CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2009 ANNUAL MEETING / RENCONTRE ANNUELLE 2009

CMESG GCEDM

York University
June 5 – June 9, 2009

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33rd Annual Meeting
York University
June 5 – June 9, 2009

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ACKNOWLEDGEMENTS / REMERCIEMENTS

The organisational work for our 2009 annual meeting at York University enabled us to have another memorable meeting. Our local organisers, Margaret Sinclair and Walter Whiteley, with the exceptional help from the Faculty of Education and York University, managed to do everything with utmost efficiency and pleasantries, and we thank them. We would also like to thank the the Fields Institute and MITACS for financial support for this conference. Finally, we extend our thanks to the guest speakers, working group leaders, topic session and ad hoc presenters, and all the participants for making the 2009 meeting a stimulating and worthwhile experience.

L'organisation locale de notre rencontre annuelle de 2009 à l'Université York nous a permis d'avoir une autre rencontre mémorable. Nos organisateurs locaux, Margaret Sinclair and Walter Whiteley, avec le support exceptionnel de la Faculté d'éducation et l'Université York, ont réussi à organiser la rencontre avec une grande efficacité et nous les en remercions. Nous aimerions aussi remercier Fields Institute et MITACS pour l'appui financier qu'ils ont fourni à la conférence. De plus, nous aimerions remercier les conférenciers invités, les animateurs de groupes de travail, les présentateurs de séances thématiques et d'ateliers ad hoc, ainsi que tous les participants pour avoir fait de la rencontre 2009 une expérience stimulante et mémorable.

Horaire

Vendredi 05 juin	Samedi 06 juin	Dimanche 07 juin	Lundi 08 juin	Mardi 09 juin
Activité pré-colloque Pour de plus amples informations, voir : http://wiki.math.yorku.ca/index.php/Math_to_Math_Ed	Inscription 8:00 – 9:00			
3:00 – 5:30 INSCRIPTION	9:00 – 10:30 Groupes de travail	9:00 – 10:30 Groupes de travail	9:00 – 10:30 Groupes de travail	9:00 – 10:00 Séance thématique (2)
5:30 – 6:45 BBQ *Michelangelo's*	10:30 – 11:00 PAUSE CAFÉ	10:30 – 11:00 PAUSE CAFÉ	10:30 – 11:00 PAUSE CAFÉ	10:00 – 11:00 Conversation with Lee Lorch
7:00 – 8 :00 Session d'ouverture	11:00 – 12:30 Groupes de travail	11:00 – 12:30 Groupes de travail	11:00 – 12:30 Groupes de travail	11:00 – 11:30 PAUSE CAFÉ
8:00 – 9:00 Plénière 1 De Vries	12:30 – 1:30 DÎNER	12:30 – 1:30 DÎNER	12:45 – 1:45 DÎNER	11:30 – 12:30 Session de clôture
9:00 RÉCEPTION *Michelangelo's*	1:30 – 2:00 en petit groupe	1:30 – 2:30 Plénière 2 Borba	1:55 – 2:25 en petit groupe	
	2:10 – 3:10 Discussion de la plénière 1	2:45 – 3:15 Nouvelles thèses(1)	2:30 – 3:30 Discussion de la plénière 2	
	EXCURSION	3:20 – 4:20 Séance thématique (1)	3:45 – 4:15 Séances ad hoc (3)	
		4:25 – 4:55 PAUSE CAFÉ	4:15 – 4:45 PAUSE CAFÉ	
		5:00 – 5:30 Nouvelles thèses(2)	5:00 – 6:15 Assemblée générale annuelle	
		5:40 – 6:10 Séances ad hoc (1)	6:30 REPAS *à l'université*	
		6:15 – 6:45 Séances ad hoc (2)		
		7:15 REPAS *à l'université*		

Schedule

Friday June 5	Saturday June 6	Sunday June 7	Monday June 8	Tuesday June 9
Pre-conference Activities. For more information see http://wiki.math.yorku.ca/index.php/Math_to_Math_Ed	Registration 8:00 – 9:00			
3:00 – 5:30 Registration	9:00 – 10:30 Working Groups	9:00 – 10:30 Working Groups	9:00 – 10:30 Working Groups	9:00 – 10:00 Topic Sessions (2)
5:30 – 6:45 BBQ *Michelangelo's*	10:30 – 11:00 COFFEE BREAK	10:30 – 11:00 COFFEE BREAK	10:30 – 11:00 COFFEE BREAK	10:00 – 11:00 Conversation with Lee Lorch
7:00 – 8:00 Opening session	11:00 – 12:30 Working Groups	11:00 – 12:30 Working Groups	11:00 – 12:30 Working Groups	11:00 – 11:30 COFFEE BREAK
8:00 – 9:00 Plenary 1 De Vries	12:30 – 1:30 LUNCH	12:30 – 1:30 LUNCH	12:45 – 1:45 LUNCH	11:30 – 12:30 Closing Session
9:00 RECEPTION *Michelangelo's*	1:30 – 2:00 Small Group Disc.	1:30 – 2:30 Plenary 2 Borba	1:55 – 2:25 Small Group Disc.	
	2:10 – 3:10 Discussion of Plenary 1	2:45 – 3:15 New PhDs (1)	2:30 – 3:30 Discussion of plenary 2	
	EXCURSION	3:20 – 4:20 Topic Sessions (1)	3:45 – 4:15 Ad hoc sessions (3)	
		4:25 – 4:55 COFFEE BREAK	4:15 – 4:45 COFFEE BREAK	
		5:00 – 5:30 New PhDs (2)	5:00 – 6:15 General Meeting	
		5:40 – 6:10 Ad hoc sessions (1)	6:30 DINNER *university*	
		6:15 – 6:45 Ad hoc sessions (2)		
		7:15 DINNER *university*		

INTRODUCTION

Florence Glanfield – President, CMESG/GCEDM
University of Alberta

It is with pleasure that I write the introduction to the 2009 Proceedings of the 33rd Annual Meeting of the CMESG/GCEDM. As I write these words I am reminded of vivid experiences around our time at York. I am especially reminded of the dedication of Margaret Sinclair, Walter Whiteley, and their organizing team. Margaret, Walter, and their team worked tirelessly while experiencing a very tumultuous time at York University; up to 4 months before the meeting it was not certain that we could be hosted at York. I want to offer a special thanks to Margaret, Walter, and the entire local team for all of the arrangements. We truly felt warmly welcomed to the campus and the city.

In addition to the social activities and socializing, the CMESG/GCEDM annual meetings are about the rich conversations that we have with colleagues; the program is designed for many opportunities to come to know and understand multiple perspectives around multiple topics and issues in mathematics teaching and learning.

I remember the initial planning for the 2009 meeting. The Executive made the decision to plan for six working groups at the 2009 meeting. Attendance at the 2009 meeting was once again closer to 100 than to 75, which meant that six working groups offered a greater possibility for conferring with colleagues about the ideas. The CMESG/GCEDM community is very fortunate to have individuals who are willing to say 'yes' when invited to co-lead working groups, and in 2009 our working group conversations circulated around mathematically gifted students, mathematics and the life sciences, contemporary and emergent research methodologies in mathematics education, reframing learning mathematics as collective action, studying teaching in practice, and mathematics as social (in)justice. A special thanks to the working group leaders. Working group leaders plan for conversations prior to the meeting, meet repeatedly during the meeting to ensure coherence between daily meetings of the working group, and then spend time following the meeting preparing the report for the proceedings. Thanks to Caroline, Dave, Ed, Gladys, Hassane, Hongmei, Jamie, Jo, Laurent, Lyndon, Lucie, Margo, Richard, Viktor, and Yves for sharing their time and expertise.

We were also joined by three plenary speakers – Gerda de Vries, Marcelo Borba, and Lee Lorch. Our 2009 annual meeting was started by Gerda's talk around mathematical biology; two days later, we learned of the production of mathematical knowledge in online environments in Marcelo's talk. Our conference closing plenary was a conversation with Lee Lorch; we all were inspired by stories that Lee shared of his experiences as an activist and a mathematician. Our topic sessions, the emergence of disparities in mathematics classrooms, mapping multiple worlds – imagining school mathematics beyond the grid, la dimension didactique de concepts mathématiques avancés: un exemple avec les séries, and étude des sens accordés à la relation d'égalité et au signe d'égalité dans la réalisation d'activités portant sur le concept d'égalité; celebrating 5 new PhD's in mathematics education; and participating in the ad hoc sessions ensured that we left the meeting with new insights, wonders, and curiosities about the way in which we live, teach, and research. On behalf of the entire community, I extend a sincere note of appreciation to each person who contributed to the scientific program.

Finally I wish to thank the 2008 – 2009 Executive, Brent Davis, Doug Franks, Laurent Theis, Dave Wagner, and Walter Whiteley, who planned the program; Eva Knoll and Viktor Freiman who helped with the translation of the program; and Peter Liljedahl, Susan Oesterle, and Veda Abu-Bakare, editors of these proceedings, for their dedication in the production process.

As you read through these proceedings you will be reminded of the energy in the conversations you had while at York, the food that you nibbled while in those conversations, of the new friends that you met, and of the contributions of so many individuals.

Plenary Lectures

Conférences plénières

HUMANS – WITH – MEDIA AND THE PRODUCTION OF MATHEMATICAL KNOWLEDGE IN ONLINE ENVIRONMENTS

Marcelo C. Borba
São Paulo State University, UNESP at Rio Claro, Brazil

INTRODUCTION

Much research has been conducted about the use of software in mathematics education since the 1980's. However, this research has not necessarily resulted in the incorporation of computer technology into the mathematics classroom. There are many possible explanations, including teachers' lack of familiarity with this research, or even their lack of time available to study how to incorporate results of research into the classroom. In Brazil, the situation is even more dramatic, as many teachers teach 40 classes a week or more. While this problem has not been totally resolved, another layer of "technological" problem has emerged. With the availability of the Internet in many schools and many possibilities to take online courses, teachers are now faced with pressure to participate in such courses while at the same time learning to incorporate the Internet into the classroom. In Borba (2009), I have dealt with the possible scenarios for education in the future if the Internet is "fully accepted" in the classroom. In this paper, I will deal with the possibility of using online courses for teachers, to support the use of mathematics software in high school classrooms. Toward this end, I will discuss a perspective on the use of technology based on the construct "humans-with-media", and a model of online education that I believe is coherent with this perspective. The discussion will be linked to the Brazilian context, so that the reader can understand the social environment that generates these kinds of ideas related to technology and to online teacher education.

NOTES ON THE SHORT HISTORY OF INFORMATION AND COMMUNICATION TECHNOLOGY IN BRAZILIAN MATHEMATICS EDUCATION

Distance education has been an institution in Brazil for more than 150 years (Litto & Formiga, 2009). Distance education done by regular mail, radio and TV has been important in Brazil due to the great size of the country (larger than the continental continuous USA), and also to the concentration of universities in the southeast region of the country, where cities like Rio de Janeiro and São Paulo are located. Since the end of the 1990's, the increased availability of the Internet led to research on new ways to carry out distance education, and a community of self-educated online professors began to emerge. In mathematics education, the research group GPIMEM[1] began developing online courses in 1999 (Borba & Villarreal, 2005), generating one of the first, if not the first, doctoral dissertations on online mathematics education (Gracias, 2003).

Research in this area built on previous research in information technology that was developed in Brazil. In this country, we can identify three phases of research on technology and mathematics education. The first is the "Logo period", in which some researchers, who were not necessarily associated with mathematics, introduced the ideas related to Logo that later, in the late 1980's and early 1990's, became the objects of theses and dissertations. A second phase, which overlaps on the time line with the first, focuses on the use of specific software –

[1] Technology, Other Media and Mathematics Education Research Group
www.rc.unesp.br/igce/pgem.gpimem.html

e.g. function or geometry software. Leading researchers during this phase returned from abroad in the early 90's after having conducted research on this topic in different countries, such as the USA, France and England, which contributed to making it a topic of research in Brazil as well. Implementation of computers in education in Brazil at this time was limited, as few schools had either computers available or the necessary support from special projects (usually university-based) to implement the use of software in schools. One large case, involving more than fifty schools in a poor area of the city of São Paulo, was an exception. During the period in which Paulo Freire was the Municipal Secretary of Education of São Paulo, Armando Valente and colleagues helped to implement the use of Logo in schools as part of a larger educational project. This lasted for about two years and was interrupted when new elections took place in the city.

There has been little research on Logo in the last ten years, though there continues to be quite a bit of development and research related to software, such as functions- and geometry software. The third phase is characterized by the increasing availability of the Internet. Online mathematics education courses started to become available at the turn of the century, and teachers seemed to be an audience that could benefit from some advantages of online courses, like flexibility of time and place. Busy teachers can, in some courses, choose the time to be involved in such courses, and there is less need, or no need, to move from home/work to the location where the course is being offered.

Research that began in one research group has now spread all over the country, with many different projects involving the design of different online courses. In Borba (2005), Borba and Villarreal (2005), Borba and Gadanidis (2008), and in Maltempi and Malheiros (2010), the reader can find a more complete discussion of research in online education. In one of the projects developed in this period, members of GPIMEM developed a series of online courses to support mathematics teachers in using software such as Geometricks and Winplot in their face-to-face high school classes. Besides offering the course, a research project was built around it (see Borba & Zulatto, 2010). Before discussing how teachers were able to produce knowledge about mathematics and the use of software in courses like this, I will present a perspective on the use of ICT[2] in mathematics education research.

REORGANIZATION OF MATHEMATICAL THINKING AND THE NOTION OF HUMAN BEING

Courses like the ones we offer are based on psychological theories regarding the way computers reorganize thinking (Tikhomirov, 1981), as they play a qualitatively different role than language does in writing and orality. Based on a combination of images, other kinds of visual feedback, writing and sound, and possibilities of sharing ideas in multimedia environments, this "new language" becomes a co-actor in the production of knowledge (Borba & Villarreal, 2005). Tikhomirov presents the notion of reorganization in order to emphasize that computers do impact cognition and permeate the human mind. Computers are not just juxtaposed with humans as substitutes for some cognitive functions. Computers reorganize thinking, as they are a qualitatively different memory extension when compared to regular language. Oral language and written language are different means of extending memory that help to extend our thinking, but that dialectically help to shape it and make it possible. Each new generation of computers seems to present new possibilities, and the latest one that is characterized by fast Internet access and multimedia, has generated multimodal discourses that are already qualitatively different from the oral and written ones. We have, therefore, computers as another means of extending memory which shapes our thinking with

[2] Information and Communication Technologies

multimodal possibilities. This is why exploring dynamic aspects of computer media and communication possibilities are paramount for courses based on the idea that we should not reproduce patterns that come from another media in a new media.

In the course in question, we are using a combination of video, figures and writing, but we believe that there are many ways of exploring the possibilities. One of the first ways we have been trying to explore is to incorporate, in online environments, what we have learned from what we refer to as the second generation of research on computers and mathematics education.

Coordination of multiple representations, as a means of knowing different aspects of functions and other calculus topics, was also transformed as different software came available. For instance, Confrey and Smith (1994) argued fifteen years ago about the importance of the software design for learning. In the case they analyzed, they showed how the design of Function Probe, which allows the graph of a function to be dragged, offered students new possibilities for coordinating representations, and therefore to think about functions. Being able to drag a graph and see the change in a given table of x-y values of the original function was a way of bringing transformation of functions to the forefront. Courses like ours, that use multi-representation software in online environments, are exploring the transformation of the use of software in an Internet environment.

In our research group, we believe that different interfaces in online courses will mean different ways of presenting mathematics. We have presented results about how mathematics may be transformed in courses in which the main tool is a chat room or a forum, but in this paper, we will explore the possibilities of video conferences for synchronous online problem solving.

For instance, there are online environments in which writing in a chat or in a forum is the only way for participants to communicate among themselves. Analysis was developed showing how writing shaped the mathematics discussed in a course offered to mathematics teachers. The analysis was carried out based on the idea that when technology changes, the possibilities for mathematics are also altered. This is the main idea behind the notion that a collective of humans-with-media (Borba & Villarreal, 2005) is the basic unit that constructs knowledge. Knowledge is produced by humans, together with different media such as orality, writing, or the new languages that emerge from computer technology.

It is not only the notion of knowledge that is transformed by new media. I would like to argue that the very notion of what it means to be human is changed. Agriculture has changed what it means to be human, as humankind no longer needs to migrate in search for food. The industrial revolution has also changed what we understand as human, as the need to "have more" was created. Nowadays, teenagers cannot imagine a world without cellular phones and Internet. It is part of what being human means today. Incorporating Internet and digital technologies into education is paramount if we want to humanize education. This is why it is important that teachers live experiences of online education and connect them to face-to-face use of ICT in classrooms.

ONLINE COURSES FOR TEACHERS: HOW TO USE SOFTWARE IN FACE-TO-FACE CLASSES

From 1999 to 2004, GPIMEM developed online courses basically with mathematics education content. Papers and books were assigned for participants to read prior to synchronous online meetings via chat, for the most part. Asynchronous interaction also took place. Although GPIMEM has been trying to explore mathematical topics in these online courses, it is only in the second half of the last decade that it has taken on the challenge of focusing on mathematics high school topics.

Among other courses, GPIMEM has developed an online course about how to learn and teach geometry with Geometricks, in response to a demand from mathematics teachers from a network of schools sponsored by the Bradesco Foundation[3], spread throughout all the Brazilian states. Teachers from more than 40 schools have access to teacher development courses which are administrated by a pedagogical center based in the greater São Paulo area. Following the improvement of Internet connections in Brazil, they realized that online courses were a good option, since sending teachers from different parts of Brazil to a single location to take courses was neither cost nor pedagogically effective. The cost factor is related to the size of the country, and the pedagogical consideration is related to the fact that teachers would usually participate in the courses for a short period of time, with little or no chance of implementing new ideas while still taking the course.

The first type of online course offered to teachers by the Foundation was based on a model involving little interaction between leaders of the courses and participants. This is one reason we had to overcome some initial resistance when we began teaching the course, as the “didactical contract” that the teachers were accustomed to in online courses like this was a more passive one. Our model, based on online interaction, applications in their face-to-face classes in middle and high school, and new discussion, gradually gained respect.

The pedagogical headquarters of this network of schools approached us, asking for a course about how to teach geometry using dynamic geometry software designed to teach plane geometry. As we knew from extended research on the interaction of information technology and mathematics education, availability of software and a well-equipped laboratory with 50 microcomputers – as is the case in these schools – did not imply that they would be used. After we designed and implemented the geometry course, the institution asked us to design a course for functions, as the network of schools had chosen Winplot, a free Windows based software.

Assigned problems in this series of courses designed for teachers usually had more than one way of being solved, and they could be incorporated at different grade levels of the curriculum according to the degree of requirements for a solution, and according to the preference of the teacher. Both intuitive and formal solutions were recognized as being important, and the articulation of trial-and-error and formal arguments was encouraged.

Each course, in general, would “meet” synchronously online for two hours for eight Saturday mornings over a period of approximately three months. Besides this activity, there was a significant amount of asynchronous e-mail exchange during the week, for clarification regarding the problems proposed, for technical issues regarding the software, or for pedagogical issues regarding the use of computer software in the classroom (e.g. should we introduce a concept in the regular classroom and then take the students to the laboratory, or the other way around?). I was mainly responsible for the synchronous activities and Rubia Zulatto for leading the asynchronous interactions. Pedagogical themes were also discussed during the online meetings, in particular in one session in which, instead of the students working on problems during the week, they had to read a short book about the use of

[3] The Bradesco Foundation obtains resources from Bradesco Bank and it has social purposes, as their schools are based for the most part in poor neighborhoods. Although they are a private foundation, their schools are free of charge and they develop intense continuing education activities with their teachers. It has at least one school in each one of the 26 Brazilian states and in the capital Brasília.

computers in mathematics education (Borba & Penteado, 2001). We encouraged teachers from one given school, or from different ones, to solve problems together in face-to-face or online fashion.

The headquarters of the Bradesco Foundation had already purchased an online platform that allowed participants to have access to chat, forum, e-mail, and videoconference, and allowed the download of activities as well. In our course, participants could download problems, and they could also post their solutions if they wanted to, or they could send them privately to one of the leaders of the course. The platform allowed the screen of any of the participants to be shared with everyone else. Our research group had to think not only about new content (mathematics instead of papers on mathematics education) but also how to answer questions about the interface of online environments: What difference does it make to teach online using mainly a videoconference environment instead of a chat room for synchronous interaction? This online course has generated much research with varied focuses. In this paper, I will explore how online environments (associated with the third phase of ICT research) can incorporate a medium like function software (connected to the second phase of ICT research) into continuing teacher education.

EXAMPLES OF SOLUTIONS FOR A LOGARITHM EQUATION

The online environment had an interface in which video with voice could be shown (see Figure 1). In many instances, due to problems related to limited bandwidth, we would have to close the videoconference window showing the teachers and professors (see upper

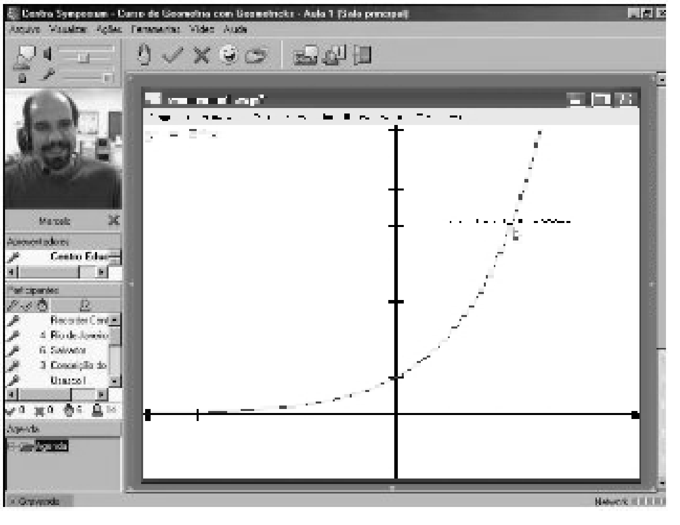

Figure 1. The online environment

left corner) and leave open only the software window that showed the actions being made either by the leaders of the course or by the teachers. One of the problems presented to the teachers was:

How can the equation $2^x = 5$ *be solved using the Winplot Software?*

It was expected that this "old problem" could become new with the idea of emphasizing the use of a different technological tool. The problem already establishes that at least one technological tool should be part of the collective of humans-with-media to solve the problem. The teachers were expected to solve the problem using the software, and many used other technologies as well, such as paper-and-pencil. In this section we will show some of the different solutions teachers presented.

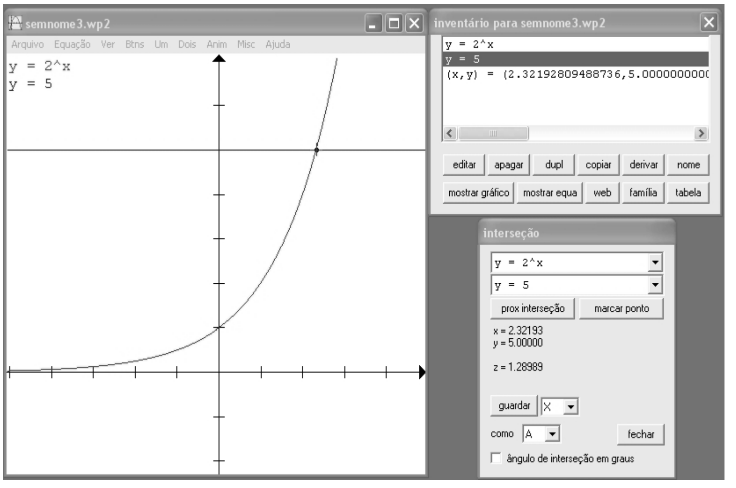

Figure 2. A graphical solution

One solution presented was to intersect $y=2^x$ with y=5. In Figure 2, it is possible to see that Winplot writes "2^x" as "2^x", that a teacher could insert the expressions for both functions, and that there is a tool in the software to find intersections of functions. In this way, teachers could find x=2.3219…. After a solution was presented, mathematics education themes were discussed. For instance, it was discussed how important it is to connect resolution of equations to functions and their intersection.

Another solution involved only paper-and-pencil, which is the usual analytical solution in which one reaches x = log5/log2. A variation of this solution did use the software to plot the function y = log(2)x – log(5), in which "log(2)" and "-log(5)" are respectively "a" and "b" in the usual linear model y = ax + b, as shown in the figure below.

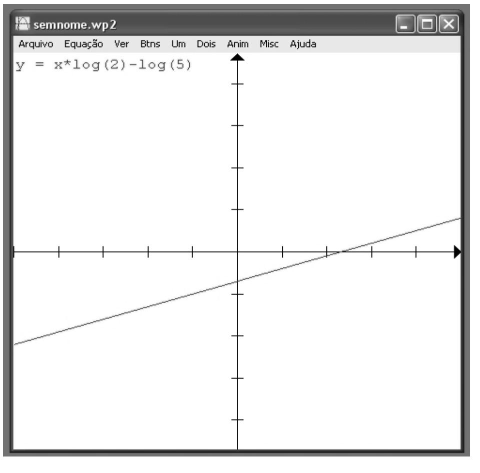

Figure 3. An analytical/graphical solution

After developing the above equation, using paper-and-pencil, the software was used to both plot the function and to argue, on a visual basis, that the intersection of this equation with the x-axis was the root, the solution for the problem.

A combination of some of the above solutions is how the next solution may be interpreted. They use the graph window to show the graph of y = log(5)/log(2) as shown below.

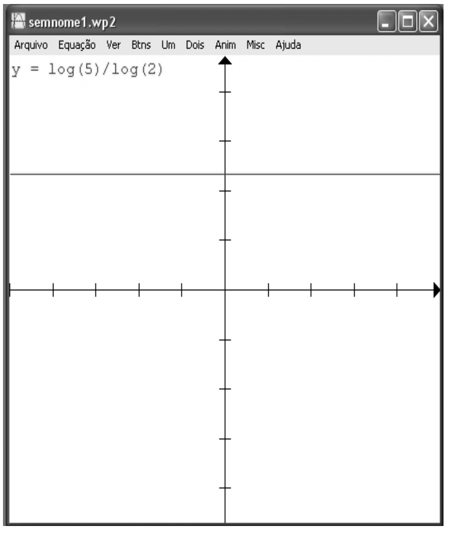

Figure 4. Paper-and-pencil and the graph window as important co-actors in this solution.

Teachers also proposed the solution below, which some argued should be the first one shown to students. The graph of $y = 2^x$ was plotted and the mouse was used to show the approximate solution of the original problem posed to the teachers. We discussed the didactical consequences of using this feature of Winplot, which makes it possible to display coordinates just by touching the curve described by the function. It was argued that the plus side of it was the possibility of discussing conceptual issues instead of the technicalities of logarithms. On the other hand, it was discussed that the limits of this solution should be shown as a means of moving on to at least some of the other solutions discussed in this paper.

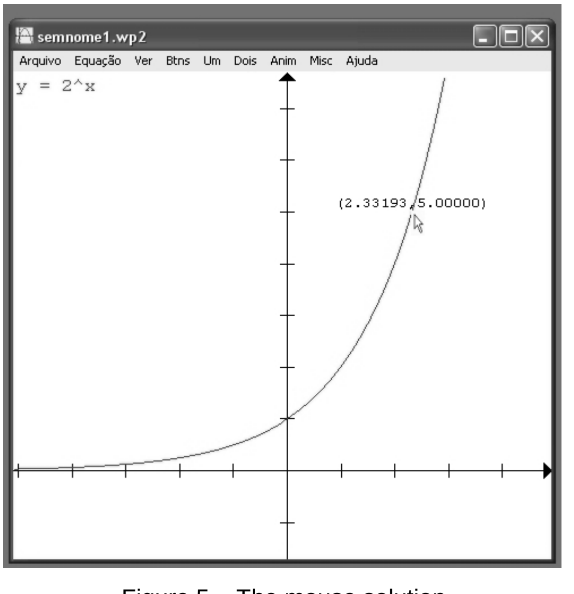

Figure 5. The mouse solution

The last solution presented in this paper used the table window. They started, as in the solution presented above, by graphing $y = 2^x$ but then they generated a table as shown in Figure 6.

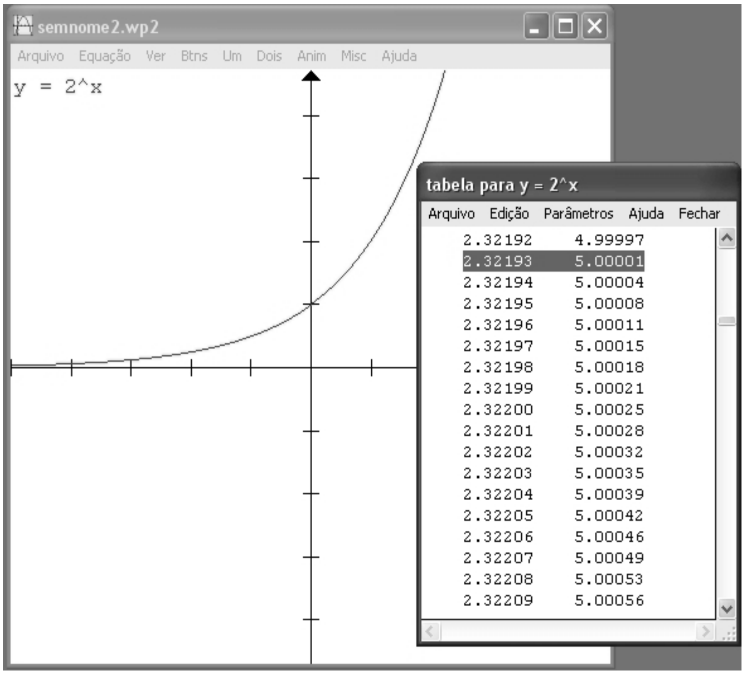

Figure 6. The interface Table plays a relevant role in the solution.

In generating the table, it was possible to see several points that belong to the function, and a better approximation of the root of the equation could be found as we restricted the interval of points displayed more and more.

This variety of solutions generated debates about the degree of formalism that should be required at different levels of schooling and about the possibilities of combining mathematical problems like the above in a computer laboratory with complementary activities in the regular classroom. There was also a significant debate about mathematics itself as teachers were discussing the validity of a solution presented. For the purpose of this paper, however, I should focus on one of the questions that was raised by one of the participants, after my address at the CMESG talk: Is it possible to learn mathematics online?

CONCLUSION

Besides the question raised above, according to my personal notes, there were two other questions: Is it possible to have teacher education online? Have participants changed their perception about mathematics as they study online? We have no data to support answers to the last question. However, with the example above and others presented in Borba (2004), Borba (2005), Borba and Villarreal (2005), Borba and Gadanidis (2008), and Borba and Zulatto (2010), it is possible to give a positive answer to the first two questions. Not only did the teachers learn mathematics, but we – who taught the courses – also learned mathematics as discussed in some of the references above. Teachers were involved in formulating answers to problems, like those shown in the last section of this paper, and often they used different online collaboration tools to ask for help or to improve a given solution. During a videoconference, they simultaneously used the chat to formulate questions about "small issues", and the "pass the pen" tool, a tool that allows participants to initiate a solution for a problem online that can then be continued by another colleague in a different geographical location.

I can also say that one thing that seems to be significantly different is the possibility to experience professional development online while having to teach a face-to-face class. Our research group does have plenty of data to support this. We can see that teachers started to use the Internet more and more, but I believe that we are only beginning to understand how multimodal discourses are changing the nature of mathematics expressed by teachers.

I believe that, at least for some, although we need to look for data to support this conjecture, some teachers used activities such as the above to recall, for instance, logarithms, a topic that they may not have taught for some time. Supported by the examples presented in this paper, as well as 11 years of research on online education conducted by GPIMEM and reviews of research from other groups, we can say that online education is a suitable means for online continuing education, even though we believe there is still much to be explored regarding the way the Internet can be of service in such courses. For instance, in one of our cooperative endeavours with other groups, we came in contact with the idea that it is possible to explore digital technology in ways that differ from the one presented in this paper. In addition to using Internet and multi-representation software to study functions (Kaput, 1992), Gadanidis and Borba (2008) have proposed that we could explore digital mathematical performance (DMP) to take advantage of the plasticity of the Internet, and incorporate applets, videos and pictures into the debate about multi-representation of mathematical concepts. The incorporation of DMP (http://www.edu.uwo.ca/dmp) into online practices is a promising direction for research. We have, until now, been exploring the idea of students and teachers becoming active actors in DMP. In the near future, I will begin analyzing how to incorporate DMP performance into online courses more systematically, and looking further into the future, we will be studying whether being a major player in a DMP may change one's perception of mathematics, and then we will be able to deal with the unanswered question raised above. I believe it will also be necessary in the future to connect the more particular discussion of Internet use in different realms of education with the broader discussion of how the Internet is changing the very notion of being human. One of the goals of education is to contribute to the formation of human beings. As the notion of human changes, we shall re-discuss the aims of mathematical education.

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Working Groups

Groupes de travail

MATHEMATICALLY GIFTED STUDENTS
LES ÉLÈVES DOUÉS ET TALENTUEUX EN MATHÉMATIQUES

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DESCRIBING STUDENTS WITH MATHEMATICAL ABILITIES

Initially, participants explained some of the issues that drew them to this working group: how to distinguish between the fruits of aptitude and hard work, whether gifted students possessed a characteristic style of thinking, how giftedness might manifest itself in terms of mathematical capability, support of giftedness through teaching and learning environments as well as activities and materials.

The designation of students as gifted is fraught with risks. The first deals with the value and benefits of separating out a special class of students. While members of the working group were prepared to recognize that some students had special abilities in mathematics, and indeed in some cases were actually working with such students, it is not easy to specify appropriate criteria. Furthermore, is it possible to have instruments and assessors, such as teachers, who can apply the criteria competently and equitably?

Indeed, there are many questions. Is giftedness necessarily a permanent attribute, or can it be temporary? How do you identify gifted students in a regular classroom? Is it through their gravitation to certain types of activities, such as solving puzzles, or asking questions? How does being a high achiever correlate with being "smart"? What is the effect of labelling on the student? What are the issues of equity and social justice that need to be considered?

Our group recognized that it is a different experience dealing with a "gifted" group compared to a regular or mixed ability group, that even in restricted groups one can discern a range of abilities, that "gifted" students are willing to work hard, devote more attention and are willing to concentrate on a task for a longer period of time; however, they often know what they want and will reject activities that they find uninteresting. Many of us recognized that some students are able to see problems differently and tend to look into the essence of a problem rather than just search for a solution.

As early as in 1894, Calkins discussed such characteristics of mathematical giftedness as powers of thought, identification, classification, comparison, and reasoning. Rosenbloom (1960) described gifted students as those who make abstractions, can generalize, go deeper, process faster, and discover for themselves what others need to be told. Krutetskii (1976) distinguished characteristics that were general, such as industriousness, productivity, persistence, active memory, concentration, motivation, from those that were more pertinent for mathematics, such as flexibility and dynamical thinking.

The modern research literature notes various characteristics of mathematical inclination, such as a desire for learning, impressive concentration, independence and originality of work, risk-taking, curiosity, and imagination, which includes the capacity to visualize and use symbolism. Their thinking is fluent, flexible, and elaborative (Williams, 1986), and they have a preference for systematic exploration, use of reasoning as opposed to application of a ready-to-use technique, generalization, as well as the study of special cases. While they may grasp concepts quickly, they may take more time to properly grasp a problem situation. Is the ability to explain one aspect of giftedness?

For very little children, according to Devlin (2007), early ability in arithmetic is not a predictor of giftedness, or at least not as good a predictor as that found by O'Neill, namely narrative skills, the ability to tell a story. The more sophisticated the preschooler's story-telling ability, the more likely the child is to do well in mathematics two years later. Other studies examined relationships between giftedness and mathematical creativity. Velikova, Bilchev, and Georgieva (2004) conducted an experiment, finding creative-productive gifted students and stimulating their mathematical creativity for real work in science. Their data led them to compare general and typological characteristics of giftedness (see Table 1 and Table 2 below).

Table 1. The comparison of the general characteristics of giftedness

"Schoolhouse giftedness"	*"Creative-productive giftedness"*
Learning of lessons and quickly adapting to the school environment.	Development of original product that has an impact on a particular audience in respect to the interest of the creator.
Motivation – success in education.	Motivation – success in the areas of interest.
Easy identification.	Difficult identification.
Stability over time.	Stability depends on abilities, interest, motivation, stimulus, etc.
High Intelligence Quotient.	Very high Intelligence Quotient.
Moderate level of Creativity Quotient.	High Creativity Quotient.
Customers of knowledge.	Creators of knowledge.
Development through well-known special programs.	Development through purposeful pedagogy interactions that follow activities of creative personalities in some mathematical area.
Paradigm: I know how to learn.	Paradigm: I know what I want and can do it.

Different investigators have identified characteristic features of gifted students: natural mathematical ability, willingness to work hard, high creativity (Mingus & Grassl, 1999); above average general abilities, high level of commitment to task, strong creativity (Ridge & Renzulli, 1981); spontaneous formulation of problems, flexibility in handling data, facility for organization, mental agility or fluency of ideas, originality of interpretation, ability to transfer ideas and to generalize (Greenes, 1981). Standard tests that emphasize speed, computation, rote memory, and spatial skills are not adequate, and one needs to look to special instruments, competition, classroom observational and non-standard investigative tasks to assess giftedness.

Table 2. The comparison of the typological characteristics of giftedness

"Schoolhouse giftedness in mathematics"	*"Creative-productive giftedness in mathematics"*
Quickly assimilating ideas of others.	The thinking is original but not quick.
Quickly going into the problem.	Slowly, step by step, building reasoning, resulting in harmonious mathematical theory.
Quickly solving the problems in school, competitions, Olympiads.	Not solving easy problems within limited time and solving very hard problems when given more time.
Quick change of interests.	The interests are solid.
Interest in non-standard problems which are solved with great investigation and technique.	Having ability for overall evaluation of the problem, to investigate many facts in the mathematical area of interest, and to create original problems.
Limited capacity for work on one problem.	Capacity for work on one interesting problem within a long period of time.

If the validity of identifying gifted students is accepted, then choices need to be made as to the extent to which they might be sequestered and with regard to the program of mathematics provided for them. Notwithstanding, somehow opportunities must be available for students demanding more mathematics.

TEACHING STUDENTS WITH MATHEMATICAL ABILITIES

Gifted students might be integrated with the regular classes, assigned to special classes within the school, or provided with extracurricular opportunities, either through a mathematics club, math fair, or contest activity. There may be extramural activities such as a mathematics circle or camp. Special schools might be set up to deal with the more exceptional among them.

Programs, too, could range from merely assigning more challenging material in the regular curriculum to providing extra material. Gallanger and Gallanger (1994) propose four approaches to curriculum modification: acceleration, enrichment, sophistication and novelty. Extra material might be an extension of the syllabus, or might consist of topics at the school level not otherwise taught, such as advanced chapters in combinatorics, transformation geometry, theory of equations or trigonometry.

Open-ended questions that promote higher-order thinking and analysis can be posed; students could be asked to analyze solutions and paradoxes and articulate underlying structure and principles (Williams, 1986; Johnson, 1993; Johnson & Ryser, 1996). Or students might be accelerated into work at a higher level, such as calculus or linear algebra. Alternatively, students might be encouraged to prepare for competitions, engage in original research under the supervision of a mentor, read books that are designed to introduce young and lay readers to various mathematical areas, or tutor other students (Feldhusen, Van Tassel-Baska, & Seeley, 1989; Parke, 1989; Westberg & Archambault, 1997; Winebrenner, 1992).

An important decision is the extent to which it is appropriate for students to work on their own or in groups. It is dangerous to assume that mathematically inclined students enjoy all mathematical tasks, work productively in any mathematics class, want to collaborate with peers, or will develop without outside intervention. They may well be frustrated or bored, or feel isolated (Cropper, 1998).

The Working Group did not come to firm conclusions on each of these matters, but rather explored them in the context of its own experiences and research. Since the group, though small, included individuals from different environments, we gained insight into how the concept and handling of giftedness manifests itself in the North American aboriginal culture, in Iran, Turkey, Eastern Europe, and Russia, and this allowed the views of each of us to be refined and enriched.

IRAN

In Grade 5, students write a preliminary round of examinations, similar to IQ tests. Some are subsequently invited to take a written examination (full problem-solving) and be interviewed. Successful candidates attend a special middle school (grades 6-8), and from here, depending on their progress, go onto a high school in the same program or else are reintegrated back into a normal high school. It is also possible to enter the special high school from Grade 8. Often, these secondary students are trained to compete in international Olympiads in such fields as mathematics, physics, chemistry or informatics; the ones who make it onto the international teams are exempt from university examinations and are accepted into the university of their choice, often the highly regarded Sharif University of Technology. It is fair to say that some students who are trained by parents or tutors to succeed in the tests find themselves out of place and may not only drop out of the special schools, but even fail in regular schools. It may also happen that some students do not perform well on the examinations, do not gain admission to the special schools, but are in fact gifted. Some parents may discourage their children from writing the examinations lest it shake their self-confidence, or out of a concern for a lack of breadth in their interests and experiences. Overall, students who are correctly recognized as gifted do fairly well in these schools, are confident, and succeed in their academic pursuits. Most North American graduate students from Iran come from such schools and Sharif University.

TURKEY

Many schools in this country have clubs for mathematics, science or informatics, and some deliver a more open-ended schedule. For example, instead of the teacher leading the program, participants, students, and teachers may bring their own problems and work on them. Sometimes, these students or senior students may modify the problems or create new ones. Thus, instead of being bound by a fixed schedule, the regime is quite fluid. There are generally no unifying national programs for gifted students, and several approaches have been identified: specialized schools, accelerated mathematics classes, enrichment classes, and differentiated instruction in heterogeneous classes.

ROMANIA

Romanian students talented in mathematics have many opportunities to strengthen their abilities, especially in high school. Although they are not trained in special classes or schools during the academic year, they can attend regular vacation camps organized throughout the country by the best mathematics teachers with local and national government support. The tradition of preparing gifted students for mathematical competitions goes back many decades. In fact, the first International Mathematical Olympiad was organized by Romania. It is supported by a competitive educational environment aided, among other factors, by excellent mathematics publications targeting good students, by translations of foreign mathematics books, by passionate teachers, and by a culture appreciative toward high achievement in mathematics.

RUSSIA

In Russia, mathematically inclined students have an opportunity to attend special mathematical schools or mathematical classes within general schools at different stages. A few special math schools in Moscow accept students aged 12-13 (grade 6). These schools also employ university professors to teach senior math classes in mathematics. Special math classes in general schools start at grades 8-10. Again, these schools are very selective in choosing their math teachers to ensure high academic knowledge. Some universities organize evening math and physics classes which prepare students specifically for advanced placement at those universities. Several mathematical competitions are offered throughout the year. The students achieving high placement in the competitions have priority in admission to prestigious Russian universities. The periodical "Kvant" and other math publications target students and teachers interested in the subject and effectively support their development.

CANADA

Support for gifted students comes mainly from a few organizations and enthusiastic individuals, such as secondary and tertiary teachers. The Centre for Education in Mathematics and Computing at the University of Waterloo and the Canadian Mathematical Society sponsor competitions and training, as well as publish material; in addition, there are many regional competitions. This provides a network that allows the identification and development of students for participation in international competitions. While country-wide data about activities for gifted students in schools were not available to participants of the working group, some experiences were mentioned, like one in New Brunswick, where schools are beginning to set up innovative initiatives on inclusive teaching and learning, thus providing various enrichment opportunities for students with special needs; this includes services for gifted and talented learners in all subjects (Government of New Brunswick, 2007).

SOCIAL ISSUES

What are the effects on a student who is designated as gifted and on his or her peers? Are there expectations thrust upon the student that may or may not be fulfilled in the future? On the other hand, is there a danger that unless students are identified for special programs, they may not be challenged by their regular schooling and for some reason fail to realize their potential?

The group spent a small amount of time on social questions arising from the identification and handling of gifted students. There are the issues of equity and opportunity. When gifted students are removed from the general population, then what is the effect on those left behind? Is there a danger of downplaying how the situation might evolve for different students? After all, giftedness does not manifest itself in any specific way or at any specific age. Some children are precocious, while others are late bloomers. Unfortunately, some who seem to show promise early on may not live up to that promise. One member of the group felt that we should be thinking in terms of “becoming gifted” rather than “being gifted”, because this is first, a more inclusive way of looking at the situation and second, suggests not only the dynamism inherent in student attainment, but the ability of many students to grow intellectually under the right circumstances, and the responsibility of the education system to provide the appropriate stimulation.

At the same time, there are serious risks in not meeting the needs of pupils who are intellectually advanced or consumed by an interest. Without appropriate stimulation, they may become alienated or discouraged, and not ultimately benefit society by their skills and abilities. So while it seems as though special treatment of some sort is necessary, we need to be cautious about our goals. Some students are quick problem-solvers, others more methodical thinkers – how do we achieve an appropriate balance between their proclivities and what is best for their mathematical development?

Indeed, it is conceivable that education of the young functions best in an atmosphere of sharing, where teachers can share their expertise and love of the subject with their students. Mathematics is a point of contact between adults and youths, where the two can operate on a basis of respect and equality, and this surely has a positive social consequence.

TEACHERS

The primary interface between pupils and mathematics is the teacher. The ability of a society to provide the right leadership and stimulation depends on the attributes of the teacher (Vialle & Quigly, 2002**)**. Unfortunately, it is the case that many teachers shy away from teaching a gifted group, perhaps being or feeling incapable of meeting the challenge of having to deal with superior students and coping with classroom interaction over which they may not feel in control. Too, it is the teachers who very often are put in a position of having to decide whether a pupil might be gifted and should be given special treatment – what is the criterion that they might use?

There are important implications for the training of cadet teachers and the professional development of established ones. First of all, teachers require a mathematical background that is sufficiently robust to distinguish the markers of giftedness from just getting good marks in standard assessments or memorizing and following procedures diligently. This surely involves their having personal experiences of problem-solving and investigation to draw upon. Second, they require the confidence to accept that some of the pupils they encounter will indeed be quicker and more intelligent than they are, but also that they have a role in nurturing whatever talent they find. Teachers must be able to train students in the skills of independent, self-directed learning (Feldhusen et al., 1989; Tomlinson, 1993), as well as to serve as “metacognitive coaches” and put more emphasis on modelling the process of problem-solving by their own example of thinking out loud, rather than just providing students with information and techniques.

Third, they need to be familiar with the resources that are available, such as journals and websites, so that they can orchestrate a program that will benefit their pupils. Some of this will involve networking, so that peers and folks outside the school are available for advice, assistance, and mentoring. All of these presuppose a level of self-confidence that many teachers lack. In contrast to other subjects, teachers of mathematics tend not to be practitioners of mathematics.

Some teachers are quick to identify giftedness for reasons that others might regard as spurious; others might feel that they have never met a gifted student. While giftedness might be difficult to define, communication among education practitioners might help us work towards a consensus as to what giftedness entails. One member of the group suggested that giftedness might turn on the answer to the question, “Can the teacher learn something from the student?” It is possible that a proper assessment of giftedness requires contact over a long time, as the teacher needs to understand how a given student thinks. Instead of having a new teacher each year at school, perhaps pupils need fewer teachers, each for several years. This allows a dynamic to be created between the teacher and the class and allows the teacher to get to know the student in a way not possible over a single year.

This also allows for a more consistent educational regime for a student, who does not have to adjust to a new teacher situation more frequently.

Teachers of mathematics should provide a safe environment for both genders in the classroom and make an effort to recruit girls to participate in advanced courses and extracurricular activities related to mathematics, science, and technology (Davis & Rimm, 1994; Smutny, 1998).

RESEARCH ON ABILITY GROUPINGS

While the literature addresses grouping according to ability, unfortunately there is no coherent vision supported by the data. Results are controversial and, in assessing them, we need to be careful to be aware of the context. Theoretically, ability grouping should increase student achievement by reducing disparity among students so that teachers can provide instruction that is of the right difficulty for most of their students. Nevertheless, there is a compelling argument against it lest it foster academic elites and transgress democratic norms (Slavin, 1986). This can be mitigated if students spend much of their time in heterogeneous groups, are grouped flexibly according to performance rather than age, and are in a setting where they can proceed at their own pace. Well-controlled studies at regular schools support a regime where subjects are stratified into levels that students work through in their own time (comprehensive non-graded plans). Research on class ability grouping supports this practice, especially when there are two or three groups. According to Slavin (1986), the positive effects are slightly greater for low-achieving students. Slavin articulates various principles, basically that students should be removed from heterogeneous groups sparingly, depending on increased opportunity for learning and on specific material being taught, and on the understanding that there is frequent assessment and flexibility in reassigning students.

Research is not unanimous as to who benefits most from ability grouping. Some found that higher achievers are supported the best (Ireson, 2002); others noted that both high and low achievers benefited (Kulik & Kulik, 1992; Ireson, Hallam, Hack, Clark, & Plewis, 2002); still others noted that it harmed education (Boaler, 1997). This uncertainty also applies to affective outcomes, such as level of self-esteem, attitudes towards school, and possibility of alienation (Oakes, 1985; Kulik & Kulik, 1992; Hallam & Deathe, 2002).

Placement of students is problematic. Some students may feel that they belong to a higher level and want to be given more demanding work to improve their status and examination prospects, while others might prefer a lower level that allows them to improve their understanding and provides work according to their needs (Ireson & Hallam, 2005).

It is also worth remembering that the quality of instruction in a particular classroom may often have a stronger influence on achievements than the initial placement (Gamoran, 1991, 1992). Small (2009) gives examples of mathematical tasks within a differentiated instruction approach that allows struggling students to be successful, and proficient students to be challenged. We discuss a few mathematical problems in the next section.

THE ROLE AND QUALITIES OF MATHEMATICAL PROBLEMS

It is hard for a session at CMESG to proceed without some actual mathematics being put on the board for discussion. We agreed that a good mathematical problem is at the core of mathematical instruction, especially for gifted students. Questions need to be open enough to provide for a variety of approaches yet remain focussed on a set of ideas. It helps if questions lead to more advanced ones that force students to appreciate the essence of the mathematical situation and generalities.

Many bright students do not reveal all the details of their solution and it requires an astute marker to decide what deficiencies it might have or whether it is essentially valid. Teachers need to train their students to write convincing solutions, including all important stages with justification, but also to know when routine details can be left out. There are often many ways to solve a problem, from the mundane, where a set procedure is followed to its conclusion, to ways which highlight underlying structure and relationships, or which involve a non-standard perspective or surprising elegance.

We put a few problems up to be deconstructed.

1. If 16 × 16 × -16 = 8 × 8 × p, what is p?

A pupil might take this at its face value, multiply out the left side and divide by 8 × 8 = 64 (with or without a calculator) and get *p*. A simpler version of this is to solve the problem by a cancellation of terms, dividing the 8s into the 16s on the left. However, there are more “tactile” ways to conceive of the situation.

For example, one has to divide by 2 twice to cut the two 16s down to 8s, and therefore multiply the last sixteen by 2 twice (i.e. by 4) to “break even” and derive the value of *p*. Is there an approach that might be considered as “gifted”?

2. The trapezoid ABCD is given with AB||DC. Let the diagonals AC and BD intersect at E. What can be said about the triangles ADE and BCE?

This is a more open-ended question and demands of the solver some understanding of the situation as to what sort of thing one might say. A characteristic of mathematics is that each problem is a self-contained microcosm. Everything that we have to deal with is on the table, so that what is given needs to be parsed for its significance and possibilities. For example, noting the equal opposite angles at E, it is conceivable that the triangles are congruent; a little exploration indicates that this is not necessarily so. Is it reasonable to consider their areas? If so, how can we describe the areas? Should a formula be brought into play (e.g. the base-height formula or Heron’s formula)? Should coordinates be introduced? How can giftedness be disentangled from background and experience in assessing how students handle such issues?

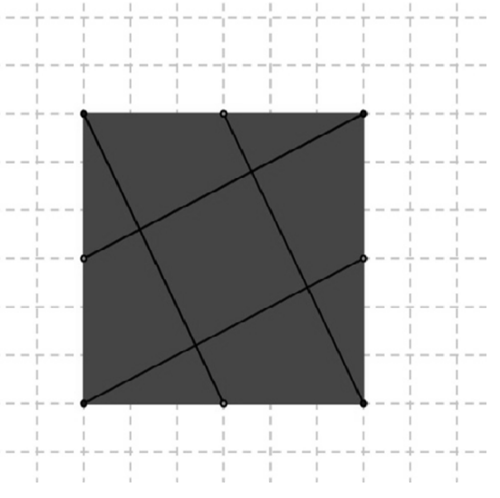
One insightful solution is to note that [ADE] = [ADB] − [AEB] = [CAB] − [AEB] = [BCE] (triangles on the same base between the same parallels), so that the triangles in question have equal area. (Here, the square brackets denote area.) A little deconstruction indicates that there are a few connections that the solver needs to make, having to do with recognizing the connection between the parallels and equal area, and relating the given triangles to triangles on a common base. Is this sort of recognition a marker of giftedness?

A more structurally inclined student might realize that ratio of areas stays invariant under affine transformations, and therefore consider a shear that fixes AB and shifts DC into a position that makes the trapezoid equilateral. In this special situation, the two triangles are indeed congruent by symmetry, and so have equal area. Equality of area (but not congruence) is then carried back to the original trapezoid.

3. Prove that the sum of two consecutive odd primes can always be written as the product of three (not necessarily distinct) integers exceeding 1. (Consecutive primes are primes for which every intermediate number is composite. An example of a pair of consecutive primes is (23, 29), whose sum is 52 = 2×2×13).

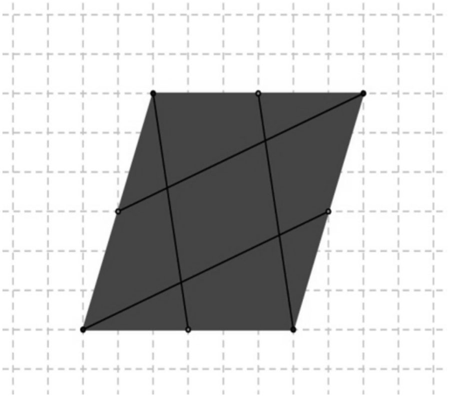
Here again it is necessary to take apart every particular of the situation. A successful solution depends on the realization that the sum of two numbers is twice their average, and that the average of two numbers lies between the numbers. That the primes are odd signifies that their average is in fact an integer. This problem was given to a 10-year old who took probably half a minute to solve it; is it a mark of giftedness that these pieces fell together for him?

4. Let ABDC be a square and E, F, G, H be the respective midpoints of the sides AC, CD, DB, BA. The lines AF, BE, CG, DH bound a quadrilateral IJKL. What is the ratio of the areas of the square ABCD and the inner quadrilateral?

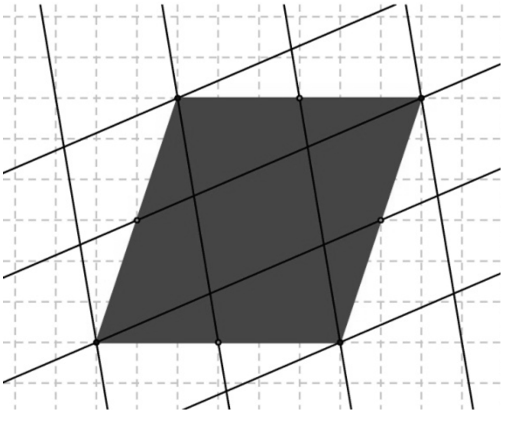


This problem admits a calculation-based solution. If the sides of the given square have length 2 and we suppose that the inner figure is a square, then its sides have length $2\sqrt{5}/5$ and we find that the desired ratio is 5:1. But how do we know that the inner figure is a square? This can be explored with dynamic software, or justified by a proof that can be more-or-less complex. However, the solution can be found by exploiting the structure, both to use suitable symmetries to determine that the inner figure is a square, and to insert the situation into overlapping tessellations based on the given square and the inner figure as cells.

It can be asked what happens if we replace ABCD by a rectangle or even a parallelogram. One might conjecture that the ratio remains 5:1.



Proving this computationally is more complicated, while a structural argument taps into the essence of the situation and retains its simplicity. Indeed, the square can be transformed into a parallelogram by transformations that preserve the ratio of areas. The following picture shows the overlapping tessellations and provides a structural insight about this general result:



5. Show that the sum of the distances from any point within an equilateral triangle to its sides is a constant, independent of the point.

While we did not do this, the result can be illustrated using dynamic geometry software. However, if one notes that the three sides of the triangle are equal and that the triangle can be partitioned into three triangles whose bases are the sides and whose heights are the distances from the sides to the inner point, as measured by the perpendicular to the sides from the point, then a straightforward areas argument is available.

One can now explore what other figures have the same property. Does it hold true for a square? A rectangle? A polygon with all sides equal? Is there an analogue for a solid figure? What is the most general planar or solid figure for which the property holds?

CONCLUSION

While we may not have answered the entire list of questions below, the group agreed that they are worth considering by any educator planning their mathematics instruction at both global curricular as well as local classroom levels.

- To what degree should gifted kids be separated from, or integrated into, regular classroom instruction?
- When is it appropriate to set up gifted classes or schools?
- Can the standard curriculum be modified in a way that supports maximum progress of gifted students and at the same time is accessible for the rest of the students?
- Can the materials designed for gifted students be used for general class instruction?
- Who should attend gifted programs: students who look for enrichment or students that are extraordinarily talented?
- What should gifted programs deliver: more advanced material or broader material?
- Is it possible to regulate the interaction between gifted and regular students in a classroom in such a way that regular students accept that some of their peers are talented while gifted students develop their ability without building their ego at the expense of others?
- What is a proper award policy for math competition participants?
- Should bright high school students take university courses?
- In what way can schools and universities productively collaborate in order to support gifted kids?

Some of these questions were discussed in recently published results in the *ICMI 16 Study on Challenging Mathematics In and Beyond the Classroom* (Barbeau & Taylor, 2009), and at the ICME-11 Congress (*Topic Study Group on Mathematical Giftedness and Discussion Groups on Creativity and Competitions*, 2008, http://tsg.icme11.org). A newly published book entitled *Creativity in Mathematics and the Education of Gifted Students* sets up a research agenda (Leikin, Berman, & Koichu, 2009). International conferences about these issues draw the interest of hundreds of math educators worldwide who are willing to share their experiences and to collaborate. The next (6th) International Conference, "Creativity in Math Education and Education of Gifted Children", will be held in Riga in 2010 (http://www.igmcg.org/).

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MATHEMATICS AND THE LIFE SCIENCES

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The working group report includes text contributed by Richard Barwell, Hongmei Zhu, Jane Heffernan and Miroslav Lovric.

Mathematics plays an increasingly important role in many areas of the life sciences – for analysis, for theorising, and for representing information. It follows that life scientists benefit from training in mathematical thinking. But what mathematics should they learn? What should the nature of their mathematical education be? And how is education for the life sciences shaped by mathematics?

The aim of this working group was to explore the above questions and several related ones, through working on specific examples of the use of mathematics in the life sciences. These examples included:

- Medical imaging: the construction and analysis of digital images of parts of the human body in order, for example, to identify diseased tissue.
- Epidemiology: the mapping, analysis and prediction of how disease spreads though a population.

Part of each day's work was devoted to looking at one of these examples, introduced by a practitioner of the relevant field. Through these examples, we discussed a range of issues and questions relevant to the teaching of mathematics in, for or through the life sciences, particularly at university level. An aim stated at the start was to generate some principles for the use of examples or problems linking the life sciences with mathematics.

During an initial discussion of participants' experiences of teaching mathematics in the context of the life sciences, it was clear that the group represented a range of different teaching situations, mostly at the university level. These situations included so-called 'service courses' in mathematics for undergraduates in life science programs as well as teacher education courses. Several issues were highlighted that had arisen in the context of their teaching. These included:

- the challenge of teaching mathematics to a class that includes mathematics majors, biology majors, and teacher education students;
- the fact that non-mathematics majors (e.g., biology) nevertheless often end up teaching mathematics;
- the perceived need to include more non-calculus mathematics in university courses;

- an awareness that mathematics is in some sense different for different groups of students.

We continued to explore these and other issues through work on the different examples that were shared with the group.

EXAMPLE 1: LEARNING MATHEMATICS THROUGH LEARNING ABOUT MEDICAL IMAGING (HONGMEI ZHU)

Hongmei Zhu shared her personal experiences in teaching mathematics and talked about a number of examples that she has used to inspire undergraduate teaching. Mathematics has become a key element in modern technology. In the last fifty years there has been a rapid increase in the use of sophisticated mathematical techniques in medical imaging and computer-aided disease diagnosis. We discussed examples in two areas: 1) linear algebra and computed tomography (CT) image reconstruction and 2) probability and image characterization.

CT SCANNING

CT scanners are one of the most accessible and affordable medical imaging modalities used in clinical work. Tomography refers to a picture (-graphy) of a slice (tomo-). A CT scanner utilizes X-ray technology to produce a 360 degree view of the internal structure of a slice of an object. More specifically, CT imaging is performed with a rotating x-ray tube that sends x-rays passing through the body at many angles and one or multiple linear detector arrays on the other side of the x-ray tube measure the transmitted x-rays. The values of a CT image approximate the attenuation coefficients of different materials within the scanned object. The attenuation coefficient of a material describes its X-ray stopping power. Denser materials (such as lead or bones) have greater attenuation coefficients than the soft tissues (such as water or muscle); therefore denser structure appears brighter in a CT image and soft tissues appear darker in the image.

Mathematically, the problem of CT image reconstruction boils down to approximating the attenuation coefficients within an object from these measurements outside of the object. As illustrated in Figure 1, suppose that the object is discretized into two-by-two blocks. We determine an attenuation coefficient for each block so that the sums of their values at different directions agree with the CT measurements. One intuitive way to determine these coefficients is to solve a set of linear equations iteratively so that the values of the coefficients are modified iteratively to agree with the measured sums. This algorithm is called an algebraic reconstruction technique (ART) (Funkhouser, Jafari, & Eubank, 2002).

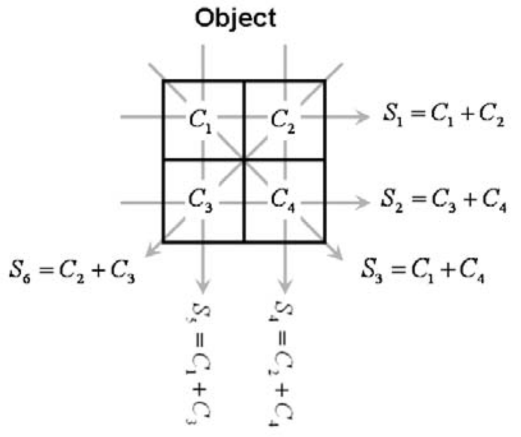

Figure 1. A simple illustration of CT image reconstruction: here, the C_i represent the attenuation coefficients of the object, which is to be determined from the CT measures, the S_i, outside of the object.

To explore how ART works, participants worked on two activities presented in a spreadsheet. For each activity, a small array representing a CT image was shown, along with totals representing the CT measures from various slices (one of which is shown below). The spreadsheet version of the activity allowed participants to enter values into the array in order to find a solution. Through this activity, we explored some aspects of how an ART algorithm could lead to a solution.

3x3 exploration

15

					40
16				13	14
					36

14

14 29 47

Outer shows horizontal and vertical totals
Inner shows clockwise diagonal totals

The ART method is simple and robust. It is, however, computationally expensive and therefore is not used clinically. More practical image reconstruction schemes based on the Radon and Fourier transforms can be found in Epstein (2008).

MEDICAL IMAGE PROCESSING

The second example we discussed is the connection of medical image processing to mathematics. We mostly focused on the use of probability distribution to characterize and manipulate a medical image (or a digital photo). Let us consider a CT image as shown in Figure 2(a) and its variations Figure 2 (b)-(e). We might use “dark” and “bright” to describe the brightness differences between Figure 2(b) and (c). We might label Figure 2(d) as being “low-contrast” and Figure 2(e) as being “high contrast” where contrast refers to the degree of differences between dark and light colours.

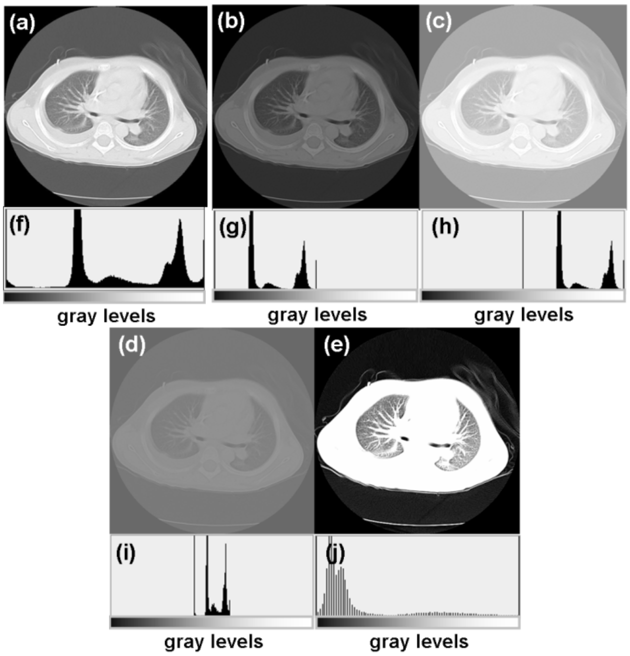

Figure 2. (a) The original CT image, (b) through (e) its variations, and (f) through (j) their corresponding histograms. (The original image is kindly provided by Dr. Paul Babyn, Hospital of Sick Children, Toronto, ON.)

A more quantitative way to describe the differences among these images is the image histogram. The histogram shows the distribution of image intensity and is often displayed in a bar graph. Consider an image with the grey levels in the range [0, L – 1], where L is the maximum number of grey levels allowed in the image. Its histogram can be defined as

$$h(g_k) = n_k, \text{ where } g_k = 0, 1, 2, \ldots, L-1,$$

where n_k is the number of pixels in the image taking the grey level g_k . The bar graph shown in Figure 2(f) is the histogram of the image shown in Figure 2(a), showing the well-distributed image grey values. The histograms of the dark and the bright variations of the CT images, as shown in Figures 2(b) and 2(c), reveal their image intensity distributions biased towards the dark and bright values, respectively. A low-contrast image as shown in Figure 2(d) can be characterized with a narrow histogram while a high-contrast image as shown in Figure 2(e) with a broad histogram.

Participants explored these ideas using standard image processing software, such as Adobe® Photoshop (www.photoshop.com) or the *GNU Image Manipulation Program (*GIMP, www.gimp.org). Such software allows users to adjust brightness, contrast, etc., of an image by manipulating the histogram associated with it. More complex manipulations are also possible. When divided by the total number of pixels in an image, denoted as *N,* the histogram can be normalized:

$$p(g_k) = \frac{h(g_k)}{N} = \frac{n_k}{N}, \text{ where } g_k = 0, 1, 2, \ldots, L-1.$$

The normalized histogram estimates the probability distribution of the occurrence of image grey values, resembling the concept of a probability distribution in statistics. By transforming one probability distribution to another using a simple function or a statistical technique, one can manipulate an image according to specific needs. For details, refer to [3]. In medical applications, for example, such manipulation may allow boundaries between different kinds of tissue to be more clearly delineated.

DISCUSSION

Discussion of the first day's activities focused on cross-disciplinary issues. Some participants felt, for example, that visual or physical models are essential to having conversations across different audiences (life sciences students, mathematics students, engineering students, teacher candidates, etc.). Such an approach can be challenging, however. There was a feeling that we would need to be experts in 'everything'. One participant asked, for example, "How do I take the preceding example and use in class?" Team-teaching is one possible strategy to deal with this issue, but may not always be possible.

We also discussed the depth of mathematics that can or should be included in such a topic. We discussed whether students need to 'know what's inside the black box' – should they understand probability density functions, or simply be able to use image manipulation software? This issue was discussed in terms of depth and interest: a good example should capture student interest while going into more mathematical depth might be of less interest to students. Participants wanted students to have both depth and interest, not just depth or just interest. Addressing both may in part depend on a good choice of examples.

THE BASIC REPRODUCTIVE RATIO

The basic reproductive ratio, R_0, is a key concept in epidemiology. Originally developed for the study of demographics, it is now widely used in the study of infectious disease, and more recently in models of in-host population dynamics (Heffernan, Smith, & Wahl, 2005).

R_0 is defined as the expected number of secondary individuals produced by an infected individual in its lifetime. When $R_0<1$, each infected individual produces, on average, less than one new infected individual, and thus we would predict that the infection will be cleared from the population. If $R_0>1$, however, the pathogen is able to invade the susceptible population. This threshold behaviour is a very useful aspect of R_0 as it allows us to determine if a disease will spread or die out and it enables us to determine what control measures, if any, may be effective in halting infection. This approach has been most important in providing guidance for public health initiatives, i.e., drug distribution, vaccine efficacy, etc.

R_0 is also used to gauge the risk of an epidemic or pandemic in emerging infectious disease. Recently, the estimation of R_0 has been used to gauge the potential threat of H1N1 (e.g. Fraser et al., 2009). In 2003, it was of critical importance in understanding the outbreak and potential danger from SARS (Choi & Pak, 2003; Lipsitch et al., 2003; Lloyd-Smith, Galvani, & Getz, 2003; Riley et al., 2003). R_0 has also been used to characterize seasonal influenza, foot and mouth disease, HIV, measles, other childhood diseases, and many other infections (see, for example, Heffernan et al. (2005) and Keeling and Rohani (2008)).

Recently, first during the SARS outbreak and now the H1N1 outbreak, the term R_0 has become known in circles other than those in academia. This past summer, Jane Heffernan, who is a mathematical immunologist, had the opportunity to teach a course focused on introducing modelling tools to high school teachers. Jane describes the course below:

> *I felt that this course should start with an application of some mathematics that they had learned previously, i.e., curve fitting, least squares, simple algebra. I wanted*

> *students to use these simple tools to understand the notions behind the value of R_0 that had been previously announced in the news on H1N1.*
>
> *To introduce R_0 to the students, I started with an introduction to two infectious diseases, measles and influenza. We then looked at some articles in the news and in the scientific literature showing how R_0 has been employed and discussed. The students were intrigued and wanted to learn more about diseases that affect them and their families.*
>
> *Next, I presented the students with a small sheet of epidemic data to see if they could see any patterns in the data or recognize any special characteristics. The students were quick to realize that the data followed an exponential increase and they determined that fitting an exponential curve to the data may allow them to determine R_0.*
>
> *They then continued to estimate R_0 from epidemic data. I presented them with a formula $R_0 = 1+r_0/a$ where a is the length of the infectious period and r_0 is the initial growth rate. They fitted an exponential curve to the data by drawing a line of best fit to log(data). The slope of this line gives r_0. With general knowledge of the disease we can approximate a, and then calculate R_0.*
>
> *We then discussed what the line of best fit actually represented. This idea is related to that of least squares. The students used a line of best fit and least squares to estimate r_0 and then R_0 for some data sets of diseases where a is known. I then provided participants with a review paper on many methods for determining R_0 (Heffernan et al., 2005). To end the class we discussed the different methods and how they could be used to determine different characteristics of a disease.*

DISCUSSION

We discussed the question: Is there something specific about *life sciences* and mathematics? That is, if 'life sciences' was replaced by accounting or engineering or some other subject, to what extent would the issues be the same, and what would seem to be specific to life sciences?

Several points were raised in relation to this question. First, it was observed that in the past biology may have been an area that students selected because it needed less mathematics than other sciences. That is no longer the case. Biology is becoming much more mathematical, involving functions, differential equations, measurement, calculus, rates of change. Participants discussed whether, in teaching mathematics for biologists, for example, it is important to know what mathematics biologists need to know. On the other hand, courses should not be limited to what students 'need': perhaps talking about 'mathematics for life sciences' is unhelpful. Gerda de Vries referred to the example about West Nile virus that she presented in her plenary lecture. Having worked on this topic, her students were better equipped to model various phenomena, not just viruses. Walter Whitely observed that life science contexts could also prompt work on mathematics that would not usually be covered, such as work on symmetries arising from studying the common cold virus, vitamin E molecules or the history of thalidomide. It may be, therefore, that when working with examples from the life sciences, such as R_0 or CT scans, the focus should be primarily on mathematical techniques, and only secondarily on specific life science contexts. More broadly, such work aims to develop critical thinking and problem solving skills.

AN EXAMPLE OF A UNIVERSITY COURSE

Miroslav Lovric talked about the course 'Mathematics for Life Science' that he designed for first-year students at McMaster University. He keeps several issues in mind when teaching mathematics in context:

- We need to introduce applications in a natural way, rather than force them just so that we can show how mathematics is used.
- Applications need to be rich both in mathematics and in the context of the discipline from which they come.
- We have to keep the integrity of mathematics, i.e., teach concepts, ideas, and techniques that are part of the natural flow of a calculus course, but for which there are no good applications (or no applications appropriate for first-year students).
- We need to abandon calling applications 'real life/world': applications may or may not be real life/ world, but they have to make sense, i.e., they have to have purpose and meaning.

Applications that Miroslav teaches in his course are of two types: shorter ones that are embedded into lecture flow (and need very little context), and longer, more complex ones, which require a half or full lecture to discuss both the context and mathematics involved. Examples of shorter applications include: allometric models (involve scaling or proportionality, such as the relationship between the volume of an animal and its surface area), basic population and growth models, or dissolution of drugs (caffeine). Examples of longer applications include: exponential growth (growth of breast cancer, with real data), optimization involving trig functions (vascular branching), dynamical systems, stability (dynamics of alcohol elimination from the body), Riemann sums (volume of a heart chamber, volume of a lake), radioactive decay (C-14 dating of organic material, potassium-argon dating of rocks), etc.

Miroslav explained how he teaches a couple of longer applications. In the case of alcohol consumption, the context involves discussion of how alcohol influences the human body, and how blood-alcohol level is measured in reality. A mathematical model is used to derive various scenarios for drinking (half a drink every hour, one drink every hour, etc.) so that students realise the negative effects of continuous drinking.

The other application Miroslav discussed was related to a sonogram of a heart chamber. The problem of calculating its volume (for the purpose of figuring out features of blood flow out of the heart) was explained in the context of Riemann sums.

BLACK AND WHITE MOTHS: THE HARDY-WEINBERG LAW

Gerda de Vries gave the following example of mathematics used in genetics. The example concerns a notional population of moths, some of which are black and some of which are white. The colour of the moths is determined by the presence or absence of melamine, which is in turn dependent on what geneticists call alleles. The allele for the presence of melamine can be denoted by M; the allele for the absence of melamine can be denoted by m. These two alleles result in three possible genotypes: Mm, MM and mm. If M is dominant, these three genotypes result in two possible phenotypes: black (from Mm or MM) and white (from mm).

It might be assumed that the m allele would disappear over time. This turns out not to be the case. To understand why not, we can model the population over time. First, we need to make some assumptions: that mating is random, that phenotype confers no survival advantage, that there is no genetic mutation and that alleles are equally distributed across both sexes of moth. With these simplified conditions, let p_n denote the frequency of the M allele in the n^{th} generation. Hence the frequency of the m allele is $1 - p_n$.

The table below shows the frequencies of the different genotypes that will arise in the next generation:

		M p_n	**m** $1 - p_n$
M	p_n **(M)**	p_n^2	$p_n(1 - p_n)$
m	$1 - p_n$**(M)**	$p_n(1 - p_n)$	$(1 - p_n)^2$

The frequency of M alleles in the next generation is equivalent to the probability of selecting the M allele from the possible genotypes that arise. The probability for MM is 1, for Mm it is ½ and for mm it is 0. Hence:

$$p_{n+1} = \frac{p_n^2 + {}^1\!/_2 \cdot 2p_n(1-p_n) + 0 \cdot (1-p_n)^2}{p_n^2 + 2p_n(1-p_n) + (1-p_n)^2}$$

$$= \frac{p_n^2 + p_n - p_n{}^2}{1}$$

$$= p_n$$

Hence, under the stated assumptions, the frequency of each allele is stable. This result is the Hardy-Weinberg Law.

The above model can be developed to include selection for different phenotypes, such as where black moths have some advantage over white moths and so are more likely to survive long enough to produce offspring. This is achieved by introducing parameters, $0 \leq \alpha \leq 1$ and $0 \leq \gamma \leq 1$, that represent the proportion of white and black moths surviving to the next generation. The frequency of M alleles in the $(n + 1)^{st}$ generation is then given as follows:

$$p_{n+1} = \frac{\alpha p_n^2 + {}^1\!/_2 \cdot 2\alpha p_n(1-p_n) + 0 \cdot \gamma(1-p_n)^2}{\alpha p_n^2 + 2\alpha p_n(1-p_n) + \gamma(1-p_n)^2}$$

$$= \frac{\alpha p_n}{(\gamma - \alpha)p_n^2 - 2(\gamma - \alpha)p_n + \gamma}$$

Where $\alpha = \gamma$, this model is the same as the simpler one above. Varying the two parameters results in non-linear equations that represent a range of different outcomes in terms of the change in proportions of black and white moths. For further elaboration, see de Vries, Hillen, Lewis, Müller, and Schönfisch (2006).

During the session, Gerda showed us an online simulator which allows parameters to be varied by moving a slider, so that effects on a graph of population can be observed.

CONCLUDING DISCUSSION

Richard Barwell offered a synthesis of some of the ideas that were discussed during the working group. First, teaching mathematics for life scientists involves working on different levels of mathematical thinking: higher-order thinking, such as problem-solving and critical thinking; mathematical strategies, such as modelling, graphing, etc.; and specific aspects of mathematics, such as calculus, probability, etc. All these aspects of mathematics are needed to interpret situations from the life sciences, as well as to interpret from the life sciences into mathematics (see Figure 3).

←INTERPRETATION→	
Higher-order thinking: Critical thinking/ problem-solving	**Life science situations**
Specific mathematical strategies: Mathematical modelling, measurement, graphing …	
Specific mathematics: Calculus, differential equations, linear algebra, geometry, probability	

Figure 3. Teaching mathematics and the life sciences.

Richard took the example of medical imaging to illustrate these ideas. In working on the mathematics of medical imaging, students need to work on higher-order thinking, including problem-solving, communicating, and using mathematical representations. Mathematical strategies used include approximation and algorithms. The specific mathematics involved is linear algebra (see Figure 4).

←INTERPRETATION→ reading back and forth	
Higher-order thinking: Critical thinking/ problem-solving/ communicating/ selecting representations/ what if…	**Life science situations:** Medical imaging
Specific mathematical strategies: Approximation, algorithms	
Specific mathematics: Linear algebra	

Figure 4. Teaching linear algebra in the context of medical imaging.

Finally, participants discussed some key principles for teaching mathematics in the context of the life sciences, as well as for selecting suitable examples from the life sciences:

- Examples should allow enough insight into mathematical 'black boxes' to allow critical thinking.
- Life science topics need to engage students and lead to questions, including mathematical questions: drawing on topical (for students) issues is one way to achieve this.
- Teaching needs to promote 'interpretation' discussions between mathematics and life science situations as well as to support student confidence.
- Such an approach requires students to do a lot of independent work (compared with 'traditional' mathematics courses).
- It is valuable to communicate and share ideas and sources with bio/life-scientist education colleagues (as this working group has shown).
- (All) school teachers of mathematics need experience of mathematical modelling.

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Report of Working Group C
Rapport du Groupe de travail C

LES MÉTHODOLOGIES DE RECHERCHES ACTUELLES ET ÉMERGENTES EN DIDACTIQUE DES MATHÉMATIQUES / CONTEMPORARY AND EMERGENT RESEARCH METHODOLOGIES IN MATHEMATICS EDUCATION

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NOTRE INVITATION

Il n'est pas possible de réaliser une recherche dans l'isolement. Les chercheures et chercheurs, confrontés à la complexité des problématiques des courants en didactique des mathématiques, développent une variété de méthodologies. Ces dernières reflètent la nature contextualisée de la recherche en didactique des mathématiques, contextes dans lesquels sont interreliés les préoccupations sociales, éducatives et les enjeux politiques de l'époque. Nous comprenons mieux la recherche en didactique des mathématiques lorsque nous prenons en compte également les préoccupations personnelles et professionnelles des chercheures et des chercheurs de même que l'engagement social qui l'initie. En effet, de multiples visées surgissent : retombées dans les milieux de la pratique, contribution au développement des connaissances, place accordée au débat. Les réformes de l'éducation, les interrogations sur la relation entre recherche et formation, sur l'articulation des expertises respectives des différents acteurs engagés dans une recherche influencent le choix des objets de recherche, des cadres théoriques et des méthodes retenues. Il convient donc d'examiner non seulement les relations entre la recherche et les méthodes, mais aussi entre les perspectives des chercheures et des chercheurs et le choix des objets de recherche, leurs cadres théoriques.

Par exemple, les travaux de Lieberman (1986), Lave (1991), Erickson (1991) et Bauersfeld (1994) ont alimenté les réflexions de chercheurs déjà impliqués dans des travaux réalisés avec les enseignants et à qui l'on doit le développement de ce qu'il est convenu d'appeler aujourd'hui la *recherche collaborative* (Bednarz, Desgagné, Diallo, & Poirier, 2001; Bednarz, 2000; Desgagnés, 1997, 2007). Des définitions, des questions et des méthodes lui sont spécifiques tout comme les conditions et les contraintes, les apports, mais aussi les limites.

- Quelles sont les problématiques actuelles? Quels types de recherche peuvent-elles initier?

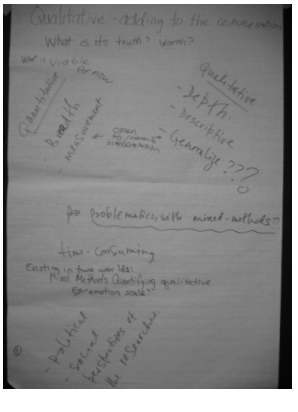
- Quelles hypothèses influencent les méthodes de recherche contemporaines?
- Quelles méthodologies innovatrices émergent de la littérature?
- Quels types de recherche sont possibles lorsque sont considérés les rôles des participants?
- Quel est le travail réel du chercheur? Le travail « idéal »? Le travail possible?
- Quelles sont les retombées des interprétations de la pratique de la classe pour la recherche et les théories en didactique des mathématiques?

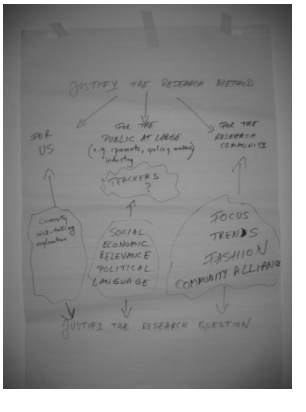
UNE SYNTHÈSE DE NOTRE PROCESSUS DE DISCUSSION

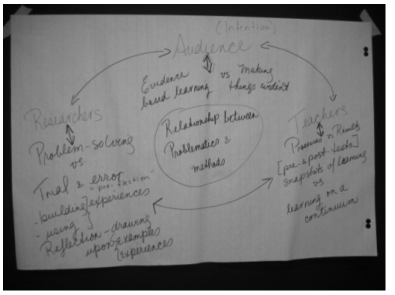
LA PREMIÈRE JOURNÉE : L'IMPORTANCE DE SITUER LA QUESTION DE RECHERCHE ET LES RÔLES DES ACTEURS

Les buts visés par chacune des rencontres permettaient d'approfondir le thème. Ainsi, la première journée visait à favoriser le contact entre les participantes et les participants, à examiner les relations entre les problématiques d'une recherche et les méthodes qu'elles initient. Durant cette première session, nous avons présenté l'opportunité pour les participants de former une communauté d'apprentissage (Wenger, 1998) en partageant leur expérience de recherche en didactique des mathématiques et en examinant les relations entre problématique et méthode. Nous introduisons la notion de problématique comme un terme qui représente la complexité de laquelle sont issus les problèmes qui guident l'élaboration de la question de recherche.

La nature des travaux de plusieurs de nos participants au début de notre groupe de travail semble accentuer l'importance de discuter de l'interprétation des données de recherche recueillies. Quelques-uns n'ont pas de cadres théorique en didactique des mathématiques parce qu'ils sont de nouveaux chercheurs dans ce champ. Ce sont leurs expériences d'apprenants des mathématiques qui les guident dans cette activité d'analyse. Les chercheures et les chercheurs sont confrontés avec les notions de réalité et de vérité dans un contexte mathématique et dans un contexte de recherche en didactique des mathématiques. À ce moment, notre description de la méthode de recherche inclus : l'épistémologie du chercheur et le cadre théorique, la question de recherche, le type de méthodologies et les méthodes qu'elles initient, les outils, et les analyses. Une variété de méthodes de recherche est évoquée allant des entrevues aux enquêtes en passant par la création de communautés de pratiques. Ces dernières favorisent l'étude de concepts mathématiques (égalité, fractal) ou encore l'exploration d'expériences de classe.







La deuxième session débute par la présentation de l'article de Hart, Smith, Swars et Marvin (2009). Cet article traite de méthodologie mixte et précise que 50% des méthodes de recherche qui se préoccupent de l'enseignement des mathématiques sont qualitatives seulement, 21% des méthodes de recherches sont quantitatives seulement, 13% des méthodes de recherche sont mixtes, c'est-à-dire qu'elles combinent des méthodes qualitatives, des méthodes quantitatives et des statistiques inférentielles. 16% combinent des méthodes qualitatives avec des statistiques descriptives. Une discussion sur ces questions émerge afin de situer nos travaux de recherche respectifs :

- Quel est le contexte de notre recherche (social, politique, etc.)?
- Quelles sont les perspectives des chercheurs?
- Quelles sont les retombées (politiques, éducatives, etc.)?
- Quels sont les impacts de la problématique sur les questions de recherche?

Plusieurs sous-groupes n'ont pas accepté d'insérer leur recherche à travers les catégories fermées de recherche quantitative, qualitative et mixte, jugée trop restrictives. Ces questions ont plutôt conduit à discuter des concepts de vérité, de valeurs, d'étendue, de mesure, de description approfondie et de généralisation. Nous avons aussi évoqué une variété de thèmes : objectivité/subjectivité, attention portée à la valeur des résultats, rôle des participants, conséquences des résultats. Puis, nous avons situé nos travaux de recherche dans le contexte politique et social du chercheur. À ce moment, nous avons identifié les intérêts de la recherche selon les publics :

- Curiosité ou explication pour les chercheurs
- Pertinence socioéconomique ou politique pour le public
- Clarification de l'objet de recherche, tendance ou alliances pour la communauté de recherche.

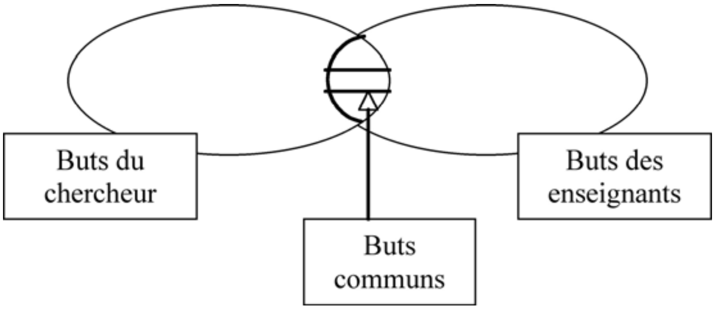
Émerge alors une discussion au sujet du besoin de situer la question de recherche et de situer les enseignants dans la recherche.

Les relations entre problématique et méthode conduisent à discuter davantage les preuves d'apprentissage et la construction de preuves pour les différents publics de chercheures et de chercheurs, d'enseignantes et d'enseignants. Nous avons aussi reconnu que chaque fois qu'intervient l'interprétation en recherche, apparaissent les questions de : vérité/réalité, subjectivité/objectivité, conception des mathématiques (corps de connaissances vs recherche humaine), rôle du chercheur et cadre théorique. Une relation entre problématique, question de recherche et méthode a créé un discours divergent.

LA DEUXIÈME JOURNÉE : INFLUENCE DES ACTEURS SUR LA DÉFINITION DE LA PROBLÉMATIQUE

La deuxième journée avait pour but d'examiner les rôles des participants dans les recherches afin de cerner le travail réel du chercheur, son travail idéal et le travail possible. Lucie a présenté quelques résultats de recherche obtenus par le biais de la méthode de recherche collaborative (DeBlois, 2009). Ce type de méthodologie, basé sur la communauté de pratiques, permettrait la production de connaissances viables. Des conditions sont toutefois nécessaires à la réalisation de ce type de recherche. Par exemple, la mise en œuvre du renouveau pédagogique et les réactions des milieux scolaires conduisent à reconnaître l'intérêt de participer à des activités qui offrent des résultats tant pour les praticiens que pour les chercheurs. Dans ce contexte, chercheurs et enseignants sont des experts qui collaborent à un but commun à travers des situations développées conjointement et des résultats diffusés conjointement. Cette méthodologie amplifie un changement dans les rôles des partenaires. Il est possible d'identifier les apports, les contraintes et les conditions de ce type de recherche. Par exemple, une alternance entre les expertises de la chercheure, des enseignantes et des enseignants se réalise. En outre, les enseignants arrivent à démystifier les résultats de la recherche puisqu'ils jouent un rôle dans cette construction. Les contraintes de la recherche présentée sont surtout liées à la nécessité de redéfinir régulièrement les buts visés, redéfinition sans laquelle chercheure et enseignants risquent de centrer leurs analyses sur des points de vue irréconciliables (DeBlois, 2009).

Nous avons posé les questions suivantes à notre groupe : Quel type de recherche est-il possible de faire quand les rôles des participants sont considérés? Quelles sont les conditions, les contraintes et les apports de ce type de recherche? Les suggestions ont consisté en l'identification des différentes relations possibles lorsque les participantes et des participants sont actifs de différentes façons. La création d'opportunités permettant d'établir des relations à partir de l'intersection entre les buts des participants et des participantes a été évoquée.



Les membres du groupe ont fait ressortir le fait que les participantes et les participants sont affectés par les chercheurs et que ces derniers sont influencés par les premiers. La présence du contexte est importante sur l'influence des relations. Les relations interpersonnelles demeurent un thème de discussion. Les questions incluent : Sommes-nous indulgents comme chercheures et chercheurs? Qu'en est-il pour les participantes et les participants? Comment nous assurer qu'il s'agit d'une expérience positive? Qui fait de la recherche sur nos pratiques de chercheures et de chercheurs?

La nature de la recherche en didactique des mathématiques considère la classe dans des sens différents (interactions, contraintes, incidents, etc.). Nous constatons qu'il est possible de réaliser des recherches en relation avec d'autres (exemple en recherche collaboration, ou herméneutique). Cela influence la problématique et l'importance d'utiliser une théorie de la complexité ou de combiner un ensemble de théorie permettant d'analyser les données. Si nous acceptons que les méthodologies émergeantes impliquent un changement de rôle des partenaires, cela a des conséquences sur la formulation des questions de recherche, la posture épistémologique adoptée, le design de recherche et la diffusion de la recherche.

LA TROISIÈME JOURNÉE : LE TRAVAIL PUBLIC DU CHERCHEUR

La troisième journée visait à développer une vision épistémologique à l'égard des méthodes de recherches. Gladys a présenté son travail de recherche sur la relation à l'apprentissage, recherche réalisée avec Theresa McDonnell, une enseignante de l'école des premières nations. Elle a partagé ses impressions sur le travail de relations interpersonnelles impliqué : changement de rôles comme co-chercheure et co-enseignante, développement d'une relation de confiance, partage des expertises, confrontation des défis (sur des concepts de racisme, colonialisme et préconceptions de ces notions), dépense des temps de visite.

La recherche de Brown (2008) est ensuite présentée et devient le point de départ de la discussion. Cette étude vise à identifier les auteurs cités en didactique des mathématiques et à identifier les publics auxquels s'adressent leurs travaux. Ces caractéristiques conduisent à développer une conceptualisation de la communauté de recherche. Brown constate: «The research is conceptualized as adopting a relatively objective eye rather than subjective "I" in positioning teachers and students in roles from which they cannot readily escape» (p. 6). Il observe que les auteurs se citent eux-mêmes, citent des philosophes décédés ou d'autres chercheurs en didactiques des mathématiques. Il observe à l'occasion des citations de scientifiques sociaux et presque jamais des citations d'enseignantes, d'enseignants ou d'éducateurs. Nous nous sommes demandés : quelles sont les conséquences de ces méthodologies émergentes? Une variété de réponses a surgi :

- Enthousiasme du chercheur pouvant influencer les résultats
- Investissement exigée par la méthodologie de la recherche
- Biais
- Révélations pour les participants notamment à l'égard du processus d'apprentissage
- Tensions relationnelles étant responsable moral et éthique dans la relation
- Équité entre les partenaires
- Augmentation des défis
- Processus de diffusion de la recherche, soumissions
- Sources de données, formulaires de consentement
- Éthique
- Langage authentique lorsque nous voulons être compris.

Nous avons offert l'article de Saha (2009) pour une lecture ultérieure sur l'éthique et la diffusion des connaissances sur la recherche des enseignantes et des enseignants. En petits groupes, nous avons tenté de répondre à la question «Comment ces implications contribuent-elles au développement des connaissances?». Nous étions préoccupés de l'influence que nous pouvons avoir sur les résultats de recherche mais avons compris que les choix faits sont parfois des choix émotifs.

CONCLUSION

L'incertitude fait partie de notre travail dans un contexte relationnel. Nous travaillons à une construction sociale. Nous ne pouvons isoler des variables mais nous avons un passé de recherche mathématique scientifique qui permet de travailler dans un monde idéal. Maintenant, nous devons être attentifs aux conséquences pour les enseignantes, les enseignants, les chercheures et les chercheurs : échanges d'idées, construction de connaissances et validation de l'enseignement. Les enseignantes et les enseignants relèvent un défi avec les chercheures et les chercheurs mais ces derniers relèvent un défi avec les premiers. Dans ce travail relationnel, nous considérons le rôle des technologies dans la publication (dynamique et non linéaire) et le processus de filtrage des idées par les revues. Nous constatons que les connaissances sont partagées par les publications et les planifications. Nous nous définissons comme une communauté en développement.

OUR INVITATION

We claimed it is not possible to do research in mathematics education isolated from contextual factors. As researchers encounter the complexity of current educational issues in mathematics, multifaceted methodologies emerge. These methodologies reflect the contextual nature of research in mathematics education, that is, they are intertwined with the social, political, and economic issues of the era. We can better understand research in mathematics education when we consider personal and professional issues and the variety of research goals inherent in social engagements such as implications for classroom practice, the development of knowledge, and engagement in academic conversations. For example, the works of Lieberman (1986), Lave (1991), Erickson (1991), and Bauersfeld (1994) contribute to the development of collaborative research, especially in the area of research with teachers (e.g. Bednarz, Desgagné, Diallo, & Poirier, 2001; Bednarz, 2000; Desgagné, 1997, 2007). This type of methodology employs specific definitions, questions, and methods with specific conditions, constraints, and contributions with limits.

Thus, it is necessary to examine the relationships between research and design, and between researchers' perspectives and how these influence their choices of research objects, theoretical frameworks, and methods. Through literature, mathematical tasks, and dialogue, we invited members of this working group to explore:

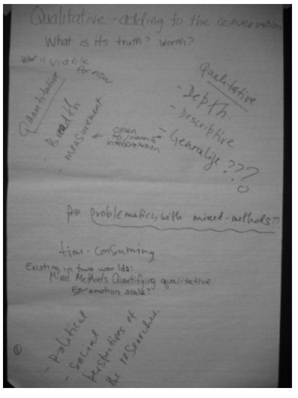
- What is the nature of inquiry in mathematics education? What is problematic?
- What theoretical assumptions influence contemporary research methodologies?
- What innovative methodologies are emerging in the literature?
- What kinds of research are possible when considering shifting roles of participants and teachers as co-researchers?
- What is the real work of the researcher? the ideal work? the possible work?
- What are the implications of competing interpretations of practice for research and theory in mathematics education?

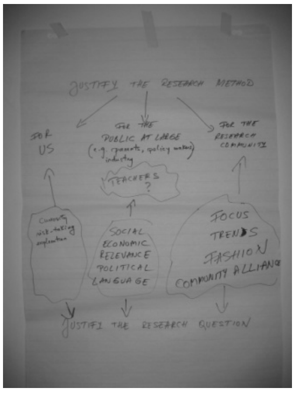
OUR DISCUSSIONS

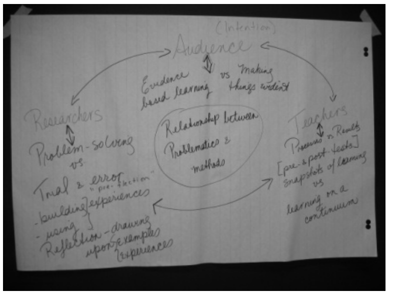
THE FIRST DAY: THE IMPORTANCE OF SITUATING THE RESEARCH QUESTION AND PARTICIPANTS' ROLES

Our goals were, for each meeting, to develop a deeper discussion on our theme. The first day we wanted to encourage connections between participants and to examine the relationship between problematics and methods of research. First, we presented opportunities for participants to form a learning community (Wenger, 1998) by sharing their research experiences about the nature of inquiry in mathematics education and by examining the relationship between problematics and method. Drawing on the French notion of *les problématiques*, we introduced *problematics* as a term to represent the complex issues that informed or led to research questions.

At the beginning of our discussion, the nature of inquiry for many of our participants seemed to emphasize the importance of interpretation during the analysis of data. Some of our participants did not have a developed theoretical framework in mathematics education because they were new scholars in this area. Drawing on their experiences of mathematics, they were challenged by others in the group to consider a subjective reality. Contemporary research designs emerged from the participants' experiences of mathematics. Participants were confronted with notions of reality and truth. At this point, our description of research included: research questions, tools, analysis, types of methodology, method, epistemology, and theoretical frameworks. A variety of methods was identified, from interviews to inquiry, through communities of practices. These methods were applied in studies about gestures in mathematics, equality, fractals, and exploration in class.







The second session began with a presentation of an overview of research in mathematics education (Hart, Smith, Swars, & Marvin, 2009). The authors found that from 1995 – 2005, 50% of the studies used qualitative methods only, 21% used quantitative methods only, 13% used a mixed method that combined qualitative methods and inferential statistics, and 16% combined qualitative methods with descriptive statistics. In small groups, we discussed these questions:

- Using the literature presented, how would you situate your work?
- What is the context of your research (social, political, etc.)?
- What are your perspectives as researchers?
- What are the issues (political, educational, etc.)?
- What is the impact of using problematic or research questions?

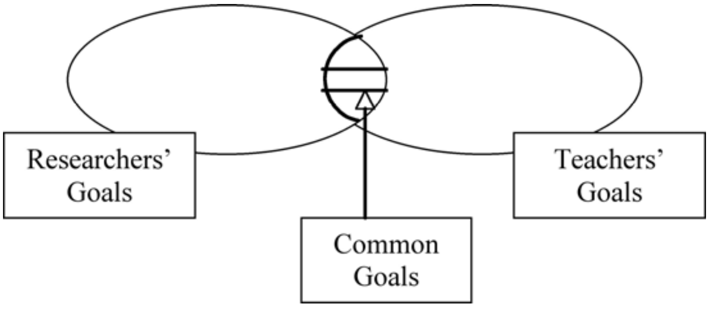
Most group members did not want to situate their research in the three categories because they judged that it was too restrictive. These questions prompted discussion about truth, worth, breadth, measurement, depth, description, and generalization. Various themes emerged during our discussion: objective vs. subjective, awareness about truth of findings, tensions vs. attention, the role of the participants, and issues arising from research findings. We situated the research work in the context of the political and social perspectives of the researchers. Participants expressed tensions in justifying the research method for researchers (curiosity, explanation), for the public (socio-economic, relevance, political, and language), and for the research community (focus, trends, fashion, community alliance). They also talked about the need to justify the research question to situate teachers within the research.

The relationship between problematics and methods led to more discussion about evidence-based learning and making things evident, and about the audience for researchers and teachers. We recognized that everything involves interpretation in research, and considered questions about trust/reality, what we believe, subjectivity/objectivity, our conception about mathematics (body of knowledge vs. human inquiry), the role of the researcher, and theoretical frameworks. A connection between problematics, research questions and methods created a divergent discourse.

THE SECOND DAY: INFLUENCE OF PARTICIPANTS ON DEFINING THE PROBLEMATIC

In the second session we examined the role of participants and considered questions like: What is the real work of the researcher? What is the ideal work? What is the possible work? Lucie presented some of her research results around the idea of collaborative methods of research (DeBlois, 2009). She suggested that a collaborative research method was based on a community of practice, that it raised the production of viable knowledge, and that it allowed for the renewal of knowledge. However, some conditions are present in this type of research. Pedagogical reforms and teachers' reactions have led school districts to provide "a la carte" training. It is therefore important that both schools and teachers participate in these types of professional development activities that provide useful results for teachers as well as researchers. Researchers and teachers are experts working towards common goals of co-location, cooperation, and co-production. This methodology involves a shift in roles. Some benefits, challenges, and conditions emerge from this type of research. For example, alternating between the expertise of researcher and of teachers occurred. In addition, teachers demystified research results because they played a role in the construction. Constraints of the research were linked with the necessity to redefine goals. Without redefinition, researchers and teachers risk focusing their analysis on an irreconcilable point of view (DeBlois, 2009).

We posed the following questions for group discussion: What type of research is possible when the participants' roles are reconsidered? What are the conditions, constraints, and benefits in using this type of research? Suggestions were that different relationships are possible and participants are not passive but active in different ways. This creates opportunities for relationships and the intersection of goals among participants becomes a priority. Group members emphasized that the participant is affected by the research(er) and the research(er) is affected by the participant. The presence of context is important in influencing relationships. Relationships remained a theme of this discussion. Questions raised through this discussion included: Are we self-indulgent as researchers? What is in it for the participants? How can we ensure this is a positive experience? Who is researching our practice as researchers?



We claimed that if we accept that emerging methodologies involve a shift in roles (e.g. participants as researchers), then there are implications for: dissemination, research questions, epistemology, and research design. The nature of inquiry in mathematics education takes into account the classroom in a different way (interactions, constraints, incidents, etc.). We assume that it is possible to do research in relation with each other (e.g. collaborative research, hermeneutic inquiry, etc.). This influences the problematic and the importance of using complexity theory, or combining many theories allow the analysis of data.

THE THIRD DAY: THE PUBLIC WORK OF THE RESEARCHER

The goal of the third day was to develop an epistemological view of research methodologies. Gladys presented her work with Theresa McDonnell, a teacher in a First Nation school, about learning relationally. She shared her impressions of working in a relation that involved: shifting roles as co-researchers and co-teachers, developing trust, sharing expertise, confronting challenging notions such as racism, colonialism, and preconceived notions, and spending time visiting.

Brown's (2008) research was presented as a starting point for discussion. The purpose of his study is to investigate who authors in mathematics education are listening to and addressing in their work, and how this characterizes how the research community is being conceptualized. Brown states, "The research is conceptualized as adopting a relatively objective eye rather than subjective 'I' in positioning teachers and students in roles from which they cannot readily escape" (p. 6). He found that authors are citing themselves, dead philosophers, and other mathematics education researchers. They are occasionally citing social scientists, and almost never citing mathematics teacher education researchers, school teachers or teacher educators. We asked: What are the implications of emerging methodologies? A variety of responses were offered:

- Imposing researcher enthusiasm
- Influencing results
- Commitment to identified research methodology
- Imposing biases
- Revelations to participants (learning process)
- Relational tensions of being a role model and moral/ethical responsibilities of being relational
- Equal partners; power
- Increased challenges
- Dissemination process "submit"
- Data sources, consent forms "submit"
- Ethical review forms "submit"
- Authentic language when writing mindfully

We offered Saha's (2009) work for further reading about the ethical act of disseminating knowledge about research on teachers to teachers. In small groups, we tried to answer the question: How do these implications contribute to the development of knowledge? These ideas were shared with the large group. We were preoccupied by influencing results but understood that research choices are emotional choices.

CONCLUSION

Uncertainty is a part of our work as we engage in a relational context. We work to build a social construction of our work. We cannot isolate variables but we come from a mathematical scientific background where we worked in an ideal world. Now, we must be aware of explicit implications for teachers and research: exchange of ideas, knowledge-building and validation of teaching. Teachers are challenged by researchers and researchers are challenged by teachers in a reciprocal relationship. We realized that we should credit, acknowledge and honour the researcher, teacher and participants. In relational work, we wonder if all voices can be attended to, if there is a role for technology in publishing research in a dynamic, non-linear way, and if the process of filtering our ideas by journal editors and reviewers is appropriate. We observe that knowledge is shaped by dissemination and planning. We are defining ourselves as a growing community.

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Report of Working Group D
Rapport du Groupe de travail D

REFRAMING LEARNING (MATHEMATICS) AS COLLECTIVE ACTION

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INTRODUCTION

Many aspects of school mathematics and mathematics education research are rooted in assumptions about learning, practice, and knowledge that are proving to be troublesome. These assumptions include beliefs that learning and knowing are (only) individual or personal phenomena that are context-free, contained and innate, and fragmented (i.e., disciplinary/competence specific). A growing body of interdisciplinary research challenges such statements and instead compels us to reframe our accounts of learning in terms of collective, situated, adaptive and interconnected action. This working group explored this area of emergent thought. On our journey, in an effort to understand collective phenomena and their implications for mathematics classrooms, we considered contemporary research in mathematics education that offers promising theoretical framings, studied and analysed video excerpts of groups of learners engaging with mathematics, and tackled mathematical tasks that prompted reconsideration of our own experiences of learning and doing mathematics.

DAY ONE

The origins of the working group topic lie in the ongoing research of Jo Towers and Lyndon Martin concerning the nature of mathematical understanding. As part of the usual group introductions we shared some of what interests us (and has interested us for many years) and which we hoped the work of the group would help to extend:

- *What are we interested in?*
- *Mathematical understanding*
- *How mathematical understanding grows*
- *Where understanding is located*
- *How its growth is occasioned*
- *How it can be observed, characterised and described*
- *How we might usefully theorise it to inform classroom practice.*

We found that our group was an exciting, interesting and diverse group of colleagues many of whom had thought about or researched around these issues albeit from very different perspectives. The group included teachers and researchers who were concerned with interaction, with the relationship between the individual and the collective, with beliefs about how we know and what we know, with teacher education, with using group work in the classroom, and with how all of this might be usefully theorised.

As a starting point for the work of the group over the three days we framed some of the issues we hoped to consider and explore together. We did not expect to, nor did we, find full and complete answers to any of these questions, but they did help to direct and focus our thinking, working and discussions. We suggested that:

> *In this working group we are interested in how understanding grows in (and for) groups of learners,*

and more specifically asked:

- *Is there a phenomenon we can distinguish as shared or collective understanding?*
- *What is its nature?*
- *How might it be observed, characterised and described?*
- *Why might we want to pay attention to it:*
    - *as a research community?*
    - *as a classroom teacher?*
    - *as a learner of mathematics*?

These four questions helped to direct our individual and collective actions for the working group. To help bring these questions to life, and to give the group some sense of how we view collective understanding, we shared two short clips of video data. The first extract featured three Canadian grade six students starting work on a problem in an interview setting. The students had been posed the problem of calculating the area of a parallelogram, a figure with which they had not worked before. The second extract involved four pre-service mathematics teaching students, training to be high school teachers, and the episode was taken from a regular university classroom session. The class was working on the 'taxi-cab problem', a well-known problem involving non-Euclidean geometry (see Krause, 1975), and the students were asked to find out what happens to different figures under taxi-cab geometry.

There is not space here to fully detail or analyse the clips (see Martin, Towers, & Pirie, 2006). However, for us, they served the purpose of highlighting that, in particular forms of group interaction, it can be extremely difficult to discern individual pathways of growth of mathematical understanding, and that, sometimes (and in both examples we offered), the growth in mathematical understanding can be significantly observed at the collective level. These key notions provided the basis for the discussion that followed in which we focused particularly on the first two of our questions: What is the nature of collective mathematical understanding and how might it be observed, characterised and described?

In the second part of the session we moved to introduce some of the ways that we think about collective mathematical actions and understanding, specifically using the lens of improvisational theory. We described how our work was influenced by the work of Susan Pirie and Thomas Kieren in the field of personal mathematical understanding, and of how we came to consider collective understanding as distinct from this. We introduced some key ideas of improvisational theory that we have found useful in considering collective mathematical actions:

- *Potential pathways*
- *Collectively created structure and "striking a groove"*
- *Etiquette and listening to the "group mind"*
- *Dependence on* others

and highlighted some of the key elements of what we saw as being improvisational in the two video-clips on which the group had been working. The group continued to discuss both these clips and our offered theoretical perspective, and these formed the basis of the closing discussion of the morning. Some of the elaborations of the proposed framing developed by the group during this closing discussion were:

- *Sense of collective purpose – committed to the problem*
- *Potential for many pathways – no way to predict how the mathematics will unfold*
- *Emergence of a collective structure – an agreed upon Image Making action*
- *Etiquette of emerging understanding – deferring to the group mind, looking for the "better" idea to further their Image Making*
- *Dependence on others – hard to identify individual pathways of growth, continually building on what is offered to further collective progress*.

DAY TWO

On day two we continued to work around the two questions: *'What is the nature of collective mathematical understanding, and how might it be observed, characterised and described?'* We also sought from the group additional questions and issues that participants felt we might want to explore during the next two sessions. Some of those offered included:

- *How might collective understanding be assessed?*
- *Where's the teacher? What is his or her role?*
- *The challenge of the detailed analysis of video required*
- *What does a human have to know in order to accomplish this action?*
- *The problem of the language we have and use*
- *Are there conditions under which learners can collaborate to create?*
- *How to engage students in collective mathematical understanding? Are there ways to facilitate?*
- *The role of the larger societal culture*
- *How does knowledge relate to understanding?*

Alongside these we offered some of our own current thinking concerning what we have identified as important to think about and to attend to (drawing here from improvisational constructs):

- *Group composition*
- *Nature of the task*
- *The role of the 'expert'*
- *The dominant member*
- *The role of the 'soloist'*
- *Resources, tools etc*
- *Cultural and historical backgrounds*
- *Presentation of the task*
- *Intentionality of actors*
- *Prior knowing and existing understandings*

- *Etiquette – how group members listen to each other*
- *Nature of involvement*
- *Degree/extent of collectivity*
- *Striking a groove*
- *Kinds of interaction – language, physical, space, etc.*
- *Necessity of all group members*

With all of these issues, questions, ideas and thoughts in mind we then moved to engage the group in some mathematical problem solving. The idea here was to see whether we ourselves would generate something identifiable as collective understanding when we worked together mathematically. The group divided into three smaller groups, and chose a problem (some offered by us, others by members of the group). It was suggested that within each group one person take on the role of observer, with the task of looking for and noting what might seem to be moments or periods of collective understanding. Some groups also chose to use video to record how they worked and then reviewed this later.

Many ideas and conversations followed from this problem solving session. These included thoughts, issues, observations and questions such as:

- *The role of tools – e.g. Blackboard and Geometers' Sketch Pad*
- *The lack of initial need for 'the collective' if everyone could 'do' the problem individually*
- *The significance of a shared piece of paper to work on*
- *The expertise of one group member, but the importance of them also not dominating*
- *The need to step back in order to find a role in which to step forward*
- *The 'soloist idea' – and the recognition that we are attracted to this idea because we are not yet willing to let go of the notion of the individual*
- *The ideas are moving, not just the people*
- *One person objectifying and synthesizing the ideas, invoking the idea for the collective*

DAY THREE

We began day three by drawing on some of the ideas and questions that had emerged from the group members during their mathematical problem-solving (as listed above) and by showing how, in our own research, we have begun to theorise and frame these issues. In particular, we worked with the questions about the 'need' for the collective and the expertise and contribution of various group members by invoking the notion of 'co-actions' (Towers & Martin, 2006; Martin & Towers, 2009). As a starting point we offered the following quote from Gerry Stahl:

> *Only individuals can interpret meaning. But this does not imply that the group meaning is some kind of statistical average of individual mental meanings, an agreement among pre-existing opinions, or an overlap of internal representations. A group meaning is constructed by the interactions of the individual members, not by the individuals on their own. It is an emergent property of the discourse and interaction. It is not necessarily reducible to opinions or understandings of individuals (Stahl, 2006, p.349).*

Followed by our definitions of improvisational co-actions:

- *Co-action is a specific kind of interaction, but whereas interaction allows for a reciprocal, complementary collaboration, co-action goes beyond this and requires mutual, joint action.*

- *Co-action depends on how an offered idea is collectively accepted by the group into the 'emerging performance'.*
- *C-oaction is the process of acting with the ideas and actions of others in a mutual, joint way.*
- *Co-action describes a "particular kind of mathematical action, one that whilst obviously in execution is still being carried out by an individual, is also dependent and contingent upon the actions of others in the group" (Martin, Towers, & Pirie, 2006, p.156).*

And what we have called the characteristics of improvisational co-action (Martin & Towers, 2009):

- *No one person driving*
- *An interweaving of partial fragments of images*
- *Listening to the group mind*
- *Collectively building on the 'better' idea (elaborated in Towers & Martin, 2009)*

We illustrated these through some short extracts of video, which focused on three students, preparing to be elementary school teachers, as they worked on some mathematical problems ranging from "Fractions, Decimals and Percentages" to "Mathematical Argument". The ideas we were introducing here provoked some intense and strong debate among the group, especially the notion of what might be meant by the 'better idea' in the context of mathematical problem-solving.

In the final part of this session we returned to our original questions and in particular to the vital one of "*Why might we want to pay attention to collective mathematical understanding?"* Ideas and thoughts offered here included:

- *Allowing you to notice what is happening for a group so as to be able to make some decisions about when to intervene*
- *The better idea is the one we go with*
- *Better' is not as bad as 'best'*
- *Better is all about maintaining the collective structure*
- *Role of the teacher is to institutionalize the pieces of knowledge that pop up in a classroom*
- *Some teachers don't see themselves as part of the collective*
- *The learner can be a group*
- *The recognition or awareness that this 'feeling' of striking a groove is possible in a math classroom might be important for teachers and learners*
- *Importance of making the learner of mathematics more than just a learner of math – it is also about being something, being with others*
- *Learning from* (not just about) teaching

Although no consensus was reached on many of the issues and questions raised we felt we had both highlighted the importance and complexity of the notion of collective mathematical understanding, enabled participants to explore their own experience of working on mathematics in a group through a new lens, and emphasised the need and potential for further research and thinking in the field. In the spirit of improvisation, our reporting back to the wider group of conference participants in the closing session took the form of non-scripted personal responses. These were prompted initially by one of the group quoting from Stahl (as above) and thereafter opened to free-flowing statements in response to what others in the group were saying.

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Report of Working Group E
Rapport du Groupe de travail E

ÉTUDE DES PRATIQUES D'ENSEIGNEMENT

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INITIAL DESCRIPTION OF OUR WORKING GROUP

The study of teacher practice is increasingly taken into account in research in mathematics education. There is increasing evidence of influences on teacher practice—such as context and situation, subject matter knowledge, beliefs, and pedagogical knowledge. From pre-service, to in-service, to graduate studies, to research, the effect of teaching practice on learning is known to be essential—such as the impact on teacher behaviour and possibly student achievement (although to date the direct relationship between teaching and learning has not been shown). The concept of teaching practice is central to research with an interest on the impact of teachers' work.

In this working group, we wanted to discuss various questions about the study of teacher practice around three dimensions: 1) the foundations of the notion of practice, 2) the reasons for studying teacher practice, and 3) the possible methodologies to do so.

1. The concept of teacher practice relies on the various perspectives, beliefs, and conceptualizations of "practice". What are the underlying (implicit) principles of the concept of practice? What are some of the different meanings the notion of practice can have in various epistemological perspectives? In what sense is teaching a practice? How is the concept of practice useful for mathematics education?
2. Why should the concept of practice be studied? What can we learn from the study of teacher practice? From the point of view of students' learning? From the point of view of in-service teacher training? From the point of view of pre-service teacher training? From the point of view of research in mathematics education?
3. Using particular elementary school level and secondary school level examples of teacher practice to explore the methods of analysing mathematics teaching practice, what dimensions should be taken into account? How can we describe teaching practice? Is it possible to refer to teacher practice without referring to a particular teacher's practice? Is it possible to refer to teacher practice without taking into account the content that is taught?

BACKGROUND

The initial discussion of the working group illuminated a number of common and shared conceptions of teacher practice. Common themes emerged, for example: i) the importance of beliefs and confidence in oneself as a teacher; ii) concerns for the selection and management of materials, resources, and mathematics content; iii) the importance of the knowledge of students and of self—used in the processes of interpretation and understanding of classroom experiences, and in making pedagogical decisions with respect to particular tasks for mathematical learning opportunities; and iv) social and cultural factors that influence and impact the teachers' classroom practices and the learners' mathematical experiences.

These emergent themes began to coalesce into three overarching conceptions: "institutional/logistics", "personal/professional/theoretical", and "classroom students" or alternatively conceptualized as "a priori knowledge", "responsiveness in the classroom", and "post-reflective turns", respectively. These categories and category titles segued into an articulation that as researchers or classroom teachers, we carry various frameworks of conceptions of mathematics teaching and learning, and that these frameworks are often evident theoretically in our thinking as well as practically in our actions.

DIFFICULTÉ À CERNER LA NOTION DE PRATIQUE D'ENSEIGNEMENT

Au cours de la première séance du groupe de travail, nous avons tenté de définir en quoi consiste la pratique d'enseignement. Dans la littérature scientifique, il se dégage consensus à l'idée que le travail de l'enseignant ne s'effectue pas uniquement pendant le temps de classe. L'enseignant travaille aussi en dehors de la classe. Étudier les pratiques d'enseignement exige alors de s'intéresser au travail de l'enseignant en classe et en dehors de la classe. Pour Wenger (1998), la pratique relève du "faire", dans ses dimensions à la fois historiques et sociales, et dans sa capacité à produire de la structure et une signification aux actions. Il en découle plusieurs caractéristiques d'une pratique :

1. La pratique relève du faire mais n'est pas réductible aux actions posées par l'enseignant et/ou observées par le chercheur. Elle doit inclure le système qui les oriente. Dans ce sens, le chercheur ne doit pas s'arrêter aux actions de l'enseignant mais tenter de remonter aux logiques de ces actions et leurs déterminants.
2. Une pratique s'inscrit dans la durée et dans une historicité. Une activité isolée, sans précédent historique, sans référent historique, sans fondement historique ne relèverait pas de la pratique.
3. Cela ne signifie pas que la pratique n'évolue pas. Au contraire, une pratique est dynamique, elle évolue dans le temps. L'étude de la pratique vise, entre autres, à « déceler le permanent dans le changement (…)» (Weil, 1996).
4. En tant qu'activité humaine, une pratique est ancrée dans un contexte social. Une pratique est socialement partagée, au moins partiellement. L'enseignant n'est pas un être solitaire. Il appartient à des collectifs de travail, qui influencent ses choix.
5. Toujours selon Wenger (1998), le concept de pratique inclut à la fois le champ de l'explicite (le langage, les outils, les documents, les symboles, les procédures, les règles que les différentes pratiques rendent explicites), et le registre du tacite (relations implicites, conventions, hypothèses, représentations sur le monde).

Une pratique produit et est le produit d'activités. Si l'on accepte qu'une activité est une séquence d'actions médiatisées et interreliées qu'une personne réalise pour atteindre un but (Radford, 2008), une pratique serait un système d'activités, médiatisées (culturellement, socialement, historiquement) et interreliées qu'une personne réalise en vue d'atteindre un but. Dans un certain sens, les actions sont aux activités ce que les activités sont aux pratiques.

Dans l'étude des pratiques, le chercheur est pris par un certain nombre de tensions qui ont fait l'objet de discussions au sein du groupe.

1. Singularité / régularité
2. L'analyse du travail de l'enseignant est située et conditionnée par un ensemble de facteurs. Le chercheur doit distinguer ce qui dans l'action de l'enseignant est singulier, de ce qui témoigne d'une certaine régularité. Il doit pour cela observer l'enseignant durant une longue période et faire varier certaines variables selon l'objet de son étude (différentes classes de même niveau ou de niveaux différents; différents objets d'enseignement, etc.). Le risque est de prendre pour générique une action spécifique de l'enseignant.
3. Particulier / général
4. Les pratiques d'enseignement sont avant tout des pratiques d'enseignants. Ce sont ces derniers qui pratiquent l'enseignement. Une pratique est donc toujours une pratique de quelqu'un. Comment alors la pratique d'un enseignant rend-elle compte d'une pratique d'enseignement au sens générique du terme. En outre, il n'y a pas d'enseignement sans un objet d'enseignement; il n'y a pas de pratique sans un objet de pratique. Comment alors la pratique d'un enseignant relativement à un objet d'enseignement rend-elle compte de la pratique d'enseignement de cet enseignant ?
5. Microscopique / macroscopique
6. Le niveau de grain d'analyse est un enjeu important dans les choix méthodologiques d'analyse de la pratique d'enseignement. La centration sur l'identification de régularités amène à utiliser un grain d'analyse assez gros. En revanche un grain microscopique rend difficile de dépasser le caractère singulier des observables.
7. Point de vue du chercheur/point de vue de l'enseignant
8. Le point de vue du chercheur seul n'est pas suffisant pour rendre compte de la pratique d'enseignement. La prise en compte du point de vue de l'acteur est primordiale.

TEACHER EFFICACY - AIMS

The aim of this section of the working group was to articulate a possible framework to encompass these expressions of the factors that reflect our own teaching practices. Encouragingly, the three conceptual frameworks selected prior to the working group meeting appeared to encapsulate the working group's expressions, thoughts, and feelings. The three conceptual frameworks are: teacher efficacy (Bandura, 1997; Tschannen-Moran & Woolfolk Hoy, 2001; Tschannen-Moran, Woolfolk Hoy & Hoy, 1998), teacher orientation (Feimen-Nemser, 1990), and teacher concern (Borich & Tombai, 1997; Fuller & Bown, 1975). The following few paragraphs will describe these conceptual frameworks in relatively general terms.

> *We teach who we are. So* **who** *is the Self that teaches? From the perspective grounded in teacher formation, that Self is the* **who** *we are 'disposed' to be, not the* **who** *external forces maintain we are 'supposed' to be (Hare, 2007, p. 143, bold in original).*

Specific beliefs that teachers carry regarding their confidence in their teaching ability, within the context of teaching, are called teacher efficacy. These beliefs also pertain to the teacher's beliefs of his or her capacity to affect student performance. Teacher efficacy has been found to be a powerful construct that appears to explain and/or predict many aspects of teaching and learning. For example, teacher efficacy is related to student achievement (Tschannen-Moran & Woolfolk Hoy, 2001, 2002), pre-service teacher behaviours and pre-service teacher preparation (Ashton, 1984; Bruce, 2005; Gordon & Debus, 2002; Watters & Ginns, 1995), in-service professional development effects (Ross & Bruce, 2007), attitudes towards children and control (Woolfolk & Hoy, 1990), and mathematics reform efforts (Smith, 1996; Wheatley, 2002). This is not an exhaustive list, however it appears research into teaching and learning has benefited from the teacher efficacy perspective, and the working group's thoughts and reflections on teacher practice appear to align with this framework.

Feimen-Nemser (1990) suggests that the 'orientation', or complex mix of orientations, held by the pre-service program faculty influences the development of, and subsequent classroom practice orientation of pre-service teachers. Feimen-Nemser (1990) conceptualized teacher orientation with five 'extreme' cases: Academic, Technological, Practical, Personal, and Critical/Social orientations. The diversity of orientations reflected as attitudes and perspectives held by pre-service education faculty was understood to have influence, to impact, to align with, and/or to conflict with, the developing pre-service teachers' sense of self and classroom instructional behaviour. Feimen-Nemser (1990) and Cotti and Schiro (2004) noted pre-service teachers' orientations may reflect the faculty orientation to beliefs about the purpose and means of teaching mathematics. In practice, an institution or an individual teacher would express a mix of these orientations. The combination and integration of these orientations would be apparent in the focus of a pre-service education course, and subsequently may become apparent in pre-service teachers' articulations and expressions of their classroom practice, e.g., classroom management or instructional strategies.

The following is the prompt for the working group, and the descriptive paragraphs of each teacher orientation used in the working group discussion:

"Which of these most appeals to your sense of yourself as a mathematics teacher?"

ACADEMIC ORIENTATION

> *The teacher's academic preparation is vital. The knowledge of the structure, concepts, skills, and processes of mathematics is the fundamental basis for successful teaching in a secondary school math classroom. I know the mathematics and my professional treatment of mathematics determines the quality of my teaching.*

TECHNICAL (TECHNOLOGICAL) ORIENTATION

> *There are tried and true skills, processes, and steps to follow in order to be an effective classroom teacher. There are basic principles and procedures to be used by teachers to achieve specified goals. In order to be a successful teacher, it is necessary to develop proficiency in the skills of teaching.*

PRACTICAL ORIENTATION

> *The greatest source of knowledge about teaching mathematics is the experience of teaching mathematics. To become a successful classroom mathematics teacher, one must be immersed in the classroom environment as the teacher. Daily practical dilemmas and situations in the classroom provide the opportunities to develop and hone the teacher's ability to learn to teach and develop the practical wisdom of expert teachers.*

PERSONAL ORIENTATION

> *To be a good teacher one must know students as individuals. In order to select appropriate materials and tasks for student learning, the teacher must know the student's individual interests, needs, and abilities. In addition teachers must know themselves and work towards personal fulfillment and meaning as a classroom teacher. These dual goals intersect and the attention to the personal development of*

the students and the teacher creates the opportunities for quality learning and successful teaching.

CRITICAL/SOCIAL ORIENTATION

The critique of schooling in combination with a progressive social vision provides the optimal classroom environment for quality teaching and learning. A teacher's social justice focus empowers students to connect the relevance of mathematics to their personal identity and find influential experiences in the larger local and/or global community.

Teacher concerns are the perceived problems or worries of teachers (Fuller, 1969). Teaching behaviours and classroom practices for teaching and learning mathematics arising from teachers' beliefs may be related to teachers' concerns. Fuller and Bown (1975) identify a set of concerns experienced by teachers and suggest that these concerns are somewhat linear, in that teachers progress through stages of concerns. Initially, teachers experience concerns for 'self' in a focus on survival and "*one's adequacy ... as a teacher, about class control, about being liked by pupils, about supervisors' opinions, about being observed, evaluated, praised, and failed*" (p. 37). Then, stretching into the first years of teaching, teachers' concerns turn to 'task', the knowing and presenting of the mathematics content, lesson timing issues, and other instructional duties, and then, to 'impact' and being aware of the learner and his or her needs, and the evaluation of learning fairness.

These stages are also articulated within the context of power in the classroom (Staton, 1992). This sense of power in the classroom is pre-service and in-service teachers' sense of control and being controlled, their sense of power exerted by them in the classroom and exerted on them from outside the classroom. The teachers' sense of control is also related to their sense of teacher efficacy, for example, their beliefs in their ability to provide effective classroom behaviour management. Erikson (1993) suggests that there exists a continuum on which beliefs and classroom practice change at different rates, and it is possible that varying degrees of all three concerns will be evident in teachers' expressions of beliefs and classroom practice (in Muis, 2004).

The following is the prompt for the working group members, and the descriptive paragraphs of the three teacher concerns used in the working group's discussion:

"Which of these most accurately represents the concerns you have in your mathematics teaching at this point?"

SELF

I feel stress about class control—that is, classroom management and discipline and dealing with student behaviours in my class. I think about having to master all the content of courses I am to teach, and not knowing the answers to the mathematics questions my students will ask. I am concerned about being evaluated by supervisors, like my Associate Teacher during practicum, and soon my Principal when I start teaching in my own classroom. I feel in survival mode and sometimes I wonder how I will ever learn to teach at all.

TASK

I worry about finding the appropriate teaching method. It is vitally important to find the right materials too, like the right activity, task, computer applet, and using the smartboard or graphing calculators. For each topic I worry about how I am going to teach it. I know the math I am to teach, I am just unsure how I should teach it.

IMPACT

> *Assessment is the first thing I think of and I worry about finding and then using the right assessment tool (for example: rubrics, achievement categories, checklists, performance criteria, marking schemes). I look for the right task and materials to adequately measure student achievement. Fairness and equitable assessment is important because I work to continually recognize the social and emotional needs of my students as well as their intellectual development.*

DISCUSSION PROCESS IN THE WORKING GROUP

During the working group session, the participants were asked to read the paragraphs of teacher orientation from the perspective of a classroom teacher, and to reflect upon and draw a picture of their own individual teacher orientation, using intersecting circles where size, position on the paper, and overlap with the other circles, would indicate the interactions and connections of the five orientations. Participants then shared their diagrams with each other and the group as a whole. The paragraphs for teacher concern were then read, from the perspective of the classroom teacher. A short response was written to the prompt, "*What is your teacher concern? Give an example to support your claim.*" This was then shared with the working group. Teacher efficacy was explored through the Teachers' Sense of Efficacy Survey (Tschannen-Moran & Woolfolk Hoy, 2001). The working group as a whole discussed the implications of our own teacher efficacy with regard to the 'study of teacher practice'.

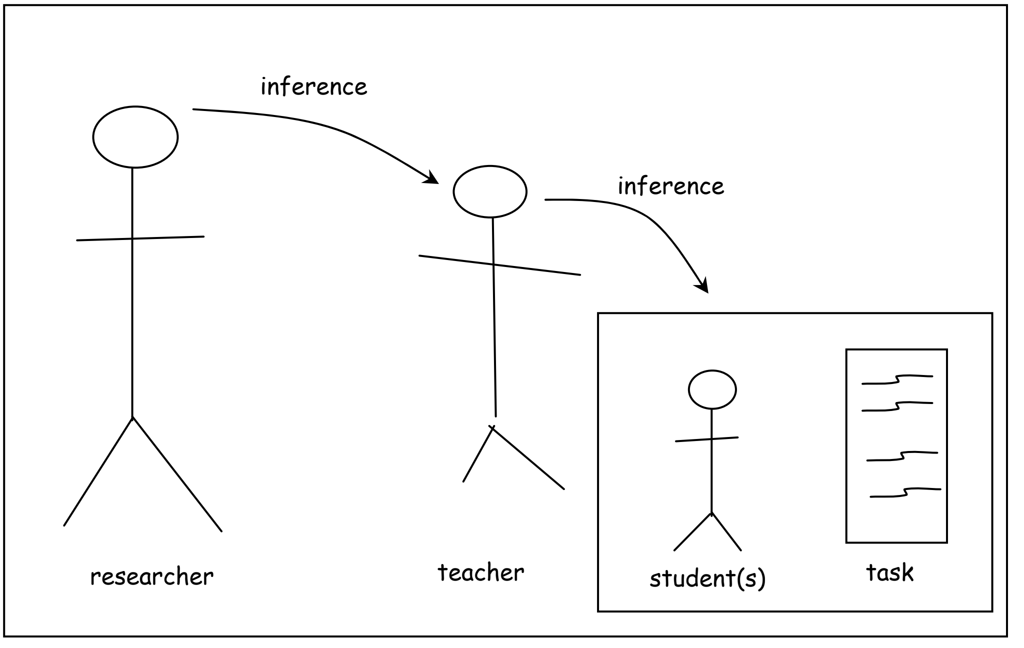
The purpose of the presentation of these three theoretical constructs was not to provide a rigid structure on which to fit all future discussion, but to provide a possible conceptual framework to deepen an inquiry and understanding of teacher practice, which might align with the themes and relationships expressed by the working group members about 'the study of teacher practice'.

PARTICIPANTS' PERCEPTIONS OF CHALLENGES IN THE STUDY OF TEACHING PRACTICE

At the end of the working group, we asked the participants to write down the challenges they believe the study of teacher practice needs to face. In the following paragraphs, we will present some excerpts of the participants' answers.

Several participants mention the complexity of teacher practice as one of the challenges of its study. For one participant, the number of variables involved constitutes a problem: "*The complexity of the factors involved in teaching practice makes the study of teacher practice challenging*". This complexity translates on one hand into the necessity of using a large quantity of data, and on the other hand, the challenge of "*making sense of abundant data*". The concept of practice itself also seems to lack clarity, as one of the participants identified "*defining what is (constitutes) practice*" as one of the challenges.

On a methodological level, one participant points out that the study of teacher practice needs to deal with different levels of inferences. "*For example, the teacher makes inferences on how students are "dealing" with the task and the researcher makes inferences on how the teacher is "dealing" with the students "dealing" with the task.*"



In the same way, the researcher needs to put aside his own point of view in order to get into the teacher's logic. « *L'effort de mettre de côté temporairement la posture du chercheur pour pouvoir rentrer dans la logique de l'acteur.* » Furthermore, teachers' and researchers' theoretical points of view are not necessarily the same, which generates a "*potential [...] of synergies or disconnects between teachers' and researchers' theoretical perspectives*."

Still on a methodological level, the tension between the individual practice of a particular teacher that has been observed and the practice of teachers in general is also seen as a challenge by several participants. On one hand, a practice necessarily has an individual part, which is difficult to generalize: « *Une difficulté est le côté propre individuel d'une pratique et donc, le peu de généralisation possible.* » For another participant, the generalization is problematic, because practice is necessarily related to a particular context: "the context is complex and an essential factor". The same participant also wonders what the researcher is seeing while analyzing a teacher's practice: "*A teacher's practice is organic, making it difficult to be able to capture any more than a snapshot of the practice (or a series of snapshots…?)*".

The tension between what is particular to a situation of a teacher and what can be considered as general appears, for one participant, both in what the researcher sees and what his aims are. Therefore, he needs to decide whether "*we are dealing with the particular or the general"* and whether *"we want to say something particular or general*". Another participant wonders how the presence of the researcher influences the practice of the teacher: "*Are we seeing a "real" picture, or a "special occasion" for the purpose of the research?*" The tension between the particular and the general is also mentioned in relation to the research methodology: if the researcher chooses a fine-grained analysis, the results will be difficult to generalize, and if the analysis is too general, results will be vague.

Finally, one participant sees one of the main challenges of the study of teacher practice in its use during teacher training: « *Un des défis de l'étude des pratiques est de contribuer à l'amélioration en alimentant la formation des enseignants et de la recherche.* » However, this use is also associated with dangers, mentioned by two participants: the "*prescription of* *practice*" and the identification of what constitutes "*good, best or effective teaching*". "*The dangers being that a province/government can take this and then say that research shows this is effective practice and so we will insist that every teacher should do this. This leads to teachers carrying out this surface level behaviour (because they are told they have to) but without a connection with their own set of beliefs, etc. The result of this can be very ineffective teaching or teachers are just replicating surface level behaviour. It also sends a signal where teachers are not being encouraged to develop their own pedagogical beliefs.*"

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Report of Working Group F
Rapport du Groupe de travail F

MATHEMATICS AS SOCIAL (IN)JUSTICE
MATHÉMATIQUES CITOYENNES FACE À L'(IN)JUSTICE SOCIALE

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INTRODUCTION

We set out to "explore the the (ab)use of mathematics in macrosociety, by drawing on artefacts that relate to mathematics at play (or should we say 'at work'?) outside academic settings. We planned to consider published texts from news media and other publicly active groups (including political, justice-oriented and profit-driven groups), and materials and information released by Statistics Canada. These sources would be complemented by the working group participants' personal experiences of mathematics in society".

Key questions for our discussion included the following:

- How is mathematics used in society?
- How is it abused?
- On what bases might we judge the use of mathematics in society?
- How is mathematics reported or mediated in other ways?
- What are possible forms for this kind of mediation?
- How is the form of its mediation influential in public perceptions of mathematics and the agendas it services?
- How is the (ab)use of mathematics resisted or mitigated, and how might it be?
- Who has responsibility for addressing the abuse of mathematics and promoting its proper use?
- Why would these people (or groups) have the authority to address the use of mathematics in society?
- How willing is the public to accept this authority?

BEGINNING THE DISCUSSION

We began our time together with us all sharing our reasons for wanting to discuss social justice and mathematics. These included: hopes to understand the world better, to understand how mathematics can be or is implicated in violence and in social justice work, and to inform our teaching and resource development. These introductions to each other were followed by some discussion about what social justice can mean.

Even though there had been working groups related to social justice in recent CMESG conferences, only two of this year's participants had participated in the group on "Mathematics, Education, Society and Peace" (Dawson & Powell, 2005) and three had been to the group on "Mathematics and Human Alienation" (Henderson, McLoughlin, & Gourdeau, 2008). So we took some time to review some of the main themes of those earlier working groups as a point of departure for extending the conversation to focus more on mathematics in society – outside of classrooms. As part of the introductory discussion of past working groups, we watched David Stocker discuss Math that Matters (see http://www.youtube.com/watch?v=BPGPcsVpjMk), which had been discussed in the 2005 working group.

These starting points (personal hopes and past CMESG work on social justice) prompted some questions. First, we wondered what fields of mathematics have been used most to teach about or study social justice? Has it been primarily statistics and probability? What other parts of the discipline could be brought in? Since our group did not focus on the classroom, the question was left open for discussion elsewhere.

Second, making a distinction between ethical decisions informed by mathematics and mathematized decisions (in which the model decides), someone asked what the difference is in terms of mathematics and morality. When reflecting on mathematics that relates to social justice, one can ask: Are you using the mathematics as a tool to help make a decision? Or, are you distancing yourself from the ethics by positioning yourself as relying on a model? We agreed that, when looking at a mathematical model, attention to the point of view of the person constructing the model helps us not lose sight of the ethics.

Third, and related to the second question, what comes first, the ethics or the mathematics? Do we start from the mathematics, looking at the social implications of mathematical work that we or others do, or ask, "I want an ethical world, what mathematics will help me get there?"

ARTEFACTS

Most of the work that followed this opening discussion centred on artefacts that related to the theme of the working group. Our first set of artefacts comprised a StatCan bulletin on trends in income disparity, and a few newspaper articles that drew from that bulletin[1]. It was most interesting to see the blatancy of the biases in the articles, which seemed to be either sloppy or sinister. The biases seemed predictable based on the reputations of the publications and the authors (especially from Linda McQuaig and Lorne Gunter), which raises questions about

[1] The Daily, May 1st 2008, *2006 Census: Earnings, income and shelter costs*, http://www.statcan.gc.ca/daily-quotidien/080501/dq080501a-eng.htm. and the following articles:
Gunter, Lorne, "StatsCan omissions stir class war", *Calgary Herald*, May 9th 2008, Final edition, page A27.
"Rich wage class war, not StatsCan," *Toronto Star*, Opinion, May 6th 2008, page AA08.
"Taxes have middle class by the throat," *Toronto Sun*, Editorial, May 2nd, Final edition, page 20.

what the intended purposes and unintended consequences of such reporting might be. Another observation was that the French articles that drew on the bulletin appeared less biased than the English articles. This observation prompted some speculation about reasons for this difference. Related to biases, it was noted that there is bias as well in the selection of what StatCan releases in its bulletins, when it releases each set of statistics, and also in the form of the bulletin.

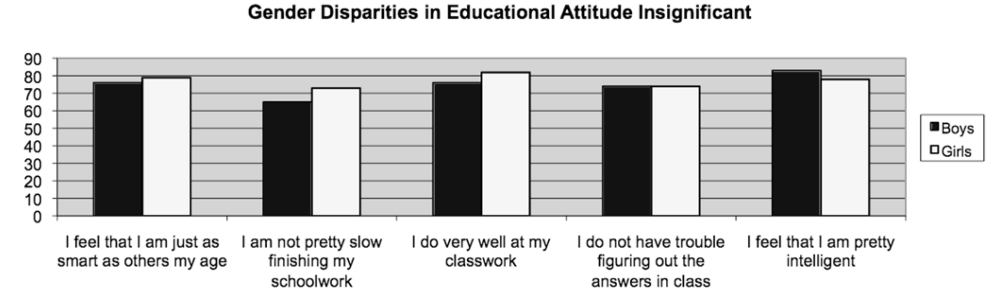
The form of the bulletin, along with the form of the articles drawing on the bulletin, directed our attention to a form of linguistics called *appraisal linguistics*, which notes that certain linguistic resources (grammatical choices, lexical choices, etc.) open or close dialogue (see White, 2003). With that in mind, the linguistic choices made by the bulletin authors obscured the fact that humans were making decisions about what statistics to present together, and what assumptions and parameters to work with in the compilation of these statistics. We wondered if this feature of the bulletin was dangerous as it encourages the view that it is neutral. Though obscuring the human choices seemed to go against the preferences of our group, it was also recognized that StatCan bulletins are valuable and deserve more faith than other statistics because StatCan is a trustworthy institution doing careful statistics with relatively little vested interests. Other aspects of the Statcan bulletin promoted dialogue, especially the regularity and pervasiveness of its release. Other features of the bulletin were less clear in terms of whether they opened or closed dialogue. For example, the release on income had very few graphs to help decipher what was presented. Indeed it had far fewer illustrations and tables than most other Statistics Canada releases. The presence or absence of graphs, we noted, can have various effects on the reader.

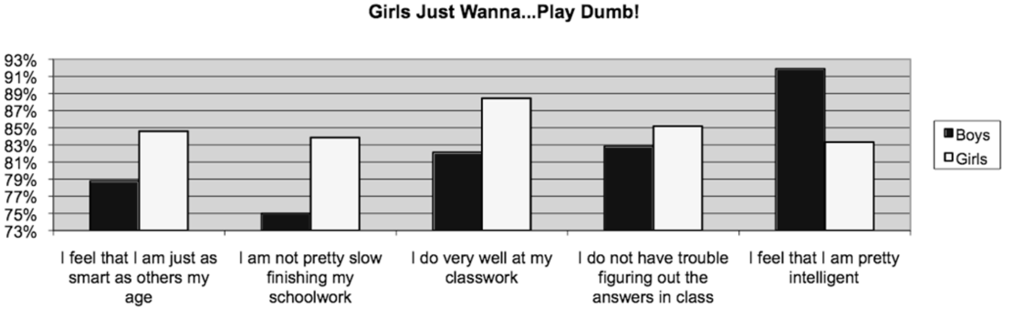
Applying appraisal linguistics to the publications drawing on the bulletin was relatively straightforward because the authors seemed to be unapologetic about representing their views as the only reasonable views. Author choices that supported such closing down of dialogue included showing only one point of view as reasonable, accusing people with other points of view of harbouring sinister motives, representing the StatCan statistics as facts (and positioning the article as a presentation of the facts). However, we noted that such blatant bias may serve to open up dialogue by inciting reaction.

Recognizing that we were looking at one set of examples as representative of extensive publications, we moved to a more general view of mathematics in the public forum. We looked at some trends in the use of mathematics in the press over the past 20 years, and discussed reasons for the changes in its use. Comparing three complete issues of the Globe and Mail from 1988, 1997 and 2005, it was noted that there was a decline in articles that drew significantly on mathematics or statistics though there was a steady prevalence of marginal use. As with the questions raised by the StatCan bulletin and articles, various interpretations were given to account for the trends.

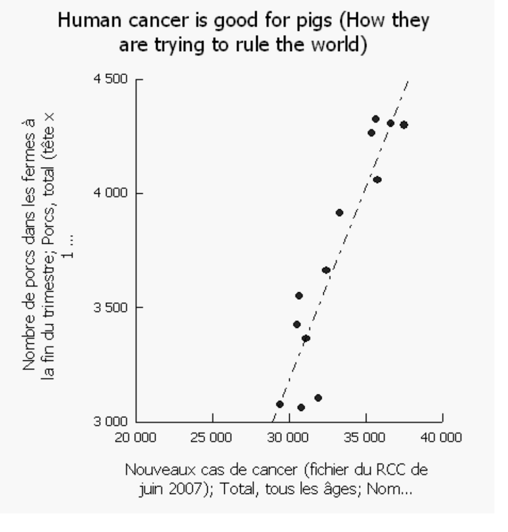
MAKE YOUR OWN ABUSE

We then gathered in the computer lab to use E-Stat learning resources to understand the abuse of mathematics better by making examples of misused and misrepresented statistics. For example, one group showed a fair representation of some statistics about student educational attitudes, and then another representation that skewed the statistics by adjusting the vertical axis of the graph and making a more rhetorical title.

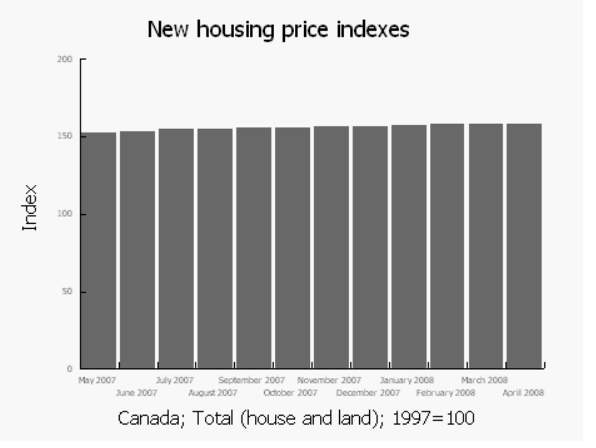




Another group showed how dubious correlations are possible by "proving" a strong correlation between cancer and pig populations. Do pigs thrive on our human suffering?



Another group showed that there was nothing to fear in the housing markets leading into the Fall of 2008. This report of stability is interesting in the context of us knowing in retrospect that a worldwide economic meltdown starting in the Fall of 2008 is widely blamed on housing market issues. Here is the very stable-looking graph the group produced using StatCan data:



MORE ARTEFACTS

On our last day, we looked at some images and videos brought to our attention by group participants. This discussion included some resources that group members were working on, for which they wanted some feedback. We also discussed two videos making their rounds on the internet: *The Story of Stuff* at http://www.storyofstuff.com, and Chris Jordan talking about his photography at www.ted.com/talks/chris_jordan_pictures_some_shocking_stats.html (see his pictures more clearly at http://www.chrisjordan.com). A significant difference between the videos was the extent to which they made claims. Annie Leonard, in *The Story of Stuff* made causal claims, whereas Jordan did not. It is easier to agree or disagree with someone who makes claims, we noticed. With this observation, we realized that we could not predict how people who are less oriented to social justice would respond to the same videos. For example, with Jordan's manipulated photographs that show the enormity of social phenomena (e.g. incarceration rates, breast implants, garbage output), he may move someone to action or may support a conclusion that one's contribution to the large phenomenon would make no difference to the overall result.

MANIFESTOS

Reflecting on the various discussions we had during our working sessions, and other discussions that related to them (e.g. discussion about the first plenary address), we brought closure by inviting the working group members to draft manifestos that would articulate a good vision for mathematics and social justice. These were written in small groups and drafted with limited time, so they are not very refined, but we include them here because they reflect well the nature of our discussions over the course of the three days. Some groups spent more time discussing than writing in the allotted time.

Just before the writing, we showed three short videos with no time allotted for discussion, to remind us all of claims about the good things mathematics contributes to society (using the following IBM advertisement:

http://www.youtube.com/watch?v=-udGE8POcZk), and about some of the human tragedy that has been aided by mathematics (using two videos of the USA dropping nuclear bombs on Japanese cities: http://www.youtube.com/watch?v=Xs3JE4WRL-8&feature=related and http://www.youtube.com/watch?v=JOyIcFrSQz4&feature=related). These were painful to show and watch.

Here are the draft manifestos:

We propose that the Canadian Mathematics Society create two bodies to conduct the following tasks:

Carrots:

1. Awards/recognition for media outlets'/reporters' specific stories that use mathematics and statistics in a responsible, accurate, and appropriate manner;
2. Awards/recognition for school boards/schools/teachers that are innovative in promoting responsible programs that encourage the teaching and learning of mathematical and statistical literacy (i.e. numeracy).

Sticks:

1. Provide 'watchdogs' who will be responsible for monitoring and publicizing stories that contain irresponsible, inaccurate and inappropriate uses of mathematics and statistics:
    a. If there is an inaccuracy, the CMS will recommend the story be withdrawn.
    b. If there is an inappropriate usage, the CMS will recommend that a 'correction' be published.

Journalists, Mathematicians, Math Educators, Engineers, Health Workers, Natural Scientists, Social Scientists …

1. We already use mathematics to change the world. Many of our technologies depend on mathematics. We should use mathematics in thoughtful ways to make the world better, in terms of ecology, social justice, economics, environmental sustainability, political justice. We will never have complete consensus on what "a better world" means, but certain injustices can be agreed upon: the degradation of the planet, extremes of poverty…
2. We need to work on both localized and globalized levels.
3. We see examples of communities that *do* work to make sure that no one starves – for example, First Nations communities in Canada and elsewhere, poor people in Brazil, etc.
4. Math can help us see things that have become invisible, things that are implicit.
5. Math needs to be contextualized and critiqued in the holistic context of the world and people. Mathematics itself should not be beyond question or beyond critique. Optimization methods, Cartesian grids and other mathematical methodologies can be the cause of suffering; it may be necessary to develop other mathematical tools…
6. Math is invisible in itself many times in formatting the world. We need to be aware of an ideology of certainty that is around mathematics – that "mathematics is the best".
7. We merge environmental issues with social issues – environmental and social justice, the "Red & Green Alliance".
8. Mathematics shouldn't be used to intimidate people. It's antidemocratic to leave mathematics to the few 'elite mathematicians'. We all have a responsibility to understand mathematics ourselves and to help other people understand it. (Parallel to Star Trek geeks who speak the Klingon language and resented subtitles in the new Star Trek movie – math geeks may resent others understanding their secret language…) Graduate maths is treated as "the real thing" – those below that level are not let in to the "priesthood" or inner circle. We should aim to open the fences that keep people

from understanding mathematics like calculus, etc. 9. Is the math we have enough to help us change the world? Maybe the content doesn't really matter, but rather thinking critically and creatively – having confidence in posing, considering and solving problems. Mathematics can give interesting, useful tools to think with and mind-expanding ways of conceiving of the world. Can mathematics be used to help people open (rather than close) their minds? 10. Real world problems: can they be used in the teaching of mathematics? Can mathematics be used in solving real world problems that students/people are concerned about? Can math be integrated with other curricular areas? 11. Mathematics and science have become excessively specialized. Doctors only know the cardiac system, the endocrine system, etc.; engineers work on optimizing traffic flow at the expense of air quality, quality of life in the city, etc. The education of experts and their social status is a big part of the problem. Consultation among experts may be a first step, but also knowledge of context in a holistic way – interrelationships, appreciation of other areas – for example, math appreciation courses, conceptual-level courses about mathematics that might not involve calculations, etc.
Any form of good practices for mathematics and social justice needs to include an emotional engagement. The emotional engagement must be positive and should allow critical thinking and one's own judgment.
Math educators have the responsibility to: • teach math so that students understand the world around them, and have the critical tools to question others' mathematical representations of the world. Journalists (and anyone reporting to the public) have the responsibility to: • understand the mathematics that others use to describe and understand the world; • point their readers to the statistics/data they use in their reporting; • reveal the models, assumptions and human agents in their mathematical and representational decisions. Mathematicians have the responsibility to: • reveal their processes when they report to others; • recognize the simplification that is necessarily a part of modeling; • not put mathematics in the service of oppression but rather to serve truth; • make models that have assumptions that fit what we see in the world, not how we think the world should be (e.g. "perfect competition"). Rejected manifesto item: • others shouldn't manipulate statistics to hide and foreground particular aspects of the data (but it's okay for us to do this, because we are sensitive/just).

All people should consider ethics as equally important as mathematical soundness at all stages of the mathematical modeling process (assumptions of abstraction/ simplification; development of the model; and use of the model-making predictions/making decisions/dissemination of results).

All people should develop a sufficient understanding of mathematics in order to:

- understand how mathematics can help explain world problems and
- understand how a critical viewpoint can enable one to develop a healthy doubt in mathematics as it is presented.

Mathematics educators have a special responsibility for ensuring members of society develop this sufficient understanding of mathematics.

INTRODUCTION

Nous avons entrepris d'explorer l'usage et l'abus des mathématiques en société et d'étudier des extraits publiés par les médias et autres groupes actifs publiquement (incluant des groupes politiques et publics, les groupes orientés vers la justice et ceux menés par le profit), en complémentant ces sources par l'expérience personnelle des participants du groupe de travail en ce qui a trait à l'usage des mathématiques en société.

Voici quelques-unes des questions qui ont alimenté notre discussion:

- Comment utilise t-on les mathématiques en société?
- Comment en abuse-t-on?
- Sur quelles bases pourrait-on juger de l'utilisation des mathématiques en société? Comment rapporte-t-on les mathématiques?
- Quels genres ou quelles formes de médiations des mathématiques trouve-t-on?
- Comment la forme de la médiation influence-t-elle la perception du public des mathématiques et comment influence-t-elle le but qu'elle sert?
- Comment résiste-t-on à l'usage abusif des mathématiques et quelles avenues s'offrent à nous?
- Qui a la responsabilité de dénoncer les abus des mathématiques et de promouvoir leur bon usage?
- Pourquoi ces gens (ou groupes) auraient-elles l'autorité de se prononcer sur l'usage des mathématiques en société?
- Jusqu'à quel point le public accepte-t-il cette autorité?

AU COMMENCEMENT

Nous avons commencé en partageant pourquoi ça nous attirait de discuter de justice sociale et de mathématiques. Certains voulaient mieux comprendre le monde, mieux comprendre le rôle actuel ou potentiel des mathématiques dans la violence ou dans le travail pour la justice sociale, ou encore avoir des outils pour développer leur enseignement ou des ressources. On a aussi discuté du sens de la justice sociale pour chacun.

Il y avait déjà eu récemment des groupes de travail sur la justice sociale au GCEDM, parmi les gens de notre groupe, seulement deux avaient pris part au groupe « Mathématiques, éducation, société et paix » (Dawson & Powell, 2005) et seulement trois avaient participé au groupe intitulé « Mathématiques et aliénation humaine » (Henderson, McLoughlin, & Gourdeau, 2008). Nous avons donc pris le temps de revoir les thèmes principaux de ces groupes comme point de départ avant de poursuivre une discussion portant plutôt sur les mathématiques en société – à l'extérieur de la salle de classe. Nous avons entre autres visionnés ensemble un vidéo où David Stocker, présentait son manuel scolaire intitulé Math that Matters (http://www.youtube.com/watch?v=BPGPcsVpjMk): un des textes dont on avait discuté en 2005.

Ces points de départ ont suscités des questions. D'abord, quels champs des mathématiques ont le plus été utilisées pour enseigner ou pour étudier la justice sociale? Étaient-ce principalement les statistiques et les probabilités? Quelles autres parties de notre discipline pouvions-nous intégrer à cette étude? Puisque nous ne portions pas notre discussion sur la salle de classe, ces questions sont restées ouvertes, à être discutées ailleurs.

Deuxièmement, en distinguant les décisions éthiques influencées par les mathématiques des décisions mathématisées (dans lesquelles le modèle fait la décision), quelqu'un a demandé quel était la différence en termes de mathématiques et de moralité. En réfléchissant sur les mathématiques en relation avec la justice sociale, on peut se demander si nous utilisons les mathématiques comme outil pour nous aider à prendre une décision? Ou est-ce que nous prenons des distances par rapport aux dimensions éthiques en adoptant la position de se fier sur le modèle? Un consensus est apparu: lorsqu'on porte attention au point de vue du créateur d'un modèle mathématique, ça nous aide à ne pas perdre de vue la dimension éthique.

Enfin et directement relié à la troisième question, qu'est-ce qui vient en premier: l'éthique ou les mathématiques? Est-ce qu'on part des mathématiques, en examinant les implications sociales de notre travail mathématique ou est-ce que nous nous disons: "Je veux un monde éthique... quels seront les outils mathématiques qui m'aideront à y parvenir?"

ARTEFACTS

Suite à cette discussion d'ouverture, la plupart de notre travail était centré autour d'artefacts reliés au thème du groupe de travail. Nous avons commencé en analysant un bulletin de Statistique Canada au sujet de l'inégalité du revenu, et quelques articles de journal qui commentaient ce bulletin[2]. C'était très intéressant de voir le biais des articles. En fait il était assez facile à prévoir selon la réputation des journaux et des auteurs (en particulier ceux de Linda McQuaig et de Lorne Gunter). Quels étaient le but de ces articles? Ont-ils eu des conséquences imprévues? Nous avons observé aussi que les articles en français étaient moins biaisés que les articles en anglais (en fait Yves avait cherché à trouver les articles les plus biaisés mais n'en avait pas trouvé en français). Nous nous sommes demandé quelle était la raison pour cette différence. Parlant de biais, quelqu'un a mentionné qu'il y avait forcément biais dans le choix des statistiques que StatCan choisissait de publier, quand elle les publie et aussi dans la forme du bulletin quotidien.

[2] Le Quotidien, 1[er] mai 2008, *Recensement de 2006: gains revenus et coûts d'habitation*, http://www.statcan.gc.ca/daily-quotidien/080501/dq080501a-fra.htm et les articles suivants:
Gunter, Lorne, "StatsCan omissions stir class war", *Calgary Herald*, May 9th 2008, Final edition, page A27.
"Rich wage class war, not StatsCan", *Toronto Star*, Opinion, May 6[th] 2008, page AA08.
"Taxes have middle class by the throat", *Toronto Sun*, Editorial, May 2[nd], Final edition, page 20.

La forme du bulletin ainsi que celle des articles qui le citaient a porté notre attention sur une forme de linguistique appelée en anglais “appraisal linguistics” ou on regarde les choix linguistiques (grammaire, lexique) en fonction de ce qu’ils ouvrent ou ferment le dialogue (voir White, 2003). Selon cette analyse, les choix linguistiques des auteurs du bulletin occultaient le fait que des êtres humains décidaient des statistiques à présenter et des hypothèses et paramètres guidant la compilation de ces statistiques. Nous nous sommes demandé si cet aspect du bulletin était dangereux puisqu’il donnait l’impression que celui-ci était neutre. Quoique le groupe s’opposait à une pratique qui occultait les choix humains, un membre a souligné que les bulletins ont une grande valeur et méritent plus de confiance que d’autres statistiques puisque StatCan est une institution de confiance, qui produit des statistiques avec relativement peu d’intérêts cachés. D’autres aspects du bulletin de Statcan encourageaient le dialogue, particulièrement ses dates de parutions prévisibles et sa couverture d’un grand nombre de faits socio-économiques. Quant à d’autres aspects du bulletin, c’était moins clair s’ils ouvraient ou fermaient le dialogue. Par exemple, le bulletin sur le revenu avait très peu de diagrammes pour aider à déchiffrer ce qu’on présentait. En fait, ce bulletin contenait beaucoup moins de diagrammes et de tableaux que la plupart des bulletins de Statistique Canada. La présence ou l’absence de diagrammes peut avoir divers effets sur le lecteur.

Ce même outil de la linguistique d’évaluation était plus facile à appliquer aux articles qui citaient StatCan. Leurs auteurs représentaient sans gêne leur point de vue comme le seul qui soit raisonnable. Parmi les autres choix qui fermaient le dialogue: accuser les gens qui ont d’autres points de vue d’avoir de mauvaises intentions, représenter les statistiques comme des faits (et l’article qu’ils écrivaient comme présentation des faits). Par contre, nous avons noté qu’un biais aussi évident pouvait servir à ouvrir le dialogue en incitant une réaction.

Après avoir regardé quelques exemples de grands journaux, nous sommes passés à une vue plus générale des mathématiques dans l’arène publique. Nous avons regardé la tendance de l’utilisation des mathématiques dans la presse écrite au cours des 20 dernières années et avons discuté des raisons qui motivaient le changement noté dans cette utilisation. En comparant trois numéros complets du Globe and Mail de 1988, 1997 et 2005, nous avons vu la diminution du nombre d’articles qui utilisait grandement des mathématiques ou des statistiques, alors que l’utilisation marginale de mathématiques était constante. Comme lorsque nous discutions des questions suscitées par le bulletin de StatCan et les articles, diverses interprétations furent données pour expliquer ces tendances.

CRÉEZ VOS PROPRES REPRÉSENTATIONS ABUSIVES

Nous sommes ensuite allés dans un laboratoire informatique pour apprendre à utiliser E-STAT et pour mieux comprendre l’abus dans les représentations mathématiques en créant des exemples de statistiques mal citées et mal représentées. Un groupe a contrasté un diagramme qui représentait les attitudes des élèves face à l’éducation à un autre diagramme qui déformait les statistiques en ajustant l’axe vertical et en ajoutant un titre rhétorique.

Un autre groupe a montré qu’on pouvait “prouver” une causalité douteuse en montrant une corrélation forte entre le cancer et la population de porcs. Est-ce que les porcs jouissent de la souffrance humaine?

Un autre groupe a montré qu’il n’y avait rien à craindre dans le marché du logement avant les événements de l’automne 2008. Ce rapport de stabilité est d’autant plus intéressant que nous savons maintenant que plusieurs blâment l’effondrement de l’économie mondiale sur des problèmes du marché du logement.

D'AUTRES ARTEFACTS

Lors de notre dernier jour de rencontre, nous avons regardé des images et des vidéos que des participants du groupe avaient apportés. Dans cette discussion des ressources auxquelles des participants travaillaient et sur lesquels ils voulaient des commentaires, nous avons aussi discuté de deux vidéos circulant sur l'Internet: *The Story of Stuff* au site http://www.storyofstuff.com, et une vidéo de Chris Jordan présentant ces photographies au site http://www.ted.com/talks/chris_jordan_pictures_some_shocking_stats.html (on peut voir plus clairement ses photographies au site http://www.chrisjordan.com). Une des différences marquantes entre les vidéos étaient la présence ou l'absence de conclusions. Dans *The Story of Stuff*, Annie Leonard faisait plusieurs relations causales, tandis que Chris Jordan n'en faisaient pas. Nous avons noté que c'est plus facile d'être d'accord ou en désaccord avec quelqu'un qui émets une opinion. Il était difficile de prédire la réaction à ces vidéos de gens qui sont moins orienté vers la justice sociale. Par exemple, les photographies manipulées de Jordan montraient l'énormité de phénomènes sociaux (e.g. taux d'incarcération, implants mammaires, ampleur des déchets). Elles pourraient inciter quelqu'un à l'action, ou encore soutenir la conclusion que la contribution d'un individu à de phénomènes si amples ne ferait pas de différence au résultat de l'ensemble de la société.

MANIFESTES

Avec en tête les discussions que nous avions eu lors de nos sessions de travail et d'autres discussions qui étaient reliées à notre thème (e.g. lors de la première plénière), nous avons tenté de clôturer en invitant les membres du groupe de travail à écrire des manifestes qui énoncerait clairement une bonne vision du rapport entre mathématiques et justice sociale. De petits groupes ont écrit ces manifestes en peu de temps. Ils ne sont donc pas très raffinés, mais nous les incluons car ils reflètent le contenu de nos discussions au cours des trois jours. Certains groupes ont passé plus de temps à la discussion qu'à la rédaction dans le temps alloué à l'exercice.

Juste avant d'écrire, nous avons visionné trois courts vidéos sans allouer de temps pour la discussion, comme rappel de ce qu'on prétends que les mathématiques contribuent à la société (voir l'annonce d'IBM au lien suivant: http://www.youtube.com/watch?v=-udGE8POcZk) et pour se souvenir aussi de tragédies humaines aidées par les mathématiques (par ces deux vidéos concernant l'usage de la bombe atomique par les États-Unis sur des villes japonaises: http://www.youtube.com/watch?v=Xs3JE4WRL-8&feature=related et http://www.youtube.com/watch?v=JOyIcFrSQz4&feature=related). C'était difficile de montrer et de regarder ces films.

Voici ces premiers jets de manifestes:

Nous proposons que la société canadienne de mathématiques crée deux organismes avec les mandats suivants:

Carottes:

1. Prix/reconnaissance pour les médias/reporters pour des histoires particulières qui utilisent les mathématiques d'une façon responsable, précise et appropriée;
2. Prix/reconnaissance pour des conseils scolaires/écoles/enseignant-es qui innovent en promouvant des programmes responsables qui encouragent l'enseignement et l'apprentissage de littératie mathématique et statistique (i.e. numéracie).

Bâtons:
 	
1. Qu'il y ait des « chiens de garde » responsables de veiller et de publiciser les articles qui contiennent des représentations irresponsables, non précises et non appropriées de mathématiques ou de statistiques:
    a. S'il y a erreur, la SCM recommandera que l'article soit retiré.
    b. S'il y a usage non approprié, la SCM recommandera qu'on publie une correction.

Journalistes, mathématiciens, enseignants de math, ingénieurs, travailleurs de la santé, scientifiques (sciences naturelles et sociales) 1. Nous utilisons déjà les mathématiques pour changer le monde. Plusieurs technologies dépendent de mathématiques. Nous devrions utiliser les mathématiques de façon réfléchie pour rendre le monde meilleur en termes d'écologie, de justice sociale, d'économie, de développement durable, de justice politique. Il n'y aura jamais de consensus sur cette définition « d'un monde meilleur », mais peut-être sur certaines injustices: la dégradation de la planète, les extrêmes de pauvreté … 2. Nous devons travailler à la fois au niveau local et global. 3. Nous avons des exemples de communautés qui fonctionnent afin de s'assurer que personne ne meure de faim – premières nations au Canada et ailleurs, pauvres au Brésil, etc. 4. Les mathématiques peuvent aider à rendre visible ce qui était invisible, implicite. 5. On doit critiquer la mathématique dans le contexte holistique du monde et des gens. Les mathématiques ne devraient pas être à l'abri des questions ou des critiques. Méthodes d'optimisation, grilles cartésiennes, et autres méthodes mathématiques peuvent causer de la souffrance; il y a peut-être besoin de développer d'autres outils mathématiques … 6. Le rôle des math en formatant le monde est souvent invisible. Nous devons garder à la pensée l'idéologie de certitude qui entoure les mathématiques – que "si c'est mathématique, c'est bon ". 7. Nous réunissons les enjeux environnementaux et sociaux – « l'alliance rouge et verte ». 8. Ne pas utiliser les mathématiques pour intimider les gens. C'est antidémocratique de laisser les math aux mathématiciens de l'élite. Nous avons tous la responsabilité de comprendre les mathématiques et d'aider les autres à les comprendre. (Parallèlement aux initiés de Star Trek geeks qui parlent la langue Klingon et qui se fâchaient de voir des sous-titre dans le nouveau film de Star Trek – les initiés des maths peuvent ne pas vouloir qu'on connaisse les secrets de leur langue…) Les mathématiques du 2[e] et 3[e] cycle universitaire sont considérées « la référence » – ceux qui n'ont pas atteint ce niveau ne sont pas admis dans la « prêtrise ». On devrait viser à ouvrir les barrières empêchant les gens de comprendre les mathématiques avancées (calcul etc.). 9. A t-on assez de math pour changer le monde? Peut-être que le contenu importe moins que de penser de façon critique et créatrice – en posant, considérant et résolvant des problèmes avec confiance. Les mathématiques peuvent fournir des outils intéressants et utiles pour la pensée et pour concevoir le monde d'une façon qui étend l'esprit. Les mathématiques peuvent-elles être utilisées pour ouvrir (plutôt que de fermer) les esprits? 10. Peut-on utiliser des problèmes du monde réel pour enseigner les mathématiques? des problèmes auxquels les élèves et les gens s'intéressent? Peut-on intégrer les mathématiques à d'autres parties du curriculum? 11. Les mathématiques et les sciences sont devenues excessivement spécialisées. Des

médecins ne connaissent que le système cardiaque ou endocrinien, etc; les ingénieurs travaillent à optimiser l'affluence au détriment de la qualité de l'air et de la vie en ville, etc. L'éducation d'experts et leur statut social constitue une bonne partie du problème. La consultation entre experts peut être la première étape, mais aussi la connaissance du contexte de façon holistique – interrelations, connaissance d'autres champs – par exemple cours d'appréciation des maths, des cours conceptuels qui parlent des mathématiques sans demander de calculs, etc.
Toute forme de bonnes pratiques pour les mathématiques et la justice sociale doit engager les émotions Cet engagement doit être positif et doit permettre la pensée critique et son propre jugement.
Les enseignants de mathématiques ont la responsabilité : • d'enseigner les math de façon à ce que les élèves comprennent le monde qui les entoure, et à ce qu'ils aient les outils critiques pour questionner les représentations mathématiques du monde émises par d'autres. Les journalistes (et tous ceux qui écrivent/diffusent pour le public) ont la responsabilité : • de comprendre les mathématiques que d'autres utilisent pour décrire et comprendre le monde • d'indiquer à leurs lecteurs les statistiques et les données qu'ils utilisent • de révéler les modèles, les hypothèses et les agents humains présents dans les décisions mathématiques et les décisions de représentation. Les mathématiciens ont la responsabilité : • de révéler leurs procédés lorsqu'ils diffusent aux autres • de reconnaître la simplification inhérente à la modélisation • de ne pas mettre les mathématiques au service de l'oppression, mais plutôt au service de la vérité • de créer des modèles avec des hypothèses en lien avec ce qui se passe dans la réalité, et non avec le monde qu'on aimerait voir (e.g. “concurrence parfaite”). Nous avons rejeté l'item suivant : • Les autres ne devraient pas manipuler les statistiques pour montrer ou cacher certains aspects des données (mais cette pratique va pour nous puisque nous sommes sensibles/justes).

Le poids de l'éthique devrait être aussi important que celui de la rigueur mathématique, à tous les stages du processus de modélisation mathématique (hypothèses d'abstraction ou de simplification; développement du modèle; et utilisation des prédictions du modèle pour faire des décisions ainsi que la diffusion des résultats).

Chacun devrait développer une compréhension suffisante des mathématiques afin de :

- comprendre comment le rôle des mathématiques dans l'explication de certains enjeux mondiaux
- comprendre comment un point de vue critique peut lui permettre de développer un doute sage par rapport aux représentations mathématiques.

Les enseignants de mathématiques ont la responsabilité spéciale d'assurer que les membres de la société développent cette compréhension suffisante des mathématiques.

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Topic Sessions

Séances thématiques

ÉTUDE DES SENS ACCORDÉS À LA RELATION D'ÉGALITÉ ET AU SIGNE D'ÉGALITÉ DANS LA RÉALISATION D'ACTIVITÉS PORTANT SUR LE CONCEPT D'ÉGALITÉ

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RÉSUMÉ

Cette présentation fait partie d'une recherche qui étudie le sens accordé au signe d'égalité et à la relation d'égalité par les élèves du premier cycle du secondaire.

INTRODUCTION

Notre intérêt porte, d'une part, sur la compréhension du concept d'égalité dans des activités numériques, algébriques et de résolution de problèmes réalisées par les élèves du premier cycle du secondaire et, d'autre part, sur les difficultés (symbolisation, représentation, compréhension, etc.) qui y sont liées lors de la réalisation d'activités numériques, arithmétiques et algébriques. L'ingénierie didactique a permis l'élaboration de situations sur la base d'analyses préalables. Elles ont été ensuite expérimentées en classe. L'ingénierie didactique sert à la fois d'outil d'enseignement et de recherche.

ÉLÉMENTS DE LA PROBLÉMATIQUE : QUELQUES CONSIDÉRATIONS

ÉGALITÉ ET RELATION D'ÉQUIVALENCE

L'analyse de certaines activités au primaire et au secondaire, qui mettent en évidence l'égalité, fait ressortir les propriétés de la relation d'équivalence : réflexivité, symétrie et transitivité, ainsi que régularité des opérations. Ces propriétés sont souvent implicites. Exemple : Pour résoudre par la méthode de comparaison le système d'équations formé par $3x - y = -1$ et $2x - y = -5$, on écrit : $y = 3x + 1$ et $y = 2x + 5$, et la transitivité permet de poser l'équation à une inconnue : $3x + 1 = 2x + 5$. La régularité des opérations permet de partir de $3x + 1 = 2x + 5$ d'obtenir $3x + 1 - 1 = 2x + 5 - 1$ et ensuite $3x - 2x = 2x - 2x + 4$. On déduit que $x = 4$, ensuite, on calcule en substituant 4 dans une des équations pour trouver $y = 13$.

RELATION D'ÉGALITÉ ET LES NOMBRES

Selon Conne (1989) dans certaines activités d'un volet différent, on utilise : le « égal » qui signifie « ça fait », puis à partir de « ça fait le égal » il y a « c'est fait par », et ensuite il y a le « ça fait la même chose que ». Aucun de ces « égal » n'est réflexif, ni symétrique, ni transitif. Dans l'égalité $3 \times 10 + 5 = 12 + 16 + 7$ ou $3 \times 10 + 5 = ½ \times (7 \times 10)$ le signe égal devient l'équivalence de procédures parce que les deux donnent le même résultat. En effet, si nous prenons 35 comme le résumé d'une procédure pour obtenir un résultat, alors « = » sera le signe d'équivalence entre les procédures. Mais alors quel sera le résultat de 35? Par exemple, « 35 » peut être considéré comme une procédure pour obtenir un nombre : $35 = 3 \times 10 + 5$; $35 = 12 + 16 + 7$ et $35 = ½ \times (7 \times 10)$. L'intérêt est de dire que le nombre que l'on a obtenu par telle procédure (calcul, mesure, etc.) est égal à celui qu'on aurait obtenu par telle autre procédure. Ceci met en évidence des relations entre procédures, et en particulier des relations d'équivalence de procédures. Conne (1989) défend l'idée selon laquelle, les notations symboliques jouent un rôle déterminant dans ces relations entre procédures. Ses recherches dans des classes de première année en Suisse romande sur les égalités lacunaires illustrent très bien cette problématique. Pour Conne, ce qui traduit «l'équivalence» est «être la même chose que», mais il faut alors la différencier de «ça fait». Ce qui traduit la pluralité des significations que l'on peut donner à «égal».

ÉGALITÉ AU SECONDAIRE 1 ET AU SECONDAIRE 2

Au premier cycle du secondaire des activités sur la simplification d'expressions sont souvent proposées aux élèves.

Exemple : Montrer que les expressions suivantes sont égales ou équivalentes :
$14xy - 3x + 2y - 7xy + 5x - xy + 9y$ et $16x - y + 2xy + 12y + 4xy - 14x + 12y$.

<table>
<tr><td colspan="2">Première façon de faire observée chez les élèves</td></tr>
<tr><td>$14xy - 3x + 2y - 7xy + 5x - xy + 9y$
$= (14xy - 7xy - xy) + (-3x + 5x) + (2y - 9y)$
$= 6xy + 2x + 11y$</td><td>$16x - y + 2xy + 12y + 4xy - 14x + 12y$
$= (16x - 14x) + (-y + 12y) + (2xy + 4xy)$
$= 2x + 11y + 6xy$</td></tr>
<tr><td colspan="2">Deuxième façon de faire observée chez les élèves
$14xy - 3x + 2y - 7xy + 5x - xy + 9y = 16x - y + 2xy + 12y + 4xy - 14x + 12y$
$14xy - 3x + 2y - 7xy + 5x - xy + 9y - 16x + y - 2xy - 12y - 4xy + 14x - 12y = 0$
$(14xy - 7xy - xy - 2xy - 4xy) + (-12y + y + 2y + 9y) + (-16x + 14x - 3x + 5x) = 0$ on a $0 = 0$</td></tr>
</table>

D'un autre point de vue, lorsque le contexte d'égalité renvoie ou demande la détermination d'une inconnue, on a deux types de conception du signe d'égalité, celle arithmétique : $x + a = b$, $ax = b$, $ax + b = c$, et celle algébrique : $ax + b = cx + d$. Il faut noter ici qu'un traitement d'équations n'est pas une simplification d'expressions. Le calcul littéral et le traitement des équations sont deux activités différentes. Dans une autre perspective, lorsque l'équation est inscrite sous une forme différente de celle à laquelle ils sont habitués (exemple : 7 = 3 + 4), bien des élèves ont de la difficulté à l'accepter et ils déclarent qu'elle est inscrite à l'envers. Cette forme peut même porter des élèves à lire l'équation de droite à gauche. Lorsque ce signe est utilisé pour représenter une équivalence arithmétique (exemple : 4 + 5 = 2 + 7), certains élèves éprouvent aussi des difficultés, proposant deux nouvelles équations plus simples à comprendre : 4 + 5 = 9 et 2 + 7 = 9. Dans ce cas, ils considèrent le signe d'égalité comme étant un séparateur au lieu d'un symbole de relation symétrique et transitive. Cette compréhension limitée persiste chez certains élèves du secondaire et affecte leur rapport à l'algèbre. Donner un sens à une équation du type $5x - 8 = 2 + 3x$ est alors difficile. Theis (2005), dans ses recherches, a montré que les élèves considèrent le signe d'égalité, au début du primaire, comme un symbole qui *« sert à séparer les chiffres et à écrire la réponse»*. Il conclue que le signe d'égalité est un obstacle à la compréhension de la relation d'équivalence. Conne (1989) quant à lui déduit que les élèves ont de la difficulté à comprendre le symbolisme. Qu'est-ce qu'une égalité ? Une équation ?

OBJECTIF DE LA RECHERCHE

L'objectif de la recherche est d'étudier, d'une part, les sens que les élèves attribuent au concept d'égalité et comment ils l'utilisent dans les activités numériques, algébriques et de résolution de problèmes au premier cycle du secondaire et, d'autre part, d'étudier les difficultés qu'ont les élèves avec le concept d'égalité et le signe d'égalité.

ÉLÉMENTS DE CADRE THÉORIQUE

Le signe égal « = » indique, en mathématiques, l'identité des expressions qu'il sépare. Par ailleurs, en faisant référence aux travaux de Kieran (1992, 1996), Van Ameron (2003) précise deux conceptions des expressions mathématique : celle procédurale et celle structurale. Le signe égal peut donc exprimer une idée de comparaison entre les expressions (vérification de leur égalité), d'affectation de la valeur de l'une à l'autre (afin qu'elles deviennent identiques), ou il peut séparer les étapes d'un raisonnement ou d'un calcul, lors desquelles on transforme progressivement, afin de démontrer leur égalité, une expression en une autre. L'égalité n'est pas un savoir spécifique qui est enseigné en 1^{e} et 2^{e} secondaire; par contre, elle intervient dans des contenus et savoirs mathématiques. En ce sens, l'égalité fait partie intégrante du milieu de l'élève en 1^{e} et 2^{e} secondaire. Le « milieu » est tout ce qui agit sur l'élève et ce sur quoi il agit et tout ce qui lui assure une rétroaction des actions qu'il produit. L'élève agit donc sur le milieu à l'aide de ses connaissances et dans le cadre des règles qui régissent une situation didactique. Le milieu doit être spécifique d'un savoir à enseigner de manière que les stratégies mises en œuvre par l'élève pour contrôler le milieu engagent et fassent appel aux connaissances visées. Par exemple, l'égalité fait partie du milieu lorsqu'il s'agit d'apprendre à résoudre une équation de la forme : $ax + b = cx + d$. Les milieux construits pour nos activités permettent des contradictions, des difficultés et des déséquilibres entraînant, par la suite, des adaptations. Le milieu est autonome afin que la connaissance y devienne une production et que l'élève puisse expérimenter le savoir acquis (Gonzalez-Martin, 2008).

ÉGALITÉ

Selon Bouvier et al. (1992),

> *Une égalité est une relation notée "=" (on lit "égal"). On écrit a = b lorsque les symboles a et b représentent le même objet mathématique. En fait, l'égalité est une notion primitive des mathématiques (comme l'appartenance); elle ne se définit que par ses règles d'emploi qui imposent que deux objets mathématiques égaux ont les mêmes propriétés. L'égalité est réflexive, symétrique et transitive; c'est donc une équivalence. Elle est aussi antisymétrique (p. 285).*

Pour Baruk (1993),

> *Une égalité est une affirmation qui utilise le signe "=" et qui ne peut être vraie ou fausse. Les termes écrits de part et d'autres du signe "=" sont les membres de l'égalité (p. 396).*
>
> *Une égalité est l'affirmation que les deux membres sont des expressions d'un même objet, nombre, vecteur, figure, etc. Si on peut, en remplaçant ces expressions par des expressions équivalentes, transformer l'égalité en identité, c'est qu'elle est vraie, sinon elle est fausse (p. 398).*

Quant à Champlain et al. (1990), « *Une égalité est une relation entre deux représentations d'un même objet mathématique. Une égalité est une proposition vraie* » (p. 97).

ÉQUATION

Selon Bouvier et al. (1992) et Reinhardt et al. (1997), le terme « équation » est mathématiquement indéfinissable.

> *Son sens dépend essentiellement du contexte ou du qualificatif qui l'accompagne (équation différentielle, équation d'une conique, etc.) Lorsque l'équation est une relation de la forme $f(x) = g(x)$ où f et g sont des applications d'un ensemble E dans un ensemble F, la résoudre c'est déterminer l'ensemble des éléments de E (appelés racines ou solutions de l'équation) pour lesquels l'égalité $f(a) = g(a)$ est vérifiée.*

f(x) et g(x) sont les deux membres de l'équation et x l'inconnue (Bouvier et al., 1992, p. 299; Reinhardt et al., 1997, p. 93).

Pour Champlain et al. (1990), une équation est « un énoncé mathématique qui comporte une ou des variables et la relation d'égalité. C'est une forme propositionnelle qui contient le symbole d'égalité» (p. 102).

ÉGALITÉ ET ÉQUATION EN 1RE ET 2E SECONDAIRE

Une équation n'est pas une égalité mais une fonction propositionnelle. Lorsqu'on remplace dans une équation *x* par une valeur donnée, on obtient une égalité, qui est une proposition. Cette proposition peut être vraie ou fausse. Quand on résout une équation, on cherche l'ensemble des valeurs de *x* qui rendent ces équations comme des égalités vraies. Égalité, équations et calcul littéral interviennent dans plusieurs activités et sont sources de difficultés pour les élèves. L'égalité qui est un concept que l'on considère naturel n'est pas si simple qu'on le croirait ! Elle intervient aussi dans plusieurs autres notions qui posent des difficultés aux élèves du premier cycle du secondaire. C'est le cas des fractions équivalentes. Certains enseignants et les manuels confondent égalité et équation.

MÉTHODOLOGIE : INGÉNIERIE DIDACTIQUE

La théorie des situations didactiques (Brousseau, 1998) servant de cadre à cette recherche, l'ingénierie didactique (Artigue, 1990) en est la méthodologie. La construction d'une ingénierie didactique se base sur le développement de trois analyses préalables qui prennent en compte trois dimensions des concepts mathématiques à étudier : épistémologique, didactique et cognitive (Artigue, 1996; Giroux, 2008; Gonzalez-Martin, 2008). L'ingénierie didactique permet aussi d'utiliser des situations qui font appel à des processus adaptatifs différents. Ces situations sont : les situations d'action, les situations de formulation et les situations de validation[1]. L'ingénierie didactique, méthodologie de recherche qui se caractérise par un schéma expérimental basé sur des réalisations didactiques en classe (conception, réalisation, observation et analyse de séquences d'enseignement), vise à provoquer une genèse artificielle des connaissances dans un jeu d'interactions entre un élève et un milieu didactique (problèmes, supports matériels ou symboliques, consignes, etc.) qui lui est antagoniste. Elle permet d'observer, mais aussi d'analyser d'une façon objective, l'apprentissage des élèves. Elle sert à la fois d'outil d'enseignement et de recherche (Artigue, 1996; Giroux, 2008; Gonzalez-Martin, 2008).

ANALYSES PRÉALABLES

Nous avons fait des analyses préalables qui prennent en compte trois dimensions des concepts mathématiques.

DIMENSION ÉPISTÉMOLOGIQUE

Cette dimension se justifie par le recours aux représentations, mais l'enseignement la réduit aux objets enseignés. La dimension épistémologique nous aide à avoir une vision extrinsèque

[1] **Situation d'action :** Situation où la connaissance se manifeste par des décisions, par des actions sur le milieu. La connaissance visée détermine la stratégie qui permet d'exercer le contrôle (adaptation de la connaissance). Il n'est pas nécessaire que l'élève soit en mesure d'expliciter la connaissance. **Situation de formulation** : Situation où la communication, à un autre « joueur » (élève), de la connaissance visée est nécessaire (adaptation d'un répertoire de connaissances pour convaincre). **Situation de validation** : Situation où une justification des formulations est nécessaire (énoncés, démonstrations). Elle permet la reconnaissance d'une conformité à une norme. Elle est suivie d'une phase d'institutionnalisation par laquelle l'enseignant fixe le statut culturel du savoir en jeu.

des objets enseignés et à être conscients de leur évolution et des difficultés qu'ils mettent en lumière (Artigue, 1996). À ce propos, nous avons fait une recherche documentaire sur l'histoire de l'égalité en vue d'une analyse historique du concept d'égalité. Ce travail a été axé sur les différents sens du concept, mais aussi sur les recherches effectuées sur le concept. Dans l'Égypte ancienne, ce signe existait déjà et symbolisait l'amitié, par opposition à deux lignes se croisant, symbole d'inimitié. Le signe « = » a été introduit en 1557 par Robert Recorde, mathématicien et physicien gallois (1510–1558), dans « Whetstone of Witte » pour épargner à tous ceux qui effectuaient des calculs (lui en particulier) d'avoir à écrire *est égal* en toutes lettres. Il semblerait que ce signe représentait la gémellité (deux lignes de même longueur) apparemment synonyme pour lui d'égalité[2].

DIDACTIQUE

Nous avons analysé le programme du premier cycle du secondaire du Québec pour identifier les contenus qui font référence au concept d'égalité. Nous avons analysé des manuels du premier cycle du secondaire du Québec en vue de faire ressortir des traces du signe d'égalité dans les activités. Ce sont trois ensembles didactiques du Québec. Ils ont été approuvés par le Ministère de l'Éducation, du Loisir et du Sport (MELS). Il s'agit des ensembles didactiques suivants : *Panoram@th*, *À vos maths !* et *Perspective mathématique*. L'égalité intervient dans plusieurs situations et activités numériques, algébriques, géométriques et de résolution de problèmes. Par contre, les manuels ne font pas mention des propriétés du signe d'égalité. Tout se passe comme si le concept d'égalité était « naturel ». Nous avons conçu un questionnaire à l'intention des enseignants en vue d'étudier leur conception du signe d'égalité. Nous avons analysé leurs réponses et enfin nous avons eu des entretiens d'explicitation avec eux à propos de leurs réponses et de leur connaissance des propriétés de la relation d'équivalence.

MATHÉMATIQUE ET COGNITIVE

En mathématiques, l'égalité est une relation binaire entre objets (souvent appartenant à un même ensemble). Les objets sont alors identiques, c'est-à-dire que le remplacement de l'un par l'autre dans une expression ne change jamais la valeur de cette dernière. Une égalité est une proposition (qu'elle soit vraie ou fausse) et elle peut s'écrire à l'aide du signe égal « = » qui sépare deux expressions mathématiques de même nature (nombres, vecteurs, fonctions, ensembles, etc.). La proposition contraire s'écrit à l'aide du symbole de différence : « ≠ ». Une autre conception du signe *d'égalité*, un signal pour faire des opérations, permet de donner du sens à des équations telles que $2x + 3 = 7$, en pensant que le second membre (7) est le résultat à obtenir. Pourtant, dans des équations du type $2x + 3 = x + 4$, certains élèves répugnent à accepter des énoncés tels que : $4 + 3 = 6 + 1$. Ils voient le signe *égal* comme un séparateur et non pas comme un symbole de relation symétrique et transitive. Il importe donc de faire ressortir d'abord le sens relationnel du signe *égal*; ensuite de distinguer le signe *égal* comme identité et comme égalité conditionnelle. Une égalité peut apparaître comme une affirmation, une définition de notation ou encore comme une équation.

SCHÉMA EXPÉRIMENTAL BASÉ SUR DES RÉALISATIONS DIDACTIQUES EN CLASSE

CONCEPTION

Nous avons distribué aux élèves un questionnaire comprenant des situations et les avons analysées par la suite. Celui-ci a permis aux élèves d'agir (Situation d'action), de formuler (Situation de formulation) et de valider (Situation de validation) leurs raisonnements. Les

[2] http://fr.wikipedia.org/wiki/Signe_%C3%A9gal

élèves ont ainsi eu l'occasion de répondre à des questions traitant de l'égalité. Le questionnaire comporte quatre parties.

Première partie « Sens et Contexte » : Elle est composée de deux questions qui ont pour objectifs de connaître les conceptions des élèves, ainsi que les contextes dans lesquels ces derniers pensent que l'on peut utiliser l'égalité.

Deuxième partie « Pourquoi le signe d'égalité entre des expressions ? » : Cette partie est divisée en trois sections. La première fait référence à un travail de décomposition de nombres et d'équivalence de procédures. La deuxième traite d'équivalence d'expressions. La troisième porte sur la transitivité et l'utilisation de l'implication. Nous nous attendons à un travail de transformation de la part des élèves, permettant de justifier et de valider les réponses.

Troisième partie « Propriétés du signe d'égalité » : Les trois premières questions de cette partie mettent en évidence les propriétés qui définissent la relation d'équivalence : réflexivité, symétrie et transitivité. Quant aux trois dernières, elles soulignent la régularité de l'addition et de la soustraction. Les questions ont pour but d'amener les élèves à fournir des explications de ces relations dans leurs propres mots.

Quatrième partie « Résolution de problèmes » : Cette partie propose des problèmes à texte que les élèves doivent résoudre. Elle articule les termes « autant » et « même que », termes dont la traduction en langage mathématique est « = ». La résolution des problèmes renvoie à des raisonnements qui utilisent la réflexivité, la symétrie et la transitivité.

Extrait de situation

Pourquoi le signe d'égalité entre des expressions ? Justifiez pour quelles raisons le signe d'égalité peut être mis entre les termes 1 et 2.

	TERME 1		TERME 2
a)	35	=	12 + 16 + 7
b)	39 – 2 + 7 – 57	=	-13
c)	8 + 2 + 26 + 2	=	37 – 5 + 12 – 6
d)	$2x + 5y - 4x - 5$	=	$3 + 6x + 11y + 2 - 10 - 8x - 6y$
e)	$6x + 3y - 4xy - 11 + 2y - 8x + 6$	=	$-3 - 6y + xy + 11y - 2x - 2 - 5xy$
f)	Si x = 11 on a $2x + 5$	=	$3x - 6$
g)	$6x - 8 = 11x + 7$	alors	$x = -3$
h)	Si $2x - 7y + 2 = 3x - 9 - 2y$	alors	$-x - 5y + 11 = 0$

RÉALISATION

Le dispositif d'enseignement des enseignants nous a permis la réalisation de séquences. Nous avons articulé la réalisation en classe au dispositif de l'enseignant. Nous avons soumis des activités aux élèves. Ces activités, articulées autour de l'égalité, ont permis aux élèves d'agir (Situation d'action), de formuler (Situation de formulation) et de valider (Situation de validation) leurs raisonnements. La réalisation et les observations dans 12 classes (8 enseignants et 180 élèves) ont favorisé l'étude du processus de dévolution et des rétroactions, ainsi que l'identification des variables didactiques sur lesquelles l'enseignant s'appuie pour faire avancer le savoir et le temps didactique. Le corpus intègre les analyses de trois manuels, les notes d'observation, les enregistrements de séquences, les productions d'élèves et les entretiens d'explicitation.

OBSERVATION

Nous avons observé et enregistré des séquences d'enseignement. Nous avons récupéré les productions des élèves, procédé à des entretiens d'explicitation en vue de les analyser.

ANALYSE DE SÉQUENCES D'ENSEIGNEMENT

Elle permet de faire le lien avec la réalité de la classe, les raisonnements et les contenus mathématiques. Nous présentons les questionnaires des enseignants et les activités qui ont été proposées aux élèves.

QUESTIONNAIRE 1 À L'INTENTION DES ENSEIGNANTS

Le premier questionnaire à l'intention des enseignants comporte deux parties. La première partie « Enseignement et signe d'égalité » est composée de six questions. Elles ont pour objectifs de connaître les conceptions des enseignants, ainsi que les contextes dans lesquels ces derniers ont recours au concept d'égalité. La deuxième partie « Traces du signe d'égalité » donne l'occasion aux enseignants de nous informer sur les choix qu'ils effectuent ainsi que sur les ensembles didactiques qu'ils utilisent dans la conception de leurs cours.

Partie I : Enseignement et signe d'égalité

a) Au mieux de vos connaissances, pouvez-vous nous indiquer les différents sens qui peuvent être accordés au signe d'égalité ?
b) Comment utilisez-vous le signe d'égalité dans votre enseignement ?
c) Dans quels contextes utilisez-vous le signe d'égalité ?
d) Citez quelques situations dans lesquelles vous utilisez le signe d'égalité.
e) Concrètement, avez-vous des supports pour illustrer le signe d'égalité ? Lesquels ?
f) Que représente le signe d'égalité pour les élèves selon votre expérience d'enseignant?

Partie II : Traces du signe d'égalité

a) Pensez-vous qu'il est important de faire référence aux sens de l'égalité dans l'enseignement de l'égalité ? Si oui pourquoi ? Si non pourquoi ?
b) Quels ensembles didactiques utilisez-vous dans votre enseignement?
c) Selon-vous les différents sens du signe d'égalité sont-ils pris explicitement en compte dans ces ensembles didactiques ?

QUESTIONNAIRE 2 À L'INTENTION DES ENSEIGNANTS

Le deuxième questionnaire à l'intention des enseignants permet à ceux qui le souhaiteraient d'apporter des modifications à tel ou tel aspect du questionnaire des élèves. Il a pour but de bonifier les activités des élèves et de les rendre plus pertinentes dans le cadre de cette recherche sur le concept.

Série 1 : De façon générale pour quelles raisons avez-vous voulu réduire ou à modifier le questionnaire ?

1. C'est pour alléger le questionnaire.
2. C'est parce qu'il y a des questions qui se répètent. Si oui lesquelles ?
3. C'est parce que certaines questions sont ambiguës. Si oui lesquelles ?
4. C'est parce que certaines questions sont trop difficiles. Si oui lesquelles ?
5. C'est parce que le contenu mathématique de certaines questions va au-delà des attentes. Si oui lesquelles ?
6. Précisez-nous d'autres raisons qui ne sont pas citées ici.

Série 2 : Pouvez-vous nous dire de façon spécifique pour quelles raisons proposez-vous de modifier ou d'enlever des questions ?

Les raisons peuvent être liées aux contenus mathématiques, à l'articulation de certains contenus mathématiques ou à d'autres raisons qui peuvent être lien avec les réponses de la Série 1 par exemple. **Partie I –; Partie II – ; Partie III – ; Partie IV –**

BILAN DES RÉPONSES DES ENSEIGNANTS

Partie I : Enseignement et signe d'égalité	**Enseignants**
a) Au mieux de vos connaissances, pouvez-vous nous indiquer les différents sens qui peuvent être accordés au signe d'égalité ?	Résultat de…, Équivalence, Égal à Égalité que 2 identités sont égales : $2x + 3 = 4x + 5$ ou $10 + 2 = 2 \times 6$
b) Comment utilisez-vous le signe d'égalité dans votre enseignement ?	Place entre deux éléments qui valent la même chose : 3 + 8 = 11 ou 3 + 8 = 14 – 3,…Dans les équations, Membre de gauche est égal à membre de droite, Dans le sens de relation d'équivalence
c) Dans quels contextes utilisez-vous le signe d'égalité ?	Montrer le résultat d'une opération arithmétique, Indiquer que 2 choses sont identiques Ex : taille de Julie = taille de André Algèbre; Arithmétique; Géométrie; Statistique
d) Citez quelques situations dans lesquelles vous utilisez le signe d'égalité.	Pour comparer deux choses, Dans les équations de droite, Trouver l'inconnue dans une équation : $2x + 4 = 8x - 3$ et x?
e) Concrètement, avez-vous des supports pour illustrer le signe d'égalité ? Lesquels ?	Non, Référence à la balance lorsqu'on isole des variables, Lorsqu'on ajoute 3 dans chaque membre d'une égalité pour conserver l'égalité, Pas vraiment
f) Que représente le signe d'égalité pour les élèves selon votre expérience d'enseignant?	Résultat de….En algèbre les élèves devront voir que l'égalité correspond à l'équivalence. La phrase qui revient « c'est égal à.. »
Partie II : Traces du signe d'égalité	
a) Pensez-vous qu'il est important de faire référence aux sens dans l'enseignement de l'égalité? Si oui pourquoi? Si non pourquoi ?	Oui, Attention aux mots qui peuvent mêler, Expliquer le pourquoi du pourquoi aux élèves n'apporte rien, Expliquer le sens de l'égalité dès le départ, Équivalence, Résultat de peut correspondre à une équivalence, Deux parties d'une équation sont équivalentes, Probablement, Il faut le savoir, Pas enseigner à l'université, Si c'est les fractions, il faut connaître les différents sens pour bien comprendre
b) Quels ensembles didactiques utilisez-vous dans votre enseignement?	Panoramaths Point de vue
c) Selon-vous les différents sens du signe d'égalité sont-ils pris explicitement en compte dans ces ensembles didactiques ?	Non, Plein de signes dans les volumes, mais pas d'explications du sens Pas du tout, Je ne le savais même pas qu'il y avait différents sens
d) Si vous avez des commentaires, n'hésitez pas à nous en faire part!	Informer des résultats de la recherche Si c'est important de connaître

BILAN DES RÉPONSES DES ÉLÈVES : APERÇU GÉNÉRAL ET TENDANCE DES RÉPONSES

Partie I : Sens et Contexte
- Annonce un résultat ou l'égalité entre deux termes
- Dans des équations mathématiques

Partie II : Pourquoi le signe d'égalité entre des expressions ?
- Car 35 est le résultat de l'opération du terme de droite
- Car -13 est le résultat du terme de droite.
- Les deux séries d'opérations donnent le même résultat.
- Les expressions sont équivalentes (aucun élève n'a prouvé par des manipulations ou transformations algébriques).
- Les expressions sont équivalentes (aucun élève n'a prouvé par des manipulations ou transformations algébriques).
- En remplaçant x par 11 on obtient le même résultat des deux côtés (aucune preuve).
- Très peu de réponses cohérentes.

Partie III : Propriétés du signe d'égalité
- Cela est évident. Ils utilisent des exemples (3=3).
- Lorsque l'on inverse les deux variables, l'égalité est maintenue.
- Les expressions sont équivalentes.
- Le résultat est vrai mais sans trop savoir pourquoi.
- Le résultat est vrai mais ils ne fournissent pas d'explication.
- Le résultat est vrai mais aucune explication.

Partie IV : Résolution de problèmes
- 102 fleurs. La majorité utilise la transitivité sans en être conscient.
- La majorité trouve 74 en faisant l'addition.
- La majorité trouve 13 en faisant la moyenne de roches et en faisant une soustraction par la suite.

SYNTHÈSE

À la suite de l'expérimentation, quelques constantes se dégagent pour les élèves et les enseignants. L'égalité sert à donner le résultat d'une opération. Elle est utilisée dans les équations. L'égalité traduit une équivalence, mais les enseignants ni les élèves ne peuvent dire ce qu'est la relation d'équivalence. Pour eux l'équivalence veut dire que des objets sont pareils. Il y a une confusion entre égalité et équation auprès des enseignants et dans certains manuels. Pour certains enseignants ignorent la conception arithmétique et la conception algébrique de l'égalité. Les enseignants connaissent des différents sens des fractions et ils sont surpris d'ignorer les deux conceptions de l'égalité. Ces derniers ne trouvent pas opportun d'enseigner les sens de l'égalité. Ils constatent que les sens et les conceptions de l'égalité ne sont pas explicites dans les manuels et sont conscients qu'il faut tenir compte des sens et conceptions de l'égalité dans le choix des activités. Certains ont les mêmes conceptions que les élèves et ils ignorent les propriétés de réflexivité, de symétrie et de transitivité. Quant aux élèves, l'égalité annonce un résultat et traduit l'équivalence. Ces derniers portent souvent toute leur attention sur le résultat final et non les procédures. Étant habitués qu'aux activités pour lesquels il y a une réponse numérique à trouver, ils ont parfois été déroutés par certaines questions où l'on demande à l'élève de justifier et que la simplification donne une expression algébrique. Certaines activités se sont révélées plus difficiles que prévu. C'est le cas des activités des parties II, III, et IV. Les élèves reconnaissent l'équivalence des expressions sans pour autant faire les transformations. Les justifications pour les activités numériques paraissent faciles, mais quand on passe aux expressions algébriques, cela devient difficile. Ils ont la difficulté à expliquer les questions de la partie III et à raisonner pour résoudre les problèmes.

CONCLUSION

L'ingénierie didactique comme méthodologie s'est révélée pertinente. Elle a permis l'élaboration d'activités sur la base d'analyses préalables. La préparation et l'expérimentation d'activités nous ont permis de pointer des contenus et leur gestion. Par ailleurs, les questionnaires des enseignants a permis d'explorer leurs conceptions.

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THE DIDACTIC DIMENSION OF ADVANCED MATHEMATICAL CONCEPTS: AN EXAMPLE WITH SERIES

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CONTEXT

The starting point of this research is the author's PhD dissertation (González-Martín, 2006) on the concept of improper integrals. This research constructed an alternative teaching sequence for the concept of improper integrals through a didactic engineering design. One of the main characteristics of this didactic engineering was to reconstruct this new concept using students' knowledge about definite integrals and series. However, we discovered that our students' understanding about series was very fragile. For instance, when we asked our students whether the following implication was true: "$\sum_{n=1}^{\infty} f(n) < \infty \Rightarrow \int_1^{\infty} f(x)dx < \infty$", we obtained the following answers: "*it's true* […] *since the addition of the terms of a series is a part of the integral. So, if a part converges, therefore the integral converges*", "*it's true* […]*, since the integral is the limit of the sum of the partitions*", or "*in the integral, the series is implicit*".

All of these answers made us suspect students' understandings of series were not very complete. During our experimentation with improper integrals, we continued paying some attention to our students' conceptions about series and we arrived at the following hypotheses:

- Students do not adequately master the results concerning series.
- There are no relations established between the criteria of convergence for series and for improper integrals.
- There is no visual image associated with the concept of series.

At the present time, we want to investigate students' understandings of the concept of series. To do so, we aim to analyse first how this concept is presented and later, to analyse its historical evolution. This paper shows our preliminary results about the first issue: how this concept is presented to students.

In Québec, the concept of series is first introduced at the post-secondary level (at ages 17-18 years, during what is called *collégial* or *cégep* studies). Consequently, we have chosen to analyse how the concept of series is presented to students in their pre-university courses[1].

BACKGROUND

From a literature review on the topic of the teaching and learning of series, two main observations can be made: first, there is very little research that focuses on the concept of series (the concept usually appears implicitly in works about sequences or convergence), and

[1] Currently, a parallel study is being conducted in the UK, at the University of East Anglia, at the undergraduate level. See, for instance, Nardi, Biza & González-Martín (2008, 2009).

secondly, the authors have used diverse approaches to study this concept. For the purpose of this article, we will only refer to some of the major works.

One of the pioneer works about the notion of convergence (Robert, 1982) already pointed out that the exercises used in teaching do not guide the students towards constructing an adequate notion of the convergence of numerical series. It was also pointed out in the 80's (Fay & Webster, 1985) that Calculus textbooks show little or no relation between improper integrals and series (González-Martín arrived at similar results in his PhD research, 2006).

Regarding ways to improve the understanding of series, some authors have argued that the use of visual reasoning could be advantageous for students (Alcock & Simpson, 2004; Bagni, 2000; Codes & Sierra, 2007). Bagni (2000, 2005) also suggests that the use of historical examples could improve the teaching of infinite series and help to overcome the conception that "infinite terms implies infinite sum".

Finally, some authors have studied certain difficulties that are linked in particular to the notion of series. Kidron (2002) identified three main difficulties: the process–object duality, the use of potential infinity versus actual infinity, and the differences between students' concept definition and their concept image for series. She also pointed out the difficulties which arise through the use of symbolic notation, in accordance with Mamona (1990), who also highlighted students' confusion between sequences and series.

Keeping in mind the difficulties documented in research and outlined above, we will pay particular attention to whether or not textbooks take them into account when introducing the notion of series.

THEORETICAL FRAMEWORK

Our objective in the long term is to construct a sequence of activities designed to improve students' understandings of the notion of series. Didactic engineering is one of the methodologies privileged by research. This methodology includes constructing activities, while at the same time researching and collecting data. Before constructing an activity through didactic engineering, a thorough analysis of three dimensions of the targeted mathematical concept is required. Artigue (1992) points out that these three dimensions are not coincidental and that they are parallel to the classification of the obstacles:

> *This classification is in no way original. It follows naturally from the systemic perspective which has been explicitly adopted. It is not surprising therefore to find that this classification is parallel to that put forward in 1976 by G. Brousseau in the study of obstacles (p.47).*
>
> *The analysis of the constraints proceeds by distinguishing three dimensions:*
>
> - *The epistemological dimension associated with the characteristics of the knowledge at stake.*
> - *The cognitive dimension associated with the cognitive characteristics of those who are to be taught.*
> - *The didactic dimension associated with the characteristics of the workings of the educational system (p.47).*

Following this classification, we are currently analysing the *didactic* dimension of the concept of series. In order to do this, we have considered three main sources of information: official programs, textbooks, and teacher practise. These three elements actually define what in some works (see, for example, Chevallard (1991)) has been identified as the *institutional* dimension. For the purpose of this article, we are analysing how the institution considers and presents the concept of series in the pre-university courses given in Québec.

To develop our analyses, we are also considering Duval's (1995) theory of the registers of semiotic representation. According to this theory, to understand a mathematical concept, it is necessary to coordinate at least two different representations of this concept. He also identifies three activities linked to semiosis: the formation of a representation within a register, the treatment of this representation in the same register in which it was created, and the conversion to another representation in another register. We are particularly interested in whether the institution introduces the notion of series using different registers, specifically the graphic register, and whether the institution encourages the use of the three activities linked to semiosis in the students.

OBJECTIVES

Considering the three dimensions distinguished above, the main research question that will guide our project over the next years is the following: taking into account these three dimensions for the concept of series, what are the difficulties and obstacles that appear during its learning at the post-secondary level?

As stated above, we are presently considering the institutional aspect and our sources of information are the official syllabi, textbooks, and teaching practises. The questions we aim to answer at this stage are the following:

- How does the *institution* refer to the concept of series?
- Do textbooks take into account research results?
- Is there an evolution in the way in which textbooks introduce series?
- Which representations and applications are privileged?

The following sections describe some of the data we have obtained from each of the three sources, together with the methodology used.

SERIES: THE SYLLABUS

For this source, we wanted to develop an analysis grid in order to perform an analysis of the content. However, when we had access to the indications given by the official program, we discovered that in Québec, the official program for the *collégial* level concerning the contents for Calculus is very succinct (Figure 1).

The contents on series are limited to: sequences, series, and l'Hôpital's rule; convergence tests; alternating series; power series; Maclaurin series and Taylor series. There are no suggestions about how to develop this content or about the kinds of representations to be privileged.

SERIES: THE TEXTBOOKS

To have a clear idea of the evolution of the textbooks used to teach series, we decided to choose a sample of textbooks used in Québec over the last fifteen years. In many cases, the textbooks were just a newer edition of the original textbooks, but the chapter about series had not undergone substantial changes. In these cases, we just considered the earlier edition. Our final sample included seventeen textbooks appearing in the official programs of several post-secondary establishments in Montréal. For our research, we only considered the Montréal area, under the supposition that, being the biggest city in Québec, the other cities probably follow what is done in Montréal. Once our sample was chosen, we created an analysis grid, taking into account both quantitative and qualitative aspects. Some of these aspects are the following:

- Number of pages, pages for series, ratio.
- Number of graphs and drawings, type, ratio.
- Role of these graphs and drawings.
- Semiotic activities explicitly requested of the student.
- Reasons given to justify the introduction of the new concept.
- Number and type of historical references.
- Applications in real life.
- Types of exercises and problems.

201-203-77 Pondération : 3-2-3

Unités : 2,66

CALCUL DIFFÉRENTIEL ET INTÉGRAL II

PREALABLE(S)

201-103-77

CONTENU

Techniques d'intégration

Rappel et approfondissement des notions de dérivation et de primitivation des fonctions transcendantes. Substitutions algébriques. Substitutions trigonométriques. Intégration par parties. Fractions partielles.

Théorème fondamental, intégrale de Riemann et applications

Calcul d'aires de toutes sortes. Volumes de révolution (axes verticaux et horizontaux). Intégrale impropre.

Suites, séries et règle de l'hôpital

Suites. Séries. Règle de l'Hôpital. Tests de convergence. Séries alternées. Séries de puissances. Séries de Maclaurin, séries de Taylor.

Figure 1

At the time of the presentation at the CMESG conference, most of the compiled data was quantitative. We will describe this data here, with a glimpse of the qualitative data we are analysing at the present time. The seventeen textbooks that form our sample are the following:

A Robert (1992)	*B* Ayres & Mendelson (1993)	*C* Charron & Parent (1993)	*D* Beaudoin & Laforest (1994)	*E* Wild (1995)	*F* Swokowski (1995)
G Anton (1996)	*H* Massé (1997)	*I* Charron & Parent (1997)	*J* Ouellet (2000)	*K* Hughes-Hallet & Gleason (2001)	*L* Bradley et al. (2002)
M Dominguez & Rouquès (2002)		*N* Thomas et al. (2002)	*O* Charron & Parent (2004)	*P* Ross (2006)	*Q* Amyotte (2008)

Most of these textbooks concentrate on integration and sequences and series, and have a separate volume for continuity and derivatives. It is not surprising that the average space consecrated for series in our sample is 10.64%.

As we are interested in the use of visual representation, we first counted the number of graphs and other images per page. The following table shows these data: for every textbook, the first row indicates the number of pages used for series; the second row shows the number of graphs and of other images, and the third row indicates the average number of visual representations per page:

A 21.5p 0 / 2 0.09	*B* 20.5p 1 / 2 0.15	*C* 38.5p 1 / 2 0.08	*D* 59p 2 / 1 0.05	*E* 37.5p 0 / 7 0.19	*F* 40p 2 / 6 0.20	*G* 64p 2 / 5 0.11	*H* 51p 3 / 8 0.22	*I* 41p 2 / 3 0.12
J 37.5p 1 / 2 0.08	*K* 13.5p 2 / 3 0.37	*L* 53.5p 7 / 10 0.32	*M* 28p 2 / 7 0.32	*N* 49p 5 / 17 0.45	*O* 54.5p 2 / 11 0.24	*P* 64p 15 / 41 0.88	*Q* 50p 2 / 11 0.26	

It is surprising how the use of visual representations is quite limited, relative to the space given to series. Being interested in the use textbooks make of the visual representations they display, we later analysed how many of these representations appear in the theoretical development of the concepts. The following table summarizes these results:

A 21.5p 0 / 2 0.09	*B* 20.5p 1 / 2 0.15	*C* 38.5p 1 / 1 0.05	*D* 59p 2 / 1 0.05	*E* 37.5p 0 / 7 0.19	*F* 40p 2 / 2 0.10	*G* 64p 2 / 1 0.05	*H* 51p 3 / 8 0.22	*I* 41p 2 / 2 0.10
J 37.5p 1 / 2 0.08	*K* 13.5p 2 / 3 0.37	*L* 53.5p 7 / 4 0.21	*M* 28p 2 / 3 0.18	*N* 49p 5 / 10 0.31	*O* 54.5p 2 / 10 0.22	*P* 64p 15 / 14 0.45	*Q* 50p 2 / 11 0.26	

Of course, more important than the number of images is the use of these images. In order to analyse the role of images in the theoretical explanations, we defined three categories:

- Non-conceptual (NC): does not relate to a mathematical concept (e.g., a portrait, a photo). Its use is merely decorative.
- Bland-conceptualised (BC): represents a mathematical concept, but is used to remind the student of something he should know (e.g., the graph of a function in a margin, to remind the student of its shape).
- Conceptualised (C): is used to explain a concept, or to illustrate one step in a proof. It is intended to help the student understand a notion.

The following table shows the use of these different kinds of visual representations to explain the theoretical concepts in each textbook. The first row shows the number of NC graphs (g) and other images (i), the second shows the number of BC graphs and other images, and the third shows the number of C graphs and other images (in bold):

A - 0g 2i -	*B* - - **1g 2i**	*C* - - **1g 1i**	*D* - 0g 1i **2g 0i**	*E* - 0g 4i **0g 3i**	*F* - 0g 2i **2g 0i**	*G* - - **2g 1i**	*H* 0g 7i 1g 1i **2g 0i**	*I* - 0g 1i **2g 1i**
J 0g 1i 0g 1i **1g 0i**	*K* - 0g 1i **2g 2i**	*L* 0g 1i 5g 0i **2g 3i**	*M* - 0g 3i **2g 0i**	*N* 0g 4i 0g 1i **5g 5i**	*O* 0g 8i 0g 1i **2g 1i**	*P* 0g 1i 6g 6i **9g 7i**	*Q* 0g 8i - **2g 3i**	

In general, as we can see, very few visual representations of series appear in textbooks. In particular, very few visual representations are used to explain the concept of series (which could be a paradox, taking into account the different representations possible for this concept, see Figure 2).

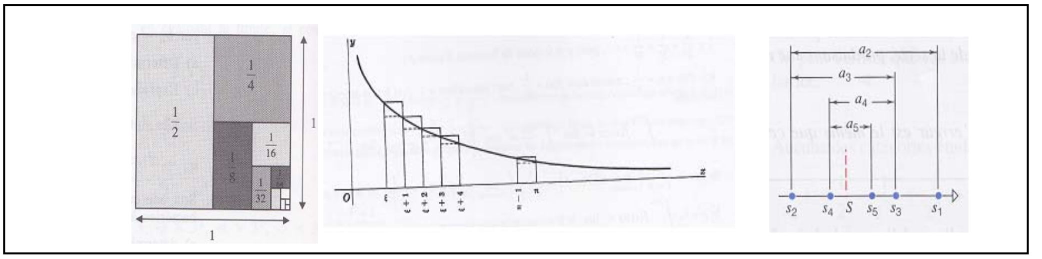

Figure 2

Actually, all the textbooks which use C graphs use them to illustrate the Integral Test, illustrating when the behaviour of a series coincides with that of the associated improper integral. However, this is usually the first occasion in which a visual representation of a series as a graph is displayed, and the authors take for granted that the students will instantly interpret that some rectangles under a curve represent the sum of a series. However, our previous research (González-Martín, 2006) reveals that this is not the case, and many university students are not able to recognise these rectangles as the sum of a series. Moreover, no semiotic activity is explicitly asked of the students regarding the graphic register in any of the seventeen textbooks.

Regarding the use of examples and the privileged registers, our analysis shows that none of the textbooks offer an example given only in the graphic register. Only eight textbooks offer mixed examples (using visualisation and the algebraic register at the same time), with an average ratio of 0.0302 of mixed examples per exclusively algebraic example (meaning that in the whole sample, the average proportion is 3.02 mixed examples for every 100 algebraic examples).

Finally, regarding applications, we considered the number of problems with a context in each textbook. Six of the textbooks do not offer any problem with a context, and we cannot say that this is a characteristic of old textbooks, since the years of publication of these textbooks are: *A* (1992), *B* (1993), *H* (1997), *J* (2000), *M* (2002), and *Q* (2008). The average ratio of problems with a context in the sample is 0.03135, meaning that, on average, each textbook offers 3.135 problems with a context out of every 100 problems.

We are at present analysing most of the qualitative elements of the textbooks (ways to introduce series, types of historic references, kinds of applications shown, etc.) and these aspects will be the source of future works.

SERIES: TEACHER PRACTICE

Our study aims to interview at least six teachers from the *collège* level, with the intention of inquiring about how they introduce the concept of series, the registers used, and the examples and applications given, among other aspects of the concept taught. The following excerpt of an interview with a teacher gives a glimpse of the data that we are in the process of collecting. This teacher has more than five years experience teaching series and uses the textbook *O*.

He acknowledged that series are usually seen at the end of the semester, so they do not have much time to study them, since the objective is to study Taylor series. Asked about whether he considered that the textbook fulfilled the teacher's and the students' needs, he replied: "*Teachers' needs… Yes. Yes, I think it's well written for sequences and series. Well, regarding students' needs, it's the eternal question, why do we teach that?…*"

To introduce the concept of series on the first day: "*I introduce the whole thing with the motivation that certain functions are not integrable*", and the kind of everyday application he gives is the following: "*I like to give an example that talks to them. It's the idea of having two glasses and that a ray of light is reflected, but not completely…*", so it is reflected infinitely between the two glasses. He acknowledges that he does not give other examples of applications: "*Probably I give one or two from time to time, but no, I don't offer many examples.*"

We also asked him whether he tries to make his students aware of the paradox of adding infinite terms. His reply was: "When you say 'the paradox', what do you understand by a paradox?" We prompted: "The paradox of adding infinite terms", and his reply was: "Where do you see a paradox?" Later he completed his response by adding: "No, strangely they understand relatively well… I don't have the impression that they don't feel it."

Asked about his practise, he acknowledged that he does not ask his students to produce any visual representation for series, and that for him, the key graphic element that illustrates series is integrals. He also explained that most of the exercises required are focused on the application of convergence tests. Regarding a question about the main points that students should retain about series, his answer was: "Taylor, that's a big part […]. The idea that an infinite sum of objects can give something finite […]. Well, everything that's linked to integrals." So it seems that series do not have their own status as a targeted concept—they are just tools that enable a better understanding of other concepts.

Finally, when asked about his impression of his students' understandings, he said: "*Unfortunately, I have the impression that the students… they don't understand series very well. They succeed at applying the convergence criteria… but that's mechanical. They learn by heart, it's pure association: this gives this, that gives that. I think they're going to just understand the easy examples* […]*, they understand the ideas, but I don't have the impression that when the function is outside of the easy cases, they see it. They are mixed: sequences, series, converges, diverges*". So even if he first said that he believes that students understand series, he finally acknowledges that he doubts they really understand the notion, and that it is reduced to algorithmic tasks.

We are still analysing the protocols of our interviews, but at the moment, we have the impression that most teachers follow their textbooks, so most of their conceptions seem to reflect the patterns that we found in our analysis of textbooks.

FINAL REMARKS

If we refer back to our main research questions, our preliminary results provide us with some guidelines. Regarding whether textbooks take into account research results, first, we have to stress the fact that few research projects focus on this concept and that there is a variety of approaches. This may be one of the reasons why there is no impact on the development of textbooks; therefore, the approach used to present the concept is still quite "traditional". We also feel that it is important to underline the fact that the difficulties identified by research relating to the understanding the concept of series are usually not explicitly taken into account in the textbooks (nor in teacher practises).

With regards to the evolution in the way in which textbooks introduce series, our data reveals that even if in some cases there are more images, the ratio of images per page is still quite low, and series are still usually introduced as a mathematical object that fulfils mathematical needs (integrals, Taylor series, writing numbers with infinite decimals, etc.). Furthermore, the tasks that the students are requested to perform are usually reduced to the application of criteria and theorems.

Regarding the representations and applications which are privileged, we have seen that the register used almost exclusively is the algebraic register, and no semiotic activity is asked of the students in the graphic register; the proposed examples are almost always algebraic. The use of the activities relating to semiosis in the graphic register is not encouraged by the institution.

Of course, we are aware of the fact that we need to finish our analyses to draw some general conclusions. But the information that we are obtaining from our partial analyses leads us to believe that the institution (reflected in the official programs, textbooks, and teacher practises) seems not to sufficiently consider some difficulties identified by research pertaining to the understanding of the concept of series, even some of the difficulties that might appear during the introduction of the concept.

The concept of series seems to be portrayed exclusively as a mathematical tool required for other mathematical needs and its representation is reduced to algorithms; no visualisation is used and few applications are shown.

We hope that this partial summary of results will allow us to begin reflecting on the consequences that this presentation of series has on students' learning. We also hope that, once our analyses are finished, we will be able to provide some recommendations to practitioners.

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THE EMERGENCE OF DISPARITIES IN MATHEMATICS CLASSROOMS

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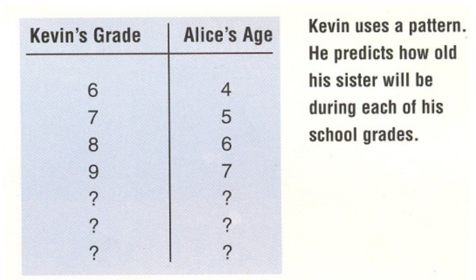
Teachers and students in mathematics classrooms quickly come to know which students perform well in mathematics and which do not. This occurs even in contexts where the students are together for the first time. In our research we investigate the emergence of these disparities from a theoretical perspective, based on the work of Basil Bernstein, that examines their construction in the context of the social practices of the mathematics classroom rather than students' cognitive dispositions. In this paper, I introduce some of our theoretical concepts and use them to analyze two classroom episodes (taught by "Mr. White" and "Mr. Black") that illustrate how classroom practices can become an advantage/ disadvantage for students. Bernstein makes us aware that not only is classroom discourse quite often structured by implicit rules, but that not all students have equal access to these implicit rules. But as these rules allow for the recognition of legitimate classroom discourse, and also the production or realization of such discourse, implicit rules can become a disadvantage for students.

RECOGNITION OF CLASSIFICATION PRINCIPLES

According to Bernstein (1996, 1971) 'discourses of knowledge' which children learn in school reflect social power relations. **Classification** is, in Bernstein's framework, one of the principles that translates power relations into the space of classroom practice. Classification refers, in the context of schooling, to categorizing areas of knowledge within the curriculum or what is taught (cf. Bernstein, 1996, p. 100). Strong classification means that strong boundaries between topics and subject areas are maintained. For example, a traditional mathematics curriculum has strong boundaries as few connections are made to other disciplines. A project-based mathematics curriculum, on the other hand, has weaker boundaries, as mathematics teaching is integrated with other subject areas inside or outside school. This means that everyday knowledge and subject knowledge are less insulated from each other. According to Bernstein, weak or strong boundaries establish a relation to everyday knowledge and therefore students' contributions in class can appear more or less appropriate. For example, when Mr. White starts a unit on T-tables, he shows a T-table in the textbook to his students (see Figure 1). The table refers to Kevin's grade and the age of his younger sister Alice. The textbook activity seems weakly classified as it not only displays a table, leaves blanks and asks for a pattern, but it is also related to Kevin and Alice, their ages and school grades. When the teacher copies this table onto the blackboard (see Figure 2), nearly all the context information disappears. Kevin and Alice remain as headers, but their meaning as grades and ages, or their connection as two siblings, is not mentioned by him anymore. When he asks his students to copy this table into their notebooks and to try to "fill in the blanks" for homework, he indicates this strong classification as well.

Classificatory principles generate and regulate rules that allow an individual to orient and recognize special features which distinguish a given context from another one, e.g. everyday knowledge from subject knowledge in a mathematics class. Having command of a

recognition rule means being able to *locate* classroom discourse and to recognize the classificatory principles of the discourse in the mathematics class. **Realization rules** on the other hand refer to being able to behave appropriately and to understand the *production* of legitimate text (cf. Bernstein, 1996, p. 32ff.). It is one thing to know that you are a student; it is another thing to be able to behave like a student. Using algebra in problem solving is part of the expectation of being a high school mathematics student; most high school students recognize this. However, not all of them are able to realize the behaviour of effectively employing algebra in problem solving.



Kevin's Grade	Alice's Age
6	4
7	5
8	6
9	7
?	?
?	?
?	?

Kevin uses a pattern. He predicts how old his sister will be during each of his school grades.

Figure 1. Textbook Mathquest 2000, p. 8

Kevin	Alice
6	4
7	5
8	6
9	7

Figure 2. The T-table as copied by the teacher onto the blackboard

EMPIRICAL INSTANCES - FROM FIRST MATHEMATICS LESSONS

In the following I will present two cases which illustrate how the implicit transmission of classificatory principles in the mathematics classroom becomes an advantage for some students and a disadvantage for others. The school context of these cases is quite distinct: the first case represents an inclusive (un-streamed) school system, whereas the second case comes from a selective context, where students are streamed into different schools based on their performance in elementary school.

THE CASE OF MR. WHITE

Mr. White teaches grade six in Nova Scotia. The students come from different elementary schools in the area. Some of them have been in the same elementary class, but not many. Others might know each other as they have gone to the same elementary school, but for many the new peers are not familiar. Mr. White is the home room teacher for this class. This means he is teaching them not only math but most other subjects, for example, language arts, science and art. Mr. White is with his home class for more than seventy percent of his weekly teaching.

WORKING OUT T-TABLES

In the first mathematics lesson of the school year Mr. White starts a unit on T-tables. When textbooks are handed out to students, he shows a T-table in the textbook to his students (see Figure 1). He asks them to copy this table into their notebooks and to try to “fill in the blanks” for homework.

At the start of the next lesson Mr. White organizes the students into groups of four and tells them to share the results of their homework with their group members. He asks them: "*What are your three answers? How did you get them?*" After the students have worked in their groups for a few minutes Mr. White asks for the attention of the class. He calls for a volunteer to fill in the blanks of the T-table he has sketched on the board (see Figure 2). Alicia volunteers to go to the board and fills in the missing numbers in the T-table. While Alicia is writing on the board the other students in the class are attentive; they watch what Alicia is writing. Transcript 1 begins when Alicia finishes writing.

TRANSCRIPT 1

58 ***T-White:*** *Is there anybody from her group, as well as Alicia, who can tell us how those numbers fit in the way they do? What did you do? [waits]*
59 *[Max raises his hand.]*
60 ***T-White:*** *OK.*
61 ***Max:*** *Added one on each time.*
62 ***T-White:*** *Which side are we talking about? The left side or the right side?*
63 ***Nick:*** *Both sides.*
64 ***Max:*** *Either or both of them. Because Kevin, in one year he's in grade six and Alice is four years old. So the next year, he's going to be in grade seven and she is going to be five years old. So you add one on both groups.*
65 ***T-White:*** *So in other words, you're adding down, adding one. Is this what you mean? If you started, if you started here you just add one to get to ten.*
66 ***Max:*** *Yeah.*
67 ***T-White:*** *You just add one to get eleven. You just add one to get to twelve? Is that what you did?*
68 ***Max:*** *Yes.*

In the video it is evident that most students lose interest once Alicia has completed the table on the board. For them the "filling in the blanks" task is complete. They do not recognize that something more is expected in Mr. White's question in line 58. Note that he makes no reference to the everyday context, only the numbers. The task has become more strongly classified than in the textbook presentation. Max, however, has not recognized this shift and his explanation in line 64, while perfectly coherent, is based on the given "real life" context of the textbook. Saying "*So the next year...*" refers to the familiar real life phenomenon that in each year we are a year older. The reason Max provides is contextual, or "mundane" as Bernstein (1996) would comment. When Mr. White pretends to rephrase Max's statement, "*So in other words ...*" (65), he cuts out Max's reasons and contextual references. Max does not seem to mind that the teacher does not respond to his reasons, but confirms with "*Yeah*" (66), that this is the idea he had in mind.

TRANSCRIPT 2

69 ***T-White:*** *So what did you did, what did you do over here?*
70 ***Max:*** *The same thing. I had the number seven because I knew she was two-*
71 ***T-White:*** *This one here?*
72 ***Max:*** *Yeah.*
73 ***T-White:*** *Yes.*
74.1 ***Max:*** *Because I knew she was two years younger than the grade he was in.*
74.2 *So then I just added one on [the numbers?] from there.*
75.1 ***T-White:*** *I have a question. This can come, the answer may come from any*
75.2 *group. You may look at the T-Table here or you may look at the one you've*
75.3 *created in your notebook. Can anybody figure out or tell me the relationship*
75.4 *between the left side of this T-Table and the right side of the T-Table.*

76 *[Max is the only student who raises his hand.]*
77 ***T-White:*** *OK.*
78.1 ***Max:*** *The difference between the numbers, there's a difference of two on*
78.2 *each number.*
79 ***T-White:*** *A difference of two. How do you mean difference?*
80 ***Max:*** *There is, one is two higher.*
81 ***T-White:*** *So in other words, this one is two higher.*
82 ***Max:*** *Yes.*

Transcript 2 is the continuation of the conversation between the two. Mr. White shifts the focus to the right side of the table (69). Max describes the relationship between the two columns, again phrased in terms of the context (70, 74.1). Evidently, he has still not recognized Mr. White's shift in the classification of the task. Even though Max has just given the relationship between the columns, Mr. White raises it as a new question for the class (75.3, 75.4). However, the other students do not respond. It might be that they do not recognize that this conversation is important, or they cannot realize the necessary behaviour to participate. Or they may be able to participate but choose not to until the expectations become clearer.

In line 78 there is an interesting shift in Max's language. He has dropped all reference to the context. But Mr. White's response suggests that he has left out too much. It does not matter what the numbers refer to, but it does matter that they are ordered, that one is two more than the other. Max has recognized that the context no longer matters and can realize a purely numerical formulation of the pattern in the numbers. Through this conversation the classification of the task has become clear to Max and to some other students, who now begin to engage in the conversation. But Mr. White is now making another shift, from number patterns to functional relationships, that will eventually be described symbolically in what he calls a "*little tiny equation*".

RETROSPECTIVE TRANSMISSION OF EVALUATION CRITERIA – THE CASE OF MR. WHITE

One striking element of this episode is that what it means "to work out" a T-table changes. What is considered in Mr. White's class as a legitimate mathematical answer only becomes evident through the whole class discussion, in which Max is almost the only active participant. First, it was appropriate to just "fill in blanks", in what appears to be a weakly classified activity. This can be read as an encouraging way to begin the year, with homework the majority of students will be able to do. "Filling in blanks" puts the emphasis on procedure, but not necessarily on a mathematical procedure.

After Alicia has correctly filled in the table on the board, the teacher shifts the focus to numbers and their *pattern*. This is an important moment in the course of the lesson as the classification is suddenly much stronger. However, many students (judging from their body language visible in the video) do not recognize the importance of this moment. Max is prepared to describe not only patterns but also the relationship between the left and the right column, which will be Mr. White's next focus; however he has not recognized the stronger classification. Hence, his contextual argument of a constant grade/age-difference is not acknowledged by Mr. White. Since for Mr. White the instructional discourse is strongly classified, the contextual argument is not considered a legitimate contribution to the evolving instructional talk and is consequently ignored. Note that the classificatory principle is not made explicit on this occasion; the irrelevance of the context is communicated only implicitly.

When Mr. White directs the class to consider the relationship between the left and the right column he introduces another important shift. He strives for a reformulation in symbolic terms: $X - 2 = N$ is the "*little tiny equation*" he seeks in this case. Mr. White shifts from a procedural level "Fill in the blanks" to a more abstract level where a functional relationship is expressed entirely symbolically. In the equation $X - 2 = N$ the decontextualised nature of mathematics is manifest. All connection to the embedding context has been removed. This shift, however, has again been accomplished implicitly. The importance of the process of formalization is not reflected in the teacher's talk. It is as if the teacher tries to move to the formalization as smoothly and gradually as possible, to make it look like the most natural and unproblematic thing.

However, making the problematic look unproblematic might have an effect on the stratification of achievement. Max has behaved in a way that is likely to have produced an impression in the minds of both his teacher and his peers that he is mathematically strong. But what he has is not just mathematical understanding, but also the recognition rules and realization rules that let him negotiate this teacher's recontextualisations of the task from everyday life to mathematics and from concrete numbers to a symbolic relation. And he has been willing to risk being involved in a public discussion early in the school year. Self-confidence and understanding have long been linked to success in school mathematics, but I claim that recognition and realization rules also play a significant role, and reveal a richer picture of the emergence of disparities in mathematics classrooms. In this case, precisely because the teacher's recontextualisations are made implicitly, it is only those students like Max, the ones with the appropriate recognition and realization rules, who can be successful in this lesson. Those students, in contrast, who do not have the appropriate recognition rule, might be lost in the implicitness of the classification principle, and the potential of the classroom situation for their learning is low.

THE CASE OF MR. BLACK

The next piece of empirical data is from an inner-city 5th grade mathematics classroom in Germany. It is the very first lesson after the summer holidays and it is the students' first day in the new school. In most federal states in Germany, primary school ends after 4th grade. From 5th grade on, the students are grouped according to achievement and assumed capacity. Those students who did best in primary school attend the Gymnasium. Mr. Black is a specialist mathematics teacher in a Gymnasium and his 5th graders come from different primary schools. As in Mr. White's class, the teacher and the students do not know each other. Mr. Black starts the lesson by announcing that he wants to test if the students know how to count to 20, something that students would be expected to know in grade 1 in a German elementary school. It turns out that in fact he is introducing a strategic game, which is known as "the race to 20" (Brousseau, 1997). Transcript 3 begins at this point.

TRANSCRIPT 3

3.1 ***T-Black****: Nicole, okay. So you think you can count till 20. Then I would like*
3.2 *to hear that.*
4 > ***Nicole:*** *Okay, one two thr-*
5.1 > ***T-Black:*** *Two, oh sorry, I have forgotten to say that we alternate,*
5.2 *okay?*
6 ***Nicole:*** *Okay.*
7 ***T-Black:*** *Yes? Do we start again?*
8 ***Nicole:*** *Yes. One.*
9 ***T-Black:*** *Two.*
10 ***Nicole:*** *Three.*

11.1 **T-Black:** *Five, oops, I've also forgotten another thing. [Students' laughter.]*
11.2 *You are allowed to skip one number. If you say three, then I can skip four*
11.3 *and directly say five.*
12 ***Nicole:*** *Okay.*
13 ***T-Black:*** *Uhm, do we start again?*
14 ***Nicole:*** *Yeah, one.*
15 ***T-Black:*** *Two.*

Both continue 'counting' according to the teacher's rules. Nicole is patient with her new teacher; she accepts that he discontinues her counting and tries to play the game. In the end Mr. Black states "20" and Nicole loses the game. In a joking way he declares that Nicole was not able to count to 20. He asks if there are other students who really can count to 20. During the next 7 minutes of the lesson, eight other students try but lose against the teacher whilst an atmosphere of students-against-the-teacher competition is developing. While 'counting' against the teacher, the tenth student (Hannes) uses notes that he has written and is the first to win against the teacher.

TRANSCRIPT 4

226.1 ***T-Black:*** *Yeah, well done. [Students applaud.] Did you just write this up or*
226.2 *did you bring it to the lesson? Did you know that today?*
227 ***Hannes:*** *I observed the numbers you always take.*
228.1 ***T-Black:*** *Uhm. You have recorded it, yeah. Did you [directing his voice to*
228.2 *the class] notice, or, what was his trick now?*
229 ***Torsten:*** *Yes, your trick.*
230 **T-Black:** But what is exactly the trick?

After Hannes wins, the student-teacher interaction turns in a different direction, interrupting the "game". Mr. Black seems to be surprised that Hannes noticed his strategic numbers. He asks (226) if Hannes wrote these numbers down or had brought them from home. The latter seems to be an odd assumption, unless Mr. Black plays this game every year and remembers an older sibling of Hannes who might have seen the game before. We will discuss an alternate explanation below, analyzing why Hannes' answer, which is related to the home and therefore a mundane domain, will not get the same credit as the contribution of one of his peers. Mr. Black calls Hannes' strategy a "*trick*" (228.2), and Torsten calls it "*your trick*", suggesting the teacher is introducing games and tricks here. Mr. Black overhears this comment and focuses on the "trick" now.

During the next 5½ minutes the teacher guides the mathematical analysis of "the race to 20". In a whole class discussion, he calls 17, 14, 11, 8, 5 and 2 the "most important numbers" and writes these numbers on the blackboard. He makes no attempt to check whether the students understand the strategy for winning the race. Instead, he introduces a variation: you are allowed to skip one number and you are also allowed to skip two numbers. The students are asked to find the winning strategy by working in pairs.

After 10 minutes, the teacher stops the activity and asks for volunteers to 'count' against him. The first six students lose, but the seventh student (Lena) succeeds. Again the students applaud and Mr. Black asks "*please tell us how you've figured out what matters in this game*". Lena first notes that she and her partner "*figured it out as a pair*" and that "*We found out the four most important numbers and that, in addition, the other must start if you want to win*." Mr. Black gives her a hint, "*Do you want to start from the end?"*, which she ignores. She then outlines her strategy:

> *Okay, well if the other starts then he must say one, two or three. Then you can always say four. [Mr. Black writes 4 on the blackboard.] When the other says five, six or seven, then you can say eight. [Mr. Black writes 8 on the blackboard.] And when the other says nine, ten or eleven, then you can say twelve. [Mr. Black writes 12 on the blackboard.] And when the other says thirteen, fourteen or fifteen, then you can say sixteen. [Mr. Black writes 16 on the blackboard.] And then the other can say seventeen, eighteen or nineteen and then I can say twenty.*

Transcript 5 is Mr. Black's response.

TRANSCRIPT 5

436.1 ***T-Black:*** *Yeah, great. What I appreciate particularly is that you have not*
436.2 *only told us the important numbers, but also have explained it perfectly and*
436.3 *spontaneously. Yes, this is really great. Often, students just say the result, they*
436.4 *haven't the heart, but you have explained it voluntarily. That's how I want*
436.5 *you to answer.*

Mr. Black not only acknowledges her answer (436.1), but praises her explanation as an outstanding contribution ("*really great*", 436.3). He even advocates this as a model of responding ("*That's how I want you to answer.*", 436.4-436.5).

RETROSPECTIVE TRANSMISSION OF EVALUATION CRITERIA – MR. BLACK

The everyday, fun context of playing a game is misleading in this lesson. The classification of content is strong in this classroom; only mathematical approaches and explanations are valued. Nevertheless, the classificatory principles are not made explicit. It is rather astonishing that some students, such as Lena, seem to sense these principles and strive successfully for a semantically and syntactically correct mathematical explanation. How can they know the classificatory principles of a practice they are only starting to participate in? Where did they tacitly acquire the necessary recognition rule for this type of mathematical instruction, which is quite typical for the Gymnasium?

When comparing Mr. Black's talk with Hannes to his talk with Lena, it becomes clear that Lena is presented as a star player. Space is given for her reasoning, and when guidance is offered it is not necessarily expected to be followed. Independent thought and progress are recognized and accepted. The star player's answer is not only declared to be a legitimate contribution, but praised as a mathematical argument and a model response in the mathematics class. Because Lena recognized the classificatory principles of the discourse in Mr. Black's class and produced a legitimate text according to these implicit principles, her answer is praised and she receives a privileged status in the class. Hannes, in contrast, uncovers only "a trick" that allows him to win the game against the teacher. He is still operating in the context of a game, which is associated with out-of-school activities, not mathematics. The numbers Hannes noticed do not convince the teacher that Hannes has engaged in an appropriate mathematical activity; therefore his contribution does not get full recognition.

Other volunteers, like Nicole, who engage in the game and lose against the teacher, also show that they have not yet recognized the activity as a mathematical problem disguised as a student-teacher competition. Among the silent students there might be those who do not recognize the true nature of the game (like Nicole), and some who do but cannot yet realize a legitimate answer (as Lena eventually did). All students who do not notice the implicit hints of the teacher, or cannot decode them, remain in uncertainty about what is expected or how to produce an acceptable answer. As in Mr. White's classroom, silence can be a wise behaviour under these circumstances.

The teacher keeps the students in the dark about some essential aspects of the mathematical teaching that is going on. Although some students who read between the lines of Mr. Black's talk may well identify characteristics and criteria of the pedagogic practice they are participating in, Mr. Black transmits these characteristics and criteria only implicitly, only noting them when they occur.

DISCUSSION

Part of succeeding in school mathematics is behaving according to what is classified as school mathematics. Two steps are needed to do this. First it must be recognized that the context *is* school mathematics, not a real situation involving a boy and his sister, or a game played for amusement. Then behaviours that are appropriate to that context must be realized. Doing these things may be less straightforward for some students, whose backgrounds have not provided them with the appropriate recognition and realization rules.

We have seen in these two cases that some students start in a new school with a new teacher possessing the appropriate recognition and realization rules, and others do not. These rules are introduced implicitly and when they are finally made somewhat explicit (as in Mr. Black's praising of Lena) this may be only sufficient for some students to recognize that she has done something special to become a star player, but still not to have the realization rule to become a winner themselves. As Bernstein notes:

> *...we may have the recognition rule which enables us to distinguish the speciality of the context but we may still be unable to produce legitimate communication. Many children of the marginal classes may indeed have a recognition rule, that is, they can recognize the power relations in which they are involved, and their position in them, but they may not possess the realization rule. If they do not possess the realization rule, they cannot then speak the expected legitimate text. These children in school, then, will not have acquired the legitimate pedagogic code, but they will have acquired their place in the classificatory system. For these children, the experience of school is essentially an experience of the classificatory system and their place in it. (Bernstein, 1996, p. 17)*

By looking at the early lessons of the school year, in a context where teachers and students do not know each other, we explore how Bernstein's theoretical framework helps to reveal how it is that teachers and students quickly decide who will be successful in school mathematics and who will not. The two cases I have discussed here suggest that one element in the emergence of disparity is the implicitness of the classificatory principles applied in the classroom. But this implicitness may be necessary or desirable from another perspective, or perhaps simply unavoidable. Eliminating implicitness cannot be the way of addressing it as an element leading to the emergence of disparity. However, recognizing the kinds of rules that are left implicit in mathematics classrooms, and the kinds of recognition and realization rules that are needed to appear successful in such a context, is a step towards preparing teaching approaches that develop appropriate recognition and realization rules in students who do not already have them.

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MAPPING MULTIPLE WORLDS: IMAGINING SCHOOL MATHEMATICS BEYOND THE GRID

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The problem with our schools is that they aren't focusing on this place. They're focusing on the curriculum. Trying to get through the math. Yet it's not about here. It's not about this place. (Guujaaw, Council of the Haida Nation President)

How might we teach mathematics with place in mind? How does our view of place structure how we see our worlds and see mathematics? Our paper addresses these questions and brings into conversation two areas of research: culturally responsive education and place-based education with philosophical texts on language use and historical readings of mapping. This paper is organized in two parts. We first conceptualize what culturally responsive mathematics education might be, its connection to place and how this can be represented mathematically. We then explore philosophical and historical ways in which place, and our relationship to it, has been represented and articulated through maps and mapping.

CULTURALLY RESPONSIVE MATHEMATICS EDUCATION

In considering culturally responsive pedagogy we draw upon literature that provides a context for developing a model of education for diverse groups that incorporates connections to culture and community, respects and is responsive to indigenous knowledge systems and epistemologies, and is rooted in relationships and places. The term culturally responsive education was first introduced in the 1970's, and although not new it has, over the past year, received increased attention as seen in recent academic publications (Averill et al., 2009; Castango & Brayboy, 2008; Geer, Mukhopadhyay, Powell & Nelson-Barber, 2009).

Culturally responsive pedagogy is conceptualized by the Assembly of Alaska Native Educators (1998) "to provide a way for schools and communities to examine the extent to which they are attending to the educational and cultural well being of the students in their care" (p. 2). In association with the Alaska Rural Systemic Initiative[1], Alaska Native educators articulated a set of culturally responsive standards with the underlying belief that a focus on local environment, language and culture can support culturally-healthy students, educators, schools and communities. Culturally responsive pedagogy involves strategies that include an

> *in-depth study of the surrounding physical and cultural environment in which the school is situated, while recognizing the unique contribution that indigenous people can make to such study as long-term inhabitants who have accumulated extensive specialized knowledge related to that environment (Assembly of Alaska Native Educators, 1998, p. 3).*

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[1]The Alaska Rural Systemic Initiative is an educational project begun in 1995 that involves the University of Alaska Fairbanks, the Alaska Federation of Natives, and funding from the National Science Foundation.

Culturally responsive pedagogy then brings significance to learning in local cultural contexts and uses this context as a way of connecting students with ideas and knowledge systems, and with community practices. Barnhardt and Kawagley (2005) emphasize how the teaching methods of mainstream schooling have not recognized or appreciated indigenous knowledge systems that focus on relationships and the importance of inter-relationships and interconnectivity. The work of the Alaska Rural Systemic Initiative challenges a monocultural education system to include and respect knowledge systems and pedagogical practices from cultural traditions other than what has become accepted as formal schooling. For mathematics education this means elucidating the interconnectedness of mathematics within local, regional, national, and global contexts, toward acknowledging indigenous epistemologies in making mathematics meaningful to, for, and with students and communities.

Certainly improving student learning, participation and achievement are goals of culturally responsive curriculum and pedagogy. However, Averill, Anderson, Easton, Te Maro, Smith and Hynds (2009), in their development of models for culturally responsive teaching note, "*raising the social consciousness and developing cultural competence are also key goals of culturally responsive teaching*" (p. 160) and should be considered a significant aspect of research on culturally responsive education. Yet, how non-indigenous teachers develop cultural understandings (Dion, 2009) and how indigenous educators may enact culturally appropriate teaching (Yazzie-Mintz, 2007) is still not well understood. Raising the social consciousness of educators, decolonizing teacher education, developing partnerships between non-aboriginal and aboriginal educators (Averill et al., 2009), and working with educators to recognize the funds of knowledge (Civil & Bernier, 2006) and cultural resources that students and teachers bring with them to the classroom is significant work.

PLACE-BASED EDUCATION

Culturally responsive pedagogy is responsive to the cultural environment in which students and schools are situated. In rural aboriginal communities this has particular significance where many community members and students have historic or current connections to the land. A sense of place leads to an understanding of historical, cultural, emotional and genetic links to one's surroundings. It offers possibilities for experiencing the deeply interconnected nature of the human and non-human worlds. Smith (2002) describes teaching practices that focus on place to help students become and stay connected to their local contexts in order to better understand the global. A pedagogy of place or place-based education then strives to help students develop a sense of place that is grounded not only in knowing and understanding communities, neighbourhoods, or local regions but also in understanding the interrelationships between our local places and other places in the world (Cajete, 1999). Orr (2004) offers five reasons for the importance of local place: 1) we are place-based creatures shaped by the locality of our birth and upbringing; 2) with calls to protect certain places we are more aware of efforts to preserve place; 3) we have increased understanding that it will take local solutions to solve global issues; 4) we know that a purely global focus can obscure the local events and what happens to particular people in certain settings; and 5) we realize that the global economy is not ecologically sustainable – most of the "successful economies" are also destructive of place, people and ecologies (cf. pp. 160-162).

Gruenewald (2005) suggests that a culture-based education could be synonymous with place-conscious education. He states that "*place is a concretization of the abstract notion of culture common in educational discourse*" (p. 7). But he questions what it is in culture that educators should be responsive to and offers place as a way of answering this question. He further offers place-based education to all students when he says "*place-based education is for* *everyone; everyone has the right and the responsibility to know, to belong to, and to live well in the places he or she inhabits*" (p. 8). Others such as Smith (2002) and Sobel (2004) agree, advocating place-based education as a way to foster social responsibility while at the same time enriching learning for rural and urban, aboriginal and non-aboriginal learners.

Sobel (2004) further discusses how mapping (representing physical place through creating maps) has the potential for connecting students, mathematics, culture and community. Mapping is a way of getting to know a place–of learning or re-learning home – and of learning about home from historical, social, cultural, and physical perspectives (Sobel, 1998). Maps can be used to help change people's minds, to dream of possibilities, to record history, or to celebrate places and events. Maps can be opportunities for connections between place and selves.

MAPPING AS A WAY OF SEEING AND BEING

We began to think about *mapping* as an activity that could foster mathematical ways of thinking in the context of learners' local communities and indigenous cultures. But mapping in itself raised paradoxical problems. What did we mean by a map? Which mapping traditions were we interested in drawing upon? And whose mapping traditions counted? First Nations have had ways of mapping land and sea for thousands of years; would it be better to work with those, or perhaps with both traditional and contemporary modes of mapping?

Similar questions regarding the paradoxical gifts and costs of modern mapping for indigenous peoples are being raised in other contexts as well. For example, Tobias (2009, 2000) has worked extensively with ways of mapping traditional indigenous knowledge of land occupancy and use, as maps that will be recognized in a court of law in connection with land claims cases. His books have been used not only in Canada, but also in the US, Taiwan, Australia and other jurisdictions, where aboriginal cultural traditions do not necessarily fit with the cartography needed for land claims processes (Hume, 2010).

This line of thought led us to a consideration of the square grids that are the basis for most contemporary maps and for coordinates of latitude and longitude.

We began to see how thoroughly mainstream culture in countries like Canada is founded on a basis of orthogonal grids as a primary tool for thinking and making things. Looking around at almost any room, building, street, town plan or map that is part of modern life, we began to notice the prevalence of square corners and square grids, almost to the exclusion of all other geometries. Everything from windows and doors, floor and ceiling tiles, gratings, and sidewalks to building lots, farm plots, street layout and geographic boundaries tends to be normalized to the orthogonal and the grid. The same geometry is used as a tool for thinking and organizing information: tables, charts and graphs based on the grid offer an organizational and explanatory power that we teach in schools and use to make sense of the world. Students all over the world are required to learn ways of thinking, learning and designing so that they can function competently in this culture.

Immersed in our own cultural milieu, it is important to notice nevertheless that not all cultures nor even all historical periods of 'Western mainstream' culture have been so thoroughly imbued with the geometry of the orthogonal grid. We want to draw attention to the strangeness of our cultural reliance of the grid, without suggesting that this geometry is 'bad' or should be banned. On the contrary, we recognize the tremendous power that grids, charts and coordinates offer, and we do not want to exclude learners from this knowledge. Our aim is to draw attention to a 'monoculture' of the grid, and to point out that this is not the only powerful, culturally valid way of thinking and knowing about the world. In working toward a more culturally responsive mathematics education, we are suggesting that a 'multiculture' of representational geometries might offer more and better tools for all of us to think with, and at the same time show greater respect towards indigenous and non-mainstream cultural traditions.

Grids, used for mapping the physical world or the more abstract Cartesian plane, offer a powerful mode of naming, locating and showing relationships. Mapping grids give us the power to name and find locations regardless of their context and local, situated qualities, by treating all points as equal and all metrics as standard. GPS is the most recent version of this powerful mode of mapping; by 'laying a grid over the earth' from a series of satellites, the GPS system more or less casts a net of numbered squares over the hills and valleys, oceans, forests, deserts and cities of the earth. Without needing to walk or sail over the terrain on the face of the earth, we can use GPS to pinpoint any particular spot as if we knew it intimately.

There is a colonialist history and epistemology in the act of 'laying a grid' indiscriminately over the earth, whether it is achieved by GPS technology or by older means like Mercator-projection maps. If a gridded map is projected as a single, uniform metric of interpretation over everything in a way that ignores local conditions and cultures, then it can be disrespectful to local circumstances and conditions.

It is no accident that both the Mercator map and the Cartesian plane were developed in the great period of European rationalism and colonial expansion. Both sprang from a similar impulse: a desire to know, map and claim spaces as part of the Enlightenment project for prosperity through trade and technology. That project has been spectacularly successful in its own terms, developing science and industry in ways never before dreamed of, and establishing world-wide empires of trade and political control. The view from outside the seats of empire looks darker however, and in a post-colonial world, the damage done by colonizing, industrial empires is also clear.

It is important to note that cultures outside the European Enlightenment project also developed maps, and that these maps were not necessarily rectilinear. For example, European medieval maps were not based upon an even metric of gridded squares. Medieval concepts of space identified spiritual and temporal 'hot-spots' (shrines, grottoes, cathedrals, markets, castles, wells), and these received greater emphasis than other places on maps of the time. Medieval maps of the world are often centred on Jerusalem, which is shown large and in detail on the top of a hill, and other cities, countries and oceans are arrayed around it in an order that reflects both physical distances and the relative significance of those places with regard to Jerusalem (Williamson, 1986).

With the Renaissance, there occurred a sharp change to European mapping traditions and to the human construction of cities, roads and territory. Protestants and Catholics warred over competing conceptions of the relationship between divine and secular worlds and between Catholic hierarchies of power and Protestant individualism and more democratic impulses. When Protestant troops conquered formerly Catholic cities built on the medieval plan, there was a concerted effort to erase the medieval structure of spiritual 'hot spots' (shrines, grottos, reliquaries) and overlay an even, rational and decidedly more democratic grid plan on the city (Davis, 1981). The two approaches to city structure and planning indicated more than simply a preferred pattern of street layout; they implied two contrasting conceptual templates for thought and social structure. The mode of thought that produced the first gridded mappings was just as radically different from the mode of thought that produced 'hot spot' maps centred on Jerusalem. We do not wish to imply that one of these two cultures is intrinsically better and ought to win out over the other, but rather that these offer different ways of being in the world, different tools to think with. A comparison and consideration of both modes might offer both culturally responsive and maximally flexible ways of knowing the world.

Non-European cultures have developed informative, useful maps for navigation that did not depend on square grids either. Marshall Islanders' maps of prevailing winds and currents used a basket-like array of woven sticks and shells to mark out wave patterns and islands, an accurate mapping based on close observation of local conditions which allowed for safe navigation of the open waters of the Pacific. Local indigenous maps of the contours of the Greenland shoreline carved around the periphery of a piece of wood gave navigators a wraparound, handheld picture of the coast that did not depend on north-south orientation (Harmon, 2003; Abrams & Hall, 2005).

Maps could be tangible and graphic, like the examples above, and they could also be narrated through poetry and story. Two famous European examples are the verbal map of the route to Troy from Ileum in the Homeric tradition (which guided Heinrich Schliemann to the ruins of Troy in the early 20th century) (EMuseum, 2010), and the verbal map of Leif Erikson's island-hopping navigational route from Iceland to Vinland in the Nordic sagas (which, followed literally, allowed Norwegian archaeologists Helge Ingstad and Anne Stine Ingstad to discover the ruins of the Viking settlement in L'Anse aux Meadows, Newfoundland in 1961) (Parks Canada, 2010). Other accurate narrative mappings and histories passed down generations through oral cultural traditions include Inuit accounts of encounters with Henry Hudson and Australian Aboriginal mappings through Songlines.

Postmodern theorists have begun to address the differences between representations of space in terms of grids and those based on 'hot spots'. Theorizing of these contrasting approaches has led to a bifurcation of cultural types to account for these very different ways of approaching a mapping of the world, named variously as 'visual vs. acoustic-tactile', 'linear vs. complex', or 'striated vs. smooth'. McLuhan's (2005) theorizing of 'linear vs. acoustic-tactile' cultural/technological spaces and Deleuze and Guattari's (1987)'striated vs. smooth' cultural spaces are two complementary and quite detailed accounts of the tendency of cultures to bifurcate into one of these two approaches to space.

A consideration of these issues may lead us to a multiculture, rather than a monoculture, in terms of representation of geographic and cultural spaces. We urge an exploration of the variety of representations in the development of more culturally responsive modes of mathematics education.

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New PhD Reports

Présentations de thèses de doctorat

A STUDY OF UNDERSTANDINGS OF COMBINATORIAL STRUCTURES

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INTRODUCTION

Combinatorics is a part of the curriculum in schools and universities in North America as well as in many other parts of the world. Its connection to other branches of mathematics and other fields of science makes combinatorics an important subject in the mathematics curriculum. In addition to its different applications, combinatorics offers a unique educational opportunity in mathematics: without much prior knowledge of mathematics one can solve many creative, interesting, and challenging combinatorial problems.

In this paper, I present a brief report on my doctoral dissertation. I first describe the motivation and context of my study and I present the research questions. Then I describe the study and findings regarding students' understandings in two situations. The first situation is of understanding something new for the first time. The second situation is when a previous understanding is proven to be inadequate or inappropriate, and there is a need for the previous understanding to be modified or even replaced by a more complete and appropriate understanding. I also describe the theoretical and methodological contributions and the limitations of my research.

THE STUDY: MOTIVATION, CONTEXT, PURPOSE, AND RESEARCH QUESTIONS

During the last year of my master's degree in mathematics I was invited to teach a discrete mathematics course, called MACM 101, in the computing science department of Simon Fraser University. Over 200 students were enrolled in this class. Teaching this course provided me with an opportunity to observe many students and how they learned combinatorics. The students enrolled in MACM 101 are generally mathematics, engineering, or science majors, who are usually quite interested in mathematics. After I started my teaching position at Langara College, I started teaching a course called Finite Mathematics, which includes combinatorics. While teaching Finite Mathematics, I had another opportunity to interact with students, this time more closely in a smaller classroom setting. The students' backgrounds in the Finite Mathematics course varied, but most students were more interested in social sciences and humanities. Some students wanted to study commerce, while others were pre-service elementary school teachers. The mathematical background of most of the students in Finite Mathematics was not very extensive. From my experience of teaching these classes of varying sizes to students with different backgrounds, I realized that combinatorics is a difficult topic for most students to grasp. I found many students had particular difficulty in combinatorics in the following areas:

- recognizing the appropriate combinatorial structure;
- making connections between different concepts;
- interpreting the problem statement correctly;
- using the formula correctly;
- verifying the solution.

Observing these difficulties motivated me to examine students' understandings of combinatorics in more detail. In particular, I had the following questions:

12. How do learners understand a new combinatorial structure? How do they approach a new concept?
13. What are the obstacles in understanding a combinatorial structure? What are students' main difficulties in solving combinatorial problems? Are there obstacles and problems specifically related to combinatorics?
14. How do learners modify their existing knowledge to obtain a better understanding?

There are very few studies in mathematics education related to combinatorics. Hence, it is not difficult to identify and discuss many gaps in the existing research in this area. The previous studies all suggested that students have difficulty in solving and understanding combinatorics, and identified some of these difficulties and obstacles. Hence, studying the previous research helped me answer the second question. However, to the best of my knowledge, there is no study that directly deals with students' understandings of combinatorics and how to make students aware of their own understandings. Hence, the first and last questions remained unanswered by previous studies.

APPROACHING A NEW CONCEPT

Students are expected to understand and use mathematical definitions in most post-secondary mathematics courses. In combinatorics, like other disciplines of mathematics, there are formal definitions for specific structures. To examine students' understandings of a new concept, I presented students with a definition that they had not seen before, and a set of related tasks. Through this set of tasks, I examined how students connect this new definition to what they have learned before, and how they place this structure within the general schema of the previous familiar combinatorial structures. In particular I observed students' use of examples, formulae, and graphical representations, through the lens of the connections they had made between the new definition and their existing knowledge.

The definition was given as follows:

> *A trization of a set of n distinct elements is a placement of these elements into 3 different cells, with k_i objects in cell $i, i = (1,2,3)$, and $k_1 + k_2 + k_3 = n$. The order of objects in each cell does not matter.*
>
> *The number of trizations of a set with n elements with k_i objects in cell $i, i = (1,2,3)$, is denoted by $T(n : k_1, k_2, k_3)$.*

Not only was this definition one that the students had not previously encountered, it does not exist in the literature, and so students could not seek help in understanding it from online or library resources. Three of the eight students tried to google the definition and said that it did not exist in the context of combinatorics. Furthermore, I wanted the language of the definition to be familiar to the students and to have a direct relation to the concepts that were previously explored in their class. However, because I was not teaching their class, I tried to create the definition based on the language used in the notes provided by the instructor. Students were given this definition a few days before the interview and were asked to reflect on it and try to understand it in any way that they could. On the day of the interview they were given a set of tasks that were directly or indirectly related to trizations. In what follows, I briefly describe the results of this study, concentrating on students' use of examples and representations, i.e. algebraic (formulae) or graphical.

For the purpose of this study, I categorize students' examples in a learning situation into three different types: active examples, passive examples, and learner-generated examples. Active examples are the ones that learners create without an external prompt or direction, to help them understand a concept better. Passive examples are examples that are presented by the teacher or the textbook, to illustrate an instance of a concept, so as to aid students with understanding of that concept. Learner-generated examples (LGE) are examples that are generated by students when they are prompted by their teacher or another external source. The terminology for learner-generated examples is borrowed from Watson and Mason (2005) who described and analyzed this class of examples in detail.

The data revealed that learners generally did not use active examples to understand and approach a new combinatorial structure. However, they used the examples of the concepts that they had seen in the class extensively (passive examples). It was also revealed that when the learners were asked to generate examples (LGE), they had some difficulty in going beyond the trivial examples and being creative.

Most participants used some graphical approach in their initial understanding of the definition of a trization. However, they did not accept their graphical understanding as adequate. Their image of understanding mathematical structures required an algebraic representation, i.e., a formula. The data also revealed that participants relied heavily on rote memorization and application of formulae rather than on understanding them. Memorization of formulae helped students to solve some problems very efficiently and correctly, but when they could not remember a formula correctly, they could not proceed. Also when they could not identify a formula because of the unfamiliar structure, they had difficulty approaching the problem. On the other hand, many participants were successful in finding a formula for a trization after they had formed an understanding of the concept.

I found that participants had difficulties making the connection between the new concept and their existing concept image of the related concepts. The only instance when the participants did recognize a connection was through the use of formulae. A few participants recognized a trization as "multiple combinations" or its relation to the binomial theorem when they found the formula for counting the possible trizations. However, none of the participants was able to make any connection without the use of a formula. Furthermore, another anticipated connection $T(n:k,n-k,0)=\binom{n}{k}$ was not made by participants.

Considering that the participants were students with generally strong mathematical backgrounds, the findings revealed that they relied heavily on the use of formulae to solve the tasks presented to them. They often had difficulty approaching problems that did not provide them with a recipe and a ready-to-use formula for its solution.

MODIFICATION OF AN INADEQUATE UNDERSTANDING

Students, especially in post secondary-education, often have strongly formed mathematical preconceptions. These preconceptions are sometimes in agreement with the mathematics community's conceptions and sometimes they are in conflict. Students' preconceptions that are in conflict (misconceptions) with that of the general mathematical community (scientific conceptions), can create obstacles for their learning. Nussbaum and Novick (1982) claim that:

> *When a student retains and continues to use his preconception to interpret classroom information, he is likely to give it meaning which differs from or even*

> *conflicts with the meaning intended by his teacher. It is possible that the learner is not even aware of this gap and that he is perfectly satisfied with his own interpretation, thinking that such was also his teacher's intention (p.184).*

To help students recognize and reflect on their misconceptions, as a starting point for a more complete and refined understanding, I created a methodology called mediated successive refinement. Mediated successive refinement emerged from a seized opportunity. The opportunity arose when all but one of the students in my Finite Mathematics class responded to a problem with substantially the same inappropriate solution. Their reasoning was so persuasive that even the single student with the correct answer was convinced! Rather than simply correct their mistaken reasoning, I decided to invite them to reflect on their inappropriate solution by asking them to generate an example of a problem whose solution was their solution. Later they were given a chance to reflect on their collective responses and solve the problems their peers generated. I later realized that mediated successive refinement could go further as a pedagogical methodology, and could be used as a research tool to gather data as well.

Mediated successive refinement consists of different cycles, in each of which the students interact with their teacher (the mediator), or with each other, to refine and re-examine their thoughts on a particular question or structure. A brief description of the cycles in mediated successive refinement can be found in Figure 1.

Mediated successive refinement helps students deal with their misconceptions by first recognizing their misconceptions and reflecting on not only their own concept image, but their peers' concept images as well, through a set of problems that they generate collectively. This process began as an instructional tool; however, I also used it as a research tool to answer the following questions.

How do learners modify their existing concept image to obtain a better understanding? In particular:

- How can the learners become aware of their inappropriate understandings? In particular does mediated successive refinement help students to become more aware of their own understandings?
- What can the method of mediated successive refinement reveal about the process of change in students' understandings? What are the changes that can happen in students' understandings?
- How does this instructional intervention influence students' understandings of the concept?

When a common misconception is revealed, we can use mediated successive refinement to help students achieve a better understanding of their misconception and modify it to reach a better understanding. There may be many factors that trigger a change in students' concept images. One of these triggers is to encourage learners to reflect on their own inadequate concept image by asking them to embody that image through example-generation. Furthermore, by reflecting on the examples generated by their peers they get yet another opportunity to become more aware of their own understanding. It provides learners with an opportunity to re-examine their understanding by asking them to modify their generated problems. The data revealed that every student had some kind of change in their concept image through the employment of this methodology. In fact, for most students some kind of learning event did happen and their concept image was refined; hence they achieved a better understanding. However, a few students also replaced an inappropriate concept image with another inappropriate concept image.

Teacher	Students
The First Cycle Teacher poses a problem (The initial problem) →	Students attempt to solve it. A particular incorrect solution appears to occur frequently. (The initial solution)
← ***The Second Cycle*** Teacher explains the initial problem and clarifies the solution, then asks students to provide examples of a problem whose solution is the initial solution. →	Students present their examples (The generated problems)
← ***The Third Cycle*** Teacher collects the generated problems and redistributes them among students, then asks students to reflect on some of their peers' generated problems, identify those that are not solved by the initial solution, and modify them so that the solution now applies. →	Students discuss and analyze their peers' examples.
← Teacher summarizes problems and their solutions.	

Figure 3

Another finding of this study was the high level of participants' interest in the tasks. Even though participation in this study was completely voluntary, many students took part and completed all the tasks. They appreciated the opportunity to interact with their peers through their generated examples and described their overall experience as positive.

The general objective of this study was to examine students' learning and understandings of combinatorial structures in post-secondary education. In the course of this study, I identified different factors that could paint a picture of students' understandings of combinatorial structures and examined the different kinds of change that can happen in students' concept images. Furthermore, I developed the methodology of mediated successive refinement to help students examine their own understandings and refine their concept images by comparing and contrasting their concept images with that of their peers. This methodology was also utilized as a research tool for gathering data for examining the gradual change in students' concept images. This study contributed to two aspects of understanding, the theoretical aspect and the methodological and pedagogical aspect. I briefly recap these contributions.

THEORETICAL CONTRIBUTIONS

As mentioned before, in addition to its application in different fields of science and mathematics, combinatorics can be an interesting topic to be explored in the curriculum. However, many students find it challenging not only to solve combinatorial problems, but also to verify their solutions. Sometimes a solution to a combinatorial problem can be very convincing and elegant, but incorrect. Showing that incorrectness could prove very challenging.

There are many valuable books and papers written about problem solving, strategies, pedagogy, etc. In this study, my goal was not to re-examine or reiterate those, but to examine students' development of understanding and concept image through what seemed to be their methods of solving elementary counting problems. To the best of my knowledge, there is no research that specifically pertains to the examination of concept images and understandings of students in combinatorics. I examined the creation of a new concept image and the modification of an existing one. In both studies, the data confirmed that students do face many challenges and difficulties such as the errors that were presented in the previous research. Moreover, I developed a framework to explain and examine different aspects of students' understandings of combinatorial structures.

REVEALING CONCEPT IMAGE

A major contribution of this work is the development of a framework that can paint a picture of students' concept images and their understandings for the educator and researcher. The framework consists of examination of different factors while students are asked to solve some problems. These factors are: students' uses of different kinds of examples (active, passive, and LGE), different kinds of representations (graphical and algebraic), their use of and reliance on formulae, and the connections they make to their previous knowledge.

EVOLVING CONCEPT IMAGE

Another contribution of this study is an examination of the change in students' concept images. The investigation of their maturing concept images revealed that there are different levels of change in the concept images. These different levels of change are as follows:

- Persistent concept image
- Inappropriate concept image without improvement
- Inappropriate concept image with improvement
- Fragmented concept image
- Flexible concept image

This allows a more in-depth examination and analysis of the change in students' concept images.

PEDAGOGICAL AND METHODOLOGICAL CONTRIBUTIONS

According to Batanero, Navarro-Pelayo, and Godino (1997), one of the important aspects in research in the domain of teaching and learning combinatorics is understanding students' difficulties in this field and identifying variables that might influence their difficulties. By using mediated successive refinement, I encouraged learners to reflect on different aspects of their own understanding of a particular structure and its relation to other structures that they had encountered previously. In addition, I provided them with an opportunity to observe and reflect on the examples that were generated by their peers. I also utilized their responses to identify some of the difficulties and challenges that learners face in understanding combinatorial structures.

Use of learner-generated examples provides many beneficial opportunities for the learner. However, research shows that generating examples is not an easy task for students. In their interviews, Hazzan and Zazkis (1999) noticed some difficulties for participants in generating their own examples. Dahlberg and Housman (1997) also noted that some students were reluctant or unwilling to generate examples, were unsure of their responses, and needed the confirmation of the interviewer. The data from my study also confirmed students' difficulties with generating examples, both in the tasks that required them to do so and in other tasks in which example generation could have helped their understanding.

I believe that by asking students to generate their own examples, in practice such as was done in mediated successive refinement, they can overcome their difficulties with the use and generation of examples. The ability to generate examples enables learners to learn mathematics as a constructive activity and it additionally enables them to examine their own understanding of the concepts. This exercise helped students to practice being critical in relation to the problems they were solving. Because they knew the problems were designed by their peers, they felt that they could think about the structure of the problem with a critical eye. This is a very important pedagogical aspect of these tasks. This method seems to help students achieve a better understanding. At the very least it helps them to think about, and compare and contrast their peers' and their own inadequate understandings in a critical way.

Mediated successive refinement also served as a tool for gathering data. The dual role of mediated successive refinement as a pedagogical and a methodological tool arose from the dual role that I played as the teacher and researcher in the class. Researchers have previously employed learner-generated examples as a methodological tool to gather data and they have proven it to be a successful and effective methodology (Bogomolny, 2006; Zazkis & Leikin, 2007). Mediated successive refinement takes example generation one step further, exploring students' thoughts on their peers' examples as well as their own.

LIMITATIONS AND FURTHER RESEARCH

There are a few limitations to each of these two studies. The first limitation in both studies was the sample size. In the first study, there were only eight students who volunteered from a class of over 200 students. These were students who were more confident in their mathematical abilities and had a considerably stronger background in mathematics. Hence, the findings of this study may not be representative of a typical similar classroom. In the second study, the class size was small, which is what was needed to perform the study.

Another limitation was the timing of the interviews. Combinatorics was taught in the last weeks of the course, and by the time that the students had enough information to participate in the study, it was the end of the semester. Hence, there was no time to have a follow-up interview and re-examine students' understanding of trizations at a later time, after their initial interview.

For future research, it would be interesting to examine not only initial understandings, but also how these understandings evolve through time. Also one could examine how mediated successive refinement can be employed to examine the change in students' inappropriate concept images of trizations.

Other directions for future research are the examination of the effect of mediated successive refinement as opposed to another instructional tool such as a classic problem-solving session and an examination of how each contributes to students' awareness of their problematic conceptions and to refining their misconceptions.

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CARING FOR STUDENTS AND CARING FOR MATHEMATICAL IDEAS IN AN ELEMENTARY CLASSROOM[1]

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It is a Friday morning, early. Karen is sitting at her computer near the window. The lights are off, but the spring sun is peeking through the slits between the vertical blinds. The halls are quiet, though the backpacks hanging in the hallway across from the classroom are hints that students have snuck in and out quickly for recess.

At the bell, students spill into the classroom, pulling on their indoor shoes. I prepare my notebook and sit down at the back of the classroom. On the board there is a Venn diagram. The overlapping circles are titled "I'm wearing blue jeans," "My shirt has red on it," and "My shoes are mostly white." As they go to the board to place their magnetic nametags in the appropriate spaces, the students are chatting with friends. Karen is putting on her FM system microphone as she walks towards the board.

This excerpt sets the scene for a typical day in one Grade 6 classroom in a public school in Alberta. It is a fictionalized collage of real moments from my 4-month case study in Karen Marks's classroom. School mathematics is part of everyday life for Karen and her students. In this example, she uses a Venn diagram as an administrative tool to take attendance. Karen and her students also use the diagram to discuss the finer points of membership in the labelled groups as they place the names of absent students and students who do not belong to any of the groups.

But there is more underneath, within, and around moments of teaching such as the Venn diagram excerpt. Karen is passionate about mathematics and she infuses the school day with the subject. At the same time, Karen attends closely to her students. The discipline of mathematics emphasizes generalizations such as theorems and formulae, while the practice of teaching is firmly grounded in the unique interactions amongst teacher and students. Teachers must be interested in both the particular and the general as they plan for learning, taking into account the actual students in their classrooms and comprehensive considerations such as prescribed curricula. There is an active tension as teachers traverse personal and pedagogic landscapes. I am interested to see how Karen works through this tension every day, caring both for her students and for mathematical ideas. More broadly, the questions guiding my study are:

- How does a teacher in an elementary classroom care both for students and for mathematical ideas?
- What are the complexities of caring both for students and for mathematical ideas?
- How are the complexities related to and shaped by the subject of mathematics?

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AN ETHIC OF CARE[2]

I use the word *care* in a deliberate way. I am pointing toward the work of Nel Noddings (2003), but there are many other common meanings for care. Some of them are in keeping with an ethic of care and some are not. For example, "taking care" or "caretaking" have to do with maintaining the status quo, while an ethic of care is more dynamic. "Being careful" has to do with appearances (careful work) and attending closely. Though attention is part of an ethic of care, it is but one way that people maintain or deepen their relationships with others. It is these relationships that are at the heart of an ethic of care.

I begin with Nel Noddings' notion of an ethic of care as the theoretical basis for my research. She describes care as a relationship between two people, but also extends it to care for ideas, plants, and animals. She claims that a caring relationship is characterized by engrossment, motivational displacement, and receptivity. Engrossment involves attending to the person being cared for by being involved and present as well as by listening. The carer's goals become secondary as he or she imagines the reality of the one being cared for; this is motivational displacement. Receptivity is also essential to a caring relationship; the carer is ready for and open to caring. I use the term *profound attention* to group these three characteristics that focus on the carer.

For Noddings, reciprocity is another feature of the caring relationship. Reciprocity traces the movement of care—the give and take of the relationship. Each person can be the giver or receiver of care. Care is not unidirectional; when the person receiving the care—for example, a student—acknowledges and recognizes the care, he or she completes the relationship. The giver of care receives something in return. The reciprocity at work in day-to-day classroom interactions nurtures a sense of connection between teacher and student.

RESEARCH APPROACH

This case study is informed by two academic traditions: phenomenology and narrative inquiry. My descriptions of Karen Marks' experiences of care are central to this study. I also examine the vocabulary around and the etymology of the word *care*. Attention to personal experience and a focus on language are essential to phenomenology. Equally important to this study are my relationships with the participants (Karen and her students) and their relationships amongst themselves. I attend to the contexts of their lives, their identities, their histories, and their imagined futures. These aspects are at the heart of narrative inquiry.

Each of these academic traditions offers a different approach to the research questions. With a phenomenological approach, I consider the phenomenon of care. One of the orientations of phenomenology is to "borrow" the experiences of others (van Manen, 1997, p. 62). In addition to being with Karen in her classroom during her teaching, I met with her weekly for conversation. During these talks, we often selected a particular experience from the previous week to discuss and write about. From these talks and texts, I have borrowed three accounts to explore how Karen cares both for students and for mathematics. I present these three accounts in the section titled "Mathematical sites of care".

From a narrative inquiry perspective, I attend to a three-dimensional inquiry space (Clandinin & Connelly, 2000) as I consider interaction, continuity, and place. The dimension of interaction refers to personal and social aspects of Karen's care for her students and for

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[2] *Care* est un mot qui se traduit de plusieurs façons vers le français. On peut dire *s'intéresser à*, *être concerné par*, *être attaché à, se soucier de*. Toutes ces définitions font parties d'une éthique de la sollicitude.

mathematics. Continuity names the importance of time: past, present, and future. The places where Karen and her students find themselves—the classroom, the school, and the wider milieu—also affect the possibilities for care. There are some traces of these three dimensions in the opening excerpt, but they are more evident in the section titled "A day in the life of Karen's classroom".

The traditions of phenomenology and narrative inquiry offer meaningful approaches that are in keeping with my research questions. I think of these approaches as different ways to walk in the world. In each world, I attend differently to aspects of care. My gait changes slightly, my noticing shifts, and the texture of my writing is altered. Though I do not think that the two worlds I walk in are mutually exclusive, each has a special atmosphere or way of seeing that I describe in the sections that follow.

A DAY IN THE LIFE OF KAREN'S CLASSROOM

The opening section of this paper is an excerpt from a much longer fictionalized collage called *A day in the life of Karen's classroom*. It describes the events of one ordinary school day. The events did happen during my time in Karen's classroom, but I selected and rearranged the moments for the collage in order to highlight the shape and contours of caring both for people and for ideas in a classroom. Because the entire text of *A day in the life* is too long to share in this space, I offer instead a general discussion of the day-to-day activities of Karen and her students.

Karen's care for her students and for mathematics is marked by profound attention (engrossment, motivational displacement, and receptivity). By attending profoundly to her students as they communicate with her and with one another, Karen is able to respond dynamically and improvise her teaching by drawing on her knowledge of current and previous students, the contexts of students' lives, prescribed curricula, mathematics, and much more. As she attends profoundly to mathematics, Karen thinks about the sense students are making of mathematical ideas and about the connections students are making within the mathematics. She invents ways to explore lines of thinking. In other words, Karen continually re-examines her assumptions about what students understand as she reads their work and attends to their questions and explanations.

Reciprocity—the possibility of attention being returned—differentiates human care from care for animals, plants, and ideas. It is through reciprocity that care is completed as the carer and cared-for respond to one another. While maintaining that there is something different and special about human care, I claim that a sense of reciprocity in feelings of accomplishment and pleasure can also come from working on mathematics. The reciprocity at work between Karen and her students is often taken for granted in classroom interactions. Karen sets a task that she has carefully crafted or selected, such as the Venn diagram task in *A day in the life*. The students show their care for her (and for mathematics) by working on the task and answering questions. I give special consideration to the reciprocity that one student, Mariah, experienced in her relationships with Karen and with mathematics. Mariah's relationship with mathematics is strained, but she still tries to respond to the tasks that Karen sets. Mariah shows that she cares for her teacher as she aligns her work with Karen's interests. This alignment is not always easy; the reciprocity is at times tenuous. When Mariah becomes frustrated, Karen responds by helping, by speaking privately with her, or by letting her figure it out for herself, depending on the situation. Mariah also encounters frustration as she negotiates social norms about "intelligence", age, and fairness. Mariah has academic difficulties and is a year older than her classmates. Karen draws on her relationship with her daughter, who also had academic difficulties, to help her care for Mariah. Karen and Mariah maintain a reciprocal caring relationship that is coloured with tension and marked by the traces of a mother-daughter bond.

Proximity is essential to care in Karen's classroom, though it is not one of the characteristics of care described by Noddings nor is it addressed meaningfully in the research and philosophical literature around care. I propose extending the boundaries for considering care to include proximity. In the classroom, I noticed Karen's physical proximity to students. When they work together, she sits or kneels next to them. She looks at their faces and monitors their responses. She enters into a personal relationship with them, one that I describe as a caring relationship. She responds to their questions and ideas. This proximity creates a physical, personal, and intellectual intimacy as Karen works to be close to them, to who they are, and to what they might be thinking. Mental proximity is also at work in the classroom, especially with mathematics. Karen often speaks about conceptual connections between mathematics and other subject areas. Karen holds mathematics in her mind and brings it forward in the classroom. One student jokes that Karen is "mathing" them.

As I observed the way of life in Karen's classroom, I saw the depth and complexity of care among the individuals begin to unfold. Profound attention, reciprocity, and proximity sustain the caring relationships amongst Karen and her students. Caring for ideas is rooted in experiences of caring for people, though the response that sustains care for mathematics may be elusive or it may be blocked by the structure of the school subject matter. In order to explore care for mathematical ideas more deeply, I chose three classroom episodes to draw attention to aspects of care that are of particular interest in mathematics. Karen and I composed accounts to describe these classroom moments. We explored indifference in mathematics through labelling angles, mistakes in the context of long division, and conjecturing in a probability game.

MATHEMATICAL SITES OF CARE

In the first classroom episode, student Emily wonders why the symbol used to label a straight angle is actually an acute angle (and not a straight line). The symbol for all angles is indifferent to the type of angle it is applied to; an acute angle $\angle$ stands in for all angles. Emily's question about why a straight angle was labelled with an acute angle symbol indicates that she was prepared to write |ABC so that the symbol matched the shape of the particular angle. Emily's query causes Karen to re-evaluate the symbol and its connection to what it symbolizes. Karen writes, "[Emily] wondered why this symbol ($\angle$) is used to indicate 'angle' even when the angle itself looks nothing like that." As she cares for Emily's mathematical learning, she is caring for mathematics through Emily. Karen also cares for mathematics directly as she stops to think about Emily's question and enters into her own relationship with mathematics, though that entrance is only evident in the slight pause in the rhythm of conversation between Emily and Karen.

In terms of the arbitrary and the necessary as described by Hewitt (1999), I think that work on the necessary aspects of mathematics provides more possibility for a caring response than does work on the arbitrary aspects. The arbitrary aspects of mathematics (social conventions such as symbols and labels) have interesting historical facets, but it takes a different kind of work to find pleasure (or pride or accomplishment) in this domain. The reciprocity that I see as being essential to caring for people is at work in caring for the necessary aspects of mathematical ideas (the properties and relationships that can be worked out based on what one already knows). The reciprocity that Emily was used to experiencing as she worked on the necessary aspects of mathematics was blocked by the arbitrary, indifferent symbol for labelling angles, but Karen's care for Emily and for the mathematics sustained the relationships amongst them.

Mistakes in mathematics provide another rich landscape for examining care. How a teacher responds to a student's mistake may reveal tensions between caring for the person who makes the mistake and caring for the ideas of mathematics. In the second classroom episode I relate in my dissertation, students Khalil and Anne share with the class their computational mistakes in long division and the strategies they use to correct them. As she invites students to look at their work and their mistakes, Karen shapes what it means to care for mathematics. She reiterates the primacy of making sense, a form of care, as she recognizes that the students have been working on mathematics and that they are able to re-evaluate that work by drawing on their knowledge of mathematics. She describes the importance not only of finding errors but also of analysing where those errors come from. Mistakes are part of doing mathematics in Karen's classroom; they are a serious consideration without being high-stakes. The indifference that is part of mathematics becomes an asset; the mathematics is not dismayed when students make mistakes. Though there are often emotional or psychological issues at play around student mistakes, Karen works at shaping an ethical ideal of caring both for one another and for the mathematics at hand. Together, Karen and her students show that school mathematics can provide a safe place to make mistakes, to deal with frustrations in a healthy way, to persevere, and to care.

In the third classroom episode, students Aidan and Thomas play with the evenness and oddness of multiplication in a probability game. Although the two boys are not partners in this game, nor are they particularly good friends, they share their conjectures with one another and explain their thinking. They refine. Karen and I listen to what they say. We notice and respond to their mathematical thinking. They listen to the questions we ask. No one tells the *right* answer and it is not obvious. As Aidan and Thomas clarify their conjectures, they improve them. It is evident from their faces, the tone of their voices, and their excitement in understanding that Aidan and Thomas receive something from mathematics. Their work with one another expands their mathematical and relational possibilities. Care for people and care for mathematical ideas can nourish each other.

Throughout my dissertation, I inquire into care in both the immediate milieu of Karen's classroom as well as in wider contexts such as mathematics and mathematics education. Two forms of care, caring for people and caring for mathematical ideas, are explored deeply through *A day in the life of Karen's classroom*, the three classroom episodes, and my discussion of the meanings that can be brought forward from field texts, the research texts, and related scholarly literature. Both forms of care involve emotional, intellectual, and moral work; they are full of tensions, congruities, and complexities. As I reflect on the moments and sites of care I have offered from Karen's classroom, I claim that care is a legitimate way to consider teaching and learning mathematics, though this approach may involve emotional costs as well as intellectual and moral challenges.

INTERPLAY BETWEEN MATHEMATICS AND CARE

Karen's care both for her students and for mathematics involves emotional, intellectual, and moral work. This work is subtle and difficult to communicate. Caring work has long been regarded as natural for women (Rodríguez, Peña, Fernández, & Viñuela, 2006). It is perhaps for these reasons that many people underestimate the work of care. Care extends far beyond a personal quality to be respected and serves as a way to reconceptualise teaching, learning, and living while honouring experience and work. By describing moments from Karen's classroom, I draw attention to the work of care in classrooms and to the complexity of caring both for people and for ideas. The emotional, moral, and intellectual work of care has been taken for granted, especially in elementary schools. Naming this care as valuable, difficult, joyous, challenging, and complex work recognizes the professional responsibilities of teachers as well as the depth of the personal practical knowledge (Clandinin & Connelly, 1996) they bring to those responsibilities.

I also claim that to care for people and to care for ideas are not separate forms of care, but that they are linked in complex and sometimes supporting ways. For example, Emily's mathematical learning was blocked by accepted mathematical conventions for labelling angles. When this happened, it was the care between Karen and Emily that sustained Emily's care for the mathematics. When Aidan and Thomas conjectured together, their care for mathematics nurtured the relationship between them as people. The two forms of care are not independent, but they interplay in complex, and sometimes synergistic, ways.

Though I stand firmly in my claims, the work of this dissertation has also led to some wonderings. I wonder especially about what mathematics has to offer to an ethic of care. I have new questions about the place of mathematics in schools, especially elementary schools. If mathematics can provide a fruitful and safe context for the work of care, how might teachers and other educators take advantage of this context in their work with students? How might parents, administrators, and curriculum developers participate in and sustain this work? How might teachers and students draw on the reciprocity available through caring for mathematics to stimulate other forms of care? How might mathematics educators use an ethic of care to work with students on mathematics and to engage in mathematics education research? Karen and her students have shown me that care is full of possibility for emotional, intellectual, and moral work alongside people and ideas. Karen's care both for students and for mathematical ideas offers fresh ways to think about curriculum contexts, future research, the practices of teaching, and educational policies.

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GLIMPSES OF INFINITY: INTUITIONS, PARADOXES, AND COGNITIVE LEAPS

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Commemorating Georg Cantor's contributions to the theory of mathematical infinity is a plaque that reads: "*The essence of mathematics lies in its freedom*" (Aczel, 2000, p.228).

Infinity in mathematics has a contentious history, much of which is connected to the 'freedom' of conceiving of more than one description of infinity. My research centres on the interplay between two types of infinity: *potential infinity* and *actual infinity*. According to Fischbein (2001), *potential infinity* can be thought of as a process which at every moment in time is finite, but which goes on forever. In contrast, *actual infinity* can be described as a completed entity that envelops what was previously potential. The old arguments and refutations that dismiss the idea of actual infinity survive in today's learners and are resilient, coercive, and often problematic. Accordingly, a guiding theme in my research was to explore not only what an individual *knows* about infinity, but also what an individual *is willing to* learn about infinity.

MY RESEARCH, IN SHORT

My dissertation consisted of three separate yet related studies, linked together by a common theoretical underpinning and the following overarching research questions: (1) What can be learned about university students' emergent conceptions of infinity through their engagement with geometric tasks and paradoxes? (2) What are the specific features of accommodating actual infinity? (3) What are the cognitive leaps connected to the idea of mathematical infinity?

My studies followed different methodologies, addressing different 'sub-questions', and were designed to delve progressively deeper into university students' understanding of infinity. Common to each of my methodologies was the invitation for participants to address the tasks first naïvely and then following a discussion of the normative resolutions. The first study, *Intuitions of 'Infinite Numbers': Infinite Magnitude vs. Infinite Representation*, explored the naïve and emergent conceptions of infinity of undergraduate university students, as manifested in their engagement with a series of geometric tasks. My second study, *Paradoxes as a Window to Infinity*, examined approaches to infinity of two groups of university students with different mathematical backgrounds: undergraduate students in liberal arts programs and graduate students in a mathematics education master's program. Data for this study was drawn from participants' engagement with two well-known paradoxes discussed below: Hilbert's Grand Hotel paradox, and the Ping-Pong Ball Conundrum. As a follow up, my final study, *Accommodating the Idea of Actual Infinity*, sought to provide a refined account of both naïve and sophisticated conceptions of infinity. This study considered the conceptions of mathematics majors, graduates, and doctoral candidates, as they engaged with the Ping-Pong Ball Conundrum and one of its variations.

Participants' conceptions were interpreted through an integration of well-known theoretical frameworks: Hazzan's (1999) 'reducing abstraction', the APOS Theory (Dubinsky & McDonald, 2001), and Tall's (1980) 'measuring infinity'. Briefly, these theories may be connected in the following way. As an individual engages with novel problem-solving situations, he or she may attempt to make sense of new mathematical ideas by *reducing the level of abstraction* of those ideas (Hazzan, 1999). There are different ways to reduce levels of abstraction; they include personalizing formal expressions (such as 'I can find' rather than 'there exists'), choosing to work with an entity that is less complex than the one in question (such as working with a number rather than a set of numbers), and relying on familiar techniques to cope with unfamiliar entities (such as relying on number operations rather than group operations). Hazzan (1999) observed that this last strategy was common for learners who were faced with problems for which an understanding of the mathematical entities involved were not yet constructed. Relating this to the concept of infinity, Tall's (1980) notion of 'measuring infinity' can be interpreted as an attempt to reduce abstraction by relying on familiar techniques. Tall (1980) suggested that when learners address infinity in a geometric context (such as the number of points on line segments) they may extrapolate experiences with (familiar) finite measurements and apply them (erroneously) to infinite entities. The notion that 'longer means more' is common in finite measurements and if applied to sets of points to conclude that a longer line segment must have more points serves as an example of relying on the familiar to cope with the unfamiliar – an example of reducing abstraction.

The APOS (Action-Process-Object-Schema) Theory (Dubinsky & McDonald, 2001) postulates a framework for interpreting individuals' conceptions of mathematics. The theory suggests that a learner's understanding of a mathematical entity begins at an *action* stage, which is recognised by a need for an explicit expression to manipulate or evaluate. This action may then be interiorised to a *process*, which is recognised by qualitative descriptions that may describe actions though not execute them. Eventually, an individual may encapsulate this process to a 'completed' *object*, upon which he or she may then apply other actions or processes. The actions, processes, and objects which may be connected to a particular mathematical entity are said to be coordinated in a mental *schema*. Dubinsky, Weller, McDonald, and Brown (2005) proposed an APOS analysis of infinity, suggesting that a learner may construct meaning for infinity as a process or as an object. They relate these two conceptualisations to the ideas of potential infinity and actual infinity, respectively. The APOS Theory is connected to Hazzan's (1999) perspective by the observation that a process conception may be interpreted as on a lower level of abstraction than an object conception. As such, an individual's reliance on notions of potential or 'measuring' infinities served as indication of attempts to reduce levels of abstraction of, and thus cope cognitively with, the idea of actual infinity.

As I mentioned, my interest was in participants' emerging conceptions – that is, in what they could, or were willing to, understand about actual infinity. My analysis attended to changes in participants' expressed ideas as they tried to make sense of formal properties. In addition to focussing on specific conceptions elicited by the particular presentations of infinity in each of the individual studies, I also sought trends that emerged across the studies, and that transcended the different contexts. In the remainder of this paper I present one of the threads that weaved through my thesis, transcending my studies: the interplay between intuition and formality. To illustrate this thread, I begin with a brief discussion of the paradoxes used in my research. Prototypical excerpts from participants' naïve and informed responses are discussed and used to motivate a call for separating intuitive from formal knowledge when addressing actual infinity. I conclude by highlighting a few of the results and contributions of my research.

PARADOXES OF INFINITY

The tension between potential and actual infinity has inspired some of the earliest and also some of the most well-known mathematical paradoxes. The paradoxes presented here – Hilbert's Grand Hotel and variations of the Ross-Littlewood 'super-task' – can be resolved by considering the entities involved as instances of actual infinity and abiding by the conventions introduced by Cantor in his theory of transfinite numbers. In particular, it is Cantor's approach to cardinality comparison of infinite sets that is necessary. Two infinite sets are said to have the same cardinality, or 'size', if and only if there exists a one-to-one correspondence between the two sets. This definition is key to each of the paradoxes in this section, as I briefly illustrate below. More detailed resolutions can be found in Mamolo and Zazkis (2008).

HILBERT'S GRAND HOTEL

> *The Grand Hotel has infinitely many rooms and no vacancy. If only one person is allowed per room, how can the hotel accommodate a new guest?*

Unlike in a hotel with finitely many rooms, in the Grand Hotel, 'no vacancy' does not bar a new guest from being accommodated. The idea is simply to free up an already occupied room by rearranging the accommodations. This can be done by having the guest in room one move to room two and displace the person there. This guest moves from room two to room three. The guest in room three moves to room four, and so on. Since there are infinitely many rooms, each guest can displace his neighbour, and leave the first room vacant for the new arrival. The resolution relies on the one-to-one correspondence between the set of guests and the set of occupied rooms after the shift. Such a correspondence guarantees that the cardinalities of the two sets are the same, and so even when each guest has moved to his neighbour's room, there are still enough rooms for all.

THE PING-PONG BALL CONUNDRUM

> *An infinite set of numbered ping-pong balls and a very large barrel are instruments in the following experiment, which lasts 60 seconds. In 30 seconds, the task is to place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time (working more and more quickly), balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. At the end of the 60 seconds, how many ping-pong balls remain in the barrel?*

Resolving this paradox involves coordinating three infinite sets: the in-going ping-pong balls, the out-going ping-pong balls, and the intervals of time. Although there are more in-going ping-pong balls than out-going ping-pong balls at each time interval, at the end of the experiment the barrel will be empty. The key to the resolution of this paradox is the one-to-one correspondence between any two of the three infinite sets in question. Given these equivalences, at the end of the experiment, the same amount of ping-pong balls went into the barrel as came out. Moreover, since the balls were removed in order, there is a specific time for which each of the in-going balls was removed. Thus at the end of the 60 seconds, the barrel is empty.

THE PING-PONG BALL VARIATION

> *Rather than removing the balls in order, at the first time interval remove ball 1; at the second time interval, remove ball 11; at the third time interval, remove ball 21; and so on… At the end of the experiment, how many balls remain in the barrel?*

This paradox begins in much the same way as the original. In one minute, an experiment involving inserting and removing infinitely many ping-pong balls from a barrel is carried out. However, the distinction lies in the fact that this variation calls for the removal of balls numbered 1 at time one, ball number 11 at time two, ball number 21 at time three, and so on. Thus, despite the one-to-one correspondences between all of the infinite sets in question, at the end of the 60 seconds infinitely many balls will remain in the barrel. In this experiment there is no time interval wherein balls 2 to 10, 12 to 20, 22 to 30, and so on, are removed. The seemingly minor distinction between removing balls consecutively versus removing them in a different order has a profound impact on the resolution of the paradoxes: while in one instance subtracting infinitely many balls from infinitely many balls yielded zero, in the other it yielded infinitely many.

SEPARATING THE INTUITIVE FROM THE FORMAL

Fischbein, Tirosh, and Melamed (1981) suggested that intuitive interpretations are active during an individual's attempts to solve, understand, or create in mathematics. However primary intuitions, which are generally rooted in everyday life and previous practical experience, may hinder students' functioning in a new mathematical field (Tsamir, 1999). Tsamir suggests "*instructors should be attentive to the relations among formal and intuitive knowledge and to the conflicts which may arise in the mismatching applications of these different types of knowledge*" (pp. 231-2). Dubinsky and Yiparaki (2000) suggested that using 'real life' intuitive contexts to teach evaluation of mathematical statements can be more harmful than helpful. They observed that "*the conventional wisdom to teach by making analogies to the real world can fail dramatically*", and advised, "*to remain in the mathematical realm*" (p. 283). Relating this advice to infinity – a concept for which no 'real world' analogy can do justice – my research suggests that the ability to clarify a separation between intuitive and formal knowledge is an important leap toward accommodating the idea of actual infinity.

This section begins with a glance at some of the trends in participants' naïve responses to Hilbert's Grand Hotel and the Ping-Pong Ball Conundrum. It then examines some of the struggles participants faced in trying to bridge their intuitions with formal properties. The section concludes by presenting an argument in support of encouraging learners to separate their intuitive from their formal knowledge when addressing properties of actual infinity.

INTUITIONS OF INFINITY

The naïve ideas and intuitive strategies that emerged during participants' engagement in my research are consistent with those observed in prior research (e.g. Fischbein et al., 1981; Tall, 1980). Participants related the idea of infinity to endlessness, relied on prior experience with number and measurement, and remained in most cases unaware of the inconsistencies between competing intuitions and also between naïve and formal notions.

Connecting the idea of infinity to an 'endless', 'on-going' entity was prevalent in participants' responses to each of the paradoxes. The intuition of 'endlessness' corresponds to the idea of *potential infinity* – a process conception, with respect to the APOS Theory. Describing infinite entities in terms of the process required to establish those entities surfaced in the responses of participants regardless of their level of mathematical sophistication. Intuitions of potential infinity emerged, for example, in participants' descriptions of a Grand Hotel with "an always increasing number" of rooms, and in their resistance to the idea of actual infinity that is manifest in a 'completely filled' hotel. Responses to the Ping-Pong Ball Conundrum also focused on endless processes, such as halving the time intervals. Both liberal arts undergraduate students and doctoral candidates in mathematics objected to the time limit of

60 seconds "since the time interval is halved infinitely many times… the 60 seconds never ends".

As participants attended to the comparison of two or more infinite sets, an intuition of infinity that extrapolated measuring properties of numbers emerged. Tall (1980) introduced the idea of 'measuring infinity' as a metaphor to describe learners' intuition that a longer line segment would have more (infinitely many) points than a shorter segment. Although Tall's perspective focuses on geometric entities, I would like to suggest that the notion of 'measuring infinity' extends further. For instance, intuitions of 'measuring infinity' appeared in participants' responses to the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation as they attended to the different rates of in-going and out-going ping-pong balls. A common resolution to the Ping-Pong Ball Conundrum suggested that "*the process of putting balls in at a higher rate than taking balls out*" would result in a barrel that contained infinitely many balls and from which a 'smaller' infinite number of balls was removed. Attending to the different rates of in-going and out-going balls evoked arguments of a "*bigger infinity*" since there are "*9× more balls in the barrel than out of the barrel at all times. At the end of the 60 seconds there are 9∞ balls in and ∞ balls out*." Attending to the measurable entity of a rate of change and deducing from it ideas of "larger" and "smaller" infinities serves as an example of an intuition of 'measuring infinity'.

The intuitions of potential infinity and of measuring infinity surfaced as competing ideas despite the inconsistency of a 'never-ending' entity that may be 'smaller' or 'larger' than another 'never-ending' entity. A common trend in participants' conceptions is illustrated in Joey's response to the Ping-Pong Ball Conundrum. Joey, a mathematics major, reasoned inconsistently that an infinite number of time intervals is endless, but an infinite number of ping-pong balls could be exceeded by a larger infinite amount. He wrote:

> *"I will never reach 60 seconds. So the experiment should never end, really. Meaning I have an infinite number of ping-pong balls, and yet there are more in the barrel."*

Similarly, Kenny, a liberal arts undergraduate student, argued that the ping-pong ball experiment "will continue into eternity and the number of [ping-pong] balls will be infinite in the barrel". With respect to time, Kenny imagined an endless, potential infinite, however, with respect to measuring the amount of balls, Kenny imagined a large, unknown number. The flexible use of these incompatible notions, which were elicited by different presentations of equinumerous infinite sets, illustrates a hazard of relying on an intuitive understanding of a counterintuitive concept, and motivates the significance of a cognitive leap away from the intuitive.

ATTEMPTS TO COORDINATE INTUITIVE AND FORMAL KNOWLEDGE

As naïve conceptions were challenged by some of the formal properties of actual infinity, the conflict between an intuition of potential or measuring infinity, and the normative properties of actual infinity was realised by some participants. A trend that emerged as a consequence of this realisation involved participants' attempts to appreciate formal properties on an intuitive level, and 'bridge the gap'. In resonance with observations by Fischbein (1987), participants who attempted to reconcile intuitive and formal understandings tended to adapt the formal notions to establish consistency with their intuitions.

The common strategy in participants' attempts toward reconciliation involved what I refer to as 'shifting the process'. Shifting the process occurs when there is a change in aspect, or quality, of infinity to which participants attribute a process. This is recognised, for example, in Eric's consideration of Hilbert's Grand Hotel paradox. Eric, a liberal arts undergraduate, initially reasoned that "*you could keep on adding people forever to fill*" Hilbert's Grand Hotel because the rooms in the hotel "*would go on forever*". When this conception was challenged by the normative resolution to the paradox, Eric refined his idea of the hotel. In his attempt to reconcile the intuition of an endless process with the idea of a completely full hotel, Eric explained:

> *"Although the infinite rooms are infinitely full, it makes space for you by making one of those rooms free. I was first troubled by the idea of one 'last' person not having a room, but then I realized that the last person would ask me to shift rooms, and so on, so there would be a constant rotation."*

The conflict in a hotel that should 'go on forever' but that is 'full' was resolved for Eric through his introduction of the idea of an infinite "rotation" of guests. The infinite process in Eric's conception shifted from adding guests to the process of moving them. Similarly, Clyde's approach to Hilbert's Grand Hotel demonstrated a shift in processes, when he reasoned that the new guest would get "*sound sleep while everyone else has to continue to shift rooms infinitely*." As Clyde addressed the normative resolution to the paradox, the infinite process in his conception was shifted to the transformation of moving guests, despite the fact that each guest moves only once. Clyde's response is also interesting because in addition to attempting to bridge intuition with formal properties, he discerned that the problem was 'unrealistic' and that the solution was reasonable despite his intuitions. Such discernment is important; it will be explored further in the following section.

SEPARATING THE INTUITIVE FROM THE FORMAL

Aczel (2000) shared the story of Rabbi Ben Zoma, a rabbi who strove through meditation to witness the robed figure of God as He had appeared to Moses. As Rabbi Ben Zoma achieved his goal, his experience was so intense that he allegedly "*glanced at the infinite light of God's robe and lost his mind, for he could not reconcile ordinary life with his vision*" (p.27). While my participants' reactions to infinity were not so severe, they did face considerable frustration trying to reconcile intuitions stemming from 'ordinary life' with an understanding of infinity, and in the end they were unable to do so in a way that would yield consistent conclusions.

The concept of actual infinity is so far removed from our experiences that relying on intuition can be treacherous, even when those intuitions develop from experience with Cantor's theory; that is, even when the reliance is on what Fischbein (1987) termed 'secondary intuitions'. A case in point is Dion's attempt to resolve the Ping-Pong Ball Variation. Dion held a master's degree in math education and a bachelor's degree in math. He had demonstrated a solid understanding of corresponding infinite sets and by applying prior knowledge had easily resolved the Ping-Pong Ball Conundrum. However, when he instinctively applied this knowledge to the Ping-Pong Ball Variation it resulted in a conflict. Dion recognised immediately the similarities between the variation and the original experiment, identifying the one-to-one correspondences between appropriate sets. He argued that the two cases would yield the same result – an empty barrel, because "*after you go [remove] 1, 11, 21, 31,…, 91, etc, you go back to 2". He described a "strong leaning*" toward Cantor's theory and expressed "trouble" with the idea that the two paradoxes – similar as they were – would yield inconsistent results. Dion's experience with these paradoxes exemplifies a situation where an appropriate formal understanding might still be led astray by the influence of intuition, in particular, the intuition that 'anything' minus itself should be zero.

The one participant who consistently resisted being led astray by intuition was Jan - an undergraduate student in mathematics. Jan demonstrated a very profound understanding of actual infinity, and an important idea that she often returned to regarded the separation of intuition and formal knowledge. When addressing the Ping-Pong Ball Conundrum, Jan realised that through the experiment

> *"we seem to have obtained a strictly increasing function (namely, number of balls as a function of number of time steps) that is bounded below by zero, but that is 'discontinuous at infinity', and somehow equals zero 'at infinity'."*

She observed that "*intuitively, it seems that the number of balls SHOULD blow up to infinity (though intuition frequently fails us when it comes to the infinite)*".

Similarly, when considering the Ping-Pong Ball Variation, Jan clarified a distinction between her intuition that "*the infinity of balls put in is somehow greater than the infinity of the balls removed*" and the normative property of "*the indeterminacy of the 'quantity' infinity minus infinity*". She went on to reflect that the indeterminacy of transfinite arithmetic "*is another case where the intuition we've learned from the physical world fails us when it comes to the infinite*". Jan's sophisticated understanding of actual infinity seemed to hinge on her awareness that "*it is nearly impossible to talk about it [infinity] informally for too long without running into entirely too much weirdness*".

RESULTS AND CONTRIBUTIONS HIGHLIGHT REEL

Separating intuitive from formal knowledge emerged as one of the 'cognitive leaps' required to accommodate the idea of actual infinity. As participants tried to make sense of the abstract and counterintuitive nature of actual infinity, the persuasive intuitions and prior mathematical knowledge that influenced their ideas surfaced in sharp contrast to normative properties. Many participants experienced considerable cognitive conflict as they attempted to reduce the level of abstraction of actual infinity to make accessible the inaccessible. Participants' determined attempts to reconcile finite reality with the infinite emphasised their desire to appreciate on an intuitive level an entity which is totally beyond the reach of our finite intuitions. In contrast, participants who were able to clarify a separation between their intuitive and formal knowledge achieved what many minds throughout history have strived for: a glimpse of the deep and mysterious nature of infinity.

In addition to a leap from the intuitive to the formal, my research identified other cognitive leaps facing learners, such as the leap from the philosophical to the mathematical. A common trend observed in participants' responses was a resistance toward accepting infinity as an entity with mathematical properties. Instead, philosophies that connected the idea of infinity to eternity, or to all-encompassing entity, or to the unknowable, emerged throughout. These philosophical perspectives have yet to be expressed explicitly in learners' reasoning with more 'standard' infinite set comparison tasks (as in e.g. Fischbein et al., 1981), yet they may have a tacit influence on learners' resistance toward encapsulating actual infinity as an object. The philosophies mentioned were persuasive lines of reasoning, and were, for some participants, the preferred argument even after instruction. Accordingly, one of the contributions of my research was in identifying paradoxes as beneficial research tools for eliciting participants' ideas, provoking cognitive conflict, and clarifying perceptions and intuitions that might present obstacles in adopting a 'conventional' understanding of actual infinity. Through the use of paradoxes, a refined understanding of learners' intuitions was achieved, extending prior knowledge regarding the tacit influences that contribute to learners' conceptions of infinity.

My research also offers a first look at learners' conceptions of transfinite arithmetic and the associated cognitive leaps. Distinct properties of transfinite arithmetic are illustrated in Hilbert's Grand Hotel (transfinite addition) and the ping-pong ball paradoxes (transfinite subtraction) and relate to one of the contributions of my research: the identification of features in learners' conceptualisation of infinity which go beyond the APOS description of encapsulation. Understanding actual infinity as a cardinal includes conceiving of a completed object that describes 'how many', and which may be acted upon in the sense of the APOS Theory. A subtlety related to acting on infinity was brought to light by Dion's and Jan's responses to the Ping-Pong Ball Conundrum and its variation. Although Dion had 'leapt' to the realm of mathematics and could conceive of infinity as 'how many', his understanding of infinity nevertheless lacked one of the fundamental features that contributed to Jan's profound understanding: the knowledge of *how* infinite cardinals are dealt with. An important contribution of my research identifies the necessity of understanding properties of transfinite arithmetic in order to accommodate the idea of actual infinity. Furthermore, my research lays the foundation for an extension to the theoretical framework of the APOS Theory. The APOS Theory connects a learner's ability to apply actions to a mathematical entity to his or her encapsulation of that entity as an object. However, this framework overlooks the different ways in which actions may be applied. It also neglects to consider what, if anything, can be inferred about an individual's conceptualisation based on *how* that individual applies actions and *which* actions are applied. My research suggests a refinement of the APOS Theory which includes a consideration of *how* actions are applied, and it opens the door to future investigations regarding the extent to which this refinement may be appropriate.

My research opens the door to future investigations into the use of paradoxes as a research tool; to the specific conceptual challenges associated with transfinite arithmetic, in particular the indeterminacy of transfinite subtraction; as well as to learners' understanding of the 'domain-dependence' of arithmetic in general.

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EXPLORATION DE PRATIQUES D'ENSEIGNEMENT DE LA PROPORTIONNALITÉ AU SECONDAIRE EN LIEN AVEC L'ACTIVITÉ MATHÉMATIQUE INDUITE CHEZ LES ÉLÈVES DANS DES PROBLÈMES DE PROPORTION

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ORIGINE DU QUESTIONNEMENT ET PROBLÉMATIQUE

Ma recherche de maîtrise, réalisée dans un autre contexte que celui du Québec, soit au Brésil, questionne la pertinence de l'enseignement de la proportionnalité au regard des stratégies développées par les élèves avant et après enseignement (Oliveira, 2000). Nos observations, réalisées auprès de 494 élèves de secondaire I à secondaire IV nous ont en effet amenée, d'une part, à montrer que pour résoudre des problèmes de proportion (différents types de problèmes directement proportionnels et inversement proportionnels étaient ici considérés) la plupart des élèves de secondaire I disposent déjà, avant tout enseignement de la proportionnalité, de différentes stratégies de résolution permettant effectivement de solutionner les problèmes. D'autre part, cette même recherche tend aussi à montrer que les élèves qui ont suivi un certain enseignement de la proportionnalité (secondaire II à secondaire IV) sont davantage centrés sur les données numériques des problèmes présentés, commettant des erreurs liées à une utilisation non-contrôlée d'un certain algorithme. La diversité des stratégies utilisées par les élèves avant tout enseignement et les erreurs rencontrées après enseignement nous ont amenée à nous interroger sur les pratiques d'enseignement de la proportionnalité, au moment de son introduction, en regard notamment de la place attribuée par l'enseignant aux connaissances des élèves.

Plusieurs recherches ont contribué à de considérables avancées sur l'apprentissage de la proportionnalité (Oliveira, 2000; Vergnaud, 1991; Levain, 1997). Ces travaux sur l'apprentissage de la proportionnalité nous permettent en effet d'avoir une idée des stratégies auxquelles les élèves ont recours dans la résolution de différents types de problèmes, et ce avant même tout enseignement, ainsi que des difficultés des élèves au regard de différents types de problèmes de proportion. Certaines de ces recherches éclairent aussi l'analyse des situations et des variables susceptibles d'influencer l'engagement des élèves.

Des séquences d'enseignement tenant compte de ces résultats de recherche ont été développées par Brousseau (1981) avec des élèves au primaire, par Vergnaud (1983) et Rouchier (1991) au secondaire, ainsi que par Gnass (2000) avec des élèves en difficulté d'apprentissage au secondaire. Elles montrent le potentiel de certaines situations/séquences d'enseignement pour l'apprentissage de la proportionnalité. Elles nous permettent de mettre en évidence le rôle clé des situations proposées et des interventions conduites par le professeur au moment de l'introduction de la proportionnalité. Mais que sait-on de l'enseignement de la proportionnalité en classes régulières, des pratiques d’enseignement menées dans ces classes lors de l'introduction de la proportionnalité?

Nous avons répertorié trois études portant sur l'analyse de pratiques d’enseignement de la proportionnalité en classe (Hersant, 2001, 2004; Adjiage & Pluvinage, 2007). Ces recherches viennent éclairer, d’une part, les pratiques de la proportionnalité dans des environnements spécifiques, utilisant un support informatique. Elles nous permettent de mettre en évidence dans ce cas, en lien avec l'utilisation d'un logiciel, la façon d'interagir du professeur, sous les aspects de dévolution, d'explication et d'institutionnalisation (Hersant, 2001) ainsi que l'activité mathématique induite chez les élèves (Adjiage et al., 2007). La seconde étude d'Hersant (2004) nous permet de caractériser une pratique spécifique d'enseignement que l'auteur appelle « cours dialogué », sous l'angle de la progression du savoir chez les élèves. Ces études spécifiques réalisées en France portent donc sur des approches particulières et n'apportent pas d'éclairage sur l'enseignement usuel de la proportionnalité en lien avec l'apprentissage des élèves.

Cette absence de recherches conduites sur les pratiques usuelles d'enseignement de la proportionnalité nous a amenée à vouloir en savoir plus sur la pratique effective des enseignants en classe ordinaire, au moment de l'introduction de ce concept. Par ailleurs, en retournant à notre questionnement de départ, celui soulevé par les changements qui semblent s'opérer dans les stratégies des élèves avant et après enseignement dans des problèmes de proportion, nous cherchions à en savoir plus sur ces pratiques usuelles en lien avec l'activité mathématique qu'elles étaient susceptibles d'induire chez les élèves. Remarquons à cet effet, comme le précisait Perrin-Glorian, qu'il n'y a pas en général dans la recherche portant sur les pratiques des enseignants depuis plusieurs années, de travaux concernant les effets réels de ces pratiques sur les élèves (Bednarz & Perrin-Glorian, 2003). Notre objet de recherche découle donc de cette absence de connaissances face aux pratiques usuelles d'enseignement de la proportionnalité, en lien notamment avec les apprentissages qu'elles sont susceptibles d'induire chez les élèves. Une analyse de ces pratiques nous permettra de mieux comprendre comment elles interfèrent sur l'activité mathématique des élèves dans des problèmes de proportion, au moment clé de l'introduction de ce concept. Plus précisément, notre objectif de recherche a été celui de « *Décrire, analyser et interpréter des pratiques « ordinaires » d'enseignement de la proportionnalité en classe de mathématiques au secondaire, au moment de l'introduction de ce concept, et ce en lien avec l'activité mathématique induite chez les élèves dans des problèmes de proportion* ».

UN BREF APERÇU DU CADRE THÉORIQUE

La compréhension des pratiques d'enseignement passe par une certaine déconstruction faisant intervenir différentes dimensions de celles-ci, dont deux plus particulièrement: la planification des situations d'enseignement et le déroulement proprement dit de ces situations en classe (la pratique effective en classe), les deux étant interreliées. Selon Robert (2001) les pratiques en classe désignent :

> *« [...] tout ce que dit et fait l'enseignant en classe, en tenant compte de sa préparation, de ses conceptions et connaissances en mathématiques et de ses décisions instantanées, si elles sont conscientes (les séances en temps réel) » (p. 66).*

L'ensemble de ces composantes (la séance en classe, sa préparation, les choix didactiques de l'enseignant, ses conceptions et connaissances, ses décisions dans l'action) caractérise sa pratique. Les pratiques d'enseignement peuvent être observées à partir de différents points de vue: celui de la complexité (en analysant différentes composantes de cette pratique, leur imbrication, en lien avec différents temps de travail de l'enseignant), celui de la cohérence de cette pratique (on cherche ici à mettre en évidence les choix de l'enseignant, les principes sous-jacents à travers sa planification, l'action en classe), celui du jeu des contraintes susceptibles d'intervenir comme domaine de justification possible de cette pratique.

Pour atteindre notre objectif de recherche, nous puiserons à différents cadres théoriques qui serviront de base à l'analyse, dont: la perspective **ergonomique** (Rogalski, 1999; Robert & Rogalski, 2002), et la théorie des **situations didactiques** de Brousseau (1998).

Dans la double-approche que nous préconisons, la perspective ergonomique, nous permettra d'aborder l'analyse de la pratique de l'enseignant en prenant en compte l'enseignant comme acteur professionnel dans l'exercice d'un métier, son activité, ses actions, notamment à travers les gestes professionnels qu'il met en place autour de tâches précises.

La théorie des situations didactiques, quant à elle, nous permettra de caractériser davantage la pratique d'enseignement en classe autour d'un savoir spécifique, la proportionnalité, notamment à travers la dévolution des situations proposées à l'élève, l'institutionnalisation, ou encore le contrat didactique. Nous pouvons par exemple nous poser les questions suivantes : l'élève est-il amené à prendre en charge la résolution des problèmes proposés? Comment l'activité de l'élève est-elle reprise dans les retours collectifs, dans les institutionnalisations locales que l'enseignant en fait? Comment l'enseignant gère la progression du savoir en classe? Quel contrat didactique s'installe-t-il entre l'enseignant et les élèves? Induisant quel apprentissage chez les élèves?

Tableau 1. Perspectives d'analyse des pratiques d'enseignement : différents angles d'entrée

Deux perspectives d'analyse des pratiques complémentaires	**Une perspective ergonomique**	**Une perspective didactique (théorie des situations didactiques de Brousseau)**
Les concepts clés considérés	-Gestes professionnels de l'enseignant -Activité de l'enseignant	-Contrat didactique -Dévolution -Institutionnalisation
Une lunette particulière permettant de documenter la pratique d'enseignement de la proportionnalité dans l'action	Du point de vue de ce que fait l'enseignant dans l'action, en essayant d'inférer les principes qui le guident…	Du point de vue de la manière dont l'enseignant gère la progression du savoir dans la classe

MÉTHODOLOGIE

Pour comprendre les pratiques d'enseignement de la proportionnalité, en lien avec l'activité mathématique induite chez les élèves, deux classes de secondaire II comprenant 34 élèves chacune (13-14 ans) et leurs enseignants (que nous nommerons Maurice et Jacques) ont été suivis pendant une séquence d'enseignement portant sur l'introduction de la proportionnalité, donnant lieu à une observation systématique des séances en classe. Ces observations ont été complétées par des entrevues avec chaque enseignant et par l'administration individuelle d'un test écrit aux élèves, en amont et en aval de la séquence d'enseignement, portant sur la résolution de différents types de problèmes proportionnels et non proportionnels.

Pour atteindre notre objectif, nous avons centré nos observations sur différents aspects de la pratique : la planification de l'enseignant et les situations proposées dans celle-ci, la pratique réelle en classe et les situations alors proposées, la manière de les approcher, les interactions enseignant – élèves – savoir *in situ*, les choix didactiques liés à la planification et la réalisation en classe, les principes qui guident l'enseignant, les apprentissages des élèves qui y prennent place. Pour cela, et ce pour chacun des enseignants, nous avons eu recours à différents modes de collecte de données : l'observation en classe, le recours à des entrevues avec l'enseignant, à un test écrit passé aux élèves portant sur la résolution de problèmes de proportion, et le recours à des documents écrits complémentaires (traces des situations, problèmes proposés, des notes de cours). Les différentes sources de données viennent, de manière complémentaire, couvrir différentes dimensions de l'analyse des pratiques d'enseignement de la proportionnalité, au moment de leur introduction (sous les aspects planification et réalisation en classe) et l'activité mathématique induite chez les élèves dans la résolution de problèmes de proportion. Plus précisément, notons que *les entrevues* réalisées avec chacun des enseignants (sur la planification, mini-entrevues en cours d'observation, après l'ensemble de la séquence) nous ont permis d'aller chercher la rationalité de l'enseignant, ce qui le guide dans la construction de sa séquence, dans son action en classe, les principes sous-jacentes. De manière complémentaire, *l'analyse des notes de cours* préparées *a priori* pour les élèves nous a permis d'identifier les choix didactiques explicités par l'enseignant dans un document écrit destiné aux élèves, tout comme il nous a permis d'observer de quelle manière ce qu'il nous avait annoncé lors de l'entrevue sur la planification s'est actualisé dans ce document écrit. *L'observation des séances en classe*, à son tour, nous a permis dans un premier temps d'observer l'activité effective de l'enseignant et de repérer les choix didactiques qu'il faisait. Dans un deuxième temps, de remarquer de quelle manière le travail des élèves avait été pris en compte par l'enseignant et de quelle façon ce qui avait été annoncé lors de la première entrevue était mis en œuvre dans une action effective.

Après que nous ayons compris la pratique d'enseignement en classe du point de vue de l'activité enseignante, les test écrits expérimentés auprès des élèves nous éclairaient sur l'activité mathématique induite, à la lumière de la manière dont les élèves faisaient des mathématiques (résolution de problèmes de proportion, reconnaissance des situations proportionnelles) avant tout enseignement formel de la proportionnalité et après celui-ci, en ciblant en quoi cette manière de faire change après enseignement. Quelles traces de l'activité mathématique développée en classe retrouvons-nous dans la production écrite des élèves?

RÉSULTATS

Pour analyser la pratique d'enseignement de ces deux enseignants (Maurice et Jacques) lors de l'introduction de la proportionnalité en classe de secondaire II, nous nous sommes appuyées sur les deux cadres d'analyse mentionnés précédemment. Les brefs résultats que nous présenterons maintenant portent sur l'analyse de la pratique de Maurice[1] à travers le concept de « cohérence » et de rationalité sous-jacente. D'abord, le regard sur les pratiques à travers le concept de cohérence permet de rendre compte des raisons, de la logique qui guide l'enseignant à travers les invariants et les différences. La cohérence traduit ainsi l'idée d'une activité de l'enseignement non-aléatoire, non-arbitraire, et donc l'idée de choix stables, repérables, dans différents contextes, à différents moments. Ensuite, le regard à travers les gestes professionnels nous informe sur l'activité de l'enseignant (la pratique en action en classe) en lien avec la tâche donnée aux élèves.

LA PRATIQUE DE MAURICE À TRAVERS LE CONCEPT DE COHÉRENCE

L'analyse de la pratique de *Maurice* conduite sous l'angle de la cohérence nous renseigne sur la rationalité de l'acteur (les principes sous-jacents qui le guident). Cette analyse a été faite selon trois moments de la pratique : la planification de la séquence, les notes de cours données aux élèves (support écrit) et les séances en classe (pratique en action). De l'analyse de ces trois moments se dégage une cohérence qui est présente de manière constante. Elle est

[1] Par manque d'espace, nous avons choisi de ne présenter que l'analyse de la pratique de Maurice.

explicitée en termes du savoir en jeu, de son rôle (d'enseignant) face à l'élève et de ses choix didactiques.

Dans ce qui concerne le *savoir en jeu,* nous pouvons noter que Maurice organise sa séquence à partir d'une certaine progression du savoir, d'un certain découpage des séances et des choix des problèmes[2]. Concernant son *rôle face à l'élève,* nous pouvons noter que Maurice présente un certain souci quant à la structuration de la démarche des élèves à travers les marches à suivre qui sont présentées. Cette structuration de la démarche et les marches à suivre qui sont associées démontrent une prise en compte des erreurs des élèves *a priori*. Les *choix didactiques* sont explicités par l'enseignant, entre autres, par des attentes. L'explicitation de ces attentes apparaissent dans les trois moments de l'analyse soit, dans l'entrevue, dans les notes de cours données aux élèves et dans les séances en classe. Notons que lors de l'explicitation de ces attentes, un accent est mis sur la notation qui doit être utilisée ainsi que sur la structuration de la démarche des élèves (marches à suivre) et sur le savoir en jeu.

D'une manière plus précise, la cohérence de la pratique de Maurice peut être vue à partir de l'interaction entre les trois moments présentés précédemment. Dans ces termes nous soulignons une cohérence entre *l'action et le discours sur l'action*, une cohérence *dans la pratique en classe* et une cohérence *dans le temps*. Dans ce qui suit, nous aborderons chacun de ces aspects.

Une cohérence entre l'action et le discours sur l'action (propos tenus en entrevue) : à ce moment, la cohérence est explicitée à travers la manière dont Maurice pense[3] la planification, la construction des notes de cours et l'organisation des séances en classe. Dans l'action cette cohérence est repérée à travers le discours de l'enseignant au moment où il rend accessible les principes sous-jacents qui guident sa pratique et les choix didactiques faits.

Une cohérence dans la pratique en classe (à un temps donné de l'action) : dans l'action en classe, cette cohérence apparait à travers un certain pattern dans la structure de la leçon, à travers une certaine manière de faire pour introduire un nouveau contenu, et enfin à travers l'explicitation aux élèves des attentes (accent mis sur la notation, l'écriture des unités et des marches à suivre) qu'il maintient tout au long de la séquence sur la proportionnalité.

Une cohérence dans le temps (à différents moments) : cette cohérence peut être observée à travers la forme que prend chacun des moments. Par exemple, la planification est pensée selon une certaine avancée du savoir mathématique. La planification s'actualise par la suite dans les notes de cours et dans la pratique en classe.

Autant que la cohérence, les différences entre les différents moments nous informent également sur la pratique de Maurice. Par exemple, la prise en compte des erreurs des élèves se manifeste différemment dans l'entrevue par rapport aux notes de cours et à la pratique en classe. Plus précisément, dans l'entrevue Maurice annonce qu'il prend en compte les erreurs des élèves lors de sa planification, ce qui pourrait nous laisser penser qu'il va choisir les problèmes de manière à provoquer ces erreurs pour ensuite intervenir plus en profondeur en classe. Contrairement à ce que nous avons pensé initialement, cette prise en compte des difficultés des élèves, dans les notes de cours et dans les séances en classe, prend la forme de mises en garde où il annonce aux élèves les difficultés existantes concernant la notion étudiée et les prévient des erreurs possibles. À ce moment, son objectif est d'éviter que l'erreur ait lieu. Cette manière de faire se différencie de celle annoncée lors de l'entrevue et, à son tour,

[2] Cette organisation de la séquence pourrait être faite autrement.

[3] Dans ce texte, le mot « penser » prend la signification des choix faits par l'enseignant pour organiser sa séquence d'enseignement, choix qui guident, d'une certaine manière, ses prises de décision.

nous confirme certaines caractéristiques de la pratique d'enseignement de Maurice, notamment celles de structurer les démarches des élèves et de faire des mises en garde.

ACTIVITÉ MATHÉMATIQUE INDUITE CHEZ LES ÉLÈVES

Pour aborder cette question de l'activité mathématique induite chez les élèves, une analyse didactique sera conduite, qui emprunte à certains concepts de la théorie des situations didactiques de Brousseau. Nous reviendrons donc sur la pratique d'enseignement, mais cette fois à la lumière des concepts de dévolution, d'institutionnalisation et de contrat didactique issus de la théorie des situations didactiques (Brousseau, 1998).

- *Le processus de dévolution* : Nous avons pu observer que dans une seule séance, parmi les sept observées (la 4e séance), on retrouve une certaine interaction entre l'enseignant et les élèves. Mais même avec une certaine interaction, il n'y a pas pour autant dévolution des problèmes proposés aux élèves, la résolution des problèmes étant entièrement prise en charge par l'enseignant. Dans les autres séances, il n'y a jamais de dévolution de problèmes, ou de tâches faites par les élèves. Leur activité consiste à mettre en pratique *a posteriori* un certain savoir enseigné précédemment, lors des exercices, des problèmes donnés en classe sur cette matière. En classe, les problèmes sont présentés et résolus collectivement par Maurice, les questions posées aux élèves restent au niveau de la résolution de calculs sans qu'un engagement dans la tâche soit sollicité. La résolution reste donc sous le contrôle de l'enseignant qui propose le problème, oriente la résolution et valide la réponse.
- *Le processus d'institutionnalisation* : Le mode de fonctionnement en classe témoigne d'une institutionnalisation constante du savoir qui vient de l'enseignant et qui est faite en début de séance. Le savoir, codifié en lien avec les exigences du programme d'études, est présenté par l'enseignant aux élèves selon une certaine progression déterminée par lui (introduction de la notion de rapport, de taux, de la notion de proportion, de ses propriétés…des outils, table de valeurs, graphiques, propriétés des proportions, permettant de contrôler si une situation est proportionnelle ou non…). On ne peut donc pas dire dans ce cas qu'il y ait passage des connaissances des élèves à un certain savoir institutionnel, caractéristique du processus d'institutionnalisation. Ce que l'on observe au cours des leçons est davantage de l'ordre d'une mise en application par les élèves d'un certain savoir institutionnel introduit au préalable par l'enseignant. Dans certains cas, une interaction apparaît avec les élèves, suite à un travail des élèves sur des exercices ou des problèmes en classe.
- *Le contrat didactique* : L'analyse des séances en classe a révélé peu d'interactions avec les élèves. La seule séance dans laquelle des interactions sont présentes est la 4e. Dans ce cas, le mode d'interaction de l'enseignant en classe est caractérisé par un pattern d'élicitation (Voigt, 1985) quand il revient sur la résolution du problème (il cherche alors à arriver à l'énoncé d'une certaine réponse/solution attendue). C'est l'enseignant aussi qui présente le savoir, qui statue sur les erreurs, et qui énonce des manières de faire (sous forme de marches à suivre). Le mode de fonctionnement en classe prend ainsi plutôt la forme d'un « style institutionnalisant », pour reprendre ici l'appellation de Sarrazy et Roiné (2007). Dans ce style d'enseignement, c'est l'enseignant qui introduit le contenu, l'explique, qui résout, qui justifie, qui valide. L'enseignant prend une place très importante, pour ne pas dire toute la place dans cette négociation des mathématiques. Une certaine culture de classe en mathématique émerge donc de cette analyse : la pratique mathématique, les manières de faire et d'argumenter sont en quelque sorte initiées, structurées, contrôlées par le discours de l'enseignant. Ces manières de faire sont entre autres associées, nous l'avons vu, à des marches à suivre (on approche le problème d'une certaine façon, on l'explique d'une certaine façon).

À partir des résultats du test écrit final des élèves, nous avons pu noter que ceux-ci s'approprient les attentes du professeur. Après enseignement, on observe que les élèves vont majoritairement utiliser une manière de traiter les problèmes proportionnels, passant par l'énoncé à l'écrit d'une proportion suivie d'une résolution de celle-ci (le produit croisé comme procédure de résolution apparaît souvent). Or, cette manière de traiter le problème a été la marche à suivre privilégiée lors de l'enseignement. Nous pouvons remarquer aussi la présence d'une écriture des unités (notation) dans les résolutions des élèves. Même si cette notation était déjà présente chez les élèves avant enseignement, elle l'est toujours après enseignement. Nous avons pu noter toutefois que le mode d'écriture a changé. Au départ, les élèves présentaient plutôt une rédaction de la démarche en mots qui était liée au contexte du problème. Après enseignement, elle devient davantage abrégée et plus proche de celle utilisée en classe (associée à une écriture sous forme de proportion). Ce changement chez les élèves nous montre que l'attente explicitée par Maurice sur la manière d'écrire la proportion a été intégrée par les élèves. Le recours systématique à une telle écriture, même dans les cas où elle n'est pas pertinente, illustre le peu de contrôle qu'exercent les élèves sur la proportionnalité.

EN GUISE DE CONCLUSION

À partir des entrevues faites avant enseignement, nous avons pu observer les éléments qui guident l'enseignant dans la construction de la séquence sur la proportionnalité. Pour Maurice, c'est avant tout ce qui est enseigné qui vient donner une certaine forme à cette séquence. Le savoir mathématique de référence, conceptuellement analysé par cet enseignant, prend une place importante au sein de sa planification. Cette dernière est complètement pensée à l'avance par l'enseignant et repose sur une certaine analyse conceptuelle préalable qu'il explicite dans l'entrevue : la progression des contenus est pensée, il cerne les préalables, il anticipe les difficultés des élèves, il explicite des raisonnements clés… Pour Maurice, les élèves apprennent à travers une réutilisation constante des savoirs développés en classe, d'où l'importance attribuée à l'idée de progression d'une étape à l'autre dans son enseignement. Les savoirs donnent donc ici leurs formes aux pratiques d'enseignement et d'apprentissage, ces dernières étant abordées par les contenus. Les problèmes occupent une place importante dans cette progression, à travers ces notes de cours et la pratique en classe. Ces problèmes sont présentés d'une manière graduée et viennent appuyer cette progression du savoir.

Cohérente avec sa façon de penser *a priori* la progression du savoir, l'action en classe de Maurice fait ressortir les mêmes étapes, le même enchaînement prenant forme autour des contenus. L'enseignant prend une place très importante dans ces séances en classe, alors qu'il expose le savoir, propose des marches à suivre, fait des mises en garde. Cette exposition du savoir est donc en quelque sorte contrôlée par l'enseignant qui structure et organise les démarches des élèves à travers les marches à suivre induites dans l'action (et dans les notes de cours). Une certaine pratique mathématique se développe en classe autour de ces contenus de manière à approcher ceux-ci, à éviter les erreurs, c'est-à-dire une pratique destinée pour l'enseignant à éviter que les élèves se mélangent. Cette pratique a comme conséquence un faible niveau d'interaction entre l'enseignant et les élèves. Dans cette pratique en classe, Maurice manifeste toutefois le souci d'attribuer un certain sens au contenu, et cela en introduisant des exemples et en les verbalisant en contexte.

Dans la production écrite des élèves, on retrouve une appropriation de ce travail privilégié par l'enseignant, par le biais du recours à une procédure et à une notation qui n'étaient pas présentes avant enseignement. Le recours non-contrôlé par l'élève de ces outils conduit à une certaine perte du sens attribué à la proportionnalité, perte de sens que l'on observe dans la résolution de problèmes inversement proportionnels et le jugement porté sur des situations non-proportionnelles. L'utilisation non appropriée d'une marche à suivre (écriture de la proportion, utilisation des propriétés) conduit à un non-contrôle dès que l'on sort des problèmes usuels de proportion.

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LE DÉVELOPPEMENT D'UNE PENSÉE CRITIQUE ENVERS LES JEUX DE HASARD ET D'ARGENT PAR L'ENSEIGNEMENT DES PROBABILITÉS À L'ÉCOLE PRIMAIRE : VERS UNE PRISE DE DÉCISION

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Cette thèse doctorale étudie de quelle manière un apprentissage des probabilités qui contribue au développement d'une pensée critique envers les jeux de hasard et d'argent peut influencer une prise de décision envers une éventuelle participation à ces activités. Elle s'intéresse donc à la mobilisation des ressources mathématiques dans une perspective de citoyenneté. Puisque le gambling chez les jeunes est un phénomène mondial en pleine expansion, il importe de les outiller dans un but préventif, et ce, dès l'école primaire. Pour ce faire, six situations d'apprentissages ont été proposées dans une classe de quatrième année du primaire. La chercheure, qui était également l'enseignante de cette classe, a en outre proposé des activités permettant de circonscrire les jeux de hasard et d'argent. Les résultats suggèrent que les contextes mathématique, socioculturel et personnel issus de notre modélisation ethnomathématique déterminent la perspective dans laquelle se situent les apprenants.

This dissertation studies how learning about probability, that contributes to the development of critical thinking about gambling in children, can influence decision making toward eventual participation in these activities. It therefore focuses on mathematical mobilization of resources in a citizenship approach. Because teenage gambling is prevalent worldwide, it is important to provide them with tools for gambling prevention as early as elementary school. For this study, six learning situations were proposed in a fourth grade classroom. The researcher, who was also the teacher, proposed some extra activities in order to define gambling. Results suggested that mathematical, sociocultural, and personal contexts were the determinants of which perspective the students were situated in.

UNE PROBLÉMATIQUE SOCIALE

Les jeux de hasard et d'argent font partie de la culture populaire de nombreuses sociétés (Korn & Shaffer, 1999). Avec le développement des moyens de communication, ceux-ci connaissent une vague de popularité à l'échelle mondiale, portés par le succès des tournois de poker télévisés, des casinos en ligne et des loteries interactives (Chevalier & Allard, 2001). Pour certains, ces activités mènent à une dépendance dont il est difficile de se défaire et dont les conséquences sont néfastes tant pour le joueur que son entourage (Ladouceur, 2000). Le cas des jeunes, qui sont aussi sensibles à la popularité de ces jeux, mérite une attention particulière. Certains pays, dont le Canada, interdisent ces jeux aux mineurs (Chevalier & Allard, 2001). Cependant, il semble que les jeunes puissent tout de même y jouer. Chez les jeunes, la participation à des jeux de hasard et d'argent débute avant l'âge de douze ans (Griffiths & Wood, 2000; Gupta & Derevensky, 1998; Ladouceur, Dubé, & Bujold, 1994; Tremblay, Huffman, & Drabman, 1998; Wynne, Smith, & Jacobs, 1996). À cet effet, une importante étude de Ladouceur et al. (1994) réalisée dans la région de Québec auprès de 1 320 élèves de niveau primaire de 8 à 12 ans, révèle que 86 % des élèves ont déjà gagé de l'argent et que 37 % d'entre eux ont misé un objet important pour eux. Certains enfants auraient même gagé de fortes sommes pour leur âge, alors que d'autres, plus de 40 %, jouent au moins une fois par semaine. Parmi ces enfants, les loteries étaient les activités les plus populaires, suivies du bingo, des jeux de cartes, des paris sportifs, des paris sur des évènements spécifiques, des machines vidéos poker ou des machines à sous et d'autres jeux mêlant l'adresse et le hasard.

Les travaux de Felsher, Derevensky, et Gupta (2003) qui portent sur les loteries révèlent que l'âge moyen des enfants jouant aux loteries instantanées à gratter est de 10 ans, tandis qu'il est de 11 ans pour les billets de tirage et de 12 ans pour les billets de loterie sportive. Nonobstant leur degré d'implication dans les activités de gambling, cette dernière est la loterie la plus populaire parmi les jeunes de tous les âges (Felsher et al., 2003). Les loteries forment une première introduction aux jeux de hasard et d'argent. Puisque les jeunes se perçoivent invulnérables et que les risques associés à la loterie sont perçus comme négligeables, leur popularité, alimentée par la promotion et la publicité, est très forte.

Puisque les jeunes ne possèdent pas les connaissances nécessaires pour évaluer les probabilités de gagner et les risques de dépendance associés au jeu, il revient à l'école de jouer son rôle et de les outiller. Le moment préventif doit s'instaurer dès l'école primaire (Crites, 2003). Il ne suffit toutefois pas de leur faire acquérir des connaissances : un mouvement préventif à l'égard des jeux de hasard et d'argent doit s'appuyer sur le développement de compétences mathématiques et de compétences citoyennes comme la pensée critique et la prise de décision.

LE DÉVELOPPEMENT DE COMPÉTENCES CITOYENNES COMME MOYEN PRÉVENTIF

Legendre (2004) nous rappelle qu'une compétence est complexe, car elle est le produit d'une organisation dynamique de ses composantes. La mobilisation des ressources d'une personne témoigne davantage d'une compétence que d'une accumulation de connaissances. C'est par l'orchestration et l'utilisation de ses ressources qu'une personne peut manifester sa compétence. À cet effet, ten Dam et Volman (2004) nous informent qu'une compétence citoyenne vise à ce que chaque personne, membre d'une communauté de pratiques sociales, participe de façon critique et responsable à la pratique en question. Chaque membre d'une société participe démocratiquement à son évolution. Il est ici question, entre autres, de faire des choix et savoir pourquoi ces choix sont faits, respecter les choix et les opinions d'autrui, en discuter, former sa propre opinion et la faire connaître (Halpern, 2003; Paul & Elder, 2001; Swartz & Perkins, 1990). La pensée critique est intrinsèquement liée au processus de prise de décision, notamment par l'utilisation de critères pour former son jugement, par l'emploi d'une réflexion métacognitive du processus de pensée ainsi qu'une sensibilité au contexte (Guilbert, 1990; Lipman, 2003; Paul & Elder, 2001). Une des composantes de la pensée critique est l'utilisation de connaissances générales et disciplinaires, telles les mathématiques. Ces connaissances sont donc mobilisées lors de prises de décisions éclairées et elles contribuent par le fait même au développement identitaire et citoyen de la personne.

UNE ARTICULATION ETHNOMATHÉMATIQUE POUR L'ÉTUDE DES STRUCTURES PROBABILISTES

Dans le cadre de notre projet de recherche doctorale, un modèle ethnomathématique inspiré par Mukhopadhyay et Greer (2001), a été développé afin de construire des situations d'apprentissage des structures probabilistes (Vergnaud, 1990). Ce modèle présente trois contextes distincts : un contexte socioculturel, un contexte citoyen et un contexte mathématique. Le contexte socioculturel constitue le point de départ de l'apprentissage par l'étude d'un objet. Cette étude peut conduire à chercher des réponses du côté des mathématiques. Dans ce cas, le cas, le contexte mathématique propose une décontextualisation de l'objet par une modélisation mathématique. Les résultats dégagés sont ensuite repris en contexte initial, afin d'étudier les implications de ses résultats sur l'objet. Les apprentissages mathématiques réalisés peuvent ainsi servir d'assises à une pensée critique ou bien à une prise de décision en contexte citoyen. Ce modèle vise donc à prendre en compte la complexité d'un objet ou d'un phénomène afin de donner sens aux apprentissages en cours de réalisation. En ce sens, il peut également être perçu comme un outil théorique pour la mise en place d'approches interdisciplinaires.

D'une part, l'apprentissage des structures probabilistes a été étudié sous l'angle de l'aléatoire. À cet effet, les travaux de Piaget et Inhelder (1974), Green (1988), Fischbein, Sainati Nello, & Sciolis Marino (1991), Watson et Kelly (2004), Falk et Wilkening (1998) ainsi que ceux de Jones, Thornton, Langrall, et Tarr (1999) se sont attardés à étudier la compréhension que les enfants ont de certains phénomènes aléatoires. D'autre part, des recherches ont porté sur l'enseignement des probabilités au primaire (Brousseau, Brousseau, & Warfield, 2002; Medici & Vighi, 1996). En fait, les structures probabilistes se déclinent sous trois facettes : les probabilités théoriques, les probabilités fréquentielles et les probabilités subjectives (Caron, 2004; Konold, 1991). Certains de ces travaux, entre autres, ont permis d'orienter la recherche vers les conceptions probabilistes des élèves. Ainsi, les travaux de Fischbein et Schnarch (1997), Konold (1991), Brousseau (2005), Shaughessy (1992), Amir et Williams (1999), et de Kahneman et Tversky (1972) ont permis d'identifier des conceptions des élèves. Ces conceptions sont en fait des systèmes explicatifs que les élèves se donnent afin d'expliquer un résultat. Par exemple, des élèves vont expliquer le résultat d'un lancer d'un dé par la façon dont il a été lancé (Amir & Williams, 1999). Nous avons classifié ces conceptions selon le type de raisonnement employé : probabiliste, déterministe ou affectif (Savard, 2008). La complexification conceptuelle (Larochelle & Désautels, 1992) de ces conceptions permet alors aux élèves de poursuivre leur apprentissage.

Dans le cadre de ce projet, nous avons cherché à utiliser les mathématiques en réponse à un problème social, le gambling chez les jeunes. Nous avons cherché à répondre à cette question : comment un enseignement qui favorise l'apprentissage des probabilités contribuant au développement d'une pensée critique peut-il influencer la prise de décision d'une éventuelle participation à des jeux de hasard et d'argent chez les élèves rencontrés? Cette question comporte quatre sous questions de recherche:

1. Comment se développe la compréhension des structures probabilistes?
2. Comment se complexifient les conceptions probabilistes, déterministes et affectives?
3. Quel est l'apport de cet apprentissage sur le développement d'une pensée critique à l'égard des jeux de hasard et d'argent?
4. Quels sont les arguments déployés lors d'une prise de décision face à une éventuelle participation à un jeu de hasard et d'argent?

UNE EXPÉRIMENTATION DIDACTIQUE AUPRÈS D'ÉLÈVES DU PRIMAIRE

Nous nous sommes inspirées du modèle ethnomathématique pour élaborer un modèle qui tienne compte des contextes socioculturel, mathématique et citoyen. Ce modèle théorique a servi de base pour construire des situations d'apprentissages faisant intervenir des situations de jeu, un développement mathématique et un développement citoyen. Une séquence d'enseignement composée de six situations d'apprentissage a été proposée à 27 élèves d'une classe de quatrième année du primaire (9 et 10 ans). La chercheure était également l'enseignante titulaire des élèves de cette école située en banlieue de Québec. Un questionnaire a été proposé avant et après l'expérimentation didactique, laquelle s'est déroulée sur six mois. Trois vignettes proposant des situations fictives de jeu ont été proposées à la toute fin de l'expérimentation didactique. Les situations d'apprentissage ont été filmées et les propos ont été retranscrits en verbatim. Un pseudonyme a été attribué à chaque élève afin de conserver leur confidentialité.

L'encodage a été réalisé à l'aide logiciel Atlas/ti. Les données ont été interprétées à l'aide du modèle d'interprétation des activités cognitives de DeBlois (2003). Ce modèle vise à décrire la dynamique impliquée dans la compréhension d'un élève en contexte d'apprentissage scolaire. Il prend en considération les représentations de la situation par l'élève, des procédures qu'il a utilisées, ainsi que des attentes provoquées par le contrat didactique. Les coordinations entre ces composantes peuvent conduire à une compréhension partielle ou généralisable en d'autres contextes.

LES RÉPONSES SUGGÉRÉES PAR CETTE RECHERCHE

En réponse à la première sous question de recherche qui cherchait à savoir comment se développe la compréhension des structures probabilistes, les résultats obtenus nous indiquent que c'est par l'emploi d'un vocabulaire comme certain, possible et impossible, pour ensuite discuter de l'équiprobabilité en se basant sur la forme du dé, puis par le dénombrement du nombre de cas possibles ce qui favorise le passage du qualitatif au quantitatif et par finalement par la construction du rapport du nombre de cas favorables sur le nombre de cas possibles. C'est également par la comparaison entre les fréquences relatives obtenues lors d'une expérimentation, puis par la prise en compte de la variabilité des résultats, pour ensuite comparer les fréquences obtenues et les probabilités théoriques. C'est par le traitement des informations issues du contexte socioculturel, ensuite par l'emploi de probabilités théoriques lors de l'expérimentation (valider les informations) qui a permis la construction de l'aléatoire. En fait, c'est par le passage par les élèves d'un contexte socioculturel à un contexte mathématique puis un contexte personnel.

En réponse à la deuxième sous question de recherche qui s'intéressait à la complexification des conceptions probabilistes, déterministes et affectives, nos résultats ont montré que c'est par la prise en compte de l'aléatoire, de la variabilité des résultats et de l'indépendance entre les tours que les conceptions se complexifient. Cette prise en compte signifie en fait l'abandon d'un raisonnement déterministe au profit de l'utilisation d'un raisonnement probabiliste. Cet abandon s'effectue par la confrontation du domaine de validité des conceptions par une pensée critique.

En réponse à la troisième sous question de recherche portant sur l'apport de cet apprentissage sur le développement d'une pensée critique à l'égard des jeux de hasard et d'argent, nos résultats ont mis en lumière que cet apprentissage a contribué au développement d'une pensée critique. En effet, l'apprentissage a permis, entre autres, d'évaluer le domaine de validité des conceptions par la discrimination entre l'adresse et le hasard pour discuter de l'illusion de contrôle et la tricherie. C'est par l'utilisation de connaissances mathématiques comme arguments, ainsi que par le questionnement envers le fonctionnement des jeux de hasard et d'argent (validité de l'objet utilisé et son équité) que la formation du jugement s'est effectuée. Nous avons aussi constaté que l'apport de cet apprentissage consistait à développer un point de vue éthique envers le gambling par la structuration des connaissances envers les jeux de hasard et d'argent, par la discussion sur l'espérance des gains, par la conscientisation envers les dépendances possibles et par l'identification des motivations à jouer.

En réponse à la quatrième sous question de recherche qui a étudié les arguments déployés lors d'une prise de décision face à une éventuelle participation à un jeu de hasard et d'argent, les résultats obtenus nous montrent que les arguments reposent sur l'affectivité manifestée envers la situation de jeu, ou bien sur l'aspect éthique de l'activité proposée, ou les conceptions affectives à l'égard du hasard ou finalement sur l'aspect mathématique de la situation.

Les réponses amenées par ces sous questions contribuent à répondre à notre question de recherche qui cherchait à savoir comment un enseignement qui favorise l'apprentissage des probabilités contribuant au développement d'une pensée critique peut influencer la prise de décision d'une éventuelle participation à des jeux de hasard et d'argent. En effet, l'influence de cet apprentissage s'est manifestée par le développement d'un raisonnement probabiliste, par une complexification des conceptions affectives, déterministes et probabilistes envers le hasard, par le développement d'une pensée critique et par la prise en compte des éléments du contexte socioculturel proposé pour se positionner.

CONCLUSION

Cette recherche a voulu mettre en lumière la construction et l'utilisation de connaissances mathématiques et leur mobilisation en contexte citoyen. Les résultats suggèrent que les contextes mathématique, socioculturel et personnel issus de notre modélisation ethnomathématique déterminent la perspective dans laquelle se situent les apprenants. Ainsi, lors d'une prise de décision en contexte socioculturel, il s'agit de se situer en contexte mathématique pour générer et évaluer des alternatives à l'aide des probabilités, ce qui permet d'utiliser ou de construire des savoirs mathématiques pour prendre une décision en contexte citoyen. D'autres recherches sont toutefois nécessaires afin d'étudier l'applicabilité de ces contextes en d'autres situations de la vie courante.

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Ad Hoc Sessions

Séances ad hoc

PROFESSIONAL LEARNING OF BEGINNING MATHEMATICS TEACHERS USING LESSON STUDY

Paul Betts, *University of Winnipeg*
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Designing professional development programs for mathematics teachers is a serious challenge (Hill & Ball, 2004), especially for beginning teachers (Smith & Ingersoll, 2004). Some qualities of effective teacher professional development design are starting to emerge from the literature: implementation within a collaborative environment (Rogers et al., 2007), locating the professional development within the classroom contexts of the teachers (Cobb et al., 2003), providing sustained support over time (Desimone et al., 2002), and incorporating time for professional reflection within the program (LaChance & Confrey, 2003). Lesson study reflects these qualities, in a mathematical context, because it involves a network of teachers working together to develop, implement and reflect on lessons (Taylor, Ferrell, & Hopkins, 2007). Originally developed in Japan, lesson study has been found to be an effective professional development approach within North America (Putnam & Borko, 2000).

We shared our experiences in an action research project exploring the professional learning of first-year middle-years mathematics teachers participating in a lesson study project. Following principles of collaborative narrative inquiry, the participating teachers were regularly invited to work with themes generated by the researcher. At the time of this ad hoc, we still faced the challenge of representing the professional learning experiences of these teachers.

Our goal was to invite others to engage with selections from our data. We provided sets of quotes drawn from the beginning, middle, and end of the year-long project, which featured discussions by teachers of the potential effectiveness of math problems they were considering. These quotes suggested potential trajectories of professional learning for four teachers.

In the data we presented, a repeated teacher concern involved the length of time that a single problem would take students. For example, one quote that generated considerable discussion was:

> *The other thing that is good is that they [the 5 pizza-theme problems] are quick. Kids aren't going to be spending 20 minutes, half an hour on a single problem. They spend a couple of minutes, couple of minutes, couple of minutes.*

Ad Hoc participants conjectured about the reasons the teachers might have for wanting to use problems that would yield quickly to students. Could it be because of the difficulties of implementing inquiry-based problems? Have management issues limited potential growth in their instructional approach? Have they found a positioning that navigates the tension between meeting content goals and engaging students in rich learning processes? The Ad Hoc participants' comments led us to realize that these teachers were prioritizing their time management so that they could all, as one teacher expressed it, "*stay on the same page*." These teachers valued the supportive conversations with each other that were only possible when they were all doing the same problems in their classroom.

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THE ART OF EXPLORATION

Sean Chorney
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The activity of exploration, although it has obvious manifestations in the "real" world, is not such a common activity in a school environment. So it was, that "for this occasion", we discussed the idea of what exploration in a classroom environment might look like – and what it might accomplish. The evolution, however (for I felt obliged to tell), of this idea had earlier roots in problem posing.

Some questions arose:

- Was the idea of implementing student questioning, or problem posing, simply educational rhetoric?
- What empirical evidence have we seen since Brown and Walter's (1983) seminal work, The Art of Problem Posing?
- The idea of students being more involved in their responsibilities in the mathematics classroom has an appeal, for if students engage in a mathematical topic, and have the opportunity to question what the relationships or the connections are, then it seems to make sense that the students are more involved, more interested, more willing to take risks, conjecture and be like mathematicians. If this is true, which seems to be an accepted thesis from many researchers (Abramovich & Cho, 2008; Silver, 1994), how does one engage the students in such an activity?

Hawkins (2001) says that the process of problem posing is much more "organic" and that things have to develop into problems out of exploration. He *"…makes it plain that goals in problem solving cannot be imposed by external input, but must be evolved, fabricated, set out of the antecedent, ongoing activity of the learner*" (p. 116).

What can a teacher do to engender exploration? Does exploration require an object? Is it necessary to be able to move, reorient, squeeze, etc. an object, and was this something Papert (1993) was on about with his "objects-to-think-with"? And does context satisfy this apparent need of an object? A lot of questions, a lot of contemplation. It is indeed a good starting point and there is still a lot to explore.

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REPRESENTATIONS OF MATHEMATICS AND MATHEMATICIANS IN MAGAZINE ADVERTISEMENTS

Jennifer Hall
University of Ottawa

The discussions in this session were based on my recent research, which involved investigating how advertisements in mainstream magazines represented mathematics and mathematicians. I selected a year's worth of issues for six monthly magazines that had a variety of target audiences: *Parents* (mothers with young children), *Popular Science* (men interested in science and technology), *Report on Business* (male business executives), *Runner's World* (male and female 'serious' runners), *Seventeen* (pre-teen and teenage girls), and *Vogue* (women interested in haute couture). Then, I looked through every page of each magazine, searching for advertisements that featured representations of mathematics or mathematicians. I only selected advertisements that I perceived had either an explicit or implicit message about mathematics or mathematicians. This method of selection resulted in only 12 relevant advertisements: four in *Report on Business* (two for banking/financial services, two for luxury cars), five in *Popular Science* (four for mathematics DVDs, one for a computer processor), one in *Parents* (building blocks), one in *Runner's World* (banking), and one in *Seventeen* (deodorant). There were no relevant advertisements in *Vogue*.

In the session, I first gave a brief oral description of each advertisement and posted a copy of each advertisement on the chalkboard. The participants were invited to come up to the chalkboard to conduct their own analyses of what representations of mathematics and mathematicians they thought each advertisement was making. We had a productive group discussion about the participants' interpretations of the advertisements. In some cases, there were contradictory interpretations made by participants about an advertisement, such as the Ban® deodorant advertisement from *Seventeen*, which featured a picture of a young woman in front of a chalkboard filled with mathematical symbols coupled with the phrase 'Ban stress'. Some participants interpreted this as meaning math was stressful, whereas others thought it meant that math was a challenge that could be overcome. Generally, the participants raised many of the same issues about the advertisements that I found in my own analysis, such as a lack of people in the advertisements, particularly women; inappropriate usage of mathematical symbols; and the predominance of negative, stereotypical messages about mathematics/mathematicians. In most cases, mathematics was portrayed as a difficult, boring, and stressful subject that was not relevant to people's day-to-day lives and lacked human interaction. Furthermore, the lack of advertisements that featured any sort of representation of mathematics/mathematicians speaks to the invisibility and marginality of the field in the general society.

Although magazine advertisements may be viewed as a more marginal type of documentary source, the representations therein align with those in other media sources, such as television shows and movies. Furthermore, the fact that even a 'marginal' documentary source presents such negative, stereotyped images of mathematics/mathematicians highlights the pervasive nature of these views in popular culture. This raises the question of how we, as educators, may counteract the deeply entrenched negative views of mathematics in society and help our students to garner positive attitudes toward the subject area.

MATHEMATICS FOR TEACHING AND LEARNING

Ann Kajander
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The notion of pedagogical content knowledge has been further refined recently to include a body of mathematical knowledge specific to teaching (Ball, Thames, & Phelps, 2008). Identifying mathematical errors in student work, analyzing models and using them to justify the development of generalisations, and supporting the making of connections within mathematics requires a specialised understanding on the part of the teacher, and addressing such content during teacher preparation has shown a measureable increase in teacher capacity (Kajander, in revision).

At the *Canadian Mathematics Education Forum*, a working group on the topic of elementary mathematics for teaching examined aspects of teachers' knowledge (Kajander & Jarvis, accepted), and crafted a Policy Statement outlining recommendations for teacher content-knowledge preparation in Canada (see Appendix), which was circulated during the current session. One area of concern raised in the discussion was the question of suitable faculty backgrounds for teaching mathematics-for-teaching courses to preservice candidates. Currently such courses, if offered at all, are housed within mathematics faculties. Arguing for specialised content implied that faculty involved with such courses need a sensitivity to and awareness of the specialised nature of such content and how it might best be approached. Additionally, a one-page policy is, by its very nature, brief, and cannot address all of the issues. It is hoped that the Policy will be useful to mathematics educators, professional mathematics organisations, school board personnel and others, by encouraging further examination of the topic by policy-makers. The Policy Statement, which aligns with US recommendations (National Council on Teacher Quality, 2008) is offered as a starting point for individuals wishing to argue for more attention to specific content preparation for elementary teachers in Canada.

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APPENDIX: POLICY STATEMENT FOR CANADIAN ELEMENTARY (K-8) TEACHER MATHEMATICS CONTENT DEVELOPMENT

Teachers of elementary school mathematics believe that all of their students should succeed in mathematics, and now is the time to support such a stance. However, fundamental changes in mathematics teaching and learning place significant demands upon teacher capacity. Teachers need a specialized mathematical knowledge in order to enhance students' conceptual understanding as well as to support students' confidence in, as well as their capacity for, continuing to learn mathematics.

While it is essential that teachers have a deep, well interconnected, understanding of the concepts that students in elementary school will encounter, significant evidence indicates that there is a body of mathematical knowledge that is particular to the work of elementary teaching. Knowledge of how mathematical understanding may develop in children, as well as of the models, representations and practices that support students' mathematical development are essential. Teachers need to quickly recognize the mathematical significance of their students' thinking in order to determine and respond to misconceptions, and through appropriate questioning or provision of alternate tasks, move students forward. Appreciating and responding to alternative student approaches requires deep and flexible conceptual understanding on the part of teachers, as well as the ability to unpack students' thinking in order to recognize generalizable strategies or identify student misconceptions. Also needed is a deep and flexible understanding of operations, and the ability to make multiple connections to models, contexts, and essential mathematical activities such as reasoning and problem solving. This specialized knowledge does include highly *mathematical* components, such as the understanding of multiple models and representations, including appropriate use of manipulatives and technology, the mathematical knowledge needed to design and select classroom tasks useful in supporting student learning, as well as the mathematical insight required to make appropriate choices when anticipating and addressing student responses. Such fundamental aspects of effective elementary mathematics teaching need to be addressed prior to, or concurrent with, other teacher preparation courses.

Many elementary teacher candidates suffer from a lack of confidence and even anxiety about mathematics, often largely due to initially weak conceptual mathematical understanding. Confidence and a willingness to continue mathematical self-development can and should be supported in preservice teacher education programs. It is recommended that approximately 100 hours be provided for preservice teachers in specialized **mathematics knowledge for teaching** (for example, two 50-hour courses, or three 36-hour courses) which should *not* be replaced by more general undergraduate mathematics courses, as the content differs. Faculty members teaching these foundational courses should have specific mathematics education interests and background, which may require special attention when such courses are offered through mathematics departments. In addition, a similar amount of instruction in mathematics methods (content specific pedagogy) should be provided. These recommendations are particularly crucial for teachers of grades 5 to 8. Ideally, these recommendations should be further strengthened by appropriate teacher mathematics and mathematics education courses being taken over multiple years of an undergraduate degree program. Concerted efforts towards standardization of certification requirements of mathematical competence for teaching, as part of preservice program requirements across Canada, are highly recommended.

Both new and experienced classroom teachers should be encouraged to seek on-going professional development opportunities such as mentorships and mathematics-for-teaching courses, and such efforts should be supported and rewarded much more formally at the school, school board, and provincial levels. In particular, continuing education courses focused on mathematics-for-teaching should be provided for in-service teachers, and previous mathematical background, such as a prescribed number of general undergraduate mathematics courses, should *not* be used to preclude teachers from enrolling in such courses. The development and support of specialist mathematics teachers in grades 5 to 8 is strongly recommended.

In summary, elementary teachers require specialised mathematical knowledge for effective classroom teaching, and the development of such knowledge is a critical component of teacher competence as required to effectively support improved student learning and outcomes.

GEOGEBRA INSTITUTE OF CANADA

Zekeriya Karadag, *University of Toronto*
Oana Radu, *Memorial University*

Participants: Christian Bernèche (Simon Fraser University), Sean Chorney (Simon Fraser University), Warden Egodawatte (OISE/UT), Viktor Freiman (Université de Moncton), Shiva Gol Tabaghi (Simon Fraser University), Geoff Roulet (Queen's University).

This ad hoc session focused on the formation of the GeoGebra Institute of Canada and the possible expectations from such an institute. The meeting had two sections: (1) introducing GeoGebra and the core team of the GeoGebra Institute of Canada to the audience and (2) discussing future expectations in an inspiring environment. We must admit that the ad hoc sessions of the CMESG are a great opportunity to develop new ideas and advance existing plans.

By giving a brief demonstration of GeoGebra, the first author illustrated the relationship between the behaviour of a function and the slope of the tangent passing through a point on the function on a dynamic worksheet. The importance of visual and dynamic learning in students' conceptual understanding of mathematical concepts was emphasized.

GeoGebra was described as an online, free and dynamic learning environment supported in 46 languages. These features are seen as particularly beneficial within the Canadian context since Canada is known as the most multicultural and multilingual country in the world. Consequently, promoting GeoGebra and having a Canadian Institute would be particularly important.

After the first 10 minutes of introduction, the second author explained the vision and mission of the International Geogebra Institute (IGI) which includes: developing free workshop materials, organizing training workshops for teachers, furthering the GeoGebra software, designing and implementing research projects, facilitating research collaboration between various institutes, planning GeoGebra conferences and facilitating the alliance between researchers and practitioners.

The names of the individuals working on establishing the GeoGebra Institute of Canada were introduced to the audience. This team includes Dragana Martinovic (University of Windsor), Viktor Freiman (Université de Moncton), Daniel Jarvis (Nipissing University), Philipe Etchecopar (Cégep de Rimouski), Oana Radu (Memorial University) and Zekeriya Karadag (University of Toronto). The First International GeoGebra Conference was advertised as being held in Austria in July 2009.

At the end, the participants were offered the opportunity to share their thoughts on GeoGebra and their expectations from such an Institute. Viktor and Geoff provided deeply inspiring suggestions for the future by pointing out explorative features of the GeoGebra environment.

THE ROLE OF ROTE: WHAT IS MEANINGFUL PRACTICE?

Wes Maciejewski
Queen's University

As a new instructor who wants to “get it right”, I often feel overwhelmed by the education research literature, advice of colleagues, expectations of students, et cetera. I want my classroom practices to rest on a solid research-based foundation; certainly, I want what is best for my students. I have always maintained this desire, and I was initially drawn to the study of education by this. I did not, however, expect what I did find when I arrived into the field. The “Math Wars” were—indeed, are—in full swing in parts of North America, a battle I was largely ignorant of as being fought so vehemently. It is not my intention to bring up this tired, old war. I do, however, feel that one aspect of learning mathematics, practice—here taken to mean “pencil and paper” drills—has been stigmatized heavily in the crossfire. Without being properly explored, the notion of practice and its role in learning mathematics is now used to distinguish which side of the war you fight for: those who favour routine practice and emphasis on technical ability are deemed “traditionalist”; those in favour of discovery methods employing group interactions and explorations, “reformists”.

My naive, outsider's perspective is that this polarization is unfortunate, for both the educators and, perhaps more importantly, the students. Unfortunate even more so since there are arguments in the literature that support both sides of the debate as necessary. Two quotes which, when contrasted, exemplify the contradictions mentioned here are:

> *Nothing flies more in the face of the last 20 years of research than the assertion that practice is bad. All evidence, from the laboratory and from extensive case studies of professionals, indicates that real competence only comes with extensive practice. In denying the critical role of practice one is denying children the very thing they need to achieve real competence (Anderson, Reder, & Simon, 1995),*

and,

> *[t]he really effective teachers, of course, have always known—at least since Socrates—that examples, drill, and overt reinforcement are quite effective in producing a desired behavior; but precisely because they were good teachers they also knew that generating understanding was a worthier educational objective than merely modifying behavior (von Glasersfeld & Steffe, p. 95).*

It should be noted that the term “practice” in the first quote and “understanding” in the second are not defined explicitly. This may be at the heart of the debate. As Star (2005) writes, “*...these terms* [procedural and conceptual knowledge] *suffer from an entanglement of knowledge type and knowledge quality*” (p. 408). This entanglement, I believe, is the cause of the divide.

What or whom is a new instructor to believe? The camp with the louder weapons? Can't both sides be right, end this war and stem the flow of unnecessary student casualties?

It was the intention of this ad hoc session to explore the pragmatic implementation of practice in mathematics instruction. I offered this session with the expectation that, upon its completion, I would have an understanding of what practice I should expose my students to.

This, of course, did not happen. Far more—or perhaps less, since we hardly left the discussion of what practice *is*—happened and I am that much the better for it.

The session was well attended by a variety of CMESG participants—Ph.D. students, mathematicians, educational researchers—which made for a lively discussion. No general consensus emerged on any of the topics discussed and it became clear at the end of the session that more discourse is required. Getting a handle on what good practice is and how it should be implemented may have profound implications for math learners and educators.

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MATH FOR ELEMENTARY TEACHERS: WHAT DO WE/SHOULD WE TEACH IN "CONTENT" COURSES?

Susan Oesterle
Simon Fraser University

The theme of this ad hoc session was the preparation of elementary school mathematics teachers. This topic is of interest to me both as an instructor of a mathematics content course for prospective elementary teachers, and also as a graduate student, as the inherent tensions in teaching this course have become the focus of my dissertation. In the session I described the current situation in the province of British Columbia, and invited participants to share their approaches and priorities for teaching these courses at their own institutions.

In BC, the College of Teachers requires elementary teachers to have only one post-secondary mathematics course for certification. Most institutions offer a mathematics content course within their Mathematics departments, specifically aimed at preservice elementary teachers. Future teachers may also opt to take a mathematics methods course from Education departments during their certification programme, though this is not required. There are many who feel this is an inadequate level of preparation. As well as making recommendations to government for increasing these requirements, the BCcupms (a committee of representatives from all Mathematics departments, from all provincial post-secondary institutions) has recently struck a subcommittee in order to formulate guidelines for those who teach the mathematics content courses (see http://bccupms.ca/mfee.shtml). The hope is that these guidelines will help instructors of these courses make the most of the very limited time they have available.

As a member of this subcommittee I was party to many interesting discussions on what is done, and what should be done in the specialised mathematics content courses for prospective elementary teachers. Typically the content covered in these courses includes the fundamentals of number and operations, including fractions, along with basic geometry and a substantial amount of problem solving. However there is considerable variability between these courses as offered by different institutions and different instructors. The lack of sufficient time to cover all of the topics in the elementary curriculum, the typically weak arithmetic skills of the students, and their negative attitudes towards mathematics in general, all pose challenges for the instructors of these courses.

In this ad hoc session, I invited the group to consider the following questions: Given that we don't have enough time to address all of our students' needs, where should our priorities lie? What is the most important thing we can give them in a "content" course? What assumptions or expectations underlie your choice? Discussion was lively, with participants describing the requirements for teacher preparation in their provinces and sharing their thoughts on priorities. Most spoke in favour of addressing affective issues as a priority, including building confidence and creating an interest in mathematics, in order to encourage students to continue their professional development as mathematics teachers beyond the course. In this same vein, there were suggestions that students be provided with memorable experiences, including activities that allow them to confront cognitive obstacles, reconstruct mathematics themselves, or shed light on children's thinking. Unfortunately, given the short amount of time we had, we were unable to do more than scratch the surface of the issues around teaching these mathematics content courses.

INTERDISCIPLINARY DOCTORAL WORK: A JOINT EXPRESSION OF MATHEMATICS AND EDUCATION RESEARCH

Tina Rapke
University of Calgary

This ad hoc session discussed the interdisciplinary nature of my research. I am currently engaged in doctoral work that deals with the relationship between learning and doing/creating mathematics, and with the attendant pedagogical implications for the classroom. Given the nature of the research, it is imperative that I consider and engage in both original and significant mathematical and educational research. The work will tell the story of my journey of learning and creating mathematics in the field of graph theory. My research will not be presented as the sanitized version of mathematical solutions that one might find in text books or journals. All of the dead ends, wrong turns and frustrations will be included. It will borrow its style from Lakatos' book "Proofs and Refutations" (1976), taking the form of a fictionalized dialogue between a student and supervisor. I am pursuing the following research questions: How could a mathematician's learning/doing contribute to understanding students' learning of math? What does it mean to learn math? What does it mean to do math?

I plan to respond to the research questions by employing an interpretive duoethnographic methodology (Norris, in press; Sawyer & Norris, 2005). Typically, duoethnography exposes two perspectives on a phenomenon by having co-researchers/co-authors present their own perspectives which, through the process of analysis and writing, may merge or blend. In a typical duoethnography, these two voices would be provided, and articulated orally or in print, by two individuals. As the research described here is a sole authored work, the two voices will be articulated by a single author but drawn from multiple sources (and experiences), including the dissertation supervisors. The dissertation will involve an original and unique adaptation of the duoethnography methodology that introduces the use of fictional dialogue as a teaching and learning device. Furthermore, fictionalizing allows me to open up, rearticulate, challenge, and reconstruct my experiences in order not simply to re-tell the complex process of creating new mathematics, but to expose and interpret the pedagogical heart of learning/doing/teaching mathematics.

The conversation in the ad hoc session went on to explore what other graduate students are doing at other universities (including York University). Many interesting references were shared, including some dealing with ethnodrama. Overall, we talked about what a dissertation that would be considered both mathematical and educational might entail.

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ONLINE ENVIRONMENTS FOR COLLABORATIVE MATHEMATICAL EXPLORATION

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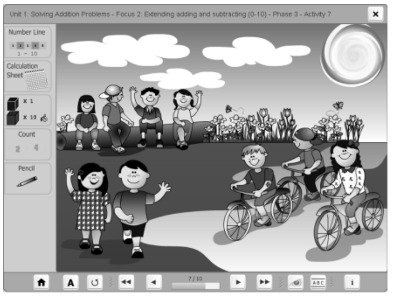
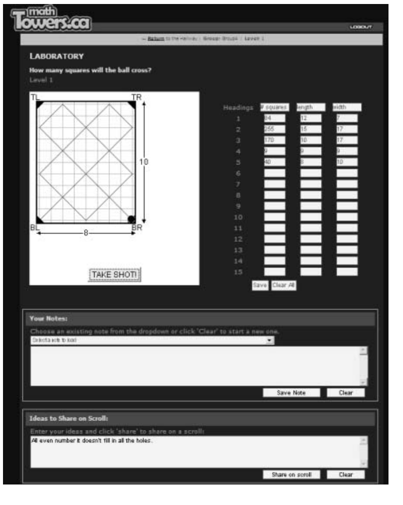
The value of student-to-student interaction in mathematics learning has been reported in research publications over the past three decades (Miller & Brownell, 1975), and present professional- (NCTM, 2000) and government- (Bruce, 2007) led mathematics education reforms emphasize peer interactions and the ensuing mathematical talk. At Queen's University we have developed and mounted two online resources designed to support collaborative mathematical investigations and math-talk: *Web-based Mathematics Activities for the Early Years* (http://resources.educ.queensu.ca/early_years_math/) and *Math-Towers* (www.math-towers.ca).



Our Early Years resource has been developed in collaboration with the Instituto de Informática Educativa (IIE) at the Universidad de La Frontera, Temuco, Chile and is an English translation and Canadian adaptation of their *Unidades de Aprendizaje con apoyo TIC para Matemáticas*. Colourful graphics present objects which can be moved around, grouped, and counted. While solving problems, children can employ tools such as a number line, hundreds chart, counting blocks, and a pencil for marking off items. When displayed on a large screen or, better on an interactive whiteboard, groups of pupils can engage in collective exploration and mathematical conversations. While the website presents suggestions for teachers and possible questions to solve, users are free to pose their own problems and manipulate the objects as they wish.



Math-Towers, a website designed for Grades 7 to 9, carries the interaction theme a step further, providing an online tool by which users can exchange ideas as they engage in investigations. Upon entering *Math-Towers* students are greeted by the Lord or Lady of the castle and are presented with a teacher selected mathematical challenge. Pupils are then provided with a laboratory containing interactive tools for investigating the problem, a notepad for recording their observations, and a link to the Scroll communication system. Together, class members solve aspects of the problem and climb the tower, finally returning to re-join the Lord or Lady and attempt to meet the original challenge.

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MULTIPLE DEFINITIONS OF A MATHEMATICAL CONCEPT

Angela Smart
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For my ad hoc session I intended to focus on one particular finding I came upon while conducting my master's thesis research. In particular, that was the many different definitions of the term "angle" I came across in my research, as well as some of the pedagogical implications of having multiple definitions of a mathematical concept.

I began the session by asking the participants to write down a definition of an angle with the hopes that as a group we would end up with a number of different variations. Once we shared our definitions and observed the differences between them, it was clear that the manner in which we each defined an angle depended on the context in which we were thinking about an angle. For example, one person defined an angle in a very formal mathematical context, similar to the types of definitions found in advanced mathematical textbooks. In contrast, another person provided a definition that was more similar to what would be used in an elementary school classroom.

Using this initial investigation of angle definitions as the setting for our discussion, I shared with the group some of the results I obtained when I conducted the same exercise with a class of first year university mathematics students. The results I shared made up a small component of my master's thesis research. I also shared some angle definitions I had found in different mathematical resources, such as textbooks and glossaries. Using the results of my investigation with my university class and my personal search of mathematical resources, as well as the angle definitions we had each individually written in the session, we discussed as a group the contexts that the different definitions represented. For example, some of the definitions distinctly defined an angle as a shape made up of particular properties. A number of definitions defined an angle in terms of a dynamic motion such as a turn or a rotation. I presented to the group the idea of different categories or themes of the angle definitions. In particular, I hoped to demonstrate that the angle definitions from these samples could be categorised into three different contexts, namely: dynamic angle definitions, angle as measurement definitions, and angle as shape definitions.

I ended the session by asking the group about possible pedagogical implications of these different definitions and definition context situations, such as how these differences may cause difficulties for students or for teachers. As a group, we also brainstormed some other concepts in mathematics that have similar multiple definitions and/or contexts. One example we came up with was the use of the word "line" in mathematics. Overall, this session provided a valuable opportunity to hear the thoughts of other mathematics educators in regards to these findings.

TECHNOLOGICAL PEDAGOGICAL CONTENT KNOWLEDGE IN SECONDARY MATHEMATICS EDUCATION

Dorian Stoilescu
OISE, University of Toronto

This presentation discusses the Technological Pedagogical Content Knowledge model. Based on the advantages and limits that this model has, I build a rationale for using a pedagogical model that facilitates the integration of educational technology in teaching and learning mathematics.

INTRODUCTION

Researchers argue that computers are not making a difference in teaching, rather, a difference is made by the pedagogical methods that teachers use when instructing with computer technologies (Clark, 2001). Learning is most effective when pedagogical strategies are carefully developed, with computer technologies used as mediators of learning and incorporated into the pedagogical strategy. Consequently, adequate pedagogical models are necessary in order to explicitly describe and support the use of technology. Of these, Technological Pedagogical Content Knowledge (TPACK), developed by Mishra and Koehler (2006), is one of the most well-known. TPACK is an extension of Pedagogical Content Knowledge (Shulman, 1986), defined as a systematic approach of joining technical expertise in teaching with pedagogical content knowledge.

TPACK is an emergent form resulting from the intersection of technology, pedagogy and content. This model considers the context as an important aspect. Technological aspects are:

- Technology knowledge (TK): fluency of understanding information technology broadly;
- Technological content knowledge (TCK): understanding technology in specific topics;
- Technological pedagogical knowledge (TPK): understanding how technology can shape teaching.

Some limitations that this model might have are that it was originally designed by studying graduate students working with professors. It was based on only one study. However, this model was redesigned and tested in different contexts and domains: arts, science, mathematics, literacy, social studies, preservice and in-service teachers, and physical education. TPACK should be disseminated and more discussions and research are required in order to clarify the opportunities that the framework offers. After studying a group of preservice teachers in mathematics, Niess (2005) recommends that TPACK should offer four important aspects: 1. An overarching conception of what it means to teach a particular subject while integrating technology in learning; 2. Knowledge of instructional strategies and representations for teaching particular topics with technology; 3. Knowledge of students' understandings, thinking, and learning with technology; and 4. Knowledge of curriculum and curriculum materials that integrate technology with learning.

FINAL CONCLUSIONS

These pedagogical models were designed for mathematics teachers already involved in integrating technology in teaching mathematics. Therefore, these findings and recommendations might not be valid for teachers who are inexperienced in using technology. Also, as was mentioned, TPACK findings are limited to the specific topics, curriculum and technological products used in this study. This framework might be used for both in-service and preservice teachers. In addition, a new pedagogical model designed for supporting technology and participatory action research will be provided. Most significantly, it can lead to valuable insights into improving the incorporation of educational technology into math education.

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THE VEDIC APPROACH TO MULTIPLICATION OF BINOMIALS AND FACTORIZATION OF QUADRATIC TRINOMIALS

Krishna Subedi, *Simon Fraser University*
George Ekol, *Simon Fraser University*

We argue that the Vedic approach to multiplying two binomials and factoring trinomials reduces the level of difficulty faced by students. This idea mirrors Hazzan and Zazkis's (2005) notion of reducing abstraction.

Reducing abstraction refers to the tendency of students to relate unfamiliar mathematical concepts to more familiar ones in order to grasp the concepts (Hazzan & Zazkis, 2005). We assume that students are already familiar with arithmetic algorithms by the time they begin algebra. The Vedic approach involves the same algorithms for multiplication (and factorization) of arithmetic expressions and algebraic expressions. We present an example for multiplying two binomials (and verifying the answer) which gives a solution with relatively fewer steps and less demand on the students. This in a nutshell is the principle behind Vedic mathematics (Tirtha, 1965; also see http://www.vedicmaths.org). Our main argument is that this approach reduces the level of difficulty faced by students.

Example: The use of *Ûrdhva Tiryagbhyam Sutra,* (meaning vertical and crosswise method) in multiplying two numbers, 32 x 36, gives the product in a single, simple step of work.

$$\begin{array}{r} 3\ 2 \\ \underline{\times\ 3\ 6} \\ \mathbf{11_2 5_1 2} \end{array}$$

STEPS

i) Multiply vertically in the ones column: (6 x 2) = 12. Record the 2 and carry the 1. ii) Multiply crosswise and add the two products: (6 x 3) + (2 x 3) = 18 + 6 = 24. Add the carry: 24 + 1 = 25. Record the 5 and carry the 2. iii) Multiply vertically in the tens column: (3 x 3) = 9. Add the carry: 9 + 2 = 11. The answer is 1152. The same vertical and crosswise method can be used for algebraic multiplications. For example $(3x + 4)(2x + 3)$:

$$\begin{array}{r} 3x + 4 \\ \underline{*\ 2x + 3} \\ 6x^2+17x+12 \end{array} \quad \text{and} \quad \begin{array}{l} 4 * 3 = 12 \\ 3x * 3 + 2x * 4 = 17x \\ 3x * 2x = 6x^2 \end{array}$$

Checking the answer is one of the important components in doing algebra. Conventional methods for checking the solution to this problem involve factorization techniques which students often find difficult. The Vedic method however uses the sutra called *Gunita Sammuchhaya Sammuchhaya Gunita* (meaning "product of the sum of the factors = sum of the coefficient of the product"), which students can easily apply to verify their solutions as shown below.

From above, $(3x + 4)(2x + 3) = 6x^2 + 17x + 12$. Consider the coefficient of each term, (3 + 4) (2 + 3) = 6 + 17 + 12, which gives 35 = 35 (True).

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THE EFFECT OF DGS ON STUDENTS' CONCEPTIONS OF SLOPE

Shiva Gol Tabaghi, *Simon Fraser University*
Ami Mamolo, *York University*

In this ad hoc session, we presented the first instalment of a broader study which investigates university students' conceptualisations of static and dynamic geometric entities. We offered a refined look at the conceptualisations of two groups of students – one group which was taught using Dynamic Geometric Software, and the other using a 'traditional' approach. We used both APOS Theory (Dubinsky & McDonald, 2001) and Sfard (2008) to interpret learners' understandings of the slope of lines.

Based on our analysis of the students' responses to post-test questions, we suggest that the use of *Sketchpad* can help students develop a strong proceptual understanding of slope that emerges somewhat independently from the process-oriented conception inherent in the "rise-over-run" formulation of slope. This proceptual understanding corresponds to a descriptive narrative about the conceptual object of slope. The students in Class A, who had interacted with a dynamic graph, were able to use a descriptive narrative to distinguish positive and negative slope, to evaluate the steepness of lines, and to solve story-based problems. We argue that they were able to solve these problems because of their ability to see slope as an object, an ability that they had developed in their interactions with *Sketchpad*, in which they directly acted on the graph to create different slopes (rather than acting on, say, the rise or run values of the line). The students in Class A solved these problems very differently than students in Class B, who depended far more strongly—and sometimes superfluously—on process-based conceptions.

This report sets the stage for a look forward to how DGS may influence learners' process-object conceptualisations of other geometric representations of algebraic equations.

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MATHEMATICS TEACHERS' BELIEFS AND PRACTICE OF INTEGRATING COMMUNICATION

Svitlana Voytsekhovska
Queen's University

The purpose of the ad hoc session was to share with my colleagues the 'draft' ideas about my PhD research. My intention was to receive feedback regarding my research questions and methodology. The idea of my PhD study is built upon my master's thesis which describes one mathematics teacher's experience of integrating reading in Grade 9 mathematics classrooms. My master's thesis shows that a mathematics teacher developed his own ways of incorporating reading strategies and experienced difficulties while doing it. Based on these findings, I propose to extend my master's thesis in order to find a 'bigger' picture of integration of communication modes in Ontario mathematics classrooms. I define communication as a process of expressing and understanding mathematics ideas by using *written communication* (reading and writing) and *oral communication* (speaking and listening) (Morgan, 1998).

Researchers and educators have been emphasizing that all teachers, regardless of content specialization or grade, are responsible for developing and extending students' abilities to read, write, speak, and listen, within the context of discipline-based courses, not only within the confines of Language Arts programs (Lapp & Flood, 2008). Explicit support for integrating communication in the mathematics classroom appears in the National Council of Teachers of Mathematics *Standards* and in the Ontario mathematics curriculum. Even though communication is promoted in the curriculum documents and supported by educators, the degree of integrating communication varies among teachers (Lester, 2000; Remillard, 2005). Studies show that secondary teachers are less likely to value and include communication elements in their program. However, there is little known about to what degree mathematics teachers in Ontario value the inclusion of written and oral communication modes, and what challenges they face in their efforts to implement these modes. The purpose of my PhD study is to discover and describe the extent to which mathematics teachers in Grade 9 implement communication modes in their regular classrooms.

I will use a mixed-method approach. The quantitative data will be collected through a survey of Ontario secondary mathematics teachers who teach Grade 9 (n≈2,000). The survey instrument will include questions related to teachers' beliefs, strategies, and difficulties in implementing communication modes. This data will present a broad picture of how communication is implemented in mathematics classroom in Ontario. The qualitative data will be collected through classroom observations and interviews to obtain rich descriptive data from teachers (n≈6) who have a range of experiences with implementing communication modes.

Ad hoc session discussions were focused on the ways of clarifying research questions and developing a reliable questionnaire for the survey. It helped me to evaluate some of my research questions and my methodological approach to the study.

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Plenary Conversation

Conversation plénière

A CONVERSATION WITH MATHEMATICIAN AND ACTIVIST LEE LORCH – PROFESSOR EMERITUS, YORK UNIVERSITY

Martin Muldoon
York University

Professor Lee Lorch has over 65 years of experience as a research mathematician and an activist inside and outside the university systems in the US and Canada. For decades Lee has been an activist on issues of inclusion, and non-discrimination for women, visible minorities, and the obligations of professional societies to uphold the highest standards of civil rights and support for all their members. In recognition of his contributions, he is a life-time member of the Association of Women in Mathematics, the National Association of Mathematicians, and the Royal Society of Canada, and has received an MAA award for service to the community. The conversation will be moderated by Emeritus Professor Martin Muldoon (York University) who was a PhD student of Lee.

In his introduction of Lee, Walter Whiteley recalled the story of how, when Lee married before going overseas with the US Army in World War II, his wife Grace lost her job with the Boston School system because of a regulation against the employment of married women as regular teachers.

Martin Muldoon, who moderated the session, recalled how Lee's family had been profoundly affected by their role in the struggle for school integration in the 1950's including such landmark events as the 1954 Supreme Court decision "Brown vs. Board of Education" and its aftermath, and the 1957 confrontation at Central High School in Little Rock.

Lee said that he would like to speak about the life and times of a few women mathematicians, most of whom he had known personally, with particular reference to the societal circumstances in which they found themselves.

The first mentioned was Mary Cartwright (1900-1998) who was to head Girton College, then the leading women's college at Cambridge University. She was a Reader (not a Professor) in Mathematics, though she was honoured in other ways, not least as the first woman mathematician elected a Fellow of the Royal Society. In her very direct way, she had ascribed both her career success and her failure to marry to the deaths of so many of her male contemporaries in the Great War.

Aldona Aleskeviciene (1936 -) was the first Lithuanian woman to receive a doctorate in Mathematics. As a village girl, it was expected that she would be a schoolteacher, but when she went to study in Vilnius she was advised to pursue a scientific career in mathematics. Her example led to a great increase in women's participation, a trend that recent economic problems have reversed.

Today, there are many women in prominent positions in the mathematical establishment. There have been two women presidents of the French Mathematical Society (Mireille Martin-Deschamps and Marie-Françoise Roy), two women presidents of the Canadian Mathematical Society (Katherine Heinrich and Christiane Rousseau) and two women presidents of the American Mathematical Society (Julia Robinson and Cathleen Morawetz). But there are lots of obstacles to progress, some masquerading as objective. For example, when there was the possibility of layoffs of faculty here at York University, the most recently hired would have been the first to go. One administrator described the reminder, that in mathematics this would have had a devastating effect on the number of women faculty, as a “dirty argument”. Fortunately, the layoffs did not have to be implemented.

Other policies have hindered women’s progress. The City College of New York system (including Hunter College, then for women only) had an official policy of not appointing its own graduates to teaching positions to avoid “urban provincialism”. But since the free tuition policy there had attracted those who could not afford to study elsewhere, the policy discriminated against the impoverished. Lee recalled being told by an administrator at City College, where he taught in the period 1946-49, that he should be a teacher at a rich school, where the students would not be infected by his social ideas!

Lee displayed a photo of himself with three African-American women, a colleague (Evelyn Boyd Granville) and two of his students (Vivienne Malone Mayes and Etta Falconer), at Fisk University (Nashville, Tennessee) where he worked in the period 1950-1955. Fisk was a “Traditionally Black” college with no tradition of students going on to graduate work in Mathematics. Five of Lee’s Fisk students (four of them women) went on to do PhDs in Mathematics. See Mayes (1976) and Kenschaft (1981). Lee explained that the students were aware of his work in the community and responded by being good students. Even then, he had to take some extraordinary measures to convince them to apply to graduate school, and they had to go where they would not be victims of discrimination.

Sofia Kovalevskaya (1850-1891) was a “woman of the sixties” (meaning the 1860’s). In Tsarist Russia, a woman passed from being the property of her father to being that of her husband. Kovalevskaya was one of the few women of privilege who rebelled against this state of affairs by a marriage of convenience to a husband willing to give her permission to go abroad to study. She was tutored by Weierstrass in Berlin but, as a woman, was unable to register as a student. Through the influence of Weierstrass, she was able to obtain her PhD from the University of Göttingen. Later, she was appointed to what was to become the University of Stockholm, a position she held until her untimely death at the age of 41. Her contributions to mathematics were only one side of this fascinating personality. She also wrote novels and was politically engaged. In fact, after leaving Russia, she was to participate with her husband and sister in the Paris Commune (1870). See Koblitz (1993) for more information on Sofia Kovalevskaya.

Lee spoke of his interactions with and memories of Olga Aleksandrovna Ladyzhenskaya (1922-2004, Leningrad) and Olga Arsenievna Oleinik (1925-2001, Moscow), both worthy followers of Kovalevskaya, who as she, worked in partial differential equations.

In answer to a question from Florence Glanfield, Lee explained that his involvement in civil rights arose from his experience of the 1930’s Depression and the rise of fascism in Europe.

Walter Whiteley recalled participating with Lee in an MAA (Mathematical Association of America) panel on Mathematics and Ethics. A feature of that discussion was the importance of ethical behaviour by the professional organizations. An example was the issue raised by Lee (Lorch, 1951), when the MAA held a meeting which included an official dinner in a hotel that, as was well known to the organizers in advance, refused to admit Black members. As a more recent example, Lee reminded us that when the Fields Institute selected its first 33 Fellows in 2002, the list consisted entirely of white males. This was rectified the following year, after vigorous protest by Lee and others.

We thank the ABEL (Advanced Broadband Enabled Learning) program at York University for making a video record of the session. It may be viewed at: mms://windows.stream.yorku.ca/faculty/abel/as/leelorch.wmv

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Appendices

Annexes

APPENDIX A / ANNEXE A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

· The influence of computer science on undergraduate mathematics education
· Applications of research in mathematics education to teacher training programmes
· Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

· Developing statistical thinking
· Training in diagnosis and remediation of teachers
· Mathematics and language
· The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

· Logo and the mathematics curriculum
· The impact of research and technology on school algebra
· Epistemology and mathematics
· Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

· Lessons from research about students' errors
· Logo activities for the high school
· Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

· The role of feelings in mathematics
· The problem of rigour in mathematics teaching
· Microcomputers in teacher education
· The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

· Methods courses for secondary teacher education
· The problem of formal reasoning in undergraduate programmes
· Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

· Teacher education: what could it be?
· Natural learning and mathematics
· Using software for geometrical investigations
· A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

· Using computers to investigate work with teachers
· Computers in the undergraduate mathematics curriculum
· Natural language and mathematical language
· Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathémathiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Univerité Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education - Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Dicrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d’enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l’(in)justice sociale

APPENDIX B / ANNEXE B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathematiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity

APPENDIX C / ANNEXE C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting	ED 204120
Proceedings of the 1981 Annual Meeting	ED 234988
Proceedings of the 1982 Annual Meeting	ED 234989
Proceedings of the 1983 Annual Meeting	ED 243653
Proceedings of the 1984 Annual Meeting	ED 257640
Proceedings of the 1985 Annual Meeting	ED 277573
Proceedings of the 1986 Annual Meeting	ED 297966
Proceedings of the 1987 Annual Meeting	ED 295842
Proceedings of the 1988 Annual Meeting	ED 306259
Proceedings of the 1989 Annual Meeting	ED 319606
Proceedings of the 1990 Annual Meeting	ED 344746
Proceedings of the 1991 Annual Meeting	ED 350161
Proceedings of the 1993 Annual Meeting	ED 407243

Proceedings of the 1994 Annual Meeting	ED 407242
Proceedings of the 1995 Annual Meeting	ED 407241
Proceedings of the 1996 Annual Meeting	ED 425054
Proceedings of the 1997 Annual Meeting	ED 423116
Proceedings of the 1998 Annual Meeting	ED 431624
Proceedings of the 1999 Annual Meeting	ED 445894
Proceedings of the 2000 Annual Meeting	ED 472094
Proceedings of the 2001 Annual Meeting	ED 472091
Proceedings of the 2002 Annual Meeting	*submitted*
Proceedings of the 2003 Annual Meeting	*submitted*
Proceedings of the 2004 Annual Meeting	*submitted*
Proceedings of the 2005 Annual Meeting	*submitted*
Proceedings of the 2006 Annual Meeting	*submitted*
Proceedings of the 2007 Annual Meeting	*submitted*
Proceedings of the 2008 Annual Meeting	*submitted*
Proceedings of the 2009 Annual Meeting	*submitted*

Note

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.