste programme. Je renvoie à mon livre *La règle, le compas et le divan* où j'analyse un peu plus la notion. Disons ici simplement que la pensée de Lacan qui souligne le caractère créatif des délires interprétatifs et soutient la thèse d'une même nature entre cette créativité et la connaissance scientifique, s'inscrit dans un courant. Une autre ligne de force en est la critique par Deleuze et Guattari de l'analyse du cas du président Schreber par Freud. Dans *L'anti-Oedipe*, ceux-ci reprochent à Freud d'avoir occulté dans son étude de la paranoïa du président Schreber l'étonnante imagination créative de celui-ci, présente à l'évidence dans le texte de la confession du président Schreber. Pour ces auteurs, Freud est passé là à côté de l'essentiel et le souci de scientificité de Freud y serait pour quelque chose. Deleuze était particulièrement sensible à cette question et y avait été amené par sa réflexion sur le plaisir et le désir où la construction d'un décor, d'un contexte, d'une mise en scène joue un rôle central (dramatisation, machines désirantes).

Le positivisme de Comte, de Mill, de Renan, puis le néopositivisme du début du 20e siècle en voulant s'en tenir « aux faits » et effacer le phlogistique, l'éther, et autres entités dites occultes, ont coupé la connaissance d'une de ses sources fondamentales. Un auteur comme Bruno Latour en soulignant que les faits sont faits, qu'ils sont imbibés de valeurs et ne peuvent pas en être séparés (les « faitiches »), s'inscrit également d'une certaine façon dans ce courant.

Le constat des psychiatres Sérieux et Capgras dans leur traité de référence du début du 20e siècle sur les folies raisonnantes que la validité logique du discours ne renseigne en rien sur la santé mentale du patient est évidemment porteuse d'interrogation si on ouvre le parallèle avec la science elle-même. C'est le social qui fait donc la différence d'où l'idée de Lacan de la science comme paranoïa réussie.

Working Groups

Groupes de travail

Report Of Working Group A
Rapport du Groupe de travail A

Learner-Generated Examples

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Anne Watson, *Oxford University*
Rina Zazkis, *Simon Fraser University*

Participants

Maha Belkhodja	Frédérique Gourdeau	Immaculate Namukasa
Eli Brettler	Dave Hewitt	David Pimm
Peter Brouwer	Siobhan Kavousian	Jérôme Proulx
Laurinda Brown	Dave Lidstone	Elaine Simmt
Aldona Businskas	Peter Liljedahl	Margaret Sinclair
Stewart Craven	Julie Long	Tom Steinke
Sandy Dawson	Calin Lucas	Luc Tremblay
Lucie DeBlois	Doug McDougall	Dave Wagner
Jean-Lou De Carufel	John Mason	Vicki Zack

Prologue

Give an example of an irrational number between 0 and 1. And another, and another, and 10 more. How were these examples found/constructed/generated?

In this workshop participants were offered a sequence of mathematical tasks to work on in small groups, or by themselves. All the tasks involved the construction of mathematical objects in some sense, or exploration of what it means to work with 'learner-generated examples' (LGEs). Some explanation of terms is necessary here. By 'mathematical object' we mean something we can make sense of mathematically, an expression, a question, a class, a number, an equation, a diagram, a proof—anything which in mathematics can be the focus of attention. By 'construction' we mean building, creating, generating such an object rather than being given one by an authority such as a teacher or textbook. We are using the phrase 'learner-generated example' to show that when a learner creates an object it is an example of some class of such objects, and for this purpose we sometimes refer to the workshop participants as 'learners' because they were often in the position of having to construct such objects.

The purpose of the workshop was for all the participants, including the three leaders, to explore the power, purposes and pedagogic implications of asking learners to construct their own examples. Some of the tasks, and the thinking behind them, arose during the writing of a forthcoming book by Anne Watson and John Mason. Rina Zazkis and Nathalie Sinclair both had draft copies of this book, and Nathalie had been using this with practicing teachers. However, the workshop was far from being a presentation of fully-formed ideas, rather it was an arena in which all of us could work authentically to gain further insight into the issues from our own perspectives and starting points, for our own working contexts. In this sense, the workshop contributed to praxis and every participant's story will be different. This is an attempt to summarise what was spoken in the public domain.

It is widely acknowledged that mathematics is learned by engagement in mathematical activity that invites the construction of meanings which are mediated, in classrooms and textbooks, by conventional distinctions and discourse. In usual teaching, the objects of attention are given by authorities and learners may or may not see in them what the teacher intends. Thus mathematics lessons might appear to involve observing a sequence of presented things whose genesis is a mystery and which may seem to be unconnected and incoherent. As a student once said to Bateson 'we know you are offering us examples, by we don't know what they are examples of' (Bateson, 1973). We are wondering if asking learners to construct their own examples of objects on which to work mathematically is possible and useful. Certainly as teachers we have found it to be a powerful way to engage learners, and teachers with whom we have worked (experienced and new) react as if this approach makes sense to them also. The literature supports this idea as well (e.g., Dahlberg and Housman 1997). There is a substantial literature on students posing their own problems or test questions, generating their own examples for inductive reasoning, and contributing 'owned' examples for motivational purposes (such as using their own heights for statistical analysis). In this sense, the focus of this workshop was not new but was part of an attempt to bring these kinds of activity under one umbrella and see what the world looks like if we ask 'can *all* types of mathematical object be learner generated?'

Task 1:

Each participant was asked to say their name and give a number between 99 and 100. Then they were asked to discuss in small groups how they chose their number.

This task was intended to be introductory on several levels. It proved to be unexpectedly rich socially, pedagogically and mathematically. Socially, of course, it got people talking but, in addition, it illustrated that the business of choosing a number also contained an element of self-presentation. Some people were aware that their choices would give an impression to others of their mathematics and this influenced the way they chose numbers, as well as their emotional response to the task. For some, the element of competition was mathematically fruitful in pushing them to expand their example space, while for others it was inhibiting. In addition, for some there was an attempt to 'guess what the teacher is thinking' and to try to anticipate what the workshop leaders might be looking for. In fact, we were looking for discussion of these reactions! Mathematically, the richness of the task came from the fact that people were working on 'number' in creative and imaginative ways—the property of 'richness' was in the ways people worked, not in the task itself which also had the potential to become superficial. One pedagogic observation was that it was the discussion which made the task rich, but several people reported that the way they had chosen their number was not at all superficial, hence showing that individuals can enrich a task by the way they impose personal constraints and private rules. For example, some 'learners' used their knowledgeable reactions to numbers involving 9s and 1s to offer numbers which expressed a sense of 100 as a limit; one gave a number which had already been said 'ninety-nine and a half', but in a different form ('ninety-nine point five') to focus on representation; one used a number he had used earlier in the day for something else and offered '100 minus one over root 2'. This last offering was hotly debated since some people thought it was an operation rather than a number. One table engaged in an exploration about the likelihood of a number randomly selected between 99 and 100 being rational or irrational; another table offered research evidence to claim that people tend to choose numbers towards the upper end of an interval.

The responses to this task offer fragments towards a theory of LGEs:

1. **Learners select from a personal, task-specific example space influenced as much by the situation and their expectations of it as by their prior knowledge. This space is likely to be a proper subset of what they know.**
2. **A 'good' example has to be seen to be an example of something.**
3. **Examples are examples of something. A mathematical object may be a member of multiple classes; what is seen by one learner to be an example of class G may be seen by another to be an example of class H.**
4. **Sharing examples exposes a range of possibilities which may not have occurred to others, but one person cannot necessarily make use of the examples given by another. New examples are more likely to be taken up by other learners if there is proximal relationship to someone's existing example space, and if they are seen to be useful or interesting in some way.**

Task 2:

Write down a pairs of numbers which differ by 2.

… *now write down another pair of numbers which differ by 2*

… *now write down another pair of numbers which differ by 2.*

This task was done individually and then discussed in groups. The difference between writing privately and giving publicly was deliberate, since the pressures of speaking out had been raised earlier. It was interesting that for some people this was an entirely new task, but for others the extended example spaces developed in the previous task were used for this one. One participant said that she found herself using numbers for task 2 she would never have chosen if she had not just heard the discussion of task 1. Some reported trying to find interesting pairs, but the interpretation of 'interesting' is personal. For some, (1, -1) was interesting since it straddles zero which gave it a singular status; others developed generating sequences which would produce as many new pairs as were required; others had personal, social or emotional reasons for choosing certain numbers. For most, the decisions about how to create second and third pairs were mathematical in some way, such as creating a pattern, or going to some extreme, or stating a generality of some subclass of possibilities. This led to further fragments:

5. **Learners can employ personal constraints and goals which can make a task more interesting for them. Constraints and goals may have the same effect, although differently expressed.**
6. **The effect of having a generalisation expressed can be to energise or to de-energise. Sometimes it is more useful to hold the generalisation until a particular is needed; other times exploration of particulars is interesting enough to postpone, or put aside, generalisations.**

Task 3:

Write down a time at which the hands of a clock are at 90 degrees.

… *now write down another time when the hands of a clock are at 90 degrees*

… *now write down another ….*

This task is, for many, intrinsically more mathematically interesting than the previous task, but is of the same generic structure. We did not offer a plenary discussion of the mathematical findings, although plenty of time was spent on the maths. Instead we asked participants to compare the two 'similar' tasks, since both ostensibly ask for LGEs and push learners beyond their immediate offerings.

Theoretical fragments:

7. **Asking for more than one example can force exploration of a class of objects, and more particularly of mathematical structures as learners find out what is possible.**
8. **It makes a difference how familiar the class is; how many dimensions of possible variation it has; how easily it can be explored; whether there are finitely or infinitely many possibilities …**
9. **If an example is not easy to find, learners may be pushed towards generalisation which suddenly gives access to multiple examples.**
10. **Limitations and possibilities may be perceived differently by teachers and learners.**

Task 4:

'Teacher' says to one learner:	*7 squared take away 1 is … ?*
Learner replies:	*48*
Teacher:	*6 times 'what' is 48?*
Learner:	*8*
Teacher:	*now you make up one like that for me*
Learner:	*10 squared take away 1 is … ?*
Teacher:	*99*
Learner:	*9 times 'what' is 99?*
Teacher:	*11*

This task continued to be passed between participants around the room until someone said 'n squared take away 1'. This task was offered as an example of the kind of repetition which can help learners participate in mathematical structures through speech patterns. In the literature on problem posing, researchers have shown that learners usually copy the kinds of questions asked by their teachers, and this is often taken to be a negative result. In this task, we try to show that such copying can be a positive feature of classrooms when used to demonstrate generalities through LGEs. However, learners may not see the same generality which the teacher intends, and there was discussion of how to handle this in the classroom without being insensitive to learners. An example arose in which the 'teacher' tried to start another 'round the room' sequence.

Teacher: *what number must I take away from 4 to get -4?*
Learner: *8*

Learner: *what number must I take away from 5 to get -2?*

The teacher had intended to work on ' n – (-n) = 2n' but the learner had apparently taken it to be an example of more general subtraction of negative numbers. Discussion of this seemed to converge towards:

11. **Learners may not sense the same generality that the teacher intends; being 'wrong' can be uncomfortable unless a classroom ethos has been strongly established that there are always other possible generalities and these may have to be put to one side.**

Task 5:

Find a data set of seven items which have a mode of 5, median of 6 and mean of 7.

… and another
… and another

Alter your set to make the mode 10, the median 12, and the mean 14.

Alter your set to make the mode 8, the median 9 and the mode 10.

Discussion of this task led to the observations that, for many, it had not felt like an example-generating situation until a decision had to be made about whether and how to vary the set to form new ones. The scope of possible variation, and the ranges of numbers it was possible to have, and their relationships, were all significant. It was generally agreed that this had led to significant engagement with the structures of these averages—though, as with all tasks, not for everyone. Supplementary questions about the smallest possible data set for which these averages can have any pre-assigned values, and how much choice there is for data-sets, were provided but also arose from the mathematical activity. One participant said that 7 data points had not provided him with enough challenge, so he had decided to limit himself to 5. It was important that the participants used a range of different images and methods, such as 'balancing' or 'algebraic', to develop their examples, because this gave rise to the first of the following fragments:

12. **Learners' example spaces are structured in individual ways, and these structures depend also on the initial entry into the space, different images possibly giving access to different relationships.**
13. **'To find' questions are asking for LGEs, but the number of members of the example space may be limited to one. Thus 'finding' can be seen as imposing a sequence of constraints on possibilities until only one example fits all the constraints.**
14. **Creating an LGE can lead learners to engage deeply with the meaning of concepts; varying constraints can expose assumptions.**
15. **Creating examples can involve searching through dusty corners of knowledge, or creating new things from known things.**

Task 6:

We watched a video of a teacher, a student of Nathalie's, who has decided to include LGEs as a central feature of his teaching. He was working with a class who was by then used to participating in a range of ways. He asked them to choose numbers between 75 and 100 and to design prisms which would have that number as their volume. After some discussion about what was meant by 'prism' students started work. Using the 'another and another' prompt he encouraged students to generate more than one idea, and some chose to do this by adapting or transforming their first ideas. Participants spent some time exploring the task themselves. We then watched on video three students showing the rest of the class their ideas at the end of the lesson, including one who decided to assign an area of 1 to his cross-section, and discussed what had happened. Some issues which emerged were:

- Choice of total volume was a good idea to engage learners and ensure a variety of possibilities, but the choice of number significantly affects the nature of the task.
- The student who chose a cross-section of 1 may have done it as a joke, but it showed the power of using extreme examples.
- Had the teacher thought about why only boys were demonstrating their answers? Had he thought about the purpose of sharing results?
- Indeed, there was extended discussion of whether sharing results was ever useful unless learners had a purpose for listening to each other, such as comparing methods, deciding which is most powerful, seeing if there were generalisations to be made, and so on.

The teacher's view is that these new-to-him strategies he is using have led to significant improvement in the participation of students and his own enjoyment of teaching.

Pedagogic theoretical fragments arising from this video and discussion included:

16. **Listening to learners includes giving up detailed advance planning.**
17. **Listening to learners involves giving up authority to learners for the direction of the lesson, and depending on the intrinsic warrants for validity within mathematics, expressed through discussion and mathematical activity, such as exemplification and counter-exemplification**
18. **There may be an optimal level of constraint; too much constraint may stifle creativity or willingness to engage.**

Giving up authority creates ethical implications for whether and how the teacher, or the whole class, handles 'wrong' answers (Chazan & Ball, 1999).

Task 7:

Give[1] an example of an arithmetic sequence, and another, and another.

Rina has for some time been asking learners to give examples of concepts, or classes of objects subject to certain constraints, as a regular feature of her teaching (Hazzan & Zazkis, 1997, 1999; Zazkis & Liljedahl, 2002). She offered us some transcripts arising from clinical research interviews with students who were learning about arithmetic sequences. The aim was to ask students to give numbers which would appear in a given sequence, and to ask them also if particular numbers would or would not appear, and why. Thus students' understanding of the generality expressed by the sequence could both be explored and would also develop within the interview. Below are the specific excerpts presented in the working group.

Excerpt #1

Interviewer:	Okay. I would like you to look at a different sequence, and it is 17, 34, 51, 68 and so on. And I would like to ask you about the number 204. Is it an element of this sequence?
Dave:	If it's a multiple of 17 it is . . .
Interviewer:	And if it is not a multiple of 17?
Dave:	Then it shouldn't be.
Interviewer:	So this will guide your decision.
Dave:	Um hm.
Interviewer:	So 204 is indeed 17 x 12. . .
Dave:	Then it's in. . .
Interviewer:	It's in. Can you please give me an example of a big number which is in this sequence?
Dave:	17,000.
Interviewer:	Another one.
Dave:	17,051.
Interviewer:	Okay. What makes you believe that 17,051 is an element of this sequence?
Dave:	(pause) 17,000 is 1,000 x 17 and that's a multiple of 17. . .
Interviewer:	Um hm. . .
Dave:	I also know that 51 is a multiple of 17, and so it's the 3 x 17, so I add 1,003 17's, I've still got a multiple of 17, it's still going to be in there.

Excerpt #2 — Considering sequence 8, 15, 22, 29…

Interviewer:	Okay, so can you give me an example of a number that you believe is not in the sequence and example of a number that you believe is, or could be in the sequence?
Leah:	Um, I don't think 714 would be in the sequence, um, a number that could be, I would just pick a number that hasn't a factor of 7, so like 511 possibly, or something
Interviewer:	And you are saying possibly because. . .
Leah:	Just, I just picked a number that wasn't, didn't have 7 as a factor.

Excerpt #3 — Considering sequence 17, 34, 51, 68, …

Interviewer:	How would you decide whether 204 is an element in this sequence?
Eve:	Okay, (pause) okay I guess I would use 204 and divide by the first number in here, because it looks like, when I'm looking at this sequence it looks like um all these numbers are multiples of 17, so if 204 is a multiple of 17 which means that it will also occur in this sequence, so in order to be a multiple of 17, 204 divided by 17 must give us a result of a whole number and no decimal places.
Interviewer:	Okay. . .
Eve:	So 204 divided by 17, that gives us 12, okay it's 12, this whole number, so it's a

number in this sequence.
Interviewer: Okay. Can you please give me an example of any 4-digit number in this sequence?
Eve: I could just randomly pick any, okay. . .
Interviewer: Yes, please pick any, but convince me that it is in the sequence.
Eve: Okay. 17, um, (pause) I just keep on adding 17 to get um this sequence up 85, 102, 119, 136 and 153. . .
Interviewer: Yeah, this is a pretty hard work. . .
Eve: Yeah. . .
Interviewer: If I want a 4-digit number, it will take you quite a while to get that. . .
Eve: Oh, you want a 4-digit number. . .
Interviewer: Yeah. . .
Eve: Umm, (pause) I don't know how to do this.

Excerpt #4 — Considering sequence 8, 15, 22, 29, …

Interviewer: So 704 is not divisible by 7, none of these elements in this sequence you believe will be divisible by 7, so can you draw conclusions from what you have now?
Sally: It's, it's um very possibly in this set.
Interviewer: Um hm. What, what will convince you?
Sally: (laugh) Well just because it's not divisible by 7, doesn't mean it's in the set, right?
Interviewer: Can you give me an example of a number that you know for sure that is <u>not</u> in this arithmetic sequence?
Sally: Um hm, um 700. . .
Interviewer: Another one. . .
Sally: Um, 77.
Interviewer: Okay. And how about 78?
Sally: It may be in the set, but it's not divisible by 7. . .
Interviewer: (laugh) So 77 you're sure is not, 78 you're not sure.
Sally: Right.
Interviewer: 79?
Sally: Could be. . .
Interviewer: Could be. 80?
Sally: Could be. . .

Excerpt #5 — Considering sequence 2 ,5, 8, 11, …

Interviewer: Could you please give me an example of a number, and I would like a relatively big number, like 3-digit number or 4-digit number, that you're sure will be listed in this sequence [2,5,8, ...] ?
Sue: Mmm, okay, I'm guess it has to be a multiple of 3, because it's common difference, so um 333, maybe?
Interviewer: So you think that 333 will be listed in this sequence?
Sue: I think so.

[…]

Sue: Hmm, wait a minute, 360 is a multiple of 3, yet I just said that it didn't go in, right...
Interviewer: You did. . .
Sue: So then this might not go in there, I don't know, um, (pause) I'm not sure (laugh). I think I'll have to guess a couple, I'll have to do trial and error to figure it out.
Interviewer: And what do you mean by trial and error here?
Sue: Like um, I'm going to start with pick a couple of numbers that I think would work and then put it back into this formula. . .
Interviewer: Okay. . .
Sue: To see if I get a whole number. . .
Interviewer: For?
Sue: For N. (The interview excerpts are from Zazkis & Liljedahl, 2002.)

After reading excerpts from several transcripts, participants were invited to continue the transcript by imagining the next few interchanges. They were then asked to provide another continuation, and then another continuation, thus using the LGE strategy on our own learning about the use of examples as a way to engage with mathematical structures. During our discussion of these tasks the following arose:

19. The concept of 'sufficiency' relates to whether the properties the learner is using are those the teacher imagines are operating. Asking about exemplification both ways, ('give me an example' and 'is this an example?') gives a structure to explore this further.

Task 8:

Participants formed groups to produce teaching sequences which used LGEs, either using a strategy employed already in the workshop or inventing 'new' ones. Thus everyone had experience of thinking through the pedagogic issues. Teaching sequences could be to teach us something new, or to look at some old knowledge in new ways. Tasks presented are left for readers to contemplate:

- Make up a problem whose answer is ${}_7C_3$ ('7 choose 3')—these were then accepted or rejected by the 'teacher' saying 'I like it' or 'I don't like it'.
- Choose a number between 9 and 19. Make triangles with integer sides, the lengths of which add up to that number, and another, and another ….
- Make up a number system which is based on some system of (cultural?) values.
- Ask learners for an example of a quadrilateral, then ask for one with a constraint that excludes the quadrilateral just drawn, then another, and so on. What is the longest string of examples which you can create using this structure and any class of objects?
- Give me a number bigger than 1; bigger than 2; … bigger than 17 000, …
- Construct 'greater than' and 'less than' statements about the numbers of pens each pair of people at a table of four have. Pass the statements to an adjacent table and they have to interpret the statements to put people in order of their possession of pens.

Finally, each participant constructed a statement about what had struck them about the content of the workshop. Many of these were restatements of the theoretical fragments reported above, so only a few further reflective thoughts will be summarised here:

The learners' choice of examples tells the teacher how they are classifying and what they have as prototypes (Participants referred to Women, Fire and Dangerous Things *by George Lakoff).*

Errors provide opportunity for growth in a safe environment.

LGEs are only useful if the teacher is working on strategies for reacting to and valuing what is generated, and how to handle surprises.

Yes-examples and no-examples are both of value, cf. concept-attainment.

LGE tasks give a way to move students 'out of the box', to be inventive, to feel safe while on the edge of not-knowing.

Asking for LGEs to be constrained in some way forces awareness (Gattegno).

Shifts from algebra to number, or number to algebra, change the nature of the exploration.

Sometimes LGE tasks cause one to look for particulars, sometimes for patterns and generalities.

LGE tasks are cognitive and constructive.

Epilogue

The workshop concluded with offering participants several additional tasks for exploration. These tasks are from Watson and Mason (2005, hopefully). We included some of them here for the readers' perusal.

Inter-Rootal Distance

We have decided to call the horizontal distance between neighbouring roots of a polynomial function the *inter-rootal distance*.

Imagine a quadratic equation with two real roots. What families of quadratic curves have the same inter-rootal distance?

Inter-Rootal Distance Constrained

Find three different examples of quadratics whose roots are 1 and 2.

Finding primes

Find a prime number which cannot be expressed as $4k \pm 1$ for any positive integer k.

Find a prime number which cannot be expressed as $6m \pm 1$ for any positive integer m.

Find a prime number which cannot be expressed as $8n \pm 1$ for any positive integer n.

Self-Perpendicular Hexagons

Draw a hexagon with each pair of opposite edges perpendicular to each other.

Acknowledgment

Many thanks to John Mason for scribing all the sessions.

Note

1. Not all LGE tasks need to be of this form—but we have found that it can be particularly fruitful. Look in Watson & Mason (forthcoming) and Bills, Bills, Watson and Mason (2004) for many more.

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Report Of Working Group B
Rapport du Groupe de travail B

Transition to University Mathematics

Tom O'Shea, *Simon Fraser University*
Peter Taylor, *Queen's University*

Participants

Malgorzata Dubiel	Ralph Mason	Graham Rankin
Joel Hillel	John Grant McLoughlin	Walter Whiteley
Tom Kieren	Mahdokht Naghibi	

Introduction

The description of the group as stated in the program was as follows:

Past CMESG working groups have examined a number of issues related to the content and pedagogy of lower-level university mathematics courses. In this year's working group we would like to explore how mathematicians and mathematics educators at the university level have created or might construct courses or mathematical experiences that would address the problem of the "transition to university mathematics." This might consist of innovations within traditional introductory university mathematics courses or the creation of new courses that attempt to paint a non-traditional but in fact much more authentic picture of mathematics. In a sense, the theme of the working group is the general question of what exactly should be the experience of students in their first encounter with a university math department. Participants are encouraged to bring to the discussion examples of topics, activities, or pedagogy they have used that might foster the transition. Or they may like to identify experiences or projects others have written about or developed that would be suitable to include in the proceedings of the workshop.

Day1

Peter initiated the discussion by circulating an open letter he had written to his colleagues which had just appeared in the MAA FOCUS. It outlined a series of changes he wanted to make in the first-year Arts and Science calculus and linear algebra courses at Queen's, making them more lively, more exploratory, discussion-oriented and problems-based. He pointed out there would have to be much less "content" under such an approach, at least measured in the traditional way. However he explained that "cutting out certain topics" was not the way to think about what needed to happen. Rather he outlined four essential steps in the process:

1. Let go of *all* the traditional content. At this stage nothing is sacred.
2. Get a firm hold on the big ideas of the subject, at least in your own mind.
3. Design or beg, borrow or steal a collection of "good" exploratory problems which embrace and do justice to this collection of big ideas.
4. Include (and insist on mastery of) the technical topics that support these problems. These define the (traditional) content of the course.

The word "good" in point 3 above of course needs to be understood. Or at least there needs to be an operational consensus among the teachers of the various sections of the course about the meaning of the word. Peter is wont to call them "works of art."

A number of points were raised in discussion. First there must be general agreement within the department, not only among the teachers of the course but among who will teach these students in future courses. Secondly those in other departments who will teach these students in future courses need to be comfortable with this approach and to honour it, where appropriate, in their own courses. The consensus was that colleagues would accept this provided the problems were interesting and of high quality. Finding these certainly constitutes the big challenge. Peter promised to give a progress report next year.

Joel described a "transition" course at Concordia focussed on mathematical thinking. An edited course outline (obtained from the Concordia website) follows:

Objectives:

This course is meant primarily for students who intend to pursue some concentration in mathematics or statistics at the university level. It has been the experience of our department that students often have difficulty with university level mathematics, even if they have done well in CEGEP and high school. University courses tend to be more theoretical and they require a particular language and style that is not familiar to the students. Many students find it difficult to know what the "rules of the game" are: why all these proofs?; what is it that you are expected to know about proofs?; how much emphasis should you put on the "how to do things" rather than on the "why something is true"? MAST 217: Introduction to Mathematical Thinking is not a course designed to teach you much of new mathematical content (although there will be new material in the course that you will be responsible for). The course's aim is to lay a foundation which will help you in all of the mathematics courses which you take at university. We hope to let you in on some of the "tools of the trade" of the mathematician. The topics that we will be discussing include: how proofs work and different styles of proofs, the difference between mathematical and everyday language and logic, the roles of examples and counter-examples, the transition from the finite and infinite, and different techniques of problem solving. The mathematics that will be used to illustrate the above topics will be based on the real number system, geometry, and functions. Most of this material will be familiar to you from CEGEP mathematics and it will be re-discussed in class to the extent that is needed.

Assignments & Portfolio:

You will be given regular assignments consisting of several problems each.
Your "portfolio" will include a selected set of 5 or 6 of the assigned problems - you will include in the portfolio your initial solution to each problem and any subsequent attempt at improving your solution.

Final Grade:

The grade in the course will be based on the following:

· 50% on a final exam
· 30% on two class tests
· 20% on the assignments and the "portfolio"

References:

Mathematical Thinking (by J.P. D'Angelo & D.B. West, Prentice-Hall (1997))
Doing Mathematics: an introduction to proofs and problem solving (by S. Galovich)
Mathematical Proofs (by A. Hallenberg)
Proofs Without Words (by R. Nelsen, Mathematical Association of America (1993))
How To Read and Do Proofs (by D. Solow, John Wiley & Sons (1990))
How To Prove It (by D.J. Velleman, Cambridge University Press (1994))

Joel commented on the value of the portfolio, and its feasibility if the number of students is around 20 to 25. He noted, however, that the workload associated with recent class sizes of 40 have made it necessary to be selective in choosing problems to place in the portfolio. There was general agreement that portfolios are desirable, but they can result in too heavy a workload (for example, not feasible for a class of 300 in a linear algebra class). Poster presentations were suggested as an alternative.

Joel reported that the curriculum committee at Concordia saw the need for a course like this, for general students. Because it was optional, it met little resistance from members of the mathematics department. As the course proved itself, the department agreed to move accept it into the mathematics major program and is now being pressed to accept it into the honours program. The course has become a good candidate for Concordia's "breadth requirement."

Ralph noted that the Consumer Mathematics 12 course in Manitoba high schools requires each student to present a portfolio and undergo an interview, and that this has proved to be very successful. The costs are about equivalent to setting and marking a traditional examination.

Walter wondered why a calendar can mandate course content, but not pedagogy, for example, open-ended problem solving. The group noted that we need to explain to students the constraints of the system, that is, we have exams to measure basic skills and knowledge, but we also need means to evaluate important objectives around understanding, appreciation, and problem solving.

Ralph described a course recently developed at the University of Manitoba. The course description, (subsequently obtained from the U of M website) is as follows:

136.102M Mathematics in Art (3)

Specific theory, structuring systems, and mathematical methods and principles used in works of art from various historical periods and contexts will be explored in relation to Euclidean and non-Euclidean geometries. Topics include: linear perspective; shapes, patterns, balance and symmetry; ratio, proportion and harmony; and order, dynamics and chaos. The course will be one half art and one half mathematics, team-taught by faculty from the School of Art and the Department of Mathematics. This course is also given in the School of Art as 054.102. No prerequisite.

Textbook: Lecture Notes available at the Bookstore

Ralph noted that he was on a committee to develop a course focussing on Mathematics and Literature that proposed to use the book "The Parrot's Theorem" as an organizer for the course.

Day 2

Tom O'S. began the session by describing an initiative taking place at Simon Fraser University. Relevant excerpts from the SFU document follow (continued on the next page):

Recommendation 1: New WQB Graduation Requirements

We recommend that the following University-wide graduation requirements be implemented for students admitted to SFU for the Fall 2006 semester:

- 6 credits of courses that foster writing abilities (W courses), including at least one course in the upper division, preferably within the student's discipline;
- 6 credits of courses that foster quantitative abilities (Q courses);
- 24 credits of breadth

To fulfill these requirements, students must obtain a grade of C- or better in the courses in question.

Definition

To qualify as Quantitative/Analytic (or 'Q' for short), a course must have either quantitative (numerical, geometric) or formal (deductive, probabilistic) reasoning as part of its primary subject matter, or make substantial use of such reasoning in practical problem solving, critical evaluation, or analysis.

Interpreting the Definition

Mathematics courses already required in Math, the Sciences, Engineering, Business Administration and Economics, and statistics courses required in Social Science programs clearly qualify as Q courses, as do the symbolic logic courses offered in Philosophy.

Courses currently offered in programs such as Engineering Science, Physics, Chemistry, Biology, Business, Economics and other Social Science programs that contain a significant math or stats component also would be eligible for Q designation.

A third type of course eligible for Q designation will be designed especially for students in the Humanities and Fine Arts. The goal of such courses will not be simply to nurture traditional math skills. Such courses will aspire to the greater challenge of deepening the understanding and appreciation of quantitative and formal reasoning, their ubiquitous utility, and their creative potential. We view such courses as focusing on the relation between (a) concepts and structures communicated through numbers and other systems of abstract representation (such as formal languages, programming languages, geometries, graphs) and (b) fostering students' ability to engage more effectively with the subject matter of their respective programs and practical everyday situations. Such courses need not focus primarily on quantitative or formal reasoning methods, but should give significant exercise to such techniques through model building and problem solving, both in class and in course assignments.

The floor was opened to a discussion about what the third type of course might contain and how it should best be taught.

Concern was expressed about the nature of students taking such a course, and the danger that students who have previously been exposed to university math courses may silence those who have not.

Joel mentioned a course at Concordia that had been developed in the spirit of the Q course notion. [A course description, along with an edited course outline, from the Concordia website follows:

MATH 215 - Great Ideas in Mathematics (3 credits)

Mathematics is used to unravel the secrets of nature. This course introduces students to the world of mathematical ideas and mathematical thinking. Without being overly technical, that is, without requiring any formal background from the student other than high school mathematics, the course delves into some of the great ideas of mathematics. The topics discussed range from the geometric results of the Ancient Greeks to the notion of infinity to more modern developments.

Textbook: *Journey Through Genius*, (by W. Dunham).

Content: The course will consist of three modules of approximately 4 weeks each.

Module 1: Euclidean Geometry
Module 2: Numbers
Module 3: Infinity

References in addition to the text references will be made available for each module.

Evaluation:

· Class participation - 15%

· Papers - 40%

We will require 2 papers of 5-6 pages each. They will be written individually on topics connected with two of the three modules. Each will count for 20%, and one will be due after 8 weeks, the other after 12 weeks.

· Assignments - 45%

For each module, the class will be divided into groups of 5 students. There will be 3 assignments per module, which will be a group effort with common grading for all members of a group. Each group will be responsible to write up and hand in each assignment and the written submission is to be signed by each group member. In addition each group should be prepared to discuss every assignment during class time reserved for this activity.

Tom Kieren recommended including notions of infinity. Walter mentioned the value of the book "Stories About Sets" by Naum Iakovlevich Vilenkin (translated from Russian, now out of print). Ralph recommended exploring Goldbach's conjecture as a rich topic for students. John argued to include problems with multiple solutions. Walter posed the question of what are the "big ideas" that should be included. Peter responded that it would be worthwhile to consult Nathalie Sinclair's recent PhD dissertation on aesthetics and mathematics. Malgorzata recommended combinatorics problems and coding. Walter wanted to build an understanding that models, paper-folding, etc., are not peripheral, but are vital, to mathematical thinking. Peter stated that technology should be a component of the course, naturally integrated. Walter insisted that groupwork was necessary. Peter described some of his Mathematics and Poetry course in which a take-home examination was submitted in groups, varying from 2 to 4 students.

The notion of proof was recommended, but participants differed in their interpretation of what this meant. Ralph, for example, wished to distinguish mathematical reasoning from proof or "proving." Walter wished that students would be required to pose questions about their assignments, perhaps in the form of "What if…?" Malgorzata suggested extended explanations of particular problems. John suggested the course might consist of four components: history, the development of numbers, problem solving, and biographies of mathematicians. In such a course students would present their ideas to peers, perhaps through posters. He cited his own research centred on the notion of "convenors."

Walter recommended a a lecture by Maurice Hirsch delivered to Howard University in 1993 called "Myths of Mathematics" to help convey the notion of mathematics as a human creation.

Malgorzata noted that Yale has recently prescribed "Q" courses that are similar to SFU. This led to a discussion of whether such courses should have a required content. Joel commented that Concordia's course does have content expectations. Peter felt that it was sufficient to build a course around four good problems, perhaps three weeks each, with a focus on deep understanding. John suggested that one should identify a theme that would characterize each such course, e.g., aesthetics. Walter suggested Geoff Week's book "The Shape of Space" (ISBN 0824707095). Ralph suggested Lynn Steen's "Why Numbers Count: Quantitative Literacy for Tomorrow's America (Literacy Series)" and that statistics be included.

John demurred and suggested that we should reconsider using any textbook for the course, arguing that it immediately implies content, and that the intent is to shape students' attitudes. Peter responded that we should focus on problems. There was general agreement that we should be more open and responsive to students' reactions. Walter suggested that by posing the questions: "What do you want to know by the end of the course? What will you be doing in five years?" we can see that prescribing a textbook would inhibit flexibility in designing appropriate activities. Joel commented that he was part of a study to determine how students viewed their future with regard to subjects being studies…students were clear for most subject, but not for mathematics. John mentioned that his experience with teaching tradesmen (e.g., carpenters) was very rich, implying the need to draw on the mathematical experiences of the students themselves. Walter suggested consulting Hoyles & Noss's work in England on mathematics used for professions. Tom K. cited his experiences with carpenters and pipefitters. John emphasized the importance of context. Walter recommended looking at the work of Michael Roth at the University of Victoria in which he noted how one interprets graphs depends on the background of the interpreter. Joel maintained that it was important to start with different experiences in mathematics from any previously experienced.

At this point, Graham summarized the variety of topics and concerns that had been discussed in the group, and observed that each one might take a whole session. He argued that we have to consider changes in the secondary school curriculum, and noted the reduction of content in Japan. We need to consider the different learning styles of students, and not try to fit one particular approach to all. John noted the existence of an education series in Halifax focussed on how high school teachers can make the transition easier.

Day 3

Peter suggested that the group might like to explore the issue of the "politics" within a university/department regarding the introduction of the kind of course considered over the previous two days. Walter commented on the need to take seriously in university teaching the idea that findings from mathematics education research are important , e.g., creating a teaching environment where conversation is encouraged. So the problem is how to create an atmosphere where the mathematics faculty member can construct courses in non-traditional ways. Joel mentioned the demographic problem in which no new faculty were hired for a number of years. This has resulted in few faculty members between the ages of 40 and 60, that is few people with tenure who have the inclination to teach such courses. Malgorzata suggested the key political issue is how to describe courses that do not have "content" in ways that are acceptable to others.

Peter pointed out that calculus and linear algebra are the *de facto* transition courses for most high school students in Canada. Joel suggested that since we cannot reconstruct these courses, there is a need for a prior course to serve as a transition. But Malgorzata argued that we can modify them suitably. Peter pushed the notion by saying that we can remove *all* of the content, replace it with "big ideas," and find good problems that involve those ideas. There was general agreement that this model would be appropriate for all courses. Ralph pointed out that there would no longer be <u>a</u> transition course. Malgorzata cited an example at the University of Wroclaw in Poland where they reconstructed an entire program because the old one was unsatisfactory. John noted that most students take only one mathematics course, and asked what is the "transition" course would transition to. It's really a terminal course. He cited McMaster's medical school and its problem-based curriculum as a model for students who specialize in mathematics.

John argued that students entering university now are different from in the past, for example, some now complete high school through distance learning. A university classroom environment may be a shock, and the problem is even more complex in thinking about "transition" for such students. Joel noted that the CEGEPs in Quebec were intended to provide transition experiences for students, but he has found their students still do not have confidence in their ability when taking university courses, and this suggests another kind of transition. John suggested that the universities may want to introduce a new type of teaching position, for example, a 5-year contract to teach courses. Malgorzata said it was better to replace sessionals with limited-term faculty as a way to ensure those hired are good teachers.

Peter proposed a model: 1st and 2nd year courses should be "light and lively." They should be problem-based using an enquiry approach, with reduced content. Those who need content will get it anyway, others will enjoy the experience. Will it work?

Tom K. said it would, based on his experience as a student in a linear algebra course, and on a course he and Ralph designed in the past. But there is no guarantee that colleagues will buy into it. John asked about colleagues outside the mathematics department, and what would be the expectations of incoming high school students.

The session wound down with a general discussion about what big ideas might be incorporated into such a course and what would be the role of technology and past experiences of students.

Postscript

(Several weeks after the meeting, Ralph Mason wrote the following response to a preliminary draft of the above report. His musings very effectively portray the dilemmas that we faced.)

Any need for courses that offer a transition to university mathematics suggests a gap between university students' actual initial readiness and the readiness expected of those students that is built into existing university mathematics courses. Such a gap could be addressed by altering students' readiness (changing high school math) or by altering expectations of university students (changing university math). But we didn't gather to discuss those options. Instead, we discussed what mathematics could be offered to bridge that gap between the two elements.

Clearly, the idea of a transition to university math suggests that we could implement something more effective than sudden immersion in university mathematics, as represented metaphorically by throwing children into the deep end of the swimming pool, and represented in practice by first-year linear algebra and calculus. But there are many ways to ease children's entries into the deep end of a swimming pool.

0. Don't worry about it. Allow kids to stay in the wading pool as long as they want, or take up canoeing or water-skiing whatever their deep-water skills.
1. Throw them in. But put them in life jackets first. They won't be able to swim in their life jackets, but they won't drown. They'll splash around and have fun, and eventually go to the edge of the pool and climb out. Ignore that they've never actually swum, or learned to swim. Everyone survived, and had a good time, and we can give them a certificate attesting to completing a Deep-End Experience.
2. Children have learned to climb ladders, and they know how to jump. Have them go up on the five-metre tower, and then prevent them from backing down. The only way to progress from there is by jumping in. (Some students may need to be prodded or even pushed, or perhaps just removed from the pool for their failure to enter the deep end.) The natural selection principles of sink or swim take over from there.
3. Children are safe in the shallow end. Direct them to wade in, and keep going. When wading no longer works, it's sink or swim again. Anyone who gets to the far end of the pool advances, and the rest aren't our concern.
4. We could teach them only to swim well enough to get to the side of the pool any time they needed to.
5. We could have shallow-end swimming competitions, and only let those who reach the finals in those competitions come into the deep end.
6. We could teach the children to tread water or train them in drown-proofing. Either way, they're learning behaviors that don't enable them to accomplish much in the deep end and don't lead to advanced activity such as water polo or scuba, but they'll be able to keep their heads above the surface.
7. We could teach them to enjoy swimming, and to enjoy getting better at it, all in the shallow end of the pool. Then we could move the lessons to the deep end.
8. We could teach them to enjoy swimming, and change the deep end of the pool to be a place where enjoyable things happen for those who get better at swimming. Recreational and commercial scuba diving, water polo, synchronized swimming, competitive swimming, pearl diving and surfing could provide contexts in which swimming (and getting better at it) could be purposeful, well-defined, and rewarding.

The analogies suggested by Choices 0 and 8 are out of bounds for our group. They involve changing the nature of undergraduate mathematics (in very different ways). And Choice 7 is out of bounds too. Its analogy suggests that students' prior experiences in the shallow end of the swimming pool (i.e., school, especially high school) could be made effective enough to obviate the need for transitional mathematics. Our job was to consider the role of special transition courses, not change school math or university math. To think about the potential nature of transitional math courses, let's look at the other options.

Choice 1 parallels the criticism of traditional mathematicians of efforts to design Math-and-Art classes for Fine Arts majors. Would persons who attain such a certificate be any more prepared for traditional university mathematics? Certainly their skill-sets and learning strategies would not be advanced significantly, but their confidence might have recovered from any past traumas. Perhaps it is misleading to encourage students to think that mathematics at university is safe and undemanding.

Choices 2 and 3 are analogies that suggest that students' prior experiences are inadequate or inappropriate. They are relatively passive options, differing only in the process by which students are inducted into university mathematics. Whether students are pushed into university math or enter it simply as a continuation of earlier experiences seems not wholly critical to the difficulties of students when they actually get into university mathematics. Although being good at climbing ladders or wading (in mathematics, the parallel could be arithmetic by symbol manipulation, or calculator button-pushing) can get children into the deep end, the skills aren't relevant to successful behavior in that environment—swimming (in mathematics, such skills might be – arguably – deductive reasoning and progressive abstraction). Do our transitional math courses need to teach people to think mathematically, beginning virtually from scratch because their pre-university experiences focused only on skills that lead students to that kind of experience?

Choice 4 implies survival math: the minimum amount of math necessary for adult life, but not nearly enough mathematics for learning additional content or complex skills that build upon mathematics. Whether or not there is a need for survival math, mathematical literacy in a minimalist sense of the term, the need is not one for university to address.

The analogy in Choice 5 implies a systemic, or institutional, response to having a majority of students unprepared for university math. Using a selection procedure would enable us to limit the number of persons who get in over their head. Could we offer transition mathematics for some of those whose scores suggest near but not complete readiness? For the educational design of transition courses for those not quite ready for university math, we must ask who we would want to select among the non-winners for entry into these courses: those whose technique is well-schooled but whose ability limited their achievement on the test, or those whose technique is flawed in such a way that coaching could make a difference, or those who have adequate technique and talent but not motivation, or the opposite. Interestingly, such distinctions tend not to be available to us when all we have is their swim-times from their last competition.

And that leaves us with only one choice left—and no, that choice doesn't approach our needs. In fact, I see Choice 6 to be reminiscent of traditional tutoring processes and remedial courses aiming to get marginal candidates to qualify for year-one university math. The students are taught to manipulate symbols and memorize techniques in isolation from the meanings of those symbols and techniques. It is a technically demanding set of capabilities they are taught, and it may result in test scores that grant access to first-year mathematics. But it's not mathematics, any more than drown-proofing is swimming. If such an instructional emphasis does offer an effective transition to university math courses, we would need to reconsider whether university math courses themselves are mathematics.

So where do we fit our working group's efforts to organize courses that serve a transitional role? I suggest that all our conversational moves that were truly transitional in intention can be considered to have something in common with the list of eight analogies offered above. If none of these analogies seem to offer a viable image of our intentions and our strategies, of course we must be ready to question the analogy-space itself. I would welcome alternative metaphors as positions for looking at our efforts from the outside. But the limitations of the swimming-pool analogies may be an invitation to question our premises. Can we presume that the problems that students have with university math are due to a deficit in the students' readiness, a deficit that could be addressed by more of what may be the problem itself? Can we not think outside of our tradition of practice when we are looking to achieve goals that our practices have never achieved, never espoused? I would characterize that tradition of practice as:

- the provision of sequenced abstract and symbolic mathematical content (rather than the collaborative development of mathematical experiences inclusive of the processes of abstracting and symbolizing)
- accepting an institutional intention of creating and identifying an academic elite who qualify for further courses and programs (rather than intending to create general success by the majority of students followed by selection of further courses in mathematics based on students' interest and identity).

In these last dichotomies I see ways to make sense of many of the well-intended elements of our conversation that didn't address our official goal of conceptualizing a transitional mathematics course for year one of university. And for assisting in the development of these dichotomies I feel intense gratitude to my conversation partners and (especially) the working group leaders. Perhaps we should look at Choices 7 and 8 in the above list after all.

L'intégration de l'application et de la modélisation dans les mathématiques au secondaire et au collégial

Integrating Applications and Modelling in Secondary and Postsecondary Mathematics

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This Working Group conducted its discussion both in French and in English. To reflect this fact, the report has been written using both languages.

Introduction

CMESG has previously hosted two Working Groups on applications; they were on 'Applications of mathematics for high school students' in 1980, and on 'Applied mathematics in the secondary school curriculum' in 1999. Their reports appear in the proceedings of these meetings. Internationally, there have been many conferences and meetings that had some or all of their focus on applications and modelling. The ICMEs (International Congresses on Mathematical Education) have hosted working and topic groups with this focus. Since 1983 ICTMA (International Conferences on the Teaching of Mathematical Modelling and Applications) has held biennial conferences. In February 2004, ICMI (International Commission on Mathematics Instruction) held its 14th Study entitled 'Applications and Modelling in Mathematics Education'. The discussion document (ref. 1) for this study contains an extensive bibliography. From the proceedings of these conferences and from other articles and books we now have a wealth of information on applications and modelling in mathematics education.

Nature, caractéristiques et visées de la modélisation mathématique

Le travail du groupe a démarré avec un échange à partir du document de discussion issu de l'Étude 14 de la CIEM. L'objectif était d'en arriver à une compréhension commune des concepts de *modélisation* et d'*application* avant d'envisager les implications de leur intégration dans l'enseignement des mathématiques. Ce premier échange, s'il était nécessaire, n'en fut pas pour autant suffisant : les discussions qui suivirent à travers les trois séances de travail nous ont régulièrement ramenés à préciser à nouveau ce que chacun d'entre nous entendait par l'emploi de ces deux mots. Cette section tente de synthétiser les idées principales qui ont émergé de ces discussions.

La simplification fut rapidement identifiée comme un des éléments-clés du processus de modélisation. Sur cet aspect, il convient de citer Blum et Niss (ref. 2) qui, tout en reconnaissant qu'il y a bien simplification dans la modélisation, mettent en garde contre une vision naïve du produit de cette simplification :

« The modeling process does not merely yield a simplified but true image of some part of a pre-existing reality. Rather, mathematical modeling also structures and creates a piece of reality, dependent on knowledge, intentions and interests of the problem solver. »

Cette réflexion peut conduire à questionner ce qu'on entend par *réalité* et en quoi cette vision peut affecter la représentation qu'on se fait de la modélisation et de l'application (voir la contribution de Stephen Campbell en Annexe B). Mais il reste que toute simplification dans une modélisation est fortement déterminée par le but pour lequel le modèle est construit et par les contraintes qui pèsent sur les ressources qui seront utilisées dans l'élaboration et l'utilisation du modèle.

Quelques diagrammes ont été proposés pour représenter le processus de modélisation. Celui proposé par Jacques Bélair (voir Annexe A) met en évidence le caractère itératif du processus, vite reconnu par le groupe comme un des attributs fondamentaux de la modélisation. On a même suggéré que le nombre de fois où le cycle est parcouru puisse être considéré comme une mesure du degré d'authenticité de l'application. Cette suggestion pourrait être vue comme une extension d'une idée présente dans le document de discussion du groupe d'étude 14 :

> If need be (and more often than not this is the case in 'really real' problem solving processes), the whole process has to be repeated with a modified or a totally different model.

De toute façon, l'idée de raffiner ou même de redéfinir le modèle à partir des résultats d'une validation avec la « réalité » n'est pas quelque chose de nouveau dans la littérature. On la retrouve dans les manuels classiques traitant de modélisation mathématique (ex. Bender, ref. 3), et elle est systématiquement présente dans les écrits d'aujourd'hui qui traitent de l'intégration de la modélisation dans l'enseignement des mathématiques. Par exemple, le diagramme suivant est tiré d'un livre de Dossey, Giordano, McCrone and Weir (ref. 4) qui s'adresse aux enseignants de mathématiques soucieux d'intégrer la modélisation dans leur classe :

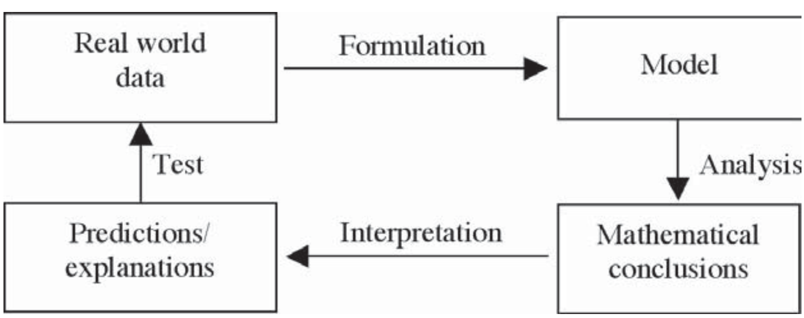

Figure 1. Le processus de modélisation comme système fermé (Dossey *et al.*, ref. 4)

Tout en préservant le caractère cyclique du processus, le schéma de Berry et Davies (ref. 5) prend pour origine l'énoncé du problème, tel qu'il se présente dans le monde réel, sans le lier au départ à la disponibilité ou à l'utilisation de données.

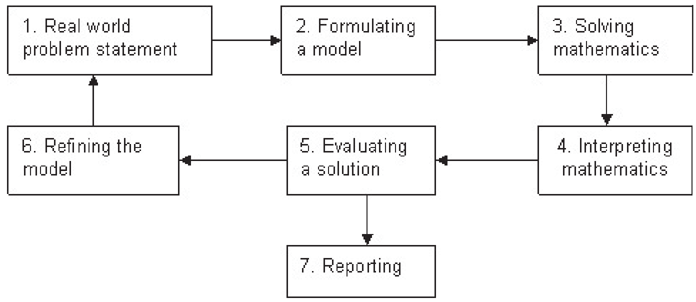

Figure 2. La modélisation comme processus cyclique (Berry and Davies, ref. 5)

La description donnée dans le document de discussion de l'Étude 14 va un peu plus loin dans cette direction en incluant dans le processus de modélisation la formulation du problème et la définition d'un *modèle réel* : ce modèle intermédiaire (non mathématisé) consiste en une simplification d'une situation-problème réelle en fonction des connaissances et intérêts de la personne en charge de sa résolution. Lors de cette étape, si cela est jugé pertinent, des données réelles sont recueillies et colligées pour obtenir plus d'information sur la situation. La décision de recueillir ou non des données, tout au moins initialement, semble liée au fait que la situation se prête à une *modélisation directe* (« forward modelling ») ou *inverse* (« inverse modelling »); dans l'Annexe B, Stephen Campbell apporte des précisions à ce sujet. Et ce serait seulement après que le problème et le modèle réel aient été définis qu'il conviendrait de se prononcer sur la faisabilité et l'adéquation d'un modèle mathématique. Et de le formuler, le cas échéant.

En établissant un parallèle avec le développement logiciel, certains membres du groupe ont fait valoir la nécessité d'une validation préliminaire du modèle mathématique avant d'aborder la résolution mathématique du problème ainsi modélisé. À nouveau, cette idée est bien documentée dans la littérature. Bender (ref. 3, p. 7) recommandait en effet de prendre à ce stade un peu de recul pour tester l'utilité éventuelle du modèle, en évaluant la disponibilité des données requises et la possibilité de les utiliser dans le modèle. Dossey *et al*. (ref. 4, p. 116) préconisent de vérifier que le modèle réponde bien au problème, qu'il soit utilisable sur le plan pratique et qu'il soit en accord avec le bon sens. Une réponse négative à l'un de ces tests commande la révision et la reformulation du modèle.

Le groupe s'est aussi penché sur la notion de *modèle mathématique*. Pour certains, un tel modèle correspond inévitablement à un système d'équations. D'autres étaient prêts à inclure les diagrammes, graphiques et autres représentations mathématiques. Et les plus « radicaux » d'entre nous proposaient d'étendre au matériel de manipulation la notion de modèle mathématique. Cette discussion nous a ramenés aux fonctions possibles du modèle : décrire, expliquer, prédire, établir une norme, élaborer, etc. D'un point de vue strictement fonctionnel, le matériel de manipulation pourrait bien mériter d'être reconnu comme une forme particulière de modèle puisqu'un tel matériel peut aider, lors de la résolution de certains problèmes, à décrire, expliquer et même prédire. Mais c'est la nature concrète de ce matériel qui semble l'exclure de la classe des modèles.

> A mathematical model is an abstract, simplified, mathematical construct related to a part of reality and created for a particular purpose" Bender (ref. 3). "A mathematical model of a situation is the result of a translation into mathematics of some of the objects, data, relations and conditions involved in the situation. (Blum *et al*., ref. 1)

Comme le matériel de manipulation peut difficilement être associé à un élément du langage mathématique ou à toute autre représentation abstraite (puisque sa raison d'être dans l'apprentissage des mathématiques est justement de concrétiser l'abstrait), il apparaît plus raisonnable de ne pas l'inclure dans la classe des modèles mathématiques. Néanmoins, puisqu'un tel matériel peut aider à combler le fossé conceptuel entre certaines situations réelles et leur formalisation dans le langage mathématique, le groupe a examiné sa contribution au développement de compétences de modélisation; les résultats sont rapportés plus loin.

Il a été plus simple d'arriver à une vision commune de la notion d'*application*. Spontanément, le groupe a rejeté tout problème qui ressemblait, de près ou de loin, à un déguisement ou à une mise en scène d'un problème purement mathématique, comme on en trouve souvent dans les manuels scolaires. Pour qu'une *situation-problème* puisse recevoir la dénomination d'*application*, elle devrait provenir d'un problème authentique, issu du monde réel, et pour lequel l'utilisation de mathématiques pourrait conduire à une solution. Et pour que cette application puisse s'inscrire dans le développement de compétences de modélisation chez l'élève, elle devrait lui laisser le soin de déterminer lui-même quelles mathématiques utiliser.

Modelling Apprenticeship

In its discussions the Working Group used the French word "apprentissage". In this report the term 'apprenticeship' rather than 'teaching and learning' is used because teachers may find it meaningful to view the modelling process within an apprenticeship context. Modelling often develops nonlinearly and in many cases the teacher is faced with situations that were not pre-planned. In such an apprenticeship, is it possible to become proficient in some of the activities independently of the others? Taking for example the schematic developed by Berry and Davies reproduced in the previous section, and looking in a regular mathematics class, one can certainly identify some of the activities in the schematic. There is much 'solving mathematics', some 'interpreting mathematics', some 'evaluating a solution', and some 'reporting'. However placing constant emphasis on these four activities at the expense of the others usually limits the scope of applications to ones which are artificially contrived to highlight specific mathematical concepts or techniques. The other activities 'real world problem statement', 'formulating a model' and 'refining the model' are generally absent. In the section on 'Manipulatives' we suggest that activities using these tools can provide students experiences in developing a problem statement, in formulating a model and in refining the model. Manipulatives can therefore offer an opportunity to introduce students to some parts of the modelling process starting at an early age. In the section on 'Technology' it is suggested that the availability of computers and other technological tools substantially open both the field of "real world" problems that can now be tackled by students and the set of modelling approaches that can be used. The view that it is possible to become proficient in some modelling processes independently of the others is put to question by Lesh and Yoon (ref. 6):

> In mathematics education research, it is common to assume that learning to solve 'real life' problems involves three steps:
>
> 1. First, students should learn prerequisite ideas and skills (in decontextualized situations).
> 2. Next, students should learn certain problem-solving processes & heuristics.
> 3. Finally (if time permits), students should learn to use the preceding ideas, skills processes and heuristics in messy 'real life' situations.
>
> Such views treat ideas, skills, heuristics, metacognition processes, values, attitudes, and beliefs as separate entities. In contrast, modelling perspectives consider models as including heuristics, metacognition processes, values, attitudes, and beliefs which are inseparable from the constructs and conceptual systems they embody, all of which develop in parallel and interactively.

The working group identified, deliberated, and contrasted three components of a modelling situation. The first was establishing the context of the problem, the second was seeking out the mathematics to solve the problem, and the third was validating the mathematical model and its results.

The context of the problem should be authentic and motivating, it can use data generated through in class experiments, produced by the teacher, or from other sources. The challenge is to find authentic situations which engage all the students. If the situation is too far removed from the knowledge and experience of a student, he may be turned off by its details or perceived complexity. Various suggestions for finding problem situations were put forward; seek situations that can be modified to the experiences of different students; outline a problem and get students to modify it to reflect their own situation; select news items where quantification and analysis need to be performed to reach published conclusions; choose problems that deal with extremes (smallest, fastest, most desirable, etc.) as these properties are an attraction for many students. It was suggested that minimizing the amount of information given to the student maximized the creativity that the student would apply to the situation. It is often the case that such rich modelling situations limit the level of mathematics used by the student. It is a quite a challenge to propose rich modelling situations where secondary students use and stretch their knowledge of mathematics. However, technology may remove some of these limitations at the secondary and post secondary levels. These concerns provided a focus on the second component, namely, seeking out the mathematics to solve the problem. In the majority of mathematics classroom situations the average student can guess the mathematics that can be used to solve a problem. Textbook and test problems reflect the contents of the chapter or chapters, even final examinations are used to test the applications of specific mathematical knowledge which is outlined by the course and the teacher. Students rarely find themselves in a situation where they are encouraged to seek different numerical and graphical representations, make assumptions to satisfy different mathematical formulations, and then suggest what mathematics may or may not be appropriate. Students are rarely exposed to situations where they need to communicate their assumptions, the mathematics they have used and the results they have obtained. A rich modelling situation will provide different mathematical possibilities, each with its sets of assumptions. Thus one should expect that different groups of students may produce alternative models for the same situation. As part of the validation process, students should be encouraged to compare the models produced. The teacher, as a member of the class and as the leader of the apprenticeship program, has the important role of explaining her choice of mathematical model while emphasizing the strengths and weaknesses of all the models, including her own. This is also an opportunity to bring new mathematics into play.

Because the group discussed validation in different contexts, it reappears in this section. Validation is central to modelling, it forces the modeller to return to the original problem, to look for inconsistencies in the results and to seek ways to redress them, to repeat the process, and if this is not possible to use data which is consistent with the original problem. The inconsistencies and limitations in the results may arise from, for example, a misinterpretation of the problem, too restrictive a set of conditions imposed by the mathematical formulation, approximations used in the solution process, etc. Validation also forces the modeller to look for bounds of reasonableness in the results. For the group, the use of additional data (historical or new) for validating a model also raised the question "Is consistency equivalent to validity?"

In summary the Working Group concluded that modelling situations require a revision of the role that mathematics is to play. Namely, modelling is not to be seen as an extension of mathematics but the 'real world' situation is to be central and mathematics is to be used as one of the ways to better understand the problem.

Les technologies et la modélisation mathématique

L'intégration des technologies dans la classe de mathématiques ouvre considérablement l'espace des problèmes accessibles à l'élève. D'une part, il lui est désormais possible d'avoir accès par internet à des bases de données authentiques (comme celles de Statistique Canada ou d'Environnement Canada) à l'intérieur desquelles il peut choisir une quantité d'informations pertinentes pour documenter les aspects de la situation réelle qui l'intéressent. D'autre part, il peut aussi compter sur des logiciels qui l'aideront à traiter cette masse de données.

L'expérience du cours « Mathematics for Data Management » qui est offert en douzième année en Ontario (voir la contribution de Chris Suurtamm et Geoff Roulet en Annexe C) montre bien qu'en misant sur des bases de données existantes et des logiciels de traitement statistique, il devient possible d'amener les étudiants à effectuer des recherches autour d'une question qui les préoccupe, à poser des hypothèses, à choisir, extraire et analyser des données réelles, et à valider leurs hypothèses en mettant à contribution les mathématiques qu'ils jugent nécessaires.

Dans ces conditions typiques d'une modélisation inverse, le développement du jugement critique dans la validation du modèle peut devenir un des enjeux principaux de l'apprentissage. En effet, le modèle le plus précis dans la façon dont il s'accorde aux données actuelles peut être le pire pour l'extrapolation ou la prédiction de nouveaux phénomènes. Cette prise de conscience devient essentielle quand on prend en compte, par exemple, la facilité avec laquelle il devient possible d'appliquer une régression : le cas rapporté d'une application d'une régression quadratique sur un ensemble de données liées au crime devrait se doubler d'un scepticisme lorsque le modèle ainsi construit amène à conclure à une éradication prochaine et totale du crime… et même à un nombre négatif de méfaits pour les années à venir !

Les outils technologiques peuvent aussi se révéler d'une grande richesse pour développer des capacités de modélisation directe. Celles-ci continuent d'avoir leur raison d'être, autant dans la résolution de problèmes d'optimisation que dans l'étude de phénomènes complexes. En effet, la complexité de certains phénomènes peut provenir de la combinaison sur un grand nombre de variables de forces, règles et relations relativement simples à décrire. Ces relations peuvent être appelées à évoluer dans le temps, en fonction d'événements discrets, de décisions ponctuelles, de délais dans les effets, de seuils d'activation, etc. L'approche par la simulation constitue alors une approche privilégiée pour appréhender à un niveau global l'évolution dans le temps du comportement d'un système dynamique à partir de la modélisation à un niveau local de ses multiples composantes et de leurs interactions mutuelles à travers les dynamiques qui les régissent.

Un logiciel comme Stella, avec son interface iconographique, se prête à une telle modélisation (voir Annexe D) : on le retrouve utilisé aussi bien dans des départements universitaires de physique ou de biologie que dans certaines écoles secondaires nord-américaines ou européennes. En utilisant une métaphore avec l'hydraulique, qui n'est pas toujours évidente pour le novice, ce logiciel ramène à un niveau plus concret l'abstraction qui caractérise la modélisation. Une fois surmontées les difficultés de familiarisation avec cette façon de décrire les relations, l'emploi du logiciel favoriserait l'identification de similitudes structurales entre des problèmes, qui ne sont pas toujours perceptibles dans leur formulation en langage symbolique (Doerr, ref. 7).

Si ce type de logiciel de simulation permet d'outiller l'étudiant dans la modélisation et la résolution de problèmes complexes, son intégration dans l'apprentissage des mathématiques n'est pas exempte de risques, comme l'ont fait remarquer les participants du groupe. D'une part, la discrétisation dans le temps peut demeurer implicite et même obscure pour l'étudiant. Une initiation par le tableur à la modélisation des systèmes dynamiques pourrait constituer un préalable intéressant (Kreith et Chakerian, ref. 8). D'autre part, en attribuant à l'outil la responsabilité de reconstituer le modèle du système à un niveau global, on semble décharger l'étudiant de cette vision globale. Mais peut-être convient-il ici de reconnaître que cette perte de vision globale est inhérente au traitement de la complexité; on pourrait y voir un parallèle avec le passage d'une programmation procédurale à une programmation orientée-objets.

D'ailleurs, certains auteurs (ex. Wilensky, ref. 9) reprochent à un logiciel comme Stella de coller de trop près, par l'utilisation d'équations aux différences et de variables décrivant un agrégat d'objets, à une modélisation classique par équations différentielles; ils soutiennent que le maintien d'une telle approche s'instaure en obstacle à l'utilisation du logiciel par des élèves. Ils leur préfèrent l'approche purement orientée-objets de logiciels comme StarLogo où l'on modélise directement au niveau de chacun des invididus d'une population (ou éléments d'un système), dans l'explicitation des actions et interactions possibles avec les autres composantes du système : cette approche serait plus intuitive et se prêterait mieux à une validation du modèle. Mais, ayant le défaut de ses qualités, elle s'éloigne du langage normalement utilisé dans la modélisation mathématique.

Évidemment, le choix des outils mis à contribution dépend des buts visés par l'intégration de la modélisation et de la simulation dans le cours de mathématiques, et, de façon plus fondamentale encore, des finalités associées au curriculum mathématique. On peut viser le développement d'une capacité à décrire un phénomène, à en expliciter les relations. On peut chercher à étendre des capacités de résolution de problèmes en intégrant de nouvelles approches pour identifier une solution ou prédire l'effet de certaines décisions. On peut aussi vouloir expliquer différents phénomènes en faisant ressortir les mécanismes et concepts sous-jacents. Ou on peut simplement chercher à donner sens aux mathématiques enseignées en leur associant un contexte d'émergence ou d'application.

Pour voir comment une même situation peut donner lieu, selon la technologie utilisée, à différents modèles qui permettent d'atteindre différents objectifs, le groupe a examiné un problème classique de minimisation de distance (voir Annexe E). Ce type de problème se retrouve typiquement au niveau collégial dans les cours de calcul différentiel et intégral, et un peu plus récemment, au niveau secondaire avec l'utilisation de la calculatrice graphique. Par son caractère parfait (route parfaitement rectiligne, aucune contrainte particulière), ce problème ne peut certes pas être associé à une application authentique. Mais il se prête quand même, et de façon intéressante, à différentes approches de modélisation, qui, selon l'outil technologique utilisé (calculatrice graphique, logiciel de calcul symbolique, logiciel de géométrie dynamique), feront faire appel à des concepts mathématiques différents. Ces approches ne sont pas équivalentes sur les plans de l'efficacité pour résoudre le problème, de la compréhension du phénomène sous-jacent, du potentiel de transfert à d'autres problèmes, de la complexité des concepts mathématiques mis à contribution, de l'appréciation de la tolérance de la solution optimale à la variation de la variable cherchée, etc. Mais prises conjointement, ces différentes approches peuvent s'éclairer mutuellement, et l'utilisation de la technologie facilite le passage de l'une à l'autre.

Finalement, la conception de « learning objects » (voir Annexe F) a été évoquée comme une occasion pour les étudiants en enseignement des mathématiques qui les élaborent de développer des compétences de modélisation et, par conséquent, d'apprendre à les repérer chez leurs futurs élèves.

Manipulatives

We use the term manipulatives to cover all types of tactile objects, games, and include simulation visual environments provided by technology. Manipulatives provide opportunities to experience some of the characteristics of forward modelling (see Campbell, Appendix B); more specifically they provide an opportunity to explore a situation, to develop a mathematical model, and to experience new mathematics. Campbell also points to the difficulties of the term 'real world' that appears in many modelling publications. An important property of a class manipulative is that once the teacher and the class have all played with a manipulative, they have common ground on which to develop ideas and models. This common ground is real for everyone. Both the teacher and the students can raise question to be investigated in that environment. When a question is raised everyone either understands it straight away or can return to the manipulative to get further understanding. This is the first stage of any mathematical modelling scenario, posing and/or understanding a real world problem statement. Most manipulatives used in a mathematics classroom are introduced to assist in the students understanding of new mathematics or for the reinforcement of known mathematical concepts. Thus the building of mathematical models is central to the use of manipulatives. A good mathematics teacher will ensure that every student makes the transitions from the manipulative through various stages to the mathematics. Van den Heuvel-Panhuizen (ref. 10) provides details of evolving situations of the bar model that students are exposed to in learning percentages. The author is careful to order the shifts in understanding that the students must make to develop their understanding of percentages.

From the point of view of modelling, manipulatives have their limitations. In general they can be used to illustrate only one or two mathematical concepts and they rarely provide the range of experiences found in more general mathematical modelling. However they are widely available in elementary mathematics classroom and can provide experiences of modelling activities that are not normally part of their mathematical experiences. The group also discussed manipulatives as a means for students to acquire new mathematical language. Certainly these situations allow students to move from the concrete to the abstract and vice versa. The challenge for the student is to develop the ability to leave the concrete situation and to work solely within the abstract mathematical language.

The transition from arithmetic to algebra involves the development of a new symbolic language, and a generalized way of thinking. Zack and Reid (ref. 11), have described their students use of centi-cubes to explore, develop, and visualize concepts of symbolic notation, generalization and algebra. Certain manipulatives elicit more mathematical modelling than others. For example, the Brock Bugs game reported in the CMESG Proceedings (ref. 12) challenges students to find mathematical models that provide winning strategies. Developing a card simulation of the well known Car and Goats problem (see Appendix G) requires the identification of the appropriate data for a decision model. An important property of these two manipulatives is that students can access them at different mathematical levels.

In conclusion, teachers with little modelling experience or who do not see the benefits of modelling are prepared to use manipulatives in their mathematics classroom as they can match the mathematics content to the curriculum. Such manipulative experiences do provide useful apprenticeship in some of the mathematical modelling components that are often not part of the student's mathematics classroom experience.

Conclusions

Participants in this working group brought a whole range of experiences and knowledge in applications and modelling. The focus of the discussions was different on each of the three days, and it ranged from the philosophical to the practical. We hope that, because of the scope of topics, all participants came away with new questions, points of view, and understanding. The discussions and examples of the use of technology addressed important and up to date issues in the teaching and learning of applications and modelling in secondary and post secondary mathematics.

Annexe / Appendix A

Modeling in Mathematical Biology: More than Solving Equations

Jacques Bélair, *Université de Montréal* and *McGill University*

It has been said that as much as physics dominated the sciences in the first half of the 20th century, so will biology will be the dominant science of the 21st century. This dominance will be achieved, in part, by an unprecedently high penetration of mathematical techniques in the development of the discipline. These techniques, in turn, will be often introduced in the curriculum in a modelling context, especially for the audience of biologists.

In the absence of an elegant law "of motion" like Newton's in classical mechanics, much more emphasis is placed on the derivation of the model, most often, but not always, taking the form of a systems of equations. In the diagram below, which represents the mathematical modelling process, much effort has to be invested in the horizontal arrow on top, where the translation from biology to mathematics is performed. Solving the equations, either numerically or qualitatively (most often both, but only rarely analytically), occurs entirely within the realm of mathematics, which lies on the right-hand side of the diagram.

The feedback depicted in the box refers to the comparison of the mathematical results with both the original data or experiment *and* the confrontation of these results with the mathematical derivation. This feedback is often presented as one of the defining features of the field of Systems Biology: this iterative comparison of mathematical results leading to new experiments and refinements in the model, at a pace accelerated by technological advances, is what makes the field so promising in the road to an understanding of the *dynamics* behind the Human Genome.

This iteration has been present in Mathematical Biology, for quite some time, on the study of systems on a different anatomical scale. An illustration is provided by the series of papers listed in the bibliography, where progressively more sophisticated models of blood cell regulation are derived and analysed.

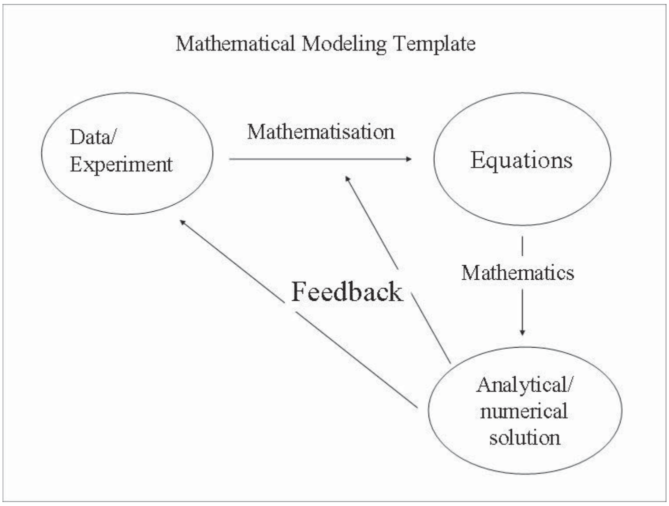

Figure A-1. Schematic representation of the mathematical modelling process

The "real world" is on the left-hand side; all mathematics takes place on the right-hand side. The current challenge in biology, and the fundamental difficulty of the modelling process, involves the horizontal interactions, first in the derivation of the model, and in the feedback process from the solution to the data and the refinement of the model. A more detailed discussion of these topics can be found in the bibliography (refs. 13–17).

Annexe / Appendix B

Forward and Inverse Modelling : An Introduction

Stephen R. Campbell, *Simon Fraser University*

The discussion document for the ICMI Study Group 14 on "Applications and Modelling…" offers that phrase "to denote any relations whatsoever between the real world and mathematics." Of course, we would all like to think that we know what is meant by "real world" and "mathematics." But do we? These matters are not as straightforward as they may seem. Here it will suffice to note that there are differing philosophical frameworks as to the nature of the world and mathematics, and what they tell us about our own human nature in relation to the world. What is real, and what is ideal? For some, what is real is what exists independently of our experience, the real world itself, while for others, what is real is just the very lived reality of our experience. For some, what is ideal are the ideas that we have about the world, while for others, mathematical ideas that can capture the essence of the world speak to mathematical realities that are deeper than the world as it appears to us. The meaning of mathematical modelling and applications are bound up with these different points of view. I have discussed these kinds of philosophical issues in greater detail elsewhere (e.g., Campbell, refs. 18–20). Rather than try to further articulate and contribute to this dialectic between the real and ideal from a philosophical perspective, my contribution to the ICMI Study Group was to explore this issue in terms more familiar to professional modellers: forward and inverse modelling.

Oddly, to date, this fundamental distinction between forward and inverse modelling has been virtually absent in the research literature on mathematical modelling and applications in mathematics education. One way to raise the bar in this area would be to become apprentices to the profession, or at least become familiar with the discourse.

Basically, forward modelling is the process of using an idealised version of the world, based on our understanding of that world and limited by our ability to recreate or to mathematise it, to simulate potential manifestations of the world within certain restricting conditions. Forward modelling can be manifested physically, as in the case of physical models, say model airplanes, and the efficacy of such physical models tested in conditions simulating the purpose such a model is intended to resolve, say determining drag parameters with respect to velocity and wing shape in a wind-tunnel. With regard to idealised mathematical models, simulations of real world conditions take the form of computation. The efficacy of the simulation (typically a process) and the model (typically an object) is evaluated by comparing the outcome of the simulation with the reality the model is specifically intended to emulate or explore. Sometime idealised models and simulations using them predict the real observed behaviour of the world with remarkable accuracy, as is the case with quantum electrodynamics. So much so, in fact, that one may be tempted to suggest that the idealised mathematical model is itself the true reality governing the world and/or our experience of it.

Unfortunately, there are many areas of the world and human experience where our mathematical understanding falls short. That is to say, we are unable to forward model all the kinds of situations and solve all the kinds of problems that we would like. Put yet another way, we are lacking in fundamental principles from which we can accurately deduce all of our observations of the world in all contexts that may serve our purposes. In such cases, inverse modelling comes to the rescue. Inverse modelling is the process of inverting data obtained from actual real world observations into idealised mathematical models that can potentially account for those observations. Because of entropy, inverse modelling appears to be purely mathematical, as there is no physical counterpart to it, as is the case with forward modelling. We can inverse model the conditions that led to spilt milk, based on measurements of the resultant splashes, but we cannot physically, so far as we know, run such an experiment in reverse. A more sophisticated example of inverse modelling is to try and determine the location and magnitude of source dipole generators within the brain that are postulated to give rise to measurements of voltage differentials on the scalp.

Linear algebra provides an exemplary way to illustrate the differences and relations between forward and inverse modelling. Let a vector $\mathbf{x}$ represent fixed or constrained input parameters of the model. Let $\mathbf{A}$ represent a coefficient matrix of the dynamic equations governing/constraining the processes operating upon the model (these equations, in turn constitute a mathematical model of our understanding of the physical processes involved). Then forward modelling can be seen as deterministically computing a unique solution to the change of state $\mathbf{b}$ of the model as follows:

$$\mathbf{Ax} = \mathbf{b}$$

Inverse modelling requires interfacing with unconstrained and empirically derived data, typically acquired through observations and measurements of one kind or another. In this approach one does not have a unique predefined model of some aspect of the world x, rather one would like to *derive* such a model on the basis of empirical data $\mathbf{b}$ acquired from the world by observation and measurement. In this case one inverts the *same* dynamic equations governing the system, $\mathbf{A}^{-1}$, and then *attempts* to calculate:

$$\mathbf{A}^{-1}\mathbf{b} = \mathbf{x}$$

A major difficulty here, referred to as "the inverse problem," is that all measurements contain a degree of inaccuracy and noise, so that in actuality, $\mathbf{b} = \mathbf{b'} + \mathbf{e} + \mathbf{n}$, where $\mathbf{b'}$ represents ideal measurements, e represents the intrinsic error within the actual measurements and $\mathbf{n}$ represents noise. A further problem is associated with inverse modelling. As is usually the case, the number of possible models that can account for sparse and imprecise data is infinite, and, in some cases, subsets of model solutions can vary significantly. In such cases, context and problem dependent heuristics, typically based on unquantified or qualitative knowledge and experience of the modeller, can serve to prune less plausible possibilities.

Aforementioned differences and relations between forward and inverse modelling are summarily expressed in the following table.

Forward Modelling	Inverse Modelling
Ideal => Real	Real => Ideal
Physical and Computational	Computational Only (Entropy)
Theory-Driven	Data-Driven
$\mathbf{Ax = b}$	$\mathbf{A^{-1}b = x}$, where $\mathbf{b = b' + e + n}$
Deterministic	Stochastic

A more simplistic and perhaps more accessible way to get the general idea of how forward and inverse modelling approaches can be similar and yet differ is to consider the problem of fitting a polynomial equation to a set of data points. The forward modelling approach would be to start with a polynomial that seemed theoretically reasonable, and evaluate how well that polynomial modelled the given points. The inverse modelling approach would be to start with the data points, and derive a polynomial using those points. Compare, for instance, the forward process of using a line to generate a synthetic set of data points with the inverse process of using an actual set of data points to produce a(n optimal) line running through them. Linear algebra works quite well here.

See (Campbell, ref. 21), for further articulation and discussion of the distinction between forward and inverse modelling, particularly with regard to previous literature from research in mathematical applications and modelling in mathematics education.

Annexe / Appendix C

Applications and Modelling in Ontario

Chris Suurtamm, *University of Ottawa*
Geoff Roulet, *Queen's University*

During the discussion at the CMESG working group on modelling, several definitions of modelling were discussed. In the discussion in this paper, we will lean on a definition of modelling presented as the following sequence of steps: problem identification, production of a simplified real model, translation into mathematics, manipulation of the mathematical model, re-interpretation and testing in the real setting, and possibly refinement through a repeat of the cycle. This description of modelling mirrors and expands upon schemes presented by others arguing for mathematical modelling as part of the school curriculum (Blum & Niss, ref. 22; Galbraith, ref. 23; Hersee, ref. 24; Swetz & Hartzler, ref. 25) and specifically agrees with the definition of modelling presented at the recent ICMI-Study 14: Applications and Modelling in Mathematics Education (ref. 1) in Dortmund, Germany.

In a paper presented at the ICMI Study 14, we suggested that secondary school mathematics curriculum documents in Ontario have, for the past two decades, called for classroom practices involving both modelling and applications (Roulet & Suurtamm, ref. 26). The paper outlined our observations, based on on-going research, of teachers' interpretations and implementation of the applications and mathematical modelling themes of this new curriculum. It was evident that, in incorporating mathematical modelling in the classroom, teachers' conceptions of the discipline greatly influence the materials they choose to use and how they employ them. In fact, many teachers have difficulty implementing mathematical modelling as it does not necessarily fit with their view of mathematics or with their experiences of learning mathematics.

However, we also noted that teachers appeared to be adopting and implementing a modelling approach in a new Grade 12 course, Mathematics for Data Management (ref. 27, p. 48). This course requires students to participate in a culminating project in which they investigate a problem. In this course, students define a problem that they would like to investigate; determine how they would investigate the problem; collect or find data that will help them address the problem; analyze the data and present the findings. Examples of topics that students generate include relationships between: literacy and fertility rates; the amount of money spent on baseball teams' salaries and their ranking in the playoffs; and education spending and national wealth (Brock

University). Statistics is explored in much more depth than in previous Ontario curricula along with new topics such as: iteration, database organization, flowcharting, graph theory, networks, and coding. The expectation is that students will be engaged in extended tasks that involve the manipulation of "large bodies of information" (ref. 27, p. 49) supported by the use of technology. Ontario mathematics curricula had never before presented such large tasks to students or required teachers to think about how to support their pupils as they tackle such wide-ranging projects. This course received extensive support through the creation of resources and professional development opportunities (Brock University, ref. 28; Dalrymple & Dilena, ref. 29; Roulet, ref.h 30; Statistics Canada, ref. 31).

Although adoption of mathematical modelling in other secondary courses may appear patchy, there is strong evidence that teachers are adopting mathematical modelling within the Mathematics of Data Management course. There may be several reasons for this, many stemming from the challenges that this course poses for teachers. These challenges include course content that is new to teachers, the management of assessment through a culminating problem, use of a new software package, Fathom™, to manage and analyze data; and a course structure that is unlike other math courses that they have taught. To prepare for teaching the course, teachers explored content that was unfamiliar to them through workshops and learning environments that modelled the types of activities suggested for use in the classroom. Teachers therefore experienced first hand the value of learning and teaching through modelling and experienced the construction of new knowledge by engaging in their own investigations. This construction and the challenge of exploring questions for which there can be no definitive answers presented an opportunity for both students and teachers to grapple with a new conception of mathematics. We argue that the adoption of mathematical modelling as a central curriculum theme requires activities that allow teachers to personally experience modelling in unfamiliar content domains. Our attendance at the ICMI-Study 14 conference brought us in contact with other research that supported our argument (Makar & Confrey, ref. 32). Our attendance at the Conference also helped us gain a new perspective on Ontario's position along the continuum of the implementation of modelling. From the research presented on mathematical modelling at ICMI-Study 14, it appears that modelling is occurring in very isolated cases in classrooms around the world. In many jurisdictions, modelling is not evident in the curriculum, textbooks, or resources. Large-scale assessment that tests facts and procedures through multiple choice questions also hinders the adoption of modelling. However, in Ontario, modelling is embedded as a system-wide focus in secondary school mathematics education. Modelling, presented in the curriculum as the core activity of the inquiry process by which mathematics is to be developed, occurs, to some degree, in all classrooms in the province. We recognized that this level of support for modelling was rare.

Because of this unique position of Ontario, we proposed to write a paper on Modelling in Ontario for the upcoming ICMI-Study 14 Volume. The paper has been accepted and presents a discussion of Applications and Modelling in the Ontario curriculum and in Ontario classrooms (Suurtamm & Roulet, ref. 33). We suggest that several factors have helped to support the development and implementation of modelling. Since 1980 the vision of a modelling curriculum has been shared and nurtured by both the leadership of the mathematics education community and the Ministry of Education of the province. The present curriculum, issued in 1999/2000, is the product of leading mathematics educators in the province who share this common vision and have enacted it in their classrooms. Positive progress on both the official intended curriculum and implemented curriculum has been possible because of the synergy that comes from the common vision of these progressive educators that is imbedded in curriculum documents, and is supported by the provision of technology, professional development and resources. Naturally, the process of implementing a modelling curriculum has met with some obstacles. For instance, not all teachers share this vision and do not readily accept the full implementation of modelling, but rather incorporate only some limited application activities. Obstacles such as this are being addressed in a variety of ways. Our discussion describes the "levers" that made and continue to make the development and implementation of a modelling curriculum possible and the "barriers" and obstacles that have been faced and are being addressed (Burkhardt, ref. 34). We hope that the messages in our research are valuable to all jurisdictions seeking to implement a systemic focus on modelling and applications in mathematics.

Annexe / Appendix D

Modélisation de systèmes dynamiques avec Stella

Le logiciel Stella, dont la première version remonte à 1985 (ref. 35), permet de modéliser et d'étudier par la simulation les systèmes dynamiques qui caractérisent plusieurs problématiques contemporaines (Hannon et Ruth, ref. 36). La modélisation se fait par la construction progressive d'un diagramme où interviennent quatre types d'objets :

· des *réservoirs* pour représenter la quantité ou la grandeur associée à une variable;
· des *flux* (ou pompes) qui vident ou remplissent les réservoirs, représentant le taux de variation dans le temps d'une variable en fonction d'autres variables;
· des *convertisseurs* pour représenter une constante ou paramètre ou une autre variable qui affecte ou est affectée par d'autres composantes du système;
· des *connecteurs* qui relient les différents objets pour spécifier les échanges d'information (entrées, sorties).

Ces différents objets servent à représenter aussi bien des systèmes mécaniques ou électriques que biologiques, écologiques ou économiques. Les relations entre les variables peuvent être exprimée à partir d'équations (souvent simples) ou à partir de tables de valeurs.

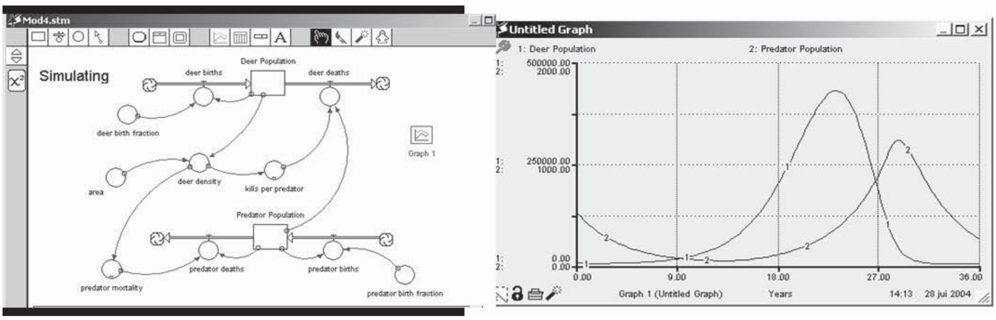

Figure D-1. Modélisation et simulation avec Stella d'un système proie-prédateur

Chaque sous-système réservoir-pompe agit comme intégrateur numérique par rapport au temps. En combinant l'interaction dans le temps de ces différents sous-systèmes, le logiciel permet de simuler le comportement du système et d'en générer les graphiques souhaités. Et il se prête aisément à l'ajustement itératif du modèle, par la sélection et la modification des différents objets ou des liens qui les unissent. La modification peut inclure l'ajout ou la suppression d'objets, le changement de direction des connecteurs, ou la modification des formules et tables.

Annexe / Appendix E

Un Même Problème, Différents Modèles, Différents Apports

Énoncé du problème : « Vous organisez l'une des étapes d'un rallye qui va de la ville G à la ville H. Différents postes de ravitaillement (essence, nourriture, etc.) sont placés le long de la route VW. Vous devez obligatoirement passer par un de ces postes. Pour minimiser la distance à parcourir, à quelle distance du point V le poste devrait-il se situer ? »

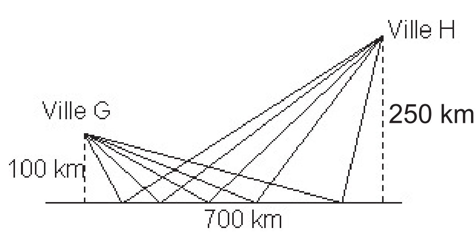

Figure E-1. Représentation du problème

L'approche par le calcul différentiel amène à déterminer la distance x qui minimise

$$\sqrt{100^2 + x^2} + \sqrt{250^2 + (700 - x)^2}$$

et donc à résoudre, en passant par la dérivée :

$$\frac{x}{\sqrt{100^2 + x^2}} + \frac{(700 - x)}{\sqrt{250^2 + (700 - x)^2}} = 0$$

Après quelques manipulations algébriques (ou l'utilisation d'un système de calcul formel), on trouve x = 200. Mais ce que l'équation comme modèle ne communique pas clairement, c'est la sensibilité de la distance totale à une variation légère de x. Car, selon l'énoncé, il faut passer par un des postes de ravitaillement. Or rien ne nous assure qu'il y en ait un (ou qu'il soit possible d'en avoir un) exactement à x = 200.

Ici, un passage par une représentation graphique de la fonction à minimiser (avec une calculatrice graphique ou un tableur) s'avère plus éclairant.

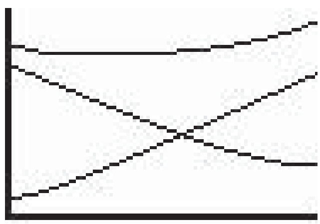

Figure E-2. Représentation de la fonction à minimiser sur une calculatrice graphique

La courbe du haut, représentant la somme des deux fonctions irrationnelles, montre une relative stabilité dans un intervalle important autour du minimum. Sur le plan pratique, on pourrait ainsi considérer comme solution adéquate toute valeur x située dans l'intervalle [150, 250].

Une exploration avec un logiciel de géométrie dynamique permet d'arriver aux mêmes conclusions en affichant les mesures ou en les reportant sur un graphique.

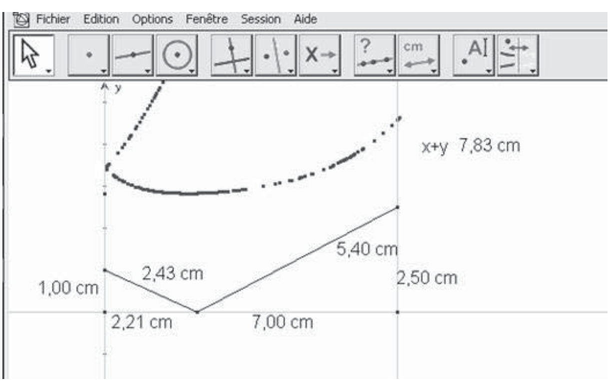

Figure E-3. Construction de la fonction à minimiser avec Cabri Géomètre

Mais là où une modélisation avec un tel logiciel s'avère nettement plus intéressante, c'est lorsqu'elle permet, par l'exploration dynamique du problème, de faire ressortir des invariants ou des propriétés géométriques qui dirigent vers la solution ou permettent de l'*expliquer*. Pour ce problème, l'introduction du symétrique OH' du segment OH par rapport à l'axe de la route, permet d'identifier la position optimale du point O cherché comme celle qui fait en sorte que G, O et H' soient alignés.

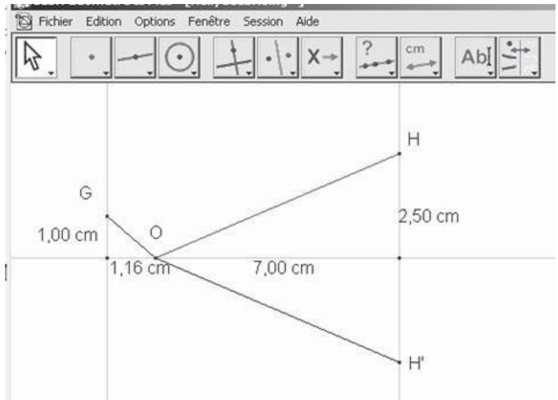

Figure E-4. Exploration géométrique avec Cabri Géomètre

À partir de là, la solution peut être trouvée par un simple recours au modèle proportionnel appliqué sur les triangles semblables : $\frac{x}{100} = \frac{700 - x}{250}$.

Annexe / Appendix F

Mathematics Learning Objects and Other Rich Modelling Experiences

Four years ago the Department of Mathematics at Brock University implemented an innovative core mathematics program called MICA (Mathematics Integrating Computers and Applications). A summary of the initiative can be found in Brock Teaching Journal (ref. 37). In the first year MICA course (Math 1P40) students develop an interactive computer based project. Mathematics majors normally select a mathematics problem that can be explored using technology. Future mathematics teachers normally choose to develop a Mathematics Learning Object. Wiley (ref. 38) provides the following description of learning objects "the fundamental idea behind learning objects: instructional designers can build small (relative to the size of an entire course) instructional components that can be reused a number of times in different learning contexts. Additionally, learning objects are generally understood to be digital entities deliverable over the Internet, meaning that any number of people can access and use them simultaneously (as opposed to traditional instructional media, such as an overhead or video tape, which can only exist in one place at a time)." At present Math 1P40 students use Visual Basic (ref. 39) as the programming language and the result of their work is interactive but is provided on CD ROM and not on the Web. Early projects using Java are available on the web (ref. 40)

The student projects in Math 1P40 have all the properties of a rich modelling experience. The students select a problem of their choice. They develop a model to meet their objective. They program the model. This involves revision of the model and of their objectives, new programming, etc., thereby cycling through different loops many times, testing and retesting, validating again and again. Programming is unforgiving, the logic and the program must be correct to achieve ones goal. Not only does each student project speak (the instructor can run the CD) for itself, but the students produce a report outlining the project's strengths weaknesses, the original aims, the modified aims and how these are met in the final program. Faculty are amazed at the time and effort students are willing to devote to their project. Projects undertaken by some future teachers start to address questions such as, what is the best interactive activity to introduce the mathematical concept, what are the best activities to reinforce understanding, how does one develop computer based testing of mathematical concepts, etc. etc?

In the summer of 2003 and under an EDUSOURCE (ref. 41) and other Brock University grants, a group of professors and students was assembled to develop mathematics learning objects. The aim was to focus on a mathematical concept and to develop an engaging computer environment where students could learn the concept. The group met with a number of elementary and secondary school teachers to generate a list of mathematical concepts their students were having difficulty with and also to explore their willingness to use them in class. Mathematical topics were selected and the mathematics students (future teachers), developed a Power Point presentation that outlined in detail each of the screens that were to be programmed. Some of the general guidelines for each Learning Object were, the user must never be in doubt what to do next, the number of words on each screen was to be minimized, the number of screens were to be small (less than 15), the mathematics must be developed in a fun environment, etc. The presentation was made to the whole group, changes were suggested and a new presentation date set. Once the group had accepted the development it was given to a student computer programmer. From that stage on changes were not to be undertaken unless the programming was much harder than anticipated. There is a tendency to always want to improve and this is a major reason for incomplete objects. This way the group was able to achieve more than originally projected. The results of this project (ref. 42) are six mathematics learning objects at different levels of education. For the future mathematics teacher, the development of a learning object had all the components of rich modelling experiences.

Annexe / Appendix G

Modelling the Car and Goats Problem

The Car and Goats problem also referred to as the Monty Hall problem (see for example Barbeau (ref. 43)) can be developed as a manipulative and used as an exercise of the first few characteristics of a modelling situation. My experience is that, with the proper introduction, the problem engages every student in the class. A simple form of the problem is, "A winner of a game show is asked to select one of three doors behind which there is a prize. Behind one door there is a car, behind the other two a goat. The winner selects a door. The game show host then opens one of the other doors that reveals a goat and invites the winner to change her choice of doors. Should the winner change her choice to the other unopened door?"

Here are my suggestions for developing the problem in a class setting. For young students, we start with a class discussion about what we understand by a simulation. As most of the students have played computer games, they already have a superficial understanding of the concept. Young students also need to experience that results of a simulation stabilize with increasing repetitions. To do this students are placed in pairs and given an identical spinnor, designed with a number different sized pie shapes, each coloured uniquely. We agree that, when the spinnor is spun many times, the colour with the largest pie shape is likely to occur more often, and that the occurrence of a colour will be in proportion with the size of the pie shape (How do we measure size?). Our aim is to generate some quantitative value for each of the colours. Students spin the spinnor a number of times, record their results on a class spreadsheet, and are asked to reflect on their results and the results of the class as a whole. The class now has the necessary background to explore a simulation of the Car and Goats problem. Each pair of students is given three cards, one with a picture of a car on its face and two with pictures of a goat. Each pair of students describes, in writing, what they would do to simulate the game show and to advise the winner. Should she stick with her choice of door or is it worthwhile to switch to the other door? Having discussed their proposed simulation with the teacher they try to perform it. In most cases students return with a modified simulation, they found an error, a problem, or a better way to do it. The students note their changes. Data from similar models are recorded together on the class spreadsheet. Different models may produce data that do not agree with the rest. The class now validates the models and discusses their results. For some students the results are counter intuitive. It is certainly the case for most university students who are also required to develop a probability model. I have modified this activity for presentation to a large number of mathematics teachers in a conference setting. Divided into groups they were supplied with a dice each with a different decision mechanisms (for example, 1 you change your choice of door otherwise you do not change, or, 1 and 2 you change otherwise you don't). The data provided intermediate results between the two previous alternatives, namely that of never changing and that of always changing. Using a spreadsheet the results were plotted as a function of the decision probability. When graphed they provided interesting insights. A search on the web will reveal a number of Internet sites that explore this problem.

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Report Of Working Group D
Rapport du Groupe de travail D

Elementary Teacher Education: Defining the Crucial Experiences

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Jean Dionne, *Université Laval*

Participants

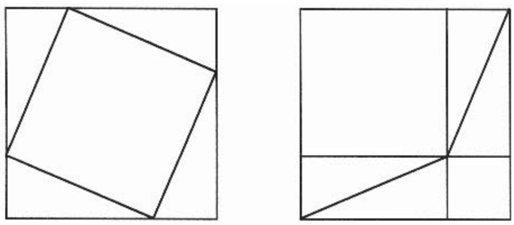
Renée Caron	Leo Jonker	Luis Radford
Natasha Davidson	Kathy Kuboto-Zarivnij	Gladys Sterenberg
Brent Davis	Gisele Lemogne	Laurent Theis
Florence Glanfield	Steve Mazerolle	Steve Thornton
Bernard Hodgson	Cynthia Nicol	

The goal of this working group was to find a way to define experiences that might be crucial for pre-service teachers in the limited contact time available in most elementary education programs. We did not wish to generate an exhaustive list of desirable features of such programs, which has been done in previous working groups. While such lists may be intellectually satisfying, frustration may result in their application due to lack of time, inadequate student preparation in mathematics, poor attitudes to mathematics, and so on. Rather what we hoped to attain was a framework for choosing the experiences to be provided to teachers to help us decide which ideas might be most important to include in mathematics pedagogy or curriculum courses. As well we hoped to discuss what types of problems might be most suitable to create such desirable experiences.

Participants were asked to bring along some personal vignettes of experiences they have had teaching such courses: one successful and one not so successful. This sharing of our personal experiences was used to begin our discussion, in an effort to begin to frame our ideas and ground them in our own experiences.

Beginning with comments about bad experiences, a number of examples were shared. Natasha cited students being proud of their poor numeracy skills and her frustration over being unable initially to influence this. Claire also commented that it was difficult to convince students that their own mathematical knowledge was important in teaching. Bernard commented that it is important to have special mathematics courses for education students rather than their taking regular university math courses. A number of participants in the group taught such courses and agreed with this comment. Kathy cited a project in which a mathematician was involved in the practicum experience, but noted the problem of continuity; after the project ended the level of discourse deteriorated. She questioned how continued support from school boards can be encouraged. Ann gave an example in which an in-service Grade 5 teacher was unable to recognize a correct answer that was framed by a student in a sophisticated way and, even in dialog with Ann, unwilling to agree that a different format of the answer (a pattern rule) might be correct. Claire continued with related ideas, giving examples of students weak in a given topic such as decimals who were greatly helped in their understanding in their education courses, but later in higher level courses such as algebra were not able to continue and extend their new understandings. She also discussed a student who helped her neighbour with her new-found knowledge of fractions, but very procedurally. The student did not remember *how* she was taught, she only remembered the procedures. She valued and thus taught only the final outcome of her own learning activities. Kathy commented also that perpetuating the interest in developing deeper mathematical understanding at the in-service level was important. In summary, many of the negative experiences had to do with inadequate teacher mathematics knowledge, or valuing procedural over conceptual understanding.

When the group began sharing positive experiences, some common themes also emerged. Florence explained that her most exciting moments came when the students questioned ideas themselves and thus took charge of the curriculum themselves by bringing out questions and examples. Steve shared examples in which the students took pride and ownership of their activities. Leo discussed the success students had going out into the schools and doing enrichment—knowing they have to teach it encourages an interest in deeper understanding. Steve added that *really* understanding feels totally different. Jean gave an example with measuring and the creation of new units in which collectively the group 'invented' decimal numbers. Such experiences encourage students to realize that they know more than they think they do, and that they really can do mathematics. A very strong example came from Bernard in which he shared the following diagrams:



Bernard explained that he began a mathematics course by simply drawing the diagrams on the board and saying nothing. After a period of uncertainty students begin playing with the diagrams, and eventually the idea of Pythagorean Theorem emerged. Other group members applauded this degree of courage in setting the standard for student self direction! This agrees with Brent's conviction that the answer is present in the collective and that we only need to encourage the ideas to emerge.

Deepening the mathematical knowledge of teachers was seen as an important aspect. In summary, some of the important initial ideas that emerged from the early discussion were that teachers do know some math but they need convincing that they do and pride in themselves, and that many of the ideas are already present either in themselves or in the group.

Moving into small groups, we began to challenge the central question of the working group: *How can we define crucial experiences in elementary mathematics education?* Many thoughts emerged from this discussion which group members shared on the board. The following text box shows some of the ideas generated.

confidence – knowing when you know and when you don't

able to ask – curiosity

math as a way of reflecting on the world

simple knowledge may be rich with many representations

sustainability - develop communities

translation between different symbolic representations

ask – get answers – more questions – define a project

procedural knowledge vs depth

ability to access - understanding of & ability with elementary math

effectively participate in a production of knowledge

independence creativity

commitment to student initiated problems and solutions

generalization of experience

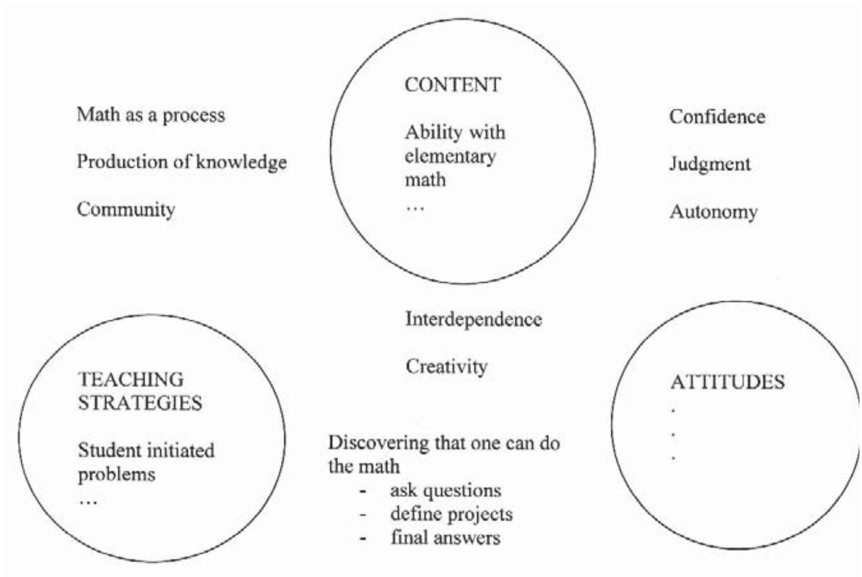
math as a process

math as a cultural matter

math as a community of practices

judgment – understanding – confidence – knowing when to doubt

We ended the first day's discussion with this collection of thoughts, which Jean and Ann attempted to group as a way to frame the discussion to follow. The diagram below was our initial grouping of the attributes of a crucial experience.



Steve then shared with us the Australian *Standards for Excellence for Teachers* which contained the dimensions of "Professional Knowledge," "Professional Attributes," and "Professional Practice." We noticed a parallel between what we wanted the outcomes of our crucial experiences to be, and qualities of excellent teachers.

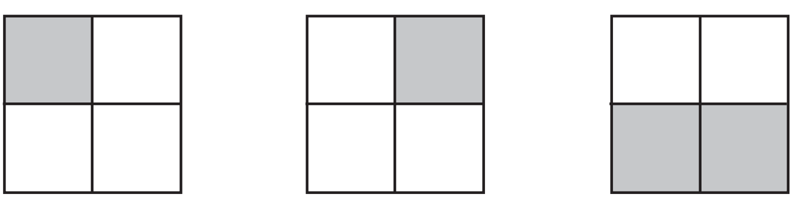
It was important to push farther than defining the outcomes of these experiences; we wanted to define the experiences themselves more exactly. Pushing ahead with the notion that what we wanted was present in the collective, group members again broke into small groups with the challenge of creating a particular example of a classroom idea for teachers that might create a crucial experience. As some of these examples might be of interest for classroom use, an outline of them follows.

Examples Generated by Small Groups

Incomplete Patterns

The goal of this activity was to emphasize the importance of one's rationale for an answer over the answer itself. Creativity and communication were goals, as well as having to learn about teaching by trying to understand the rationales of other participants' solutions. It was also hoped that attitudes were involved as the activity could be engaging and de-emphasizes the concept of a "right answer."

The instructor generates an incomplete pattern. For example,



The students are asked to individually come up with a possible next three figures in the pattern, *with rationale.* The students then exchange their proposed diagrams with another group member who must try to determine the reasoning used to generate the second set of three items in the pattern, and continue the pattern with three more, again with rationale. Afterwards, group members share their reasoning.

This activity was successful in the working group setting, with three very different patterns emerging and differing rationales.

Understanding Means

In this activity, a situation is proposed in which a student is asked to find the mean of a list of numbers and he does so (numerically). The answer is correct. The question addressed to the student teachers for discussion is “Does this student understand the concept?” The idea we want to be discussed is does the student really know what is happening—that is, could they do it with boxes of pencils without calculation. To augment the discussion, students teachers could be given boxes of pencils (include a full box, an empty box, and several partially filled) and ask them to discuss and show what the concept of mean really is. In this activity, it is hoped that students gain a deeper notion of what is meant by conceptual versus procedural understanding.

Magic Number Games

Another group presented some ideas for games with numbers, to focus on teacher attitudes, and overcome their avoidance of math. For example, various magic tricks were suggested, such as with a penny and dime. The teacher leaves the room and a student hides one coin in each hand—the class decides which coin in which hand. Then the teacher is invited back in and is allowed to ask questions involving math operations to decide which coin is where. For example, “multiply the number in your right hand by …,” and so on. Then the students play with a partner. The game can be expanded to use other coins.

Stories

The use of contextual stories was discussed by a group, and several examples were presented. Having pre-service teachers create their own stories was also thought to be valuable. For example, the example of “Lucy and the Licorice Factory” was given, in which machines are available to create licorice of lengths one to six. But then the length six machine breaks. The question is, what other machines can be used to make several pieces which can be attached together to make length six? The question can be expanded to ask what are the *primary* machines needed to make any length up to six?

The book *Oral Story Telling in Mathematics* by Michael Shapiro was mentioned by Florence as a great resource.

Many Answers to Fundamental Questions

In the final example, content was used as the starting point. This group felt that the other two aspects, which we have loosely dubbed attitudes and pedagogy, fall into place when students are engaged powerfully in the content. It was suggested to use a question such as “What is multiplication?” or “Why can’t you divide by zero?” as a starting point. Several methods were suggested to promote thinking about the question. Students could be asked to make a poster to explain their answer, and then share these. They could discuss how the carious ranges of interpretations in the posters come into the curriculum and when. Alternately, they could be asked to generate three answers to the question, such as by using the internet, asking parents or other places to generate a richer answer.

•••

Brent summarized his current criteria for a good task to the group as variable entry, open-routed, and open-ended. We felt that many of the examples given achieved these ideals.

Using the examples, the group then set about more formally to attempt to define what crucial experiences are, and to identify the conditions under which such experiences could occur. Participants were challenged to think about these two questions overnight and, on the third day, we began to collect and organize everyone's thoughts. Initially, we sought to gather everyone's ideas on the blackboard in hopes that some common themes and a frame would emerge. Some of the ideas follow.

The question was raised as to whether beliefs and attitudes arise naturally, or whether they need to be made explicit. For example, teachers might benefit from explicitly thinking about how questioning or another particular method helps them to understand a concept better or more deeply. It is also important that students realize they do actually have something to learn, as in the example of the means and the pencil boxes.

Group members had several comments about the activities themselves and the mathematics involved. It was felt that tasks must be rich enough in mathematics and be authentic mathematical activities. Students should have some control of the activity—and an important balance exists between authoring and authority. Student teachers themselves should experience the positions of both student and teacher, and the education class must be a pedagogical model of later lessons. Activities should be framed to be variable entry, open-routed, and have the potential for surprise. There should be some sense of an appropriate product.

Separate from the crucial experiences are a list of "must-do" items of a different nature. These are things like negotiating types and formats of lesson plans, as might be required for practicum experiences, and preparing students to negotiate the reality of the school experience. The idea of a "linear" lesson plan could be contrasted with the idea of lesson "preparation" which is less linear. While such things are necessary, they were separated out from the group's evolving notion of a crucial experience.

The issue of mathematical content involved in crucial experiences was also discussed. Several group members mentioned that research is beginning in terms of "watershed concepts," that is the idea that understanding certain critical mathematical ideas such as place value, operation, and generalization deeply can be extremely important to teachers.

Looking at the idea of important content more deeply, we discussed whether a common subset of important mathematical content could be agreed upon. After each participant created a short list, we were in fact able to group our content ideas. These included arithmetic operations and relationships such as the figurative substrate of number, approximation, mental computation, equivalence, part-whole relationships and rational numbers, and place value. The ideas of space and continuity brought in the need for a sense of geometry and space and the operations on them, the ideas of constancy, change and movement, and the concept that numbers can mean different things. The importance of mathematics as problem solving and the ability to translate fluently among different representations was included. Also the connection between concrete materials and algebraic formulations as in moving from a pattern to a general formulation was agreed to be important. Finally, the sense of mathematics as a human endeavor that involves kinship, stories, seeing the forest through the trees, and just plain fun was included.

At this point, our definition of crucial experiences had evolved to include two main components: watershed concepts and what we termed "authentic mathematical activities." The criteria for the latter include personal meaning, creation of new understanding, and experiencing mathematics as a process.

An improved framing of our final understanding of crucial experiences was proposed by Steve to include three main areas as in the list below. We feel this summary accurately conveys the main ideas of the understanding attained by our group.

Crucial Experiences in Elementary Mathematics Teacher Education

Watershed Concepts

... such as ...

- figurative substrate of number;
- place value;
- equivalence;
- continuous/discrete.

Authentic Mathematical Activities

... including ...

- creation of knowledge from collective;
- trust in one's own mathematical understanding;
- problematization;
- process (mathematics as something that has to be *done*).

Meta-Ideas

... including ...

- relationships (kinship);
- communication (telling stories);
- generalization (forest through the trees).

Throughout our discussion, resources were mentioned to support our creation of teacher education classrooms rich with crucial experiences. We thank Florence for providing most of these ideas. Suggested resources include:

- the NCTM video clips of classrooms, under "Reflections";
- The TIMMS videos;
- The Randal Phillip video *Lenses on Learning* available through the Pearson Professional Development Catalog.

Report of Working Group E
Rapport du Groupe de travail E

La technologie dans l'enseignement des mathématiques: regard critique sur le discours et la pratique

A Critical Look at the Language and Practice of Mathematics Education Technology

André Boileau, *Université du Québec à Montréal*
Geoffrey Roulet, *Queen's University*

Participants

André Boileau
Olive Chapman
Jean-François Maheux
Patricia Marchand
Geoffrey Roulet

Introduction

> I have suggested that the absence of a suitable technology has been a principle cause of the past stagnation of thinking about education. The emergence first of large computers and now of the microcomputer has removed this cause of stagnation. (Papert, 1980, p. 186)

In the decade and a half since the publishing of Papert's *Mindstorms: Children, Computers, and Powerful Ideas*, computers have become a common physical presence in Canada's classrooms (Bussière, et al., 2001, Mullis et al., 2000). The initial excitement, and in some cases trepidation, that met their arrival appears to have subsided. The computer's slide into the background of schooling and the relatively high comfort level with respect to information technology felt by mathematics education researchers may be partially responsible for the small number who came together to discuss issues around mathematics education technology. Are there any significant technology-centred issues left to debate?

When data gathering shifts from counting computers to reporting actual usage in mathematics lessons a somewhat different picture emerges. Information technology has become pervasive in our daily lives, but computer and calculator use in Canadian mathematics lessons is much less frequent (Mullis et al., 1998, 2000). The impact of computers has not reached the levels imagined in the early 1980s. Papert, along with his prediction, provided a caveat.

> The computer by itself cannot change the existing institutional assumptions that separate scientist from educator, technologist from humanist. Nor can it change assumptions about whether science for the people is a matter of packaging and delivery or a proper area of serious research. To do any of these things will require deliberate action of a kind that could, in principle, have happened in the past, before computers existed. But it did not happen. (Papert, 1980, p. 189)

It would appear that Papert provided an appropriate warning. Significant assumptions and issues concerning technology and mathematics teaching and learning continue to remain open for examination. During our nine hours together the members of Working Group E explored a variety of these concerns.

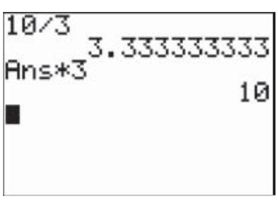
Wishing to ground discussions on real classroom experience, we initiated the conversation concerning issues by looking at examples of student use of computers and graphing calculators in the learning of mathematics. Each example surfaced one or more issues which the group then isolated and explored. In what follows we have attempted to present some of our deliberations, providing descriptions of the information technology applications, the issues involved, and our conversations concerning these.

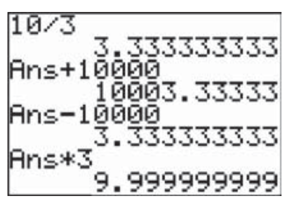
Persuasion versus Compulsion

Over the past two decades, professional associations for mathematics teachers have issued documents recommending the use of computers and calculators in teaching and learning. In some cases provincial curriculum documents have weakly echoed this call. “Alternative strategies for teaching and learning mathematics using the computer should be explored” (Ontario Ministry of Education, 1985, p. 20). The writing team for Ontario’s new curriculum took a much stronger approach and issued draft courses with multiple compulsory applications of computers and graphing calculators. Although the political process of obtaining Ministry of Education approval resulted in a reduction in the number of cited applications, the new Grades 9 and 10 curriculum still contains 76 references to information technology including many that make it clear that computer and calculator utilities must be employed. For example, in Grade 10 students are to “determine some properties of similar triangles (e.g., the correspondence and equality of angles, the ratio of corresponding sides) through investigation, using dynamic geometry software” (Ontario Ministry of Education, 1999, p. 31). These curriculum demands have been accompanied by increased support in the form of funding for the purchase of graphing calculators and provincial licences for computer software. Teachers are definitely making increased efforts to employ technology (Lock, 2001), but does compulsory adoption always lead to productive classroom experiences? Teacher’s clearly need support to understand the potential impact of technology in their classrooms and how to go beyond rather trivial applications to meet curriculum dictates.

L’utilisation de la technologie n’est pas transparente

On entend souvent dire que l’utilisation de la technologie peut nous aider à sauver du temps. C’est parfois vrai, mais il ne faut pas oublier qu’une utilisation éclairée nécessite aussi qu’on lui consacre du temps. Par exemple, le recours à la calculatrice nous permet de faire nos calculs aisément et rapidement, mais il nous faut connaître un peu son fonctionnement interne si on veut expliquer certains phénomènes, comme celui illustré par les écrans ci-dessous d’une calculatrice TI-83 Plus :





L’écran de gauche nous montre que, lors de la division de 10 par 3, la calculatrice affiche un résultat inexact (bien que d’une grande précision) : c’est compréhensible, car on ne peut « stocker » ou afficher une infinité de décimales. Mais comment expliquer que, lorsqu’on multiplie ce résultat par 3, on retrouve apparemment un résultat exact? Et si, entre la division et la multiplication, on ajoute puis on retranche un nombre comme 10000, le résultat final n’est plus exact (voir l’écran de droite) : comment expliquer tout ceci? Pour bien comprendre ce qui se passe, il faut connaître comment les nombres sont représentés dans une telle calculatrice (représentation dite « en virgule flottante »), et comment se font les calculs et l’affichage...

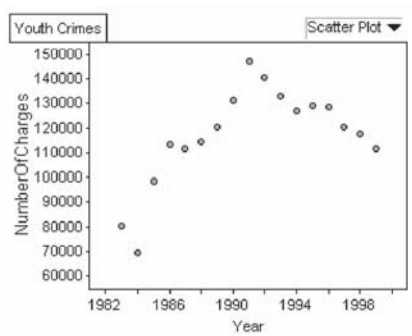
Pursuing the Doable

Once an official curriculum demands the use of information technology there arises a need to provide support sufficient to ensure that all teachers can and will take the necessary implementation steps. This can lead to "cookbook" style directions for teachers and students with accompanying truncated learning opportunities.

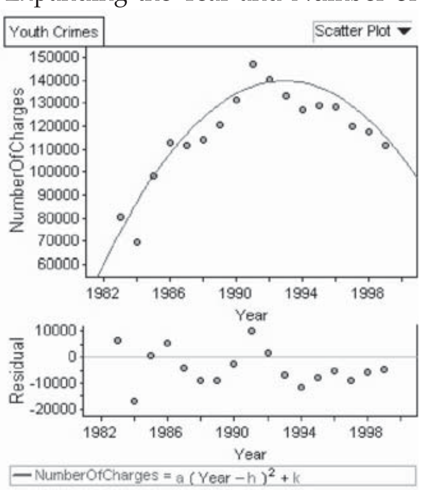
In Ontario, Grade 10 students, while studying quadratic functions, are expected to "collect data that may be represented by quadratic functions, from secondary sources (e.g., the Internet, Statistics Canada) ... fit the equation of a quadratic function to a scatter plot ... and compare the results with the equation of a curve of best fit produced by using graphing calculators or graphing software" (Ontario Ministry of Education, 1999, p. 26). The Fathom software (Key Curriculum Press, 2002), provided for all schools in the province, makes this task relatively simple and the *Parabola Power* (Professional Development Center, Key Curriculum Press, 2001) provides teachers and students with detailed instructions. Students are directed to capture the data for 1983-1999 youth crime using the Statistics Canada E-Stat facility and to produce a scatter plot.

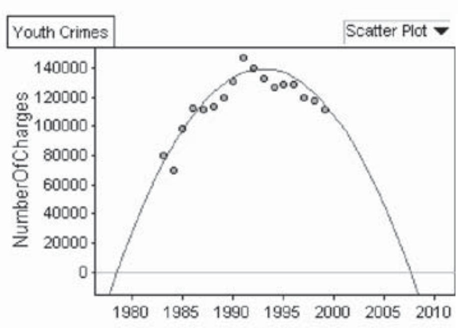
Youth Crimes

	Year	NumberOfCharges
1	1983	80525
2	1984	69448
3	1985	98271
4	1986	113027
5	1987	111731
6	1988	114079
7	1989	120277
8	1990	131155
9	1991	146906
10	1992	140378
11	1993	132983
12	1994	127095
13	1995	128809
14	1996	128542
15	1997	120208
16	1998	117542
17	1999	111474



The scatter plot does have a somewhat parabolic pattern and the activity instructions continue with: "The number of youth crimes goes up and comes back down in a roughly symmetrical way. One way to quantify this variation is to fit a parabola through the points" (Professional Development Center, Key Curriculum Press, 2001, p. 3). Following this advice students plot and adjust a parabola and find the quadratic function of best fit. But, why a quadratic model? Is there an underlying reason to postulate a second order relationship between year and the number of youth crimes? The activity instructions do not raise such questions. Students are not asked to explore and critique this model. Posing the simple question, "What was the state of youth crime prior to 1983?", leads to an obvious problem. Expanding the Year and Number of Charges limits of the graphing window shows that prior to1978 the number of youth crimes should have been negative. Similarly in about 2007 a negative number of youth crimes can be predicted. What interpretation is to be given to negative counts of crime?





Le statut des mathématiques expérimentales dans l'enseignement

Dans l'enseignement traditionnel, un énoncé mathématique ne pouvait être que vrai ou faux, et un énoncé vrai ne pouvait qu'avoir été démontré ou non. L'arrivée de la technologie a rendu non seulement possible mais facile l'utilisation de techniques expérimentales en mathématiques, avec pour conséquence une diversification du statut des énoncés mathématiques : un énoncé mathématique peut maintenant être « fort probablement vrai », car « confirmé par une vérification expérimentale sérieuse ». Les logiciels de géométrie dynamique constituent une illustration fort éloquente de ce phénomène : des élèves encore fort peu habiles à produire des démonstrations peuvent néanmoins se prononcer avec conviction et de façon généralement exacte sur la véracité d'énoncés en géométrie élémentaire.

Les enseignants doivent cependant identifier clairement le statut des énoncés mathématiques, ne serait-ce que pour les deux raisons suivantes:

- Si une étude expérimentale peut apporter une certitude raisonnable, seule une démonstration peut mener à une compréhension véritable. Il y a lieu de sensibiliser les élèves à ce phénomène.
- Même une expérimentation sérieuse et fort convaincante peut nous mener à des erreurs, et il est très difficile de prévoir quand un tel phénomène va se produire.

Nous illustrerons ce deuxième point par l'exemple suivant, basé sur une suite définie par récurrence comme suit (Muller, 1989):

$$s_{n+2} = 111 - \frac{1130}{s_{n+1}} + \frac{3000}{s_n s_{n+1}} \; (avec\ s_0 = \tfrac{11}{2}\ et\ s_1 = \tfrac{61}{11})$$

Si nous utilisons une calculatrice, un chiffrier, ou tout autre outil technologique utilisant une représentation en virgule flottante des nombres, pour calculer les premiers termes de cette suite, nous aurons une « certitude raisonnable » que cette suite converge vers 100. Pour illustrer ceci, nous voyons ce qui se passe quand nous demandons à Maple de calculer, avec une précision de 100 chiffres décimaux, les 150 premiers termes de cette suite :

```
5.500000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
5.545454545454545454545454545454545454545454545454545454545454545454545454545454545454545454545454545
5.59016393442622950819672131147540983606557377049180327868852459016393442622950819672131147540983598
5.63343108504398826979472140762463343108504398826979472140762463343108504398826979472140762463342953
5.67464862051015096304008328995314940135346173867777199375325351379489849036959916710046850598643780
...
100.0000000000000000000000000000000000000000000000000000000000000000000146146978998573673279429204756
100.0000000000000000000000000000000000000000000000000000000000000000000008768818739902286196803321910
100.0000000000000000000000000000000000000000000000000000000000000000000000526129124393530461810077796
100.0000000000000000000000000000000000000000000000000000000000000000000000031567747463581492208698592
100.0000000000000000000000000000000000000000000000000000000000000000000000001894064847813372757526612
100.0000000000000000000000000000000000000000000000000000000000000000000000000113643890868726526701832
100.0000000000000000000000000000000000000000000000000000000000000000000000000006818633452119799664622
100.0000000000000000000000000000000000000000000000000000000000000000000000000000409118007126998383003
100.0000000000000000000000000000000000000000000000000000000000000000000000000000024547080427610423136
100.0000000000000000000000000000000000000000000000000000000000000000000000000000001472824825656151396
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Par contre, nous pouvons demander à Maple de faire les calculs de façon exacte, en utilisant une représentation fractionnaire, et d'afficher à la fin le résultat sous forme décimale (toujours avec 100 chiffres):

```
5.500000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
5.545454545454545454545454545454545454545454545454545454545454545454545454545454545454545454545454545
5.59016393442622950819672131147540983606557377049180327868852459016393442622950819672131147540983606 6
5.6334310850439882697947214076246334310850439882697947214076246334310850439882697947214076246334310 85
5.67464862051015096304008328995314940135346173867777199375325351379489849036959916710046850598646538 3
...
5.99999999999178465978288101272082323447730035444895616648463132052489445934400179830757737983746340 8
5.99999999999315388315239147007084115302824764704236932593492341971992064274304954361798403974270307 0
5.99999999999429490262698638209849385490767701148406619311864651970297701693100061126493315183904458 9
5.99999999999524575218915079784096215716856166012410461007325441565853067507225988324717183113077248 5
5.99999999999603812682428919224632305009309956720486297739488657016524533941647158725307789079466033 7
5.99999999999669843902023881347762192685459416721337424247528965553013964083982373009823497191418565 0
5.99999999999724869918353083063344839909495553050553819033012400343414364274273398484362264115984480 1
5.99999999999770724931960797418673717877509555819754996670391406483103425748142102271893706668413958 9
5.99999999999808937443300591505760286563118806510033801719353837744005323906566627240720535819212316 9
5.99999999999840781202750442220216110293834815147479982391905739556756046668430686819797679808283230 2
```

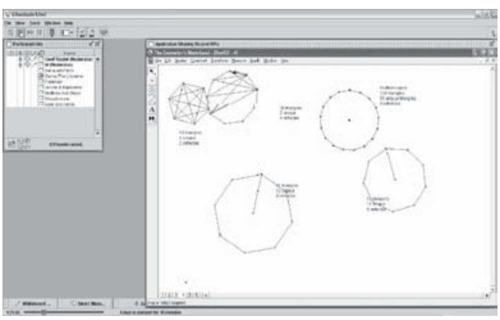
On voit maintenant que la suite semble croître vers 6, mais nous sommes devenus plus méfiants : une simple vérification expérimentale, si poussée soit-elle, n'est plus suffisante pour nous convaincre de quoi que ce soit à propos de cette suite. On peut cependant continuer d'utiliser la technologie (*Maple*, en l'occurrence) pour chercher à mieux connaître cette suite et émettre diverses conjectures. De cette façon, on pourra arriver à énoncer la conjecture

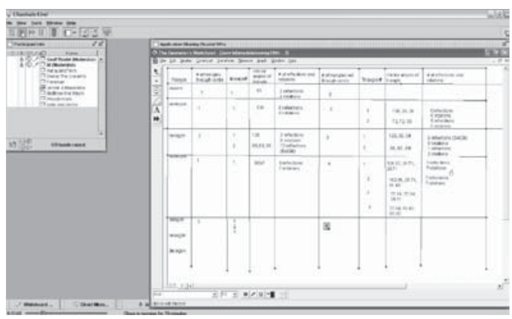
$$s_n = \frac{5^{n+1} + 6^{n+1}}{5^n + 6^n}$$

qui pourra ensuite être démontrée par induction, ce qui nous permettra finalement de prouver que la suite tend bien vers 6. Subséquemment, on pourra chercher à comprendre plus profondément pourquoi le calcul numérique de telles suites est si problématique, et encore là *Maple* pourra nous être d'un grand secours...

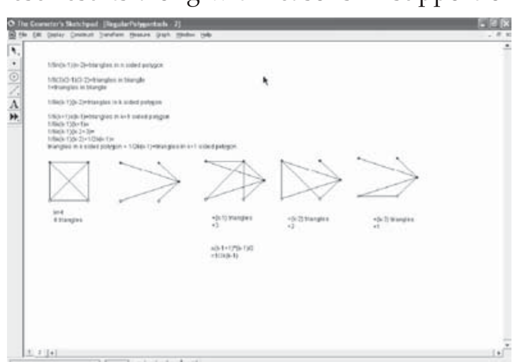
On the other hand, technology supported experimental approaches to mathematics can motivate the study of traditional proof methods. In a recent study (Roulet, Mackrell, Taylor & Farahani, 2004) students in a senior Geometry and Discrete Mathematics course (Ontario Ministry of Education, 2000) employed The Geometer's Sketchpad to explore the question "How many different triangles can you make by joining only the vertex points of a polygon?". A regular polygon construction tool, provided for the class, and The Geometer's Sketchpad transform utilities helped student pairs to categorize triangles and organize their data for particular polygons. But, technology, in the form of dynamic geometry software, could provide not further support in this exploration. Individual polygon cases could be adjusted, but there was no way to dynamically move from an n-gon to (n+1)-gon.

In addition to The Geometer's Sketchpad the project employed Elluminate Live!, Web-based software that supports desktop sharing. Student pairs, when demonstrating their work and sharing conjectures, could display their constructions on the screens of all computers in the room. Beyond this, if any other students had suggestions to offer, control could be passed to them so that they could manipulate the original sketch. With this visual communication channel in place, the potential for collaborative learning was greatly increased. Patterns in the triangle counts and conjectures for the number in the general n-gon case were shared.





Complexities in the sketches and problems with counting led to a variety of conjectured patterns and general formulas. Sharing of these results along with reasons in support of individual hypotheses highlighted the conflicting views and motivated the class to return to the problem. Some pairs, seeking to strengthen their arguments, resorted to careful checking of previous cases and production of additional data. Others, without teacher prompting, decided to explore a general case and worked on the construction of a proof. For this they reached back to the discrete mathematics section of the course and developed a sophisticated proof by mathematical induction.



Tendance à stéréotyper les utilisations de la technologie

De nos jours, pour des raisons en partie très valides, on assiste à une tendance à l'uniformisation. Ainsi, par exemple, la multitude foisonnante d'ordinateurs incompatibles du passé (Apple, Radio-Shack, Commodore, Atari, Adam, Sinclair, etc.) a été remplacée par un système d'exploitation quasi-monopolistique (Windows).

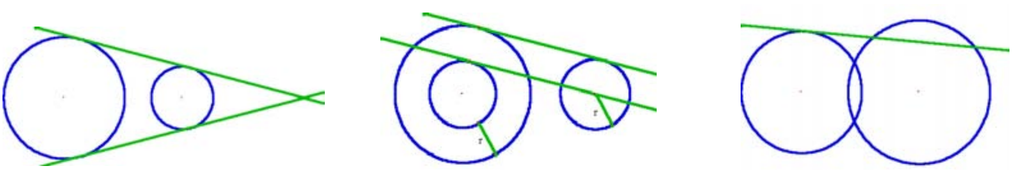
Il en est de même dans le domaine des logiciels utilisés dans l'enseignement des mathématiques. Dans certains cas, le nombre de logiciels disponibles s'est considérablement réduit: on a qu'à penser aux tableurs, avec Excel comme quasi unique représentant. Dans d'autres cas, il existe encore une bonne variété de logiciels d'un type donné, mais tous fonctionnent de façon très semblable: on pourrait citer en exemple les traceurs de courbes et les logiciels de géométrie dynamique.

Dans le cas des traceurs de courbes, cette standardisation s'est accompagnée d'un phénomène digne de mention : la montée en importance de l'étude du rôle des paramètres définissant des familles de courbes du type $g(x) = af(b(x+h)) + k$. La plupart du temps, cette étude se fait de façon purement expérimentale, ce qui est compréhensible en raison de la complexité d'une démarche déductive, que l'on peut résumer formellement par l'égalité ensembliste suivante:

$$\{(x, g(x)) : x \in Dom(g)\} = \{(\tfrac{1}{b}x - h, af(x) + k) : x \in Dom(f)\}$$

Cependant, on peut se demander si la popularité d'une telle étude des paramètres n'est pas plus due à la facilité de l'entreprendre avec les traceurs de courbes disponibles qu'à l'importance intrinsèque du sujet à l'heure actuelle. Par contraste, il est surprenant de voir que l'impact du choix du système d'axes sur la représentation graphique obtenue ne fait pas l'objet de tant d'attentions [Il y a déjà eu des logiciels de tracé de courbes mettant l'accent sur cet aspect (voir par exemple le logiciel *CARAPACE* (Boileau, Kieran & Garançon, 1996)), mais ils n'ont guère eu de descendants.], alors qu'on pourrait soutenir qu'il s'agit là d'un thème d'une importance au moins comparable.

Passons maintenant au cas des logiciels de géométrie dynamique, tel Cabri-géomètre et The Geometer's Sketchpad. Tous ces logiciels fonctionnent de manière semblable, alliant le dynamisme des figures à la fluidité des mouvements de souris. Mais nous allons voir que cette façon de faire, bien que généralement satisfaisante, comporte aussi des faiblesses. Pour illustrer ceci, nous allons construire les tangentes communes à deux cercles (voir la figure de gauche ci-dessous).



Une façon de faire est soustraire le rayon r du petit cercle au rayon R du grand cercle pour se ramener au problème de tracer les tangentes à un cercle passant par un point donné : les tangentes communes recherchées s'obtiendront ensuite aisément comme droites parallèles (voir la figure du centre ci-dessus).

Cette construction pose cependant problème, précisément en raison du dynamisme permis par ces logiciels : si en modifiant le rayon du petit cercle celui-ci en vient à devenir plus grand que l'autre, alors la construction précédente cesse d'être valable, comme on peut le voir dans la figure de droite ci-dessus.

En fait, ce qui manque, c'est de pouvoir « dire » au logiciel qu'on veut travailler non pas sur tel cercle, désigné par un geste, mais sur tel cercle, désigné par une propriété (comme celle d'avoir un rayon supérieur à tel autre cercle).

La discussion précédente met en lumière la délicate recherche d'un équilibre entre deux manières de communiquer avec nos outils technologiques: textuellement (en tapant des commandes au clavier) et gestuellement (par des mouvements de souris et des clics de boutons). Les logiciels de géométrie dynamique ont largement choisi de privilégier une communication gestuelle [Encore une fois il y a déjà eu des logiciels de géométrie privilégiant une approche textuelle (le logiciel Euclide, écrit en Logo par J.-C. Allard et C. Pascal, par exemple), mais il semble que cet héritage se soit presque complètement perdu.], mais on a pu constater les problèmes apportés par l'exclusivité de ce choix, notamment dans la gestion des cas de figure.

Nous avons tenté de créer un prototype de logiciel de géométrie dynamique (il s'agit de ProEuclide) intégrant à la fois les approches gestuelles et textuelles: chaque objet créé gestuellement ajoute une commande textuelle correspondante, et la liste de ces commandes peut être modifiée textuellement par la suite. Dans le cas de la construction de tangentes communes à deux cercles, nous pouvons ajouter textuellement une condition pour identifier le plus grand et le plus petit cercle, et poursuivre gestuellement la construction. L'encadré ci-contre montre le résultat final (légèrement modifié pour une meilleure lisibilité) : *les commandes tapées textuellement sont en italique*, le reste ayant été inséré automatiquement suite à des commandes gestuelles de l'utilisateur.

```
PointLibre "P1"
PointLibre "P2"
Cercle "C1","P1","P2"
PointLibre "P3"
PointLibre "P4"
Cercle "C2","P3","P4"
if Rayon("C1") > Rayon("C2")
then
  Soit "Cmax", "C1""
  Soit "Cmin", "C2"
else
  Soit "Cmax", "C2""
  Soit "Cmin", "C1"
end if
MACRO "Tangentes-grand-petit",
          "Cmax",        "Cmin",        "-
```

Dans la même veine, on peut constater la désaffection croissante de l'utilisation de la programmation (dont Logo, dont on ne parle presque plus) pour enseigner les mathématiques, peut-être parce qu'elle utilise une approche essentiellement textuelle. [Il y a eu quelques tentatives pour créer des environnements de programmation intégrant une approche gestuelle, mais sans jamais fondamentalement réussir à simplifier la tâche de manière significative.] Pourtant, dans le contexte de la formation initiale des maîtres en mathématiques à l'UQÀM, nous avons pu constater l'efficacité d'une approche combinant les approches gestuelle et textuelle (sous la forme d'une légère programmation) pour créer des graphiques précis et des environnements d'exploration dans le cadre de logiciels comme Word ou Excel. [Pour un exemple d'un environnement combinant des approches gestuelle et textuelle pour tracer des graphiques mathématiques, le lecteur pourra consulter le site web : http://www.math.uqam.ca/_boileau/LangageGraphique.html.]

Mathematics, Pedagogy and Technology: Balance in Teacher Education

Beginning secondary school mathematics teachers need technology supported teaching and learning experiences, but these are difficult to provide within the limited time available in teacher education programs. Most faculties include a sampling of technology applications within their mathematics instructional methods course, possibly augmented by an optional course dealing with information technology and education in general.

The issue of classroom use of information technology is connected to all other aspects of curriculum: course content, student learning activities, teacher planning, and classroom management. Workshops that focus on accessing software and calculator features do not provide models of effective classroom applications. Working through typical secondary school technology supported activities is a step in the right direction, but still does not provide new mathematics teachers with authentic investigation experiences. There is a need for mathematical exploration activities that present challenges to beginning mathematics teachers and at the same time illustrate and address parallel curriculum issues. Queen's University has for some years run a "problem of the week" activity for mathematics teacher candidates. Recently the Problem of the Week committee has been posing questions that invite technology application and increasingly participants have been submitting problem solutions in the form of electronic output from computer software presently available in Ontario schools. Hopefully, through these technology supported investigations, teacher candidates will develop the skills and more importantly the inclination to bring computers and calculators into their future classrooms.

One major exception to the blended curriculum methods - technology applications approach occurs at l'Université du Québec À Montréal, where those preparing to be secondary school mathematics teachers take courses with a focus on information technology applications in their subject. Other faculties may wish to consider something parallel to the UQAM program described in the next section.

Les tic dans la formation initiale des maîtres en mathématiques à l'UQÀM

Décrivons en terminant les cours portants spécifiquement sur les Technologies de l'Information et de la Communication que doivent suivre les futurs enseignants de mathématiques qui sont inscrits dans la concentration *mathématiques* du *Baccalauréat en Enseignement Secondaire* à l'Université du Québec à Montréal. En tout, il y a quatre cours de trois crédits, s'échelonnant sur quatre ans :

- *Progiciels dans l'enseignement des mathématiques I (MAT1812 - session 1)*
  Ce cours vise à apprendre à utiliser les outils technologiques essentiels, ainsi qu'à comprendre les bases et les limites de leur fonctionnement. On y apprend à :
  · utiliser une calculatrice graphique (TI-83 Plus);
  · mettre en page des textes mathématiques (avec Word);
  · produire des graphiques mathématiques précis (avec LangageGraphique.doc);
  · se servir des capacités numériques, graphiques et interactives d'Excel;
  · utiliser un logiciel de géométrie dynamique (Cabri-géomètre).
  [Pour plus de renseignements sur les deux cours « Progiciels », vous pouvez consulter le site http://www.math.uqam.ca/_boileau/Progiciels.html.]
- *Applications pédagogiques de l'informatique dans l'enseignement et l'apprentissage des mathématiques (MAT2812 - session 2)*
  Ce cours de didactique porte sur l'utilisation des outils technologiques dans l'enseignement des mathématiques au secondaire. Plus particulièrement, il vise à analyser les différentes façons d'intégrer des outils technologiques dans l'enseignement et l'apprentissage des notions-clés du programme de mathématiques aux cinq niveaux du secondaire : le sens du nombre, l'algèbre et les fonctions, la géométrie et les probabilités et statistiques. Une importance est accordée au rôle que peuvent jouer les outils technologiques dans l'apprentissage des concepts et procédures, et dans la résolution de problèmes. À l'aide d'une analyse des manuels utilisés le plus souvent dans les écoles québécoises, soutenue par celles des concepts mathématiques sous-jacents et des processus cognitifs mis en jeu par des apprenants, les étudiants de ce cours explorent le comment et le pourquoi d'une intégration des activités et des tâches utilisant de la technologie dans le programme de mathématiques au secondaire.
- *Progiciels dans l'enseignement des mathématiques II (MAT3812 - session 4)*
  Ce cours vise à apprendre à programmer les outils technologiques essentiels vus lors du premier cours, ainsi qu'à sensibiliser les étudiants à certaines problématiques, telle la gestion de la composante expérimentale en mathématiques. On y apprend à :
  · programmer une calculatrice graphique (TI-83 Plus);
  · utiliser *Visual Basic pour Applications* à des fins mathématiques (Word, Excel);
  · produire de petits films mathématiques 3D (POV-Ray et QuickTime);
  · créer des pages web mathématiques simples (Netscape Composer);
  · se servir d'un système de calcul symbolique (Maple).
- *Explorations des mathématiques à l'aide de l'informatique I (MAT4812 - session 8)*
  L'objectif principal de ce cours (qui ne s'est jamais encore donné) est d'utiliser la technologie pour explorer des situations mathématiques ou pour créer des ressources à cette fin, destinées

à des élèves de niveau secondaire. Il est prévu d'utiliser un site web qui agira comme plaque tournante pour que des maîtres en exercice puissent suggérer des projets qui seront réalisés par nos étudiants, projets qui seront par la suite déposés sur ledit site afin de les mettre à la disposition du milieu scolaire.

University Expectations and Models

University mathematics departments are sending out mixed messages concerning the connections between information technology tools and the doing and learning of mathematics. In Ontario, new curriculum guidelines (Ontario Ministry of Education, 1999, 2000) state that, "The development of sophisticated yet easily used calculators and computers is changing the role of procedure and technique in mathematics" (p. 3, 5). Pupils in graduation year university preparation courses are required to use graphing calculators. One year later, some of these students are enrolled in first year university programs where they meet course outlines stating, "You may use only non-programmable, non-graphing calculators for the tests and the final examination. I reserve the right to disallow any calculator" (Carleton University, School of Mathematics and Statistics, 2004). Secondary school teachers, knowing that their students will meet such restrictions, are reluctant to make technology a core aspect of their programs.

Other institutions deliver messages that are more encouraging. Visitors to the Web-site maintained by the University of Western Ontario, Department of Applied Mathematics read that "Computers play an important role in the research activity of the Department" (2004) and are informed that high powered Hewlett Packard calculators with computer algebra systems are required for completion of course work. University faculty who value the use of technology in mathematics study need to ensure that their views are known at the high school level and must encourage their university colleagues to acknowledge the ITC experience and skills of newly arrived undergraduate students.

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Topic Sessions

Sessions thématiques

L'enseignement des probabilités dans les écoles primaires du Québec

Renée Caron
Intervenante en enseignement des mathématiques

Point de départ

Dès le début des années 60, de nombreux changement sont introduits dans l'enseignement des mathématiques au Québec. Ces changements vont dans différentes directions qui sont parfois divergentes. Dans les années 1970, les leaders québécois dans le domaine de l'enseignement des mathématiques vont entreprendre une démarche plus convergente et c'est dans ce contexte que les propositions du colloque de Royaumont tenu en 1971, auxquels participaient des mathématiciens et éducateurs de plus d'une dizaine de pays, auront un écho. Nous considérerons ici, plus particulièrement, les propositions qui concernent l'enseignement des probabilités et de la statistique.

En plus de l'idée de moyenne, déjà présente dans les programmes antérieurs, et des problèmes liés à la lecture de graphique et de tableaux, le programme québécois de 1980 proposait des activités d'estimation et de vérification expérimentale de « certains cas de probabilité connus ». On proposait l'introduction de ces activités au 2^{e} cycle du primaire (9-12 ans).

À titre de conseillère pédagogique, j'ai travaillé, avec les enseignantes et les enseignants, à mettre au point un ensemble d'activités favorisant l'apprentissage des probabilités chez les jeunes des différents niveaux du primaire. L'expérience nous a en effet très vite appris que le jeune de la fin du primaire pouvait difficilement donner un sens aux résultats obtenus expérimentalement, sans les connaissances et l'expérience que pouvaient lui procurer les activités sur la combinatoire.

En effet, sans le support qu'apporte cet outil d'analyse, les élèves risquaient de s'en reporter au pouvoir de leur incantation ou à leur malchance pour expliquer les résultats obtenus et nos activités risquaient de maintenir et d'amplifier même les préjugés reliés au hasard. Le *premier principe* qu'il nous est apparu important de respecter fût donc celui qui veut que les enfants construisent leurs connaissances à partir d'expériences concrètes dont ils peuvent discuter les résultats, les organiser et en tirer les relations mathématiques pertinentes.

Avec les plus jeunes

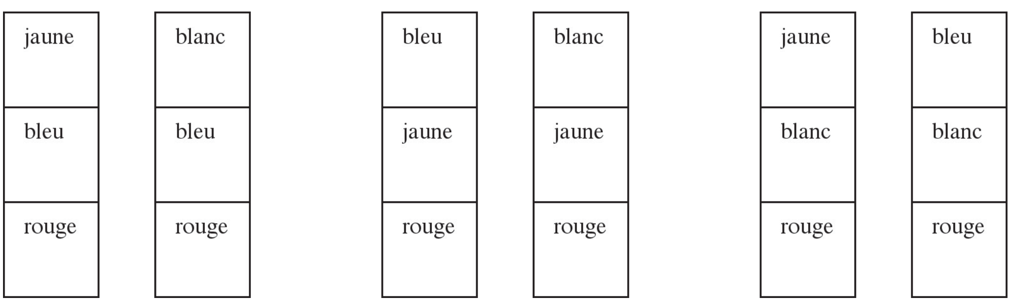
Nous avons commencé avec les jeunes élèves du début du primaire (6–7ans) et leur avons proposé un problème qui consistait à déterminer le nombres de tours de trois étages qu'ils pouvaient faire avec des cubes emboîtables de quatre couleurs différentes (rouge, bleu, jaune et blanc). Les élèves étaient répartis en équipe de 5 ou 6 et disposaient d'une quantité de cubes suffisante pour construire un assez grand nombre de tours à partir desquelles nous croyions et nous espérions qu'ils puissent construire la solution. Voyons comment se débrouillent des jeunes de cet âge pour produire une solution à ce problème.

Au début, chacun construit les tours auxquels il pense, sans tenir compte de ce que ses camarades font, mais les cubes viennent très vite à manquer alors que les enfants ont encore en tête des idées de tour à construire. Ils viennent assez rapidement nous voir pour nous demander d'autres cubes. C'est alors que nous leur proposons de mettre de l'ordre dans ce qu'ils ont fait dans leur groupe pour ne garder qu'une tour du même modèle. Cette démarche leur permet, bien sûr, de récupérer quelques cubes, mais elle leur permet surtout de porter un regard plus systématique sur ce qu'ils ont construits: avant d'éliminer une tour, il faut être certain qu'on en a au moins une autre identique et, pour le faire, on va devoir les regarder plus attentivement et prendre conscience que la tour dont le 1er étage est rouge, le 2e, bleu et le 3e, jaune n'est pas la même que celle qui a le 1er étage rouge, le 2e, jaune et le 3e, bleu. Cette activité de classification leur permet de se familiariser avec la structure que la solution va prendre ultérieurement.

On va donc devoir procéder à un classement plus systématique et ce qui va ressortir, d'abord, ce sont les quatre tours d'une seule couleur. La conclusion se présente comme une évidence : si on dispose de cubes de quatre couleurs différentes, on peut construire quatre tours d'une seule couleur. On peut récupérer les cubes qui nous ont permis de faire ces tours parce qu'on n'a pas besoin de les avoir devant les yeux pour se les rappeler, on n'a qu'à prendre le résultat en note. Il nous reste donc à mettre de l'ordre dans les tours de deux et de trois couleurs. Les élèves choisissent de mettre de l'ordre dans les tours de trois couleurs qui leur semblent plus faciles à classer, probablement parce qu'aucune couleur ne se répète.

On décide donc de procéder par étage, on va d'abord regarder le premier étage des tours et on va les classer selon que cet étage est rouge, bleu, jaune ou blanc. Quand on a tenu compte de ce premier étage, il faut regarder si on a bien construit toutes les tours dont le premier étage est rouge. Si le premier est rouge, le deuxième peut être bleu ou jaune ou blanc. On aura donc des débuts de tours ROUGE-BLEU, ROUGE-JAUNE et ROUGE-BLANC. On peut compléter chacune de ces tours de deux manières différentes, ce qui nous donne, pour la tour avec un début ROUGE-BLEU, les possibilités ROUGE-BLEU-JAUNE et ROUGE-BLEU-BLANC. On aura aussi les tours ROUGE-JAUNE-BLEU et ROUGE-JAUNE-BLANC, ainsi que les tours ROUGE-BLANC-JAUNE et ROUGE-BLANC-BLEU.

Voici une illustration des 6 tours regroupées plus ou moins comme elles l'étaient sous les yeux des élèves.



Il est possible de faire la même chose avec les tours dont le premier étage est bleu, jaune ou blanc. Les élèves ont donc classé et reconstruits leurs tours de façon à voir les trois autres séries de six tours. Ceci nous a permis d'ajouter les 24 tours aux 4 premières dont la possibilité nous était apparue évidente.

Mais il en sera tout autrement pour les tours de deux couleurs. Les élèves éprouvent des difficultés à trouver un classement qui va permettre de toutes les construire. Pour eux, la solution de considérer le premier étage d'abord, puis le deuxième et enfin le troisième leur semble plus difficile à appliquer. Quelle couleur vont-ils répéter, celle du premier étage ou celle du deuxième étage? D'autres voient la chose autrement. Si le premier étage est rouge, est-ce que ce sera le deuxième ou le troisième qui sera aussi en rouge? Il leur apparaît plus difficile de classer les tours même s'ils en ont un certain nombre devant eux. Ceci nous montre bien que la maîtrise du modèle, dont semblaient faire preuve les élèves lorsque les tours étaient constituées de cubes de trois couleurs différentes, était plus apparente que réelle. Probablement qu'en laissant mûrir l'idée, ils auraient pu construire le reste de la solution après quelques temps. Mais nous avons choisi de faire une pause pour leur donner la possibilité de prendre conscience qu'en résolution de problèmes, il arrive qu'on ne puisse trouver qu'une partie de la solution, alors on précise ce qu'on a trouvé et on en prend note pour pouvoir ultérieurement continuer nos recherches.

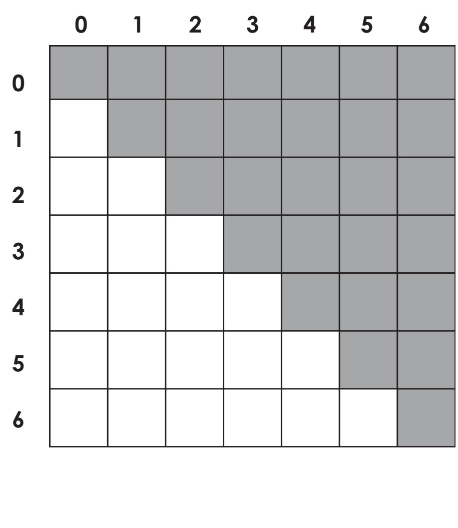
La partie du problème où il n'y a pas de répétition, comme dans le cas des tours de trois couleurs différentes, semble assez facilement accessible à des enfants de 6 ans. Nous l'avons proposé à des enseignantes de 1re année dans un autre contexte avec un matériel différent et cela ne semble pas voir causé de difficulté.

Ces activités permettent aux élèves de faire eux-mêmes une sorte d'inventaire de tous les cas possibles avant d'introduire les classifications dans des tableaux à double entrée ou des diagrammes en arbres dans lesquels ils pourront d'abord classer les éléments de leur solution avant que ces tableaux ou ces diagrammes deviennent des outils pour faire l'inventaire des tous les cas possibles. Il est à remarqer que le « dénombrement de résultats possibles d'une expérience aléatoire », qui était absent du programme de 1980, figure dans le programme de 2001.

Mais encore là, il ne faudra pas tenter de les énumérer en ordre du premier au dernier ou demander aux élèves de compléter un tableau à double entrée ou un diagramme en arbre. Par quel résultat commencer ? Il faudra justement que les élèves extraient ces résultats d'une expérience aléatoire réelle. En compilant et en classant les résultats que chacun à obtenus, on éliminera ceux qui reviennent plus d'une fois, mais il sera aussi beaucoup plus facile de repérer ceux qu'on n'a pas obtenu expérimentalement mais qu'on aurait pu obtenir. La structure qui permet de dénombrer tous les cas possibles se dévoilera assez naturellement aux yeux et à l'esprit des enfants.

Cette habileté à concevoir une structure d'organisation pour dénombrer et classer l'ensemble des résultats possibles leur permettra progressivement de résoudre des problèmes liés à cette structure. Parmi les autres problèmes que nous avons proposé aux enfants de cet âge, il y a aussi celui d'un enfant qui a perdu une pièce de son jeu de domino et qui, ne la retrouvant pas, veut s'en fabriquer une autre mais ne sait pas laquelle fabriquer.

À partir de jeux de dominos dont on a enlevé une pièce, on peut fournir aux élèves une autre occasion d'explorer la structure d'un jeu qu'ils connaissent et de développer leurs habiletés combinatoires. Suite à cette activité, ils pourront aussi expliquer pourquoi il n'y a pas 49 dominos mais seulement 28, chacun des dominos constitué de deux ensembles de points différents peut en effet occuper deux places dans le tableau à double entrée, mais cela ne fait quand même qu'un seul domino. Comme le domino ayant 0 point dans un de ses carrés et 1 point dans l'autre est déjà placé dans le deuxième carré de la première ligne, il ne peut l'être dans le premier carré de la deuxième ligne. On se retrouve donc avec toute une section du tableau à double entrée dont les carrés ne peuvent être remplis. Cette section a plus ou moins la forme d'un triangle comme celle de la section qui est complétée.



Toutes ces activités plus ou moins préalables aux véritables expériences aléatoires vont offrir la base structurelle sur laquelle l'enfant pourra éventuellement s'appuyer pour évaluer véritablement ses chances d'obtenir un résultat précis. Ainsi, quand il choisira un nombre avant de lancer deux dés, il pourra réfléchir au nombre de possibilités d'obtenir ce nombre et le comparer aux nombres de cas possibles.

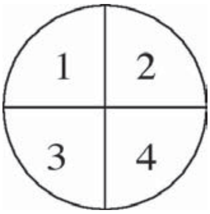
Dans l'expérimentation de ces activités ou d'autres semblables, un *second principe* est ressorti de façon tout aussi évidente à propos du matériel. Le matériel utilisé doit d'abord permettre aux élèves d'obtenir suffisamment de donnés pour imaginer, créer ou construire une solution qui tienne compte de la situation dans son entier. Comme dans le cas des tours avec des cubes, la construction de la solution complète repose sur les capacité des enfants à imaginer les parties manquantes de la solution. Pour les élèves, c'est leur premier pas vers l'abstraction dans ce domaine.

Avec les plus grands

Dans le cadre d'une expérimentation faisant appel à la coopération, nous avons été amenés à identifier un *troisième principe* qui tient autant pour les activités en probabilités et statistique, que pour tous les autres domaines d'apprentissage, à savoir qu'il n'est pas nécessaire, qu'il n'est même pas souhaitable, que tous les élèves fassent la même expérience pour assurer une discussion fructueuse sur les concepts mathématiques. Nous avons proposé à un groupe d'élèves de 6^{e} année (11-12 ans) d'explorer un ensemble de problèmes portant sur la probabilité de divers événements tels que:

Quelle est la probabilité d'obtenir deux faces paires en lançant deux dés?

Quelle est la probabilité que le trombone s'arrête dans la région 2 de la roulette illustrée plus bas ?



Quelle est la probabilité d'obtenir 3 piles ou 3 faces en lançant une pièce de monnaie 3 fois ?

Chaque problème était proposé à deux équipes de la classe. Les élèves étaient invités à faire les expériences à plusieurs reprises avant de se prononcer sur la probabilité de l'événement. Ils étaient ensuite invités à échanger leurs résultats avec leurs camarades de l'équipe ayant travaillé sur le même problème et à dégager des conclusions de l'expérience. Un porte-parole devait ensuite présenter leur expérience à toute la classe de même que leurs conclusions sur la probabilité de l'événement recherché.

Suite aux présentations, les élèves étaient invités à commenter les conclusions proposées par les membre des différentes équipes et à expliquer en quoi les résultats étaient comparables. De toutes les expériences sur les probabilités que j'ai proposées à des élèves de cet âge, cette expérience s'est avérée la plus fructueuse en raisonnements mathématiques et en conclusions pertinentes de la part des élèves. Après coup, il est toujours facile d'expliquer que puisque les élèves disposaient d'un ensemble d'expériences et de résultats plus grands, il était normal que leurs réflexions soient plus riches. C'est bien vrai, mais combien de fois hésite-t-on avant de proposer de telles activités aux élèves de peur de créer use confusion dans leur esprit ?

Les besoins des enseignantes et des enseignants

L'implantation d'activités portant sur les probabilités ne se fait encore que très lentement. Les enseignantes et enseignants sont souvent hésitants à engager les élèves dans l'exploration de telles activités. En effet, il n'est pas toujours facile d'amener les élèves à réaliser que les résultats qu'ils ont obtenus expérimentalement ne sont pas nécessairement ceux qu'ils auraient « théoriquement » dû obtenir. Il n'est pas toujours facile non plus d'animer une activité faisant appel à la manipulation de dés dont chaque lancer est la source d'un bruit plus ou moins percutant.

C'est pour cela qu'il importe de rechercher des situations et un matériel qui vont permettre d'expérimenter les activités de probabilités dans un climat de sécurité et de calme suffisant pour encourager les enseignantes et les enseignants à en prendre le risque.

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Feedback

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Some throat clearing …

In deciding what to write for the *Proceedings*, I have decided to elaborate on some of the activities and themes we worked on within the session rather than report my interpretations of what actually took place. All activities I mention here did take place within the session (and in fact some additional activities also took place which I have decided not to comment upon here), however I have taken the liberty to expand on some of my thoughts which lay behind these activities, whereas in the session I offered little of this preferring to provoke thought in those present. End of throat clearing.

I am teaching.… I see/hear a student do something relating to mathematics.… I make a choice about how to respond (which may involve the choice not to say or do anything).…

What informs my decisions about how to respond? The nature of what was observed will be a factor, but what a student does is not the sole defining factor on the nature of a response. Sometimes I observe a teacher who makes different choices, in response to something a student has said or written, compared with how I might imagine myself responding. Our beliefs about teaching and learning mathematics partly informs the way in which we respond as teachers, to such situations. As we continue educating ourselves our awareness of pedagogical situations develops and we begin responding in different ways to how we used to respond.

Through examining our responses to students we can begin to examine the awareness and beliefs which inform our decisions. I will start by considering a particular scenario and consider different forms of feedback which may be offered to someone in a learning situation. The situation does not involve explicit mathematical content (although I would argue that there is considerable geometry involved in the activity) however I suggest it can act as a metaphor when thinking about more explicitly mathematical situations.

Imagine a student blindfolded sitting at a table with a wastepaper bin somewhere the other side of the table.

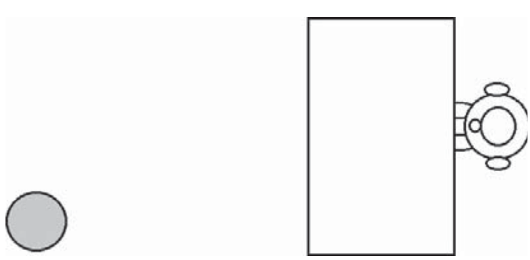

FIGURE 1. A bird's eye view of a blindfolded student sitting at a table with a wastepaper bin the other side of the table.

I will offer several scenarios relating to this situation and reflect upon teaching and learning issues for each in turn.

A. The student is asked to throw a beanbag into the bin. Initially the student may complain because they do not know where the bin is.

After complaining, the student has a choice of whether to engage in this activity or not. If not, then the student opts out and may seek other things to occupy their time. If they decide to engage and throw the beanbag, they will have no awareness of where the bin is and so their attempt will not be a particularly informed one. The bin could be anywhere as far as they are concerned and so they may base the way they throw the beanbag on factors such as how comfortable the throw is on the throwing arm, or maybe they may imagine some throws they have done at other times in other contexts, such as sport, and try to replicate those. To an observer, it may appear as if the student has thrown the beanbag in a random manner.

At times students are asked to engage in some mathematics when they feel they have no idea how to proceed and would not necessarily spot when they have done something 'correct'. Such times can lead to students opting out of the activity and finding other things to do which are not related to mathematics. So those students could become disruptive in the classroom and difficult for the teacher to manage. Even if the students decide to engage in the activity, they may not have relevant awareness which would inform how to proceed in an appropriate mathematical manner. They may appear to do things which seem completely random to a teacher. This is because their actions will be informed by something other than the mathematics which the teacher considers relevant. Without any further feedback on their actions, the students may just continue and have no sense of whether they are doing something relevant or whether they have or have not completed the original challenge.

B. The blindfolded student throws the beanbag. Someone else in the room, whom I shall call Teacher 1, gives feedback to the student in the form of either "*Yes, it is in the wastepaper bin*" or "*No, it has missed the bin*."

Now the student gets some feedback. However, the feedback only informs the student of whether the stated goal has been reached or not. If the beanbag missed the bin, there is no information within the feedback which might inform future throws, other than not to repeat the same throw. Even if the beanbag did go in the bin, the student will not have educated their awareness further for a similar activity in the future where the bin is repositioned. The success gets attributed to luck rather than skill.

Sometimes mathematics work is marked with ticks and crosses. It is deemed to be either right or wrong. In this sense the feedback is similar to scenario B. However, the situation within mathematics might have different attributes. For example, with throwing the beanbag there are limitless possible ways of throwing the bag and where on the floor it might end up. It is this infinite set of possibilities which makes the *yes/no* response of such limited value. This is not always the case within some areas of the mathematics curriculum. For example, a student may be aware that a regular triangle is called either *equilateral* or *isosceles* but cannot recall which one. In this case a *yes/no* response might be very helpful. Thus, when there are limited options a *yes/no* response can be useful. However, it only is useful to inform a student to try an alternative option, not which option to choose. So there is little education of the student's awareness which they use in selecting choices. This does not matter so much if it concerns a convention or a name (which I label as *arbitrary*, see Hewitt, 1999). In these situations, it is not a matter of educating awareness but of assisting memory. In a sense, a *yes/no* response provides a check for a student as to whether they have memorised successfully.

When properties or relationships are involved in an activity I suggest it is not just a matter of memorisation but it is about mathematical awareness (Hewitt, ibid). Suppose a student uses awareness appropriate to the task even though they arrived at an incorrect answer through a relatively small error. Being informed that the answer is wrong might spur the student to examine their work in a way where they can educate their own awareness. A *yes/no* response can be helpful for a student who is prepared and motivated to examine their own thinking with a level of detail which can be provoked knowing that something is wrong. It would not be so helpful for a student who used largely inappropriate mathematical awareness for the task.

C. The blindfolded student throws the beanbag and another person, Teacher 2, offers feedback as to where the beanbag has landed with respect to the bin.

This type of feedback is informative in that it can be seen as providing non-judgemental information about where the beanbag has landed. It is significant whether the information is provided as to where the bin is with respect to the beanbag, or where the beanbag has landed with respect to the bin. In the former case, the feedback takes on the form of an instruction—for example, throw the beanbag more forward and to the left—which a student might then try to follow. In the latter case the student is left with needing to do some work to translate the feedback into how the next throw might be changed. With this subtle but significant difference they are told the consequences of their previous actions rather than being told how to act in the future.

An extension of this issue concerns a principle of whether a teacher believes that their role is to help a student use as little thought as possible to achieve an end result or whether they want a student to have to think in order to achieve an end result. Is it the result or is it the thinking which is important? If the former then the choice of activity and the choice of feedback is such that the level of challenge is small in order to maximise the chance of achieving the end result. If the latter then the choice of activity and the nature of feedback is such that the level of challenge is made high whilst still making the task feel achievable for a student using the awareness they possess. If the level of challenge is too high then a student may feel it to be impossible for them, and so give up thinking and start guessing instead.

D. The blindfold is taken off the student, who can now observe where their beanbag lands.

Here the role of the teacher is in setting the activity and allowing a student to directly observe the consequences of their actions, rather than feeding back those consequences. The student can now see the goal—the bin—as well as seeing the consequence of a throw. This direct observation reveals more information than that which could have been provided in the previous scenario. The challenge of translating the visual observation into the physical action of muscles needs to be recognised. For example, I have recently observed my two-year old daughter, Tamsin, attempt to throw a model aeroplane down the hall. She throws and it goes up in the air above her head. She throws again, and it still goes above her head. She throws again and it lands behind her. Just because a student can observe directly the undesirable consequences of something they have done, it does not imply they know in what way they need to change what they did in order to achieve a desired result.

E. Having observed the student miss the bin with several attempts, Teacher 3 invites the student to engage with some imagery. The teacher says, "*Imagine you have just thrown the ball into the bin successfully and a video was taken of this throw. Now play the video backwards so that the ball comes out of the bin and returns to your hand as your hand goes backwards towards your start position. Now physically act out the video being played forwards.*"

This form of feedback is not a direct comment on the throw which has taken place but might be considered as an appropriate offering given the throws which have taken place. Thus, the teacher has observed some unsuccessful throws and considers what could be offered to the student which might help with the task. In this case it might initially appear to the student as if the teacher is offering something which is a distraction. There requires a willingness on behalf of the student to take their immediate attention away from their next throw and engage with the imagery being offered. This requires some will-power and it requires the student to have trust in the teacher that the potential payoff from what the teacher might offer is worth giving up the more immediate desire to continue throwing. How does a student gain that trust in their teacher? It seems to me that there may (or may not) be an initial sense of giving a new teacher the benefit of the doubt but, whatever the initial feeling, it is through the quality of the learning experiences a student has with a teacher that the student will decide whether to place trust in that teacher in the future.

In contrast to the beanbag and bin activity, I now offer a different activity. A collection of pentominoes and tetrominoes (Figure 2) are stuck on a board using blu-tack.

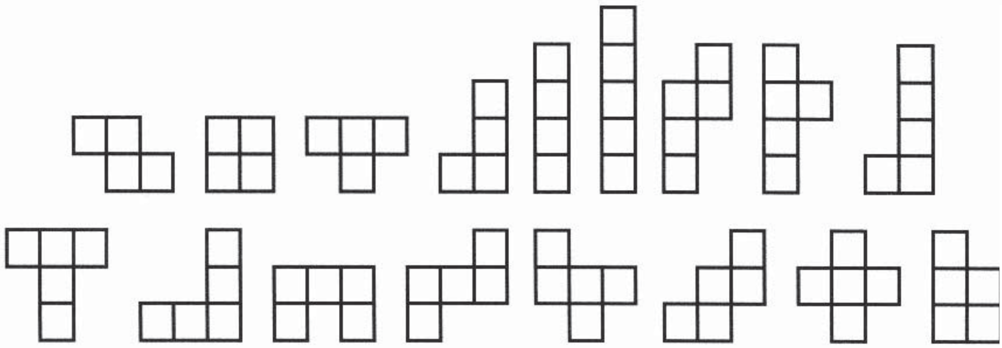

FIGURE 2. A collection of pentominoes and tetrominoes.

A student is asked to choose one of these shapes and place it on a different part of the board. I look at the shape and state whether I like that shape or not according to an unsaid rule that I am applying. If I like the shape it stays. If I don't like the shape it goes in a reject pile. The task for the students as a whole is to try to bring over all the shapes I will like whilst trying to get as few rejections as possible. So, for example, after three shapes have been offered to me the situation might be as follows:

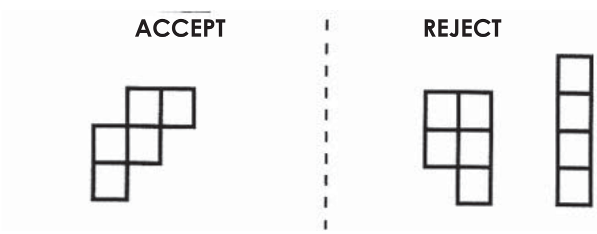

FIGURE 3. Early stages of the activity.

Which shape would you choose to place before me next? The feedback I am offering is *yes/no* yet I suggest the feedback feels more helpful for the task than when teacher I offered *yes/no* for the beanbag and bin activity. In the pentominoes and tetrominoes activity there are times when a *no* can be just as useful as a *yes*. For example, if you are thinking of two possible rules which are consistent with my acceptance or rejection of the shapes in Figure 3, you might deliberately try a shape which fits in with one of your rules but not the other. Such a choice will mean that, whether I say *yes* or *no*, one of your rules will be discounted from then on.

The *yes/no* responses are sufficient to provide much mathematical activity in finding properties of shapes, thinking about logic in terms of *if… then* statements, and also considering issues such as how you ever know for sure whether a rule you are considering is the 'same' as the rule I am using. So *yes/no* feedback can help generate considerable thought in the shapes task, whereas it did little for the beanbag and bin task. The choice we make in the nature of our feedback is concerned with our awareness of mathematics as well as our awareness and beliefs about pedagogy.

I will finish by offering the following scenario:

As part of a small group activity, you observe two students trying to decide which was the lowest out of the decimal numbers 0.853 and 0.358. One student says 0.853 is the lowest, the other thinks 0.358 is the lowest.

As a teacher, what might you do in such a situation?

On the next page, I offer six different responses that you might make as a teacher and invite you to consider one at a time and catch your feelings about each suggestion. Once you have caught your feelings, work a little harder to account for why you have those feelings. I suggest that such examination can help you articulate some of your beliefs and frameworks which inform the way in which you work with students. I strongly suggest that you work seriously on one response before even reading the next response. To help discipline yourself I recommend you get a sheet of paper where you can cover up all the responses except the one which you are currently working on (and, of course, those that you have already worked on). To help a little with your discipline I have started the scenarios on a new page. So do not read further until you have got your piece of paper and do not look at the following page.

Scenario 1

Ask one student to explain to the other why they thought their decimal was the lowest and vice versa.

Scenario 2

Invite them to do two subtractions:

0.853 – <u>0.358</u>	and	0.358 – <u>0.853</u>

Scenario 3

Explain what each of the digits are 'worth'.

Scenario 4

Ask them what each of the digits are 'worth'.

Scenario 5

Comment that they both cannot be right (and wait to see whether they begin to discuss the maths involved).

Scenario 6

Write both decimals on the board, one under the other, and reveal columns one number at a time from left to right:

0.
0.

0.8
0.3

0.85
0.35

0.853
0.358

Reference

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Body, Tool, and Symbol: Semiotic Reflections on Cognition

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Abstract

Although 20th century psychology acknowledged the role of language and kinesthetic activity in knowledge formation, and even though elementary mathematical concepts were seen as being bound to them (as in Piaget's influential epistemology), body movement, the use of artefacts, and linguistic activity, in contrast, were not seen as *direct* sources of abstract and complex mathematical conceptualizations. Nevertheless, recent research has stressed the decisive and prominent role of bodily actions, gestures, language and the use of technological artefacts in students' elaborations of elementary, as well as of abstract mathematical knowledge (Arzarello and Robutti 2001, Nemirovsky 2003, Núñez 2000). In this context, there are a number of important research questions that must be addressed. One of them relates to our understanding of the relationship between body, actions carried out through artefacts (objects, technological tools, etc.), and linguistic and symbolic activity. Research on the epistemological relationship between these three chief sources of knowledge formation is of vital importance for a better understanding of human cognition in general, and of mathematical thinking in particular. In the first of this paper, I discuss the roots of the reluctance in Western Thought to include the body in the act of knowing. In the second part, echoing current debates in mathematics education, I discuss, from a semiotic viewpoint, the importance of revisiting cognition in such a way as to think of cognitive activity as something that is not confined to mental activity. In the third part, I present a developmental overview of the theoretical foundations of my research program and of the research questions that my collaborators and I are currently investigating.

Résumé

Même si la psychologie du 20e siècle a pris en compte le rôle du langage et de l'activité kinesthésique dans la formation du savoir, et même si on a reconnu leur importance dans l'émergence des concepts mathématiques élémentaires (comme c'est le cas dans l'épistémologie génétique de Piaget), le corps, l'utilisation d'artefacts et l'activité linguistique, par contre, n'ont pas été considérés comme sources *directes* des conceptualisations mathématiques abstraites. Cependant, des nouvelles recherches ont mis en évidence le rôle central du corps, des gestes, du langage et des artefacts technologiques dans l'élaboration du savoir mathématique tant élémentaire qu'avancé (Arzarello et Robutti 2001, Nemirovsky 2003, Núñez 2000). Il y a, dans ce contexte, un certain nombre de questions qui doivent être étudiées. L'une de ces questions a trait à la relation entre le corps, les actions médiatisées par des artefacts (objets concrets, outils technologiques, etc.) et l'activité linguistique et symbolique. La recherche sur la relation épistémologique entre ces trois sources principales de la formation du savoir est d'une importance vitale pour comprendre la nature de la cognition humaine en général et la pensée mathématique en particulier. Dans la première partie de cet article, je me pencherai sur les racines qui ont amené la pensée occidentale à exclure le corps dans l'acte de la connaissance. Dans la deuxième partie, je discuterai, du point de vue sémiotique et à la lumière des débats actuels en éducation mathématique, de l'importance de repenser ce qu'on entend par cognition. Dans la troisième partie, je présenterai une courte vue historique des fondements théoriques de mon programme de recherche et des questions que nous étudions présentement.

1. Introduction

In one of the first episodes of *Star Trek*, Captain Kirk arrives at a strange planet. He does not see any signs of life. However, his detector keeps telling him that there are some forms of life in the surroundings. After a while, he realizes that life's signs come from the interior of some small transparent containers placed on a table. It turns out that these containers hold pure life: small brains that emit some color as they speak. They tell Captain Kirk that they are sophisticated forms of life that, to evolve, gave up body and became pure brains.

This science fiction story encapsulates in a clear way one of the cornerstone ideas of Western Thought, one in which the body is merely a hindrance with no relevance for our endeavours to attain knowledge. Since we humans have not yet found a way to divest ourselves of the body, we have created characters who give body to this idea. One of Captain Kirk's crew illustrates this point very well: Mr Spock is indeed the most clear-cut example of a logical thinker; emotions and body do not play any role in the way he thinks and calculates.

The *Star Trek* story is a futuristic version of the kind of rationality that Plato envisioned in the period of turmoil that followed the Peloponnesian war. The defeat of Athens led to a questioning of its traditional values and Plato's epistemology is indeed a response which attempts to salvage the aristocratic values. Politically, it was formulated as a kind rationality that opposes change. What is knowable is only that which does not change—something that Plato designated by the word *eidos* (essence). And to know it, we have to give up the body. In the *Phaedo* (65a-65b, p 47) Simmias is asked to determine who, among all sorts of men, would be able to attain true knowledge. Is it not him—Plato has Socrates ask—who

> pursues the truth by applying his pure and unadulterated thought to the pure and unadulterated object, cutting himself off as much as possible from his eyes and ears and virtually all the rest of his body, as an impediment which by its presence prevents the soul from attaining to truth and clear thinking? Is not this the person, Simmias, who will reach the goal of reality, if anybody can? (Phaedo, 65e–66a)

He then continues: "we are in fact convinced that if we are ever to have pure knowledge of anything, we must get rid of the body and contemplate things by themselves with the soul by itself" (*Phaedo*, 66b–67b).

Knowledge, for Plato, was only possible through reasoned discourse, through *logos*. The 17th century rationalists changed logos for a mind endowed with "powers" or "faculties", such as the faculties of understanding, memory and imagination—faculties that God granted to men (Descartes, *Meditation,* IV.9). When, in the First Mediation, Descartes asks the question: "What am I?" he answers: "A thinking substance". "I am anything but mind" (Descartes, *Mediation*s, II.15).

For Descartes, to know something amounted to having a distinct apprehension of the thing to be known. "I cannot be deceived in judgments of the grounds of which I possess a clear knowledge." (Descartes, *Mediations,* V.15). And apprehension and the distinctiveness of things were not ensured by the senses. Thus, to explain how bodies and external things become known, Descartes says that "bodies themselves are not properly perceived by the senses nor by the faculty of imagination". True knowledge is ensured, Descartes continues, "by the intellect alone; … [things] are not perceived because they are seen and touched, but only because they are rightly comprehended by the mind" (Descartes *Meditations*, II.16).

The mind or the spirit apprehends and knows things by the Rules of Reason, by rules expressed in the rules of logic, so that rationalists like Leibniz claimed that what we know is known not through the senses but by reason alone. Thus, all truths contained in arithmetic and geometry can be known by considering what we already have in our mind through reason, without having recourse to truths learnt by experience. This is why we can make these sciences in our own office, even with our eyes closed, for we do not need the eyes or the other senses (Leibniz, *New Essays concerning Human Understanding*).

The conception of a thinking mind governed by the cold rules of logic has served as the Western paradigm of thinking. All the attempts to reduce thinking to logical calculations belong to this paradigm. Empiricism, of course, has been a traditional contender of rationalism. Thus, opposing the rationalist trend, Hume argued that ideas are impressions that we receive from external objects.

To a large extent, the history of the 20th century pedagogy of mathematics is the history of a pedagogy that aimed to develop either a rationalist or an empiricist epistemology. The rationalist pedagogy of the early 20th century was a pedagogy focused on the development of logical thinking, abstraction, and rigor. Euclid's *Elements* was the text book and the model to follow. The rationalist epistemology re-emerged many decades later, embracing, with Bourbaki, a structural view of mathematics. In opposition to this pedagogy we find, also in the early 20th century, an empiricist one attached to the belief that the origin of our knowledge starts with our senses. Instead of focusing on rigor and proof, geometry, for instance, was taught as an experimental discipline, following Compte's positivism. But the pedagogy of mathematics was also based on the belief in a *continuity* between the sensual and the intellectual. Reason, it was assumed, picks sensual knowledge up and transforms it into abstract thinking. This mix of empiricist and rationalist pedagogy was drawing on the anti-dogmatic posture of the Enlightenment, which put the individual at the very core of knowledge. If something can be known, it can neither come from authority nor from what someone else says. It has to be known by the individual directly. Between the object of knowledge and the individual nothing could be interposed—except his/her sensual impressions. Several years later, theorizing the role of the senses along the lines of logic-mathematical structures, Piaget continued the Enlightenment tradition.

Indeed, following Kant—who attempted to achieve reconciliation between empiricist and rationalist trends—Piaget emphasized the role of sensorial-motor actions. If, however, body and artefacts played an epistemological role in his genetic epistemology, it was only to highlight the logical structures that supposedly underlay all acts of knowledge. The semiotic function, as Piaget called it (which includes *representation*, i.e situations in which one object can stand for another; *imitation* where sounds are imitated, evocation, etc.) was the bridge between the sensual and the conceptual, between concrete schemas and their intellectualized versions. This is why "operations [i.e. reflective abstracted actions] can sooner or later be carried out symbolically without any further attention being paid to the objects [of the actions] which were in any case 'any whatever' from the start." (Beth & Piaget 1966, pp. 237–238). As a result, for Piaget, signs and symbols were in the end merely the carriers and the expressions of a thinking measured by its rational structural features. The emphasis on the rationalist part of Piaget's work is well articulated by one of his collaborators, Hermine Sinclair, who, after explaining Piaget's reasons for avoiding empiricism and rationalism and presenting his genetic epistemology as a third possibility, says: "His [Piaget's] proposal of a third possibility is nearer to the rationalist than to the empiricist hypothesis" (Sinclair, 1971, p. 121).

Is there another way in which to conceptualize the relationship between sensorial-motor actions and signs? In the next section I deal with this question.

2. Revisiting Cognition

Last summer, when in the introductory talk of a PME research forum held at the University of Hawaii, Nemirovsky presented a list of research questions and argued that mathematics educators should tackle them soon, some participants had the impression that the questions to which Nemirovsky was referring were already answered by Piaget's epistemology. Two of the questions on Nemirovsky's list were the following:

> What are the roles of perceptuo-motor activity, by which we mean bodily actions, gestures, manipulation of materials, acts of drawing, etc., in the learning of mathematics? How does bodily activity become part of imagining the motion and shape of mathematical entities? (Nemirovsky, 2003)

Nemirovsky's questions are motivated by recent research which has stressed the decisive and prominent cognitive role of bodily actions, gestures, language and the use of technological artefacts in students' elaborations of elementary, as well as of abstract mathematical knowledge (Edwards, Robutti, & Frant, 2004; Arzarello and Robutti, 2001; Núñez, 2000). To give but one example, Susan Goldin-Medow (2003), Kita (2003), and Roth (2001) have shown how gestures become key elements in mathematics, the sciences and ordinary intellectual activity.

In fact, what I find new in Nemirovsky's list is not the questions therein included, but rather the concomitant invitation to revisit our own conceptions about cognition. To say it in Piagetian terms, there is concrete evidence emphatically suggesting that the semiotic function is much more than a bridge between sensorial-motor and intellectual activities. Or to say it in other terms, it is no longer possible to conceive of intellectual activity as the "natural" *prolongation* of practical sensorial-motor intelligence. From an educational perspective, it then becomes urgent to cast intellectual activity in new conceptual terms, in terms capable of including body, tool, and symbol—even in its more "advanced" manifestations. Intellectual and sensual activities are different sides of the same coin. They constitute the dialectical unit of thinking. As Parmentier remarks, even an abstract symbol bears a kind of contextuality in that "a symbol necessarily embodies an index to specify the object being signified"; reciprocally, every contextual signifying act relies on a certain generality, for "an index necessarily embodies an icon to indicate what information is being signified about that object" (Parmentier, 1997, p. 49).

In this context, there are a number of important research questions that must be addressed. One of them relates to our understanding of the relationship between body, actions carried out through artefacts (objects, technological tools, etc.), linguistic and symbolic activity. Research on the relationship between these three chief sources of knowledge formation is of vital importance for a better understanding of human cognition in general and of mathematical thinking in particular.

With regard to algebraic thinking—which has been the focus of my research program—the fundamental problem is to understand the way in which processes of symbolizing and meaning production relate to kinaesthetic activity and the artefacts employed therein. As our previous results suggest, highly complex algebraic symbolism cannot incorporate the students' kinaesthetic experience in a direct manner. The severe limitations of a direct translation of actions into symbols require the students to undergo a dynamic process of imagining, interpreting and reinterpreting. The students have to pass through a dialectical process between (concrete or imagined) actions, signs and meanings. However, little is still known about this process. Further research needs to be conducted at the theoretical and experimental level.

In the next section I provide an overview of my previous results and of the research question that will lead our forthcoming research.

3. Some previous results

We were led to notice the role of kinaesthetic activity and artefacts in the students' elaboration of mathematical conceptualizations in the course of observations that we began in a systematic way in 1998, in a longitudinal classroom based research program. We were studying the students' processes of meaning-making in generalizing tasks, conducted within traditional technology (pencil and paper). Analyzing hours and hours of videotaped lessons, it became apparent that the students' production of meaning was deeply sustained by natural language. A fine-grained analysis of the episodes made it possible to pinpoint two different key functions of language to which the students resort when they still cannot master the symbolic algebraic language and nonetheless must deal with mathematical generalizations. These were the deictic and the generative functions of language. The deictic function refers to the rich arsenal of linguistic terms (called deictics) with which the students can designate objects in their ongoing spatial-temporal mathematical experience (e.g. *this*, *that*, *here*, *there*, *top*, *bottom*, *before*, *after*). Deictics are ubiquitous in everyday communication. The generative action function refers to the linguistic terms that allow the students to convey the idea of generality. It is used by the students to express the idea of generality as a potential action that can be reiteratively accomplished. For instance, the students frequently use adverbs such as *always* to signify something that can be repeated forever (details in Radford, 2000). The generative action function can also appear in more subtle ways: instead of linguistic adverbs such as *always*, the students often use *rhythm*. They coordinate the flow of words with indexical or iconic gestures in order to produce rhythm and convey the idea that a pattern continues forever.

The deictic and the generative action functions of language empower the students with means for expressing the idea of generality—something that algebraists do using letters which stand for mathematical variables. The problem is that when the students are required to move into algebraic symbolism, they have to face the situation of expressing their mathematical experience through a semiotic system that does not possess deictics, adverbs, terms for generative actions or rhythm. The lack of such rich resources leads the students to a fundamental problem that struck me in a profound way. It is a problem of semiotic designation of objects: they have trouble designating, through algebraic symbolism, the number of e.g. circles or toothpicks in *Figure n*, that is, a non-specific figure identified only by its position in a sequence. I called this semiotic-cognitive problem the *positioning problem* (Radford, 2000, p. 250). *Figure n* cannot be seen, so reference to the number of circles or toothpicks that it contains can only be made *indirectly*, through signs.

However, this was not all. As our analyses progressed, we realized that the students were resorting to another semiotic system: gestures. Indeed, the students were continuously pointing to concrete figures in the sequence under study or imitating with some shapes of the figures with their hands. These gestures were not merely ancillary aids to communication. They appeared as *crucial* parts of their mathematical experience. Our students were more than cerebral thinking substances: their mental activity seemed, indeed, to be going beyond their internal cerebral processes and to be reaching the social world of body and artefacts. They were thinking *with*, and *through*, language, body and artefacts. This observation led us to search for ways to theoretically account for the role of body, tools and symbols in cognition.

At the end of his life, Vygotsky became more and more interested in the role played by the meaning of words in children's formations of cognitive functions, such as attention and perception. The problem about perception interested me the most. Our students were asked to deal with a general object which, because of its general nature, could not be perceived as one perceives a chair. In a way, their gestures and the whole semiotic activity that they were displaying were an attempt to supply the unperceivable general with something concrete. Following Vygotsky's work, I endeavoured to work out a theoretical account that could integrate the role of gestures, speech, symbols, and artefacts into the students' production of meaning. Certainly, Vygotsky's work is very rich, but the phenomenology of experience remained sketched in it only in very broad terms. I turned then to Edmund Husserl.

Husserl's phenomenology sets the basis for explaining how we become conscious of the things that we perceive. It seeks to explain the role of subjective intentions in the progressive apprehension of what is *there*. Husserl elaborated his account of how we become conscious of something in terms of noetic-noematic structures, but the problem of the conceptual object that we attend to in our phenomenological experience was subsumed into a rationalist idealism that was incompatible with the anthropological account that I wanted to offer. Merleau-Ponty's work was instrumental in my research in order to elaborate the role of language and body in perception, and so was the work of the epistemologist Marx Wartofsky, which I discovered through my readings of Michael Cole's papers and the work carried out at the Laboratory of Comparative Human Cognition in California. I came across the papers of David Bakhurst, of Queen's University, a great specialist on a philosopher who in turn became an important influence on my work: Evald Vasilevich Ilyenkov.

These authors (as well as many others that I have not mentioned here) led me to suggest that students' acquisition of a mathematical concept is a process of becoming aware of something that is already there, in the culture, but that the students still find difficult to notice. The awareness of the object is not a passive process. The students have to actively engage in mathematical activities not to "construct" the object (for the object is already there, in the culture) but to *make sense* of it. This process of meaning-making is an active process based on understandings and interpretations where individual biographies and conceptual cultural categories encounter each other—a process that, resorting to the etymology of the word, I call *objectification*. To learn, then, is to objectify something (Radford, 2003). Now, to see the object, to become aware of it, teachers and students mobilize all sort of tools, symbols, words, gestures, etc. These are *semiotic means of objectification*. Knowledge acquisition requires one to become aware of abstract relations that cannot be fully indicated in the realm of the concrete but that, at the same time, cannot be noticed but through concrete objects, gestures, actions, and symbols.

I would like to end this short summary by mentioning one of the problems that we are currently investigating: the problem of *the disembodiment of meaning*. As a result of the contextual nature of actions and of the aspectual view deriving from language and signs, gesture and perceptual activity, a spatial-temporal relationship is created between the individual and the conceptual object leading to what we have termed an *embodied meaning*. This embodied meaning has to become somehow *disembodied* in order to endow the scientific conceptual object with its cultural, interpersonal value. This disembodiment is very difficult to accomplish for the students, as suggested by the following example (for a more detailed account see our research reports in the PME27 and 28 Proceedings). In a Grade 11 classroom activity, we wanted to start exploring the role of kinesthetic actions and semiotic activity.

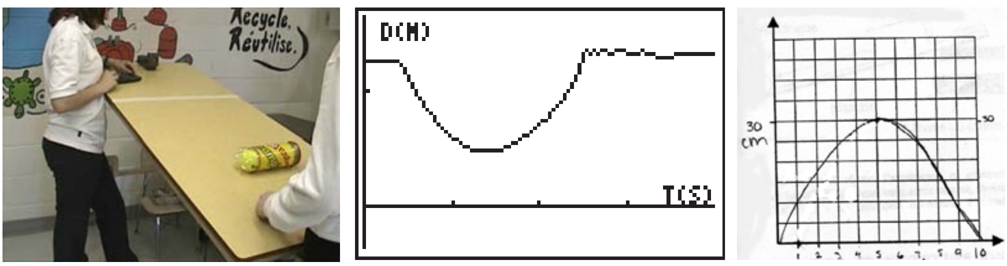

FIGURE 1. To the left, the students propelling the cylinder from the bottom of the ramp. The CBR can be seen at the top of the ramp. The second picture shows the graph produced by the calculator. The third picture shows the one drawn by the students.

The students were asked to make a graph of the relationship between the time spent and the distance traveled by a cylinder propelled from the bottom of a ramp (see Figure 1). Then the students carried out the experiment using a TI 83+ calculator connected to a Calculator Based Ranger (CBR) and were asked to compare their graph to the one produced by the CBR and calculator. Since the CBR was placed on top of the ramp, the calculator produced a convex parabola. The students drew a concave parabola and had difficulties understanding why the initial point of the graph was not at the point (0,0). As the transcript analysis reveals, for them, the bottom of the ramp is an important place.

The bottom of the ramp and the beginning of the cylinder motion orient the students' perceptual activity and become the centre of their mathematical experience. This point (that we have termed the *origo*, using the expression coined by K. Bühler, see Radford, 2002), is confused with the mathematical origin of the Cartesian graph. The distinction between these two origins (the mathematical and the *origo*) is central for the disembodiment of meaning. In this example, to disembody meaning means to realize that the mathematical origin (defined by the position of the CBR) does not necessarily coincide with the place from where the experiment starts.

Let us summarize the general aim of the research program that we are conducting in light of the previous discussion. We are investigating the dialectics between the students' kinaesthetic and artefact-mediated activity and their processes of symbolizing and meaning production. One of the research goals is the following: To investigate the role of bodily and artefact-mediated (concrete or imagined) action, perception, and linguistic activity in algebraic symbolism and in the formation of meaning.

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Standards for Excellence in Teaching Mathematics

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Abstract

Mathematics education in Australia has, in recent years, undergone many of the same changes evident in mathematics education throughout the world. Curriculum documents have emphasized mathematics as a creative endeavour, have placed high value on problem-solving and mathematical thinking, and have promoted a technology-rich environment for mathematics learning. Australian-produced teaching resources for mathematics have a high profile, and are well-regarded, both nationally and internationally.

Yet when one looks more closely at the actual practice in mathematics classrooms, such as in the TIMSS 1999 video study (Hollingsworth, Lokan and McCrae 2003), it is often dominated by a rule-based, instrumental approach, in which skills take precedence over understanding, and breadth of content takes precedence over depth. Translating the rhetoric into practice remains a critical issue for Australia's teachers.

The Australian Association of Mathematics Teachers (AAMT) has, during the past four years, undertaken a major project to publish Standards for Excellence in teaching Mathematics in Australian Schools (AAMT 2002). These Standards have been developed using teacher focus groups and teacher work samples, to ensure that they have practical validity and are owned by the profession. The Standards identify characteristics of excellent teachers of mathematics in terms of their professional knowledge, their professional attributes and their professional practice.

Mathematics Education in Australia

School education in Australia is a joint Federal Government and State and Territory Government (the term State is used to include Territories in the remainder of this paper) responsibility. Essentially the States administer school education within their borders, define curriculum frameworks and expectations, provide support to enable teachers to undertake professional development, fund schools in both the public and private sectors, and employ teachers to work in the public sector.

The Federal Government provides funding from taxation to the States to support their role, and directly funds private schools. The Federal Government also develops national policies and strategies for education, and conducts various reviews and enquiries into education.

While the States are primarily responsible for curriculum development and support, the level of specificity varies from State to State. However in all States each individual school must interpret the centrally developed document for its own context. In recent years States have begun to develop curriculum documents that take into account both traditional content domains such as mathematics, science, and English, and alternative documents such as the New Basics Project (Education Queensland 2001) that focus on generic skills such as life pathways and social futures, multiliteracies and communications media, active citizenship, and environments and technologies.

Increasingly in recent years the Federal Government has played a less direct but no less influential role in setting curriculum priorities through the development of a national literacy and numeracy plan (DEST 2004), which requires all States to report performance of students against agreed national benchmarks. Although these benchmarks do not constitute the curriculum, they naturally impact upon what teachers teach and how they choose to assess students in their classes.

Twenty years ago Stephens (1984) identified seven key issues impacting on mathematics education in Australia. These were:

- Cuts in education expenditure;
- Downgrading or disappearance of positions of State-wide responsibility for mathematics education;
- An acute shortage of qualified teachers of mathematics;
- The lack of a coherent national project in mathematics education;
- The gap between traditional tertiary-oriented senior secondary mathematics courses and the needs of young people who are, in increasing numbers, remaining longer at school;
- Interstate isolation and rivalry that prevents the effective sharing of resources; and
- The need to reconsider the content and teaching methods of mathematics to take advantage of major technological changes.

One could argue that all of these remain significant challenges in 2004. One might add to these two issues particularly pertinent to the current educational climate. These are the need to ensure that where school curricula focus on generic skills they also retain a rigorous mathematics content base, and the problem of the potential mismatch between State-wide testing of numeracy and approaches to teaching and learning mathematics promoted in contemporary curriculum documents.

The Australian Association of Mathematics Teachers (AAMT)

AAMT is a national professional association, made up of Affiliated Associations from each Australia State and Territory. It has some 5500 members from all levels of education from early childhood through to the tertiary sector. Given the structure of education in Australia, particularly with State-based curricula, maintaining a national focus and profile are significant challenges for AAMT. It does this through a variety of activities and projects, which include:

- Marketing of resources not readily available in conventional bookstores;
- Publication of three journals, one at each of the primary, middle school and senior secondary levels;
- Provision of national professional development such as a biennial conference and Internet-based discussions;
- Conducting of a national student activity, which encourages students to undertake an extended mathematical project;
- Providing informed and critical response to Government enquiries and liaising with Federal Government departments; and
- Conducting projects that promote excellent mathematics teaching and learning.

Standards for Excellence in Teaching Mathematics in Australian Schools

The Australian Association of Mathematics Teachers *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT 2002) were developed over a period of three years as a Strategic Partnerships with Industry Research Grant, in which Monash University was the research partner and AAMT was the industry partner. The methodology involved extensive consultation with teacher focus groups, with input and advice from the broader mathematics and mathematics education community. The development of the *Standards for Excellence* is set within a national and international context in which professional standards have become an increasingly important element in describing and promoting excellent teaching (NCTM 1991, Commonwealth of Australia 2003).

The AAMT *Standards for Excellence* outline three domains in which excellence in teaching mathematics is evident: professional knowledge, professional practice and professional attributes. Professional knowledge includes knowledge of mathematics, of students and of how children learn mathematics. Professional practice includes creating an effective learning environment, planning for learning, teaching in action, and incorporating appropriate assessment in teaching. Professional attributes include personal attributes such as enthusiasm for teaching mathematics, a commitment to personal professional development and adopting community responsibilities such as promoting mathematics.

The *Standards for Excellence* are intended to serve at least two major purposes: enabling a transparent and defensible method of accrediting teachers of mathematics as highly accomplished teachers, and providing a framework for effective professional development. They thus provide a description of one high-level step along a teacher's professional journey, and a vision of teacher identity at this point. The following three snapshots provide a glimpse of how these Standards have been used to frame professional learning and assessment.

Standards as a Framework for Assessment in Pre-service Teacher Education

In the study described below pre-service teachers were required to reflect on their knowledge and practice in the context of the *Standards for Excellence* described above. The study involved fifteen students who were in their final year of training to teach mathematics to high school children. The students undertook one subject of 36 hours duration in which they looked specifically at how students learn mathematics, at mathematics curriculum, and at different approaches to teaching mathematics. They also undertook a four-week period of Professional Experience in a secondary school.

It is noteworthy that many, but not all, of the students involved in this class were mature-aged students, who already had varied life experiences and a strong sense of personal identity. These students had a strong sense of why they wanted to become teachers and what they hoped to achieve. In general they "wanted to make a difference". They were also very aware of their own experiences as students in mathematics classes, and while they had been successful, they felt that their school experiences had not engaged them, and had not promoted the development of deep mathematical understanding. In the words of one student "I don't think I will make a very good maths teacher, because I have just begun to realise that I don't really understand anything I learned at school—I was just good at it."

While the traditional assessment tasks undertaken by the students in this subject, such as lesson-planning and micro-teaching have immediate and obvious practical value, it is debatable to what extent they promote life-long learning or assessment described by Boud (2000) as sustainable assessment, nor to what extent they promote the development of teacher identity (Shulman 2002). Yet for these students, this is their only pre-service experience in mathematics education, hence it is critical that they are well positioned to become life-long learners of the art and craft of teaching mathematics.

In 2003 a new assessment task, based on the AAMT *Standards for Excellence* described above, was introduced. This task required students to develop a focused portfolio and to attend a 20-minute individual interview. During the interview they were asked to explain their rationale for including parts of the portfolio, and to evaluate their knowledge of, practice of, and beliefs about, teaching. Each pre-service teacher was asked to answer three questions, one focusing on their own perceptions of their knowledge against the *Standards for Excellence*, one describing and reflecting in a focused way on their teaching during Professional Experience, and one discussing a critical issue in mathematics education.

Two mathematics educators interviewed the students, made notes during the interview, referred to the portfolio for any further clarification, and provided feedback within thirty minutes of the completion of the interview. Students were informed that the interview process was an experiment, and that it was being used as an attempt to make the portfolio more focused. Each student also agreed to have the interview taped for future reference.

As might be expected in any assessment task, there was a wide range of student responses and levels of performance. A few students were unprepared, had done little reading, and did not focus their answers or portfolio. At their best, however, the interviews were remarkable. They showed a capacity to be reflective of their own teaching, to be critical and constructive and to ask informed questions of the status quo. They provided a vivid and tangible image of pre-service teachers developing a very strong sense of teacher identity.

John focused on professional knowledge in his discussion of the *Standards for Excellence*. He drew parallels between a constructivist approach to teaching and his background in human communication theory. He noted that a key principle of communication was that "the receiver makes the message", and concluded that it was thus the teacher's role to know his students, their culture and their idiom well enough to enable each student to make the message in a productive way.

Melissa described how, in teaching fractions to a year 7 class, her supervising teacher had asked her to split the class into three groups based on results in a pre-test. On reflection she felt that, while they had worked diligently through the work assigned, the most advanced students had not been challenged in any significant way, and that, in general, the lowest achieving students remained the lowest achievers. However one student who had been placed in the lowest achieving group was able to complete the post-test with only one error. This was exciting for both the student and his teacher, who had not expected such a result. In reflecting on her experiences with, and reading about, setting students based on their perceived ability levels in mathematics, Melissa concluded by saying "I haven't got an answer, I'm still sitting on the fence".

The pre-service teachers in the study had thought deeply about their teaching, about what they had read and talked about in their academic studies, and about how it related to their practical experience. Like Melissa and John they did not provide glib answers, but saw knowledge of teaching as developing through reflection over a long period of time. As Melissa said "putting it all together (for the interview) … touched on layers of other issues".

The interviews provided strong evidence of developing teacher identity, in particular characteristics such as scepticism, the capacity to reflect on experience to link theory and practice, and a sense of self as a learner. The pre-service teachers' core beliefs about teaching, and about themselves as teachers, were challenged. They recognised their existing professional knowledge and highlighted their shortcomings; they evaluated their own and their supervising teachers' practice honestly and critically; they revealed a developing sense of what they valued in learning.

However the most surprising outcome was the sense of community generated through the process. The pre-service teachers emailed each other after the interview to discuss their feelings about the task. This was an entirely self-motivated undertaking—I had not asked them to share their reflections and had expected that, like every other assessment task I had ever set, students would just be glad that it was over. On learning of this email exchange, I requested a copy with names removed, and the students were happy to provide their reflections.

> Probably the most I got out of the whole process was how analysing, reading articles and reflecting continued to challenge me about my teaching... While I was preparing for an assessment item, I think I got more out of the exercise than the mark Steve gave me.

These pre-service teachers saw the exercise as an important part of their on-going development as teachers of mathematics. They saw themselves as part of a community, and were keen to share their experiences and thoughts with others. Unprompted, they thoughtfully evaluated the validity of the interview process and made links with assessment practices beyond their current course. In this sense the portfolio and interview did "double duty" (Boud 2000) by focusing on both the immediate and the future, by transmitting what is valued as well as making judgements, and by giving students the reflective skills to attend to their on-going development as excellent teachers of mathematics.

Standards and Teacher Professional Learning

The use of professional standards to frame and guide teachers' professional learning was recognised in the *Report of the Review of Teaching and Teacher Education* in its Agenda for Action (Commonwealth of Australia, 2003), which recommends that "professional learning opportunities provided by employers of teachers, higher education institutions and teacher professional associations be directed to the achievement of the standards to be established for advanced teaching competence…".

The program *Engaging with Excellence in Mathematics Teaching: Creating Excellence in the Learning Environment* was a series of teacher professional development workshops held during the middle part of 2004, developed conducted jointly by the Australian Council for Education Research (ACER) and the AAMT.

The catalyst for the partnership was the release of some classroom videos from the *1999 Third International Mathematics and Science Study* (TIMSS) Video Study. The videos are acknowledged as an outstanding resource for teacher development, especially when the software tools and other resources that are associated with them are taken into account. Given the finalisation of the AAMT *Standards* it was agreed that these would be used as the framework for the teachers' investigations. Standard 3.1 deals with the learning environment, and this is the focus area for this professional development program, given that the videos capture learning environments (physical, intellectual, emotional) in mathematics classrooms in a variety of countries. In the professional development program, participants used the AAMT *Standards* to:

- self identify their learning needs in mathematics;
- analyse, describe and discuss the learning environments represented in selected videos;
- express their particular learning goals for this program in terms of the learning environment of their classroom; and
- monitor progress and celebrate success.

The pilot program concluded in August 2004. Preliminary analysis of feedback from the 12 or so participants—all of whom had no previous detailed exposure to the Standards — is that they found the document useful to very useful in identifying their professional learning needs. 'The Standards self-evaluation form in particular was identified as a most useful instrument.' (Peck etc al.; 2004)

They reported that the Standards assisted them in working with colleagues who were not in face-to-face sessions and as a means for focussing their learning. 'Overwhelmingly, all teachers felt that they had significantly improved their awareness and appreciation of the AAMT Standards and were able to identify ways that their practice (or that of their colleagues) had moved closer to the Standards'" (ibid.).

Standards and Assessment of Highly Accomplished Teachers of Mathematics

In 2003/2004 the AAMT conducted a pilot program to assess teachers of mathematics as Highly Accomplished Teachers of Mathematics (HAToM). This program was intended to develop and trial a rigorous and defensible model that could be sustained into the future. This model was reviewed and finalised in the early stages of the project. It was based on clear principles that the assessment process would be:

- rigorous and valid;
- adaptable to and applicable in all teaching contexts;
- fair to all candidates no matter what their teaching situation;
- equally accessible to teachers across the country;
- controlled by the candidate insofar as this is possible; and
- oriented towards contributing to professional growth of the candidate.

The model required candidates to:

- respond to unseen questions that simulate teaching decisions through an *Assessment Centre*;

· submit a *Portfolio* of their work and achievements as a teacher consisting of a Professional Journey (reflective essay), a Case Study of one of two students' learning, an example of Current teaching and Learning Practices, Validation (report of a classroom observation or video of their teaching) and Documentation (awards, references, testimonials etc.) ; and
· take part in an *Interview*.

The model involved a 'team consensus approach' to assessment. Individual assessors accumulated evidence from what the teacher had presented to make holistic judgements directly against each *Standard*. Assessors then met to reach consensus about whether they had identified sufficient evidence in to be confident that individual *Standards* have been met. To be recommended to receive the HAToM award the teacher had to meet all ten *Standards*.

An independent external evaluator used a 'participant-observer' methodology to report on the project. Observations, document analysis and interviews with/feedback from participants of all kinds provided the data for an extensive Evaluation Report. Some key findings follow.

The Assessment Model

The TSAEP found that the Assessment Model works — candidates are validly and reliably identified as HAToMs. Importantly, the assessment process is able to discriminate among teachers, as not all candidates were successful in meeting all ten Standards. Furthermore the model and the associated guidelines provided to candidates are transparent and flexible in allowing teachers to exercise some control over the form of their submissions.

The Assessment Strategies

All three components of the assessment appear important to provide a sufficient picture of the candidate's knowledge, skills and attributes. Although requiring all components may lead to some redundancy of information about how a candidate meets the *Standards*, there are other benefits such as increased internal reliability of data.

The Assessment Centre tasks developed were searching but fair, and candidates' responses predictive of their ability to meet the array of *Standards*. A sample question from the Centre is provided below.

This simulated teaching in action decision scenario is posed to teachers of Middle School students to elicit responses that would demonstrate the teacher's ***community responsibility****.*

You run into a parent of a child you had a year or 2 ago and she tells this story. The teacher concerned is a very experienced and traditional teacher - a colleague you have known for several years. This is what they say:

As you know, Amy is in now in Year 5. She has been learning about division. For homework the other night she was doing loads of practice exercises set by the class teacher of the type

23/7 to yield an answer of the type 3r2, the remainder of 2 being identified by the "r".

She said to me that this doesn't make sense … she thinks the answer should be 3 groups and 2 out of 7 pieces for another group. I said she could talk to her teacher about it.

She did discuss it with the class teacher, only to be told she got it wrong and she was "corrected".

Using your knowledge of Mathematics and the learning of Mathematics respond to the following questions.

1. What is surprising about the context?
2. What might you say to the parent and the child?
3. What strategies might you provide to the teacher professionally to deal with this situation?

The Portfolio provided critically important evidence about a candidate's knowledge, capability and commitment as a teacher of mathematics. Importantly, the experience of assembling a Teaching Portfolio was considered by candidates as a worthwhile professional development experience in its own right. It provided an opportunity to compile a broad picture of one's teaching and to reflect on this towards one's further development as a teacher.

The Interview was an important part of the assessment process, with benefits both for those who achieve accreditation and those who do not meet all *Standards*.

The Candidates' Responses

Candidates felt positively about the assessment process, despite at times feeling frustrated or anxious, with reservations about the time available and the hard work required. Several stated that they felt the process helped to affirm their status as a good teacher and that it was valuable and confidence boosting. Indeed, in collecting documentation in the form of references, some candidates were genuinely and pleasantly surprised at how highly thought of they were in some quarters. They universally felt that the assessment process had helped to provide them with informed feedback about their teaching and an opportunity to document an accurate picture of their teaching. In other words the experience of undertaking the assessment was valuable in its own right.

The Assessors' Responses

The experience of assessing candidates was also highly positive for the Assessors. When asked about the assessment process, they used words such as 'revelatory', 'delighted', 'overwhelmed', 'amazed', 'impressed', 'inspiring', 'uplifting', 'humbling' and 'valuable' to describe their main reaction. They were impressed with the ability of the process to reveal the very high quality of the candidates' work as teachers, and were grateful for the opportunity to learn about talented colleagues in an 'interesting' way. They felt that assessing the candidates was a good learning process, as it forced them to reflect on their own professional status in relation to the *Standards*. In other words the assessment process proved to be an excellent professional development experience for the Assessors.

While the pilot project was successful in contributing to the development of a reliable and valid assessment process, its future sustainability is highly dependent on funding or incentives to teachers. This is highly uncertain, however there appears to be an increasing move towards recognition and celebration of teaching as a profession, in which the development and implementation of professional standards is a key element. Perhaps the greatest benefit of conducting the Teaching Standards Assessment Evaluation Project is that it locates a valid assessment and accreditation model firmly in the hands of the profession.

Conclusion

Professional Standards of Excellence have the potential to significantly enhance the professional agenda of teachers of mathematics at all levels in Australia in the coming few years. They provide both a model and framework for teacher professional learning and valid criteria against which to evaluate excellence in teaching. However the extent to which these Standards can impact upon the day-to-day experiences of children in Australian schools depends very much on the ownership of the Standards by teachers of mathematics themselves. Developing this sense of ownership is perhaps one of the key questions facing AAMT in the coming years.

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New PhD Reports

Présentations de thèses de doctorat

The AHA! Experience:
Mathematical Contexts, Pedagogical Implications

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> *Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated.*
>
> – Andrew Wiles[1]

Suddenly, it's all illuminated. In the time it takes to turn on a light the answer appears and all that came before it makes sense. A problem has just been solved, or a new piece of mathematics has been found, and it has happened in a "flash of insight" (Davis & Hersh, 1980, p. 283), in a moment of illumination, in an AHA! experience. From Archimedes to Andrew Wiles, from mathematicians to mathematics students, the AHA! experience is an elusive, yet real, part of 'doing' mathematics. Although it defies logic and resists explanation, it requires neither logic nor explanation to define it. The AHA! experience is self-defining. At the moment of insight, in the flash of understanding when everything seems to make sense and the answer is laid bare before you, you know it, and you call out—AHA!, I GOT IT! However, the AHA! experience is more than just this moment of insight. It is this moment of insight on the heels of lengthy, and seemingly fruitless, intentional effort. It is the turning on of the light after six months of groping in the dark.

Literature is rich with examples of these AHA! experiences, from Amadeus Mozart's seemingly effortless compositions (Hadamard, 1945) to Samuel Taylor Coleridge's dream of Kubla Kahn (Ghiselin, 1952), from Leonardo da Vinci's ideas on flight (Perkins, 2000) to Albert Einstein's vision of riding a beam of light (Ghiselin, 1952), all of which exemplify the role of this elusive mental process in the advancement of human endeavours. In particular, scientific advancements are often associated with these flashes of insight, bringing forth new understandings and new theories in the blink of an eye. On a larger scale, the advancement of science in general serves as a nice metaphor for the AHA! experience. Over long periods of time science seems to progress at a steady rate, albeit an exponential one. Upon closer examination, however, what is revealed is a field that moves along in fits and spurts, with long periods of much activity and little progress punctuated with occasional flurries of advancement (Gardner, 1978). Even nature seems to take its cue from this phenomenon. Evolution, at first thought to be a slow and steady progression over time, is now being seen as lengthy periods of inactivity with occasional bursts of reorganization (Johnson, 2001). The AHA! experience is everywhere, from human endeavours to nature itself, and it punctuates the knowledge of the world around us.

As natural a part of our thinking processes as the AHA! experience is, for the most part it remains a mystery. Somewhere within the far reaches of our unconscious some mechanism is at work which allows our mind to produce these flashes of illumination, of this there is no doubt. Yet, these mechanisms remain hidden from us, and they may always do so. This does not prevent us from theorizing about its workings, however. From Gall's suggestion that mathematical creativity resides in a bump at the back of the scull (Hadamard, 1945) to

Thom's catastrophe theory (Saunders, 1980), there is no shortage of hypotheses as to where this sudden appearance of an idea comes from. In the midst of all these theories, however, there are very few definitive answers. What is actually known about the AHA! experience can be distilled down to one concise statement: illumination occurs after a long period of conscious effort followed by a period of unconscious work (Hadamard, 1945; Poincaré, 1952). All else stems from this one understanding.

We know, for example, that the sudden appearance of an idea may occur to us in our sleep, upon waking, during conversation, or in the bath (Hadamard, 1945), but this is only a refinement of the understanding stated above with regards to where and when illumination occurs. We know that it is important to think hard and to think broadly about something if we wish for an AHA! to occur (Ghiselin, 1952; Perkins, 2000; Root-Bernstein & Root-Bernstein, 1999), but this is only a refinement with regard to the nature of the conscious work. We know that not all ideas that come to us in the flash of illumination are correct (Hadamard, 1945), but this is only a refinement with regard to the nature of the products of illumination. Clearly there is room for more to be known.

For the last three years I have been working on filling some of this room, to learn things about the AHA! experience that are not yet known. To do this I have restricted my efforts to the general context of mathematics and the specific context of mathematical problem solving. In particular, I have been interested in distilling out the very *essence* of the AHA! experience within this context to determine what, if anything, about these experiences is common, and what sets them apart from other mathematical experiences.

Methodology

To pursue this question I embarked on three separate studies. The first of these studies was specifically designed to examine the effect of an AHA! experience on students who are resistant to mathematics and who have a phobia of mathematical content as well as of the learning of mathematical content. Although this study was initiated with the hope of enhancing an understanding of the cognitive aspects of the AHA! experience, it concluded with my learning more about the affective aspects of the AHA! experience.

In the second study the survey that Jacques Hadamard used in his empirical study, some 50 years ago, was resurrected for the purpose of soliciting anecdotal accounts from contemporary mathematicians. This usage of Hadamard's survey produced a rich data set of accounts from some of the world's most prominent and well-respected mathematicians. These data were analysed for two purposes. The first of these was to distil from the mathematicians' accounts the very essence of the AHA! experience. The second was to formulate some conjectures as to what sorts of environmental and situational conditions are necessary for the occurrences of AHA! experiences. These conjectures were then used to help structure the third study, a study designed to answer the question as to whether conditions surrounding a problem solving environment could be structured to increase the occurrences of AHA! experiences among students. This third study necessitated the creation of a new form of student journaling for the purposes of recording and tracking these experiences.

Conclusions

The main conclusion from this pursuit is the assertion that the essence of the AHA! experience in mathematics in general, and in problem solving in particular, is in the affective domain. That is, what sets the AHA! experience apart from other mathematical experiences is the affective component of the experience, and ONLY the affective component. This is counter to the general understandings of the nature of the AHA! experience. In the literature, AHA! experiences, along with their cadre of extra-logical processes, have always been dealt with in the context of the cognitive domain. That is, it has always been assumed that what makes these experiences extraordinary is the hidden cognitive processes that produce these extraordinary ideas. I assert that this is not the case. What serves to make the experiences extraordinary is the affective response invoked by the experience of an untimely and unanticipated presentation of an idea or solution, not the mystery of the process, and not the idea itself. I support this assertion along several fronts, culled from both the data and the literature presented within my dissertation.

Before I begin this discussion, however, it needs to be acknowledged that, clearly, there is a cognitive component to the AHA! experience. After all, it is the arrival of an idea that punctuates the phenomenon. The argument that follows does not deny this. What the argument does do, however, is to assert that, while the cognitive component of the AHA! experience is inconsequential to the differentiation of the AHA! from other cognitive experiences, the affective dimension is not. I begin by establishing the role that the affective dimension plays in this distinction. This is then followed by a more lengthy discussion detailing how the cognitive dimension plays a relatively little role.

Through the first of my focused studies it was shown that the emotive response created by AHA! experiences is capable of instantly transforming even 'resistant' students' beliefs and attitudes about mathematics (McLeod, 1992) as well as their beliefs and attitudes about their ability to do mathematics. This transformative effect, alone, makes the affective response invoked by the AHA! experience very different from the responses invoked by other mathematical experiences. The literature on affect and mathematical experiences indicates that success and feelings of accomplishment contribute to a change in beliefs and attitudes. However, it is suggested that the change they produce is minute, and long periods of sustained and successive success are required to create significant change (Eynde, De Corte, & Verschaffel, 2001). The study presented in my dissertation shows that AHA! experiences can produce changes in beliefs and attitudes very quickly—in the time it takes for an insight to be verified. This indicates that the affective responses to a single AHA! experience is much more powerful than the affective responses to an instance of success in mathematics (not consisting of an AHA!).

Having established the contribution of the affective dimension to the distinction of the AHA! experience, I now argue the cognitive dimension's role is relatively insignificant. The most obvious evidence for this comes from the fact that nowhere in the data were the details of the idea central to the presentation of the phenomenon. This was true for both the students and the mathematicians. In the few situations that the details were presented, their contribution to the account of the experience was inconsequential. This is consistent with the words of Poincaré (1952) wherein he clearly states that the details are unimportant:

> I must apologize, for I am going to introduce some technical expressions, but they need not alarm the reader, for he has no need to understand them. I shall say, for instance, that I found the demonstration of such and such a theorem under such and such circumstances; the theorem will have a barbarous name that many will not know, but that is of no importance. What is interesting for the psychologist is not the theorem but the circumstances. (p. 52)

As such, the cognitive products of the AHA! experience, although clearly not absent, do not contribute to description of the experience. This is further supported by the existence of false AHA!'s. False AHA!'s are occurrences of the AHA! experience wherein the products of the experience (the ideas) turn out to be incorrect. However, neither the occurrence or the intensity of the AHA! is diminished by this lack of correctness. That is, the idea is irrelevant to the phenomenon.

In fact, the only time that ideas or solutions are mentioned at all by the mathematicians is in the context of describing the AHA! experience as the sudden appearance of an idea. These descriptions can be sorted into two cases. The first case speaks of ideas in the context of how and when they come to mind as expressed by Dan J. Kleitman:

> And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping.

The second case speaks of the significance or importance of the ideas that come to mind. This is demonstrated by Wendell Fleming's comment:

> What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place.

The first of these cases is no different than the situations already described. The second case, on the other hand, seems to place the nature of the idea in a much more central role. However, significance in the face of an AHA! experience is not a measure of the quality of an idea but rather a 'sense' that is invoked by the way in which the idea comes to mind. I referred to this as a 'sense of significance' and compared it to the 'sense of certainty' that also accompanies the phenomenon. As such, even the significance of the idea is an affective response to the AHA! experience.

Further evidence of the insignificance of the idea to the contribution of the phenomenon of the AHA! can be found in the occurrence of group AHA!'s. In particular, I am referring to one particular group AHA! In which all three members of the group had the same idea, at the same time, and in the same flash of insight. That is, to all intents and purposes the three members of the group had the same cognitive experience. In the end, however, only two of the group members viewed the event as an AHA! experience. The third member did not. She identified the moment of insight in her journal, but saw it simply as another idea coming to mind. Furthermore, in comparing the description of the event among these three group members the only distinction between the accounts lay in the affective descriptions of the experience. The two members who claimed to have had an AHA! experience gave accounts rich in affective descriptions, while the third member's descriptions were completely devoid of any affective elements whatsoever. I argue that the reason that this occurred is that although the three students all had the same idea come to them at the same time, and in the same way, their affective response to that idea varied. This is not dissimilar from people's varying responses to events in general. Any time a group of people share in an experience—whether it be a movie, a concert, a play, an accident, or *an idea*—they will all have different affective responses to that experience. Some will like it, some will not, some will be indifferent, and so on.

Finally, I offer the very succinct, and definitive, comment from Henry McKean:

> No, I don't find it different from understanding other things in life!

McKean's view is that the understandings gained from AHA! experiences are no different than the understandings gained from other sources. This is not only in keeping with the arguments posed above, but also brings forth the question of the origins of ideas in general. Hadamard (1945) argued this very question in terms of the antechamber of the mind. He posed that, if pushed back far enough, the origins of every idea, even of every spoken word, is the product of the mysterious and wondrous workings of the mind. As such, at the level of origin the ideas produced by AHA! experiences are no more extraordinary than the origins of any other idea, or even of this sentence.

Together all this evidence speaks to the very *essence* of the AHA! experience. An AHA! experience is an affective response to a cognitive event, and like any other affective response, it differs in intensity depending on the individual as well as the situation. What makes it special, is not the idea itself, but rather the way in which the idea comes to us, with "characteristics of brevity, suddenness, and immediate certainty" (Poincaré, 1952, p. 54).

Note

1. From the movie 'The Proof', produced by *Nova* and aired on PBS on October 28, 1997 (Nova, 2003).

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The Issues of Learning the Equals Sign for Grade 1 Students

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Context

During their first steps in the learning of arithmetic, many children have difficulties in understanding the meaning of the equals sign. Instead of conceiving this symbol as an indicator of a numerical equality, they consider this sign as an indication to perform an operation or to write an answer (a.o. Carpenter & Levi, 2001; Saenz Ludlow & Walgamuth, 1998). This conception has important implications on these children's abilities to solve certain addition tasks. While the conception of the equals sign as an operator allows the children to solve "2 + 5 = __", important difficulties appear when they have to assess number sentences like "8 = 7 + 1" or complete equations like "__ = 2 + 3". These children are unable to find the correct answer or they choose to read backwards. This last strategy, while efficient for addition number sentences, will cause important problems when the number sentence contains a subtraction. Because this operation is not commutative, the backwards reading strategy will lead to a wrong answer.

Important difficulties also show up when these children are asked to complete equations like "3 + 5 = __ + 2" (Falkner et al., 1999; Saenz-Ludlow & Walgamuth, 1998; Shoecraft, 1989). In this situation, a common error of children who regard the equals sign as an operator is to designate "8", which is the sum of the numbers preceding the equals sign, as the unknown number. This error is not only common with first grade children, but can also be frequently found with older children. In a research involving 752 primary school children, the success rate for both first grade and sixth grade students who were asked to complete "8 + 4 = __ + 5" was below ten per cent

The conception of the equals sign as an operator also plays an important role in the learning of algebra. As Bodin and Capponi (1996) point out, this conception has been clearly identified as one of the main obstacles in the transition from arithmetic to algebraic thinking.

Even if children's difficulties to understand the equals sign and the implications of their conceptions are well documented, the equals sign isn't taught explicitly in most classes. This situation is partially due to the conception of mathematics textbooks, which attach almost no importance to the teaching of the equals sign. This symbol appears in first grade of elementary school, often without further explanation. This situation can be illustrated by three examples, drawn from textbooks issued in Quebec and Luxembourg. *Clicmaths*, a French Canadian textbook tries to explain the different components of a number sentence in the following terms: "2 + 4 = 6 is a number sentence representing an addition; 2 and 4 are the term of the number sentence; 6 is the sum of the terms 2 and 4; + is the symbol that represents an addition" (Bordier, 2001, p. 35). While this explanation addresses the meaning of almost every component of a number sentence, the meaning of the equals sign is not explained.

Intersection, another first grade textbook, tries to define what the equals sign means in the following way: "The equals sign (=) is not only used to perform an algorithm or to give an answer. It may also be used to give information" (Lebel & Pagé, 2001, p. 92). The example used by the authors of this textbook to illustrate the last described meaning of the equals sign refers to even and odd numbers: "12, 13, 14, 15, 16 is a series of numbers. Even numbers = 12, 14, 16. Odd numbers = 13, 15" (ibid, p. 92). It seems obvious that an explanation like this one, which does not refer to the presence of the same quantity on both sides of the equals sign, may reinforce children's beliefs about the equals sign as an operator.

In the only first grade textbook for Luxembourg children, no explanation of the meaning of the equals sign is offered. At the beginning of the year, children have to compare numbers and use the ">", "=" or "<" symbols to do so. Later, the equals sign is introduced in the first additive tasks, but no explicit teaching is provided to the children. Therefore, we believe that the explicit teaching of the equals sign is underestimated, both in Quebec and in Luxembourg.

Methodology

The aim of our research was to describe the development of first grade children's understanding of the equals sign. To be able to observe this development, we chose to involve these children in a constructivist teaching experiment, during which the participants had to work on the meaning of the equals sign in various additive number sentences.

At the beginning of our research, we conducted a pre-test involving eleven children of a first year class of an elementary school in an urban district of Luxembourg. During this test, the children were asked to assess whether different number sentences were correct and they had to complete various equations. Some of these number sentences and equations had already been dealt with in class, while others were unknown to the children. For instance, the children had to indicate whether "3 + 4 = 1 + 6" is a correct number sentence and they had to complete "7 = __ + 5". Finally, other questions assessed the children's procedural understanding of equivalence and equality.

The result of the pre-test allowed us to select six children for our teaching experiment. Two of these children were low achievers, two were medium students and two more were assessed as strong students. The equals sign was taught to these children in six to nine individual sessions during half an hour each. Each session was videotaped and the transcription allowed us to analyze the children's reasoning. About ten days after the end of the lessons, we conducted a post-test, during which all the participants in the teaching experiment were asked to answer questions similar to those in the pre-test.

Activities

Several underlying principles guided the elaboration of the activities presented to the participants in our teaching experiment. First, the children had to work on two types of tasks during the whole teaching experiment. The first type of tasks asked the children whether a given number sentence is right or false. When an error was detected in a number sentence, children had to modify it in a way to obtain a correct number sentence. In a second type of activities, the participants were asked to complete various equations.

Second, we used different kinds of equations and number sentences. At the beginning of our sequence, we worked exclusively on "a + b = c" and "a = b + c" number sentences. Later, the "a + b = c + d" structures, more difficult for the children to understand, were introduced.

Third, in order to facilitate the establishment of a link between mathematical symbolization and concrete representation, each number sentence or equation was represented with concrete objects at the beginning of the work on each type of structure. The aim of the gradual withdrawal of this concrete representation, which took place later in the sequence was to make children able to work on number sentences and equations from the mathematics symbols only.

Fourth, the unknown number in the different equations was represented in two different ways. In a first setting, children had to add into a transparent plastic bag as many objects as necessary to make two collections equal. In a second setting, the unknown number was represented by a non transparent box which contained the right number of objects. Children were told that the same number of objects is present on both sides and then had to find out how many objects were in the box (fig. 1). This second setting was more difficult for the children to work on, because they could not see and manipulate the objects that represented the unknown.

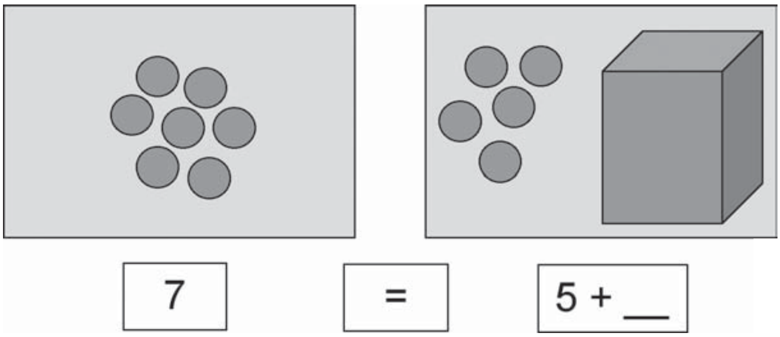

FIGURE 1. Example of an activity

Finally, two different types of representations were used throughout our sequence. We will use the example of the number sentence "2 + 5 = 7" to explain the differences between both kinds of representation. This number sentence can be illustrated in at least two different ways. In a first setting, someone could have 2 marbles in the left hand and 5 marbles in the right hand, and have 7 marbles altogether. In this setting, the 7 marbles do not exist independently from the 2 and 5 marbles, but represent the sum of the two sub-groups. We will refer to this constellation as an "inclusive" representation. In a second setting, a person could have 2 green and 5 red marbles, while another person has 7 marbles. Both persons then have the same number of marbles. This constellation will be referred to as a "comparative" representation, because in the concrete representation, the 7 marbles are not physically the same as the 2 and 5 marbles.

The same difference could apply to "a + b = c + d" number sentences. For instance, an inclusive representation of "4 + 2 = 3 + 3" could imply the transfer of one of the marbles of the collection of 4 marbles to the collection of two marbles, as illustrated in figure 2.

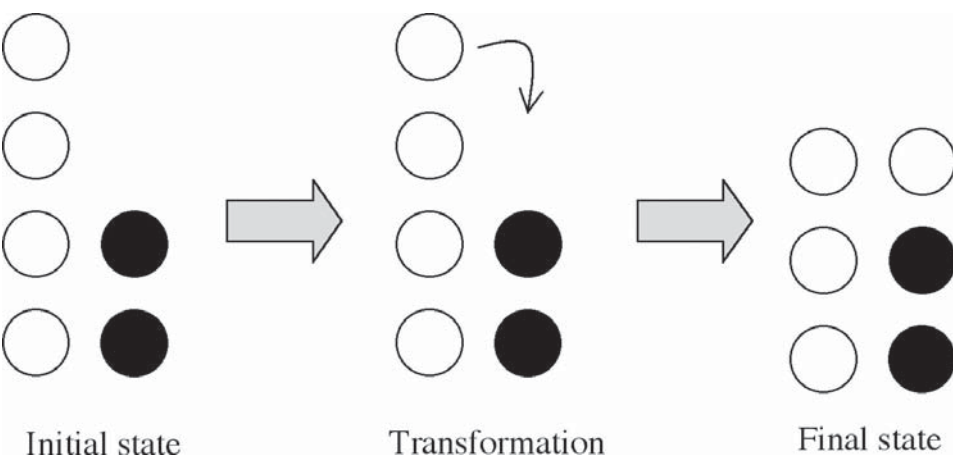

FIGURE 2. Concrete representation of "4 + 2 = 3 + 3" in an inclusive setting

Results

a) Initial conception of the equals sign as an operator

All the children we interviewed during our pre-test considered, to different extents, the equals sign as an operator. Caroline, our weakest participant in the teaching experiment, had no doubt about her explanation: “You use the equals sign when you calculate. When you have finished your addition, you write the equals sign and the answer.” Caroline accepted only “a + b = c” number sentences. When she was asked to assess “5 = 5” and “7 = 3 + 4”, she transformed this number sentence into “5 + 5 = __” and “3 + 4 = 7” respectively.

Melissa, the child evaluated as a medium achiever in mathematics, offered a more ambiguous explanation. When asked about the meaning of the equals sign, she refered to the meaning of the equals sign as an operator and as an indicator of a relation: “The equals sign means that it is equal. You have to put another number after this sign.” However, as soon as Melissa was asked to assess different number sentences, her conception of the equals sign referred clearly to that of an operator, because she refused to accept all “a = b + c” and “a + b = c + d” number sentences.

Mathieu, who has no or very few problems at school, already managed to assign a more differentiated meaning to the equals sign. Just as Melissa and Caroline, he did not accept “a = b + c” and “a + b = c + d” number sentences, but he used the equals sign as an indicator of a relation when he was asked to assess “5 = 5”: “You can use the equals sign because there is the same number on both sides of the sign”.

b) Progress made by the children throughout the teaching experiment

All of the children we worked with during our teaching experiment made significant progress in their understanding of the equals sign. However, this progress was not the same for everyone. While Mathieu and Melissa were able, at the end of the sequence, to complete “a + b = __ + d” number sentences, Caroline, which was the low achiever of the group, had much more difficulties with certain types of equations and number sentences.

c) Difficulties maintaining a conception of the equals sign as an indicator of a relation

Even if the children we worked with were able to make significant progress in their understandings of the equals sign, the development of a new meaning is a significant cognitive obstacle. Three kinds of observations lead us to this conclusion: They are unwilling to accept a new meaning of the equals sign at the first lesson; they constantly return to a conception of the equals sign as an operator throughout the teaching experiment and the post-test; carried out only a few days after the last lesson showed worse results than at the end of the teaching experiment.

During our first activity, when we tried to explain the meaning of the equals sign as an indicator of an equality relation, the children were very reluctant to accept that new meaning. Melissa offered the strongest opposition to our explanation and stated that it was simply not true. “I do not believe that the equals sign means what you say.” Mathieu’s and Caroline’s opposition to our explanation was more subtle. They accepted our explanations, but insisted on reading backwards the number sentence “8 = 5 + 3” we worked on at that moment. This strategy allowed them to accept this kind of equation while remaining coherent with the conception of the equals sign as an operator.

Even later on in the teaching experiments, all of our participants showed a tendency to return to a conception of the equals sign as an operator on several occasions. The children then used two kinds of strategies that allowed them to work with a conception of the equals sign as an operator: reading backwards, and transforming the number sentence into an “a + b = c” structure.

While the reading backwards strategy is used by all the children, the reasons for its use are different. For Caroline, a real misunderstanding of the equals sign justified the use of this strategy. During the second lesson, she read backwards “7 = 3 + 4” and explained: “You cannot read that way (7 = 3 + 4), because this would make you think that the 7, equals, and the addition only afterwards." However, when Mathieu and Melissa used the reading backwards strategy, their understandings of the equals sign may be adequate. For Melissa, it seemed more difficult to read "a = b + c" sentences: "I don't like to read like that. If we turn this around, it will be easier to read." Mathieu justified the use of the reading backwards strategy in a similar way. After reading from right to left, he often added that the direction you read the sentence makes no difference, because the same number is on both sides of the equals sign.

At several occasions, children also transformed an equation into an "a + b = c" structure, which allows them to stay coherent with a conception of the equals sign as an operator. Two examples may illustrate the use of this strategy, which appeared in various situations. During the second lesson, Mathieu and Melissa were asked to complete the equation "7 = 2 + __". Both thought that "9" must be the unknown number. Their answer indicates that they used the sum of "7" and "2", which represents a transformation of the equation into an "a + b = c" structure. Much later in the sequence, Caroline was asked to assess "5 + 3 = 6 + 2". She believed that this sentence is false and transformed it into "3 + 3 = 6 + 2" By doing this, she made the number following the equals sign correspond to the sum of the numbers that precedes it.

These situations were especially frequent at the introduction of a new type of number sentence or concrete representation. At these moments, children were already able to conceive the equals sign as an indicator of a relation in certain situations. However, as soon as a slight change in the structure of the number sentences or the concrete representations was made, they were unable to transfer their knowledge about the equals sign to the new situation.

Finally, the post-test session, held only about 10 days after the last lesson, revealed that the participants in our teaching experiment were unable to perform in the same way than at the end of the teaching experiment. This phenomenon was especially visible with Caroline, whose results were barely better than those of the pre-test. While Mathieu and Melissa showed much better progress, they had difficulties to maintain their conception of the equals sign as an indicator of a relation in certain situations they had managed to resolve at the end of the teaching sequence.

d) The importance of the ability to use various additive strategies

The ability to use various additive strategies seemed to be an important factor that distinguished Caroline, who was confronted with a series of difficulties during the teaching experiment, from Mathieu and Caroline, who made easier progress. First, the ability to deal with certain additive strategies appears to be a prerequisite for the learning of the equals sign. Second, being able to chose between several additive strategies, according to the question asked, is an advantage for children who learn to conceive the equals sign as an indicator of a relation.

Already during the pre-test, Caroline had certain difficulties to resolve additive tasks, presented as a concrete representation. For instance, in one task, Caroline was presented five objects and a non transparent box, which contained three more objects. We told Caroline that the five objects and those in the box equaled 8 and asked her how many objects were in the box. Caroline told us that there had to be five objects in the box. This answer makes us believe that Caroline somehow understood that there had to be an equality somewhere—she believed that there had to be the same number of objects in the box than in front of her—but that she has difficulties to apply this relation in a concrete setting. These difficulties may be one of the factors that could explain Caroline's poor performance during the teaching experiment, especially as they continued to appear throughout the teaching experiment. Another interesting observation is that Caroline's ability to use different additive strategies did not improve during the lessons on the equals sign. At the post-test, she even made more errors in the additive tasks where she had to work with concrete representations. This leads us to believe that without adequate understanding of additive structures, it is difficult to learn to conceive the equals sign as an indicator of a relation.

The ability to use various and complex additive strategies to complete equations seems to be another determining factor for the learning of the equals sign. While Caroline often used an erroneous strategy in these tasks, Mathieu's and Melissa's strategies were much more elaborated. For instance, neither Melissa nor Mathieu hesitated to use the concrete representation that was supplied in certain situations. When asked to complete "5 + __ = 7", Mathieu separated the seven objects in the concrete representation into to collections of five and two. Caroline, on the other hand, always refused to use this strategy, even after we explicitely explained it to her.

Mathieu and Melissa were also comfortable with the use of more complex strategies. When faced to equations which represented commutativity, they were able to use this characteristic to complete the task. For instance, when we asked Mathieu to complete "4 + 5 = 5 + __", he immediately knew the answer: "It must be 4, because I just copied on the other side of the equals sign."

A further characteristic that distinguishes Melissa in particular is the use of subtraction in certain situations. When asked to complete "3 + __ = 7 + 2", the subtraction allowed to find the answer: "It is 9 on the right side, so I just have to subtract 3 and I know it must be 6." This complex strategy, that was only used by Melissa, is another indicator of her excellent understanding of additive structures.

Discussion

Several conclusions may be drawn from our research. First, if the equals sign is not taught explicitly, children will develop a conception of this sign as an operator. In the class where we conducted our research, the equals sign had not been explained by the teacher and all the children believed they had to write an answer after the equals sign. This finding is coherent with other work on the understanding of the equals sign: several researchers confirmed the presence of the conception of the equals sign as an operator. Furthermore, this conception is not only held by first grade children, but also by much older students. Carpenter et al. (2003) support the necessity of an explicit teaching of the equals sign: According to these authors, the understanding of the equals sign is not simply a process of maturation, but has to be addressed more directly.

Second, it seems possible to make children change their conception of the equals sign, under certain conditions. This result contradicts those obtained by Denmark et al. (1976), but is coherent with more recent studies (a.o. Carpenter & Levi, 2000; Falkner et al., 1999), in in which it is argued that even first grade children are able to develop a different understanding of the equals sign.

Even if it is possible to make children change their beliefs about the equals sign, they are reluctant to do so and try to stick with their initial conception. We described different strategies the children used, which can also be found in the literature. Saenz-Ludlow and Walgamuth (1998, p. 185) for example found that it is very difficult to make children believe that the equals sign is an indicator of a relation: "The dialogues and the arithmetical tasks on equality indicate these children's intellectual commitment, logical coherence and persistence to defend their thinking unless they were convinced otherwise."

Other researchers also discovered that, as with our participants, the conception of the equals sign as an indicator of a relation is not stable. Baroody and Ginsburg observed that, after having taught the equals sign to grade one to three children, they were able to use the equals sign as an indicator of a relation in some situations, but, in others, they still believed this sign must be used to write an answer.

Furthermore, Carpenter and Levi (2000) observed that children's new understanding of the equals sign was not stable. After having taught the equals sign to first grade children, many of them returned to their initial conception of this sign several months after the end of the teaching sessions. Therefore, Carpenter et al. (2003) recommend to continue to use non-conventional number sentences throughout the year.

An important issue is also raised by the importance of the understanding and the use of different additive structures in the learning of the equals sign. In our research, those children who made good progress in the teaching experiment also showed an excellent understanding of these strategies. On the other hand, Caroline, who used only a very small variety of these strategies was unable to make significant progress. We then ask ourselves whether the introduction of the equals sign should be delayed. Several arguments may support this proposition. First, as the conception of the equals sign as an operator is very difficult to change, it could be useful to avoid its appearance at the beginning of elementary school. On the other hand, it could be useful to allow children to develop their ability to deal with additive structures before being confronted with the equals sign. Denmark (1976) and Labinowicz (1985) are also in favor of the postponement of the introduction of the equals sign. Denmark (1976) suggests to replace the equals sign by a different symbol at the beginning of elementary school while Labinowicz (1985) suggests to introduce the equals sign only in second grade, while working on different number and additive tasks in first grade.

Is the only postponement of the introduction of the equals sign a viable solution? Saenz Ludlow and Walgamuth's (1998) research seems to indicate that, even when children are confronted later to the equals sign, they have difficulties to understand this symbol as an indicator of a relation. In their research, the equals sign was taught to third grade children who did not have to use the equals sign before when they had to work on additive tasks. During their first and second grade, additions and subtractions were written horizontally, without the equals sign. However, even these children showed a tendency to conceive the equals sign as an operator. The learning of this symbol as an indicator of a relation is therefore also an important cognitive obstacle and the only postponement of the introduction of the equals sign is not sufficient to achieve a better understanding of this symbol.

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Ad Hoc Sessions

Séances ad hoc

Measuring the Impact of a Mentor Program

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In 2002 Pacific Resources for Education and Learning (PREL) received funding for five years from the National Science Foundation (NSF) teacher enhancement program to implement Project MENTOR (Mathematics Education for Novice Teachers: Opportunities for Reflection). Project MENTOR staff work with 4-member teams of mentors drawn from departments and ministries of education and institutions of higher education in the 10 U.S.-affiliated Pacific island communities of American Samoa, the Commonwealth of the Northern Mariana Islands, the Federated States of Micronesia (FSM, which includes Chuuk, Kosrae, Pohnpei, and Yap), Guam, Hawai'i, the Republic of the Marshall Islands, and the Republic of Palau. Project MENTOR established a mentoring program for novice teachers aimed at developing in novice teachers the knowledge, skills, and dispositions necessary to become effective teachers of mathematics.

The Project has a number of areas of desired impact:

- Increase novice teacher (0–3 years experience) ability to plan, implement and assess instructional sequences that reflect a standards-based approach;
- Develop novice teachers' and mentors' abilities to reflect critically on their practices;
- Develop mentors' abilities to effectively mentor novice teachers;
- Develop mentors' abilities to provide effective professional development that contributes to the professional growth of novice teacher;
- Increase mentor and novice teacher mathematical knowledge;
- Increase leader and novice teacher collaboration.

The Project created a number of strategies in an attempt to accomplish the impact desired. These strategies included at least the following:

- Annual week-long institute provided by Project staff for mentors that focuses on (1) mathematical content and pedagogical knowledge, (2) mentoring skills and techniques, and (3) assessment strategies;
- Project staff undertake bi-monthly video and/or telephone conferences with mentors, and where feasible, deliver local workshops and demonstration lessons for mentors and novice teachers;
- Annual week-long institute provided by mentors for novice teachers that focuses on (1) mathematical content, (2) pedagogical knowledge, and (3) and classroom assessment strategies;
- Monthly seminars provided by mentors for novice teachers that focus on the classroom experiences of the novice teachers;
- Monthly observations of the novice teachers by the mentors.

The tools for measuring impact selected by the Project are the following:

- Mathematics Content Test administered to novice teachers each year for the three years they are involved with the Project;

- Novice Teacher Questionnaire administered each year (questionnaire focuses on attitudes and dispositions towards the teaching of mathematics);
- Novice Teacher Survey administered after the yearly institutes that focuses on the support provided to novice teachers by the mentors;
- Mathematics Content Test administered to mentors at the beginning and end of the Project;
- Mentor Questionnaire administered at the beginning and end of the Project;
- Institute assessment instrument completed by mentors at the conclusion of the yearly Project staff-led institutes;
- Group interviews of mentors by island community regarding the observations made by mentors of the novice teachers.

The presentation sought input regarding the match among the focus of impact, strategies chosen, and the tools designed to measure impact. Participants made several suggestions for alternate ways to measure impact that may be better suited to the setting of the Project.

Ten Dimensions of Mathematics Education— A Framework for Improving Mathematics

Douglas McDougall
OISE/University of Toronto

A significant challenge for school leaders is creating a collaborative working environment for staff and community that focuses on shared strategies for school improvement in mathematics. This session introduced a framework (Ten Dimensions of Mathematics Education) for the improvement of mathematics instruction in elementary schools. The session also introduced strategies to assess and support change with respect to the ten dimensions.

The ten dimensions of mathematics education framework distinguishes between traditional and standards-based mathematics instructional approaches, using a four level continuum. This conception developed from a review of 134 mathematics education research studies conducted for the Impact Math project (McDougall, Lawson, Ross, MacLellan, Kajander, & Scane, 2000). In 2002, a further 20 studies on mathematics change were analyzed to provide over 150 studies in total (Ross, McDougall, & Hogaboam-Gray, 2002).

The ten dimensions are: 1. Program Scope and Planning; 2. Meeting Individual Needs; 3. Learning Environment; 4. Student Tasks; 5. Constructing Knowledge; 6. Communicating with Parents; 7. Manipulatives and technology; 8. Students' Mathematical Communication; 9. Assessment; and 10. Teacher's Attitude and Comfort with Mathematics. Within the rubric, Level 1 describes an approach that has been common within traditional classrooms while Level 4 represents the full implementation of Standards-based teaching. The descriptions in the cells of the rubric summarize typical teacher behaviour at four levels of implementation of math education reform. This rubric was developed over several projects in which we observed teachers, mainly elementary, in their mathematics classrooms (McDougall et al., 2000; Ross, Hogaboam-Gray, & McDougall, 2000; Ross, Hogaboam-Gray, McDougall, & Bruce, 2001–2002; Ross, Hogaboam-Gray, McDougall, & LeSage, 2003; Ross, McDougall, & LeSage, 2001).

The ten dimensions framework can be used to provide information to teachers and administrators for personal and school growth in elementary mathematics and to self-identify strengths and possible areas for improvement. Once teachers have placed themselves on the rubric, one or two dimensions can be identified as having less reform-based progress and teachers and administrators can work together to identify strategies to improve mathematics practice in those dimensions. Individual teacher areas of improvement can be clustered to form the school improvement plan.

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To JUMP or not to JUMP

Tom Steinke, *OCCDSB/OMCA*
Stewart Craven, *Toronto District School Board*

From time to time, mathematics education programs emerge in classrooms, schools, or boards of education. These programs are generally created by well meaning individuals who may or may not know and understand mathematics education and the nature of current research in this area. One such program, JUMP (Junior Undiscovered Mathematics Prodigies) was initiated in Toronto as a one-on-one tutoring program for students in the junior grades who were falling through the cracks. Based on the tutors perceived successes and the interest expressed by some teachers and administrators, the JUMP program was converted into a workbook program to be delivered in classes with a great deal of additional support. For example, JUMP instructors and tutors were to orchestrate at least two lessons a week in a number of classrooms. However, it seems that the program at this point is being spread to other boards and schools through the sale of the "workbooks," with a minimal amount of teacher training.

Very little formal research has been done to date. All evidence of success is anecdotal with the exception of a Master's paper by Kaitlyn Hughes that investigated shifts in student attitudes toward mathematics over a 3 or 4 month period. Although some universities have been approached to research the efficacy of the JUMP program, there has been very little interest.

This ad hoc session was created to bring attention to the existence of programs such as JUMP and to start the conversation about the role of the research community when such programs appear. These are some of the observations and questions that were raised at this session:

- Should the "mathematics education community" be on the lookout for programs and play a proactive role in initiating research projects to judge their efficacy?
- Who or what comprises "the mathematics education community"?
- If research is to be done, it must be credible; hence, who are the potential researchers?
- Should researchers be concerned only about giving advice to Ministries, Boards of Education, or schools about what constitutes "effective" mathematics teaching?
- If the Ministry is interested in this data, should it fund the research?

Following are some of the research questions that might help educators better understand the efficacy of programs such as JUMP:

- Is JUMP effective? If so, for whom and in what way?
- What is the long-term reality of students who have participated in the JUMP program?
- Is JUMP sustainable or generalizable? (The sustainability of programs that rely on a large number of volunteers over a extended period of time is a particularly important question, and there may be political implications.)
- Does the JUMP program help students make connections between strands and transfer applications outside the classroom?
- Does the JUMP program improve student attitudes toward doing mathematics?
- Under what conditions do programs such as JUMP arise? Which needs are they fulfilling, and for whom?

Undergraduate Students' Errors That May Be Related to Confusing a Set with Its Elements

Kalifa Traoré, *Université de Ouagadougou*
Caroline Lajoie, *Université du Québec à Montréal*
Roberta Mura, *Université Laval*

In a recent study of students' difficulties with the ideas of normal subgroup and quotient group (Lajoie & Mura, 2004), it was observed that several students misunderstood the very nature of the elements of a quotient group: They seemed to think that the elements of a quotient group G/N were elements of G instead of subsets of G. Believing that a weakness in set theoretical prerequisites might be a factor contributing to this misunderstanding, Lajoie and Mura have started a new research project aimed at examining students' difficulties with the first concepts of set theory and their use in elementary group theory. In the ad hoc group we discussed a few types of errors observed during a pilot study for this project and argued that all the types of errors presented could be related to confusing a set with its elements. All the data were collected from the work of students in one course on logics and set theory. The students were mathematics or computer science majors and had already passed a first course in abstract algebra. Some of the errors observed may seem rather surprising among such a population.

In the ad hoc group, excerpts from students' work were presented that illustrate the following types of errors.

T1: When there are two pairs of nested braces, one of the pairs can be deleted.
Example: $\cup\,(\{\phi, \{\{x_1, x_2\}\}, \{\{y_1, y_2, y_3\}\}, \{\{x_1, x_2, y_1, y_2, y_3\}\}\}) = \{x_1, x_2, y_1, y_2, y_3\}$.

T2: Confusion between belonging and inclusion.

T2.1: If all the elements of A belong to B, then A too belongs to B.
Example: since each of ϕ, $\{x\}$ and $\{y\}$ belong to $\{\phi, \{x\}, \{y\}, \{x, y\}\}$, then $\{\phi, \{\mathrm{x}\}, \{y\}\}$ also belongs to $\{\phi, \{x\}, \{y\}, \{x, y\}\}$.

T2.2: $A \in B \Rightarrow A \subset B$.
Example: since $F \not\subset G$, thus F is certainly not an element of G.

T2.3: If all the elements of A are subsets of B, then A too is a subset of B.
Example: We see immediately that $\{\{x\}, \{y\}, \{x, y\}, \phi\} \subset \{x, y\}$.

T3: Confusion between the union of two sets and the set consisting of those two sets ($A \cup B = \{A, B\}$).
Example: $a \cup c \cup x \cup a = \{a, c, x\}$.

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Expressing Mathematical Fluency

David Wagner
University of New Brunswick

My goal in this ad hoc presentation was to prompt dialogue about the word *just* as it was used by a Grade 11 mathematics student in the following utterance: "And you just change it to two square root five, right?" I described the setting of the utterance in terms of my research agenda and asked the same question I asked the students in the researched classroom: "What does it mean when Jessye says *just*?" The participants in this ad hoc session responded with insightful interpretations and reflections on implications for training mathematics teachers. Their response was helpful to me in my ongoing interpretation of the classroom-based research project.

Research Setting

I had a nineteen-week conversation with thirty-two 15- and 16-year old students and their teacher in a pure mathematics class that I co-taught. I directed their attention to features of their language practice in the mathematics classroom. By stimulating their critical language awareness (c.f. Fairclough, 1992), I hoped to develop a better understanding of a student perspective on mathematics learning and, together with the participants, discover a range of possibilities available to mathematics learners and teachers within their classroom discourses.

Overview of Relevant Research

Generally, the students in this conversation resisted my interest in language by withholding significant participation. However, the exceptions to this passive resistance were generative. Three streams of the larger conversation exemplify some possibilities associated with attention to language. One of these began when I asked the participant students what Jessye meant by the word *just* in the quotation shown above. The students argued about the significance and dangers of using "de-emphasizers" such as *just* to support minimal articulation. The vagueness (c.f. Channell, 1994; Rowland, 2000) of the word *just* can be used to gloss over aspects of the mathematics being discussed. The word represents fluency, as it can replace an elaborate description of a series of processes and procedures that are not central to the task at hand. Some of the students in the researched classroom took exception to teachers using de-emphasizers because what is simple and fluent for teachers is not necessarily simple for students. Other students argued that de-emphasizers are inevitable in mathematics communication.

Questions Raised

What is the effect when a teacher uses the word *just*? When a student uses it?
Does audience matter?
What is the effect of being aware of the role of the word *just*? For teachers? For students?
Which verbs tend to follow the word *just*? ("just go," "just do," *just* with an implied verb …)

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Panel Discussion

Table ronde

What is Canadian about Canadian Mathematics Education?

La didactique des mathématiques au Canada se distingue-t-elle de ce qui se fait ailleurs dans le monde ?

David A. Reid
Acadia University

The panel was asked to address a number of questions, including this one: "*Is there anything distinctive, "Canadian", about mathematics education in Canada?*" My first reaction is that only Canadians would ask such a question. Can you imagine our colleagues in the US, or France, or the Netherlands, wondering if there is anything distinctive about their approach to mathematics education?

Let me address this question further by asking you two trivia questions:

- What country bordering Denmark and France had four participants at CERME (the conference of the European Group for Research in Mathematics Education)? [Hint: Germany had 25.]
- What is unusual about the academic years 1996–1997 and 1997–1998?

Canada is the country bordering Denmark (Greenland) and France (St Pierre et Miquelon) that had four participants at CERME. Which shows that even if you have a conference just for Europeans, Canadians come anyway. And what is unusual about the years 1996–1998 is that Canada has always had a member on the PME IC *except* for those two years (see Table 1). The Canadian presence at PME has been strong in other ways. The President has been a Canadian and in many years there were two Canadian members on the IC. The maximum allowed is three.

You may also have seen a list that was sent out in early May 2004 by the organisers of ICME. It was a list of the

Les membres de la table ronde ont été invités à considérer plusieurs questions. L'une de ces dernières était: « *La didactique des mathématiques au Canada se distingue-t-elle de ce qui se fait ailleurs dans le monde ?* » Ma première réponse est : « Seulement les Canadiens poseraient une telle question. » Peut-être le fait que nous avons posé la question indiquait quelque chose à propos de la didactique des mathématiques au Canada. Nous n'avons pas une tradition établie en didactique des mathématiques comme les français, les américains, les hollandais, etc.

Pour commencer à répondre à cette question, je voudrais poser deux questions supplémentaires:

- Quel pays limitrophe avec le Danemark et la France a eu quatre participants à CERME (la conférence de l'association européene pour la recherche dans l'éducation de mathématiques) ? [Un conseil : L'Allemagne en a eu 25.]
- Qu'est-ce qui fut différent dans les années scolaires 1996-1997 et 1997–1998 ?

La réponse à la question 1 est « Canada ». Ce qui prouve que même si vous avez une conférence juste pour les Européens, les Canadiens viennent de toute façon. Les années scolaires 1996–1997 et 1997–1998 sont exceptionnelles parce que le Canada a toujours eu un membre sur l'IC du PME *excepté* ces deux années (voir Table 1). La présence canadienne à PME a été forte d'autres manières. Le président a été un Canadien et, pendant beaucoup d'années, il y avait deux membres canadiens à l'IC. Le maximum permis est de trois.

D'autres indications: en début mai 2004,

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<tr><td colspan="4">N. Hercovics</td><td></td><td colspan="4">J. Bergeron</td><td colspan="2"></td><td colspan="4">C. Gaulin</td><td colspan="6"></td><td colspan="3">V. Zack</td><td colspan="3">S. Dawson</td></tr>
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TABLE 1. Canadian Members of the IC of PME / Membres canadiens de l'IC de PME

ten countries with the largest number of registrations at that time. Canada was on the list in ninth place. If you take into consideration population however, Canada had the sixth highest number of registrations (see Table 2). And if you don't count the three host countries (Denmark, Sweden and Norway), Canada had the third highest number of registrations, after Australia and the UK.

le Canada a eu 55 participants inscrits à ICME, soit le neuvième plus grand nombre. Si on met ces chiffres en relation avec la population de chaque pays, le Canada se retrouve au sixième rang (voir Table 2). Si vous ne comptez pas, les pays d'accueil (le Danemark, la Suède et la Norvège) le Canada est au troisième rang, après l'Australie et le Royaume Uni.

Pays/Country	Registered participants inscrits	Population (estimation, 6 Mai 2004)	Inscriptions/ Registrations, par/per 10 million
Danemark / Denmark	113	5 397 189	209.4
Norge / Norway	84	4 563 884	184.1
Suède / Sweden	106	8 878 839	119.4
Australie / Australia	90	19 887 840	45.3
Royaume Uni / UK	108	60 247 766	17.9
Canada	**55**	**32 464 240**	**16.9**
Etats Unis / USA	314	292 611 203	10.7
Allemagne / Germany	68	82 426 319	8.2
Japon /Japan	54	127 333 349	4.2
Chine / China	59	1 293 533 754	0.5

TABLE 2. ICME inscriptions/registrations, par/by population

Speaking of PME and ICME, Canada is also one of the few countries that has hosted both ICME and PME.

What is my point? That Canadian mathematics educators participate internationally. In significant numbers and significant ways. That is part of my answer to another question the panel was asked: *"How does our situation compare to what is happening in other countries?"* Mathematics education in Canada seems to be more active, or perhaps just more international in focus, than in many other countries.

You may have noticed something odd about my claim that Canadians play a signifi-

À propos de PME et d'ICME, le Canada est également l'un des rare pays qui a accueilli ICME et PME.

Ainsi, une réponse à la question « *La didactique des mathématiques au Canada se distingue-t-elle de ce qui se fait ailleurs dans le monde ?* », est que les didacticiens des mathématiques canadiens participent internationalement dans des nombres significatifs et des manières significatives.

Les données présentées ci-dessus fournissent également une réponse à d'autres questions que les membres de la table ronde ont été invités à considérer:

cant role on the international mathematics education stage. Of the eight Canadians who have served on the PME IC, six are from Quebec, including the president. And while it is true that Canada has hosted both PME and ICME, it is equally true that Quebec has hosted both PME and ICME. So perhaps it is more accurate to say that mathematics education in Quebec seems to be more active, or perhaps just more international in focus, than in many other parts of Canada.

I'd like to now change my focus and perhaps to touch on some answers to the question, *"What are the issues and problems facing mathematical education in Canada?"* It is clear that the issues and problems facing mathematics education in Canada vary depending on where you are. For example, in some parts of the country, like Atlantic Canada, students' performance on international assessments are worrisome. As you can see from Figure 1 the range of scores within Canada is about as wide as between the US and Japan, two well known cases for comparison. Atlantic Canadian students do about as well as those in the UK and US, well below the international averages of about 50 and 520. Canada as a whole is just above the international average, and has almost exactly the same scores as Australia. As far as I know TIMSS and PISA results don't cause any great anxiety in Quebec and Alberta.

« *Comment notre situation se compare-t-elle à celle qui prévaut dans d'autres pays ?* » La didactique des mathématiques au Canada semble être plus active, ou peut-être animée d'un esprit plus international, que dans beaucoup d'autres pays.

Mais, il y a deux faits intéressants:

- Des huit Canadiens qui ont servi sur l'IC du PME, six sont du Québec.
- Et tandis qu'il est vrai que le Canada ait accueilli PME et ICME, il faut dire que les deux réunions étaient au Québec.

On peut dire que la didactique des mathématiques au Québec semble être plus active que dans d'autres régions du Canada. (Peut-être que ça change : Le canadien qui sert maintenant à l'IC du PME est de l'ouest du Canada.)

Les membres de la table ronde ont été également invités à considérer la question : « *Quelles sont les questions auxquelles la didactique des mathématiques tente de répondre* au Canada *et quels sont les problèmes auxquels elle fait face ?* » Au Canada, les questions et les problèmes se posant à la didactique des mathématiques changent selon l'endroit où vous êtes. Par exemple, dans certaines régions du pays, comme la région atlantique, la performance des élèves sur des évaluations internationales sont un souci (voir figure 1). Les élèves de la région atlantique sont en dessous de la moyenne internationale. Le Canada, dans l'ensemble, est dans la moyenne.

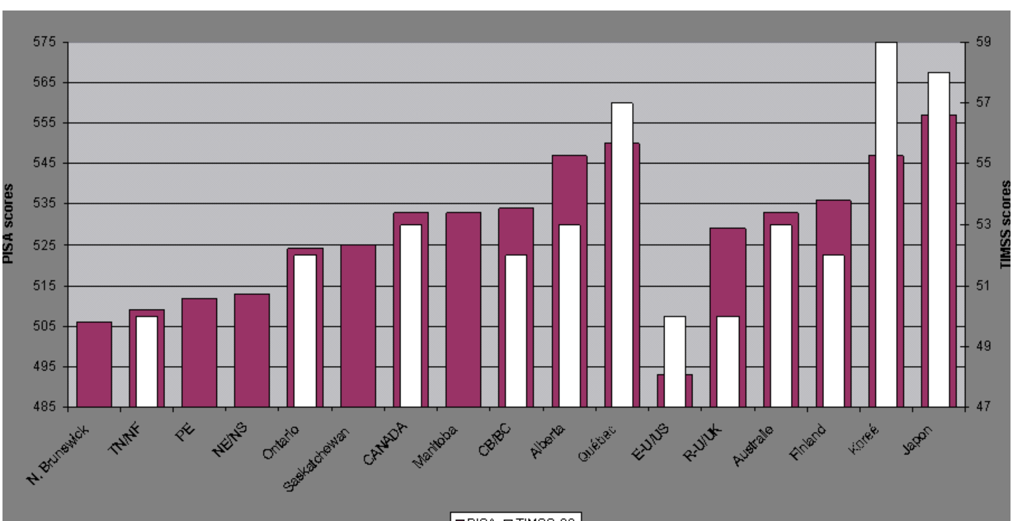

FIGURE 1. Performance des élèves canadiens / Canadian students' performance TIMSS-99 & PISA

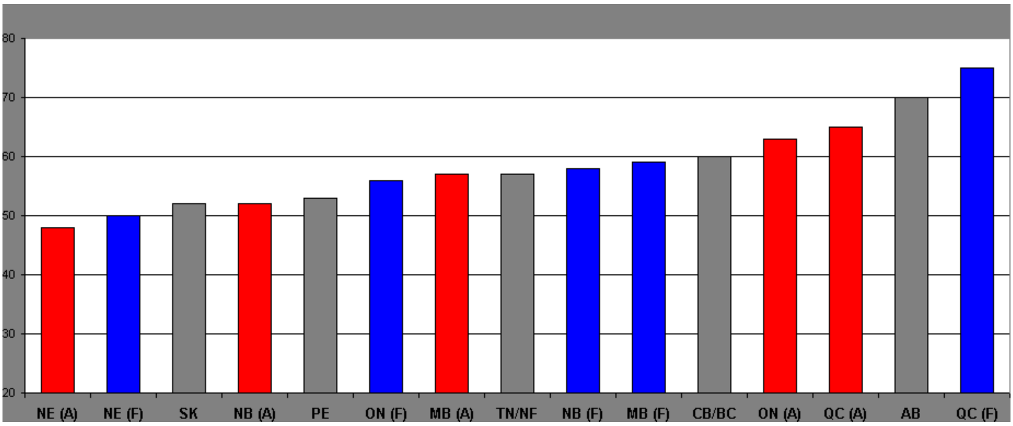

FIGURE 2. Pourcentage des élèves de 13 ans atteignant le niveau 2 ou en haut dans le contenu mathématiques / Percentage of students achieving level 2 or higher on mathematics content PIRS / SAIP 2001

The results from the mathematics content test of SAIP show a similar picture (See Figure 2). The Maritime provinces are the short bars on the left and Quebec and Alberta are on the right. There is an interesting pattern in the linguistic breakdown. Five provinces had sufficiently large populations of anglophones and francophones to allow comparison: Nova Scotia, New Brunswick, Quebec, Ontario, and Manitoba. In all but Ontario the French speaking students performed better on SAIP's content test than the English speaking students.

The difference is even more striking in the Problem solving results (see Figure 3). Again francophones outperform anglophones in Manitoba, Quebec, New

Les résultats du PIRS, examen en contenu mathématiques, montrent une image semblable (voir Figure 2). Ils indiquent aussi que les questions et les problèmes changent aussi selon la langue que vous parlez. Cinq provinces ont des populations suffisamment grandes d'anglophones et de francophones pour permettre une comparaison. Au Manitoba, au Québec, au Nouveau Brunswick et en Nouvelle-Écosse, les élèves francophones ont mieux réussi cet examen que les élèves anglophones.

Il y a aussi des différences importantes de performances entre élèves anglophones et francophones sur l'examen PIRS en résolution de problèmes (voir Figure 3). Encore, les francophones ont mieux réussi que les anglophones (au Manitoba, au Québec, au

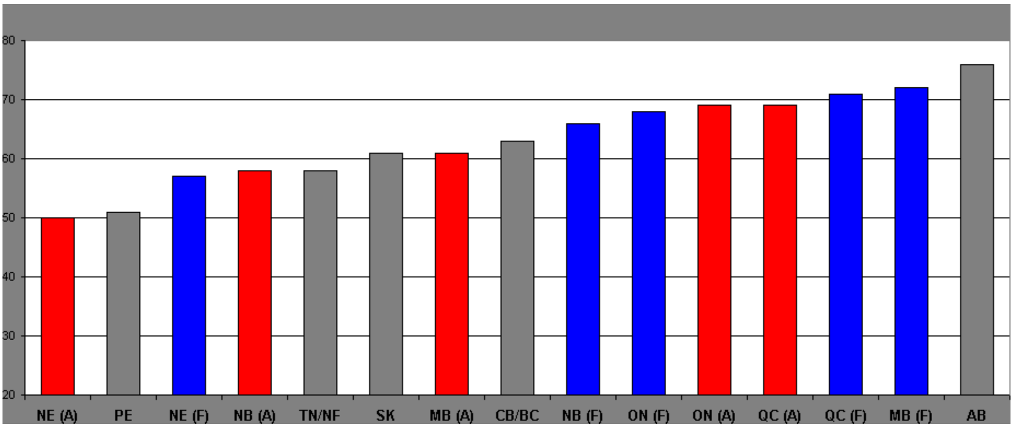

FIGURE 3. Pourcentage des élèves de 13 ans atteignant le niveau 2 ou en haut dans la résolution de problèmes / Percentage of students achieving level 2 or higher on problem solving PIRS / SAIP 2001

Brunswick and Nova Scotia. 72% of Manitoba francophones achieved Level 2 or above, the second highest percentage of students, while Manitoba anglophones were ninth at 61%.

This suggests that having two languages and ten provinces *has an impact* on the issues in mathematics education in Canada. The underlying cause for this impact is not immediately obvious. It seems worthwhile to do some research on this, and some has been done, looking at differences in curriculum.

Recently I have been looking at the preparation of Elementary school teachers to teach mathematics in our teacher preparation programs. There are definitely some striking differences within Canada in teacher preparation.

Figure 4 shows the number of contact hours students in programs for future Elementary school teachers have in mathematics content and teaching methods in the different provinces. You can see that there is a lot of variation in when and if these student study mathematics beyond high school. Some are required to student mathematics at the university level before entering their teacher preparation programs (diamonds). Other have to study mathematics as part of the program (stripes). Still others don't have to study mathematics at all. And the total number of hours, as you can see, varies considerably, even within one province.

Nouveau Brunswick et en Nouvelle-Écosse). 72% de francophones du Manitoba ont atteint le niveau 2 ou plus haut. C'est le deuxième plus haut pourcentage au pays, alors que les anglophones du Manitoba étaient au neuvième rang à 61%.

Ceci suggère que le fait que le Canada ait deux langues et dix provinces *a un impact* sur les questions d'éducation en mathématiques au Canada. La cause fondamentale de cet impact n'est pas immédiatement évidente, mais il est clair qu'ici se présente une occasion pour des recherches sur les effets de la culture en didactique des mathématiques (peut-être unique au monde).

Récemment j'ai étudié la préparation des futurs enseignants pour 'enseignement des mathématiques au primaire dans nos programmes de formation. Il y a certainement quelques différences importantes au Canada dans la formation professionnelle.

La figure 4 montre le nombre d'heures de cours en contenu de mathématiques et en didactiques des mathématiques dans les programmes pour les futurs enseignants du primaire dans les provinces. Il y a beaucoup de différences. Certains étudiants étudient les mathématiques au niveau universitaire avant de suivre leur programme de formation (losanges). D'autres doivent étudier les mathématiques en tant qu'élément du programme (diagonales). D'autres n'étudient pas du tout de mathématiques à l'université. Et le nombre total d'heures change considérablement, même dans une même

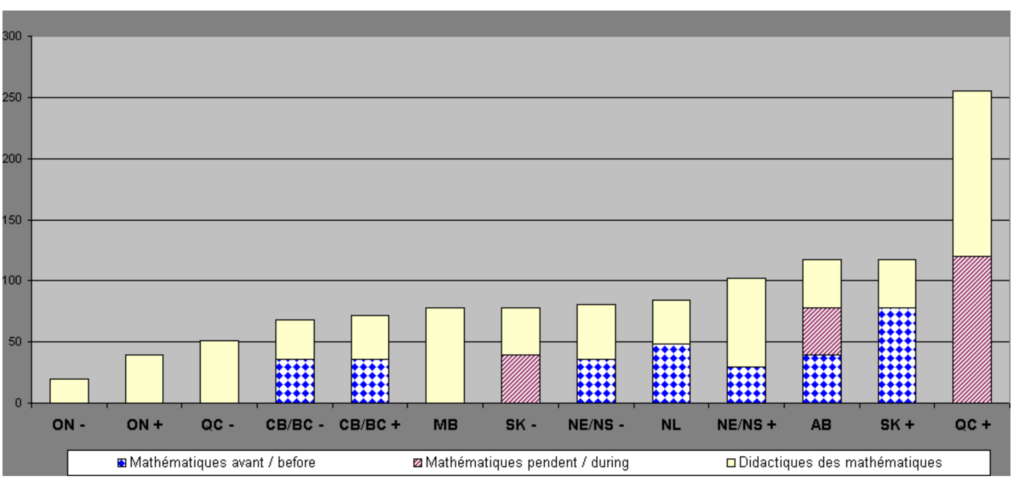

FIGURE 4. Heures du cours requises pour les futurs enseignants du primaire / Course hours required for future elementary school teachers

The biggest difference is in Quebec, between francophone universities (high) and anglophone universities (low). Across Canada the university with the greatest number of contact hours for math or math methods had over *ten times* as many hours as the university with the smallest number of hours.

On a methodological note, this doesn't cover every teacher preparation program in Canada. In some cases I ignored small programs. In others I doubted the accuracy of the data but didn't have time to double check, and there were also cases like Calgary where the structure of the program resisted this sort of quantification.

Does the wide range of number of contact hours in the preparation of future Elementary school teachers matter? I don't know, but it seems like it might be worth doing some research on. Figure 5 might suggests some research questions, like:

- Why do Ontario students (especially anglophones), do relatively well on tests when their teachers have very little pre-service training?
- Does the high level of preparation of francophone teachers in Quebec somehow account for the success of

province. La plus grande différence est au Québec, entre les universités francophones (beaucoup d'heures) et les universités anglophones (moins d'heures).

Au Canada, l'université avec le plus d'heures de cours en mathématiques, ou en didactique des mathématiques, a *dix fois* plus d'heures que l'université avec le moins d'heures.

Notez qu'il y a des limitations à propos de mes donnés. Cette étude ne couvre pas toutes les universités canadiennes. Quelques-unes avaient de trop petits programmes. Dans d'autres cas, j'ai douté de l'exactitude des données mais n'ai pas eu de temps pour vérifier. Il y avait également des cas comme l'université de Calgary où l'élément « mathématiques » ne pouvait pas être dégagé de l'ensemble du programme.

Est-ce que l'éventail du nombre d'heures de cours importe ? Il pourrait être intéressant de faire d'autres recherches, d'explorer plus loin. Peut-être la figure 5 peut donner des idées, comme :

- Pourquoi les élèves de l'Ontario réussissent aux examens comme TIMSS quand leurs enseignants ont peu d'heures de formation avant que leurs carrières commencent ?
- Est-ce que le niveau élevé de préparation des enseignants francophones au Québec explique le succès des élèves francophones

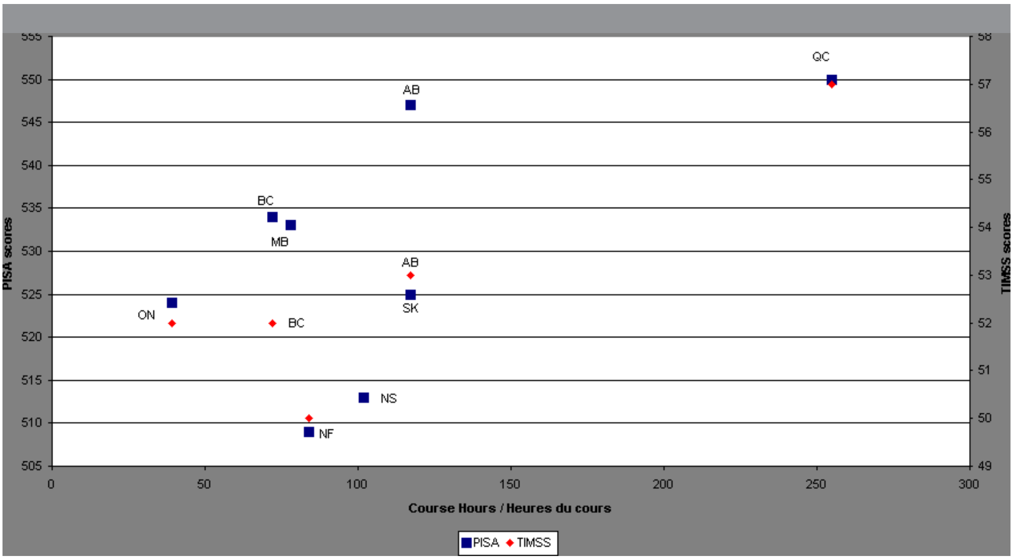

FIGURE 5. Heures du formation des maîtres en mathématiques / Hours of teacher preparation in mathematics versus Timss & PISA

francophone students outside of Quebec?

In conclusion I would suggest that it is important for Canadians to continue to participate in international mathematics education groups, but also that within Canada there is such a range of approaches, conditions and results that we could learn a great deal about fostering mathematics education by making some honest comparisons between the conditions in our regions, provinces and language groups.

Most strikingly, there seem to be things happening in Quebec that are very different from what is happening in the rest of Canada. Perhaps it is time for those of us in English Canada to stop waiting for our colleagues from Quebec to come to us and let us know what they are doing in our language, and time for us to start asking questions in theirs.

en dehors du Québec?

En conclusion, il me semble important que les Canadiens continuent à participer aux groupes internationaux d'éducation en mathématiques. Mais également, puisqu'à l'intérieur du Canada il y a une grande gamme d'approches, de conditions et de résultats, nous pourrions apprendre beaucoup sur la façon de stimuler l'éducation en mathématiques. Il faudrait faire des comparaisons honnêtes entre les conditions dans nos régions, nos provinces et nos groupes de langue.

Un fait saisissant : il semble y avoir des choses se produisant au Québec qui sont très différentes de ce qui se produit dans le reste du Canada. Peut-être il est temps pour nous au Canada anglais de cesser d'attendre que nos collègues du Québec viennent vers nous pour raconter leurs expériences en anglais. Nous pourrions aller vers eux et assister à leurs colloques en français, comme les quatre qui ont assisté à GDM en 2004.

What are the Issues and Problems Facing Mathematical Education in Canada?

Eric Muller
Brock University

Members of the Panel were asked to respond to a number of questions which they had received ahead of time. In this report for the CMESG Proceedings I have selected to comment on only one of the questions, the one on which I spent most of my Panel presentation time. Nevertheless some of the points I discuss address other questions. The views that I express reflect my position as a professor of mathematics. They are coloured by my experiences teaching courses for future teachers at all school levels, by my use of technology in undergraduate mathematics education and also by my work on various expert panels for the Ministry of Education in Ontario. Unfortunately I have only second hand knowledge of what happens in the rest of Canada.

The question that I have chosen to comment on is: 'What are the issues and problems facing mathematical education in Canada?' To me this question was by far the most important. At the start of the 21st century, mathematics education as an intellectual pursuit for all Canadians is a fragile beast. It is a beast because mathematics raises apprehension within the great majority of the population, and because mathematics is one of two main academic factors that contribute to students failing to graduate out of the Ontario school system. I suspect that the latter is true in the majority of the other Provinces as well. In Ontario the other academic factor identified as a barrier to students completing their school education is literacy. The Ontario Ministry of Education will soon be releasing two reports from task forces on students at risk, one is proposing ways to address the problem of students at risk in numeracy, the other in literacy. Mathematics education for all Canadians is fragile because mathematics is only *believed* to be necessary for all citizens. Most Canadians believe that they cannot do mathematics and by default continue to accept the stereotype that mathematics is for the few. Their experiences in mathematics classrooms at most educational levels were mostly negative, and they have rarely experienced the necessity of mathematics in their lives. The layperson seldom hears significant and convincing arguments about the necessity for all individuals to develop life long learning skills such as number sense, spatial sense, a problem solving capacity, an ability to make decisions in non deterministic situations, etc. Jeremy Kilpatrick (2003) (ref 1) suggests that these are areas that provide intellectual reasons for people to learn mathematics. Individuals asked to document their understanding of the nature of mathematics turn to examples in arithmetic or geometry, and rarely refer to experiences in areas of mathematics that are not easily tested or are not merely the implementation of procedures. Kilpatrick also describes an evolution of school mathematics curricula during the second part of the 20th century. For the first part of the century, mathematics curricula reflected a clear division between primary and secondary education, where in the primary, the curricula emphasized the practical, while in the secondary the emphasis was on the intellectual. In the second part of the 20th century, with the growth of education from the few to all, the mathematics curricula adapted by moving some of the intellectual components into the primary grades and by inserting practical components into the secondary grades.

This has serious implications for the mathematics education of elementary and secondary school teachers, implications that have not been thought through. Although there have been changes in Ontario educational requirements for future elementary school teachers there have been no evolution of requirements in mathematics education. Mathematics requirements for future secondary school mathematics teachers have not evolved to meet the needs of curricula that call for more applications and modelling. In summary the first major issue facing mathematics education in Canada in the 21st century arises from the fact that the great majority of the population sees mathematics as not for them, nevertheless they are prepared to support a compulsory mathematics education in schools. How long can this support last? Could a small change in beliefs and opinions see mathematics moved from the compulsory to the optional side of the school's discipline ledger?

The second major issue facing mathematics education in Canada in the 21st century is technology. Technology will have a far greater impact on all levels of mathematics education than most mathematics teachers and professors are prepared to accept. John Conway (1997) (ref 2) in his article concerning the uncertain future of today's mathematics departments writes: "We have to embrace technology, I don't mean just tolerate it; embrace it and celebrate it. The professional mathematics community must adapt and learn how to best incorporate technology into instruction. With the existence of powerful, inexpensive computers, I see mathematics departments rethinking their entire curriculum. Otherwise we are out of business." I have previously parallelled the impact of transport technology on society, with the impact of technology on mathematics education. Here are some parallels:

1. Modes of transportation provide individuals with choices on how they will reach their destination. In this case many parameters come into play such as time, costs, reason for travel etc. Technology will provide individuals alternative ways to meet the required level of mathematics education that is necessary to achieve their employment goals. For example, individuals who need to apply some specific mathematics now have the choice to do some of manipulation themselves, or can leave all the detailed algebraic and computational work to the technology, or use powerful simulation softwares.
2. Evolutions in transport technologies have had enormous impacts on infrastrutures and on those who depended on them. More recent examples have been the development of aircraft designs and abilities. The airport in Gander, Newfoundland had its heyday when all transatlantic flights had to fuel up to make the crossing. Mirabel was first opened when all flights to Canada from the East had to land in Montreal. Changes in airline partnerships and opening of borders soon made it obsolete as travellers could get to their destination in Canada without stopping in Montreal. Technology can now do all, or nearly all, well structured mathematical questions that students meet in their first two years of undergraduate mathematics. How long is it before other departments demand that their students concentrate on developing their mathematical concepts and leave the rest of the mathematics to technology, thereby avoiding a majority of the technical mathematics presently in the first two years of undergraduate mathematics.
3. National airlines developed an economic model that worked well as long as there was a group of business travellers who were willing to pay a heavy price. With competition and deregulation this model is failing them. How long will the 20th century mathematics education model which has Calculus as its destination survive? Lynn Steen (2003) (ref 3) notes the tremendous growth of knowledge in the mathematical sciences during the last half century, an expansion which includes linear programming, bioinformatics, string theory, simulation, etc. Steen reflects on the lack of response by mathematics school and university curricula. He writes:

   > Paradoxically, during this same half-century the role played by calculus in education, has expanded enormously, often irrationally. Today the mathematical focus of secondary school is calculus, not mathematics. Even as the mathematical sciences have

> built avenues of intellectual exchange extending in many different directions, the signposts of society still direct everyone to enter mathematics on the traditional highway of calculus. The question that we must confront at the start of this new century is whether this traditional route still serves mathematics well or whether it might not be prudent to explore other options through which students can enter the world of mathematics.

The third major challenge for mathematics education in Canada is the existing situation that, whereas mathematics crosses language and provincial boundaries, mathematics education finds it difficult to do so. Within Ontario, even though the school mathematics curricula are the same for the two language groups, there is great inequity in student and teacher resources available for the two groups. There is a real and understandable desire by the Francophone schools to develop materials and other resources in an independent way. This was evident to me this year when two separate Ministry Task Forces were formed to study students at risk in mathematics—one for Anglophone schools, the other for Francophone schools. I was invited to join the English panel. The only interaction between the two Panels occurred after each group had developed its report and I asked to read the French report. I was able to note differences in recommendations and emphases between the two reports. Anglophone teachers do not access rich resources developed or accessible to Francophone teachers, and vice versa. It is a shame that the many wonderful 'popularization of mathematics' resources developed in Quebec and in France never reach the rest of Canada.

This is the first year that I have been in a truly bilingual working group. CMESG must work harder to ensure that it occurs more frequently.

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Unique Features of the Body of Mathematics Education in Canada: A Brief Sketch

Thomas E.Kieren
University of Alberta

Outsiders to the Canadian Mathematics Education Study Group often ask us, "How did this group get to be so different from other groups interested in mathematics education. We often suggest that our nature is owed to a unique conjoint interest and collaboration between those who consider themselves practicing mathematicians and those who consider themselves mathematics educators; to the deep interaction-promoting nature of the structure of our meetings; and to the unique personalities of our organization's leaders especially David Wheeler and their continuing influence. I would like to suggest here that the nature of our organization is also related to some of the unique features of the nature and practices of mathematics education in Canada. In this brief sketch I will highlight four such features.

If one looks at the history of mathematics curriculum development particularly at the secondary school level one is struck by the long term interest in curriculum development (taken broadly) by members of the capital-M Mathematics community in Canada. While this is less the case now than in the past, curriculum development and even text book development involved unique *collaborations between interested practicing mathematicians and high quality school teachers of mathematics*. Like the deliberations of CMESG, such curriculums reflected the interest in the practices of the mathematics community as well as the mathematical practices of students in school. That is, just as in CMESG deliberations we do not separate a concern for the nature of mathematical practices from the practices appropriate for learning and teaching mathematics, such curriculums did not divorce a concern for the nature and application of mathematical ideas from the nature of mathematics knowing, especially in young persons. In fact, I think it is a time to renew our commitment in this country to curriculum practices that at once consider the personal, the collective and the cultural/mathematical aspects of mathematics knowing and the collaborations necessary to bring this about.

In our meetings of CMESG we always consider both mathematics and the mathematical as we contemplate the problems of mathematics education in Canada. I think this reflects a long standing and continuing interest in and the valuing in *our curricula of not only the content of mathemat<u>ics</u> but also the mathematic<u>al</u> processes of reasoning, problem solving, explaining, justifying and proving which form such an integral part of the living body and practices of mathematics*. As David Reid suggests in his part of this discussion, it is just in the places where this conjoint interest is developed in the curriculums most strongly, Quebec and Alberta, that students seem to achieve well by national and international standards.

Especially over the last 40 years or so, the distinguishing curriculum development features discussed above have been supported by an active mathematics education research community. The ongoing work of this community has always been supported by and integrated into the deliberations of CMESG, and to a certain extent at least has formed an important part of mathematics education in Canada more generally. If one looks across the work done by individuals across the various universities in Canada both in Quebec and in other places, one is struck by the number of long term programs of research that have been carried out in this country. That is, *the research has focused on large problems of mathematics and its learning, research which did not separate an interest in the nature of mathematics and the mathematical from the nature of possible appropriate learning and teaching approaches*. While this work has been very theoretical in its basis, it has not been separated from practices of mathematics knowing and teaching in schools and even universities.

In talking about the uniquenesses of CMESG I mentioned David Wheeler. David lived out a commitment to mathematics education in Canada while being very active in and knowledgeable about the practices of mathematics education all over the world. Such a joint interest is seen in many other members of CMESG as well. For me this is reflective of the final feature of Canadian mathematics education on which I will comment—a feature I call its *double embodiment*. While dealing with our unique settings and problems and developing unique curriculums and practices, I think Canadian mathematics education has never been insular and has been open to consideration of mathematics curricular and mathematics knowing ideas from other parts of the world. Perhaps it is our bicultural nature that has occasioned us to have continuing interest not only in the work of our American neighbors, but also continuing connections to European developments and more recently to developments in mathematics education in other areas as well. I think such a non-insular approach has served us well, even if at times it seems that we know the mathematics education of other places in the world better than we know of those practices in other places and communities in Canada. Perhaps it is time that our inter-provincial practices catch up with our international ones.

Of course, mathematics education shares many if not most features with mathematics education anywhere. This "universality" is more true of mathematics education than perhaps any other aspect of school education. What I have tried to do here is to point to four aspects of mathematics education in Canada—the collaboration of the mathematics and the educational communities in developing the practices of and particularly the curricula for school mathematics; a continued emphasis both on knowledge of mathematics and about the processes and practices of the mathematical; the research foci on large, long term, problems; and the willingness to consider and transform for our own use ideas from other countries (as well as developing our own unique ideas)—which provide the potential for uniqueness in mathematics education here. So that that unique nature might continue to have value for us, it is important that they not just be historic artifacts but continue to be adapted and developed in such a way as to provide bases for continuing solid mathematics education in Canada.

Appendices

APPENDIX A

Working Groups at Each Annual Meeting

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications

- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving

· Inservice mathematics teacher education
· Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

· Mathematics and the arts
· Philosophy for children on mathematics
· The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre : Implications pour l'enseignement des mathématiques au primaire et au secondaire
· Mathematics, the written and the drawn
· Des cours de mathémathiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

· L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
· Teacher research: An empowering practice?
· Images of undergraduate mathematics
· A mathematics curriculum manifesto

APPENDIX B

Plenary Lectures at Each Annual Meeting

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching
1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics

1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning
1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling

1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. BARTOLINI BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole

APPENDIX C

Proceedings of Annual Meetings

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243
Proceedings of the 1994 Annual Meeting	ED 407242
Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	*submitted*
Proceedings of the 2003 Annual Meeting	*submitted*
Proceedings of the 2004 Annual Meeting	*submitted*

Note

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.