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STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2007 ANNUAL MEETING /
RENCONTRE ANNUELLE 2007



University of New Brunswick
June 8 – June 12, 2007

EDITED BY:
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**Proceedings of the 2007 Annual Meeting of the Canadian
Mathematics Education Study Group / Actes de la rencontre
annuelle 2007 du Groupe Canadien d'Étude en Didactique des
Mathématiques**

31st Annual Meeting
University of New Brunswick
June 8 – June 12, 2007

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Acknowledgements / Remerciements

The organisational work for our 2007 annual meeting at the University of New Brunswick in Fredericton enabled us to have another memorable meeting. Our local organisers, Dave Wagner and John Grant McLoughlin, with the expert support of Sylvie Beaulieu, managed to do everything efficiently and with a smile, and we thank them. The university offered much support and we wish to thank the faculties of Science, of Education, of Graduate Studies as well as the Mathematics Department for it.

Le travail d'organisation de la rencontre annuelle de 2007 à l'Université du Nouveau Brunswick à Fredericton a permis au groupe d'avoir une autre rencontre annuelle mémorable. Sur le plan local, Dave Wagner et John Grant McLoughlin qui, appuyé de Sylvie Beaulieu, ont su organiser la rencontre avec efficacité et bonne humeur, et nous les en remercions. L'appui de leur institution s'est manifesté de bien des manières et nous remercions les facultés des sciences, d'éducation et des études graduées, en plus du département de mathématiques de l'université.

Schedule

Friday 8 June	Saturday 9 June	Sunday 10 June	Monday 11 June	Tuesday 12 June
	9:30 – 11:00 Working Groups	9:30 – 11:00 Working Groups	9:30 – 11:00 Working Groups	9:00 – 10:00 Topic Sessions C and D
				10:00 – 11:00 Panel
	11:00 – 11:20 Coffee Break	11:00 – 11:20 Coffee Break	11:00 – 11:20 Coffee Break	11:00 – 11:20 Coffee Break
	11:20 – 12:30 Working Groups	11:20 – 12:30 Working Groups	11:20 – 12:30 Working Groups	11:30 – 12:30 Closing Session
	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	
	13:45 – 14:15 Small Group Discussion of Plenary 1	13:45 – 14:45 Plenary 2	13:45 – 14:15 Small Group Discussion of Plenary 2	
14:15 – 15:15 Discussion of Plenary 1	14:15 – 15:15 Discussion of Plenary 2			
15:00 – 18:00 Registration	15:15 – 15:45 Ad Hoc Session (1)	15:00 – 16:00 Topic Sessions A and B	15:15 – 15:45 Ad Hoc Session (3)	
	15:45 – 16:15 Coffee Break		15:45 – 16:15 Coffee Break	
	16:15- 16:45 New PhD's (1)	16:00 – 20:00 Free time to explore Fredericton trails and have dinner downtown	16:15- 16:45 New PhD's (2)	
17:30 – 18:45 Dinner	17:00 – 18:15 Annual General Meeting			
19:00 – 19:45 CMESG Opening	19:00 Dinner		19:30 Dinner	
19:45 – 20:45 Plenary 1				
20:45 Reception		20:00 Dessets and CELTIC MUSIC		

Horaire

Vendredi 8 Juin	Samedi 9 Juin	Dimanche 10 Juin	Lundi 11 Juin	Mardi 12 Juin
	9:30 – 11:00 Groupes de travail	9:30 – 11:00 Groupes de travail	9:30 – 11:00 Groupes de travail	9:00 – 10:00 Séances thématiques C et D
	11:00 – 11:20 Pause-café	11:00 – 11:20 Pause-café	11:00 – 11:20 Pause-café	10:00 – 11:00 Table ronde
	11:20 – 12:30 Groupes de travail	11:20 – 12:30 Groupes de travail	11:20 – 12:30 Groupes de travail	11:00 – 11:20 Pause-café
	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	12:30 – 13:30 Dîner	11:30 – 12:30 Séance de clôture
	13:45 – 14:15 Discussion en petits groupes – Conférence 1	13:45 – 14:45 Conférence 2	13:45 – 14:15 Discussion en petits groupes – Conférence 2	
	14:15 – 15:15 Discussion Conférence 1		14:15 – 15:15 Discussion Conférence 2	
15:00 – 18:00 Inscription	15:15 – 15:45 Séances ad hoc (1)	15:00 – 16:00 Séances thématiques A et B	15:15 – 15:45 Séances ad hoc (3)	
	15:45 – 16:15 Pause-café		15:45 – 16:15 Pause-café	
	16:15- 16:45 Thèses (1)	16:00 – 20:00 Temps libre pour explorer les rues de Fredericton et prendre un repas en ville	16:15- 16:45 Thèses (2)	
17:30 – 18:45 Souper	17:00 – 17:30 Séances ad hoc (2)		17:00 – 18:15 Assemblée Générale annuelle	
19:00 – 19:45 Ouverture GCEDM	19:00 Souper			
19:45 – 20:45 Conférence 1				
20:45 Réception		20:00 Desserts et MUSIQUE CELTIQUE	19:30 Souper	

Introduction

Frédéric Gourdeau – Président, CMESG/GCEDM
Université Laval

As always, it is a pleasure to write the introduction to the proceedings. It forces me to think back to a wonderful meeting, to the planning and dedication that went into its organization, and to the meeting itself, the conversations, the sharing which occurred.

First, I remember our initial planning, in 2006, when we had wisely (!) decided to plan four working groups, and not five as would have been needed in Calgary that same year. We figured that after record numbers in the West, we would be back to our usual smaller numbers in Fredericton. Well, once again, numbers were much higher than expected. Our local organizers Dave Wagner and John Grant Mc Loughlin contributed to this by ensuring good participation from their graduate students and from others in New Brunswick and Nova Scotia, helped in this by AARMS (Atlantic Association for Research in the Mathematical Science) which funded some graduate students attendance. The growth of the organization twinned with a great participation from Quebec did the rest.

As a result, the organizers had to adjust, and so did the working group leaders. We all know how hard it can be to lead a group of fifteen intellectuals, not all pragmatic, towards some common goal... Well, imagine thirty or more! The meeting was superbly run, once Dave Wagner had ensured large enough supplies of coffee for all! The social program was great, but I won't venture going into details as this may not look scholarly enough.

La température était remarquable, et c'est presque une canicule qui nous a accompagnés dans nos travaux. La rencontre de 2008 nous a permis de rencontrer deux conférenciers pléniers aux profils très différents, alors que Christine Stevens (projet NExT) et Rafael Nunez étaient nos invités. Les groupes de travail étaient tout aussi variés, sinon davantage, et les impressions glanées ici et là quant aux travaux des uns et des autres nous donnaient le goût d'être partout à la fois. Mais, justement, cela n'est pas possible : si vous y étiez, les Actes vous permettent maintenant d'aller au-delà de ces impressions fugitives ; si vous n'y étiez pas, les Actes sont votre fenêtre sur une rencontre des plus réussies.

Alors, que votre intérêt soit pour la vulgarisation ou la géométrie, les situations d'apprentissage ou les multiples visages du feedback, vous pourrez lire sur les groupes de travail qui s'y sont déroulés. À cela s'ajoutait les séances thématiques, liées cette fois-ci aux groupes de travail de manière assez précise, ainsi que les présentations de thèses de doctorat récentes et les séances Ad Hoc, auxquels certains rêvent sans doute déjà pour 2010 ! Le panel qui terminait notre rencontre est aussi représenté dans ces actes, ce qui n'est pas toujours le cas pour une activité de cette nature : merci à toutes et à tous. .

I wish to thank all the presenters for their contribution to the meeting. Their generosity is essential to the success of our annual meeting, and this was true in 2007 as it has been for many years. My thanks also go to Florence Glanfield, France Caron, Brent Davis, Doug Franks and Leo Jonker, fellow members of the Executive in 2006-2007: it has been a pleasure working with you. And finally, thanks to Peter Liljedahl, editor of these proceedings, for his patience and dedication in producing these proceedings. To thank him for his hard work, read on...

Plenary Lectures



Conférences plénières

Understanding Abstraction in Mathematics Education: Meaning, Language, Gesture, and the Human Brain

Rafael Núñez
University of California, San Diego

We are not able to publish a paper from the plenary at the time of printing. The content of the plenary is closely related to a recent paper:

Núñez R. (2007). The Cognitive Science of Mathematics: Why is it relevant for Mathematics Education? In R. Lesh, E. Hamilton, & J. Kaput (Eds.), *Foundations for the Future in Mathematics Education* (pp. 127-154). Mahwah, NJ: Lawrence Erlbaum Associates.

The article can be downloaded from Rafael Núñez's website at:

<http://www.cogsci.ucsd.edu/~nunez/web/publications.html>

Mathematics Departments, New Faculty, and the Future of Collegiate Mathematics¹

T. Christine Stevens
Saint Louis University

Recent changes in the way mathematics is taught at colleges and universities in North America offer many exciting opportunities to new mathematics faculty but can also pose special challenges for them. Project NExT (New Experiences in Teaching) is a professional development program for new and recent Ph.D.s in the mathematical sciences that helps them become successful teachers and scholars. A program of The Mathematical Association of America, it addresses all aspects of an academic career: improving the teaching and learning of mathematics, maintaining research and scholarship, and participating in service and professional activities. During the last fourteen years, Project NExT has helped over a thousand new Ph.D.s to make the transition from being a graduate student to being a successful full-time faculty member. Many of the early participants in the program are now emerging as leaders on their own campuses and in the mathematical community at large. We describe our experiences with Project NExT and their implications for new Ph.D.s entering the profession, for the mathematics departments that prepare and hire them, and for the future of collegiate mathematics education, with particular attention to the situation in Canada.

Project NExT² (New Experiences in Teaching)

The excellent program at this conference addresses a wide variety of important and exciting issues in undergraduate mathematics education. At the opening session, Raphael Nuñez brought insights from neuroscience to bear upon our understanding of the nature of abstractions in mathematics. During the next two mornings, working groups grappled with the design and implementation of learning situations, explored effective strategies for strengthening students' geometric and spatial reasoning, and discussed outreach activities that will engage students and non-students in mathematics. Turning our attention to concrete matters of curriculum, an informative panel outlined various approaches to teacher preparation.

These issues and innovations in the teaching of undergraduate mathematics are often especially appealing to the new Ph.D. recipients who join college and university faculties. Having grown up with calculators and computers, such new faculty members are often eager to exploit the pedagogical potential of technology. Moreover, since they have very little experience with any method of teaching, new faculty are sometimes more willing to try a

¹ Portions of this paper are adapted from the author's paper, *Helping New Mathematics Faculty to Develop into Successful Teachers and Scholars*, in CBMS Issues in Mathematics Education 14 (2007), *Enhancing University Mathematics: Proceedings of the First KAIST International Symposium on Teaching* (ed. Ki Young Ko and Deane Arganbright), 33 - 41.

² For more information about Project NExT, consult our website <http://archives.math.utk.edu/projnext/>

variety of new ideas. And when they decide to implement one of these new ideas, they often have more energy than we older faculty members do.

On the other hand, these opportunities and pedagogical innovations can also pose special challenges for new members of the faculty. With so many good ideas available, they may have difficulty selecting a focus for their efforts. Lacking much teaching experience, they may not be able to predict how students will react to a particular teaching strategy, or how much of their own time it will consume. Finally, teaching is only one of their responsibilities as faculty members, and they cannot afford to neglect the other aspects of an academic career. They must establish and maintain an active research program, and they may also be expected to serve on committees and advise students.

Thus, although taking one's first job as a full-time faculty member has never been easy, the current climate of change in undergraduate mathematics education makes it especially hard for a new Ph.D. to make the transition from being a graduate student to being a full-time member of a college or university mathematics department. To ease that transition, and to promote the improvement of collegiate mathematics education, The Mathematical Association of America (MAA) established in 1994 a professional development program called Project NExT (New Experiences in Teaching), of which I am now the director.

Project NExT serves new and recent Ph.D.s in all of the mathematical sciences, including pure and applied mathematics, statistics, operations research, and mathematics education. It addresses all aspects of an academic career: improving the teaching and learning of undergraduate mathematics, maintaining research and scholarship, and participating in professional activities. Project NExT receives major funding from The ExxonMobil Foundation, with additional funding from other foundations, corporations, and professional organizations.

Each year, about 75 to 85 new faculty members from colleges and universities throughout the United States and Canada are selected as Project NExT Fellows. Figure 1 is a photograph of some of the 2007-08 Fellows. They work at a wide range of institutions, including research universities, comprehensive regional universities, liberal arts colleges, and community colleges. What they share is an enthusiasm for teaching, a dedication to scholarly work, and an eagerness to participate in the mathematical community.



Figure 1: Some of the 2007-08 Project NExT Fellows

T. Christine Stevens • Future of Collegiate Mathematics

Each Project NExT Fellow participates in Project NExT sessions at three national meetings. The annual cycle begins with a workshop and the summer meeting of The Mathematical Association of America, which is the professional organization in the United States and Canada that is devoted to collegiate mathematics. It continues with special events at the major mathematics meeting in January that is jointly sponsored by several mathematics professional organizations, and it concludes at the MAA meeting the following summer. During the academic year, the Fellows are linked by a very active electronic discussion list.

The program for the first workshop is planned by the director, co-directors, and associate co-directors. The topics and presenters are revised each year, to make sure that they reflect emerging issues in the profession, as well as our sense of the Fellows' current concerns. In 2006, it included sessions on teaching specific courses, such as differential equations and abstract algebra, and others on general teaching strategies, such as getting your students to read their textbooks or to make effective presentations to the class. There were also a plenary address on the mathematical preparation of elementary and secondary school teachers, a presentation about how to provide academic and professional advice to students, and a four-hour session on writing research papers and grant proposals. A copy of the program for the 2007 workshop at San Jose State University is included as an appendix.

The programs for the second two meetings are planned by the Fellows themselves, with guidance from the project directors. This arrangement permits the Fellows to tailor the topics to their interests, and it also gives them valuable experience in organizing sessions for professional meetings. For the mathematics meetings in January, 2007, the 2006-07 Fellows organized sessions on such topics as establishing research collaborations, mentoring graduate students, creating new courses, developing and sustaining an active student math club, and finding out what your students have learned. At the workshop in August, 2007, they shared practical suggestions about efficient strategies for handling grading and other aspects of teaching, discussed ways to increase the number of students majoring in mathematics and to involve them in their own mathematical research, learned how to write effective letters of recommendation and how to navigate pre-tenure reviews, and explored opportunities for grants.



Figure 2: A presentation by Jim Lewis (University of Nebraska, Lincoln) about teacher preparation³

³ Photos taken by Judith Covington, Aparna Higgins, and the author.

Each Fellow is also matched with a more experienced member of the mathematical community, who serves as a “mentor” or “consultant” to the Fellow. These consultants participate in the discussions on the Fellows’ electronic discussion list, offering information and advice in response to the questions that are raised. The involvement of the consultants also provides them with an opportunity to seek advice from the Fellows (particularly about matters involving technology), and it has built support for Project NExT within the mathematical community at large.



Figure 3: Some of the 2004-05 Fellows discussing their plans for the coming year

From the account given thus far, it may be tempting to think of Project NExT as a giant human database of information about teaching, research, and service. Although such a description captures some of the truth, it overlooks one of the essential features of Project NExT – a feature whose importance the members of CMESG/GCEDM will surely recognize. That feature is the sense of community that develops among the Fellows. Indeed, in preparation for this conference, I asked the Fellows from Canada to tell me which aspects of Project NExT are especially appropriate for mathematics faculty here, and their immediate reply was “networking.” Although the Fellows spend a significant amount of time at the workshops listening to presentations (Figure 2), the time that they spend in formal and informal discussions with each other and with the presenters is even more important (Figures 3 and 4). Having met and grown to trust one another at the initial workshop, they are able to use the Project NExT electronic discussion list to share their experiences, insights, successes, and failures.

Here’s how one Fellow described the value of the electronic discussion list: “Project NExT gave me confidence to try out new techniques, blending them with my own style. I could e-mail the group when something wasn’t working, and fix it that way. I wasn’t going it alone.” Another Fellow put it even more strongly: “Project NExT has absolutely changed my life. Without it, I would have worked in virtual isolation at a small school, exposed to very little.” The Fellows appreciate the fact that “at any time, day or night, there are dozens of people eager to serve as sounding boards or provide information.” At the same time, they also value the fact that Project NExT does not attempt to force any one particular curriculum or teaching strategy on them. As one Fellow put it, “Project NExT does not brainwash its new Fellows. It gives them advice and helps them to talk to each other.” One Fellow summed up the value of the Project NExT electronic discussion list by likening it to “joining the biggest and most active mathematics department in the world.” This sense of belonging simultaneously to a

group of supportive peers and to an active mathematical community enables the Fellows to continue their growth as teachers and scholars, long after their formal participation in Project NExT has ended.



Figure 4: Fellows and presenters at the annual ice cream social

Although this paper is about “Mathematics Departments, New Faculty and the Future of Collegiate Mathematics,” the discussion thus far has dealt only with new faculty. We now turn our attention to mathematics departments and the future of collegiate mathematics. Whatever the future of collegiate mathematics is, the people in Figure 1 are the ones who are going to take us there. Long after people my age have retired, these faculty members will be teaching students, developing new mathematics, and leading mathematics departments. Now in its fourteenth year, Project NExT has served a total of over one thousand new mathematics faculty. They work in about 600 different colleges and universities, and, as you can see from the map in Figure 5, seven of them are in Canada.



Figure 5: A map showing the distribution of the Project NExT Fellows (with the author)

There is no way to know exactly what these faculty will be doing in the coming decades, but we do know something about what their departments want them to do. Each application to Project NExT must be accompanied by a letter from the department chair that not only promises financial support for the applicant's travel expenses, but also describes the ways in which the new faculty member and the department would benefit from the applicant's participation in Project NExT. These letters shed considerable light upon the direction in which mathematics departments are heading. According to the incoming Fellows' department chairs, they will be developing new courses and programs in applied mathematics and statistics, involving undergraduates in research, using technology in teaching, promoting active learning, and re-designing courses so that they will better serve under-prepared students from rural areas. The department chairs also want the Fellows to maintain their research programs and connect the department with trends in the larger mathematical community. On top of all that, they want their new faculty to become "truly great" teachers and "gently encourage" older members of the department to try new teaching techniques. The lofty ambitions of the department chairs represent, of course, only their educated guess about the future. As a 1995-96 Fellow who is now a chair pointed out, "The most important contributions will probably be in areas I cannot anticipate. Project NExT has opened doors for me that I could not anticipate when I first became involved."

Lessons of Project NExT

Although Project NExT was designed for the benefit of new mathematics faculty, to help them develop into successful teachers and scholars, it also functions as a professional development program for the more experienced mathematicians who run and participate in it. Through my work with Project NExT, I have learned several important things about mathematics departments, new faculty, and the teaching of collegiate mathematics.

The first is that the future is in good hands. There are wonderful new Ph.D.s entering our profession who are committed in a very serious way to both research and teaching. We can count on them to grapple not only with the problems in mathematics education that were discussed at this conference, but also with new issues that we cannot now foresee. At the same time, they will continue their scholarly work, and they may well forge a new balance that truly integrates teaching and research.

Another thing I learned is that those of us who are established faculty members are not as nice as we thought we were. Although we may consider ourselves very friendly and welcoming, new Ph.D.s are actually very intimidated by their initial contacts with the mathematical community. When they go to conferences, they often feel lonely and very much like "outsiders." One of the significant achievements of Project NExT was to involve these new faculty in the mathematical community at an early stage in their careers – an involvement that has benefited both the new members of our profession and the profession itself. When I attend a mathematics meeting in the United States, I am always impressed by the large number of young mathematicians who are participating. They are presenting papers and organizing sessions at meetings; publishing expository, pedagogical and research papers; winning grants for research and education; and serving on the committees and governing boards of mathematics professional organizations. Indeed, many Fellows from the early years of the program are taking leadership roles in their departments, and one is currently the Director for Improving Teacher Quality for the Massachusetts Board of Higher Education. Thus, although Project NExT was designed as a professional development program for new faculty, it has also been very effective as a leadership development program.

I also learned from Project NExT how valuable it is for new faculty to be part of a national network that includes people from many different kinds of institutions. New faculty

ordinarily work in an environment that offers them few reference points beyond what is done by colleagues in their own departments or at similar schools in the same geographical area. Project NExT gives them access to information about curricula, technology, teaching strategies, and research practices at a diverse set of institutions throughout the United States and Canada. Although each individual Fellow's experiences are fairly narrow, collectively the Fellows are interested in many branches of mathematics, and they have taught many different courses in many different ways. By sharing those interests and experiences, they can learn quite a lot from each other. For topics with which they have no personal experience (such as teaching theoretical or advanced courses), they very much appreciate the advice of the national network of expert presenters and "consultants" that Project NExT provides. By giving the Fellows a context for the practices in their own departments, Project NExT enables them to become informed participants in departmental discussions about teaching and research.

Although the Fellows benefit enormously from the transcontinental perspective that Project NExT gives them on issues in teaching and research, I learned that they also crave specific information about how things work at their own institutions. They want to know who can help them with computer problems, who orders the mathematics books for the library, and how to find out about curricular requirements for students. They also want to learn about potential sources for internal grants and how their teaching and research will be evaluated. When I was department chair, I realized that it was my job to provide this information to my new faculty, and it fostered their assimilation into the department. Thus the lessons that I learned from Project NExT helped to make me a better chair.

Finally, Project NExT has given me many good ideas that I have used in my own classes. I learned, for example, how to structure group work in class so that it would be more effective, and I found out about projects that could help to engage the students in the courses that I was teaching. I discovered that distributing a grading rubric for written assignments not only makes the grading easier but often produces better student work, because it lets the students know to what issues the instructor will be paying attention. I also learned about the value of mid-term evaluations, which ask the students, about a quarter of the way through a course, for their opinion about the various ways in which class time is spent, such as lecturing, going over homework, and working in groups on problems that develop the theory or illustrate its applications. Invariably, a majority of the students wants more of everything, and reporting this fact to them can build support for "non-traditional" methods of instruction. Finally, I have found many good ideas for classroom activities on a website developed by Francis Su, a 1996-97 Fellow who has posted at <http://www.math.hmc.edu/funfacts/> literally hundreds of "Math Fun Facts" that he uses to expand his students' notions of what mathematics is. Although the target audience for these brief activities is undergraduates who are not majoring in mathematics, I used one of them (on the Euler characteristic) last year in my graduate course in topology.

Project NExT and Canada

Project NExT seeks to engage the Fellows, at the outset of their careers, in a stimulating discussion of important issues in teaching and learning, to introduce them to a professional community in which those issues can be discussed in a sustained way, and to supply them with tools that will enable them to address these issues in their own classes and their own institutions. It also seeks to develop a broad understanding of their responsibilities as faculty, so that they can integrate their roles as teachers, scholars, and advisers. These goals are certainly relevant to the mathematical community in Canada, so it is natural to ask about the relationship between Project NExT and Canadian mathematicians.

From 2002 to 2004, Canada had a program similar to Project NExT, called Project NExTMAC (New Experiences in Teaching Mathematics Across Canada). With support from the Canadian Mathematical Society and Pearson Canada, it was organized and run by two Project NExT Fellows who teach in Canada, David Pike of Memorial University of Newfoundland and Robert van den Hoogen of St. Francis Xavier University in Nova Scotia. Although the program that they designed was an excellent one, Project NExTMAC was unfortunately unable to attract enough participants to establish a national network of new faculty from different kinds of institutions. As we have seen, the creation of such a network is one of the important features of Project NExT. We have encountered the same difficulty, by the way, with some of the regional versions of Project NExT that we have been trying to develop in the United States, which sometimes fail to attract a group of participants that is large and diverse enough to generate tube rich discussions about research and teaching that the Fellows value so highly.

In the absence of a Canadian professional development program for new mathematics faculty, it is important to ask how Project NExT can best serve mathematical scientists who are educated or who teach in Canada. When I asked the Fellows who have lived or taught in Canada whether Project NExT had been suitable for them, they noted some important differences between Canadian institutions of higher education and those in the United States. They pointed out that most Canadian institutions are universities and that liberal arts colleges are much more rare in Canada than in the United States, so that some of the more elementary mathematics courses discussed at the Project NExT workshop are simply not part of the Canadian curriculum. They also felt that there is more emphasis on mathematical research in Canada, partly because of the way that research is funded. Conversely, they told me that undergraduate teaching receives less attention in Canada than in the United States.

Although all of these factors may make the mix of topics at Project NExT workshops less relevant for faculty in Canada, the Fellows felt that these differences were trumped by the fact that networking and participating in the larger mathematical community are even more important for faculty in Canada than in the United States. As one Fellow observed, “For me personally, the most useful aspect was to feel that the difficulties I was going through were not insurmountable. Many other young faculty were also confused and needed some guidance. Integration in the MAA was also fundamental.” This sentiment was echoed by another Fellow, who pointed out that “there is a great deal that could be gained from networking newer faculty who want to discuss issues of innovative teaching.” Noting that most Canadians live within one hundred miles of the United States border, the Fellows from Canada also stressed that “it would be great to have more connection across the border.” Since many Canadian students pursue graduate study or employment in the United States, the Fellows also felt that familiarity with the United States is “essential for advising students.”

These comments from the Fellows suggest that new mathematics faculty who work in Canada would find it valuable to participate in Project NExT, and thus I will close by raising two questions that might help Project NExT to serve Canadian faculty more effectively. The first is whether there are changes to Project NExT that would make it more useful to Canadian mathematics faculty and their institutions. The second is how we can effectively disseminate information about Project NExT to Canadian universities and colleges.

As I have already mentioned, I have learned a lot through my work with Project NExT. I am grateful to Project NExT and the Project NExT Fellows for what they have done for me, and also to my co-directors and associate co-directors, Joseph Gallian, Aparna Higgins, Judith Covington, and Gavin LaRose. Finally, I express my thanks to the Canadian Mathematics Education Study Group/Groupe Canadien d'Étude en Didactique des Mathématiques for inviting me to this outstanding conference.

Appendix: Program for the 2007 Summer Workshop

**Project NExT: New Jobs, New Responsibilities, New Ideas
Program for the Workshop in San Jose, California
July 31 - August 2, 2007**

TUESDAY, JULY 31

11 am - 1:15 pm	Arrival and registration
1:30 - 1:45 pm	Welcome and opening remarks T. Christine Stevens, Saint Louis University and Director of Project NExT
2:00 - 2:45 pm	Small group discussions
2:55 - 3:45 pm	<i>Teaching is a Practical Art</i> Jennifer Quinn, Association for Women in Mathematics
3:50 - 4:20 pm	BREAK
4:30 - 5:15 pm	Small group discussions
5:30 - 7:00 pm	DINNER
7:30 - 8:45 pm	<i>Joining the Mathematical Community</i> Elizabeth Allman, University of Alaska Fairbanks Bradford Chin, West Valley College Judith Covington, Louisiana State University, Shreveport Francis Su, Harvey Mudd College
9:00 p.m. - ?	INFORMAL SOCIALIZING

WEDNESDAY, AUGUST 1

7:00 - 8:15 am	BREAKFAST
8:30 - 9:45 am	<i>Selected topics in teaching undergraduate mathematics I</i> [Five simultaneous sessions] A. <i>Teaching Calculus Using Creative Hands-on Activities</i> – Julia Barnes, Western Carolina U. B. <i>What Really Happens When students Work Online?</i> – Andrew Bennett, Kansas State University C. <i>Helping Students Learn Linear Algebra</i> – Jane Day, San Jose State University D. <i>Advising Mathematics Students Academically and Professionally</i> – James Sellers, Pennsylvania State University E. <i>Experiencing Spherical Geometry: A Non-Axiomatic Approach to Teaching College Geometry</i> – Christopher Swanson, Ashland University
9:55 - 10:25 am	BREAK
10:25 - 11:40 a.m.	Panel: <i>Deciding how to teach</i> Suzanne Doree, Augsburg College, Minnesota William Higgins, Wittenberg University Teri Jo Murphy, University of Oklahoma
11:45 am - 1:15 pm	LUNCH
1:15 - 2:30 pm	Repeat of morning breakout sessions
2:40 - 3:40 pm	<i>Heroes, Foot Soldiers, and in Between: Fulfilling Our Responsibilities Towards the World of K-12 Mathematics</i> Judith Roitman, University of Kansas
3:40 - 4:10 pm	BREAK
4:10 - 5:25 pm	Panel: <i>The faculty member as teacher and scholar</i> Sheldon Axler, San Francisco State University Amy Cohen, Rutgers University

CMESG/GCEDM Proceedings 2007 • Plenary Lecture

Karolyne Fogel, California Lutheran University
Herbert Medina, Loyola Marymount University

5:30 - 7:00 pm DINNER
8:00 - 10:00 pm Social Event for 2006-07 and 2007-08 Project NExT Fellows and presenters
10:00 pm - ? INFORMAL SOCIALIZING

THURSDAY, AUGUST 2

7:00 - 8:30 am BREAKFAST
8:30 - 9:35 am FREE TIME for informal socializing, etc.
9:35- 10:05 am BREAK
10:10 - 11:25 am *Selected topics in teaching undergraduate mathematics II* [Five simultaneous sessions]
A. *Discrete Math* – Arthur Benjamin, Harvey Mudd College
B. *“You Can’t Do What You Want Without Mathematics, and You Can Do Mathematics”*: *Two Liberal Arts Mathematics Courses* – Judith Grabiner, Pitzer College
C. *Effectively Using Applied Writing Projects in Calculus and Differential Equations* – P. Gavin LaRose, University of Michigan
D. *Promoting Diversity in Mathematics: Challenges, Opportunities & Ideas for Involvement* – Herbert Medina, Loyola Marymount University
E. *Teaching Students to Prove Theorems* – Carol Schumacher, Kenyon College
11:30 am - 12:15 pm Small Group Discussions with other Project NExT Fellows (organized geographically)
12:15 - 1:30 pm LUNCH
1:35 - 2:50 pm Repeat of morning breakout sessions
2:55 - 3:25 pm Planning session for January Meetings in New Orleans
3:25-3:55 pm BREAK
3:55 - 5:25 pm Closing Session
Recognition of 2007-08 Fellows
Presentation: *Finding Your Niche in the Profession*
Joseph Gallian, University of Minnesota Duluth
7:30 – 9:30 pm Mathfest Opening Banquet
Master of Ceremonies: Donald Albers, Mathematical Association of America
Presentation: *Canonical forms: A mathematician’s view of musical canons*
Noam Elkies, Harvard University
9:30 pm - ? INFORMAL SOCIALIZING

FRIDAY AND SATURDAY, AUGUST 3 AND 4

Project NExT Courses During the Mathfest: Four-hour courses meeting in the Fairmont San Jose on Friday and Saturday, August 3 - 4.

- A. *Biological Applications and Mathematical Modeling for Undergraduate Courses* – Joseph Mahaffy, San Diego State University, 1:00 - 3:00 p.m.
- B. *Applying for Research and Education Grants at the National Science Foundation/Starting and Maintaining Your Mathematical Research Program* – Deborah Lockhart, National Science Foundation, and Ezra (Bud) Brown, Virginia Polytechnic Institute & State University, 1:00 - 3:00 p.m.
- C. *Teaching an Introductory Statistics Course* – Robin Lock, Saint Lawrence University, 3:15 - 5:15 p.m.
- D. *Teaching Math Courses for Teachers* – Dale Oliver, Humboldt State University, 3:15 - 5:15 p.m.
- E. *Undergraduate Research – How to Make It Work* – Aparna Higgins, University of Dayton, 3:15 - 5:15 p.m.

Working Groups



Groupes de travail

Outreach in Mathematics – Activities, Engagement, and Reflection

Véronique Hussin, *Université de Montréal*
Eric Muller, *Brock University*

Participants

Peter Brouwer	Cindy Grasse	Anna Sierpinska
Chantal Buteau	Frédéric Gourdeau	Christine Stevens
Mary Cameron	Jennifer Hall	Erin Tarala
Malgorzata Dubiel	Michelle Horrobin	Tara Taylor
Doug Franks	Ella Kaye	Sonja Travis
Viktor Freiman	Laura Myers	
John Grant McLoughlin	Alice Sewell	

Focus of the Working Group

Our Working Group discussions focused on two areas of outreach: the opportunities that arise naturally or that we can generate to promote mathematics with students in our own classrooms, and ways to popularize mathematics outside the classroom, in other educational settings, or with the general public.

We worked with classroom activities and examined the role that these could play in promoting mathematics, as could be demonstrated, for example, by a positive change in student attitude and engagement in mathematics. The following questions provided directions for our discussions. Which components of the activities are particularly successful in getting students to reflect on their doing mathematics and for them to develop creativity in mathematics? How does technology provide new ways to promote mathematics for students in our classrooms and for those in other settings? This Working Group report follows the sequence of activities and discussions that occurred over the three days and complements the work of two previous CMESG/GCEDM Working Groups, namely, the 1994 study on “Popularizing mathematics” (Hodgson and Muller, 1994) and the 2001 study on “Where is the Mathematics?” (Mason and Muller, 2001).

Representation within the Working Group

At the beginning, participants were given ten minutes to think about their experiences, academic background, working situations, etc. that were relevant to the topic of the Working Group. It soon became apparent that the experiences and educational backgrounds of the participants were quite varied. Indeed the Working Group brought together elementary and secondary school teachers, professors and graduate students from Departments of

Mathematics and Faculties of Education and a statistician. We all agreed that one of the important outcomes of our time together was the outreach by individuals across educational levels and classroom experiences.

What We Did

The First Day

After individual introductions, participants were divided into small groups and were asked to choose one or more activities to work on, either individually or in their group. Activities included ‘Chessboard Squares and Rectangles’ and ‘Threaded Pins’ taken from the book *Thinking Mathematically* by Mason et al. (Mason et al., 1982), three board type activities: ‘Brock Bugs’ (described in ref. 2), ‘Brock Bees’ (described in Appendix 1), ‘Brock Beavers’ (described in Appendix 1) and a selection of Internet activities (listed in Appendix 1).

While working with their chosen activities participants were asked to focus on the following “In our classes we interact with a ‘captive’ audience, our students who, by choice or requirement, participate in the activities we provide for their learning of mathematics. What is the role of these activities in promoting mathematics, as would be demonstrated, for example, by a positive change in student attitude and engagement in mathematics? What are important components of such activities? What activities are particularly successful in getting students to reflect on their doing mathematics and for them to develop creativity in mathematics? What opportunities for promoting mathematics with our students are provided by technology?” Groups were asked to select the two most significant issues that arose from their experimentation and reflection with the activities. The issues that were raised could be divided into three broad areas. The first addressed the nature of the activity, namely: care should be taken over introductory instructions as they can influence engagement; and, the introduction should be non-threatening and inviting to all, with or without mathematics. The second area focused on the transition to the mathematics, namely: the activity should take the student in a seamless way to the mathematics, and the student should find it safe and be motivated to engage in mathematical discovery; and, the activity should provide space for creativity and for theory building. The third area encompassed questions about the role of the teacher, namely: what role does the teacher play in introducing the activity and in assisting the student to make a transition to the mathematics and, at what point does the teacher unveil the mathematics inherent in the activity?

The Second Day

The first part of this day was devoted to presentations in two separate groups. In one group, Chantal Buteau, Viktor Frieman, and Jennifer Hall focused their presentations at the secondary and tertiary mathematics education levels and demonstrated computer based activities that they had either developed, participated in, or that had been implemented by their students. As these activities, programmes, etc. are so much different from any that have been reported in previous Working Groups, the presenters were asked to provide details of their presentation for this report and they form the contents of Appendix 2. In the other group, Malgorzata Dubiel, Doug Franks, John Grant McLoughlin and Frédéric Gourdeau focused on outreach at the elementary level and briefly outlined some of their experiences and reflections. After the break the Working Group reassembled and focused its discussions on technology and its potential for engaging students in mathematics.

The Third Day

Participants selected to join one of three groups, each working on a different activity but focusing on the same issues. The activities were the game of 31, the game of Brock Beavers and the computer Learning Object called Fire, Fire!! Each group aimed to flush out common experiences, components, issues, etc. within the three different activities. To help with the exploration the following questions were suggested:

1. Where in your province's curriculum could this activity be integrated? Can you envisage this activity being used in different parts of the curriculum?
2. How would you motivate the student's transition to the mathematics underlying the activity?
3. What other mathematics investigations would you generate for the students to work on after they have experienced the activity?
4. What type of student work would you structure to encourage their reflection on the mathematics you hoped they learnt?
5. What are the important components of the activity that help to motivate the transition to mathematics?

The results of these explorations are summarized by a group participant.

The Group's Reflections on the Game of 31 – summarized by John Grant McLoughlin

The two-person game of "31" requires players to alternate turns of adding 1, 2, 3, 4, 5, or 6 to a cumulative total. The player who lands on 31 wins the game. The simplicity of the game is appealing in that neither material resources nor advanced mathematics are necessary to participate in the game. The underlying mathematics provides pedagogically rich ground for analysis, adaptation, generalization, and conjecture. Beginning with analysis, players soon realize that 24 is a winning position. Regardless of the next move by the other player, it will be possible to land on 31. As our experience with the activity reinforced, the tendency is to lock in on seeking 24 without pushing further to realize that 17, 10, and ultimately, 3, are all winning positions. Hence, a game played forward becomes a fine example of working backwards as a problem solving strategy.

The game also lends itself to adaptation. Two possibilities are offered here:

- Change the available numbers and/or target number. For example, allow the players to add 1, 2, 3, 4, 5, 6, 7, or 8 while landing on 46 wins the game.
- Alternatively, select a target such as 31 while providing more numbers to choose from such as 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. (Note that the sum of the available numbers should significantly exceed the target.) The provision here is that each number can only be used once in the game. Landing on the target produces a win; however, often it becomes impossible to land on the target due to available numbers. In such cases, a person loses by playing the last permissible number to keep the sum below the target. (For example, if Player A adds 10 to 19 giving 29 while leaving only 4, 5, 8 and 9, then Player B wins.)

You may wish to consider a challenge: Design a game in which the player going second has a winning strategy. Much doing of mathematics is facilitated by an activity that is accessible to many levels of mathematical learners, including teachers.

The Group's Reflections on the Game Brock Beavers – summarized by Laura Myers

The game Brock Beavers is a probability-based game geared toward students at the secondary level where players try to be the first to get all of their logs to the end of the river. When our group first played the game, we found that the rules were a bit vague. After playing the game, however, we realized that the rules were in fact vague on purpose. Players are allowed to decide where to place their logs at the beginning of the game and they are given this choice because the probability of winning changes depending on the number of logs they decide to place where. Instantly, it was obvious that a winning strategy was directly dependent on

knowledge of probability. If, in the beginning, students did not see the link between these two factors, they would soon figure it out after playing the game once. When players found this link, they would then try to develop a winning strategy based on probability. This game also offers several levels of play, each one increasing in difficulty. Again, the level of difficulty is directly related to probability. As the game becomes more complex, students are encouraged to calculate the probability of winning before playing. Playing the game Brock Beavers made us think about the characteristics of a game (and for that matter any activity) that would be appropriate for use in a mathematics classroom. We felt that, in general, games are the most useful when they are adaptable, flexible and challenging. Moreover, if mathematics teachers want to draw the mathematics out of a game, the adaptability, flexibility and challenge of the game must be directly linked to the underlying mathematics. Based on these conditions, Brock Beavers is, in our opinion, an example of a game that would be an effective tool for reinforcing students' ability to calculate probability.

The Group's Reflections on the Interactive Computer Activity Fire, Fire!! – summarized by Frédéric Gourdeau

A group of us ventured to play a computer game: Fire, Fire!! Part of the appeal came from the comfortable air-conditioned room we were invited to go to. As we were playing, some of us joined efforts, playing in pairs, while others played alone. The speed at which we progressed seemed inversely related to our age, nevertheless we all managed to complete a reasonable exploration of the game. In any case, we were able to exchange our views after playing for a while.

In terms of curriculum, it seemed to us that the targeted levels were Grades 7 and 8 for ratios and proportional reasoning, and Grades 9 and 10 for trigonometry. We concluded that it could also be introduced at the elementary level as it was easy to use and could be attacked using either guesses or some measuring. Playing games like this one could also be used in teacher training.

To be good at this game, one has to extinguish fires as quickly as possible, which in turn means the least number of attempts. To achieve this one is motivated to devise a good strategy, and hopefully to use mathematics. In a class setting, we thought that playing as teams of two would be good: it would promote communication and may help move a trial and error approach to a more mathematical one. This could then be followed by a whole class discussion of the strategies used by some teams, thereby bringing in an external validation. Of course, one may then move to reinforce the trigonometric aspects of the game, but time was short and we did not discuss this much. It was however mentioned that polar coordinates were not far off, and that measuring the distance to a real object could be an interesting out of class activity: for example, taking the class outside to measure the distance between a given place (say, where the teacher is standing) and the middle of a tall building.

For us, there were some important components of this activity that could be applicable in other settings.

- It is easy to get involved and can lead to conjectures.
- It is open and flexible: it is not all spelled out.
- There is the possibility of changing some parameters.
- The mathematical approach is needed to play it better.

Concluding Remarks

The participants concluded that the Working Group had effectively bridged the circumstances between levels of mathematics education and between different classroom situations. The Working Group evolved into a living example of outreach in mathematics. Participants felt that the activities played an important role in this bridging as they served as a safe place for inviting engagement and participation in mathematical discovery at various levels. The activities often were the only common experience for the individuals in the group, and they provided a space for creativity and freedom. However, the activities by themselves would have been insufficient to achieve outreach to all individuals. The Group needed to generate an atmosphere that encouraged individuals to work from within their own context, and to promote idea sharing in a safe environment where individuals did not feel judged by their participation. Some suggested that the Working Group situation we had achieved would be an effective mode of operation at mathematics teacher professional development days. The outreach success within the Working Group raised the following questions, “in outreach activities, is there a model process to help us manage the large gap between levels of education?” The Appendices are a very important and significant part of this report. Appendix 2 focuses on engagement in mathematics through the use of technology. Chantal Buteau describes a new Brock University mathematics major program that is grounded in technology and applications and that engages students in their learning and communicating mathematics. She also describes its outreach to pre-service teachers and an area school. Viktor Frieman describes CASMI (Communauté d’apprentissages scientifiques et mathématiques) a web-based program that focuses on mathematics problem solving and that brings together a community of students, teachers and pre-service teachers. Jennifer Hall describes the work that Statistics Canada has done to make data of interest to students available in a format that they can readily access. This wealth of real life data, provided by Statistics Canada, has really engaged high school students as they develop their major project in the Ontario Grade 12 Mathematics of Data Management course. In Appendix 3, Michelle Horrobin describes outreach to students and parents in Math Night, and in Appendix 4, Tara Taylor describes a sequence of mathematics activities that she uses in her outreach outside her classrooms.

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Appendix 1 – Some Activities Used in the Working Group

Brock Bees

Brock Bees is a 7 by 7 version of the game of Hex introduced in 1942 by Piet Hein (see wikipedia at [http://en.wikipedia.org/wiki/Hex_\(board_game\)](http://en.wikipedia.org/wiki/Hex_(board_game))). Interactive versions are found on the web. An example that provides a choice of dimensions can be found at <http://web.ukonline.co.uk/arthur.vause/Hex.html>

Brock Beavers

Brock Beavers was developed by Dot Miners of the Mathematics Department at Brock University. The object of the game is to get all the logs, that you have distributed as you wish among your three beaver friends, to the beaver dam. Some red and white beans are thrown into the pond and the colours of the two beans that land closest to the centre of the pond determines the beaver that is to move one log to the dam.

Internet Outreach Activities

Three Canadian Internet sites were proposed and participants were asked to reflect on how the outreach using this medium may differ from that using other activities.

1. The University of Waterloo's Centre for Education in Mathematics and Computing, has developed a site for children, parents and teachers that contains "Mathematics Fun Resources and Online Games". www.cemc.uwaterloo.ca/mathfrog
2. Brock University has made the development and implementation of Learning Objects part of the experience of every mathematics student. On its site one finds exemplars of Learning Objects developed by teams of faculty and students as a summer project, and Learning Objects developed by students as part of their course work. www.brocku.ca/mathematics/resources/learningtools/learningobjects/
3. The Canadian Mathematical Society has developed a list of educational outreach activities from across Canada. www.outreach.math.ca/en/outreach-activities

Appendix 2 – Focus on Computer-Based Activities

Mathematics Students' Engagement in Technology Integrated Mica Program – by Chantal Buteau

In 2001, the Department of Mathematics of Brock University launched its innovative undergraduate Mathematics core program, entitled *Mathematics Integrated with Computers and Applications* (MICA), in which students make extensive use of technology to support their learning and understanding of mathematical concepts. All traditional courses, such as Calculus, Algebra, and Geometry, were revised using the MICA guiding principles (Brock News, 2001). Two of these are 1) to encourage student creativity and intellectual independence, and 2) to develop mathematical concepts hand in hand with computers and applications. In addition, innovative project-oriented core MICA courses were added to the program in which students learn to use computer programming to investigate their own mathematics conjectures and to simulate diverse mathematical applications. For more details about the MICA program, see (Ben-El-Mechaiekh et al, 2007, and Brock News, 2001).

Engaging students in mathematics - the MICA classroom

The MICA philosophy is to engage students in mathematics by bringing them to a point where they conjecture and raise interesting mathematics questions. They then use a programming environment (for example, at present in the MICA I course VB.NET) to test their conjectures or to explore their questions. Since, in general, students do not have experience raising open-ended mathematical questions, they need to be guided in this task. In the first MICA course we often choose the Collatz conjecture and prime numbers to prompt their mathematical curiosity. After a brief introduction to the topics, students are divided into small groups and are asked to come up with *any* questions and *any* conjectures. We write all questions and conjectures on the board and discuss their programmability, their interrelations, their interest, etc. For the session to work, the class atmosphere has to be completely non-judgmental. In the beginning students are reluctant to participate, but they soon build up their confidence and show great capacity for raising interesting questions. Much effort is spent to maintain this atmosphere during the whole course and in the following upper-year courses MICA II and III. In all these courses, the students then develop their own programs to test, explore, and visualize their conjecture, theory, and question.

Mathematics students engaging with their original personalized MICA project

An important component of MICA courses is the projects. Students individually design and implement highly interactive and user-friendly original digital environments centered on a mathematical topic of *their own choice*. There are three project types, giving an opportunity for students, starting in their first year, to identify with their future career (Muller & Buteau, 2006). See (MICA project web-site, URL) for viewing student projects, in particular all of those mentioned in this report. One project type is the exploration a mathematical conjecture or topic that the student has formulated or chosen. For example the *Structure of Hailstone Sequences* project developed by first-year student Colin P. in which he graphically explored the structure of hailstone sequences. Another project type is a simulation of a mathematical application. For example, second-year Kylie M. and third-year Matthew L. designed the *Running in the Rain* project in which they simulated, with various parameters, a person walking in the rain and counted the number of drops falling on the person's shoulder. Their goal was to empirically answer the question "Is it better to walk or run in the rain?". A third MICA project type is an interactive environment for teaching a mathematical concept, often called a *Learning Object*. For example, first-year Lindsay C. designed *Exploring the Pythagorean Theorem* for learning, exploring, practicing and playing with the Pythagorean

Theorem. Other examples of student projects are available at (MICA projects web-site, URL).

From our experience (Buteau et al. forthcoming-a), students *dedicate* themselves to this kind of mathematics activities. They take *ownership* of their work and they demonstrate much *pride* in it. Some students show their Learning Objects to their former mathematics teacher; how is that for pride! Through the nature of this *personalized original* activity, they demonstrate their engagement in mathematics and show great creativity in mathematics and in the communication of their understanding of mathematics. Also through their MICA projects, students are prompted to develop their independence in mathematical thinking (Buteau and Muller, 2006).

Mathematics MICA projects in use to engage pre-service students

As a collaborative project between Brock Department of Mathematics and Department of Pre-Service Education, MICA Learning Objects (MICA projects web-site, URL) were introduced in an elementary mathematics methods course (Grades 4 to 8), to engage teacher candidates in mathematics didactics. As part of their learning experience, teacher candidates were required to use a Learning Object and to write their reflections on their experience. A majority of teacher candidates observed that Learning Objects provided a non-judgmental environment that contributed to their engagement in (re)learning mathematics, specifically, mathematics concepts and/or skills they had forgotten or had not learned well in school. They appreciated the contexts (e.g. games), graphics, storylines provided by Learning Objects. Commenting on *Exponent Laws* Learning Object, one teacher candidate noted: “I did not know that you can discuss about mathematics”. One teacher candidate appreciated how the designer of *Money* Learning Object related decimals with Canadian money: “It so nice that the designer emphasized decimal notation in adding or subtracting money”. As well, teacher candidates noted that Learning Objects provided them with ideas for designing their own meaningful mathematics learning tasks.

Outreach - When a grade 5 class engages in designing their own Learning Object

In 2006, a MICA student, Sarah Camilleri, took on the challenge to extend her MICA course experience by doing a collaborative Honours thesis (Camilleri, 2007) project with a Grade 5 class. The aim was to engage an elementary class, from a local francophone school, École Nouvel Horizon (Welland), in designing a highly interactive, engaging and user friendly Learning Object, which was to include games, to focus on a mathematical topic, and which was to be programmed and finally implemented in the classroom (Buteau et al. forthcoming-b)). It yielded *Fractions Fantastiques/Fantasy Fractions* Learning Object. The school principal and the Grade 5 teacher were directly involved in guiding the class through their development of mathematical games and interactive lessons on fractions. The principal noted that students were very much engaged in the activity. Sarah was astonished by the Grade 5 students’ creativity in the mathematical material. It motivated her to face the difficult challenge of integrating all the ideas generated by the students together with the features of a good Learning Object.

Once the Learning Object was completed, not only the grade 5 students, but also the principal and the teacher, were amazed that *Fractions Fantastiques* so closely represented their ideas and material. The class 5 students asked to take home a copy of *their* Learning Object. And they did.

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Enrich Problem Solving Experiences Using Virtual Learning Communities: Example of CASMI Community – by Viktor Freiman

Communauté d'apprentissages scientifiques et mathématiques (CASMI, www.umoncton.ca/casmi) was created in October 2006, and has already attracted more than 5000 members (schoolchildren, teachers, and university students). Twelve challenging problems in math, science and chess are posted bi-weekly, and members can solve them and submit their solutions electronically. Pre-service teachers provide a personal feedback to each member who submits a solution. The discussion forum and archive are other important parts of the community that allow sharing and discussion among all members.

An unprecedented growth of web-based educational resources allows Klotz (2003) to affirm that in mathematics, as in other disciplines, the world-wide web is expanding our concept of the classroom itself, changing what is learned and how it is learned. This affects student-teacher relationship, and provides access to new types of mathematical activities and resources. These can be used by teachers to propose mathematical challenges to meet educational needs of all learners, by learners as they access the learning tools unavailable in the classroom, and by other persons who just want to have pleasure doing some mathematics. Several recent studies report a positive effect of virtual problem-based environments on pupils' motivation toward mathematics. The study of the NRICH project (Piggot, 2004) shows that 1) on-line resources are not suitable solely for the most able but have something to offer pupils of nearly all abilities, 2) enrichment is not only an issue of content but a teaching approach that offers opportunities for exploration, discovery and communication, and 3) effective mediation offers a key to unlocking the barriers to engagement and learning.

Another example of a pedagogically powerful virtual environment has been created within the project Math Forum (mathforum.org). It is built on the idea of interaction between members of a virtual community who interact around the services and resources participants generate

together. These interactions provide a basis for participant knowledge-building about mathematics, pedagogy, and/or technology. The interactions also contribute to what can be described as a Math Forum culture that encourages collaboration on problem posing and problem solving (Renninger & Shumar, 2002).

Our own experience with the CASMI project indicates that combining non-routine challenging problems (Sheffield, 1998) and technology-supported communication between schoolchildren and university students enrolled in initial teacher training (Renninger & Shumar, 2002) create new opportunities for New Brunswick francophone schoolchildren to develop their problem solving and mathematical communication abilities (Vézina & Langlais, 2002, Freiman, Vézina & Langlais, 2005), and hopefully attract more children to science and mathematics. We see CASMI's virtual community as a learning and teaching resource to construct bridges between the traditional classroom learning and the learning at large where learning objects are personally defined or adapted to the personal needs of learners (Jonnaert & Vander Borgh, 2004).

Our CASMI problems aim to adequately challenge every member of the community independent of age, education level or status. We try also to cover a variety of topics and contexts. All problems are split into four categories named after animals (manchot – penguin, girafe – giraffe, dauphin – dolphin, hibou – owl). Although each group represent different levels of difficulty (manchot – lowest; hibou – highest), we do not put emphasis on it and leave each member to choose a problem of interest. We encourage members to try all of the problems. Mathematical problems focus on the four strands of the New Brunswick French K-12 mathematics curriculum, namely: numbers and operations, algebra, space and shapes, and statistics and probability; and the four didactical principles: problem situation solving, mathematical communication, mathematical reasoning, and making links.

We pay particular attention to the development of a variety of communication tools that include not only the possibility for each member to send a solution, but also to propose a problem and to participate in the discussion forum. These tools help build a communication space. Along with a membership, each member has a personal password protected space (portfolio) that keeps track of personal communication (solutions and formative analysis of each solution, as well as proposed problems, personal micro-community (peers, parents, teachers), and personal information that can be modified at all times). The common space available to all members keeps track of collective knowledge: community news, new problems of the week, bank of all problems with the link to the CASMI archive, analysis of the last problems posted with some interesting solutions, discussion forum, links, and surprise box. Each solution sent via the CASMI electronic response form is analysed by university students enrolled in the mathematics didactic courses and student-experts who are members of our management team. We use common evaluation criteria: problem interpretation and definition, strategy, and execution as well as ability to communicate and reflect on the solving process and obtained results. A personal formative feedback is then put in each member's portfolio. Via the micro-community communication tool located in the personal portfolio page link, teachers can have access to their students' portfolios. In our responses to schoolchildren, we aim to give a friendly, encouraging and at same time constructive evaluation that guides them in their learning process, to improve their problem solving and communication skills and to encourage them to continue working on challenges in science and mathematics.

The management team is responsible for developing and maintaining all virtual and classroom-related pedagogical activities of the CASMI community. In the virtual part, we ensure the preparation and posting of new problems, coordination and realisation of all work related to the evaluation and analysis of the solutions, monitoring and mentoring discussions at the CASMI forum, and communication with members (collective and individual), as well as coaching and tutoring on how to use different options. The on-site activities include

seminars and workshops with teachers, school administrators, and university students as well as direct presentations in the classrooms with schoolchildren.

Setting up a research agenda is an important part of the community life. Researching any education setting is not an easy task. Study of virtual communities of learning is an even more complicated enterprise that requires constant evolution of research problems and innovative approaches to the construction and refining of the theoretical and methodological framework. The CASMI community is a new object of research and we are planning to study it on both levels – macro-level: questioning the development, functioning and impact of it as the whole system, as well as a micro-level: zooming particular aspects of the community life and featuring special cases related to the teaching and learning.

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Statistics Canada's Resources for Secondary Mathematics Teaching – by Jennifer Hall

Statistics Canada offers a wide variety of bilingual resources for teaching mathematics at a high school level. Most of the resources described very briefly below are also appropriate for either elementary school or college/CEGEP teaching.

Lesson Plans

www.statcan.ca/english/kits/courses/math.htm

www.statcan.ca/francais/kits/courses/math_f.htm

Lesson plans are provided for a variety of mathematics topics. Lessons are divided by grade level and further sub-divided by mathematical concept and Statistics Canada resource used in the lesson (e.g., Census at School, Census of Canada, E-STAT).

Datasets

www.statcan.ca/english/kits/courses/smath2.htm

www.statcan.ca/francais/kits/courses/smath2_f.htm

A wide variety of datasets, using both aggregate and individual level data, are provided for student or teacher use.

Summary Tables

www40.statcan.ca/

www40.statcan.ca/index_f.htm

Summary Tables provide an overview of Canada's people, economy, and governments. Over 500 tables are available and are automatically updated as new data become available. These tables can be searched by subject, province or territory, or metropolitan area. Tables can be downloaded directly in Excel, CSV, or HTML formats.

Community Profiles

www12.statcan.ca/english/census06/data/profiles/community/Index.cfm?Lang=E

www12.statcan.ca/english/census06/data/profiles/community/Index.cfm?Lang=F

Community Profiles provide census information (e.g., education, population, income) regarding communities (e.g., cities, towns, municipalities). These data, which are provided as totals as well as separated by sex, can be compared to another community or to the province or territory in which it is located. Data can be downloaded directly in CSV format.

Microdata

www.statcan.ca/english/kits/courses/smath2.htm

www.statcan.ca/francais/kits/courses/smath2_f.htm

Microdata provide individual-level data. The most recently posted microdata files, the 2001 Census and the 2002-2003 National Longitudinal Survey of Children and Youth, Ages 16 to 17, can be directly downloaded in CSV, Excel, and Fathom formats. Learning activities are provided to supplement these datasets. Microdata files on the 1991 Census and the 2002-2003 Joint Canada-United States Survey of Health are also provided.

E-STAT

<http://estat.statcan.ca>

E-STAT is an enormous database that contains data from CANSIM, the Canadian Socio-economic Information Management System, and the Census of Canada. CANSIM contains data from over 250 different surveys regarding socio-economic topics about Canadians, resulting in over 2 700 tables and over 36 million time series! Census of Canada data are provided for the 1986 to 2006 Censuses, as well as historical censuses from 1665 to 1871. This enormous wealth of data is free to all educators and students, but access from home requires a username and password. See www.statcan.ca/english/Estat/userpass.htm (English) or www.statcan.ca/francais/Estat/userpass_f.htm (French) for details.

CANSIM or Census data in E-STAT can be graphed directly in E-STAT, displayed in tables, or downloaded in a variety of formats.

External Statistical Sites

www.statcan.ca/english/reference/othsit.htm
www.statcan.ca/francais/reference/othsit_f.htm

Statistics Canada provides links to the websites of provincial and territorial statistics offices and other statistical organizations in Canada. Links to the national statistics offices of more than 130 countries are provided. Furthermore, links to relevant Government of Canada websites are also available.

Note: These websites are not subject to the Official Languages Act of Canada, so may not be provided in English and/or French.

Function Modelling using Secondary Data from E-STAT

www.statcan.ca/english/edu/mathmodel.htm
www.statcan.ca/francais/edu/mathmodel_f.htm

The Function Modelling page features CANSIM datasets from E-STAT that closely approximated by linear, quadratic, exponential, sinusoidal, and logistic functions. Data can be graphed directly in E-STAT or exported to a data analysis software program to perform mathematical analyses.

Note: See Function Modelling using Secondary Data from Statistics Canada's E-STAT Database (Hall, 2007) in the Ad Hoc workshop section of this publication for a full explanation.

Statistics: Power from Data!

www.statcan.ca/english/edu/power/toc/contents.htm
www.statcan.ca/francais/edu/power/toc/contents_f.htm

Statistics: Power from Data! is an excellent reference for both students and teachers, covering a vast variety of statistics-related topics. A veritable encyclopedia of statistics information, Statistics: Power from Data! also offers exercises, case studies, and an online graphing tool.

Teacher's Guide to Data Discovery

www.statcan.ca/english/freepub/12-593-XIE/12-593-XIE2007001.htm
www.statcan.ca/francais/freepub/12-593-XIF/12-593-XIF2007001.htm

The Teacher's Guide to Data Discovery supports teachers in helping students develop basic statistical skills. It provides instructions for finding interesting, grade-

appropriate Canadian datasets, choosing appropriate graph types for data display, and calculating basic statistical measures, by hand or with statistical software.

Census at School

www.censusatschool.ca

www.recensementecole.ca

Census at School is an in-class online survey project for students in Grades 4 to 12. Students anonymously complete online surveys about their lives and enter data into the national Census at school database. Some questions are similar to the Census of Canada and others relate directly to curriculum expectations, such as measurement and conversion. Students then can perform statistical analyses on their class data and compare themselves to Canadian summary results from the previous year. Census at School is also conducted in the United Kingdom, Australia, New Zealand, and South Africa so international data can also be retrieved for comparisons. More than 20 learning activities are provided, sorted by grade level and mathematical concept.

TeacherWeb Site: Math Resources using Canadian Data

www.teacherweb.com/on/statistics/math

www.teacherweb.com/on/statistique/math

Although not an official Statistics Canada site, Math Resources using Canadian Data is maintained by Joel Yan and Jennifer Hall of Statistics Canada and features lessons, activities, articles, datasets, and student projects related to Statistics Canada resources. This site focuses primarily on MDM4U, the Ontario Mathematics of Data Management course for Grade 12 university-level students. Additional resources for Census at School, E-STAT, Function Modelling, Community Profiles, and Health Microdata are also available on this site.

Note: Not all resources featured on the English site are available on the French site.

Contact Us!

Statistics Canada provides free in-class and professional development workshops at elementary schools, high schools, colleges, and universities (for students, teachers, and teacher-candidates) about all of our educational resources. Regional representatives are available for workshops in all provinces and territories. See **www.statcan.ca/english/edu/reps-tea.htm** (English) or **www.statcan.ca/francais/edu/reps-tea_f.htm** (French) for contact information. For workshops in the Ottawa area, contact Jennifer Hall (jennifer.hall@statcan.ca or 613-951-4869) or Joel Yan (joel.yan@statcan.ca or 613-951-2858). For general statistical inquiries, call 1-800-263-1136 (8:30 am to 4:30 pm EST, Monday to Friday) or email infostats@statcan.ca.

Appendix 3 – Experiences with School Mathematics Nights

by Michelle Horrobin

There is a spark of energy in the air as parents and children file into the school. Excitement, curiosity and anticipation are on the faces of young and old alike. Is it a sports event, is it a musical – no, it's Math Night!

Math Night is an annual to semi-annual event wherein we celebrate mathematics above and beyond our everyday life. Mathematical concepts that span all strands of the New Brunswick Curriculum are promoted. Activities, organized in a kiosk fashion, are student-led. Students in Grades 3 to 5 lead their own activities while students in Grades K-2 have Grade 5 peer helpers. The children take pride, ownership and enjoyment from leading an activity. To maximize student participation, the activities are led by partners in blocks of 30 minutes for the two hour evening. On average K to Grade 4 classes can have 16 to 24 students actively participating in leading activities. The Grade 5 children are more involved due to their peer helping role with the younger students, and it is not uncommon to have everyone in the class involved.

Student-led activities serve to focus Math Night on a student audience, increase attendance by both parents and students, and most importantly provide many students, if not all, an opportunity to explore the area of mathematics that interests *them* most. Finding their individual area of interest of mathematics increases their excitement and desire to learn new things in math. Math Night can often be the catalyst that decreases math anxiety and helps students to realize that math is not a scary thing.

The curriculum strands of numeration, patterns and relations, shape and space, probability, and data management are divided between the classrooms at every grade level, with some classrooms promoting more than one strand. Each classroom offers at least one activity, game, challenge or mini-lesson in their chosen strand. More often than not, classrooms will use multiple activities to promote a curriculum strand and thus involve more students in leading the activities. A balance is sought among the number of games, activities, mini-lessons and challenges that are offered. Although specific curriculum strands are assigned to individual classrooms, overlap across curriculum strands as well as grade levels can occur. A final verification of the events is necessary for optimal promotion of mathematical concepts and ideas.

In an effort to clearly articulate what we accomplish during Math Night, we provide the following examples of activities that have been used in our Math Nights in the past. Although this list is far from complete, it provides an idea of the plethora of choice in activities that we have used in the promotion and enjoyment of mathematics.

Activity & Brief Description – for Upper Elementary

Battle Ships (Mattel Games)

Using co-ordinates, children try to sink the opposition's battleship.

Can you make a full set of pentominoes? (Unknown source)

Using manipulatives, children try to create a full set of pentominoes.

Can you construct a bridge using no right angles? (Unknown source)

Using newspaper, straws and tape, children are challenged to make a structurally sound bridge that contains no right angles.

Race to 36 (I Get It – 3, p. 34)

CMESG/GCEDM Proceedings 2007 • Working Group Report

Children draw four cards from a pack. They must combine the numbers using +, -, and x to get 36. They earn 5 points per combination. First person to 100 wins.

Fill It Up! (I Get It – 3, p. 68)

Each child starts with an empty 10x10 grid and the area to be coloured is specified by the roll of a dice. The child who colours the most squares is the winner.

Name that Robot! (I Get It – 4, p.6)

Children are given a decimal number; 2.07; 0.67; etc., and use base 10 blocks to design their robot and compare it with a partner.

Leftovers Again (I Get It – 4, p. 38)

Using a chart to track remainders, each player rolls three die - multiplies two of the numbers and divides the product by the third number. The goal is to generate the largest remainder.

Palindrome Pals (I Get It – 4, p. 60)

Children are challenged to generate their own palindrome.

Tile It (I Get It – 5, p. 120)

Children choose a card from a pack numbered 0-8, and select pattern blocks with the corresponding number of sides. Children then test to see if the shapes tessellate. Tessellating shapes win points.

Name Those Sides (I Get It – 5, p. 88)

Children draw one 'area' card and one playing card. Children decide whether the number could be one of the sides of the area. If yes, the player collects a coloured tile – the goal is to fill an area that has a perimeter of 22 units.

How Low Can You Go? (I Get It – 5, p. 60)

Make a game board with place values to 100 000 000. Playing cards are drawn in turn and each player arranges the cards to generate the smallest number.

Activity & Brief Description – for Primary

Patterned Set Recognition (Teaching Student Centered Math Grade K-3, p. 43)

Using counters of any type, children are challenged to make up all of the sets for a number between 1 and 10.

Covered Parts (Teaching Student Centered Math Grade K-3, p. 50)

Using a predetermined number, a child counts out the appropriate number of counters and then covers some of the counters with a margarine tub: their partner must determine the number of counters under the tub.

The Smarties Count (Source unknown)

Each child counts the number of Smarties of each colour in their box. In small groups they compare their tallies and decide whether they are the same, similar, or very different and then propose reasons.

Counting Shapes (Interactions 1, p. 62)

Using a photocopied sheet that contains many hidden geometric shapes the children are challenged to find and chart as many shapes as they can.

Pattern Animals (Cuisenaire Co. of America Inc.)

Given the outline of an animal children use pattern blocks to fill in the animal's body. In a small group children compare and explain their results.

Au Jeu! (Interactions 1)

A game card has number combinations in squares and several numbers in the middle. Children roll a dice and make the number indicated in the box – if that number is in the middle they place one of their markers on it. The goal is to cover as many of the numbers in the middle with their markers.

One More / One Less (Source unknown)

Using a game sheet with a start and a finish children take turn to roll a dice and draw a card labeled one more / one less or two more / two less. The aim is to reach the finish first.

Waterworks (I Get It – 2, p.74)

Eleven containers that hold about 1 liter are labeled 2 – 12. The child rolls two dice and the total determines the container. The child estimates whether the container holds more than a liter, less than a liter or exactly one liter. A pre-measured liter of water is poured into the container to verify the answer.

Race to 100 (I Get It – 2, p. 90)

Placing a meter stick between two children, each player rolls two dice and adds the numbers. They choose the Cuisenaire Rods that correspond with their total and lay them on their side of the meter stick. First player to 100 cm wins.

In addition to these activities, an annual favorite is our Estimation Station. As estimation is a concept that spans all grade levels and easily captures the attention of most everyone, the Estimation Station, is always a big success. Several different items are presented for estimation. What you could find at the Estimation Table includes the perimeter of the gym using popsicle sticks, the perimeter of the gym using the height of the principal, the number of jelly beans in a small cup, the number of jelly beans in a large glass jar, and the height of the principal in mm. Students make their estimation and then fill out a ballot with their name and their answer. They place their ballot in the corresponding bin. A few prize winners are randomly selected from acceptable estimates in each the bin.

We choose the activities, challenges, and games to demonstrate that mathematics is not an isolated subject taught strictly in a classroom. The goal of Math Night is to promote mathematics that exists not only in our classrooms but also in the world around us.

Appendix 4 – Making Connections in Math: Some Specific Outreach Activities

by Tara Taylor

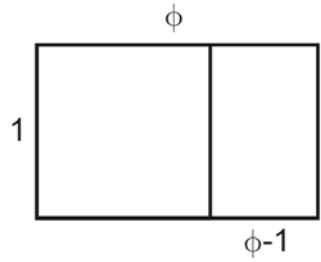
I like to show people that one can find beauty, creativity, fun and surprise in mathematics, and that mathematics is more than just numbers. In this report, I will briefly summarize a few activities that have worked well with students of any age. The activities can be modified depending on the age group. As I continue to do more outreach activities, I have learned that it is usually best to explain very little before an activity and let people make discoveries on their own. Then I will go deeper into the theory if it seems appropriate to do so. I should say that none of these activities are my own original ideas: I just find that they have worked well for me.

My own interests in mathematics generally focus on geometry, and fractals in particular. So I usually start with something that is not about numbers: a brief description of the Sierpinski Gasket as an example of a fractal. If the students have seen geometry and equilateral triangles before, I divide the students into at least nine small groups. Each group constructs an equilateral triangle on a big poster board (and they have to figure out how to do this with rulers and protractors - it is always interesting to see how they do it). Then they do the first iteration by finding the midpoints of each side of the triangle and drawing the triangle with these midpoints as the vertices. They keep doing this for at least two more levels.



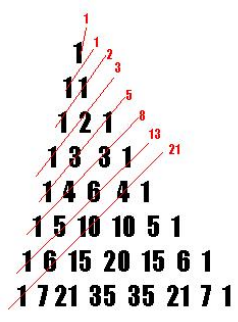
Then they colour in the remaining triangles (black in the figure above). Here they can be quite creative. If the students are younger, I provide each student with an individual copy of their own little Sierpinski Gasket up to three levels of iteration, with dots inside the triangles that they need to colour. Once the colouring is done, I don't explain what I am doing: I just start to assemble a giant version of the Sierpinski Gasket on the wall (this is why you need nine groups, or some higher power of three). Some students start to see right away what is going on, and they will explain it to their friends. Depending on the group and how much time is available, I may talk about some theory behind fractals after this. It is fairly straightforward to show that as you continue iterating (removing middle triangles), the total perimeter goes up without bound and the total area decreases without bound. So after infinitely many iterations, you have something with infinite perimeter but zero area. Then I may talk about the idea of dimension - starting with familiar objects like straight line segments, squares and cubes. If you divide a line segment in half, you get two smaller versions of the original. If you divide each side of a square in half, you get four smaller versions of the original square. Then you can see that there is a pattern that corresponds with the dimension of these objects (the number of smaller versions is equal to 2 raised to the dimension), and you can use the same pattern to find the dimension of the Sierpinski Gasket. In this case, you divide each side of the triangle in half and get three smaller versions. Thus the dimension d would be given by $2^d=3$, or $d = \ln 3 / \ln 2 \sim 1.585$, which makes sense given the infinite perimeter but zero area.

To change direction from the modern geometry, I usually follow with some activities around the golden ratio. First I hand out a sheet of paper that has five different rectangles on it and ask the students to pick their favourite. Two of the five rectangles are golden rectangles (where the ratio of the length to the width is $f = (1 + \sqrt{5})/2 \sim 1.618$). Every time I have done this activity, the most popular rectangle is a golden rectangle (but I suppose there could always be an exception!). Next I get the students to get into small groups and do some measurements. The golden ratio can be found in many different ratios on the human body. Usually height versus height to belly-button works best, but you can also check ratios like shoulder to finger-tip versus elbow-crease to finger-tip. Each group will find that the ratios are close to 1.6 on average. Thanks to the Da Vinci Code book and movie, many students will already know what is going on, but they usually haven't actually checked it, so they still find this interesting. If there is time I will show some examples of the golden ratio in art and nature (there are many good websites available).



I usually follow the golden ratio activity with something on the Fibonacci numbers (1, 1, 2, 3, 5, 8, ...). I get the students to find ratios of successive numbers, and it doesn't take long before they realize that the ratio is getting closer to f , so there is a connection between the golden ratio and the Fibonacci numbers. Sometimes I talk about Fibonacci numbers before the golden ratio: either order works well. One example of Fibonacci numbers in nature that I like to mention is that the number of petals on a flower is usually a Fibonacci number.

The last activity that ties everything together involves Pascal's triangle. I give the students a sheet with the first few rows of Pascal's triangle filled in and ask them to fill the rest. Older students have often seen it before, but it is not difficult to understand what to do. We will talk about different patterns that we see. In particular, the sums along the diagonals are the Fibonacci numbers, so that gives the connection. To show that Pascal's triangle is useful, we talk about ordering pizzas. If there are eight possible toppings, only one size, no double toppings, how many ways are there to order a pizza with no toppings? With one topping? With two toppings? Here we actually make a list to see how tedious this is. So for three toppings, instead of making a big list, we realize that this just corresponds to the eighth row of Pascal's triangle. Finally, to put it all together, I ask the students to colour in all the odd numbers. This will yield a Sierpinski Gasket, which is usually a surprise.



To summarize, there are four main themes: Sierpinski Gasket, golden ratio, Fibonacci numbers and Pascal's triangle. The order of presentation can change and some details can be left out. In my experience the students generally like these topics and like to see the surprising connections.

Geometry, Space and Technology: Challenges for Teachers and Students



Shelley Hunter, *New Brunswick, School District 41*
Donna Kotsopoulos, *Wilfrid Laurier University*
Walter Whiteley, *York University*

*2D or not 2D? That is the question.
Whether 'tis more global in the mind to suffer
The axioms and deductions of outrageous proofs,
Or to take arms against a sea of formalisms,
And by opposing, make sense of them. To glide, to turn;

No more; and by a turn to say we end
The heart-ache and the thousand unnatural fears
That mathematics is heir to — 'tis a transformation
Devoutly to be wish'd. To glide, to turn;
To turn, perchance to reflect. Ay, there's the rub,

For in that turn of half what revelations may come,
When we have shuffled off this mathematical toil,
Must give us pause. There's the respect
That makes geometry of so long math,
For who would bear the distractions and disconnect of courses in mathematics,

Th' professor's wrong, the proud student's frustration,
The pangs of "why", the delay of "how",
The insolence of PhD, and the cognitive bullying
That the student of th'unworthy takes,
When he himself might his sense make
With a box of polydron? who would Euclidean proofs bear,

To think and sweat under a weary math,
But that the dread of something higher dimensional after Flatland,
The undiscovered country from whose fourth dimension
No investigator returns, puzzles the will,
And makes us rather repeat those deductions we have
Than to try others that we know not of?

This curriculum does make cowards of us all,
And this resolve of making meaning
Is sicklied o'er with the pale cast of high stakes testing,
And math ed fori of great pitch and moment
With this regard their currents turn awry,
And lose the name of action.

3-D and then 2-D, that is the answer.*

2D or not 2D? That is the Question

Children live and learn in the third dimension. Early school geometry tends to disconnect students from physically-based experiences, creating formidable challenges later on when students are required to reason in the third dimension. One contributing factor is that many teachers are inadequately prepared mathematically and pedagogically, to *do* and to *teach* geometry and thus are unable to support students in *their* learning of geometry in space (Gal & Linchevski, 2005). Consequently, geometry curriculum is increasingly marginalized (including in university curricula) despite the growing importance of spatial information and reasoning in many areas outside of mathematics (Hoyles, Foxman, & Küchemann, 2002, p. 121).

The purpose of this working group was to explore geometry and spatial reasoning from multiple perspectives (with a focus on secondary and tertiary levels), for both content knowledge and pedagogical knowledge. Participants engaged in geometrical inquiry through key rich explorations, and collaborate with others in their domains of interest (i.e., teachers, mathematics college/professors, mathematics education researchers) to consider both the *directions* and *tools* for strengthening geometric and spatial reasoning for students.

Drawing from the famous Shakespearean soliloquy from Hamlet, our question framing the working group's deliberations and investigations was: 2D or not 2D? That is the question. The group summative report also took the form of a parody of the famous soliloquy (above). We present a summary of our discussions along with a visual narrative of our engagement with geometric inquiry. This report also ties to the larger literature that matches our working conclusions. We have attempted to weave these into both an effective record and an agenda of key points for further contemplation.

Geometry Education in a State of Flux

According to the Conference Board of the Mathematical Sciences (2001), "the visual side of geometry makes it an excellent place to explore the interplay of mathematics and cultural traditions. The visual arts of nearly every ancient and contemporary culture embody important geometric concepts and principles" (paragraph 45). Despite this endorsement of geometry, our discussions confirmed the message that "there is evidence of a state of flux in the geometry curriculum, with most countries looking to change" (Hoyles et al., 2002, p. 121).

For example, the state of flux is widely evident in the Province of Ontario. Commencing in late 1997, the province of Ontario introduced a new elementary and secondary mathematics curriculum. Implemented in stages, the new curriculum was intended to reflect wide-ranging curriculum goals, very closely modeled after those identified by the National Council of Teachers of Mathematics/NCTM (2000). With the release of the grades 11 and 12 secondary curriculums (Ontario Ministry of Education and Training/OMET, 2000), the residue of geometry was a pre-university, twelfth grade course, entitled *Geometry and Discrete Mathematics*, with a focus on proof, rather than geometric reasoning.

Prior to the twelfth grade, the learning trajectory of geometry in Ontario's new curriculum ends dramatically in the elementary panel. The gaps created by the grades 9 through 11 curriculums resulted in major difficulties for all but the most mathematically able students and thus resulted in a declining enrollment in *Geometry and Discrete Mathematics*. In the most recent review just completed, in which several members of the working group played significant roles, this course has been removed.

At the post-secondary level, the importance of geometry, particularly to those in the mathematical and engineering sciences, was not lost, but this was overshadowed by a focus on calculus, and pre-calculus. Many participants noted comparable examples from their own

academic jurisdictions. Despite the growing importance of geometry in many areas outside of mathematics, often in connection with problem solving using computers, geometry has been marginalized in many mathematics curricula (including university curricula). Numerous participants in this working group indicated that, at the university level in their institutions, geometry (Euclidian or otherwise) was virtually non-existent – or trivialized at best. This demise of geometry comes as no surprise and is a strong example of the “flux” that Hoyles, Foxman, and Küchemann (2002) identify .

Many factors have contributed to this state of flux and the ultimate demise of geometry in Ontario and other jurisdictions (both regional and institutional). One factor already discussed is the reality that many teachers of mathematics, at all levels, are often inadequately prepared to teach geometry and thus are unable to support students in their learning of geometry. Limited “geometrical pedagogical content knowledge” (i.e., knowledge of geometry in addition to knowledge of how to teach geometry) (Shulman, 1986, 1987) inadvertently further marginalizes geometry as legitimate mathematics curriculum. The geometric preparation of future mathematics teachers is also at risk (Whiteley, 1999).

Another factor contributing to the state of flux is the type of geometric knowledge typically emphasized in school curriculums. Geometry is often reduced to axiomatic theorems and proofs with limited exposure to the visual/spatial/transformative aspects of geometrical reasoning (Conference Board of the Mathematical Sciences, 2001). Concerns over the types of geometric knowledge, and most particularly geometry isolated to axioms and proofs versus broader geometric reasoning, perpetuated in school curriculums has a long history of critique (Freudenthal, 1971; Henderson, 1995; Whiteley, 1999). This debate continued in this working group.

There is a lack of consensus amongst mathematics educators and mathematicians on the sorts of geometric knowledge that ought to be taught in schools and the sorts of geometric reasoning that has the most saliency in other fields such as graphic design, computer software design, information systems, human sciences, architecture, and so forth. We identified this issue and offered some possible directions through the kinesthetic, visual/spatial connections and physical based connections of student experiences. The resulting lack of spatial problem solving ability was identified a major barrier for many students, and a source of anxiety for many educators at the elementary, secondary, or tertiary levels. As educators, we struggle to bring students back into thinking in the third dimension and higher.

“The Sun Is at the End of My Normal Vector”

Over the course of the three days, the working group explored geometric reasoning from a highly visual/spatial perspective. Technology (computers and programs for virtual explorations of space) seemed to be an appropriate companion or extension to these physical experiences. We started each of our investigations with the physical forms and objects (see the photos!). We did explore several of these with computer programs (GSP, Cabri 3-D) but these were less transparent than the physical models. The consensus was that these physical contributed to learning in essential ways, as well as offering multiple points of entry and engagement, and sources of surprise. The following visuals chronicle our investigations which resulted in the discussions described in this report.

Properties of quadrilaterals



The popcorn box problem



Symmetries of kites



2D or not 2D? What NOW Is the Answer?

As students progress in their education from kindergarten to the end of their secondary education, mathematics becomes increasingly divorced from their lived realities. Students begin initially learning kinesthetically and in the third dimension. Indeed, very little, if any at all, of the lived experience of young children occurs in the second dimension. Yet, as children progress through their mathematics education and geometry education in particular, learning is increasingly isolated to two dimensions. On the other hand, we noted the significant, even essential, role of spatial reasoning in post-secondary studies in many areas – with students ill-prepared, even at risk of losing these abilities while passing through the stages of ‘use it or lose it.’

The "Research Agenda Project" a section of the NCTM Research Committee states that a major imperative of mathematics educational researchers is the need to "*formulate a research agenda that focuses attention on critical problems of practice* [author's emphasis]" (NCTM Research Committee, 2007, p. 110). RAP proposes that, similar to Hilbert's formulation of mathematics questions in 1900, mathematics researchers need to "identify key research questions" (p. 110) that "reach consensus on the major researchable questions in each of the important research territories - a *research agenda* [author's emphasis]"

that addresses major problems of practice and is informed by the experiences and expertise of practitioners" (NCTM Research Committee, 2007, p. 110). We propose that one "problem of practice" is the "dilemma of geometry education."

As educators and researchers, a cohesive problem of practice needs to be defined in relation to geometric inquiry in classrooms – from elementary to secondary – that explores wider and more divergent views of geometric reasoning beyond axioms and proofs whilst at the same time developing deductive reasoning. The question that remains is this: Can we develop a geometry curriculum that promotes 3D reasoning but still develops deductive reasoning? The "problem of practice" here is both political and mathematical. Some mathematics educators may have little commitment to using geometry to promote "deductive" reasoning and argue that reasoning alone should be the aim.

Despite the political and mathematical divides, we suggested that three dimensional reasoning begins with rich, grounded, meaningful, and connected activities in earlier years of education, by teachers who are able to connect, extend, and respond to children's thinking. This foundational work in geometry then can and should be extended to the secondary panels and beyond – in a consistent and stable manner – meaning not divorced from students' physical experiences and relationships or from their embodied visual and spatial reasoning. Consistency across levels of mathematics education in curriculum and pedagogy is complex. Consensus with respect to the mathematics individuals ought to know varies. This leads us back to the need to establish and define the "dilemma of geometry education" through one cohesive "problem of practice."

The research and discussions in our working group suggests there is great need for significant deliberation and action in terms of how geometry, space and technology is experienced by students of all levels in mathematics curriculum. The change is complex and worthy of more contemplation, with a purposive goal of consensus. The discussions in our working group highlighted the pressing need to keep geometry education in stride with the demands of society. The experiences also show substantial interest in change and real possibilities for us to move forward. Therefore, in closing, the organizers conclude:

*2D or not 2D? That is the question.
3D and then 2D. That is our answer.*

Endnote: Resource pages for the working group are at:
<http://wiki.math.yorku.ca/index.php/CMESG>.

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The Design and Implementation of Learning Situations

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Introduction and Description

As she walked by Mr. Clark's classroom, Ms. Rochette, the principal of the school, saw Mr. Clark standing at the back of his class. He was watching his students, who were busy discussing a math problem in small groups. In one of the groups, two students were arguing about how to solve the problem but they could not reach an agreement; the third student was not sure about who was right. At lunchtime, Ms. Rochette commented on what she saw, "The students seemed very interested." "Yes", replied Mr. Clark, "but there was a group that could not agree on how to solve the problem." "So what did you do?" asked the principal, "Did you explain to them how to solve it?" Another teacher interrupted, "Of course not! He can't!" Immediately, another teacher protested, "Of course he can!" and someone else vigorously added, "He must!"

Learning situations involve designing classroom situations conducive to learning. As Mr. Clark's episode intimates, a learning situation goes beyond the choice of a good problem. But what is it exactly? What *is* and what *is not* a learning situation? How do we design and implement them?

Alors qu'elle longeait la salle de classe de M. Clark, madame Rochette, la directrice de l'école, aperçut l'enseignant à l'arrière de sa classe. Il observait ses élèves occupés à discuter en groupe d'un problème de mathématiques. Dans l'un des groupes, deux étudiants s'argumentaient sur la solution à donner au problème, sans être capables de parvenir à un accord. Le troisième élève du groupe n'arrivait pas à déterminer lequel des deux avait raison.

Durant le lunch, madame Rochette fit quelques commentaires sur ce qu'elle avait vu: «Les étudiants semblaient vraiment intéressés.» «Oui», lui répondit M. Clark, «mais il y a un groupe qui n'est pas parvenu à s'entendre sur la solution à donner au problème.» «Alors qu'avez-vous fait?» lui demanda la directrice. «Leur avez-vous expliqué la façon de le résoudre?». C'est alors qu'un enseignant intervint: «Bien sûr que non! Il ne le peut pas!». Un autre enseignant protesta aussitôt: «Bien sûr qu'il le peut!» et quelqu'un d'autre ajouta d'une façon catégorique: «Il le doit!».

Les situations d'apprentissage, qui sont le thème d'étude de ce groupe de travail, impliquent des activités de la salle de classe menant à l'apprentissage. Comme l'épisode de M. Clark le suggère – épisode dont a été témoin l'un des animateurs de ce groupe de recherche – une situation d'apprentissage va au-delà du choix d'un bon problème. Mais qu'en est-il exactement? Comment peut-on définir ce qui *est* et ce qui *n'est pas* une situation d'apprentissage? Comment les concevoir et comment les mettre en application?

Before our group tackled the above questions, we began with a round-table introduction during which participants spoke about their expectations of our group. Interests varied from the concrete to the theoretical and from specific to general. The participants came with a variety of experiences – established professors mingled with graduate students and experienced teachers discussed with beginning faculty. Drawing on these varied interests and experiences, a rich discussion ensued.

Brousseau's Theory of Didactical Situations

We began with modeling one iteration of the game, “Race to 20,” modified as a Race to 8. The object of this two-player game is to be the first player to make it to 20 (or 8 in our case). Play proceeds as follows:

- Player A chooses a number, 1 or 2
- Player B adds 1 or 2 and announces the current total
- Player A adds 1 or 2 and announces the current total
- Play repeats until either A or B say 20 (or 8 in our case)

For example, in our Race to 8, one game could proceed as follows, with Player B winning:

Player A	Player B	Total
1		1
	1	2
2		4
	1	5
2		7
	1	8

Participants were invited to play the game three times in small groups, develop a strategy, play again and refine their strategies. During the small and large-group reflection time, many interesting comments and questions were brought up:

- How might our strategy be altered if the target total number or the rules were changed?
- Might we be able to generalize a strategy for a “Race to n ”?
- Many groups used representations of thought to model the game or solution. Does one need to use a visual representation or is it sufficient to find the solution “in one’s head?”

Participants were then invited to reflect on the elements of Brousseau's Theory of Didactical Situations (1997, pp. 3-18) and identify where these elements could be seen in their interaction:

1. **Game.** The teacher selects (or constructs) a learning situation that is related to a game (such as the "Race to 20").
2. **Instruction.** The teacher talks about the game (the situation) and plays with one student (instructions about the rules of the game).
3. **Action.** In this phase, the student is playing with another student. The student acts towards a solution, trying to understand the process to win the game.
4. **Formulation.** In this phase one group plays against another group. A student is playing and representing one group against another student in the same situation. In order to win, it is not enough for one of the two students to know how to play, he/she must indicate to his/her teammates which strategy is being used. That is, the student has an implicit model and through communication with his/her teammates he/she makes explicit his/her model.
5. **Validation.** In this phase of the game, students are establishing "true statements" (theorems) which they are using for winning in an environment of controversial debate.

In our reflection, participants once again brought up some interesting points and questions:

- Brousseau's elements of Action, Formulation and Validation were identified as fluid and non-linear.
- Might we change the game so the object is to lose, that is, the first person to make it to 20 would lose the game? Also, might we change the game so the possible options are to add 1, 2 or 3? How would these alternations change the interactions?
- It may be valuable to be cognizant of the audience that is exposed to the game. For example, our working group has very different motivation levels and interest than, say, Grade 9 math students. Is this game *really* going to motivate a class of 30 14-year olds?

Didactic and A-Didactic Situations

We next invited participants to contrast didactic and a-didactic situations (Brousseau, Idem, pp. 29-34).

Didactic Situation: A didactic situation is a set of reports that settle down explicitly or implicitly between a pupil or a group of pupils, a certain "milieu" (including/understanding instruments or objects) and an education system (the professor) for purposes to promote a process of learning.

A-didactic Situation: An a-didactic situation is a situation that the pupils can and must manage themselves. The situation is selected so the pupil can acquire new knowledge but this is entirely justified by the internal logic of the situation. Once the didactical situation has been introduced, the teaching is absent in the system of the interactions of the pupil with the situation. Then, it is said there is devolution of the situation: the pupil becomes responsible, from the point of view of the acquisition of its knowledge, of the process of learning.

Again, the questions brought up by the group clarified the divide and further challenged our understandings:

- What happens if a teacher makes a suggestion or gives a hint within an a-didactic situation? Does the situation become didactic?
- The type of learning situation depends on the question "What do we want them to learn?"

- Within both situations, students must think of the learning as their “own business” and be motivated to learn and succeed. But how?
- What of the power relations between student and teacher, student and student, student and grade, and teacher and grade within didactic and a-didactic situations?

Astolfi's EXTENDED Version

Brousseau's investigation about epistemological obstacles promoted further research, and for example, Astolfi (1993, p. 319) gave a summarized version of Brousseau's model including the concept of epistemological obstacle:

1. A problem situation is organized around the overcoming an obstacle well identified in the mathematical classroom beforehand.
2. The study is organized around a situation in a concrete environment, which indeed makes it possible for the pupil to formulate assumptions and conjectures. It thus does not act of a refined study, nor as a sort of ad hoc example to illustrate a concept as might usually be found in traditional teaching situations.
3. The pupils perceive the situation that is proposed to them like a true enigma to be solved, in which they are able to invest themselves. It is the condition so that the devolution functions: the problem, although initially suggested by the teacher, becomes "their own business".
4. The pupils do not have, at the beginning, the means to construct the solution, because of the existence of the obstacle that they must overcome.
5. The situation must offer a sufficient resistance, leading the pupil to invest his/her former knowledge, as well as representations, so that it leads to questioning and development of new ideas.
6. For as much, the solution should not however be perceived as out of attack for the pupils, the problem situation shouldn't be a situation that causes a big problem. The activity must work in a proximal zone, favorable with the intellectual challenge to concern and internalization of the "rules of the game".
7. The anticipation of the results and its collective expression precede effective search to the solution, the "risk" taken by each one it is part of the "game".
8. The work of the problem situation functions thus on the mode of a scientific debate inside the class, stimulating the potential socio-cognitive conflicts.
9. The validation of the solution and its sanction is not approached in an external way by the teacher, but results from the mode of structuring of the situation itself.
10. The collective re-examination of the traveled route is the occasion of a reflexive return, in meta-cognitive context; it helps the pupils to be aware of the strategies that they implemented in a heuristic way, and to stabilize them of available processes for a new problem situation.

Several researchers found Brousseau's or Astolfi's models interesting, but when they tried to implement these structures in a mathematical class, it was too “heavy” to implement. How might we find problem situations that fit into the curriculum and that motivate students? Several approaches were made in this direction (see for example Romberg, 1994).

Research Situation in Class

Given the difficulties found to implement Brousseau's model, in France, a new group has pointed out the importance of “research situation in class (RSC)”; part of that group, have published the characteristics of a RSC (Grenier & Payan, 2003, p. 5):

1. A "research situation in class (RSC)" falls under problems of professional search. It must be close to unsolved questions. We make the assumption that this proximity

with unsolved questions – not only for the pupils, for the whole of the class, but also for the teacher, the researchers – will be determinate for the report which the pupils will have with the situation.

2. The initial question must be easy to access: the question must be "easy" to understand. So that the question is easily identifiable by the pupil, the problem must be out of formalized mathematics and, the situation itself must "bring" the pupil inside mathematics.
3. Initial strategies exist, without being essential specific prerequisites. Preferably, school knowledge necessary must be elementary and reduced as much as possible.
4. Several strategies advanced in the research and several developments are possible, as well from the point of view of the activity (construction, proof, calculation) as from the point of view of the mathematical concepts.
5. A solved question very often returns a new question. The situation does not have an "end". There are only local criteria of end.

Working group participants were invited to brainstorm answers to the following example of a Research Situation in Class (Hitt & Passaro, 2007, pp. 120-121).

A runner is following a racetrack. Inside the racetrack there is a big flagpole and a flag. The runner follows the closed racetrack, which thus enables him/her to return to its starting point. While following this racetrack, he/she never passes by the same place twice. We are interested in the distance of the runner with respect to the beginning point and the distance between the runner and the flagpole.

- Sketch a racetrack according to the restrictions.
- Which variables can be identified in this situation?
- Which relations can we find for the selected variables?
- How can we represent these relations?

After initial brainstorming and discussion by the working group participants, media clips of Grade 8 students engaging in the same activity were shown and described by Fernando. The goal of this Research Situation in Class was to eventually allow for the development of the sub-concept of co-variation as a prelude of the concept of "function". Some participant reactions and discussions are recorded below:

- In learning situations, the question must be specified without ambiguity and should eventually lead the students to "discovering" or "coming to" the knowledge desired.
- The questions in the instructions were interpreted differently by different groups of students – some did not understand that distance was to mean "shortest distance."
- Some representations or sketches done by students would not lead to the concept of "function", and were discarded by the teacher in favour of "better" ones. What message does this send?
- The teacher leads the rating (0, 1 or 2) of the particular representations and sketches and students vote on their preferences. The "winning" representations continue to be explored, while the others are discarded. Should not all representations be valued?
- What is more valuable in this situation – the diversity of representations found by students or should we re-focus our question to ask specifically about the relation between variables (to focus towards the concept of function). What exactly do we want students to learn?
- There was a lack of epistemological obstacle for the students in this example.
- Perhaps it would be helpful to organize the milieu so the student can know if his/her solution is adequate.

Learning as Conceptualized Within Socio-Cultural Perspectives

We continued by considering another way to define a learning situation. Within Socio-Cultural perspectives, learning situations rest on the idea of the Zone of Proximal Development [ZDP]. The Zone of Proximal Development is “the discrepancy between a child’s actual mental age and the level he reaches in solving problems with assistance.” (Vygotsky, 1986, p. 187). As Alex Kozulin rephrased it, the ZPD is “[t]he place at which the child’s empirically rich but disorganized spontaneous concepts ‘meet’ the systematicity and logic of adult reasoning” (Kozulin, in Vygotsky, 1986, p. xxxv; see also Kozulin, 1998).

The whole problem is to describe how this meeting between ‘spontaneous’ concepts and cultural ones occurs. Socio-cultural theorists answer that this meeting occurs through learning situations.

Learning situations can be defined by taking into account the perspective arising from the cultural practice of societal institutions such as scientific communities (e.g., mathematicians or cartographers) and the perspective of the learner. A learning situation is a path across the learner’s Zone of Proximal Development. It starts from a point A defined by the learner’s actual cognitive development and goes towards a point B defined by the cultural practice perspective (see Hedegaard, 1998, p. 123).

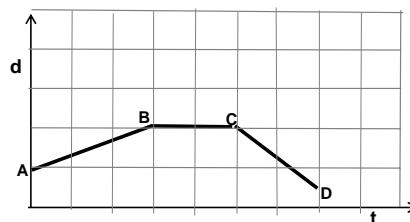
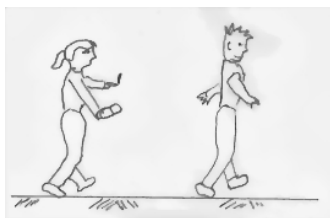
In mathematics education, the *goal* of a learning situation is the *communal acquisition* of cultural forms of thinking mathematically (Radford, 2006a). Learning is not merely building or acquiring knowledge but also to position oneself within a discursive community (Radford, 2006b).

In opposition to rationalist accounts of teaching and learning, the ‘engine’ that moves the student from point A to point B in the learning situation is not to be found in the logic of the mathematical situation – or at least not *only* there. Generally speaking, mathematical situations do not have the power to conjure up mathematical concepts by themselves, for, as historical and anthropological research shows, mathematical thinking is not only about logic but also about aesthetic and other important cultural, contextual aspects involved in human cognition.

This way of considering Learning Situations was further explored in activities with Grade 10 students as they were interacting with motion graphs and a motion detector connected into a graphing calculator (Calculator Based Ranger [CBR] with a TI-83 Graphing Calculator).

Before showing any clips of Grade 10 students interacting with the following problem, we invited participants to consider the question:

Two students, Pierre and Marthe, are one metre from each other. They start walking in a straight line. Marthe walks behind Pierre and carries a calculator plugged into a CBR. We know that their walk lasted 7 seconds. The graph obtained from the calculator and the CBR is reproduced below. Explain clearly what the variable d represents. Explain how Pierre and Marthe were able to get such a graph.



After participants had time to consider the problem, Luis shared short clips of students exploring the same problem. In the three clips, we see the following occur:

Clip 1: Group of girls discuss the problem coming to a conclusion that perhaps both Pierre and Marthe are walking and that “it doesn’t really make sense.”

Clip 2: Same group of girls come up with an idea to explain the graph then call over the teacher. The teacher comes over and asks a few leading questions to help the girls explain the graph.

Clip 3: One of the girls develops a clearer idea of how to explain the motion in the graph.

Clip 4: A boy comes over to the group of girls and shares his idea of how to explain the graph. The group stops considering their initial idea and tries to understand the boy’s.

These clips sparked an interesting and vibrant discussion in our working group. Some of the main ideas were:

- It was interesting to see the girls discuss, reject or modify their ideas to eventually come up with an explanation.
- Why did the teacher intervene? Did the teacher help too much?
- The boy who entered their group and shared his solution overrode any good discussion the girls were having. In fact, the girl who had the initial idea barely spoke while the boy was sharing.
- Perhaps the balance of power shifted to the standing boy while the girls all sat – why was the initial group structured as all girls?
- There were no sexist tendencies on the boy’s part – he was just excited to share his solution with his peers.

Conclusions and More Questions

When thinking about how our working group might conclude, we decided to return to the initial questions articulated in our description:

- What is a learning situation?
- What is not a learning situation?
- How might we design and implement learning situations?

Participants were invited to choose a theoretical framework, either their favourite one, or one that was discussed over the three days – Brousseau, Didactic/A-Didactic, Astolfi’s Extention, Research Situation in Class or Social-Cultural Approaches. Thinking within their chosen framework, participants were encouraged to reflect and discuss the above questions.

Much of the participant feedback was in the form of more questions. The theoretical frameworks, examples and student clips sparked vibrant discussion and debate over the three days, some of which is summarized below. Please note that these summary-type statements are only short snippets of more involved and deep conversations.

- What is learning?
 - Learning depends on the purpose of the teacher, or curricula, or students.
 - Perhaps learning is a change in cognitive structure, or the overcoming of a (learning) obstacle, or a change in participation, or a change in identity.
 - Is every situation a learning situation?
 - Every situation is a learning situation.
 - How is learning seen or assessed?
-
- A learning situation is socially and culturally based validation (based on Vygotsky).
 - A learning situation is overcoming an obstacle (based on Astolfi).

- A learning situation is the acquiring a new mathematical way of thinking (based on ideas put forward by socio-cultural perspectives).
- A learning situation is student validation of his/her own learning (based on Brousseau).
- A learning situation is an object of intention or conceptual understanding that is undertaken through a cycle of interpretation/re-interpretation, discovery and validation.
- A learning situation starts from what the students know then presents something that needs a resolution. There is room for solitary work within, and room to reflect on the process and the learning that occurred. A learning situation results in the ability to go further and extend.
- A learning situation helps students move into the mathematical culture. Instructors need to be aware of prior knowledge and collective understanding.
- A learning situation is a problem (for the student) that should produce engagement and encourage an investment. There might be no *a priori* known approach to find the solution and current knowledge and tools are useful.
- If there is an unbridgeable gap, there is no learning situation.

- Can individual work also be characterized as a learning situation?
- Let's consider reframing these questions and discussions in terms of educative and mis-educative experiences.
- Given a theoretical framework or perspective, perhaps we might attempt to identify what is learnable within an experience?
- How can we apply what we have talked about to real teachers with real classes and real time and curricula constraints?
- Can learning occur without cultural connection?
- What evidence do you or can you look for to show "communal acquisition"?
- Can you have learning without some resistance or some struggle to overcome?
- What of engagement, student ownership, and learning being students' "own business"?

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The Multifaceted Role of Feedback in the Teaching and Learning of Mathematics

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Introduction to the Report

As any report trying to offer a synthesis of rich activities that have happened, this report will be partial, and in both senses of the word. It will be partial as in incomplete, but also as in biased because made from the point of view of the two group-leaders who mostly attempt to offer an interpretation of what they saw happening in the sessions. Even if the participants gave us their blessings through revising the present report, there are still many ideas and learning events that took place that will unfortunately not be reported on here, be it on purpose for lack of space or misalignments with the rest of the discourse or be it because of failing to notice all the richness that happened during the sessions.

Hence, in order to offer the best possible outline of the activities, the report will be offered in four parts that in themselves we hope will provide a sense of the activities and ideas worked on. The first one will consist of a small overview of the initial intentions and mandates for this working group and the frameworks we used, as group-leaders, to structure the sessions. The second one is a short summary of the activities, offered as an orienting lens to make sense of the day by day summary offered in Part 3. Part 3, in addition to the day by day summary of the activities in the sessions, concludes with a set of previous intentions set forth at the beginning to orient these activities and a discussion of the interpretation that the working group leaders drew out of the session – interpretations that (1) were brought forth to the group the following day and (2) that helped (re-)plan the subsequent activities of the next sessions. Finally, in the fourth part, we offer some concluding thoughts and remarks concerning the activities of this working group.

Part 1: Overview of Intentions, Mandates and Orienting Frameworks of the Group Leaders

Working group D worked on the notion of *feedback*. The meaning of “feedback” is quite diverse, depending on the orientation one uses (theoretical, commonsensical, practical, etc.). Especially for us as group-leaders, since one of us, Jérôme, is a francophone and the notion of feedback is one that has many translations in French – for example, *rétroaction*, *réaction*, *contre-réaction*, *commentaires* – each of which has its own implicit meaning⁴. These diverse orientations for coming to terms with the concept of feedback were present for the group-leaders from the beginning and were brought up in many occasions during the sessions (as the descriptions of each day will try to illustrate). Unfortunately, to some extent, the notions of translations from “feedback” to some French possibilities was not exploited and looked into in depth because no francophone was present as a participant – except Jérôme!

This said, one of the mandates of the working group was to work on the notion of feedback from a practical point of view, and not from a theoretical perspective. The group worked on exploring the meaning of feedback at a pragmatic level, that is, not from drawing from theories but from experiential material and events. In that sense, the notion of feedback was to be explored at a phenomenological level (van Manen, 1997), that is at the level where the act of feedback actually happens. Our aim was to draw out a meaning and a sense, from the experience itself, of what feedback means in the action – for the one offering it and the one receiving it.

To achieve this (phenomenological) end, however, the working group leaders decided to position themselves within a specific theoretical framework to be able to lead the sessions where participants would be invited to draw on their understandings and experiences with issues of feedback (given, received, etc.) and elaborate on them. The underpinning framework orienting the group-leaders was one grounded in issues of emergence, where the outcomes of the sessions were not decided in advance and where the happenings of the sessions themselves would orient the thread to be followed throughout all the working group (instead of having the group-leaders prescribe and restrict the possibilities). From the famous quote of Varela, the path was going to be laid down while walking it. The reference framework underpinning the group-leaders’ actions and preparation was therefore away from a linearized and sequential planning that would have pre-defined the trajectory to follow in a prescriptive way, and more toward a non-linear, emergent and evolving development.

This orientation for leading these sessions is schematized in figure 1, taken from Jérôme’s doctoral dissertation (Proulx, 2007), which contrasts a linear view (‘objectives to attain’) from an emergent view (‘objectives to work on’). The group-leaders were situated in an ‘objectives to work on’ perspective.

Therefore, the preparation for the sessions was done beforehand with different activities thought in advance (more than needed) that could/would have to be adapted or brought in on an ongoing basis after each session depending on the orientation that appeared to be taken through the sessions. In a word, the working group leaders were there to push the explorations and have the participants endeavour and probe in perspectives that raised interest in them, in contrast to restricting the domain of issues to be covered following a pre-decided agenda. As one could have guessed, the work of the group-leaders was very demanding and active in order to (1) be attentive to the events of the sessions and their whereabouts and also to (2) pay attention to issues that appeared to raise interest in order to push the participants’ thinking and have them explore these perspectives more deeply.

⁴ As an interesting note, the person who translated the program description translated feedback as *rétroaction* in one place and *commentaires* in another.

Therefore, the overarching ‘objective to work on’ of the sessions was the objective set forth as the mandate of the working group, that is “the working group will attempt at clarifying the notion of ‘feedback’ and question its nature and importance in the teaching and learning of mathematics,” and was used as the starting point from which to work. In addition, depending on the issues tackled within the sessions, more specific aspects of this objective were brought to the fore for deeper explorations, with outcomes taking their own directions as they unfolded.

We felt it was significant for us to mention our orienting reference framework, as it oriented our actions and planning from the beginning of our collaboration, and mostly because it enabled the sessions to unfold as they did. Part 3 will offer, through its overview, an elaboration of how the different issues were grappled with and how the orientations towards feedback unfolded. Before that, however, we offer here a rapid overview of all the activities and issues grappled with in the sessions.

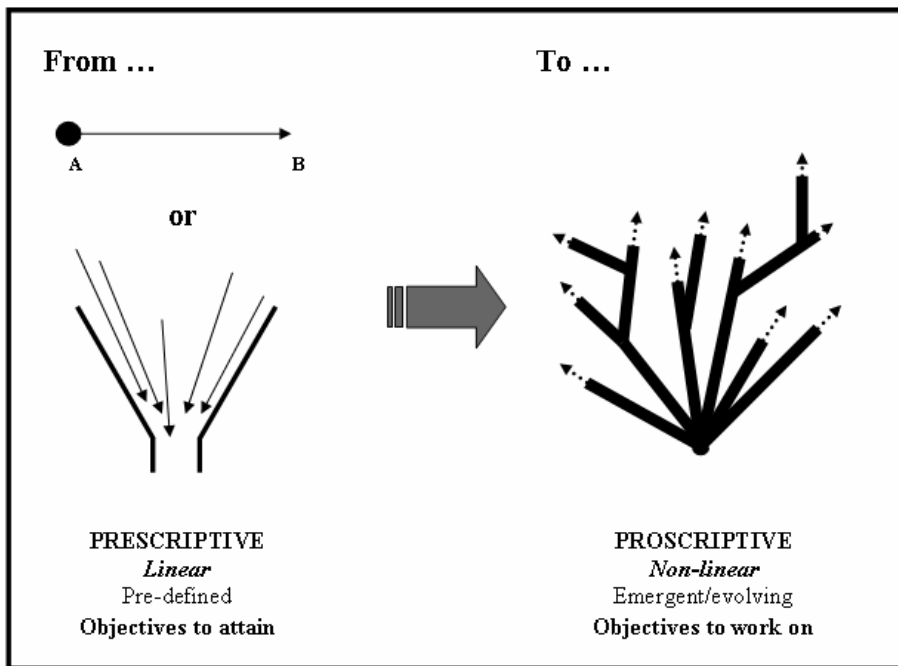


Figure 1: The framework orienting the group-leaders: from a linear to an emergent perspective

Part 2: Summary of the Activities and Issues Discussed

By way of summary, and to offer an orienting lens to the reader, we offer here a rapid overview of what has been done in the working group around the notion of feedback. The notion of feedback was looked into from the perspective of the one “giving” the feedback and from the perspective of the one “receiving” it, and to some extent the dynamic/tension it created. Issues of intentions and expectations from each of the actors in the dynamic was taken into account and explored, as one prominent comment was that these seldom align themselves, which creates important tensions and frustrations within the dynamic. That in fact led to the need to explore aspects of this tension.

Therefore, an important part of the first day was spent looking at the dynamic from the viewpoint of the person giving the feedback and all the intentions and expectations that came from the action of offering feedback to someone else. Something that emerged from this was

a sharp awareness of the negative interpretations and connotations that someone receiving feedback might associate with the experience. Another thing that emerged was how these negative connotations were seldom present in the giver's initial intentions. Hence, this complicated the feedback dynamic where different expectations about the feedback were misaligned, creating important tensions. This led to issues about the importance of taking the receiver into consideration when offering feedback.

The second day was for the most part used to look from this other/complementary viewpoint, that is, from the one of the person receiving the feedback, again with a focus on issues of intentions and expectations in this reception. More difficult this time, but still implicitly present, was again the issue of taking the other into account, this time the other being the one offering some feedback. In addition, the difficulty of placing oneself into the receiver's shoes emerged as an important, but also constricting, aspect for the one offering feedback. As participants were asked to place themselves in these shoes, they raised the complicated nature of doing that, since they have a different perspective on things. Again issues of misalignments came into play, this time arising from difficulties from the giver to be attentive to the receiver perspective and not only to interpret it from his or her own perspective – in sum, what is felt to be important, not important or useless from the giver perspective is not always well aligned with what the receiver expects, awaits or intends to receive. Therefore, creating a significantly present tension between both actors in the dynamic.

In order to wrap the issues up, day 3 was spent at exploring and discussing in deeper details these tensions and misalignments emerging from both sides of the dynamic, and what this tension meant and how could it be made sense of and, to some extent, resolved.

Part 3: A Day by Day Summary and Interpretation of the Activities and Orientations

DAY 1

The intentions of the first day was to attempt to draw out what the participants thought about when they thought about the notion of feedback. Two activities had been planned in order to have the participants endeavour and probe into this. The first one, a classic at CMESG meetings, was a personal presentation from each of the participants. The second one was related to an activity offered earlier at the 2004 CMESG meeting by Dave Hewitt (Hewitt, 2005) about feedback. In the following, we offer an overview of the activities offered and what came out of the discussions and work.

First activity: Presentation of all participants

As a first activity, all participants were invited to present themselves and to answer the specific question: "What do you think about when you think about *feedback*?" Different issues were raised in relation to feedback from the participants, but many of them were related to affect and how the feedback as an act was interpreted by the person receiving or giving it. For example, one of the issues strongly raised was how feedback is often perceived negatively as a sort of critique of one's work by the person receiving it, and was seldom seen as a constructive thing or a learning experience even if it was intended as that by the person offering it. Therefore, points were raised that it appears difficult for the person receiving it to take advantage of the feedback and build on it, as it is often discarded or simply not understood as an act that aims at contributing to the learning. (However, as some raised, not all feedback given is always intended to contribute to learning, sometimes feedback is given in a very pejorative manner and sense.) Hence, feedback is not seen for most as an opportunity to engage in a conversation with the person offering feedback, as it is taken as it is and left aside – often focusing on the mark coming with it when it is in an assessment situation.

Another example related to affect that was raised concerned teacher education and the teacher educator giving the feedback to prospective students on their work, and how one has to be careful in offering feedback because of how it could be interpreted by future teachers. This led to questions of integrity, as one is stuck within a compromising space between respecting his or her own integrity and being honest about one's thought, and being gentle and not too hard on comments given to the other. This was raised as a difficult situation, where some things sometimes need to be said and done from the perspective of the teacher educator offering the feedback, but can lead to problematic situations if the feedback is perceived as too negative (e.g., discouraging, complaints, bad student evaluations, etc.). The teacher educator is therefore within a negotiating sphere between his or her integrity and the desire to not create problematic situations, so to say. (These issues of affect in relation to feedback – how it is received, interpreted, intended, expected – that emerged from this first activity will continuously reappear and orient the discussions in the working group during the subsequent activities of each day.)

A complementary issue that would reappear as an interesting trigger point of subsequent discussions concerned the fact that what one offers as feedback is not always what the other person understands (as the famous constructivist saying says), but also not always what the other wants to hear. In that sense, this is pointing to issues of “intentions” orienting the giving of the feedback and its reception. (This will particularly become of interest in the activities of day 2.) Finally, somehow implicit in all of this was the notion of assessment and how feedback is, and is used in, assessment. As the activities unfolded, issues of assessment were intertwined in discussions of feedback and were not explicitly separated as such because of their similarity.

Second activity: Hewitt's problem

The above issues highlighted in the presentations of the participants (and many more were raised) led smoothly into the first activity offered to the group as another way of having the participants probe into their understanding of the notion of feedback. But, this time from the teacher point of view, that is, from the point of view of someone giving feedback. In groups of three, participants were asked as the first part of the activity to read the following scenario of a teaching situation (Hewitt, 2005) and to think about possible sorts of feedback that could be offered (Figure 2).

As part of a small group activity, you observe two students trying to decide which was the lowest out of the decimal numbers 0.853 and 0.358. One student says 0.853 is the lowest, the other thinks 0.358 is the lowest.

Figure 2: Hewitt's scenario

After having done that (and after the coffee break), as the second part of the activity the participants were offered six possible scenarios taken from Hewitt's article (Figure 3) and were asked to compare (distinguish, link, assimilate, contrast, etc.) them with the answers they had obtained and create a sort of classification of different sorts of possible feedback that could be offered in this situation. Each group was also asked to report on a piece of chart paper that would be presented to the group for discussion.

Very diverse classifications and distinctions were drawn out. The following tries to elaborate on each of them, however not in explicit chronological order. One of the distinctions Catherine, Egan and Jocelyn raised was between sorts of feedback concerning the “construction of truth,” that is, where does the “answer” emerges from: the teacher or the student? For example, the “truth” stays primarily in the teacher to transfer to student when actions taken are of the sort: grading, explicitly explaining the right answer, offering models to make sense of the answer, asking questions that simplifies and funnels down the range of

explanations from them (being more active). But, if the intention of the teacher is to have students simply engage in mathematical tasks, then having student obtaining the right answer and stopping the discussion appears not sufficient and the teacher is led to ask questions for students (being more active) to engage in more mathematical explorations. Hence, if students do not arrive at the right answer but are in discussions about the issues, the teacher can be tempted to walk away (being less active) because the goal or his or her intentions are achieved from having students engaged in mathematical explorations. Therefore, in similar ways as the previous group but on different terms, there is an issue of intentionality driving one's actions that is difficult to perceive from the external, but that however drives the feedback actions themselves. The orientation for the task orients the possibilities for feedback.

Jamie, Ralph and Wendy's group insisted on the fact that feedback is a relational act (something that was also present in Dasha, Marian and Morris), an act that is made in relation to someone else and for someone else. Therefore, and this will be one of the most recurring themes of the working group, the other needs to be taken into account in this dynamic as the "other" is part of this dynamic too! Jamie, Ralph and Wendy also drew out some different categories or ways of giving feedback. One of them is feedback as "judgment" when a judgment at the mathematical or personal level is made about what had been produced. Another is feedback as "attending" to the other's response, something other groups have flagged also as inviting, where the teacher listens and asks for clarifications and more explanations. Linked to this one and the "inviting" category, is one about posing more questions to delve deeper in the students' understandings, what the group has called "inquiring." A fourth category is one called "noticing" where the teacher takes time to make comments on aspects of the work and the strategies used. The next one is called "put down" and is related to feedback that belittles one's work (e.g., "Weren't you listening?" "If you look carefully, what happens here?"). The sixth one is one called "do nothing," similar to David, Gladys and Kathryn's "walk away." The seventh one raised is "instruction" where one offers directives or suggestions to the learner in order to push the thinking (e.g., "Justin... pay attention to Susan's explanation" "Let's look at simpler numbers"). Finally, the last category raised is one called "celebrating," where the teacher enters in a (genuine) discussion with the student about the issues (e.g., "Your answer makes me think of...").

All these ideas, obviously, stimulated a lot of rich discussions about issues of feedback and the links that could be seen between the different classifications given. There was a lot of *brassage d'idées!* It was very interesting in its richness and diversity. As group-leaders, this report from each group and the following discussions were rich but ... Ouf!! There was a lot of stock to take into account. However, some trends appeared to arise from the ideas and we attempted to focus on them in order to pursue some of them more deeply the next day.

Summary-outcomes (from WG leaders' perspective) and planning/previsions for Day 2

One fundamental issue that was orienting most discussions was the fact that feedback is a relational act, an act that is offered in relation to another person (or group). And this relational is also in regard to the intention (and focus, perspective) taken as one offers this feedback (e.g., the theoretical viewpoint adopted, the intention behind/orientation toward the task), which colors the nature itself of the feedback given, but that is often unseen (the same feedback action can be intended for different reasons). Therefore, all this relational act and relativity leads to the consideration of the other in the feedback dynamic, and how this person needs to be taken into account in it.

Also, and somehow complicating the issues, the perception of feedback from the point of view of the person who receives it mostly has a negative connotation and this also needs to be taken into account in the "taking the other into consideration."

Hence, these perspectives shed some useful light on issues of feedback from the teacher's perspective (or the giver), which led us to consider looking at the notion of feedback from the

student's (or receiver) perspective in the following day – the student being the very person to be taken into account in this “consideration of the other” relational act.

In addition, as working group leaders, we noticed that in the explanations given in the sessions, there seemed to be a tacit preference for offering feedback in forms of questions as a representation of more suitable feedback, and less of an appreciation for feedback in the form of plain utterances. In a word, feedback in form of questions by the teacher seemed to receive more appreciation as a form of feedback from the participants. This situation in fact motivated even more our intention as group leaders to look deeper into issues of feedback from the perspective of the student as we were wondering if students as receiver of feedback would adopt this same perspective on “good and suitable” feedback in the form of questions or would they prefer plain utterances. Again, questions of expectations from the students' perspective and behalf appeared to be a point that would be considered and explored in length, and would represent the starting point for discussion of day 2.

DAY 2

In order to explore issues of feedback from the point of view of the student, we decided to design two specific activities in which participants would have to position themselves in the receiver's shoes. In order to introduce these activities, Jérôme offered a small context of where the activities of day 1 had led, what he and Florence discussed in general, and toward where the explorations could go for this day.

Summary and introduction to the activities

Jérôme started by announcing that there did not seem to be a point in summarizing the work of the previous day, as it was too rich to be encapsulated in a short summary, but that it seemed worthwhile to flag some points that appeared to be recurrent, even if only at an implicit level, during the first activities. One of the things pointed out was the negative and commonsensical perception of the notion of feedback, as was raised by many of the participants. This perspective was explained to indeed be very present in the everyday literature and in French and English language dictionaries, for example. Inherent in most definitions of dictionary is a sense of feedback as an endeavour done in order to “improve” and “redress” a situation in order to adapt, modify and correct it. Also inherent in the definitions are the issue of control, where feedback is said to be a means to operate control over the possible outcomes and regulate them. Therefore, there are words used like “counter-reaction,” “regulating,” “predicting,” “modifying,” “retro-control,” etc. Implicit in the sense given to feedback, in its commonsense, are issues of control and of negativity – which surely can have a possible impact on the (intentions of/expectations of) persons giving and receiving the feedback⁵.

Hence, since this commonsensical sense is “out there,” and since one thing that came out strong from the discussion of the previous day was in fact that the other needs to be taken into account when offering feedback, it questioned intensely the feedback dynamic as the “negativity” perception is potentially “present” in the person receiving it. In a word, even if we as teachers saw feedback as a positive thing and as a potential learning experience, if the commonsense of feedback is negative, (how) do we take this into account as teachers? In addition, the mentioned above instance was raised about the appeared tacit preference for the participants toward “questions,” in contrast to “utterances,” as representing good forms of

⁵ Attention was also placed on a definition of feedback in relation to sound and music from the *Collins Cobuild*: “Feedback is the unpleasant high-pitched sound produced by a piece of electrical equipment when part of the signal that comes out goes back into it.” It was interesting to reflect on this as the other connotations and comments raised on the possible perceptions of feedback sometimes aligned themselves, to some extent, with this definition.

feedback to the students. And it was an issue that was leading to the activities of this second day.

Taking all this into account, the group-leaders explained to the group that as one thing that had been done during day 1 was to look at the notion of feedback mainly from the “giver” perspective, and so the activities in day 2 would attempt at looking at the notion of feedback from the “receiver” perspective.

First activity: Remembering the past?

As a first activity, the participants were invited to place themselves in the learner’s shoes, that is, in their own shoes as learners. They were invited to share, in groups of three (different groups than the day before), some of the most helpful and less helpful feedback situations they have experienced. On the basis of the discussions, each group was then asked to draw a pictorial representation of the notion of feedback, which would afterwards be shared with the group and discussed.

It is interesting to note that the directive for the task that was to talk about “the most helpful and less helpful” feedback situation was transformed *subtly* into an emotional state of “positive” and “negative” experience by the participants, bringing again to the fore the issues and connotations of positive and negative perceptions that had been there from the beginning and had been driving the discussions – not to mention the emotional twist to it. This appeared very interesting as emotions were discussed to be another of the central factors coming into play in a situation of feedback, in the relational act it represents.

In most of the representations offered by the participants, issues of contrast and similarities between helpful/unhelpful, positive/negative, and so on, were strongly present. This enabled to complexify the notion of feedback; to add to the teacher’s perspective of the previous day. These contrasts/similarities were given in relation to different aspects. One question that emerged was “negative/positive in relation to what?”, the feeling the person had when it happened?, the quality of the feedback given as a learning experience? This first distinction permitted to differentiate between the feeling stemming from the reception of the feedback and from the impact the feedback could have as a learning experience on the person. Hence, a comment raised (by Christine) was that a negatively felt feedback in the past can however have a “positive” impact in the long run as one remembers it nowadays and sees it as part of his or her luggage of experiences. Therefore, an issue that was put into question, and that was difficult to answer to, is the fact that the “negative” instances of feedback that were felt to not have been helpful raised wonders about their “negative state” since they are remembered today and appeared to have bring something to the person mentioning it. The question of helpful or not helpful appeared to lie in the moment itself, an impression that can change with time or even have a long lasting effect that is not necessarily to be seen as negative in the end as it has contributed to one’s development.

In line with this, one point noted and questioned was that of our taken-for-granted perceptions in terms of the quality of the feedback as a learning experience for the student; the negative perceptions we have of enacting a feedback in terms of control, and the positive perception we have of enacting feedback in terms of offering occasions to probe deeper. These perceptions were said to have to be questioned because they may not be what is actually happening for the learner. And therefore, as teachers we are biased by our perspective of what we perceive as good mathematics teaching and ways of offering feedback for the development of the other, which may not be that “efficient” or “appreciated” from the student point of view. And also, this represents a challenge for us as teachers because we are too engrained in our own views of what we think is to be done and is good for the other, making it difficult to

really take the other into consideration. Hence the question: “Are our actions really taking the other into account, from the learner’s point of view?”⁶.

This led to another discussion about the quality of the feedback, since quality of feedback is not really in the hands of the giver, but appears to be in the hands of the receiver. This was brilliantly summarized by an enactivist-oriented phrasing from Ralph: “The quality of the feedback is not ours to determine, even if it is ours to trigger,” questioning at the same time the entire issue of taking the other into account and deciding what is good/bad or appropriate/inappropriate feedback to offer. It is the receiver in the end that “decides” if the feedback is helpful or not. Hence, issues of quality of feedback are not cut-and-dry and can lie in many places depending on the perspective: it could lie in the “intention” of the person offering feedback, it could lie in the relationship between the giver and the receiver itself, and it could lie in the receiver’s perception of it. For example, a “teacherly-judged” bad feedback can be seen as great from the student’s perspective.

The issues of quality of the feedback being multidimensional, the reception of the feedback is also relational from the student’s point of view. This led to a complementary view of the previous day, where it was said that the teacher had to take into account and understand the student’s perspective. Hence, in the same way, the “receiver” in the act of receiving feedback should take the “giver’s” perspective into account to better understand the intentions behind the feedback offered to him or her. In a sense, in the feedback dynamic, one needs to know the person (receiving or giving) in order to understand the quality of the feedback received. This is a much more complex picture that is drawn from these reflections, one that is not simply unidirectional toward the teacher as the one who needs to take the student into consideration, but also places some weight on the student’s shoulders in this dynamic. The quality of the feedback therefore appears as a co-emergent phenomenon, where both parties are active in its realization of potential/meaning. This complex picture sheds important light on the phenomena itself from both perspectives of giver and receiver of feedback.

Second activity: Looking at data

In order to initiate the second activity, the participants were given different rectangular prisms and were asked to reflect on the question “What is the ‘base’ of a prism?” This was presented to the participants as a way to introduce the following activity, which was to examine a piece of data transcript (taken from Jérôme’s doctoral research) where this question was central.

After this initial activity, participants were offered the piece of transcript for them to read. In this piece of transcript, many instances of feedback were present, some between the teacher educator and the teachers, but also between teachers themselves. After reading it, participants were asked to reflect on the following question: “So... in relation to these events, the participant teachers in the research have explicitly expressed to have learned a lot from this situation... What does it tell us about the “feedback” situations in this transcript, and about the quality of feedback from the different perspectives (teachers, teacher educator, receiver, giver)?” (See the transcript in Appendix A.)

The outcomes of this activity were... interesting. As appeared to be normal at CMESG meetings, as participants mentioned, the attention was spent mostly on... the mathematical task and the discussion of the mathematics (!) – and not a lot on the “feedback” task itself.

⁶ This issue of possible misalignment between “teacher’s” intentions and “student’s” intentions was also discussed later on in the session in relation to our, as teachers, interest in ambiguity in mathematics and in other issues as we believe ambiguity to be a motor for reflection, deeper questioning and stronger understanding. However, as was explained, it does not mean that students see it in the same way and if one needs to take the other into account, one cannot act by discarding this issue and perspective of the student. This makes the situation very complex. And, it brought issues of integrity raised during the first day back to the forefront.

Therefore, comments emerged in relation to the mathematics in the task and issues that happened for teachers in relation to the concept of base and of prisms. We do not report on these here because they would be off topic to some extent for this report, but they were very engaging mathematically and we pursued them up to the end of the session.

Summary-outcomes (from the WG leaders' perspective) and planning/previsions for day 3

What came out strong from this second day, through having participants enter from the learner's perspective, was the inherent complexity of the dynamic of feedback and its diverse quality perspective. As was said, the quality of feedback is multifaceted and is ultimately "judged" by the receiver – that is, the one to whom it is addressed. Therefore, all this questioned the issue of "having to act along a certain way" for teachers, as this "way of having to act" is contingent and dependant on the receiver itself.

Also, though, is the fact that even if this dynamic has been explored and maybe clarified to some extent – or even complicated – and that some weight appears to have to be placed on the receiver's shoulder, it does not erase the fact that there is a tension apparent in the misalignment of the intentions/expectations of giver and receiver of feedback. Therefore, the activities for day 3 imposed themselves directly, that is, of exploring and having discussions and reflection about the sorts of spaces that potentially need to be fostered in our teaching practices to ease or bring up front these disruptions? And, another task planned for the third day was to prepare a short report to present at the closing session of the conference, offering therefore an opportunity to draw out what we all had learned from the working group.

DAY 3

Day 3 started with intense discussions on different issues about feedback. It felt to us, as group-leaders, as if there had been so many ideas stirred in the last two days that the participants felt the need to unleash and share their thoughts openly and continue various discussions of previous days. We therefore decided to postpone our specific task (i.e., of looking in depth at the sorts of spaces to ease the disruptions and how to work on the tension present in the feedback dynamic) and we all shared our thoughts about these issues. In addition, many of the comments addressed the tension itself and therefore it was preparing the ground for it, even doing exactly the task itself.

In reaction to discussions about misalignment of intentions/expectations from receiver and giver of feedback, David (and Christine we presume!) brought in a book of their daughter Lily about learning to go on the potty. What David had us realize and notice was that the feedback given, in forms of stickers representing stars that the child had to collect, was denaturing the task, to some extent, of learning to go on the potty. Paraphrasing David, the task of the child now becomes to succeed in getting a star and not at being an autonomous well functioning being. All the focus was on "being a star!" In that sense, it brought a reflection about issues of misalignment because the sort of feedback that is expected by the student can also influence the original intention of the task itself and change it for the receiver into something that it was not. Hence, the misalignment stemming from the feedback dynamic is not always a product of the misaligned intentions of the giver in relation to the receiver, but also can be in a misaligned understanding of the idea behind the task from the receiver's perspective. Coming back to the idea of asking feedback in forms of questions or of inviting to engage in more mathematical work, it is possible that it is perceived by the student as an "unhelpful" form of feedback and by the teacher as a "helpful" one, but this misalignment can come from the student misalignment with the intention of the task itself and with the intention for doing mathematics in schools thought or aimed at by the teacher (getting answers *versus* engaging in mathematical endeavours). This was quite an eye opener!

Another issue that was raised by some participants in relation to how the receiver interprets the feedback concerned if the reaction of the receiver can have an impact on the giver in

return. This issue was mainly discussed in terms of teacher education practices in universities and the student evaluations that are filled afterwards. Questions of integrity and what sorts of feedback one gives as a teacher educator appear to emerge, as one is obviously not interested in having problems with students (not to mention the administration) for offering feedback that would not be welcomed by student teachers. Even if this issue was not explored in depth, it appeared interesting to raise it out as it brings another perspective on the notion of feedback itself, which can have a returning influence on the one offering feedback, changing the dynamic from another point of view (not one of helping the student, but maybe also of protecting oneself from not being attacked in return, especially when teacher evaluations are strongly looked into and seen as important for example for promotion and tenure).

Following all these discussions, participants split in smaller groups to continue discussions in relation to different aspects of the notion of feedback, but also to come up with a sort of wrapping up of their understanding of what feedback was and what they had drawn/learned from the working group. Hence, instead of us summarizing the words of each group, two of the four thoughts that were shared at the closing session are represented here:

As teachers, we need to constantly give feedback in a variety of forms: model, explain, build a safe environment, build and maintain relationships, or to take risks, motivate, encourage. It is important to have professional conversation – dialogues about commonalities – in feedback, assessment, grades. If students have a better understanding of expectations, better able to focus on learning (*Catherine*).

Feedback is a compromise (a response to tension) between what we feel, what we want to communicate, and what the student is ready to hear. It varies on the level (school, undergraduate, graduate) and on the subject (math, education, piano lessons, etc.). (*Dasha and Rina*)

Considering the tension we feel in our roles and relationships, one of our working group members told us, “I didn’t come here with the expectation that feedback would be simplified for me.” We wish to clarify a sincere application of this ‘feedback’ with three elements: (1) the subtractive – to lose something in order to add authentic feedback; (2) the competitive – to identify what is competing for feedback time; and (3) the reflexive – the feedback returning to me that ‘feedback’ is working. In classroom practice, feedback is the assessment of learning, the assessment of growth, and not the doing of lesson plans, document creation, or the meeting of some standard or level of competency for the field. (*Jamie*)

Conclusions

Studying and exploring the notion of feedback had some sort of a schizophrenic effect on us as group-leaders. From the moment we started our collaboration to prepare the working group, we became more and more sensitized to issues of feedback and this culminated in the working group itself as we were scrutinizing each of the actions we were taking toward the participants. Exploring an issue brings one to become sensitized and pay a lot of attention to one’s own action in relation to it... and sometimes a little bit too much! We therefore became, between ourselves and toward others, a little schizophrenic!

It does not appear to be that “interesting” to offer a conclusion or a summary recapitulating all the activities of the working group, we believe that the essence of the activities were captured in the above elaborations and mostly in the reports from the participants. One legitimate question, however, that we did ask ourselves was “Have we achieved what we had said we

would achieve?” That is, have we *worked on* our overarching objective of: “attempt at clarifying the notion of “feedback” and question its nature and importance in the teaching and learning of mathematics?” From the activities and the participants account present in this report (and shared personally), we positively and proudly assert that we did.


We deeply thank the wonderful participants that were part of the working group as they, through their openness, sharing, curiosity and insights, made the working group’s richness possible, from start to finish. Thank you Catherine, Christine, Dasha, David, Egan, Gladys, Jamie, Jocelyn, Katharine, Marian, Morris, Paul, Ralph, Rina, and Wendy. Florence & Jérôme.



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Appendix A – Data Offered to Participants

The Volume Session: Discussing the Base of Prisms⁷

Volume was previously defined in the beginning of the session as a piling up of layers that creates a volume. A definition or “formula” that unfolded was *area of the base x height* which could encompass all volume formulas of prisms. However, from the beginning, we were always speaking in terms of prisms that were standing up, in the sense that if you would place a solid on the table, its base would be flat on the table and the other one on top of it (.

This brought Carole to raise the point that the orientation in which the prism is placed can create difficulties for students. Or, in other words, students can experience difficulties when prisms are positioned “standing up” or “lying down,” because in area students are use to see it in this way () with the “large” base lying down, which could bring them to have difficulties in placing the solid in another way than “lying down” with its large base on the table (). This stimulated a discussion about the fact that for a rectangular prism, it does not matter which base is chosen to create the piling up, because it will always end up with the same volume. This however provoked a strong reaction of disagreement from Gina.

Gina: In my classroom, you could not call “base” the part that is at the bottom [showing one of the non-square rectangle of the rectangular prism]. If it is lying down like this [showing the rectangular prism lying down with non-square rectangle sides on the horizontal], I expect that you tell me that these are the two bases [pointing to the two squares].

Erica: Why?

Gina: Because it’s a prism.

Jérôme: Gina, this is mathematically false, however.

Erica: Yeah.

Gina: Wait a moment, because the idea is that when you then work with this one [showing the hexagonal prism], you now say that it is a prism because these ones [pointing to the rectangular sides of the prism] are all rectangles and you still have your two others [showing the hexagons of the hexagonal prism].

Jérôme: Yes, but in a prism, you need the lateral faces to be rectangles. In this case [the hexagonal prism], you do not have the right. This [showing one hexagon of the hexagonal prism] is not a rectangle. In this case [showing the rectangular prism], I have the right.

Erica: Yeah.

[...]

Gina: You have the right [you can], but it confuses kids.

Erica: Yes, but then...

⁷ Data taken from: Proulx, J. (2007). (Enlarging) secondary-level mathematics teachers’ mathematical knowledge: An investigation of professional development. Unpublished doctoral dissertation, University of Alberta, Canada.

Jérôme: Ok yes, but then the confusion in fact, it is important that they know it however.

Erica: It is their problem [if they don't understand].


Gina: But it is important ... it is their problem?!?

Erica: Of course. You cannot teach something that is false to avoid that students get it wrong, to help them understand.

Gina: But it is not false that there are two [sides in the rectangular prism] that are the same.

Jérôme: Of course, but it does not make them the bases, it is a choice you make.

Erica: It does not make them the bases, it is a choice.

I then paralleled the situation with the case of a rectangle where both sides could be called *length* or *width* and it would not matter. Gina refused again this explanation, by saying that the small side of the rectangle (\rightarrow ) could never be called a base.

Gina: You see, I never call the small one the base.

Jérôme: I understand, but ...

Gina: You cannot do that.

Jérôme: But yes, this is exactly it, you can do that!

Carole: What is a base in fact? Because the vocabulary is important here.

Erica: The base is a pillar, it is what supports.

Jérôme: And you decide.

Erica: And so the base, this is the base [she puts the rectangular prism standing up]; the base, this is the base [she puts the rectangular prism lying down].

That prompted Gina to say and show that in a pyramid (e.g., a hexagonal one) it is impossible, since a pyramid cannot be placed with one of its lateral triangle touching the table (lying on the table) and then deciding to call it a base. Agreeing with her, I explained that she was right in the fact that it is not possible, but the reason being that a pyramid by definition requires that its lateral faces be triangles. But, for a prism the definition requires that all lateral faces be rectangles. Gina still had problems with these ideas, mostly because, in a rectangular prism with a pair of opposite sides that are squares and the rest being equal rectangles, the fact that the squares are different from rectangles makes them directly and exclusively the bases of that prism.

Claudia then suggested looking at a rectangular prism which has three pairs of different rectangular sides, and for that she took a videocassette box which had 3 pairs of different sides. This slowly brought Gina to understand and accept the idea.

Claudia: [brings the videocassette box]

Gina: Yes, that's it! With this one it is not important.

Erica: Why isn't it important with this one?

Gina: Well, because you will have, they are all rectangular. You have two [of each]...

Jérôme: But the other ones also [pointing to the other rectangular prisms on the table].

Erica: They are all rectangular also [referring to the other rectangular prisms], a square is a rectangle.

Gina: Yes, yes, yes. But this one [the rectangular prism with two opposite squares on its sides] there are two identical so when you will calculate you will say, if you want to calculate the area or anything, it is easier to see the slices in that way [pointing to when the squares are taken as the bases]. [Appearing surprised, as if she just realized] No! It is the same thing!

Jérôme: But, that it is easier is not the same thing however.

Gina: No [agreeing that it is not easier]!

Erica: It is the same thing, it is the same thing. It is not easier like this [with the prism lying down] than like this [with the prism standing up].

Topic Sessions



Séances thématiques

Mathematics Educational Neuroscience: Origins, Activities, and New Opportunities

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Abstract

Mathematics educational neuroscience is poised as the thin edge of the wedge of an emerging and potentially foundational new area of educational research. I discuss the origins and rationale pertaining to this initiative, and show how educational neuroscience forms a natural conjunction between cognitive neuroscience and educational psychology. I present an overview of activities and initiatives in this area that are particularly germane to mathematics education research. In so doing, I discuss recent activities as indicative of new opportunities for mathematics education researchers.

Introduction

The Canada Foundation for Innovation's (CFI's) New Opportunities Program, in collaboration with the British Columbia Knowledge Development Fund (BCKDF) and Simon Fraser University (SFU), recently awarded me \$500,000 to establish a state-of-the-art, and so far as I know, one-of-a-kind “educational neuroscience laboratory” in the Faculty of Education at SFU.

Data acquisition equipment in the lab highlights three computers that serve to front-end two multi-channel electroencephalographs (EEG) and an eye-tracking (ET) monitor. This equipment is housed within an audiometric suite, consisting of an observation room and with an adjoining control room. Other multi-core computers are dedicated to data analysis and interpretation.

To date, this lab has enabled my students, colleagues and I to embark upon a fascinating and challenging new journey to explore some exciting new dimensions in educational research. I will report on some of our activities below, and conclude with a brief discussion of some new opportunities we have now been afforded. It is important to add that there are other initiatives underway to help bridge neuroscience and education (e.g., Byrnes & Fox, 1988; Blakemore & Firth, 2005; Goswami, 2004; Geake & Cooper, 2003; Posner & Rothbart, 2005; Van Nes & Gebuis, 2006). My primary focus and concern here, due to space limitations, will be with my own.

Origins

For those who are familiar with my work in mathematics education, this award may come as a surprise, as my focus of research in mathematics education to this point in my career has been mainly philosophical and historical in orientation — for those who are familiar with my 'previous life' in industry engaged in seismic imaging, perhaps less so.

For me, there are connections and continuities with my industry experience and academic interests that make this radical step very natural, and to some extent, inevitable. Here, I provide some insight into the origins of my new educational neuroscience laboratory, which for reasons discussed below I also refer to as the ENGRAMMETRON.

An appropriate place to start this story would be with my convergent interests in embodied cognition (Campbell & Dawson, 1995) and cognitive modelling of preservice teachers' understandings of elementary number theory (Zazkis & Campbell, 1994a,b). My interests in embodied cognition had their origin in a long standing conviction that, although mind and body can clearly be distinguished on conceptual grounds, they are in fact, viz., ontologically, one thing, not two.

My interests in cognitive modelling had their origin in my industry experience as a knowledge engineer, using expert systems technology to model conceptual objects and processes in order to generate automated reasoning systems. Expert systems worked quite well if the objects and processes were consistent and well defined. This is a characteristic of normative systems. Natural systems, on the other hand, tend to be more complex and nuanced. Preservice teachers' understandings tend to be of this latter ilk, while our cognitive models tend to be of the former.

The work I have since done with Rina Zazkis in modelling preservice teachers' understandings of basic concepts from elementary number theory (Campbell & Zazkis, 2002; Zazkis & Campbell, 1995, 2006) has largely been based on audio recordings of think-aloud reports recorded during problem solving activities. These data helped us constrain our speculations of what was actually going on "in the head."

During my time at UCI (University of California at Irvine), I developed a method geared toward obtaining better observational control in such experiments that I called 'dynamic tracking' (Campbell, 2003a). This method combined audiovisual recording of participants engaged in a computer-based problem solving activities with computer screen video capture of those activities. This method proved very effective in capturing learning behaviours, and most notably, "aha" moments.

What was most significant to me regarding dynamic tracking, despite having much greater observational control, was how evident it became that participants' brains and bodies were reacting and responding to learning: reacting and responding, that is, in salient ways that remained only qualitatively observable, and largely invisible. This raised key questions for me: how to better observe and measure these behaviours?

It was at this point that my theoretical orientation toward embodied cognition and my empirical interests in cognitive modelling converged with the realization that recording physiological measures could provide greater insight into brain and body behaviour. First, a fundamental entailment of my "radical" enactivist view of embodied cognition, given we are the world within itself, is that changes in lived experience must manifest in some way as changes in embodied behaviour (Campbell, 2001, 2002, 2003b; Campbell & the ENL Group, 2007). Secondly, it clearly follows that cognitive modelling can be much better empirically grounded if there is more empirical data to draw upon in formulating, constraining, and testing those models. In my work with Zazkis, we used Dubinsky's APOS theory (Zazkis &

Campbell, 1996a, b) in formulating our cognitive models of preservice teachers' understanding of divisibility and prime decomposition.

Certain questions arose from this convergence of my theoretical and empirical interests. First, how might existent work in psychophysiology and the cognitive neurosciences inform the formulation and/or refinement of our cognitive models? Secondly, how might our cognitive models, derived from mathematics education research, be tested and/or refined using methods from those fields (e.g., electrocardiography (EKG) and electroencephalography (EEG))? These basic questions involve forward and inverse modelling, which have become the mainstay of the sciences (Campbell, 2004a, b).

I was further inspired in this regard by some PME colleagues at PME 26 in Norwich in 2002 (Campbell, 2006a). Schlöglmann (2002) got me thinking that the time was right to start taking the brain and the neurosciences seriously in mathematics education, and Philipp & Sowder (2002) brought my attention to eye-tracking technologies. When I returned to Simon Fraser University in after PME26 I began researching where the state-of-the-art with these technologies had progressed. It quickly became evident to me that significant progress had been made and that there were some exciting new developments in signal processing and time series analysis that I felt were situating these technologies in such a way that research in mathematics education could greatly benefit from their adoption, adaption, and application. I considered the following major factors:

First, advances in time-frequency analyses of EEG data, with concomitant improvements in recording fidelity and capacity of EEG equipment were revealing exciting new features of higher cognitive function in ways analogous to the profound insights that the stellar spectrograph had brought to the field of astrophysics.

Second, a spatial filtering technique derived from radar technology developed during World War II called "beamforming" was being successfully applied to help locate sources of EEG activities unobtrusively measured as voltage potentials on the scalp.

Third, advances in eye-tracking technology were increasing the efficiency and costs of obtaining quality measurements of eye movements and pupillary responses with minimal effort for the researcher and minimal intrusion for participants.

Fourth, the prospect of integrating these technologies into my method of dynamic tracking in an integrated and time synchronous manner addressed the limitations I had noted earlier (see above), while simultaneously merging my academic interests in embodied cognition and my industry experience in seismic imaging.

The only thing missing to bring these technologies together to augment my research in mathematics education was funding. I wrote a proposal based on the aforementioned factors and was awarded a \$500,000 CFI New Opportunities Grant in late 2004. Consequently, over the past three years I have been consumed with researching, designing, planning, implementing various components of the infrastructure which has now become a reality <www.engrammetron.net> (Campbell, 2005), and training others in their use (Campbell 2006c). At the same time, I have been doing my best to help conceptualize what kind of new discipline within educational research this kind of infrastructure can support (Campbell, 2006a; 2006b; Campbell & the ENL Group, 2007).

Activities

Designing and setting up a new laboratory from scratch, and then getting to the point of actually doing something with it, has been a very labour intensive undertaking — even more so being this lab is, so far as the author is aware, the first of its kind, at least with regard to educational research. I would conservatively estimate that at least 70% of my time and energy

over the past three years has been devoted to this process. Even if possible, it would be tedious to discuss all the seemingly endless minutiae involved, down to the levels of selecting hospital grade electrical outlets, to implementing websites, designing furniture and selecting finishes, to ensuring the ceiling won't leak. Anyone who has built a house or embarked upon major renovations will have some idea of what is involved. Nor is it possible to review the substantive research involved in seeking out different possibilities for the key infrastructure components in the lab, ranging from the audiometric suite⁸, the EEG and eye-tracking units, and their related tender processes, let alone identifying, selecting, and learning to use various software systems required for the operation of said equipment, and for analyzing data acquired from them (Campbell, 2005a,b, April). In addition, it is not only important to learn how to use these systems oneself, but also if necessary, train research assistants in their use (Campbell, in press).

There is also the challenge of attempting to bootstrap a new area of educational research from the ground up, so to speak. Fortunately, this is not quite as difficult as it sounds, given that this entire enterprise would not be possible without the aforementioned advances in EEG-related technologies in psychophysiology and vast improvements in brain imaging techniques in the neurosciences. Educational neuroscience, as a newly emerging area of educational research, has all the benefits of this groundwork. One could go so far as to say that in educational neuroscience, one stands on the shoulders of giants.

Having said that, beyond the basic activities of setting up a lab and training oneself and one's research assistants in the use of equipment and software, there are other significant challenges and obstacles to establishing educational neuroscience as a bona fide area of educational research. First, it is important to get some sense of definition as to what educational neuroscience is. As Bruer (1997) pointed out, cognitive psychology is an area that naturally concerns neuroscientists and educational researchers, especially mathematics education researchers. Nowhere is the relation between education and cognitive psychology more salient than in educational psychology. Similarly, cognitive neuroscience has emerged from the conjoining cognitive psychology and with the neurosciences, especially through the medium of brain imaging and lesion studies. To my mind, a bona fide educational neuroscience constitutes a transdisciplinary new area of research naturally emerging from the interstices of educational psychology and cognitive neuroscience (Campbell & the ENL Group, 2007). As this area requires expertise from many different disciplines, I have been developing a SSHRC "Strategic Research Cluster" called ENGRAMME (Educational Neuroscience Group for Research in Affect and Mentation in Mathematics Education), for which my lab is serving as a central hub.

With an impressive array of tools, methods, and results from psychophysiology and cognitive neuroscience to draw upon, it does not follow that the questions and interests of these well-established disciplines are the same as those of educational neuroscience. To the extent that they are considered as such, educational neuroscience could be viewed as nothing much more than an applied cognitive neuroscience. It is important to appreciate, I think, that cognitive neuroscience is predominantly interested in identifying brain mechanisms underlying cognitive function. I characterize the difference with educational neuroscience as that the latter should predominately be interested in the structures of lived experience underlying cognitive function. Thus, in an important sense, I am viewing educational neuroscience more as an educational neurophenomenology (cf., Varela, 1996), primarily concerned with "keeping learners in mind" (a double entendre I have coined as a motto for my lab).

⁸ The audiometric suite is comprised of a soundproof observation room adjacent to a control room. The observation room in this audiometric suite is also a Faraday cage, which helps to attenuate external electromagnetic noise.

Again, this is not to imply that I am advocating an idealist metaphysics akin to radical constructivism (see Campbell, 2002), but rather a more balanced mind/body metaphysics that I have long advocated (Campbell & Dawson, 1995): embodied cognition. Thus, one of the main activities I have been pre-occupied with is further developing a radical view of embodied cognition as a theoretical framework for educational neuroscience (Campbell, 2001, 2002, 2003; Campbell & Handscomb, 2007, April). What makes this view “radical,” in a foundational, root, sense, is the assumption that any change in lived experience *implies and necessitates* changes of some kind in brain and body behaviour, thus providing empirical ground for cognitive models of mathematical understanding.

Establishing a new approach to educational research, in addition to developing a soundly developed theoretical framework, typically involves the incorporation methodologies. This is most certainly the case for educational neuroscience. The approach I am taking to educational neuroscience in the ENGRAMMETRON involves the use of electroencephalograms (EEG), electrooculograms (EOG), electrocardiograms (EKG), eye-tracking (ET), to mention a few of the most notable data sets, integrated in a time-synchronous manner with more traditional audiovisual (AV) data sets.

One of the central topics the ENGRAMMETRON has been designed to research is the nature of mathematical cognition and learning. When a learner is looking at a geometrical diagram, that much is obvious. How do we know what part of the diagram a learner is looking at in any given moment? How do we gain insight, verbal reports aside, as to what they are thinking and when? One of the most intriguing areas of study in this regard concerns mathematical pattern recognition and mathematical concept formation. Toward this end, my students and I have been conducting pilot study investigations into multistable perceptions and geometric image based reasoning.

The first major project the lab has undertaken has been a study on metacognition and motivation in self-regulated learning. Data acquisition for this project is in its final stages with over a hundred participants thus far, and data analysis is also well underway. The experimental design of this project involves participants' study of a basic theorem of number theory, and as such, the large data set collected as part of this project has implications for mathematics education research as well. In this talk, I presented and discussed some preliminary results from this study.

The methods we use in the ENGRAMMETRON are particularly well suited for investigating learners' interactions with visual stimuli presented on a computer monitor. With the internet becoming so ubiquitous, we are well situated for studying the latest innovations on the web. One of these is the emergence of virtual reality environments. Here I also reported on a project initiative to implement and study various aspects of mathematics education in such an environment <www.secondlife.com>.

An important dimension of educational neuroscience to my mind is to explore to what extent we are capable of placing ourselves in brain and body states and processes that are most conducive to various aspects of learning, such as memorizing, remembering, imagining, reasoning, and general states and processes associated with "brain-storming" and other kinds of problem solving activities. One avenue into such matters could be to use biofeedback and neurofeedback techniques to explore such states a processes. As a first step in this regard, one of my students is working with me on a pilot study on learning biofeedback in a gaming environment <www.wilddivine.com>.

One contribution of the brain sciences over the past few years is growing evidence for the importance of affect in cognition and learning. It is now well known that affect can impede or improve learning. One of the major ways in which affect can impede learning is through anxiety. Anxieties can come in many forms, and two forms that we are most interested in understanding and unraveling are math anxiety and English as a Second Language (ESL)

anxiety. With another student, I have been are running a pilot study on ESL anxieties in Iranian women in Canada. We are learning much from this study that can inform future studies in mathematics anxiety.

My colleagues, students, and I are not pursuing research in educational neuroscience for the sake of pursuing research. Our aim is to make a positive difference for teachers and learners in ecologically valid environments. There is a huge divide between the neurosciences and education, and there are great differences between neurons in the brain and kids in classrooms. Our research in educational neuroscience aims to help bridge those differences through a more informed and less speculative approach to "brain-based education" that we are referring to as neuropedagogy. Closely aligned with this approach is a viable outreach program that includes various stakeholders and interest groups concerned with neuroscience and education <www.engrammetron.net>.

New Opportunities

As educators, our charge to prepare learners for the future necessitates that our work is informed by neuroscientific findings and also that practitioners are provided with reliable and validated research relating to human functioning and pedagogy. Educational neuroscience should serve as a transdisciplinary forum for researchers and educators to advance our understanding and practice in relation to the processes and interrelated functions of mentation and affect in the lived experience of teachers and learners.

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Communicating the Excitement and Beauty of Mathematics

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The Department of Mathematics at Simon Fraser University has a long history of outreach activities. And, while our big brother across town - UBC - has been focused, through math contests, workshops preparing students for Euclid, and, most recently, Math Circles, on identifying and recruiting the most gifted, SFU, true to the spirit of the school, has been creating activities designed to bring mathematics to as wide audience as possible.

During early years of the department, in late sixties and early seventies, there were the usual visits to high schools. But you could also meet enthusiastic SFU mathematicians at a table at the Pacific National Exhibition (PNE) in Vancouver, answering questions and explaining the mysteries of calculus to passers by.

In 1981, a young Australian, Kathy Heinrich, joined the department. She introduced a conference for grade 11 students, called “Mathematics Enrichment” (later renamed: “A World of Mathematics”), with math and science talks, and problem sessions. In the late 1980’s, Tasoula Berggren proposed a conference for grade 9 – 10 girls, modeled on a similar one at York University. We named it “Women do Math”. Later it was renamed “Discover the Possibilities”. This conference grew into the “Miss Infinity” workshops, offered by the Society for Women in Science and Technology (SCWIST) throughout BC. Both conferences (Math Enrichment and Women do Math) eventually lost funding and disappeared.

In late 1980s, Kathy Heinrich and I started to collect activities to motivate our students in the Mathematics for Elementary School Teachers course. Introducing these activities into the course was so successful that we decided to share them with others. The first step was a display at the 25th anniversary of SFU. We had several tables with books, math puzzles and games, probability experiments, and with supplies for constructing kaleidocycles, hexaflexagons, and flexible straw polyhedra.

This display made such an impression on then SFU president William Saywell that he gave us \$2500 to develop it further. We decided to take it to shopping malls, and the “Math in the Mall” program was born. We offered it in several Vancouver area shopping malls until 1995, when the BC government discontinued Science and Technology Week activities. Since then, we have been organizing such displays at all SFU Open House events, and on other similar occasions.

The Canadian Mathematical Society has helped to spread the word about our Math in The Mall, and featured it as a model for others. The display has inspired similar events at several other universities, both in Canada and abroad.

In 2001, SFU joined the Canadian Mathematical Society's (CMS) in organizing Math Camp for grade 9 – 11 students interested in mathematics. We continue to hold these week long camps at the SFU Burnaby campus during the last week of June. Since 2006, we also offer a shorter one at the Surrey campus. The camps continue the traditions of the conferences for high school students we were offering in 1980's. To learn more about our camps, see: <http://www.math.sfu.ca/outreach/schools/camp/>

To provide enrichment activities to high school students during the school year, we have introduced in 2004 a program called "A Taste of π ". The title and the idea of the program, as well as a great logo, originated with my colleague Veselin Jungic. Each event occupies three Saturday mornings, during which students participate in mathematics presentations, problem sessions, and, since 2007, also science presentations. The program was initially funded by an NSERC PromoScience grant, and later by the Dean of Science at SFU and the Pacific Institute for the Mathematical Sciences. Information about the program, including the logo and a long and growing list of exciting talks, can be found at the program's website: <http://www.math.sfu.ca/atasteofpi/>

Veso Jungic is also one of the creators of the "Math Girl" movies, designed to introduce students to the concepts of calculus.

A different and a very important outreach activity organized by the department and supported by the Pacific Institute for the Mathematical Sciences, has been the annual Changing the Culture conference. Inspired by the 1995 CMS Forum for Education in Mathematics, since 1998 the conference has been bringing together people interested in teaching mathematics at all levels: mathematicians, mathematics educators, school teachers from all levels, graduate students and student teachers. As the words introducing the first conference in February 1998 say, the goal of these conferences is to work together towards changing the culture of school mathematics, to allow students to experience what DOING mathematics means. The conference format: two plenary talks, a panel discussion, and workshops/discussion groups, provides a forum for reflection and exchange of ideas. The second talk of the day is always designated as a public talk and open to those who are not able to commit the whole day.

The conference website www.pims.math.ca/ctc/ chronicles the history of the event, with links to each of the Changing the Culture conferences since the series began.

While it is difficult to assess if the conference is having any impact on the teaching mathematics in BC, it has been a successful community builder, bringing together the participants, many of whom have been attending the conference year after year, and providing an opportunity for sharing ideas and learning from each other.

This brings me to the question I have been reflecting on in the recent months: what have we been learning from our outreach activities? What have we achieved? Are we reaching the audience we want to reach? Are we making a difference? Could we do better?

My work with "Math in the Mall" in the 1990s inspired me to join BC Science World's Scientists and Innovators in Schools program to bring the activities from the display to many elementary classrooms in the Vancouver area. Over the years, I have learned a lot from the teachers who have been inviting me into their classrooms through the Scientists in The Schools program. I have lost count of the number of schools I have been to, but I have probably visited between 50 and 100 classrooms over the last 15 years. All of these were memorable visits. But I soon learned that the most rewarding occasions were those on which I had an opportunity to discuss the activities with the teachers prior to the visit: How would they fit into the teacher's plans? How can they be related to the curriculum? What follow-up activities would work? And, from the discussions afterwards, I know that many of these wonderful teachers have adapted what they found useful in the visit, and used it in their own

teaching. I, on my side, have learned to initiate such discussions if a teacher was too shy or inexperienced to do so.

I have slowly come to a realization that the activities conducted in schools by people who visit the classroom on a special occasion affect the students and their teachers very differently than activities where students and teachers visit us, whether at our university, or in a shopping mall.

When enrichment activities are part of a field trip, there is an expectation of something different, exciting, and memorable. We accept that, while the experience may inspire and enrich our lives, we will be returning, at least temporarily, to our everyday life. But when a guest visits our classroom and shows us a rainbow – a colourful vision of what our everyday experience could be – a return to everyday unexciting life is so much more difficult. So, there is a potential that our well-intentioned efforts may do more harm than good.

Does it mean that we should avoid school visits? No, we should not. But we need to plan them differently than activities conducted on our “home ground”.

When we are hosting an activity, we want to make the experience interesting, educational and exciting, and of course we want to show ourselves, and our “home”, in the best possible light.

When we are guests visiting a classroom, we have one additional consideration: how we can help our guest, the teacher, to keep the rainbow, or at least a piece of it.

For many of my colleagues in the Faculties of Education this may be a “no brainer”. But it is certainly worth repeating from time to time if we want our efforts to have the greatest benefit for those we reach out to.

New colleagues have been joining the department in the last few years, and most of them happily participate in the activities of the department, and are coming with new, fresh ideas. At the same time the disconnect between secondary school mathematics and post-secondary mathematics seems as troublesome as ever. Maybe creating a seamless transition from the study of mathematics at high school to post-secondary study is a goal worthy of our energy and attention.

Conception et expérimentation de situations didactiques au préscolaire/primaire

Design and Experimentation of Didactical Situations in Kindergarten and Elementary School

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We present two teaching experiments undertaken with kindergarten students (on 'number') and elementary level special education students (on 'multiplication'), and highlight the contribution of the Theory of Didactical Situations (Brousseau, 1986) in their design. In the process, we develop some of the key notions of this theory – namely, devolution, feedback from the milieu, and didactical variables. These notions are particularly useful for prompting dynamic interactions specific to the mathematical goal and for encouraging cognitive engagement and mathematical activity among young children and students with learning difficulties. In recounting our experiences, we question the rapport of the designed situations to the contingencies of the didactical interactions and examine, as an example, the case of atypical student approaches.

Introduction

Ce texte vise à présenter deux situations didactiques originales, ayant fait l'objet d'expérimentations dans les classes, en précisant l'apport de la Théorie des situations didactiques (Brousseau, 1986) à leur conception. Trois notions clés de cette théorie, particulièrement utiles à la compréhension de l'organisation didactique des situations, sont brièvement définies. Nous détaillons par la suite cette organisation et présentons quelques conduites d'élèves pour chacune des situations.

La première situation – Le petit Poucet - est tirée d'un cahier d'activités mathématiques pour le préscolaire que nous avons développé (Giroux et Ste-Marie, 2004) dans le cadre du programme d'intervention précoce québécois Fluppy (Capuano et al. 2007). Cette situation, expérimentée dans plus de cent classes du préscolaire au Québec, vise à ce que les élèves rencontrent la nécessité de comparer et donc d'ordonner trois collections pour résoudre un problème. La résolution du problème suppose donc que l'élève établisse une relation d'ordre entre trois quantités. La seconde situation – Le jeu des étoiles - expérimentée dans une classe d'adaptation scolaire du 2^e cycle primaire (difficulté d'apprentissage), vise à une activité

mathématique sur les relations multiplicatives, plus précisément sur la relation entre facteurs et multiples.

Théorie des situations didactiques : Quelques notions de base

La didactique des mathématiques étudie les conditions spécifiques à la diffusion de savoirs mathématiques. La Théorie des situations (Brousseau, 1998) fait l'étude et modélise ces conditions en plaçant au cœur de sa théorie, le fonctionnement du savoir en situation. Elle s'attaque donc principalement à l'étude du rapport d'adéquation entre la situation et le savoir visé. Il faut préciser, particulièrement pour nos lecteurs anglophones, que la Théorie des situations didactiques établit une distinction entre connaissances et savoirs.

Les savoirs constituent la référence externe aux situations, ce sont les objets visés par l'enseignement et donc, principalement, des évaluations.

Le savoir est le produit culturel d'une institution qui a pour objet de repérer, d'analyser et d'organiser les connaissances afin de faciliter leur communication, leur usage sous forme de connaissance ou de savoir, et la production de nouveaux savoirs. (Salin, 2002)

Les connaissances se manifestent essentiellement comme des instruments de contrôle des situations et peuvent être reliées aux modèles implicites par lesquels les élèves agissent en situation.

Les connaissances sont les moyens transmissibles (...), mais non nécessairement explicites, de contrôler une situation et d'y obtenir un certain résultat conformément à une attente et à une exigence sociale. (Salin, 2002)

Dans la Théorie des situations didactiques⁹, l'apprentissage a pour finalité le fonctionnement autonome des élèves dans les situations faisant appel aux savoirs enseignés. C'est par un processus d'adaptation aux situations didactiques que les connaissances et les savoirs s'élaborent. L'ingénierie didactique, méthodologie de recherche qui se caractérise par un schéma expérimental basé sur des réalisations didactiques en classe (conception, réalisation, observation et analyse de séquences d'enseignement) (Artigue, 1996), vise à provoquer une genèse artificielle des connaissances dans un jeu d'interactions entre un élève et un milieu didactique (problèmes, supports matériel ou symbolique, consignes, etc...) qui lui est antagoniste.

Trois classes de situations sont distinguées faisant appel à des processus adaptatifs différents.

1. Situation d'action

Situation où la connaissance se manifeste par des décisions, par des actions sur le milieu. La connaissance visée détermine la stratégie qui permet d'exercer le contrôle (adaptation de la connaissance). Il n'est pas nécessaire que l'élève soit en mesure d'explicitement la connaissance.

2. Situations de formulation

Situation où la communication de la connaissance visée à un autre «joueur» (élève) est nécessaire (adaptation d'un répertoire de connaissances pour convaincre).

3. Situation de validation

Situation où une justification des formulations est nécessaire (énoncés, démonstrations). Elle permet une reconnaissance d'une conformité à une norme. Elle est suivie d'une phase d'institutionnalisation par laquelle l'enseignant fixe le statut culturel du savoir en jeu.

Notions d'ancrage à la TSD de notre travail expérimental

⁹ Pour plus d'économie, la Théorie des situations didactiques est indiquée par TSD

Notre travail expérimental s'appuie plus particulièrement sur trois notions clés de la TSD dans la mesure où ces notions sont particulièrement utiles pour élaborer des situations didactiques qui favorisent des interactions spécifiques à l'enjeu mathématique et sollicitent l'engagement cognitif ainsi que l'activité mathématique d'élèves jeunes ou en difficultés.

1. *Dévolution*

La dévolution est le processus par lequel l'enseignant conduit l'élève à engager une action sur la situation (le milieu adidactique¹⁰) qui soit produite et justifiée d'une part, par les nécessités de la situation et, d'autre part, par ses connaissances. La situation doit donc être organisée de manière à susciter chez l'élève une action qui n'est pas convenue et qui le rend donc responsable de ce que produit son action. La situation doit retourner à l'élève une rétroaction sur la justesse et la pertinence des actions qu'il a engagées. Ces rétroactions doivent être lisibles par l'élève de manière à ce qu'il puisse établir des relations entre ses décisions et le résultat obtenu.

2. *Milieu*

Le *milieu* est le système antagoniste de l'élève¹¹. On appelle "milieu" tout ce qui agit sur l'élève et/ou ce sur quoi l'élève agit, et lui assure une rétroaction des actions qu'il produit. L'élève agit donc sur le milieu à l'aide de ses connaissances et dans le cadre des règles qui régissent la situation didactique. **Le milieu doit être spécifique du savoir à enseigner de manière à ce que les stratégies mises en œuvre par l'élève pour contrôler le milieu engagent et fassent appel aux connaissances visées.**

3. *Variables didactiques*

Les variables didactiques sont les éléments de la situation didactique qui peuvent être modifiés par l'enseignant et qui affectent la hiérarchie des stratégies (et donc les connaissances) des solutions, des actions engagées par l'élève. Dans la conception de nos situations, il s'est agi pour nous de :

- Organiser la progression des connaissances des élèves du préscolaire dans une séquence didactique
- Adapter une situation à des répertoires différents de connaissances mathématiques dans une même classe (difficultés d'apprentissage)

Le petit Poucet (préscolaire)

Objectif de la séquence

La séquence vise à ce que les élèves rencontrent la nécessité de comparer et donc d'ordonner trois collections pour résoudre un problème. La résolution du problème suppose donc que l'élève établisse une relation d'ordre entre trois quantités.

Résumé de la séquence

La situation évoque l'histoire du petit Poucet qui, à chaque pas, laisse tomber un caillou pour retrouver son chemin. Le but du jeu est, pour l'élève, d'identifier parmi trois collections celle qui convient pour se rendre à l'une des trois habitations placées près d'un chemin (petite marelle) que le petit Poucet emprunte.

Dans cette situation, on retrouve à la fois un contexte ordinal (ordre des habitations) et un contexte cardinal (cardinalité des collections). En effet, l'ordre dans lequel les habitations apparaissent est l'information à prendre en compte pour identifier la collection recherchée. Aucune information sur les nombres impliqués n'est fournie. La réussite suppose que l'élève

¹⁰ Le milieu adidactique doit permettre le fonctionnement de la connaissance comme production libre de l'élève

¹¹ Il existe aussi un milieu de l'enseignant que nous n'abordons pas ici.

établit une relation d'ordre entre des quantités : « plus le trajet est long, plus il faut de cailloux et par conséquent, les trois collections de cailloux doivent être ordonnées de façon à choisir celle qui correspond au même rang que celui de l'habitation à atteindre identifiée dans la consigne ».

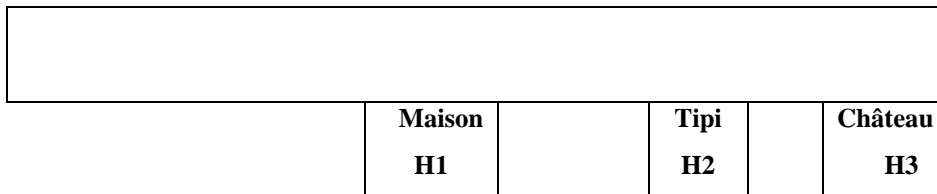


Figure 1. Dessins des trois habitations juxtaposés à un chemin (marelle sans cases)

Dévolution de la situation aux élèves

Trois types d'habitations (cabane, tipi, château) sont disposées le long d'une marelle sans cases. Le problème est soumis aux élèves :

Petit Poucet laisse tomber un caillou à chacun de ses pas sur la marelle.

 Pour se rendre à la cabane, il utilise une seule couleur de cailloux.

 Pour se rendre à la tente, une autre couleur.

 Pour se rendre au château, encore une autre couleur.

Le petit Poucet veut se rendre au château, quelle couleur de cailloux prendra-t-il ? (ou au tipi ou à la maison, selon la situation).

Situation d'action avec phases de formulation et de validation

Dans cette séquence, on ne peut véritablement référer aux trois classes de situation. Il s'agit plutôt d'une situation d'action qui comporte des phases de formulation et de validation.

Situation d'action : Chaque équipe de trois élèves s'installe à une table. Elle reçoit 3 collections de jetons de couleurs différentes. Chacune des collections comporte un nombre d'éléments qui correspond au nombre de « pas » nécessaire pour se rendre à l'une des trois habitations (par exemple, 5 jetons rouges, 8 jetons bleus et 10 jetons verts). Chaque équipe doit identifier la collection qui correspond au rang de l'habitation identifiée dans la consigne. Par exemple, si le petit Poucet veut se rendre au château et que cette habitation est la troisième, la collection à identifier est celle qui contient le plus d'éléments puisque le château, dans notre exemple, est l'habitation la plus loin.

Phase de formulation : Lorsque toutes les équipes ont fait leur choix, chaque équipe est invitée à présenter, à l'ensemble de la classe, la collection retenue et à motiver son choix.

Phase de validation : On demande aux élèves ce qu'il faut faire pour identifier les équipes gagnantes ; les élèves suggèrent rapidement lors du premier scénario de réaliser le trajet. Les collections retenues sont utilisées pour faire le trajet sur la marelle retournée, comportant cette fois, des cases.

Rouge	Rouge	Rouge	Rouge	Rouge					
Bleu	Bleu	Bleu	Bleu	Bleu	Bleu	Bleu	Bleu		
Vert	Vert	Vert	Vert	Vert	Vert	Vert	Vert	Vert	Vert
				Maison H1			Tipi H2		Château H3

La validation se fait donc par un procédé de correspondance terme à terme des éléments des trois collections ; chaque jeton étant déposé sur une case de la marelle. Un échange collectif permet de mettre en relation les différents procédés ayant servi à choisir la collection et les résultats obtenus (réussite ou échec).

Variables didactiques

Nous proposons la situation à quelques reprises en modifiant, à chaque reprise (scénario), les valeurs des variables didactiques. Ce jeu sur les variables vise à faire échec aux stratégies qualitatives et à favoriser l'élaboration de stratégies qui font appel soit à la correspondance terme à terme (scénarios 1 et 2), soit à des stratégies numériques (scénarios 2, 3 et 4).

	Scénario 1	Scénario 2	Scénario 3	Scénario 4
Nombres	5, 8, 10	9,12, 13	12, 13, 14	11, 13 , 14
Rang	3 rangs	3 ^e (cabane)	2 ^e (château)	1 ^e (château)
Matériel Collections	Jetons (couleurs différentes)	Jetons (couleurs différentes)	Jetons (couleurs différentes)	Petits jetons Gros jetons Bâtonnets

Tableau 1. Valeurs des variables pour chacun des scénarios.

Nous résumons ici brièvement les principales stratégies mises en œuvre progressivement par les élèves ainsi que des formulations qui les accompagnent. À terme, les élèves dénombrent les collections et établissent la relation d'ordre sur la base de leurs cardinalités.

1. *Ludique, non mathématique*
Par exemple : «Les bleus, parce qu'il y a du bleu sur la maison».
2. *Correspondance terme à terme entre les éléments de chacune des collections*
Par exemple : «Là, on fait une ligne pour savoir qui a le moins»
3. *Dénombrement et comparaison des cardinalités de chacune des collections*
Par exemple : «Pierre en a le deuxième moins. Moi, j'en ai le premier moins et lui, c'est le dernier moins... «Oui, il en a le moins, moi, j'en ai le plus, lui en a le moyen plus.»

Jeu des étoiles

Objectif de la séquence

La séquence sur le jeu des étoiles vise essentiellement à favoriser le passage de l'additif au multiplicatif par la mise en relation facteur /multiple.

Résumé de la séquence

Il existe une version additive et une version multiplicative du *Jeu des étoiles*; toutes deux sont des jeux de pistes (une planche représentant un tableau de nombres). Il y a 3 ou 4 joueurs par équipe. Les élèves qui ont travaillé sur la version multiplicative, présentée ici, ont tous préalablement joué sur la version additive. Le but du jeu est, pour les joueurs, d'accumuler le plus d'étoiles possible, lesquelles sont réparties sur un tableau numérique troué, en effectuant des déplacements avant et arrière déterminés par le nombre obtenu sur un ou deux dés. Les étoiles rouges valent trois points, les bleus valent deux points et, enfin, les jaunes ne valent qu'un point.

Dans la version multiplicative, les moyens mis en œuvre dans le jeu additif sont (ou devraient l'être) disqualifiés. La planche de jeu est composée d'une suite de nombres à intervalles réguliers différents de 1 et comporte 5 ou 10 lignes de 10 nombres chacune. Le joueur tire un (ou deux) dé(s) dont les faces correspondent aux 6 premiers multiples ($k \times n$ et $k = 1, 2, 3, 4, 5, 6$) de l'intervalle en jeu (n). Il doit avancer ou reculer de k cases selon le nombre obtenu ($k \times n$). La figure 2 présente une planche de jeu avec $n = 3$. Les faces des dés sont numérotées 3, 6, 9, 12, 15 et 18. Une particularité du jeu est que la case d'arrivée d'un joueur est la case de départ du joueur suivant ; autrement dit, le parcours ne se réalise qu'avec un seul pion bien que chaque joueur marque d'une couleur différente les déplacements qu'il effectue sur la planche de jeu. Chaque joueur se trouve ainsi concerné par le choix du joueur précédent car certains parcours sont plus avantageux que d'autres.

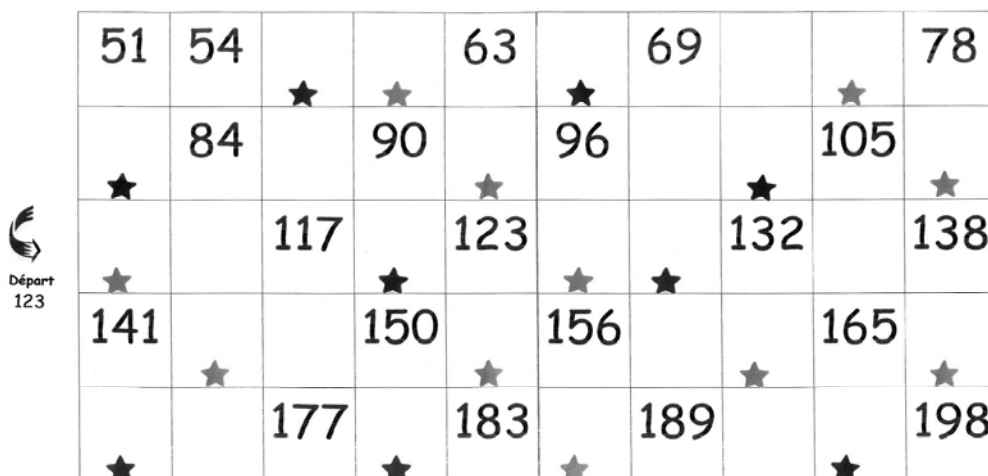


Figure 2. Planche du jeu des étoiles pour $n = 3$.

Chaque joueur doit remplir une feuille de route sur laquelle il doit inscrire pour chaque coup joué : 1) le nombre de départ; 2) le(s) nombre(s) obtenu(s) sur le(s) dé(s); 3) le nombre d'arrivée; 4) le nombre de cases qui correspond au déplacement; 5) Le nombre de points obtenus.

Case Départ	Dés obtenus	Case Arrivée	Déplacement Cases	Points Étoiles
123	12	111	4	2 pts
111	6	117	2	0 pt

Figure 3. Extrait d'une feuille de route complétée sur une planche de jeu avec $n=3$

Dans la figure 3, la relation entre facteur et multiple «s'exprime» ainsi : Le tableau est à intervalles de 3. Le nombre tiré sur le dé est 12. Le déplacement sur le tableau correspond donc à 4 cases ($4 \times 3 = 12$).

Variables didactiques

Le jeu sur les valeurs des variables didactiques permettent de : 1) tenir compte des profils scolaires variés au sein d'une même classe d'adaptation scolaire ; 2) favoriser l'élaboration

d'une stratégie multiplicative plutôt qu'additive. Les variables didactiques et leurs valeurs sont les suivantes.

Intervalle «n»: Plus l'intervalle est grand, plus les stratégies additives sont lourdes

Les valeurs possibles sont: 2, 3, 5, 6 et 7

Domaine numérique: Plus le domaine numérique du tableau de nombres de la planche de jeu est «élevé», plus les stratégies additives sont difficiles à contrôler. Les valeurs possibles sont :

Tableau à intervalles de 2 : a) 50 cases; b) 60 à 168

Tableau à intervalles de 3 : a) 50 cases; b) 51 à 198

Tableau à intervalles de 5 : a) 100 cases; b) 50 à 545

Tableau à intervalles de 6 : a) 100 cases; b) 50 à 644

Tableau à intervalles de 7 : a) 100 cases; b) 707 à 1400

Dés: 1 dé: favorise la relation facteur/multiple

2 dés: favorise le recours à la distributivité de la multiplication sur l'addition

L'emploi de deux dés favorise le recours implicite à la distributivité de la multiplication sur l'addition comme le montre la figure 4. La feuille de route qui est représentée correspond à un tableau à intervalles de 7. Les nombres tirés par l'élève sur les dés sont 21 et 14. Le déplacement de 5 cases peut être identifié ainsi : $(3 \times 7) + (2 \times 7) = 5 \times 7$.

Case Départ	Dés obtenus	Case Arrivée	Déplacement Cases	Points Étoiles
1043	21+14	1008	5	0 pt

Figure 4. Extrait d'une feuille de route d'un jeu avec deux dés ($n = 7$)

Stratégies mises en œuvre

Nous résumons dans ce qui suit, les stratégies mises en œuvre par les élèves.

1. Stratégie additive

La stratégie de base est additive. Il s'agit alors d'additionner le nombre de départ à celui obtenu sur le dé pour identifier une case d'arrivée selon un déplacement avant. Cependant, pour savoir s'il est plus «avantageux» de reculer que d'avancer, on doit ensuite effectuer une soustraction de ces nombres. En plus d'une certaine lourdeur sur le plan des calculs à effectuer, cette stratégie n'est pas d'emblée efficace si le nombre d'arrivée n'est pas inscrit au tableau. Le jeu sur les valeurs des variables didactiques vise à faire échec à la stratégie additive.

2. Stratégie intermédiaire

Une stratégie intermédiaire consiste à avancer ou reculer dans le tableau par un comptage des cases qui s'appuie sur la suite des multiples. Par exemple si, sur un tableau d'intervalles de 7, l'élève tire 35 sur le dé et que le nombre de départ est 742, un déplacement avant peut être réalisé ainsi: 7 (case 749), 14 (case 756), 21 (case 763), 28 (case 770), 35 (case 777). On peut se déplacer par l'arrière en procédant de la même manière en partant de 742.

3. Stratégie multiplicative

La stratégie optimale est celle qui engage les connaissances sur la relation multiplicative entre un nombre inscrit sur le dé (multiple : $k \times n$) et celui qui correspond à l'intervalle (n). Le déplacement dans le tableau est donc contrôlé par l'identification du facteur k . Il suffit alors d'avancer ou de reculer de k cases dans le tableau. Par exemple, si, sur un tableau d'intervalles de 7, l'élève tire 35 sur le dé et

que le nombre de départ est 742, l'élève dégage le facteur k ($35 \div 7 = 5$). Il avance et recule de 5 cases et choisit la case la plus avantageuse.

Phases de formulation et validation

D'autres activités permettent aux élèves de valider les stratégies multiplicatives et permettent d'identifier les savoirs en jeu. À titre d'exemple, des feuilles de routes semblables à celles que les élèves ont remplies lors de la situation d'action, sont remises incomplètes aux élèves. Une correction collective de la feuille complétée permet à l'enseignant d'institutionnaliser les savoirs en jeu.

Conclusion

Nous voulons souligner, en guise de conclusion, que l'engagement cognitif et mathématique, autant des élèves du préscolaire que ceux de l'adaptation scolaire, est remarquable au cours de ces séquences. Nous souhaitons également rendre compte de notre étonnement à l'égard de certaines conduites atypiques, peu prévisibles des élèves, au regard des stratégies attendues. Ainsi, certaines stratégies lourdes ont tendance à persister ; les élèves résistant à employer une stratégie plus économique bien qu'ils disposent des connaissances nécessaires pour la contrôler. On assiste donc à des processus de «sur-adaptation» qui semblent nuire à l'apprentissage. D'autres conduites atypiques sont observées chez des élèves qui modifient les règles du fonctionnement de l'activité de manière à être avantagés. Nous avons fait l'étude de telles conduites ailleurs mais le tour de la question n'est pas complétée et mérite encore notre attention (Giroux, à paraître).

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Cabri 3D: An Environment for Purposeful Mathematical Activity?

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Cabri 3D is a relatively new 3D interactive geometry software which has the potential of offering students an environment in which they are motivated to pursue their own goals using mathematical tools. This paper reports on part of an ongoing research program with grade 7 and 8 students, in which tasks and resources have been devised that engage and give support to students in pursuing their goals with the software, and in which evidence has been found of engagement with mathematics in a variety of ways.

Introduction

I first saw Cabri 3D (Bainville & Laborde, 2004) demonstrated at CabriWorld 2001. It was pretty, but I wasn't really interested. At that point I was perfectly at home moving in a 3D world and thinking in a rectilinear 3D world, but was uneasy in any 3D space not nicely defined by horizontals, verticals and rectangular prisms. Problems in 3D geometry were just too abstract.

Six years on, I'm deeply immersed in Cabri 3D. I feel a bit like I've found the starship Enterprise and am encountering "space –the final frontier." And much to my surprise ordinary everyday space has turned out to be just as interesting and exotic as space at the far-flung reaches of the galaxy. Not to mention somewhat more accessible. The vague blur which was 3D geometry is being replaced by a web of increasingly rich connections and insights.

It is this experience of an exciting and rich mathematical environment that has led me to focus my research on Cabri 3D, and my aim is to share this experience with students by finding means to enable them to participate in mathematical problem-solving and discovery in this environment.

Cabri 3D is an interactive geometry software in which a 3D environment containing objects such as points, lines, planes, and polyhedra is represented on a 2D screen. It shares with 2D interactive geometries such as Geometer's Sketchpad, Cabri 2+ or Cinderella the critical feature that objects are constructed in relationship with other objects and that such relationships are preserved when initial objects change. The software has potential as a pedagogical tool (Mackrell, 2006a), in the generation of new mathematics (Oldknow, 2006) and in enhancing students' ability to visualize (Laborde, 2007). It has been found to be a useful tool in exploring the geometry of space (Accascina & Rogora, 2006), and has won a major award for educational software in the UK (BETT, 2007), with one of the criteria being that it is intrinsically engaging to learners (Chartwell-Yorke, 2007).

What is of most interest to me is that Cabri 3D provides new areas of mathematical enquiry and the ability to model both static and dynamic structures. As such, it may offer an environment in which students can engage in meaningful mathematical activity whilst pursuing their own goals. Ainley and Pratt (2006) argue that mathematics learning is best facilitated through engaging students in tasks that are purposeful to them (“purpose”), and also necessitate the use of mathematics (“utility”). In their terms, a purposeful task, focussed on a utility of proportion, might be for the students to decide on the sizes of furniture for a doll’s house based on a sample piece of furniture.

I have certainly myself experienced both purpose, utility and an increasing awareness of mathematics when engaged in mathematical enquiry or modelling structures with Cabri 3D. Two activities have been particularly rewarding. The first activity is exploring the new mathematics of fold-up polygons (Mackrell, 2006b), which I have yet to pursue with students. The second of these is using a cartoon character (Claude) to model motion, including swinging on a swing, blowing a bubble, rowing a boat, and flying. The diagrams below show Claude in various stages of diving.

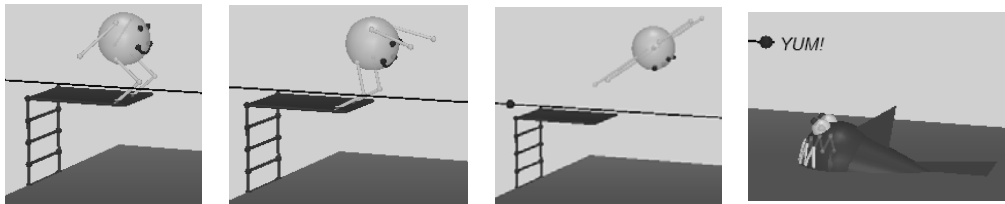


Figure 1: Modelling Diving with Cabri 3D

Here I have modelled motion as a series of transformations, connected by conditional constructions, which has required solving problems such as constructing appropriate centres and angles of rotation and creating motion tangent to a parabola. I also learned that the diver must lean out from the board before jumping! I have been deeply engaged when creating such files and have experienced frequent “aha” moments when I have found solutions to problems or made new connections.

However, I am reasonably competent at mathematics – and I find Cabri 3D challenging. I have chosen to work with grade 7/8 students: is it realistic to expect them to be able to do meaningful mathematics with Cabri 3D?

One danger is that students will avoid using the mathematical aspects of Cabri 3D. Ainley and Pratt (2006) found that students could use interactive geometry software for their own purposes (basically as a simple drawing package) without needing to encounter any mathematical ideas. However, it is less easy to represent objects in a 3D environment because any point on the screen represents a line in space, which means that a casual representation of an object using no mathematical construction may look correct from one perspective – but is likely to be wildly inaccurate when the view angle is changed. This lessens the appeal of casual construction. On the other hand, certain objects such as collections of polyhedra can be made very quickly and accurately with very little mathematical awareness. Ainley and Pratt managed to devise purposeful tasks which required engagement with the mathematical features of the software, but did not find this easy to do. I was hence aware that student tasks might need to be carefully devised.

Another issue is that the process of learning to use the software involves instrumental genesis, by which a tool, initially an object of action, becomes an instrument in achieving further action (Verillon & Rabardel, 1995) and this has been shown to be unexpectedly complex, involving not only learning about the software but also understanding the mathematics

mediated by the software (Hoyles, Noss, & Kent, 2004). This suggests, however, that simply learning to use the tools would enhance students' mathematical understanding, as all tools are mathematical. Animation, for example, can only be performed by making a point move on a segment, arc or circle: all objects which are to move need to be constructed in relationship with the moving point. In contrast, in Flash an initial object and a final object are given and the software is instructed to create intermediate images. The user does not need to understand anything about the process by which this happens. On the other hand, the mathematical understanding which results from using the tools is mediated by the specific design of the tools used (Mariotti, 2002) and hence such understanding may, initially at least, differ from conventional mathematics (Hoyles et al., 2004). I would hence need to consider carefully the way in which students were taught to use the software.

The research questions I am addressing are as follows (a) what types of task engage learners to pursue their own goals in a Cabri 3D environment (b) what type of materials and approaches give students the necessary skills to use the software effectively in performing these tasks? (c) to what extent do such tasks require the use of mathematics?

Methodology and Preliminary Design Decisions

My research currently involves two classes of grade 7 and two classes of grade 8 students, with 24 students in each class, in a programme for academically motivated and able students in an Ontario school. Students are expected to complete the normal grade 7 and 8 provincial mathematics curriculum but are provided with additional enrichment tasks. ICT facilities are relatively good, with a dedicated computer lab containing about thirty computers shared between the four classes.

There are two phases to the research reported here: the pilot study, in which only informal impressions were gained (which, however, influenced later decisions as to tasks, approaches and materials), and the research itself, during May and early June 2007, in which more extensive data was collected, with student Cabri files being submitted at the end of each session, together with brief feedback sheets. In addition, Grade 7 students submitted their final work with a description of how their figure was constructed and the mathematics involved in the figure and Grade 8 students filled in a questionnaire concerning their response to Cabri 3D and to the different learning materials used during their experience with Cabri 3D.

I acted as classroom teacher, with the regular classroom teacher either participating in class activities or engaging in other activities within the school.

During the first part of the pilot study, Cabri 3D was introduced via an optional lunchtime club in which students were given the opportunity to experiment freely. This turned out to be problematic: students often either chose tasks which involved the mathematical aspects of Cabri only minimally and were of insufficient challenge to maintain their engagement (such as creating a structure with cubes), or tasks which involved a far higher degree of skill than they possessed, which led to frustration and disengagement unless students were immediately supported in the learning required. This led to some fundamental decisions for later work: that students would first encounter directed tasks in order to gain familiarity with Cabri 3D, that subsequent open-ended tasks should have some focus and that support materials would need to be made available: the level of teacher intervention required to support students was only possible with a small group. It was also decided that the club did not give students sufficient time to become familiar with Cabri 3D and hence that Cabri 3D would be introduced as part of normal classroom activity.

A constraint in most subsequent tasks was hence that they be related to particular aspects of the grade 7 and 8 curriculum. During the pilot phase grade 8 students used ideas concerning

scale modelling to build a model house (involving ten one-hour sessions with Cabri 3D) and grade 7 students created objects using transformations (involving three one-hour sessions). During the research itself, the Grade 7 task was connected to polyhedra. The Grade 8 task had no curriculum constraint, however, as it occurred when half the grade 8's were away on a school trip.

The model house idea was suggested by the classroom teacher, based on earlier successful experience with a similar task using concrete materials. The enthusiasm with which the students engaged in this task led to the decision that further tasks would also involve construction of objects rather than exploration of mathematical concepts. Hence transformations with the Grade 7's involved creating a forest.

Decisions regarding support materials were also influenced heavily by the pilot study. Experience with handouts, teacher exposition, and Flash demos led eventually to students being provided with small Flash demos which would show them how to make a particular part of an overall structure, such as a door that opened, involving the use of a number of tools, rather than how to use a particular tool.

In the grade 7 work on polyhedra (involving six one-hour sessions) students were initially given Flash demos illustrating the dynamic "truncation" of a cube, in which either vertices or edges or a combination of vertices and edges are progressively removed. The basic process involved creating a polygon representing the result of cutting off either a vertex (a triangle) or an edge (a rectangle) and then using transformations to place an image of the original polygon at each of the other vertices or edges of the polyhedron. Students were asked to construct such a truncated cube for themselves and to then either generalize the process to other polyhedra or to look at different types of truncation. The final four sessions involved students working in pairs to design and create an interesting object which needed to have at least two planes of symmetry and to write about the mathematics involved in their object and the way they had constructed it. Finished files were submitted for assessment.

The final work with the Grade 8's developed out of the enthusiasm for motion and animation shown by students during earlier work. Students wanted dynamic rather than static objects as part of their structures, and objects such as elevators and windows that opened were highly attractive. The focus was hence specifically on creating and animating different types of motion. I worked with approximately half the grade 8 students, over four sessions. Students were introduced to various techniques for creating animated figures and were invited to create their own. Students had access to a website on which different techniques were described [note: the location of this website is being changed currently], together with some Flash demos and some Cabri files with descriptions of how they were constructed. Students were asked to fill in a questionnaire concerning their overall experience with Cabri 3D and the learning materials provided.

Results

When Grade 7 students were asked to begin the specific project of designing and creating an interesting shape, an immediate question was whether the shape needed to be based on polyhedra? The student had very much enjoyed creating shapes with transformations and many wanted to continue to create realistic shapes rather than explore polyhedra. It was decided that any shape would be permitted, providing that the final figure had two planes of symmetry. The anticipation was that this would basically restrict the shapes used to polyhedra, but this proved not to be the case. Of the twenty files submitted, eight involved realistic figures, such as a helicopter, a model of the solar system and a playground ride. Of these, three involved some form of motion. Of the twelve files involving more mathematical objects, only one did not involve motion. Students expressed enjoyment in working with

Cabri 3D and stayed well engaged for the duration of the project. Screenshots from a number of their files are shown below.

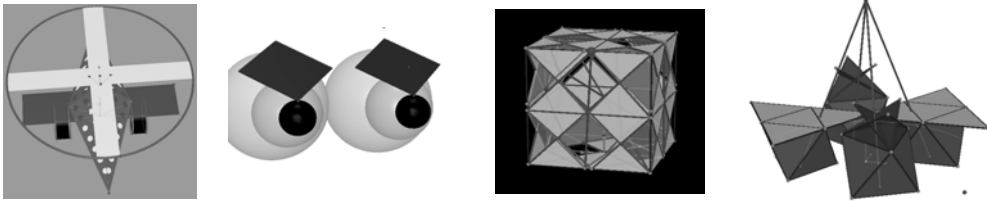


Figure 2: Grade 7 student work

The second file above was made by a highly able student who had innovated some techniques during the work with transformations and had managed to create various truncations of an icosahedron while most of the other students were still working with a cube. I was disappointed by the simplicity of his “eyeballs”: he added moving quadrilaterals in response to my challenge to make them blink. Clearly an amusing real-world object had more appeal than a more mathematically challenging object. Files will be further commented on in considering the mathematics used in their construction.

Level of engagement was more varied with the grade 8’s. Some students were well engaged with exploring animation, and creating animated figures, with eleven out of the thirteen files submitted involving realistic objects. Screenshots from several student files are shown below.



Figure 3: Grade 8 student work

The creators of the first file were particularly interested in finding ways to create their own appearance (hair) and motion (the creature rotates and also jumps up and down on a “pogo stick”). In the second file, the polyhedra rotate in highly complex ways around other hidden moving polyhedra. These files, and the third file above, involved adapting techniques given to new purposes. However, some students chose to make cosmetic modifications to existing files, or to add lettering with segments. For some, this was an engaging process of making such files their own through minor changes. The file above with the bat is an example: a file containing a sphere with moving wings was available to students. This student also created the second-to last file above, a representation of a disco light in which the sphere rotated and the segments moved, but titled the bat file with her name and “Best thing on Cabri I’ve made YET”. For others, however, such activity reflected a lack of engagement. This was no doubt partially due to the activity taking place instead of a school trip. However, the task given was also more open-ended and materials provided were more varied in their effectiveness. The website introducing motion was not written specifically for students at this level and students found the instructions difficult to follow. Downloading and exploring the files exemplifying different motions was of greater interest, with some attempts to understand how files were made by showing hidden objects and replaying the construction. However, Cabri 3D did not have a textual description of objects at that point (introduced in October 2007) and students

became discouraged when files proved difficult to understand and recreate. A few Flash demos were available for simple motion, and, for more complex motions, files were created with all objects numbered in the order of creation and with text indicating how each object was created. Students could use the “Replay Construction” function in order to watch the step-by-step construction of the file. This proved to be successful with the students who had enough facility with Cabri 3D to find and use tools without the assistance of a demo.

In the final questionnaire almost three-quarters of the grade 8 students used words such as making, expressing, doing, building or creating when describing the best aspects of Cabri 3D. Three referred to enjoyment of existing files and only one referred to “learning new things”. Students rated highly as support materials Flash demos and the files to replay, with the criteria apparent from their comments that these were easy to understand and follow.

One potential source of evidence for student use of mathematics is disappointing: Grade 7 student descriptions of the mathematics involved in their work focused on mathematics as content. Students named shapes (e.g. trapezoid, cylinder, etc), provided statistics concerning numbers of shapes or numbers of edges, etc, identified reflection symmetry (though “line of symmetry” tended to be used instead of “plane of symmetry”). Such descriptions give little evidence of the mathematics engaged in during the construction.

However, descriptions of how Grade 7 students went about constructing their objects are more informative, even though not required to be specifically mathematical. Here is an example, related to the fourth screenshot of grade 7 work above:

Our cabri project replicates an amusement park ride. We made it by starting with a circle and putting two segments in that divided the circle into perfect fourths. We then took an octahedron and placed in the middle of one segment. It was rotated around so there were four octahedrons. Then, a tetrahedron was placed on each octahedron. We redefined the segment point onto the circle so the octahedrons spun. Then, we hid all the extra lines, points, planes, and the circle. To rotate the ride, grab the point on the outside of the octahedrons and drag it.

This gives some insight into the mathematical problems that the students needed to solve. In general, student descriptions give strong evidence of the utility of transformation, and of the way in which the Cabri tools mediated their understanding of the various transformations. Almost all students describe the use of transformations in construction. The transformation was frequently specified by its initial object, the type of transformation and the location of the final object (e.g. “rotated and reflected trapezoids to fill in shape”, “translated the chimney so it would run down until the first floor”) or the purpose of the final object (“the triangles could be reflected to truncate all the vertices of the dodecahedron”). Only once was translation identified as using a vector (although students had deliberately not been introduced to the Cabri 3D option of performing translations without vectors): the highly able student who created the eyeballs “translated the cylinder along the same vector”. Several students were aware that rotation happened about an axis: no student specified an angle of rotation and some specified “around the base” or “around the center net” rather than around an axis. In the version of Cabri 3D that the students were using, angle of rotation needed to be specified by an example rather than by a numerical value: students could interpret the tool as moving a shape from one location to another rather than rotating it by a given angle. Most students did not mention the use of a tool to perform the transformation: instead of “used the rotation tool” students would say “rotated”. The central symmetry tool (which performs reflection in a point) was used more frequently than either the half-turn tool (reflection in a line) or the actual reflection tool (reflection in a plane): much of the time the use of this tool was referred to as “reflection”, fitting its function rather than its name in the toolbox. No student specified a point through which objects were reflected. There is a more fundamental meaning of transformation that permeates: the understanding that transformations are means of

construction – and, for many students, that transformation by a variable object is a means of creating motion (as with the amusement ride above).

Student files give further evidence of the use of mathematics. Such files need to be interpreted with caution, however: almost all final files contained quite complex constructions, but students had been given clear instructions as to how to make many aspects of the constructions and may have been simply following the instructions with no understanding of the underlying mathematics. The third screenshot above is an example of such a file, which represents a minor deviation from detailed instructions given in a demo. However, other files, such as the amusement park ride, give evidence of independent student use of mathematics. For example, the circle was divided into “perfect fourths” by extending the base vectors to intersect the circle. An initial octahedron was created and then rotated by the angles defined by the base vectors to create further octahedral. The students also solved the problem of making the four octahedra rotate in the opposite direction to the point controlling their motion. The helicopter file involved a large number of reflections in a plane – a transformation not introduced in creating the polyhedra, and the eyeballs used translations extensively. It is striking that in no student files were objects created according to appearance without being connected mathematically with other objects.

Although, as has been mentioned, some of the grade 8 work involved replication or cosmetic modification of existing files, the work of two students was of particular interest mathematically. These students showed an interest in methods to make objects appear and disappear, arising out of an object appearing and disappearing due to the slowness of the computers. Cabri 3D does not have hide/show buttons or macros and hence some fairly sophisticated geometric logic is required to make a hide/show slider and it was not anticipated that this would be introduced to these students. However, both students were shown how to make a basic slider which controlled the existence of a “magic” point and both were very keen to use such a slider: one made a spinning Tardis which appeared and disappeared.

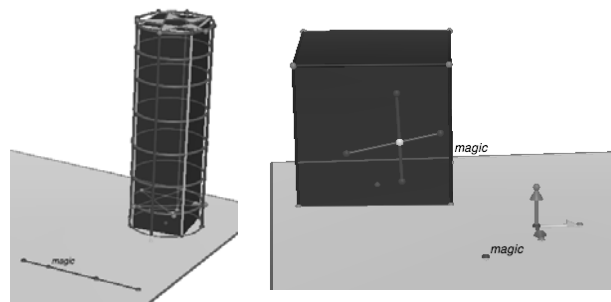


Figure 4: Grade 8 conditional constructions

The other student generalized the concept, by noticing that another way to make a “magic” point would be to have two segments defined by points animated to move around a circle: the intersection of these segments would likewise be a “magic” point, sometimes existing and sometimes not.

Discussion

With regard to the first research question, tasks involving the construction of objects with some constraints given were more effective in engaging learners than more open-ended tasks, with students showing a preference for modelling realistic objects or objects which moved.

However, some students considered that they had “made” a file when they had made some alterations (sometimes only cosmetic) to existing files.

Support materials showing how to create specific structures using Flash demos or files to replay were perceived by students as particularly effective in enabling them to pursue their own goals. Providing files in which sophisticated motion took place was problematic: students became frustrated when attempting to recreate such files and many students became engaged in modifying such files rather than creating their own.

In considering the third research question, it is evident from the cosmetically modified files that students can pursue their own goals in Cabri 3D without engaging in mathematics. The learning materials also made it possible for students to create sophisticated mathematically-based constructions without necessarily engaging deeply with the underlying mathematics. There is a major tension between providing direct support to students in pursuing their own goals in the form of clear instructions as to how to create aspects of their objects or example files to modify, and providing less support, and hence requiring students to do more of the mathematical problem-solving themselves, but potentially to become disaffected when not successful.

However, grade 7 student work showed strong evidence of the utility of transformation, and of students adapting techniques introduced to their own purposes. The use of transformation tools was found to mediate students’ understanding of the mathematics of transformations. Students universally saw transformation as an active process and as a tool for construction. However, transformations, although performed effectively, tended to be designated by initial object, type of transformation and final object rather than by the objects involved in defining the transformation. Future research will seek to find ways to make students more aware of such defining objects.

An unexpected interest was shown in conditional constructions, originally felt to be beyond the ability of the students, and one student was able to generalize the concept. This will likewise be the subject of future research.

In conclusion, Cabri 3D can indeed be an environment involving both purpose and utility for students, and as such is certainly an exciting area of research to pursue.

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New PhD Reports

Présentations de thèses de doctorat

Equity and Opportunities to Learn in Cooperative Mathematical Work

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The current mathematics education reform movement stems from a reconceptualization, built on sociocultural and situated theories of learning, of what it means to learn. Mathematical discourse and argumentation are now considered an integral part of what it means to do mathematics. Although many math reformers consider equity a top priority, more work is needed to conceptualize and operationalize equity if we are to construct better learning environments for students. Only a concrete and grounded definition of equity can help us analyse which pedagogical techniques, in which contexts, lead to more equitable learning environments (Nasir & Cobb, 2002). In this paper, I show how a concrete definition of equity, based in the learning sciences, leads to an analytic approach that helps us to understand some of the nuances of (in)equity as it plays out in mathematics cooperative groups.

Defining equity as the fair distribution of opportunities to learn

I define equity in the classroom context as the *fair distribution of opportunities to learn*, and seek to understand why within a single classroom or group, different students have such different opportunities and different learning outcomes. While equitable teaching practices certainly take into account that students have different strengths and different needs (Cobb & Hodge, 2007), too often in our classrooms we find that some students seem to be systematically denied opportunities to learn mathematics. This question may be examined at multiple levels of analysis; my research focus is at the level of classroom interactions. I analyse how equity or inequity are constructed and reinforced in students' interactions with one another. The task, in studying opportunities to learn in classroom cooperative work, is to understand how all various factors affect what students do when working cooperatively, and how what they do influences what they are able to accomplish. This analysis is grounded in theories of learning; this theoretical grounding is necessary to outline what constitutes an 'opportunity to learn' in mathematics class.

When we study opportunities to learn, it is not enough to examine the textbook, the worksheet, or the teacher's lesson plan. Quantitative approaches to measuring opportunities to learn have tended to focus on measurable quantities, like time spent in class, or evaluating classroom artifacts to determine the information encoded in them (Elliott, 1998; McPartland & Schneider, 1996). These measures are important to consider, but we need to *expand* the definition of opportunities to learn, based on what we know about how people learn. Because these quantitative measures do not take classroom interaction into account, they cannot tell us if students had the chance to experience instruction that built upon their prior knowledge, have models of expertise, get appropriate feedback on their work, or make sense of how their learning is useful (Lee, in preparation). These experiences are considered central to learning

and are not readily captured by quantitative measures at the classroom level. If we want to see opportunities to learn, we have to see how students interact with one another, the teacher, and the classroom artifacts made available for their learning.

In my dissertation study, I focused on two dimensions of group interaction that affect participants' opportunities to learn (Esmonde, 2006). I examined the *work practices* of the group – the negotiated norms for group interaction, for 'how we get things done.' It is not simply one individual's actions that determine what that person will learn, but how those actions get taken up within the group setting. In the past, much research on cooperative learning has focused on the behavior and the learning outcomes of individual students. While we have learned much from this approach, it is clear that a single individual cannot harness the power of a group interaction (Barron, 2000). After all, if one group member asks a question, another group member must be willing to answer it, and must answer it well, if the first group member is to learn from the exchange (Webb & Mastergeorge, 2003).

The second dimension of group interaction that I considered involves the *positional identities* made available to students (Holland, Lachiotte Jr., Skinner, & Cain, 2001). When students work together, their interactions do not just convey content knowledge about mathematics. These interactions also powerfully position learners as kinds of students, and kinds of people (Davies & Harré, 1990). It is important for students to develop identities as competent and authoritative knowers and doers of mathematics (Engle & Conant, 2002). Studying social positioning and positional identities requires attention to the dynamic process of interaction, as individuals take up or are pushed into varied positions in relation to one another.

Work practices and *positional identities* form the basis for the analyses presented in this paper. I argue that to examine the opportunities to learn available to group members, we must examine both the group's dynamics of interaction (the work practices that they negotiate together) and the positioning of individual group members. In this paper I will present two examples of group interaction, taken from two different groups participating in similar activities, to illustrate the analytic framework. A comparison of the two examples reveals the dynamics of negotiation for work practices and positioning, and raises fundamental questions about what equity might look like in diverse classroom settings.

Methods

The data for this study are drawn from a year-long study of three mathematics classes at a San Francisco Bay Area high school. The three classes all followed the same curriculum and were taught by the same teacher, Ms. Cassie Delack. The classes followed the Interactive Mathematics Program curriculum, Year 2. This curriculum follows the principles of reform mathematics in that students are faced with a small number of deep problems each day, they work in teams to discover and construct mathematical methods, and they engage in reading, writing, and conceptual explanations about the mathematics, in addition to learning procedures and algorithms.

Methods were ethnographic in nature. The primary data source was video of cooperative work. In an effort to illustrate the type of analysis used in the dissertation, I will present two examples of group interaction, and demonstrate the use of work practices and positional identities to frame an analysis of equity.

Analysing equity in work practices and positioning

In this paper I contrast two episodes of group interaction to give a sense of how I conducted the analyses in my dissertation by. I consider how the groups' work practices and positional identities affected students' opportunities to learn and therefore equity in the group.

Episode 1. A jigsaw presentation

In the first episode, a group of seven students was focused on preparing a presentation. Late in a unit on linear programming in two variables, the teacher had organized a ‘jigsaw activity’ to help students prepare for upcoming individual and group assessments (Clarke, 1994). The teacher split up each cooperative group of students so that each group member was responsible for joining a larger group to learn one of four challenging topics. At the end of the period, the students were to return to their usual groups of four, and give a short presentation on what they had learned.

Excerpt 1. Riley encourages Garai to explain his strategy

1	Riley	Well Garai,
2	Garai	What
3	Riley	How’d you find ₁ this ₂ feasible region? 1. <i>Taps Garai’s paper with right hand</i> 2. <i>Right hand returns to his chin, gaze towards Garai</i>
4	Garai	₃ Well ₄ , I just plotted points, (<i>inaudible</i>) and see uh, which was true? For the problem, and... that’s what I did 3. <i>Gaze towards his paper</i> 4. <i>Leans back, gaze comes up towards Riley, hand comes up to his notebook</i>
5	Riley	₅ All right so you plot a point, and see, if it works 5. <i>Gaze towards his own notebook</i>
6	Garai	Yeah
7	Riley	So like for each line ₆ you plot a point above it? And see if it works, and you plot a point below it, and see if it works, if it works then that means the line is going that way and you plot and you see where all the points, work. 6. <i>Gaze comes up towards Garai; throughout, he gestures to his own notebook, emphasizing where points are plotted</i>
8	Garai	Yeah I see where all the points are and that’s where the feasible region, where everything works at
9	Riley	₇ And- (<i>hear Candie laughing</i>) Anyone have any other ways? 7. <i>Gaze moves around the group</i>
10	Kendra	Wait what did he say? ₈ 8. <i>Eye gaze directed to Riley</i>
11	Riley	He he, like they ₉ (<i>pointing to Kendra’s paper</i>) He would plot a point for ₁₀ this line, on ₁₁ this side of the line, and on ₁₂ this side of the line (<i>explanation continues for approximately 15 seconds</i>) 9. <i>Points with finger and pencil towards Kendra’s paper</i> 10. <i>Points with finger and pencil to a line on Kendra’s paper</i> 11. <i>Points with finger and pencil to one side of the line</i> 10. <i>Points with finger and pencil to the other side of the line</i>

In this episode, the group was tasked with understanding ‘the feasible region.’ In brief, the feasible region is the set of all points in the coordinate plane that satisfy a set of constraints associated with a linear programming problem. Consider the following excerpt, from midway through the group’s discussion of the feasible region.

This excerpt was characteristic of this group’s interactions. First of all, Riley was central to the mathematical discussion. He was the one to ask Garai to explain his thinking, and he was the one to respond to Garai’s explanation. Further, when Kendra asked for clarification on Garai’s idea, she did not ask Garai directly. Instead, she asked Riley to explain what Garai had said. Riley was positioned with the authority to guide the group’s interactions (e.g., by asking for contributions from other group members), taking up a position that I called ‘facilitator.’

He was also positioned as more competent mathematically than other students in the group. Throughout the data corpus, this type of positional identity showed up repeatedly. I defined a positional identity that I called ‘expert’ for a student who was positioned as competent and authoritative, and who could convince others of his or her mathematical correctness with minimal argumentation. An expert was someone who was able to close down mathematical arguments by asserting their own strategy or solution. Experts were frequently unquestioned by their peers. Riley was positioned as expert in this interaction. (Note that I make no claims about an expert’s mathematical knowledge or expertise. Experts were positioned as such by their peers, and were treated as if they had superior mathematical knowledge. As we will see, an expert was not necessarily an effective teacher in the group.)

The other students in the group were also positioned with respect to their mathematical competence. In the excerpt, Garai was positioned as competent. This happened frequently in the group. Riley evaluated other students’ mathematical understanding, and more often than not, positioned them as competent. He was therefore not the only student in the group who was positioned positively with respect to mathematical understanding.

The group’s work practices were critically shaped by Riley’s facilitator-like interventions, as exemplified in turns 3 and 9. By turn 9, Garai had given a perfectly reasonable explanation for finding the feasible region. The group could have stopped there. Instead, Riley encouraged other students to contribute their own ideas. As a result, over the course of the 15-minute discussion, many ideas about the feasible region were included in the discussion. Almost all group members contributed, and everyone had the opportunity to consider the relationship of their own ideas with those of others. This led to a wide distribution of opportunities to learn for students in the group, because most students had the opportunity to explain their thinking (Webb & Mastergeorge, 2003) and they were able to achieve some level of intersubjectivity with the group.

This group was relatively equitable in that most students had the chance to contribute their own explanations to the group’s discussion, and most students were positioned as competent. Coupled with the intention for all group members to give presentations on the topic, and to be positioned as experts, there were powerful opportunities here for students to take on mathematical authority. The group’s discussion of a single mathematical concept included multiple correct answers, and several variations on definitions, procedures, and examples students could use. The proliferation of methods meant that students had opportunities to compare their thinking with that of their peers, supporting the construction of an intersubjective understanding within the group. Further, the facilitator’s encouragement to participate, and the acceptance of multiple ways of talking about the feasible region, contributed to the positioning of many group members as competent mathematicians.

Episode 2. Preparing an overhead presentation

The second episode contrasts with the first in many ways, although it, too, focuses on a group preparing a presentation. In this case, the presentation was to be given by one group member only, though they did not know which student would be called. This example is also taken from the linear programming unit. The group discussed in this episode was signed up to present the previous day’s classwork assignment. Of the four group members, one student had been absent the previous day and was not familiar with the assignment. The other three group members had been present for the group’s discussion of this assignment the previous day, and so were (to varying degrees) familiar with the mathematical ideas.

The assignment they were to present asked them to consider a bakery that made two kinds of cookies, but had certain constraints due to the amount of available dough, icing, oven space, and time for baking the cookies. The group had to find several examples of combinations of cookies that the bakery could make, based on the constraints, and some combinations that the bakery could not make. Consider the following excerpt, which was characteristic of the group’s conversation during this activity.

Excerpt 2. Shayenne prepares the transparency

1	Shayenne	<i>(writing on transparency)</i>
2	Riley	<i>(watching Shayenne)</i> All right... and ₁ then ₂ So the first thing we did is, ₃ we ₄ summarized it, right? <i>1. Riley’s gaze moves to his composition book</i> <i>2. Shayenne flips page in her composition book</i> <i>3. Riley leans towards Shayenne, gazes towards her</i> <i>4. Shayenne resumes writing</i>
3	Shayenne	Uh huh <i>(continues writing)</i>
4	Riley	So why don’t you write the summary.
5	Shayenne	We gotta write ₅ all this ₆ ! <i>5. Shayenne points to something in her composition book</i> <i>6. Shayenne flips pages back and forth in her book</i>
6	Riley	Just the ₇ summary! No, we don’t have to write all the information we did ₈ <i>7. Riley leans forward and points up and down at places on the composition book</i> <i>8. Riley leans back, still gazing at Shayenne</i>
5	Shayenne	₉ Okay. How you say su- Never mind. I think I spelled this right. <i>9. Shayenne continues writing</i>
6	Riley	That looks like it
7	Dawn	<i>(inaudible talk)</i> summaries?
8	Riley	<i>(after 13 second pause)</i> ₁₀ All right, so yesterday we started on a new unit <i>10. Riley turns body and gaze towards Dawn</i>

9	Dawn	¹¹ Uh huh <i>11. Turns body and gaze towards Riley</i>
10	Riley	Um, and what the question was, was- is a bakery is- was deciding how many cookies to make for the next day
11	Dawn	Mm hmm
12	Riley	And they had ¹³ all this information, like how much they had, and how much each takes (<i>continues explaining</i>) <i>13. Places left hand flat on his open composition book, points with right hand to some writing on his page</i>

In this brief excerpt, although Shayenne prepared the overhead transparency for the presentation, she did so on Riley's instructions, transferring information from her notebook onto the transparency. She did so with little input from other group members. Riley observed her work, then turned to Dawn and delivered an explanation of the task, since Dawn had been absent the previous day. He positioned himself as the expert who could instruct her in the task. Dawn's responses to him were limited to back-channel remarks, and it is not clear how much of this explanation she was taking in. (The explanation continues for quite some time after the end of this excerpt.)

Once again Riley was the group's expert, but although he tried to position himself as facilitator in this group (by encouraging contributions from other students, including Ayodele, who refused), this positioning was challenged by the others.

The group's work practices were much more individualistic than in the previous episode, with Shayenne preparing the transparency by herself. Indeed, when the transparency was finished, this group seemed to consider its work done. They did not rehearse any presentations. The main activities that the group accomplished were to prepare the transparency (done by Shayenne) and to explain the problem to Dawn (by Riley, with little input from Dawn).

Clearly, this activity was much more skewed in terms of the opportunities to learn available to students. The individualistic work practice meant that even though the task was accomplished, only Riley and Shayenne had the opportunity to benefit from it. Similarly, even though Riley verbally explained the nature of the problem they were working on, there was little effort to gain intersubjectivity with Dawn. Although this activity bears some resemblance to the first activity since both were presentation preparations, the results for group interaction and opportunities to learn were quite different.

Comparing the two presentation preparations

When we compare the findings from the two presentation preparations, we see that Riley was positioned differently in the two examples. In both cases, he was positioned as the group's expert, but his positioning with respect to other forms of authority was different. In the first example, Riley had the authority to act as the group facilitator. He encouraged other group members to contribute ideas, and was successful at doing so. In the second example, Riley was not accorded the authority to facilitate. Although he did encourage Shayenne and Ayodele to participate, they refused several times (although they did also acquiesce, when the teacher insisted that they help to write the transparency). Positioning then, is relational. Rather than considering identity as a static state, reflecting someone's internal sense of self, we benefit from considering how identity is constructed relationally in interaction.

A second critical finding was that although the two activities were organized relatively similarly – in both cases, groups were preparing presentations – the group interactions, and opportunities to learn, were quite different. A single activity can allow for multiple work

practices. While this finding may seem elementary, research on cooperative learning has often studied either activity structures, or group interactions, without considering how the two may be related. If we are to structure more equitable cooperative learning activities for mathematics students, we must understand the variability in group interaction, as well as the similarities.

The two examples also demonstrated clearly that small differences in a group's work practice can dramatically affect opportunities to learn. In the first example, when students expressed uncertainty or confusion, they were offered the chance to explain their thinking. (This was not shown in the brief transcript excerpt, but occurred several times during the group's interaction.) In the second example, Dawn was positioned as not knowing the material, and Riley gave a very directed explanation to her – yet one that did not in any way connect to her prior understandings of the problem. Riley talked, and Dawn's responses were very limited. Although she may have understood all of Riley's explanation, it could not have been clear to him whether she did or not, because her responses were limited to backchannel responses.

The analysis suggests some broader questions for future research: how do differences in group interaction come about, and how do they affect opportunities to learn? Which differences really make a difference for equity? And how can we structure more equitable group learning activities?

Connections to the dissertation

In my dissertation, I took 6 examples of presentation preparation activities, and 6 examples of a second activity called a group quiz, and analysed group work practices and positioning. I considered implications for equity based on group interactions in these episodes. I found that the details of group interaction were very consequential in terms of the mathematical ideas that were constructed on the public floor, and in terms of individual access to those ideas.

In general, I found that the presentation preparation prompted a wide variety of work practices. The individualistic work practice was much more the norm than the collaborative one. I suggested changes to the presentation preparation activity that might prompt more equitable interactions in the group.

The group quiz seemed to prompt a narrower band of work practices, and seemed to encourage students to position one or more students as the expert and leader, who would tell others how to complete the work. Facilitators were not needed in this activity, as all students were directly engaged in working and did not need to be encouraged to participate.

The dissertation study contributed to the field by bringing together research on structuring effective activities and research on group interaction, while bringing equity into the center of the conversation. If we are to make progress on supporting more equitable learning activities in mathematics classes, we must consider the nature of group participation. Learning encompasses participation in mathematical practices as well as shifts in identity, and research on equity must attend to both.

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(Élargir les) Connaissances mathématiques des enseignants de mathématiques du secondaire: une étude sur la formation continue

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Site de recherche et contexte¹³

Ma thèse doctorale porte sur l'étude d'une initiative de formation continue pour des enseignants de mathématiques du secondaire. Six enseignants de mathématiques provenant d'une large région urbaine dans l'ouest canadien ont participé à l'étude, constituée de dix sessions de trois heures distribuées sur toute l'année scolaire (comptabilisant 30 heures de formation en tout), le tout précédé de quelques visites de classes et d'entretiens individuels pour chacun des enseignants.

Les enseignants du secondaire avec qui j'ai travaillé pour cette recherche étaient très compétents en mathématiques (selon ce que j'a pu observer), c'est-à-dire ils ne faisaient pas vraiment d'erreurs ou n'avaient pas de difficultés à résoudre des problèmes et expliquer les concepts en mathématiques. De plus, ils ne semblaient pas faire d'erreurs dans l'enseignement des concepts mathématiques en classe, et ils appréciaient et aimaient beaucoup faire des mathématiques. Toutefois, leurs connaissances mathématiques étaient très procédurales et techniques, centrées autour de l'application de procédures et de faits à connaître. Ces enseignants expliquaient qu'ils avaient eu peu sinon aucune occasion de raisonner les concepts mathématiques dans leur carrière scolaire. Voici deux exemples de commentaires que les enseignants ont fait ressortir pour expliquer la nature de leurs connaissances mathématiques :

Vous savez pourquoi on n'est pas capable de résoudre par raisonnement? C'est parce que nous n'avons pas été enseignés à raisonner en mathématiques. Moi, j'ai fait copier-coller, répète et "let's go!" ... et j'ai eu 95% en mathématiques!
(Carole)

¹² Cette recherche doctorale a été rendue possible grâce au support du *Conseil de recherches en science humaine du Canada* (CRSH) et du *Fonds québécois de recherche sur la société et la culture* (FQRSC). Je suis très reconnaissant à ces organisations pour leur support généreux. Je tiens aussi à remercier ma superviseuse, Elaine Simmt.

¹³ Tel que mentionné dans ma présentation lors de la conférence à Fredericton, je ferai ici davantage état des aspects généraux de la recherche et, dû à l'espace restreint, je ne placerai pas une insistance particulière sur les cadres théoriques qui ont orienté celle-ci. Pour en savoir davantage sur la recherche elle-même et ses cadres de référence, j'invite le lecteur à consulter directement la thèse doctorale (Proulx, 2007).

Je n'ai jamais compris comment ça marchait ... Quand mes élèves me demandent pourquoi, je leur dis simplement que c'est comme ça (rires)! (Lana)

Ces enseignants avaient une forte maîtrise des formules, algorithmes et manipulations symboliques – ce que Hiebert et Lefevre (1986) appellent une connaissance procédurale – mais le sens derrière ces procédures et concepts mathématiques leur semblait peu familier (tel qu'ils l'expliquaient). Cette orientation envers les mathématiques, leurs façons de les connaître et de leur donner sens, avait aussi des répercussions importantes sur leurs enseignement, c'est-à-dire que leur enseignement était lui aussi centré de façon importante sur l'apprentissage, la mémorisation et l'application de procédures, de techniques et de calculs pour obtenir des réponses – une orientation vers l'enseignement des mathématiques que Thompson et al. (1994) ont appelé *calculational* (ma traduction). Tel que mentionné, les enseignants étaient « au courant » de cette orientation dans leur enseignement (et leurs connaissances mathématiques) – ce qui représentait une de leurs raisons principales pour participer au programme de formation continue, dans le but d'y approfondir leurs connaissances mathématiques et leur enseignement.

Ces enseignants semblaient ainsi pris à l'intérieur d'un cycle: comme étudiants ils ont appris les mathématiques d'une façon technique et maintenant, comme enseignants, ils enseignent de la façon dont ils se sont fait enseigner. Ainsi, ils reproduisent eux-mêmes ce cycle, à un tel point que les mathématiques en viennent à devenir cette série de techniques et de faits (Figure 1).

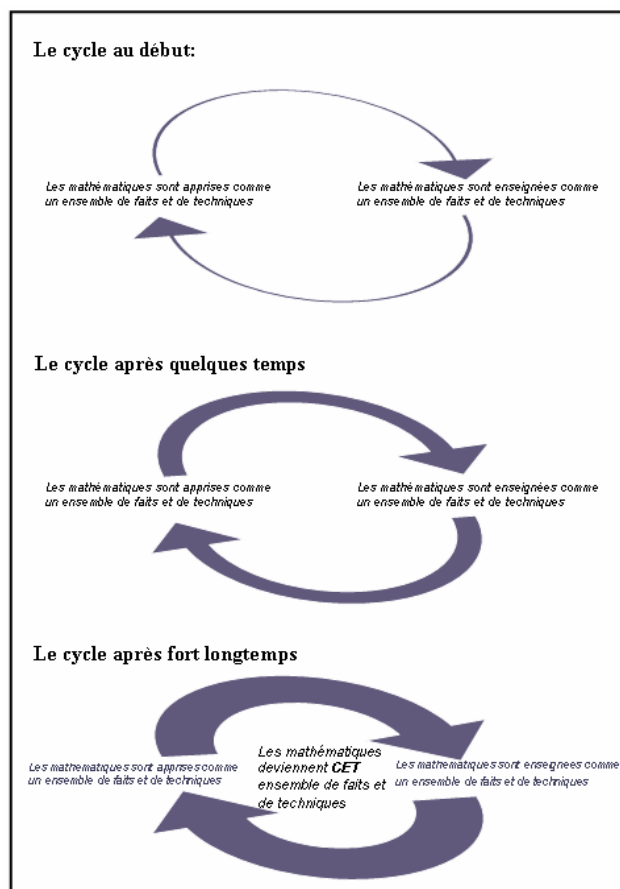


Figure 1 : Le cycle de reproduction des mathématiques techniques

L'initiative de formation continue et les objectifs de recherche

J'ai donc décidé de créer une intervention en formation continue avec l'intention de potentiellement modifier ce cycle de reproduction dans lequel les enseignants semblaient (et se sentaient) pris. J'ai ainsi mis en place une intervention centrée sur l'exploration des contenus mathématiques scolaires. Une des intentions de cette initiative était de construire et de s'appuyer sur les (importantes) connaissances mathématiques des enseignants et de les approfondir dans le but d'offrir aux enseignants des occasions d'apprendre et d'explorer les concepts mathématiques à un niveau différent qu'uniquement celui des procédures et des calculs.

La recherche a donc été orientée vers l'étude du potentiel d'une telle approche de formation continue centrée sur l'exploration des contenus des mathématiques scolaires avec les enseignants, et le type d'occasions d'apprentissage que cela leur offrait.

Le travail mathématique réalisé durant les sessions : Un exemple tiré de l'étude du volume

Dans le but de clarifier et d'illustrer le type de travail qui a été fait durant les sessions avec les enseignants, j'offre ici un exemple tiré d'une session sur le volume des solides.

Pour les enseignants, le volume se résumait en général à connaître les formules associées à chacun des solides et être capable de trouver une valeur numérique pour chacun (c'est-à-dire, appliquer ou réappliquer la formule dans des contextes différents, substituer des nombres pour calculer). Le travail du volume pour ces enseignants était ainsi transporté d'un domaine géométrique à un domaine algébrique de substitution dans des formules et de calculs, menant à une perte de la géométrie aux dépens de l'algèbre et les calculs. Une des intentions des sessions était alors d'essayer de retourner aux aspects géométriques dans l'étude du volume et d'essayer de regarder/comprendre le volume en tant que concept (et moins en tant que simple technique).

Utilisation des travaux de Janvier sur le volume des solides

Pour atteindre ce but, je me suis appuyé sur les travaux de Claude Janvier (1994a, 1994b) sur le volume des solides. Plutôt que de travailler le volume en relation avec une panoplie de formules différentes et isolées que les enseignants maîtrisaient, par ailleurs, très bien (par exemple, prisme rectangulaire: $L \times l \times h$, cube: C^3 , cylindre: $\pi r^2 h$, etc.). Le volume des prismes (et par le fait même des cylindres) a été travaillé en termes d'une accumulation de couches d'aires, reliant ainsi tous les prismes ensembles par une « formule » générale¹⁴ : « aire de la base \times nombre de couches (hauteur). » Cette façon de voir le volume en tant qu'une accumulation de couches nous a amené à considérer les solides obliques et leurs volumes respectifs, où les prismes obliques étaient vu eux aussi comme une accumulation de couches qui avaient été « tassées » sur le côté, mais qui avait le même volume qu'un prisme droit composé des mêmes couches d'aire.

¹⁴ J'ai placé le mot « formule » entre parenthèses parce que « aire de la base \times nombre de couches (hauteur) » n'a jamais été présenté dans les sessions comme une formule, mais plutôt comme une façon de percevoir le volume des prismes. Dans ce sens, ce n'était pas uniquement un changement de plusieurs formules à une seule, mais un changement dans la façon de raisonner le volume des prismes. En fait, le passage de formules diverses à une façon de percevoir le volume des prismes fait disparaître, d'une certaine façon, les formules elles-mêmes, comme le souligne le titre même du travail de Janvier : « Le volume, mais où sont les formules ? ».

L'analyse des prismes obliques, ainsi que des prismes droits, a été facilitée par l'étude d'un principe mathématique historique : le principe de Cavalieri. Bonaventura Cavalieri était un mathématicien italien et son travail est souvent relié de près au développement et à l'émergence du calcul infinitésimal en Europe dans la première partie du 17^{ème} siècle. Son principe peut être énoncé (rapidement) de la façon suivante : si j'ai deux solides de même hauteur et avec des bases placées dans le même plan, les deux solides auront le même volume si chaque fois que je fais une coupe parallèle à la base (dans le même plan) pour les deux solides, j'obtiens deux surfaces qui ont la même aire (Figure 2). Ainsi, en utilisant le principe de Cavalieri, les prismes obliques (et même tordus) pouvaient être étudiés (et comparés) par rapport à leurs volumes et associés avec un prisme droit de même base et de même hauteur¹⁵.

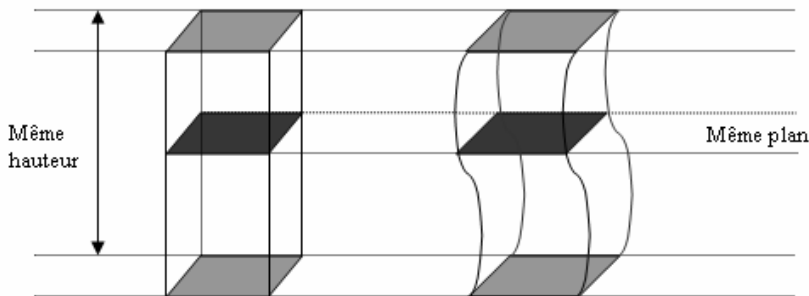


Figure 2 : Une illustration du principe de Cavalieri

En plus des prismes, les pyramides ont aussi été étudiées, surtout en lien avec leur association avec les prismes. Ceci a été fait en faisant référence à la décomposition familière du cube en trios pyramides identiques de même base et de même hauteur (la même hauteur que celle du cube duquel ils ont été décomposés). En suivant le travail de Janvier, l'étude des pyramides a aussi été prolongée par la dissection d'un prisme triangulaire en trois pyramides triangulaires non-identiques mais de même hauteur et de même base. Encore une fois, en utilisant le principe de Cavalieri, l'établissement de l'équivalence entre les trois différentes pyramides était possible. De cette façon, une pyramide était vue comme étant un tiers de son prisme associé, de même base et de même hauteur. Ce travail a aussi eu l'effet d'offrir des explications quant à la présence du "1/3" dans la formule usuelle de la pyramide, se décrivant comme la relation « qualitative » la reliant à son prisme associé. En plus, tel qu'il a été fait pour les cylindres qui avaient été définis en tant que prismes possédant une infinité de côtés, et étant donc vus en termes d'accumulation de couches, les cônes ont été définis comme des pyramides avec un nombre infini de côtés. Les cônes ont ainsi été reliés à leur « prisme » associé, ici le cylindre, par un ratio d'un tiers.

En somme, tel que mentionné, plutôt que de considérer une panoplie de formules à apprendre pour calculer le volume de solides, ce travail sur le volume a amené à prendre en compte une façon générale de faire du sens du volume et de comparer les solides de façon qualitative entre eux. Dans ce sens, ce travail amenait à percevoir le volume de solides sous l'angle d'accumulation de couches pour les prismes et d'une relation (de 1/3) entre les pyramides et leurs prismes associés.

¹⁵ Il est important de rappeler qu'aucune mesure (quantitative) n'est faite ici. Tout se situe au niveau de comparaisons qualitatives d'équivalences (ou de non équivalence selon les cas).

L'analyse des données

Même si le but de ce court compte rendu n'est pas de s'étendre sur l'analyse des données, je vais toutefois mentionner les angles utilisés pour le faire. Le premier angle d'analyse concernait les occasions d'apprentissage qui ont émergé des sessions. À travers cet angle, une attention particulière a été portée aux types d'expériences et d'apprentissages qui ont été offerts durant les sessions aux enseignants – et que ces derniers ont tiré parti de – au niveau mathématique, mais aussi au niveau pédagogique. Ce qui ressort de cette analyse de façon frappante était qu'à travers leurs explorations et leur apprentissage des aspects mathématiques (et des discussions qui ont émergé de ces dernières), les enseignants se sont aussi engagés dans des discussions pédagogiques importantes. Même si le travail dans les sessions était centré sur les mathématiques, les enseignants ont initié des discussions pédagogiques concernant la façon avec laquelle ces mêmes mathématiques pourraient être enseignées et travaillées dans leur enseignement, et comment tout cela était en lien avec leur façon d'enseigner ces concepts. Des nouveaux apprentissages et des nouvelles expériences avec les concepts mathématiques émergeaient des nouvelles possibilités et idées d'enseignement pour ces enseignants. Le travail leur a donc offert des occasions d'apprentissages importantes au niveau mathématique, mais aussi au niveau de l'enseignement et de la pédagogie.

Le deuxième angle d'analyse des données concernait « l'impact » que ces explorations semblent avoir eu sur les enseignants. Sous cet angle, une attention a été portée sur la façon avec laquelle les enseignants ont évolué à travers leur exploration des concepts, comment cela les a amenés à considérer les concepts sous un autre angle, puis comment ce nouvel angle, en retour, les a entraînés vers des nouvelles compréhensions mathématiques et des discussions et décisions pédagogiques différentes.

Le troisième angle d'analyse des données était en lien avec mes propres pratiques de formateur, en ce qui concerne le rôle que j'ai pu jouer dans les explorations, les apprentissages et l'évolution des enseignants à travers les sessions. Une attention particulière a été portée au type « d'action » que je posais à travers les sessions, dans le but d'arriver à les catégoriser et à mieux les comprendre – au niveau de leur potentiel. Une retombée de cette analyse fut le constat du caractère très actif de mon rôle de formateur pour arriver à stimuler et orienter les activités du groupe, et surtout les enseignants, vers différentes façons de faire du sens et d'explorer les concepts mathématiques travaillés – avec l'intention de permettre aux enseignants de s'éloigner et « sortir » de leur forte orientation procédurale et d'explorer d'autres avenues.

Aspects généraux tirés de ces analyses

De ces rencontres, que Frédéric aurait préféré voir se tenir à trois heures du matin plutôt qu'à dix, chacun ressortait non point meilleur, mais enrichi. (di Falco et Beigbeder, 2004, p. 14)

Le travail autour des concepts mathématiques scolaires s'est révélé être très riche, et sur plusieurs points de vue, concernant la création de (nouvelles) possibilités pour les enseignants. Un premier aspect touche la compréhension et la connaissance des concepts mathématiques qu'ils enseignent (les concepts des mathématiques scolaires). Les sessions et les explorations leur ont offert des occasions d'apprentissages pour travailler avec les concepts mathématiques sous une orientation différente qu'uniquement procédurale. En ce sens, cela leur a offert des (nouvelles) façons de faire du sens des concepts mathématiques.

Un deuxième aspect concerne les possibilités d'enseignement de ces mathématiques. L'exploration et l'apprentissage de nouvelles mathématiques a amené les enseignants à percevoir et à réfléchir sur différentes façons d'approcher ces concepts dans leur enseignement. Les explorations mathématiques ont ainsi amené les enseignants à considérer

des possibilités et des approches d'enseignement nouvelles et alternatives (qu'ils n'avaient pas pensé ou avec lesquelles ils n'étaient pas familiers).

Un troisième aspect concerne l'orientation des enseignants envers les mathématiques. Par leur exploration en profondeur des concepts mathématiques, les enseignants ont commencé à développer de nouvelles façons de « s'engager » et « d'entrer » dans l'étude des concepts – des façons qui n'étaient pas uniquement centrées sur les procédures et les calculs. En ouvrant l'étude des concepts mathématiques au-delà des procédures, la possibilité que les mathématiques *soient* plus que des procédures a émergé. Ainsi, pour les enseignants, la possibilité que d'autres sujets et concepts mathématiques soient approchés différemment que par leurs procédures était maintenant présente, représentant de nouvelles possibilités et orientations pour ces enseignants.

Un quatrième aspect concerne les nouvelles distinctions que les enseignants arrivaient à faire. Par l'exploration des différences entre une approche procédurale des concepts et une axée sur le travail des concepts sans lien nécessaire avec des procédures, les enseignants étaient maintenant plus à même de faire des distinctions entre ces approches lors de l'activité mathématique. Cela les a aussi amenés à être capables de reconnaître et distinguer des éléments spécifiques dans l'activité mathématique des élèves, par exemple ils pouvaient distinguer des solutions uniquement mécaniques de solutions à l'intérieur desquelles une compréhension mathématique semblait être déployée.

Un dernier aspect à mentionner concerne les aptitudes pour faire des mathématiques. Le travail en profondeur des concepts mathématiques semble avoir eu des « effets » significatifs chez les enseignants. Un de ceux-ci est le développement de leurs capacités à fouiller et explorer les mathématiques. Les enseignants ont développé des aptitudes pour *faire* des mathématiques, non seulement apprendre de nouvelles choses, mais aussi travailler les mathématiques à un niveau plus approfondi que celui des procédures. Ce n'était donc pas nécessairement les mathématiques apprises lors de l'activité qui avaient un effet significatif, mais le fait de faire l'exploration elle-même et de développer une capacité à fouiller en profondeur les concepts mathématiques. Un deuxième effet est en lien avec la curiosité et l'intérêt des enseignants pour entreprendre ce type de travail d'exploration. Le fait de décortiquer les concepts mathématiques a fait émerger chez certains enseignants un intérêt grandissant et une curiosité pour l'exploration des concepts mathématiques. Finalement, un dernier effet se résume par le développement de ce qui pourrait être appelé une « tendance » ou une « habitude » à faire des explorations similaires pour d'autres concepts. Cette tendance ou habitude est ainsi devenue contagieuse pour le travail d'autres concepts mathématiques, alors que les enseignants ont commencé à faire ce type de travail de façon indépendante, à l'extérieur des sessions, pour d'autres sujets et contenus mathématiques. Skemp (1978) avait préalablement discuté du déclenchement d'une tendance similaire chez les gens développant une compréhension relationnelle des concepts :

Le lien avec [la motivation des gens] est que si les gens obtiennent une satisfaction de la compréhension relationnelle, ils vont non seulement tenter de comprendre de façon relationnelle du nouveau matériel qui est placé devant eux, mais vont aussi tenter de fouiller activement et d'explorer du matériel et des domaines nouveaux, de façon semblable à un arbre qui étend ses racines ou à un animal qui explore un nouveau territoire en quête de nourriture (p. 13, ma traduction)

Ces trois effets ne se produisent toutefois pas au niveau concret de l'apprentissage de concepts ou d'aspects concernant un sujet, mais plutôt à un niveau « méta », soit concernant le développement d'aptitudes envers les mathématiques. Ceci n'est pas à négliger puisque avoir la *capacité* et l'*intérêt* pour fouiller en profondeur les concepts mathématiques et avoir une telle *orientation* envers les mathématiques apparaissent comme des qualités importantes pour un enseignant de mathématiques au niveau de sa façon d'enseigner, de planifier et de présenter les contenus et les concepts.

L'ensemble des aspects discutés dans cette section représentent des illustrations de l'évolution et l'apprentissage vécus par les enseignants à travers les sessions. La formation a permis et offert aux enseignants l'occasion de se développer de façon importante au niveau mathématique.

Au niveau de l'impact concret que ce travail peut, pourrait ou a pu avoir sur l'enseignement de ces enseignants, les résultats de la recherche doivent être vus en termes de *possibilités*. Les explorations ont offert aux enseignants des nouvelles possibilités concernant ces concepts mathématiques. Toutefois, la façon avec laquelle ces enseignants vont s'approprier ces nouvelles possibilités dans leurs pratiques ou vont tirer parti de ces dernières dépend directement d'eux¹⁶. L'important ici est que grâce au travail fait durant la formation, ils ont retiré de nouvelles possibilités et ces dernières leur sont maintenant disponibles. L'ensemble des ressources avec lesquelles ils travaillent et desquelles ils s'inspirent lorsqu'ils enseignent un sujet mathématique a été enrichi, élargi, dé-simplifié et complexifié. C'est ce que « élargir » veut dire et implique concernant l'*enseignement* des enseignants. Leurs possibilités comme enseignants ont été augmentées, leurs ressources ont été enrichies.

En ce sens, tel que la citation de di Falco et Beigbeder l'indique, à travers les sessions les enseignants ne sont pas devenus « meilleurs » – ceci serait porter un jugement sur eux – mais se sont simplement *enrichis*.

Remarques finales

L'exploration de concepts mathématiques à travers les sessions de formation continue semble avoir offert beaucoup de (nouvelles) possibilités aux enseignants concernant leur compréhension mathématique, leur enseignement de ces mathématiques, leurs aptitudes mathématiques, etc.

Un aspect particulièrement frappant dans ces explorations est que toutes ces possibilités ont émergé d'une simple exploration des concepts mathématiques scolaires, rien de plus. Cela semble très impressionnant de la part d'explorations de concepts que les enseignants connaissaient déjà, mais explorés sous une perspective différente. Je crois que les retombées de cette approche sont très intéressantes au niveau des possibilités pour la formation des enseignants et l'enseignement des mathématiques en classe : ces enseignants ont maintenant, ou ont développé, de nouvelles possibilités qu'ils n'avaient pas auparavant (envisagées) concernant les concepts mathématiques scolaires qu'ils enseignent. Les conséquences de ce travail au niveau de l'enseignement apparaissent très prometteuses puisque ce travail ouvre de nouvelles portes pour ces enseignants concernant les concepts mathématiques qu'ils enseignent. Le travail fait durant les sessions et les retombées qu'il semble avoir eu sur les enseignants pointe vers l'intérêt de porter une attention spécifique, et de travailler en profondeur, sur les concepts mathématiques scolaires dans nos pratiques de formation des enseignants. C'est en effet un aspect spécifique que cette recherche met en relief, c'est-à-dire l'importance et la pertinence de la compréhension mathématique des concepts enseignés chez les enseignants, et comment le développement de cette compréhension leur ouvre des possibilités et des occasions d'apprentissage au niveau mathématique et, surtout, au niveau de leur enseignement.

¹⁶ Et la recherche ne s'est pas intéressée à observer comment les enseignants ont évolué dans leur enseignement ou comment ils se sont (ré-)approprié, à l'intérieur de leurs pratiques, les idées travaillées et explorées durant les sessions. Ceci est le but d'un projet de recherche subséquent.

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(Enlarging) Secondary-Level Mathematics Teachers' Mathematical Knowledge: An Investigation of Professional Development

Research Site and Context¹⁷

My doctoral dissertation reports on a study of a professional development initiative for secondary mathematics teachers. Six mathematics teachers from a large urban area in Western Canada participated in the study, which was a school-year-long project constituted of ten 3-hours monthly sessions (totalizing 30 hours of in-service education), and some individual visits in each teacher's classrooms.

The secondary teachers with whom I was working with in my research site were mathematically very competent. That is, from what I observed, they did not make mistakes or experience difficulties solving problems in mathematics; they knew how, what and when to solve. Neither did they seem to make mathematical mistakes in their teaching of concepts (from my visits in their classrooms), and they claimed to enjoy mathematics very much. However, their knowledge of mathematics was very much about a set of procedures to apply and facts to know. As they explained to me and as I realized while working with them, they had seldom been asked to explain the meaning behind and make sense of concepts in mathematics. Teachers, for example, explained this issue in these sorts of terms:

Why is it that we are not able to solve by reasoning? [...] It is because we have not been educated to reason in mathematics. Me, I did copy, paste, repeat, and let's go ... and I had 95% in mathematics! (Carole)

I never understood why it worked... When students ask me why, I simply say that this is how it is! (Lana)

These teachers had a strong grasp of formulas, algorithms and symbolic manipulations – what Hiebert and Lefevre (1986) refer to as “procedural knowledge” – but the meaning behind these procedures and mathematical concepts appeared obscure or unfamiliar to them (as they personally acknowledged). This orientation toward mathematics, teachers' manner of knowing mathematics, had repercussions on their teaching in that their teaching was also focused on learning, memorizing and applying procedures, techniques and calculations to obtain answers – an orientation to mathematics teaching that Thompson et al. (1994) have labelled *calculational*. As mentioned, the teachers were quite aware of this orientation in their teaching (and their knowledge of mathematics) – and it represented one of their main reasons for participating in the program for them so that they improve their knowledge of mathematics and its teaching.

To some extent, these teachers appeared to be stuck in a cycle: as students they were taught mathematics in a technical way, and when they became teachers they continued to teach in the way they were taught. Hence, they themselves, as teachers, were reproducing the very cycle in which they were, to a point that mathematics had become for them this very set of techniques and facts (Figure 1).

¹⁷ As mentioned in the presentation at the conference in Fredericton, I mostly report here on general features of the research and do not place an emphasis on the theoretical frameworks that have guided the study, as space does not allow me to. For further elaborations on the research and its theoretical underpinnings, I refer the reader to the dissertation itself (Proulx, 2007).

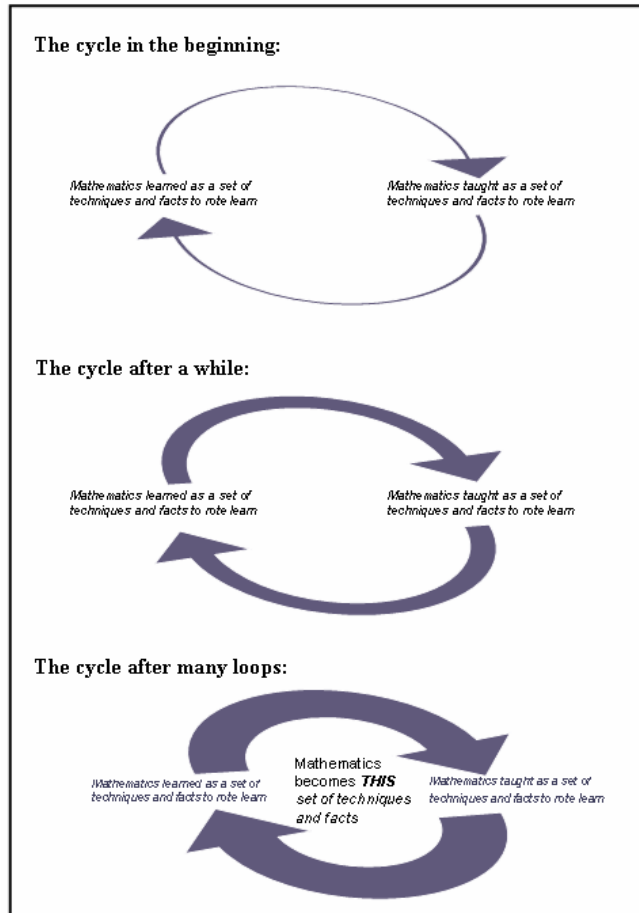


Figure 1: The cycle of reproduction creating mathematics as a set of techniques and facts

The Professional Development Initiative and the Research Objectives

The previous context led me to create an intervention of professional development, with the intention of potentially altering the cycle of reproduction in which the teachers appeared to be (and felt) caught in. Therefore, I designed an intervention focused on exploring deeply the mathematical content of the curriculum, that is, school mathematics. Through this professional development initiative, I wanted to build on teachers' strong knowledge of procedures and enlarge it to encompass more than just procedures, and in that sense to offer the teachers opportunities to experience mathematics concepts at a deeper level than one only about procedures and calculations.

The research was therefore interested in studying the potential of such an approach, focused on exploring school mathematics contents with teachers in a professional development setting, and the sorts of learning opportunities that it offered to teachers.

Mathematical Work Done in the Sessions: An Example from the Study of Volume

In order to clarify and illustrate the sort of work that was done during the sessions with the teachers, I offer here an example taken from a session about volume of solids.

For these teachers, knowing volume meant knowing the formulas associated with each solid and being able to calculate a value for it (applying the formula in different contexts, substituting numbers in them to compute). Volume was therefore transported and shifted from the geometric realm to that of algebraic substitution in formulas and calculations, leading to the loss of geometry in favour of algebra, formulas and calculations. Hence, an intention of the sessions was to go back to geometrical issues in the study of volume and attempt at looking at/understanding volume as a concept, and less as a technique.

Using Janvier's Work on Volume of Solids

In order to achieve this goal, I used and referred to the work of Claude Janvier (1994a, 1994b) on volume of solids. Using Janvier's work, instead of looking at volume of solids in regard to the diverse isolated formulas teachers knew well (e.g., rectangular prism: $L \times W \times H$, cube: S^3 , cylinder: $\pi r^2 h$, etc.), we looked at and defined the volume of prisms (and cylinders) as an accumulation or a piling up of layers of area, relating and connecting all prisms through the overarching "formula"¹⁸: "area of the base \times number of layers (height)." This perception of volume as an accumulation of area led us to consider oblique or skew solids and their volumes, where the oblique prisms were seen as piles of layers of area that had been skewed on the side, but that had the same volume as a straight prism composed of the same piles of layers.

The consideration of oblique prisms, and also straight prisms, was facilitated by the study of a historically developed mathematical principle invented by Bonaventura Cavalieri: the Cavalieri's principle. Cavalieri was an Italian mathematician whose work is often mentioned in accounts of the emergence in Europe of infinitesimal calculus and infinitesimal calculations in the early part of the seventeenth century. His principle can be (roughly) stated as follows: if you have two solids of same height and with bases placed in the same plane, both solids will have the same volume *if* each time that a cut parallel to the base is carried out (i.e., in the same plane) on both solids, it gives two surfaces that have the same area (see Figure 2). Hence, using Cavalieri's principle, oblique prisms (even twisted ones) could be studied (and compared) for their volumes and associated with a straight prism of exact same base and same height¹⁹.

In addition to the study of prisms, pyramids were also looked into in depth, primarily in regard to their relationship to prisms. This was done by using the familiar decomposition of the cube into three identical pyramids of same base and same height (a height that is also the same as the cube from which it was dissected from). The study of pyramids was also extended by looking, following Janvier's work, at a triangular prism dissected in three non-identical triangular pyramids of same height and same base. Again, referring to Cavalieri, the establishment of same volume between pyramids was possible. Therefore, a pyramid was perceived as being a third of its associated prism, with same height and same base. This work on pyramids also had the effect of offering some explanations for the presence of the " $\frac{1}{3}$ " in the usual formulas for pyramids, where it was the "qualitative" relationship linking it to its associated prism. In addition, as was done for cylinders that were defined as prisms with an infinite number of sides, hence fitting within the accumulation of layer perspective, cones

¹⁸ I place "formula" between quotation marks because "area of the base \times number of layers (height)" was never presented as a formula but mainly as a way to perceive the volume of prisms. In that sense, it was not only a switch from many formulas to one, but also a switch in the way to make sense of how the volume of prisms could be seen. In fact, this passage from the formulas to this perception of volume of prisms makes disappear, to some extent, the formulas themselves, as Janvier's title of his work points to: "Volume, but where are its formulas?"

¹⁹ Note that here there is no quantitative measurement being done, it is mostly a qualitative comparison of equivalence (or it could be of non-equivalence depending on the case).

were defined as pyramids having an infinite number of sides. It led to consider cones as being related to their associated “prism,” here the cylinder, by a ratio of one third.

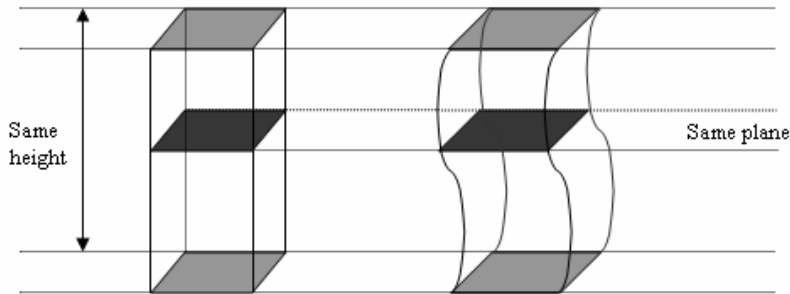


Figure 2: An illustration of Cavalieri's principle

In sum, as mentioned, instead of looking at a pile of formulas to learn to calculate volumes of solids, the work led to consider overarching ways of making sense of volume and of qualitatively comparing each solid with another. In that sense, this work led to one way of perceiving the volume of prisms in terms of piling up of layers, and of one relationship between pyramids and their associated prisms ($\frac{1}{3}$).

The Analysis of the Data

While it is not the goal of this short report to provide a deep discussion of the data analysis, I mention here some of the angles that were used to look at the data. The first angle concerned the learning opportunities that emerged out of the sessions. Through this angle, attention was closely paid to the sort of learning experiences that the sessions offered to teachers at the mathematical and also at the pedagogical level. What appeared striking was that through their exploration and learning of mathematical issues as well as the mathematical discussions that emerged from these, teachers engaged in important pedagogical discussions. Even though the work of the sessions was centered around mathematics, it led the teachers to enter into pedagogical discussions about how this mathematics could be taught and worked through in their teaching, and how it related to what and how they were teaching these concepts and topics. In that sense, from new experiences with mathematical concepts emerged new teaching possibilities and ideas for these teachers. The work therefore offered them important learning opportunities at the mathematical level, as well as at the level of teaching and pedagogy.

The second angle of analysis concerned the “impact” that the explorations appeared to have on teachers. Through this angle, attention was paid to how teachers’ understandings evolved through the exploration of specific concepts, how it led them to consider these concepts through new lenses, and how these new lenses in return led to new mathematical understandings and different pedagogical ideas and choices.

The third angle of analysis concerned myself as the teacher educator in regard to the role that I played in the teachers explorations, learning opportunities and evolutions. Specific attention was paid to the sort of “actions” that I was posing as the teacher education in the sessions – in order to categorize and understand them better – concerning the potential impact they have had on teachers. One outcome of this analysis concerned how active I was as the teacher educator in order to push (and possibly bias) the activities of the group and the teachers themselves toward different and new ways of exploring the concepts in order to get the

teachers to step away or aside from their strong orientation toward procedures and calculations in mathematics.

General Elements That Were Drawn From These Analysis

From these meetings, that Frédéric would have preferred to happen at 3 o'clock in the morning instead of at 10, each of us came out not really better, but enriched. (di Falco & Beigbeder, 2004, p. 14, my translation)

The work through school mathematics concepts revealed itself to be quite fruitful, for many aspects, in opening up and offering (new) possibilities for teachers. A first aspect concerns teachers' understanding and knowledge about the mathematics concepts that they were teaching on a regular basis (the school mathematics concepts), as the sessions and the work offered them learning opportunities to experience these mathematics concepts along different orientations than ones only about procedures. In that sense, it offered them (new) ways of understanding the mathematical concepts.

A second aspect concerns teachers' possibilities for teaching this mathematics. As teachers explored and learned new mathematics, it led them to perceive and reflect on different ways to approach these concepts in their teaching, and along new orientations that they had not thought before or simply orientations that they were not familiar with. Therefore, the mathematical explorations brought them to consider novel teaching possibilities and approaches.

A third aspect concerned teachers' orientation toward mathematics. As they explored mathematical concepts in depth, teachers began to develop new orientations to enter through concepts, orientations that were not only focused on procedures and making calculations but about the mathematical concepts themselves aside of their procedures and calculations. By opening the study of some mathematical topics to more than simply procedures, the possibility that mathematics *is* about more than just procedures arose. Hence, for the teachers, the possibility that other mathematical topics be treated differently from simply as procedures was now present, representing a new possibility and orientation for them concerning mathematics.

A fourth aspect concerned the distinctions that teachers were making. By having explored the differences in treating some concepts along a procedural orientation in contrast to one that also pays attention to the concepts themselves aside of their procedures, teachers were now more able to distinguish these focuses within the activity of doing mathematics. This led them to be able to flag and recognize specific elements in the mathematical activity of others, for example within students' work, as they could point to work solely lodged in a mechanical way of understanding mathematics in contrast to work where mathematical understanding seemed to be deployed and not being about only a sole application of procedures without meaning.

A final aspect concerned teachers' aptitudes for doing mathematics. Working on challenging pieces of mathematics had significant "effects" on teachers. One was on the development of their capacity to probe into mathematical topics. The teachers developed (greater) aptitudes to *do* mathematics, not only to learn new things about it but to work in mathematics at a deeper level than that of procedures. This is therefore not necessarily about what teachers learned through an activity, but mainly about the doing of the activity itself and the development of a capacity to dig deeply into mathematical topics. A second effect has to do with their interest in and curiosity about undertaking this kind of activity. Unearthing some mathematical concepts raised some teachers' interest in exploring more mathematics elements. Finally, another effect can be seen as the development of what might be called a habit of mind, where teachers started to think about doing similar deep analyses/explorations of concepts with other

mathematical topics. This habit of mind became a contagion toward mathematical topics that led some teachers to do these explorations on their own for other mathematical topics. Skemp (1978) had previously talked about this phenomenon concerning the development of relational understandings of concepts:

The connection with [motivation of people] is that if people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas, very much like a tree extending its roots or an animal exploring new territory in search of nourishment. (p. 13)

These three effects are not at the concrete level of learning some aspects about a specific topic, but are at a meta-level as they involve teachers' aptitudes toward mathematics. It is no small issue since having both the *capacity* and the *interest* to go deep into mathematical issues, and having an *orientation* toward addressing different mathematical topics along these lines, appears significant for a mathematics teacher in relation to how his teaching of the topics will be planned and presented to students.

All these aspects above represent ways in which teachers have evolved throughout the sessions and their explorations of school mathematics concepts. The sessions enabled and offered teachers the opportunity to develop themselves importantly at the mathematical level as mathematical doers.

In regard to the concrete impact that this work can, could or did have on the teachers' teaching, the research outcomes need to be seen at the level of *possibilities*. The explorations of school mathematics concepts in the sessions offered teachers new possibilities about these mathematical concepts. However, how these teachers will use that in their practice or will take on these newly available possibilities in their teaching directly depends on them²⁰. What is important here is that because of this professional development they have new possibilities available to them. The pool from which they draw and play upon when they teach a mathematical topic has been enriched, enlarged, unsimplified and complexified. This is what "enlarging" means and implies concerning the teachers' *teaching*. Their possibilities as teachers were augmented, their pool of possibilities was enriched.

In that sense, as the above quote points to, through the sessions the teachers did not become "better" – as it would mark a judgment on them – but simply came out *enriched*.

Concluding Remarks

The work on exploring school mathematics concepts at a deep level with teachers through professional development activities appeared to have offered a lot of (new) possibilities for these teachers: for knowing mathematics, for teaching mathematics, for being mathematical, and so on.

One implicitly striking element in these explorations is all these impact and emergence of possibilities came out from the simple exploration of school mathematics concepts, nothing more. It appears quite impressive that all this emerged only from working on school mathematics topics that teachers already knew, but explored from a different perspective. I believe that the outcomes of this approach are very interesting concerning possibilities for mathematics teacher education and mathematics teaching in classrooms as these teachers now have, or have developed, new possibilities that they did not have before concerning the school

²⁰ And the research did not look into the teachers' classrooms to observe how their teaching had potentially evolved or how they were making sense of and appropriating these issues in their teaching practices. This will be the task of a subsequent research project.

mathematics concepts they teach. The foreseeable impact concerning teachers' teaching is very promising as it opens new doors to teachers about the very mathematical concepts they are teaching in their classroom. Therefore, because of all it creates for teachers, there appears to be some interest at placing a specific attention, and working deeply, at school mathematics concepts in our teacher education practices. This is one specific aspect that this research points to, that is, to the significance of teachers mathematical understanding of the mathematics they teach, and how its development opens possibilities and opportunities for them as mathematical doers and, obviously, as teachers.

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Un paradigme d'expérimentation au laboratoire de sciences pour l'identification et l'optimisation statistique d'un modèle algébrique par interaction visuo-graphique

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Résumé

Cette recherche s'intéresse principalement à la modélisation des phénomènes physiques en sciences expérimentales. Nous avons proposé une méthode de Régression Graphico-Statistique (RGS) en sciences expérimentales qui permet de produire le modèle mathématique d'un phénomène scientifique. Cette méthode que nous avons intégrée dans un environnement d'Expérimentation Assistée par Ordinateur (ExAO), procède par itération visuelle et graphique. Elle permet non seulement d'ajuster une fonction algébrique sur un ensemble de points expérimentaux, mais aussi de l'optimiser, d'évaluer son erreur type de prédiction et d'obtenir un critère scientifique pour rejeter les points singuliers ou aberrants. Les résultats de cette recherche nous montrent que les étudiants en utilisant cette méthode, développent une attitude positive à l'égard de la modélisation scientifique. Qu'au contraire d'une méthode essentiellement algébrique, ils en comprennent mieux les fondements, ce qu'ils n'arrivaient pas à faire avec la méthode traditionnelle (Moindres carrés de Gausse-Legendre) utilisée automatiquement, et de manière aveugle, dans les calculatrices programmables et les logiciels de modélisation.

Contexte de cette recherche

Selon le Ministère de l'Éducation québécois (MEQ, 2004), l'enseignement des sciences doit privilégier des situations d'apprentissage contextualisées, ouvertes et intégratives, débouchant sur des activités diversifiées afin de donner un sens concret aux objets d'études, d'éveiller l'intérêt et de favoriser la démarche scientifique de l'élève. Dans ces situations, l'élève est amené à « *jouer un rôle d'investigation lors d'une expérimentation en laboratoire,...* » (MEQ 2004, p.272). Pour ce faire, et tel qu'exigé par le Ministère, l'élève est appelé à communiquer à l'aide du langage mathématique les résultats de ses propres expérimentations. Cependant, pour que l'élève puisse jouer ce rôle, il doit nécessairement réussir et surtout comprendre tout le processus de modélisation scientifiques des phénomènes physiques.

Situation problématique dans les écoles et les collèges

En sciences comme en mathématiques, la mise en œuvre du nouveau programme de formation à l'école québécoise révèle des aspects paradoxaux quant aux attentes qu'elle soulève.

- Ce programme demande aux étudiants d'expérimenter des phénomènes physiques, de recueillir des données, de les représenter sur un graphique, de les analyser, de les interpréter et de les modéliser algébriquement.
- Cependant, ces programmes n'incluent aucune méthode en mathématiques permettant aux étudiants de comprendre et de justifier tout le processus de modélisation algébrique.

La méthode utilisée au secondaire et collégial par les étudiants et les enseignants, consiste en l'utilisation de calculatrices programmables²¹ ou de tableurs afin d'obtenir automatiquement la meilleure courbe. Cette méthode ne leur permet pas de comprendre les rationnels sous-jacents en mathématiques qui y sont utilisés. Sur leur complexité, Beaufils (1993, p.124) confirme ces constatations en notant que:

Si l'alternative centrée sur les méthodes modernes de modélisation relève d'une épistémologie plus satisfaisante en ce qui concerne la relation théorie/expérience, elle reste problématique au niveau de l'enseignement secondaire dès lors qu'elle se place sur un plan **quantitatif et mathématique**. Elle ne peut en effet être mise en œuvre de façon immédiate du fait, en particulier, de la limitation de la complexité des modèles mathématiques et des méthodes informatiques». Dans sa thèse, Richoux (2000, p.153) a mentionné que les démarches observées chez les enseignants «comportent pour la plupart une 'confrontation' entre des résultats expérimentaux et un modèle théorique [...] et les incertitudes sur les mesures, sur les valeurs des paramètres obtenus pourtant 'un des outils privilégiés pour cette confrontation' (Guillon, 1995, p.117), ne sont ni prises en compte ni même évoquées». Elle précise ensuite, «obtenir une courbe avec un palier, une droite qui passe par l'origine, une valeur ayant le bon ordre de grandeur suffit pour valider l'accord modèle - résultats expérimentaux [...] la confrontation se réduit à une 'comparaison à vue' entre résultats expérimentaux et théoriques. (Richoux, p. 154).

Modélisation algébrique par la régression graphico-statistique (RGS)

Le passage du graphique à l'équation mathématique avec un tableur grapheur, comme par exemple EXCEL, est souvent réalisé de manière automatique et incompréhensible pour l'étudiant. Avec un système ExAO, cette compréhension, du passage du graphique à l'équation, est toute fois améliorée puisque l'étudiant ajuste lui-même, visuellement, les paramètres de sa courbe théorique afin de la superposer sur les points expérimentaux. Cette méthode qui est devenue une tradition dans les écoles et collèges²² ne permet pas à l'élève de comprendre le rationnel mathématique sous-jacent. Ainsi, en bénéficiant des avantages didactiques de l'ExAO, et pour aider l'élève à mieux parcourir et comprendre le processus de modélisation algébrique, le problème didactique auquel nous sommes confrontés découle des deux questions suivantes:

1. Comment donner à l'élève la capacité de modéliser algébriquement le nuage de points d'un phénomène en sciences expérimentales, en particulier en physique?

²¹ Les curriculums des provinces du Québec et de l'Ontario recommandent fortement l'utilisation des calculatrices graphiques programmables et les tableurs.

²² Notons ici que le niveau secondaire Québécois correspond grosso-modo au niveau collégial Français (de la 6^{ième} à la 1^{ère}) alors que le niveau collège ou CEGEP correspond au niveau lycée (terminale et bac +1).

- Comment donner à l'élève la capacité d'évaluer l'incertitude sur le modèle algébrique?

Nous voulons permettre aux étudiants d'effectuer et de réussir l'activité de modélisation algébrique des phénomènes scientifiques en utilisant les ressources visuelles et graphiques de l'ordinateur et en bénéficiant des avantages des logiciels d'ExAO. Pour ce faire, nous avons développé et conçu une nouvelle méthode informatisée à interface humaine, la Régression Graphico-Statistique (RGS, Touma, 2006). Contrairement aux méthodes traditionnelles de Gauss-Legendre qui sont programmées et utilisées automatiquement dans les calculatrices et logiciels, le niveau des connaissances préalables à la compréhension de la méthode RGS est de niveau secondaire et collégial. Cette méthode complétera les travaux et les recherches en ExAO en particulier ceux qui sont effectués au Laboratoire de Robotique Pédagogique de l'Université de Montréal par Nonnon (1986), Girouard (1999), Fournier (2003) et par Riopel (2005). Notre prototype comprend d'abord tous les éléments d'un logiciel d'acquisition de données, à savoir une fenêtre d'initialisation, un mode «vu-mètre» dans lequel nous pouvons voir les données de chaque variable sous forme numérique, un mode oscilloscope (Graphique) et un tableur.

Régression Graphico-Statistique (RGS)

Le module de Régression Graphico- Statistique comporte essentiellement trois fenêtres (Voir figure 1): la fenêtre graphique; la fenêtre des écarts et la fenêtre de l'histogramme de distribution de ces écarts

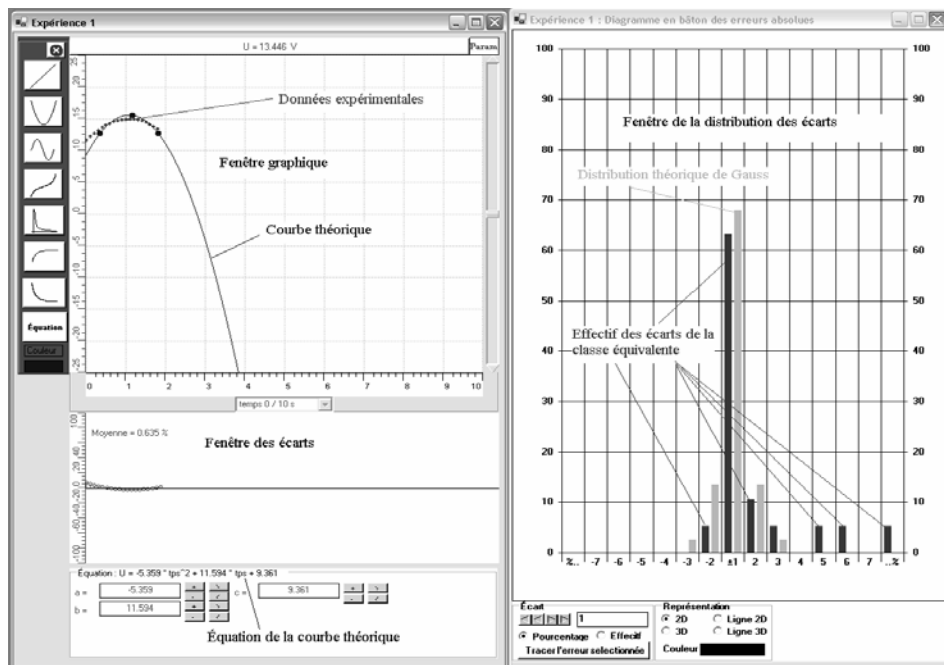


Figure 1 Les fenêtres du module RGS

Fenêtre graphique

Dans la fenêtre graphique, l'élève aura initialement:

- paramétré l'expérience, c'est-à-dire, choisi les variables, le nombre des données, la fréquence d'échantillonnage afin de déclencher l'acquisition des données;
- visualisé sous forme graphique l'interaction entre les différentes variables;
- visualisé sous forme d'un tableau les données expérimentales.

Afin de choisir le type d'équation avec lequel l'élève modélise algébriquement cette interaction de variables, nous avons créé une barre d'outils sur laquelle nous retrouvons des fonctions prédéfinies telles que: les fonctions linéaires du premier degré, du second degré, du troisième degré, les fonctions rationnelles, sinusoïdales et exponentielles. Nous lui avons aussi donné la possibilité de définir n'importe quelle fonction algébrique en cliquant sur le bouton équation.

Fenêtre des écarts

La fenêtre des écarts consistera à visualiser les écarts entre la courbe théorique et les données expérimentales afin de les réduire et les minimiser le plus possible. Notons que les écarts seront calculés en pourcentage de l'échelle de mesure de la variable à l'étude. Les points expérimentaux qui se trouvent en dessous de la courbe théorique auront un écart négatif tandis que ceux qui se trouvent en dessus de la courbe théorique auront un écart positif. L'échelle par défaut de la fenêtre des écarts est de -100% à 100% . Pour mieux visualiser les écarts, nous allons donner la possibilité de les dilater, c'est-à-dire de réduire l'échelle des écarts (Voir figure 1).

Fenêtre d'histogramme

Pour des raisons de commodité, nous avons centré la distribution des écarts à zéro. Attendu qu'en général les incertitudes de mesures en sciences expérimentales se situent en de ça de 10% de leur valeur, nous avons limité initialement l'échelle des intervalles de classe à 14% de l'échelle totale avec des intervalles de classe fixés à 1% . Puisque que la distribution est centrée à zéro, les valeurs des intervalles de classes négatives correspondront, en valeur absolue, aux écarts des points expérimentaux situés en dessous de la courbe. De même, les valeurs des intervalles de classes positives correspondront aux écarts des points situés en dessus de la courbe. Afin de conserver les écarts de toutes les données expérimentales, tous les écarts qui sont inférieurs à -7% se rempliront dans la classe à gauche «% <». De même, tous les écarts supérieurs à 7% se rempliront dans la classe à droite «% >». Toutefois, ces paramètres seront tous changés de manière dynamique lorsque l'étudiant changera la valeur de l'intervalle. La fréquence des écarts est aussi par défaut en pourcentage du nombre de points expérimentaux. Nous avons aussi affiché cinq colonnes vertes représentant la distribution théorique de Gauss. La colonne centrale indique le 68% des effectifs, deux colonnes symétriques qui indiquent le 13.5% et le -13.5% des effectifs et les deux autres qui sont aussi symétriques indiquent les 2.5% et -2.5% des effectifs. Ainsi, en même temps qu'on ajuste la courbe théorique sur les points expérimentaux, les écarts entre la courbe théorique et les données expérimentales se répartissent dynamiquement dans les classes correspondantes (de couleur rouge). Il s'agira alors de trouver d'abord l'intervalle le plus petit de la classe centrale qui contiendra les 100% des effectifs des écarts. Ensuite, il suffira de diminuer, au fur et à mesure, cet intervalle et d'ajuster les paramètres de la courbe pour optimiser son équation algébrique. Il s'agira alors de trouver les paramètres de l'équation qui minimisent le plus possible cet intervalle et qui distribuent le plus normalement possible les effectifs de ces écarts. Notons aussi que, par cette méthode, les points singuliers (ou aberrants) de cette expérience seront ceux qui correspondent aux classes des effectifs se trouvant à l'extérieur des colonnes vertes. Nous avons aussi donné la possibilité de sélectionner une classe en affichant en même temps sur le graphique les points correspondants. Cette possibilité, qui nous permet

de visualiser graphiquement la distribution des écarts, nous permettra aussi de repérer les points singuliers (ou aberrants) et de les rejeter avec un critère statistique (se situant par exemple à 3, 4 ... fois l'erreur-type). Avec le logiciel, nous donnerons alors à l'étudiant la possibilité de les éliminer.

Ainsi, pour que l'élève réussisse l'activité de modélisation algébrique d'un phénomène physique par la méthode RGS, il doit:

- **Ajuster une fonction symbolique :** Avec la fenêtre graphique, il devrait être capable de superposer visuellement un modèle fonctionnel symbolique sur les données empiriques, remplaçant ainsi le modèle empirique constitué des points associés aux données empiriques par un autre modèle mathématique sous forme d'une relation algébrique.
- **Réduire les écarts entre la fonction et les données expérimentales:** Avec la fenêtre des écarts, il devrait être capable de réduire les écarts entre la courbe symbolique et les points expérimentaux. C'est-à-dire de mieux ajuster la fonction symbolique sur les données empiriques en minimisant ces écarts.
- **Optimiser la fonction symbolique :** Avec la fenêtre de distribution des écarts, il devrait être capable d'optimiser ces ajustements en tenant compte de leur distribution statistique ce qui lui permet de prendre en compte l'occurrence des points expérimentaux pour optimiser le modèle mathématique, déterminer les points aberrants ou singuliers et évaluer l'erreur de prédiction.

Conclusion

Nous avons entrepris cette recherche dans le but de permettre aux étudiants d'accéder et comprendre la modélisation scientifique des phénomènes physiques. Pour ce faire, et contrairement aux méthodes essentiellement algébriques traditionnelles comme celle de Gauss-Legendre, nous avons développé une nouvelle méthode de régression dont la compréhension est de niveau secondaire et collégial; la Régression Graphico-Statistique (RGS). Cette méthode consiste à présenter les écarts entre les données expérimentales et la courbe théorique pour optimiser celle-ci sous forme visuelle. Cette idée est apparue fructueuse puisque, comme la méthode de Gauss-Legendre qui utilise le carré des écarts afin de les réduire de manière algébrique, la méthode RGS plus accessible permet de les réduire directement et explicitement par des opérations itératives de l'élève avec un support visuel. Pour mieux apprécier ces écarts, nous avons donné la possibilité à l'élève de les amplifier en changeant progressivement leur échelle de mesure. Pour estimer l'erreur de mesure, la méthode traditionnelle des extrêmes nous est apparue insuffisante puisque celle-ci incluait ipso facto les points singuliers. Aussi, le calcul traditionnel de l'écart type est difficile pour le secondaire et collégial. Pour résoudre ce problème, nous avons donc pensé distribuer ces écarts sur un diagramme à bandes de manière à faire apparaître leur étalement en agissant sur les intervalles de classe. Ainsi, en distribuant de cette manière les écarts, nous exerçons deux actions simultanées, à savoir l'optimisation de la courbe et l'évaluation de l'erreur-type. Ici, la distribution des incertitudes est caractérisée par deux facteurs : l'ajustement de la courbe sur les données empiriques et l'incertitude de mesure proprement dite. L'accès à l'évaluation de l'incertitude de mesure nécessite donc de réduire d'abord l'erreur d'ajustement des paramètres de la fonction symbolique de manière précise, ce que nous faisons en minimisant progressivement la valeur de l'intervalle de classes. Cette valeur minimale sera alors une estimation valable de l'erreur-type, c'est-à-dire de l'incertitude des mesures.

Les résultats de cette recherche nous montrent

Que cette méthode RGS se compare avantageusement à la méthode de Gauss-Legendre utilisée dans REGRESSI et dans EXCEL. Que les étudiants ont développé une attitude positive à l'égard de la modélisation scientifique grâce à cette méthode. Les résultats ont aussi montré que la plupart des étudiants ont pu appliquer toutes les propriétés de la méthode RGS afin de construire un modèle calculable et prédictif du phénomène à l'étude et comprendre le rationnel ou processus sous-jacent. Ils ont pu aussi déterminer l'incertitude de mesure de leur modèle algébrique. Les commentaires des étudiants et de leurs professeurs confirment ces résultats : Avec la méthode RGS les étudiants comprennent mieux le processus de modélisation des phénomènes scientifiques qu'avec la méthode traditionnelle (Moindres carrés de Gauss-Legendre) utilisée automatiquement dans les calculatrices programmables et les logiciels de modélisation.

Sur le plan théorique, le processus de modélisation en sciences exige de l'élève qu'il réalise une expérience, qu'il prenne acte des mesures obtenues, qu'il perçoive le caractère modélisable de ses résultats. En extrapolant les résultats de notre recherche, nous pourrions déduire que l'élève, en utilisant la méthode RGS, pourraient mieux comprendre et parcourir de façon plus autonome ce processus de modélisation en sciences.

Les apports de cette recherche

En mathématiques, RGS est une méthode nouvelle, originale, générale et accessible, mais néanmoins rigoureuse, pour modéliser par une fonction, linéaire ou non, un phénomène physique.

En didactique cette méthode donne accès à la compréhension de la modélisation algébrique sans avoir à recourir, comme Gauss-Legendre, aux dérivées partielles.

Au plan pratique, cette méthode permet à des étudiants, de niveau collégial et même secondaire, de pratiquer l'investigation scientifique et de mieux comprendre la confrontation de leurs résultats ainsi modélisés avec les modèles ou théories existantes.

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What Factors Influence Understanding of Written Arithmetic Problems?

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Introduction

In the teaching of Mathematics, the resolution of word problems has been around for a long time. It is possible to go back as far as the curriculum of 1861 to note the presence of word problems in Quebec (Bélanger, Gauthier, & Tardif, 1993). In 1945, George Pólya published an important paper pertaining to solving word problems and its importance in the learning process: *How To Solve It*. In 1980 the National Council of Teachers of Mathematics (NCTM) published “An Agenda for Action” a paper that called for the integration of problem solving in the curriculum. Since then in its reports of 1989 and 2000, the NCTM highlighted the necessity to grant a major role in problem solving in the curriculum. In Quebec, the competency to solve problems has become a major part of the curriculum. This competence is regarded as a goal to develop as well as a means to acquire new knowledge.

Furthermore, despite the value we put upon problem solving, despite its long time presence in mathematics courses, researchers are still interested by this class activity, particularly to understand the comprehension process involved when students are in a problem solving situation. From research in the last fifteen years, it emerges that various levels of representations play a role in the process of understanding a word problem. One of these representations is called ‘situation model’ and is based on the pupils’ real-world knowledge. The purpose of this research is to study the understanding process involved when pupils are solving an arithmetic word problem.

Theoretical framework

The process undertaken by pupils to understand problematic situations plays an important role in their problem solving abilities. This process involves the construction of representations of the problem under study. Research suggests that various levels of representation are constructed during the process of solving arithmetic word problems. Kintsch & van Dijk (1978) describe the first mental representation as the *text base* which is strongly based on the actual text. It is a primary analysis of the various notions found within the written text. The text base can lead to another level of representation, the problem model, which Kintsch & Greeno (1985) described as a more formal representation which is directly related to the mathematical question asked. Between these two levels of representation another step has been identified. This level was defined by Reusser (1990) as the *situation model*. Reusser

(1990) introduced the situation model in the process of problem comprehension in order to fill a gap in the model proposed by Kintsch & Greeno (1985) which fails to take into consideration the elements that are not essential in order to solve the problem but may help pupils better understand. The situation model was described by Reusser (1990) as an intermediate representation that is constructed by using one's real-world knowledge and personal experiences to interpret the information found in the text base. When trying to solve an arithmetic problem, the situation model is constructed when pupils take into consideration information other than mathematical information necessary for problem solving.

Among the factors which may influence the representations constructed by pupils during arithmetic word problem resolution, two categories emerge in the literature: factors related to the word problem statement and those related to the pupil himself.

An arithmetic word problem may be composed of three main categories of information: solving information, situational information and explanation information. Although many researchers have investigated the analysis of solving information of word problems after manipulating the type of problem, the order of presentation of the data or the size of the numbers, few have studied the situational and explanation information found in problems statement. However, these elements are the most likely contributing factors in the elaboration of a situation model.

The situational information plays a role in the elaboration of a context that grounds the mathematical question in a real life situation. Moreau & Coquin Viennot(2003) identified many categories of situational information: initiating events, setting information and temporal information. Their research suggests that situational information contributes to the construction of a situation model. However not all categories are treated equally by pupils.

Explanation information renders the relationship between information or the consequence of events from the text more explicit. Research suggests that these elements also influence the construction of a situation model (Moreau & Coquin-Viennot, 2003) and may increase the comprehension of the problem (Stern & Lehrndorfer, 1992).

Factors related to the pupils may also influence the construction of a situation model. Moreau & Coquin-Viennot (2003) demonstrated that arithmetic skills influenced the situation model. More specifically, we believe that three factors may play a role in the construction of the situation model: arithmetic skills, reading skills and gender.

Skills in arithmetic contribute to the strategies used by pupils to comprehend the problem. Hegarty, Mayer & Monk (1995) demonstrated a difference among pupils with strong versus weak problem solving skills in the importance given to various types of information found in a problem and in the strategies used to comprehend the arithmetic word problem. Moreau & Coquin-Viennot (2003) also observed that pupils with weaker arithmetic skills tended to give less importance to the solving information of a problem and more importance to the situational information than stronger pupils.

It is well known that reading skills play an important role in the comprehension of word problems (Helwig, Almond, Rozek-Tedesco, Tindal, & Heath, 1999; Muth, 1984; Sovik, Frostrad, & Heggberget, 1999). It is during the reading of a word problem that the first level of representation, the text base, begins to form, and evolves towards higher levels of representation. Moreover, reading skills explain a proportion of errors in problem solving (Muth, 1984).

It is also known that boys and girls do not work on arithmetic problems in the same fashion. Carr & Jessup (1997) highlighted differences in strategies used: girls rely on their memory to obtain results of a calculation and tend to be perfectionist in their work. Boys, on the other hand, are more autonomous in arithmetic and less mechanical in their approach towards problem solving. Other investigators have shown that differences in performance between

girls and boys are only present for certain types of arithmetic problems (Gallagher & De Lisi, 1994).

Few studies have published data that allow the investigation of the relationship between the construction of a situation model and performance in resolution of word problems. Stern & Lehrndorfer (1992) modified the word problems in such a way that the relationship between quantitative data was more explicit. This increased pupils' performance in problem solving. However, studies that tried to establish the relationship between the construction of a situation model and pupils' performance did not obtain significant results (Moreau & Coquin-Viennot, 2003). Although certain studies have set the ground in this field, the question regarding the influence of constructing a situation model on pupils' performance in solving word problems remains unanswered.

Objectives

The first objective of this research is to study the influence of factors related to the pupils and to the words problems on the construction of a situation model. The second objective is to look at the link between the situation model and the performance in problem solving of arithmetic word problems.

This research tries to answer those three questions:

- What is the effect of gender, reading skills and arithmetic skills of pupils on the construction of a situation model?
- What is the effect of the type of information included in the problem statement of a word problem on the construction of a situation model?
- What is the relation between the situation models and the performance in problem solving of arithmetic word problems?

Methodology

Sample

The sample is composed of 750 pupils of grade 6 elementary school. To obtain this sample, 908 pupils were solicited. They were selected from 35 classes in 17 francophone schools in Quebec. The sample comprised 354 girls and 394 boys (in two cases, data concerning the gender of the child was not documented).

Variables

In order to answer our research questions, two dependant variables were considered: construction of a situation model (questions 1 and 2) and performance of pupils in problem solving (question 3). As independent variable we considered the problem type in function of what the statement comprised, gender, mathematics skills and reading skills.

Data collection tool

The problems used for this study are word problems with two linear variations with differing rates of variation. In each problem the question is related to the intersecting point of these two relations. Each problem was elaborated into four versions in order to manipulate the type of information contained in the problem statement.

Pupils were randomly assigned one of the four versions of the problems. Each pupil was asked to solve three problems of the same category: complete (with all types of information), the version including situational information, the version including explanation information or

the simplified version composed uniquely of the necessary information required to solve the problem.

Two tasks were also proposed to the pupils. In task A, after solving the problem they were asked to identify if there were any elements in the statement that were not essential but that helped them better understand the problem. This task was inspired from a study elaborated by Moreau & Coquin-Viennot (2003). In task B, pupils were told the following: ‘here is a word problem that is not yet solved. Add one or two sentences that would help make the problem easier for other pupils to understand. These two tasks permitted to collect data regarding the construction of a situation model. Data was also collected from teachers. Each teacher was invited to give us data related to child: gender, arithmetic and reading skills.

Statistical analysis

In order to answer our first question: “What is the effect of gender, reading skills and arithmetic skills of pupils on the construction of a situation model?” we conducted a analysis of variance. For our second question: “What is the effect of the type of information included in the problem statement of a word problem on the construction of a situation model?” we proceeded with descriptive statistics and for our third question: “What is the relation between the situation models and the performance in problem solving of arithmetic word problems?” we conducted correlational analysis and linear regression.

Results

The results suggest that the pupils give greater importance to explanation information rather than situational when constructing a situation model. This is particularly the case for students with weaker arithmetic skills. It is interesting to note that pupils with weaker arithmetic skills not only give greater importance to explanation information during the process of comprehending a problem but fare better when given these problems as opposed to problems with situational information. First, this suggests that explanation information can help weaker pupils solve problems. More specifically, this suggests that pupils with weaker arithmetic skills may construct different representations as a function of the information presented in the problem. Furthermore, when we look at reading skills and the type of information present in the word problem, our results clearly indicate that problems which include situational information are better solved by pupils with strong reading skills than problems including explanation information. The opposite is true for pupils with weak reading skills. They tend to do better on problems that include explanation information as opposed to situational information. This suggests that the representations pupils build could differ as a function of reading skills.

An objective of the study was to determine whether pupils who construct a situation model fare better in problem solving. A correlational analysis assessing the relationship between the number of situational elements retained to understand the problem and pupils performance allows us to affirm that pupils who give greater importance to situational information in a problem have greater success in solving the problem. Hence, constructing a situation model with situational information presented in the problem is related to pupils’ performance in problem solving. Moreover, our analysis demonstrates that not all situational information is equally integrated into the construction of a situation model. In the current study, only the information that helped precisely situate the arithmetic problem in a context was kept by pupils in order to elaborate a situation model and consequently positively influenced the pupil’s performance.

Conclusion

Pupils with weaker arithmetic skills construct different representations as a function of the information presented in the problem. Pupils who give greater importance to situational information in a problem have greater success in solving the problem. Hence, constructing a situation model with situational information presented in the problem is related to pupils' performance. The presence of situational and explanation information within the word problem helped pupils in the construction of a situation model. The situation model influences pupils' performance in problem solving. The influence is a function of the type of information contained in the problem as well as of pupils' reading and arithmetic skills.

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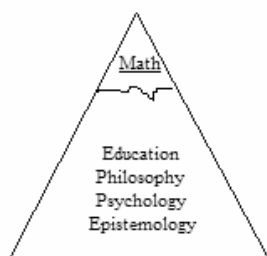
Ad Hoc Sessions

Séances ad hoc

Probabilities / Probabilités

Egan Chernoff, *Simon Fraser University*
Annie Savard, *Université Laval*

First off, we were very pleased to have such a great turnout for the session. We really enjoyed the opportunity to present, and are even more pleased to share some of the major thoughts that



helped guide our conversation, to the rest of the community. In our ad-hoc session we (1) presented and discussed (what we consider) some of the major ideas concerning probability, and (2) worked on a novel probability task to explore and extend the ideas that arose during the discussion. After the historical context was taken into consideration—through the examination of the phases of probability education—we presented the idea that, for us, the word *probability* conjured up the image of an iceberg. While we realized that (for some) this is a tired metaphor, we felt it an appropriate way to begin our session. More specifically to our discussion, the “tip of the iceberg” represented the mathematical aspect of probability, and exploring “below the surface” provided a venue for discussion of: epistemological, philosophical, psychological, socio-cultural, and educational issues inherent to probability, which makes up the bulk of the berg. This first major distinction in probability (seen in the works of Ian Hacking and Donald Gillies) led to discussion of a number of other important distinctions.

From a philosophical perspective, probability can be measured in (at least) three different ways: classical, frequentist, and subjective. However, while these three interpretations dominate mathematics education literature, the categorization of these different interpretations is not as stark as it first appears. Case in point, the notion of classical probability as subjectivist or objectivist probability is a matter of debate. Furthermore, the subjective interpretation of probability can be further categorized into the logical and personal theory of probability, while the frequentist interpretation of probability can be further categorized into the propensity and relative frequency theories of probability. All of these points led us to present the notion of whether a philosophical foundation is required for probability education?

Next we turned our attention to equiprobability, availability and the outcome approach, which are some conceptions identified by the literature in Education. Now, it is possible to distinguish some deterministic conceptions, which create learning obstacles. These deterministic conceptions attribute a causal explanation at a random phenomenon. Throwing a dice or a coin, the fate or the luck and the non-consideration at the independence between the outcomes are some examples.

Our discussion had set the stage for the second part of our session. Participants were asked to focus their discussion on the following: How “thick” must a coin be, in relation to its radius, such that the probability of landing heads, tails, or on its *side* are all equal (i.e., $P(H)=P(T)=P(S)$)? Given that (for some) the classical solution to the task is not as quickly determined, it opened the floor for lively discussions amongst the tables.

Tout d'abord, nous étions très heureux d'avoir eu une si belle assistance lors de cette session. Nous avons vraiment apprécié l'opportunité de présenter et de partager nos réflexions, mais nous avons encore plus apprécié de pouvoir échanger avec vous les courants de pensées majeures qui ont guidé nos conversations avec le reste de la communauté. Lors de cette session ad hoc, nous avons (1) présenté et discuté (ce que nous considérons) quelques unes des idées les plus importantes à l'égard des probabilités et nous avons (2) proposé un problème qui illustre certaines de ces idées. Après une brève évocation du contexte historique à travers l'examen des phases de l'enseignement des probabilités, nous avons présenté l'idée, que pour nous, le mot probabilité peut être illustré par un iceberg. Cette métaphore bien connue nous semblait appropriée pour amorcer notre session. Ainsi, la pointe de l'iceberg représente l'aspect mathématique des probabilités tandis que l'exploration de la surface cachée a conduit à discuter les aspects épistémique, philosophique, psychologique, socioculturel et didactique inhérents aux probabilités, lesquels constituent le corps de l'iceberg. Ces importantes distinctions (voir les travaux de Ian Hacking and Donald Gillies) ont orienté la discussion envers d'autres importantes distinctions.

Selon une perspective philosophique, les probabilités peuvent être déclinées selon trois approches; théorique, fréquentielle et subjective. Toutefois, pendant que ces trois approches dominent la recherche en didactique des mathématiques, la catégorisation de ces approches ne semble pas aussi homogène au premier abord. Il semble y avoir une certaine controverse autour de la propension à considérer les probabilités théoriques comme étant subjective ou objective. De plus, l'interprétation de l'approche subjective des probabilités semble être catégorisée selon une théorie logique ou personnelle des probabilités, alors que l'approche fréquentielle semble être considérée comme une tendance ou une application statistique. Ces débats nous conduisent à nous questionner sur la pertinence des fondements philosophiques des probabilités en éducation. D'importants travaux sur l'apprentissage et l'enseignement des probabilités ont permis de documenter des conceptions probabilistes telles la disponibilité, l'équiprobabilité ou l'approche par le résultat. Toutefois, il est maintenant possible de distinguer des conceptions déterministes qui peuvent constituer un obstacle à l'apprentissage. Ces conceptions déterministes cherchent à attribuer des causes à des phénomènes aléatoires. La manipulation d'un dé ou d'une pièce de monnaie, le destin ou la non-prise en compte de l'indépendance entre les tours en sont quelques exemples.

Notre discussion a introduit la seconde partie de la session. Nous avons demandé aux participants de discuter le problème suivant: Quelle épaisseur une pièce de monnaie doit-elle avoir, en rapport avec son rayon, pour que les probabilités d'obtenir face, pile ou sur son côté soient égales (i.e., $P(H)=P(T)=P(S)$)? Étant donné que la solution conventionnelle de ce problème n'est pas apparue pas au premier abord, la résolution a pavé la voie à d'intéressantes discussions parmi les participants. Nous attendons donc avec impatience la suite l'an prochain à Sherbrooke...

First Steps Toward An Archaeology Of Gesture In Graphing

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Researchers in mathematics education have taken up concepts, analytic tools and methodologies from the emergent interdisciplinary field of gesture studies and begun to adapt them to a wide range of purposes (see, for example Alibali & diRusso, 1999; Arzarello & Edwards, 2005; Núñez, 2004; Radford, Demers, Guzmán & Cerulli, 2003; Rasmussen, Stephan, & Allen, 2004; Reynolds & Reeve, 2003). Gesture can be a way of revealing unconscious aspects of concept formation and links with embodied metaphors that underlie mathematical abstractions and artifacts designed to foster the development of these concepts. Teachers and learners produce gestures in a largely unconscious way, as a byproduct of communicating and expressing ideas. Gestures produced by mathematics teachers and learners provide a rich source of data, comparable in scope to that provided by language, which can be read in terms of bodily metaphors, the construction of mathematical concepts, and the relationships among concepts.

This ad hoc session focused on a study of gestures and graphing in secondary mathematics education. In the videotaped data from the second of two pilot studies completed to date, students and teachers in three secondary schools were asked to use gestures and sounds to describe given graphs on the Cartesian plane, and their gestural descriptions were videotaped. Participants commented on their own use of gestures while viewing their own tapes during post-taping interviews. Genre analytic techniques were used as a starting point to begin to reveal a genealogy of embedded cultural meanings incorporated in the practice of graphing on the Cartesian plane in mathematics education. These meanings included culturally-bound interpretations of horizontality and verticality, high and low, left and right, which have been connected with a schematic of vertical and horizontal axes from preliterate cosmologies to present-day systems of imagery. It was conjectured that these generically embedded, unintentional meanings likely affect students' emotional and cognitive responses to (supposedly neutral) mathematical graphs, and results of this pilot study offered some support for this.

An unexpected observation from the data suggests that gestured graphs may offer a concise and accurate diagnostic tool for learners' degree of mathematical engagement and attentiveness to salient mathematical features. Furthermore, work with students on gesture and movement related to imaginative engagement with graphs may also offer effective methods for remediation, particularly at the Grade 8 level. Genre study of mathematical gestures might ground for both diagnosis and remediation of certain learning difficulties in mathematics. A follow-up study will be undertaken to test these emergent hypotheses.

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Function Modelling using Secondary Data from Statistics Canada's E-STAT Database

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Mathematics curricula across the country require students at the secondary school level to retrieve secondary data and model mathematical functions to fit the data. For example, the 2006 Revised Ontario Mathematics Curriculum (Grade 11 University course, MCR3U) explicitly states: "Students will collect data that can be modelled as an exponential function... from secondary sources (e.g., websites such as Statistics Canada, E-STAT) and graph the data" (p. 36). Similar expectations can be found for linear, quadratic, and sinusoidal functions in mathematics curricula across the country.

As noted in the Ontario Mathematics Curriculum reference above, an excellent source of secondary data is Statistics Canada's E-STAT website. E-STAT (<http://estat.statcan.ca>) is an enormous database that contains data from CANSIM, the Canadian Socio-economic Information Management System. CANSIM in E-STAT contains data from over 250 different surveys regarding socio-economic topics about Canadians, resulting in over 2 700 tables and over 36 million time series! This enormous wealth of data is free to all educators and students, but access from home requires a username and password. See www.statcan.ca/english/Estat/userpass.htm or www.statcan.ca/francais/Estat/userpass_f.htm for details. Although having access to such a vast wealth of data is certainly a great benefit to educators and students, it can also be overwhelming to find appropriate data, especially for function modelling.

Over the course of the past year, Statistics Canada has developed a new resource to assist teachers in covering expectations related to function modelling with secondary data. Our Function Modelling Using Secondary Data from E-STAT website is located at www.statcan.ca/english/edu/mathmodel.htm or www.statcan.ca/francais/edu/mathmodel_f.htm. This page features datasets from CANSIM on E-STAT that Joel Yan and I have found that are closely approximated by linear, quadratic, exponential, sinusoidal, and logistic functions over certain time periods. For example, consumption of bottled water follows a linear trend from 1995 to 2001 (see Appendix A). The number of males registered in apprenticeship programs follows a quadratic curve from 1991 to 2001 (see Appendix B). Federal debt follows an exponential curve from 1955 to 1997 (see Appendix C). The number of induced abortions for women under the age of 20 follows a sinusoidal curve from 1974 to 2003 (see Appendix D). Finally, revenue from the cable television industry follows a logistic curve from 1976 to 2000 (see Appendix E). More than 20 other datasets that follow function models are also included on this website.

It is very easy to obtain the datasets from the Function Modelling page. Two options are available: Table Number or Vector Number. If you click on the Vector Number, you will retrieve precisely the data we found that model the mathematical function type. If you wish to explore a slightly different dataset within the same topic (e.g., apprenticeship program registrations for females instead of males; any of the datasets for a specific province or territory), click on a Table Number, make your selections on the Subset selection page, and then choose Retrieve as individual Time Series (as the data are for a long period of time for only one set of specifications within one table). Retrieve as a Table is used when you select

many specifications (e.g., all the provinces, both sexes, etc. for one table) over only one or two years.

On the Output specification page (regardless of whether you selected Table Number or Vector Number), you then have several options for your data output format. You can retrieve the data in a screen output format, such as HTML or Plain text. You can retrieve the data in a downloadable format, such as CSV (to open directly in Excel) or PRN. Finally, you can make a graph of your data directly in E-STAT in one of the 15 graph types available, such as line graphs, scatter graphs, and histograms. To output your data to a data analysis program, Plain text: Table, time as rows or HTML table: Time as rows works best – simply highlight only the data and copy and paste into your software program.

As a complement to the Function Modelling website, we have also written several lessons that use the aforementioned datasets with data analysis software programs, such as Fathom or Excel. The following lessons can be found on an additional site, Math Resources using Canadian Data (www.teacherweb.com/on/statistics/math), under the E-STAT, Function Modelling and Community Profiles folder:

- Linear Modelling of the Life Expectancy of Canadians
- Quadratic Modelling of Canada's Baby Boom
- Quadratic Modelling of the Number of Males Registered in Apprenticeship Programs
- Sinusoidal Modelling of the Number of Marriages by Month
- Sinusoidal Modelling of Canada's Youth Cohorts
- Exponential Modelling of the Farm Value of Potatoes

Watch for these lessons to be translated to French and moved to the *Statistics Canada Mathematics Lessons Page* in the coming months at www.statcan.ca/english/kits/courses/math.htm or www.statcan.ca/francais/kits/courses/math_f.htm

Two articles (Hall, 2007; Yan, 2004) regarding function modelling with Statistics Canada data have been published in the Ontario Mathematics Gazette. Please email jennifer.hall@statcan.ca or joel.yan@statcan.ca with any further comments or questions. Statistics Canada provides free in-class or professional development workshops at elementary schools, high schools, colleges, and universities (for students, teachers, and teacher-candidates) about function modelling and all of our other resources. Regional representatives are available for workshops in all provinces and territories. See www.statcan.ca/english/edu/reps-tea.htm or www.statcan.ca/francais/edu/reps-tea_f.htm for contact information.

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Explanation and Proof in Mathematics and Mathematics Education

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Explanation, an important facet of proof and a significant consideration in the practice of working mathematicians, has been discussed extensively in recent literature on the philosophy of mathematics. The SSHRC project on which we are working, *Explanation, proof, and reasoning styles in mathematics: Implications for mathematics education*, is examining this recent literature with a view to identifying some of the present trends in proof and explanation and determining their possible implications for mathematics education. The project has three broad objectives: 1) to direct attention to the newest developments in the philosophy and practice of mathematics as they relate to reasoning styles and in particular to proof and explanation; 2) to determine the relevance of these developments for mathematics education; and 3) to suggest theoretical frameworks for mathematics education reflecting these innovations in mathematical practice.

As indicated, the project seeks to reach these objectives by examining critically the relevant literature on reasoning, explaining and proving in current mathematical practice. As its starting point the project takes the view that educational researchers and curriculum developers cannot foster the use of reasoning and proving in mathematics teaching without understanding what it means to reason and to prove in mathematics itself. A first step in promoting this understanding is to investigate what the most recent literature in the philosophy of mathematics has to say about the ways in which present-day mathematicians present and weigh evidence, devise proofs and judge the degree to which these proofs might be explanatory.

Our primary sources are books, articles in refereed journals, conference proceedings, and information found on the internet, including e-journals and discussion groups. We are currently building a database of the most important sources of information relevant to the above research objectives. We have already annotated the first version of this database and made it available on the Web to academic colleagues in Canada and abroad, and will update it from time to time. The current version is posted on this site: <http://fcis.oise.utoronto.ca/~ghanna/philosophyabstracts/index.htm>

As part of this project, a colloquium titled “Explanation and proof in mathematics: Philosophical and educational perspectives” was organized by Gila Hanna (OISE/UT), Niels Jahnke (Duisburg-Essen) and Helmut Pulte (Bochum) took place in Essen, Germany, in November 2006, with the participation of twenty-six philosophers and mathematics educators. Additional information about the conference can be found in the programme, posted here: http://www.uni-duisburg-essen.de/imperia/md/content/zis/programm_tagung_011106.pdf
The colloquium proceedings will appear in an edited book to be published by Springer in 2009.

The authors are also setting up a website for the upcoming ICMI Study 19, *The role of proof and proving in mathematics education*, which will take place in 2009. The discussion document is currently being written by the International Program Committee (IPC) and will be posted on the website as soon as it is completed. A call for papers will also be posted on the website in due time: <http://jps.library.utoronto.ca/ocs/index.php?cf=8>

Exploring our Embodied Knowing of the Gauss-Bonnet Theorem: Barn-Raising an Endo-Pentakis-Icosi-Dodecahedron

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When applied to the polyhedral²³ case, the Gauss-Bonnet Theorem determines a property of the vertices of the surface of a solid: if we add together the angles which have been taken away from 360° at each vertex (the angle deficit), the result will equal a constant 720° (Alexandrov & Zagaller, 1976). This result is surprising for the uninitiated, in that it is true of any polyhedron. For the case of the cube, it is easy to visualise: at each of its eight corners, 90° was removed in order to close it, giving a total of 720° .

In the planned workshop, we will explore the case of a polyhedron with no right angles, the endo-pentakis-icosi-dodecahedron (Cundy & Rollet, 1961; Conway, 1999), a polyhedron with 80 equilateral triangular faces. This will allow the participants to reflect on their embodied knowledge of polyhedra and the angle deficit property. Using 1-metre-edge-length faces, we will construct the polyhedron with the aid of a net which was developed based on the angle deficit idea (Knoll & Morgan, 1999).

The scale of the project will give the participants an experience of collaborative mathematics practice through the barn-raising (Knoll & Morgan, 1999; Hart, 2004). In addition, they will have an embodied experience of polyhedral geometry: they will be able to pace the area of the flat net and to physically enter the space defined by the polyhedron, allowing them a sense of the total angle deficit. This last experience will help to initiate reflections on the relationship between our understanding of space and our motor-control system (Lakoff & Núñez, 2000).

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²³ We are defining polyhedra as having genus 0.

Mathematical Biography

John Grant McLoughlin
University of New Brunswick

I have opened most of my courses for the past ten to fifteen years with an assignment to be submitted by the end of the first week of classes.

Assignment 1: Autobiographical Sketch

All students are required to write a brief autobiographical piece about their mathematical history. It is particularly important to identify some of your feelings toward mathematics. You are encouraged to identify one or two significant incidents or experiences that may have shaped your perceptions. The stories are personal and will not be shared with the class.

Please consider this as an invitation also to share any special circumstances (e.g. pending surgery; serious illness of family member; an issue related to learning) that may interfere with your participation in the course. You are welcome to share such a circumstance through conversation rather than writing.

This assignment will not be graded. Comments will be provided as a means of facilitating communication between individual students and John Grant McLoughlin.

I introduce the assignment by sharing a personal example from elementary school. This is done to emphasize that it is not a listing of courses that is desired but rather the sharing of one or two snapshots that affect the way one sees mathematics.

Why do I wish to share this example here with the CMESG community? I have learned so much from such a simple exercise. Recently my teaching at UNB has taken me to Bhutan and Trinidad and Tobago. In each case, these biographies have been enriching and overwhelming to read. The students (in each case, experiences teachers) express an appreciation for the opportunity to write about their mathematical stories. My insight into the educational systems and their experiences has been heightened by frequent references to corporal punishment in conjunction with mathematical errors. It is understandable why this subject does not conjure images of beauty in such circumstances. Canadian students, in undergraduate math classes and education courses, have offered insight that is also culturally relevant as they offer a sense of place and context.

Sketches are returned promptly with comments, including in some cases, encouragement to drop by for a conversation early along in the course. Conversations with people who raise serious issues (or mention high levels of math anxiety) help to humanize the mathematical experience. The essence of the biographical sketches is that the experience of teaching and learning mathematics becomes enriched through the human interaction and acknowledgement of experiences that overlay upon the coursework.

How has it affected my teaching? Practically it has required me to leave ample time in the opening week of classes to respond to the sketches. Further, it has immediately offered me a sense of the students individually and collectively. This work should not be taken lightly, however, as the invitation to write has frequently opened up more than a conversation about math. I have been mutually honoured and challenged by the respect that has been shared by others. This has been enriching in that human relationships are ultimately at the core of

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teaching and learning. The subject or course name brings us together though we know that much more than the subject is being experienced together in a learning environment.

Opportunities to expand upon these ideas with interested others are welcome. The impressive level of interest and discussion at the ad hoc session itself clarified that many in the CMESG community are keen to pursue the bridge between biography and mathematics.

This Is Mathematics; This Isn't Mathematics; But That...I'm Not So Certain About: The Possible "Emancipation" of Secondary School Mathematics From the Bonds of "Real" Mathematics

Craig Newell
Simon Fraser University

Motivation

The question that I posed at this *ad hoc* session arose from several sources. The perennial question, "What is mathematics?", has been a significant focus in my current incarnation as a graduate student. This question became vividly instantiated for me when I had the opportunity of being a sessional instructor for two "methods" courses in the same semester; one for pre-service elementary teachers and the other for pre-service secondary teachers. The stark differences I perceived in how the two classes responded to the (implicit) questions about the nature of mathematics lead to much reflection on my part. A third stimulus for this session came as I revisited William James' pragmatism and his understanding of the practical consequences of attitude and intent in asking and answering questions.

The Question and a Rationale

Mathematics has been tagged with numerous descriptors throughout its history. There is pure mathematics and applied mathematics; the ethnomathematics of D'Ambrosio; the street mathematics of Carraher; Western and [culture of your choice] mathematics of G. G. Joseph and A. J. Bishop; and the ever-present school mathematics which further differentiates into primary, secondary, and tertiary flavours. The people who discriminate among types of mathematics do not see the discipline as a monolithic structure with clearly defined boundaries. (Nor do I.) I concur with Davis and Hersh when they write, "The definition of mathematics changes. Each generation and each thoughtful mathematician within a generation formulates a definition according to his [*sic*] lights." (1981, p. 8)

In particular, I am interested in what is meant by applying 'school' as a modifier to 'mathematics'. How is school mathematics related to mathematics? Furthermore, what is the interplay between elementary and secondary school mathematics? A proposed simplistic picture of school mathematics is that it is a linear, progressive sequence in which elementary school mathematics is about developing intuition, secondary school mathematics is concerned with developing facility with symbol manipulation, and tertiary school mathematics leads the student into the paradise of Bourbakian formalism – "real" mathematics. Because of its focus on symbolic processes, secondary school mathematics seems to be more closely aligned with the formalism of "real" mathematics than with the intuitive, experientially grounded mathematical activities of the elementary classroom.

In the interests of stimulating conversation I proposed the following rather general situation. Assuming that the class of human endeavours subsumed under the term mathematics differentiates into a variety of valid and useful activities and that a role of schools is to facilitate learning, why should secondary school mathematics align itself primarily with formal mathematics? Are there not pedagogical advantages in "freeing" secondary school

mathematics from symbol manipulations and letting the students partake more freely of the other aspects of mathematics?

Dénouement

When the *ad hoc* convened, I began with William James' anecdote about resolving a dispute about a whether a man was circling a squirrel or not. (James, 1963/1910, pp. 22-23). The pragmatic resolution of "interminable metaphysical disputes" such as defining mathematics is possible.

I then presented my thesis and rationale. I made the suggestion that, if preparation for the study of formal mathematics is not the primary and sole goal of secondary school mathematics, the students might be better served with a continuation of the methods and attitudes begun in the elementary schools. In particular, the emphasis on physical experience and intuition as well as the blending of mathematics with other disciplines could be continued. Preparation for "real" mathematics could be delayed until post-secondary studies.

The participants in the *ad hoc* session responded with questions and challenges of their own. I left the session with more questions than when I started. Two, in particular, stay with me. Is secondary school mathematics a subset of mathematics? (Or does it differ in significant ways from formal mathematics?) What anecdotes or studies exist of teaching mathematics in the secondary schools using the methods and attitudes of the elementary schools?

I concluded the session with Whitehead's characterization of the role of mathematics in the history of human thought. He likened it to the role of Ophelia in the play *Hamlet* – a character essential to the play and "she is charming – and a little mad." (1967/1925, p. 20).

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Panel



Table ronde

What Courses Could or Should Mathematics Departments Offer to Graduate Programs in Mathematics Education?

France Caron, *Université de Montréal*
 Morris Orzech, *Queen's University*
 Elaine Simmt, *University of Alberta*

France Caron

Pour répondre à la question qui nous était posée, nous avons choisi de consulter un échantillon (sans aucune ambition de représentativité) des principaux intéressés par une telle offre de cours. Nous avons donc interrogé, à l'aide d'un court questionnaire, 6 étudiants à la maîtrise ou au doctorat en didactique des mathématiques à l'Université de Montréal. Parmi ces six étudiants, l'un s'intéresse à l'enseignement des mathématiques au primaire (p), quatre à l'enseignement des mathématiques au secondaire (s) et un sixième s'intéresse à l'enseignement des maths au niveau collégial (c).

Nous leur avons demandé d'évaluer d'abord sur une échelle de 0 à 4 le potentiel d'intérêt de différents cours et d'en préciser ensuite les conditions qui maximiseraient leur intérêt. Le tableau ci-dessous résume les « données quantitatives » que cette petite enquête maison a permis de recueillir.

<i>Cours</i>	<i>Étudiant</i>						Moy.
	A (p)	B (s)	C (s)	D (s)	E (s)	F (c)	
Histoire des mathématiques	3	3	2	4	4	4	3,3
Mathématiques et technologies	2	4	2	2	3	0	2,2
Applications des mathématiques	2	3	3	3	3	0	2,3
Approfondissement des maths scolaires	4	1	2	3	3	3	2,8
Raisonnement et preuve	1	4	3	3	3	3	3,0
Jeux et résolution de problèmes	2	4	1	1	4	1	2,5
Autre	4		4				

Les résultats témoignent d'une certaine variété dans les intérêts et ne permettent pas de faire ressortir clairement les cours qui répondraient le mieux aux besoins des étudiants. Comme dans bien des cas, ce sont les commentaires (les « données qualitatives ») qui se sont révélés plus éclairants. Les voici donc regroupés pour chacun des types de cours proposés :

Histoire des mathématiques

- Indispensable!!! Il faut donner un relief historique et donc mobile à une branche que trop de personnes considèrent figée.
- Surtout si l'on aborde:
 - La philogénèse des concepts abordés dans les curriculums mathématiques du primaire et du secondaire - la vie des mathématiciens m'intéresse moins.
 - L'origine, les processus ayant mené à la construction des connaissances (durée, obstacles, etc.).
 - Des pistes d'intégration du sujet dans les classes du primaire et du secondaire.
 - Une réflexion sur les avantages didactiques d'une telle intégration (ou les « risques » aussi s'il y a lieu).
- Intéressant mais plus pertinent au bacc.

Mathématiques et technologies

Surtout si...

- On y traite de:
 - la façon dont on peut utiliser les TIC pour enseigner les mathématiques au primaire et au secondaire.
 - l'évolution de l'intégration des technologies dans les cours de mathématiques.
- Le cours me permet:
 - d'approfondir ma connaissance de certains outils technologiques (ex. calculatrice à affichage graphique)... sans s'y limiter toutefois.
 - d'améliorer ma capacité à modéliser des situations par l'usage des technologies.
- On axe sur :
 - des technologies favorisant une meilleure compréhension des mathématiques et me fournissant un outil puissant d'enseignement.
 - des activités concrètes et accessibles aux élèves du secondaire, en tenant compte du temps d'appropriation de l'outil.

Applications des mathématiques

Surtout si le cours...

- Touche les applications des « mathématiques du secondaire ».
- Permet de faire des liens entre notre projet de recherche et les différentes répercussions qu'il peut avoir dans la société.
- M'habilité à faire plus de liens entre la « vie courante » et les maths.
- Me permet de mieux me servir des mathématiques en tant qu'outil pour décider et agir ; plutôt dans une perspective de modélisation que pour justifier la pertinence des mathématiques scolaires dans la vie de tous les jours.
- Revient à un niveau accessible aux élèves du secondaire.

Approfondissement des maths scolaires

- Un tel cours n'est pas nécessaire. C'est notre responsabilité de « réapprendre » ce que nous avons oublié avec le temps.
- Pertinent, surtout si le cours...
 - Ne devient pas une forme de révision des concepts élémentaires.
 - Me permet de revisiter des savoirs scolaires mais à des niveaux mathématiques plus élevés, de mieux comprendre en quoi ces apprentissages seront les assises d'apprentissages ultérieurs.
 - Me permet de mieux comprendre le choix de certaines transpositions qui ont été faites (manuel, programme) en fonction du savoir mathématique en jeu, et favorise le développement d'une pensée critique.
 - Analyse les « ruptures et continuités » avec les maths « savantes » (sous l'angle de la transposition didactique par exemple) et les impacts de ces « ruptures » sur la compréhension des concepts mathématiques à l'étude.
 - Est toujours en lien avec les nouveaux programmes de maths.
 - Aborde les opérations élémentaires et une théorie des ensembles. Il faut donc que ce soit en maths fondamentales.

Raisonnement et preuve

Surtout si le cours :

- Nous forme à détecter des raisonnements fallacieux, des preuves qui n'en sont pas, etc.
- Combine mathématiques et didactique des mathématiques.
- Dépasse la simple étude de productions d'élèves et réserve une grande partie aux théories didactiques et à leur application en salle de classe (ou en recherche).
- Propose, entre autres, des pistes didactiques pour l'apprentissage de la preuve.
- Propose une synthèse des différents types de raisonnement avec une réflexion pour leur enseignement.
- Aborde la logique formelle.

Jeux et résolution de problèmes

Surtout si le cours :

- S'intéresse aux « impacts » didactiques des jeux : outre la motivation souvent accrue (ce qui n'est pas négligeable), que retient un élève des mathématiques présentes dans les jeux?
- Me permet, par la résolution des problèmes, de construire des connaissances mathématiques, ce dont j'aurais bien envie!
- M'amène à mieux analyser mon processus de résolution qui peut très bien s'approcher de ceux des élèves en termes d'heuristiques.
- Va au-delà de la résolution et aborde également la construction de jeux et de problèmes.
- M'outille davantage pour différencier mon enseignement.

Autre

- Cours de logique.
- Tout cours qui viendrait enrichir l'offre de cours!
- Envisager le jumelage des étudiants en mathématiques avec ceux en didactique des mathématiques.
 - Augmente le nombre d'étudiants potentiels.
 - Pour l'avoir vécu, les étudiants provenant des deux départements bénéficient mutuellement de la diversité des expériences et des champs de compétences de chacun... même si l'aventure peut être tout un défi !



Le nez de la navette Atlantis vu de la station Mir © NASA

Les idées principales qui se dégagent de ces commentaires nous amènent à conclure que la plupart des étudiants aux grades supérieurs en didactique des mathématiques que nous avons interrogés se déclarent intéressés à développer, approfondir ou enrichir leurs connaissances mathématiques pour autant que ce travail leur permette de mieux comprendre les enjeux, possibilités et contraintes de l'enseignement des mathématiques, et que les cours proposés participent de façon explicite à la formulation de liens en ce sens. Autrement dit, s'ils se déclarent prêts à monter à bord d'une navette pour explorer de nouvelles zones de l'espace mathématique, ils souhaitent qu'on profite régulièrement de ce point de vue unique pour observer et mieux comprendre la terre de l'enseignement qui les préoccupe.

Pour accéder à un tel souhait, il nous semble intéressant d'envisager une forme de co-enseignement qui implique des professeurs des deux unités (mathématiques et didactique/éducation). Nous croyons qu'une telle approche, qui permettrait de conjuguer des expériences et compétences complémentaires, pourrait se révéler profitable non seulement aux étudiants aux grades supérieurs (possiblement des deux secteurs) mais aussi aux professeurs impliqués; ces derniers pourraient mettre à profit ce beau défi pour apprendre l'un de l'autre et ouvrir de nouveaux champs de collaboration.

Morris Orzech

My response to this question will be woven around three paraphrased comments or adages, adapted to the context of this panel discussion. Since the department in which I work has no direct role in graduate mathematics education programs, my comments will be in the nature of opinions. That these opinions are provisional will be apparent from the divergent conclusions I reach by considering the three quotations.

I begin with what I find a tantalizing idea of Jerome Bruner: the conjecture "that any subject can be taught effectively in some intellectually honest form to a [student] at any stage of development." What appeals to me about this idea is its suggestion that mathematics instruction does not have to be so very driven by teaching prerequisites for future courses, nor as stymied as it often is in presenting captivating mathematics to students who lack the formal prerequisites for it. Experience and reflections tells me that despite its appeal, the suggestion is a difficult one to implement. For some mathematical topics there do exist resources for realization of Bruner's "dream," often based on a historical perspective, but mathematical breadth and depth are necessary to fashion these resources into effective lessons. However, mathematical competence is not enough – even among mathematicians there isn't always interest or experience in this kind of exercise. These remarks point to the desirability of a partnership, involving mathematicians and mathematics educators, in designing and teaching a kind of course that would prepare graduate students (whose interest lies in that direction) to teach mathematics with an eye to testing Bruner's hypothesis, and to uncovering new ways of enriching the mathematical experience of students beyond what would be possible with a stacked-prerequisites approach.

The second reference point for my response to the title question is Kant's adage that "percepts without precepts are empty; precepts without percepts are blind." Preparation for teaching mathematics can easily suffer from insufficient balance between theory and reality-based reflection. Pedagogical concerns sometimes focus on social or psychological theories of learning detached from mathematical ideas or goals. Pedagogical concerns can also focus on mathematical content without reference to student learning; and even when instructors *do* take note of student difficulties and *do* modify their approach, the modifications can have an ad-hoc hit-or-miss quality that makes success or failure of little value outside a very narrow situation. The latter concern seems particularly relevant to instruction in mathematics departments, where most people lack a theoretical framework for reflective analysis of their

teaching experience. On the other hand, a graduate student who adopts an educational perspective that gives undue primacy to a particular learning theory is likely to do this as a result of their education faculty experience, rather than as a result of their mathematics department background. Of course, many mathematics and mathematics education departments have people who are savvy about *both* mathematics education and mathematics perspectives. But I believe that for preparing graduate mathematics education students who want to be teachers, consistent involvement of mathematicians and mathematics educators would in the long run benefit not only students, but both groups of faculty. In short, something mathematics departments can offer to graduate programs in mathematics education is an involvement that may develop better understanding and appreciation for the enterprise of mathematics education in *both* units.

The third statement I wish to consider is a remark (hopefully remembered well enough to not distort its essential message) from a *Mathematics Teacher* article I read. In this article a high school mathematics teacher described his (or perhaps her) desire to design engaging learning experiences for giving students a better understanding of mathematical concepts – and told about realizing that his conceptual understanding of mathematics was not adequate to the goal. My experience tells me that this situation is consistent with what drives some high school mathematics teachers to study in a graduate program. What can mathematics departments do for the further education of such people? In thinking about this question I cannot help noting that mathematics departments have usually had these people in their tutelage for four years. Admittedly, people forget things, but nevertheless my reaction is that it might be best (at least in most cases) for the mathematics education graduate students in question not to be exposed again to a strategy for teacher development that seems to be inadequate to its goals. Does this contradict my earlier comments about potential benefit in joint involvement of mathematics educators and mathematicians? Maybe so.

Elaine Simmt

In order to address the question posed for us I believe it is important to ask, what is the purpose of doctoral studies and then ask what might mathematics departments have to offer in order to address the goals of the doctorate.²⁴ I begin by referring to the Carnegie Foundation's *Initiative on the Doctorate*. Lee Shulman uses the notion of stewardship.

We view the doctorate as a degree that exists at the junction of the intellectual and moral. The Ph.D. is expected to serve as a steward of her discipline or profession, dedicated to the integrity of its work in the generation, critique, transformation, transmission, and use of its knowledge. (Shulman, quoted in Golde, 2006, p.3)

Given Shulman's conception of the role of a doctor of philosophy (in a particular discipline) we must ask, what are the elements of a graduate program that might foster the student's ability to generate, critique, transform and transmit the knowledge of the discipline and what is the knowledge of the discipline? Additionally, what are the desired skills, knowledge and habits of mind that we wish to engender in our students within their graduate programs and what elements of the graduate program foster those qualities?

²⁴ I will focus my comments on the doctorate.

There are three graduate degrees offered in education at the University of Alberta,²⁵ the M.Ed., Ed.D. and Ph.D. The M.Ed. is offered as either a course or thesis-based program and is commonly understood as a professional development degree, in part because it is a mechanism for teachers to move up the salary scale and work towards leadership positions within schools and at the ministry. The M.Ed., as we conceptualize it introduces students to the field of curriculum studies through work in foundations and inquiry and focuses on educational research in the areas of teaching and learning within particular content areas. The Ed.D. is a research degree focused on issues of professional practice and is often taken by people in leadership positions. The Ph.D. is a research degree focused on scholarship and research of educational phenomena. Students in doctoral programs study research methodology, curriculum foundations and inquiry and (in the case under question) mathematics education as a content area. In summary then, in curriculum departments at the University of Alberta doctoral programs include the study curriculum foundations, curriculum inquiry, qualitative research methods, and more specifically mathematics education.

Responding to the question what courses might departments of mathematics offer we need to first ask, what discipline is the mathematics education Ph.D. stewarding? Warren McCulloch asked, "What is a Number, that a Man [sic] May Know It, and a Man, that He May Know a Number?" For me, McCulloch's questions encapsulate the fundamental questions (but most elusive) of the discipline of mathematics education. What is the nature of mathematics and what is the nature of the human knower? At its core the mathematics education Ph.D. is an exploration of these two questions. However, those questions lead us to others that are under our stewardship: How do we learn mathematics? Why teach mathematics? How might we teach mathematics? What mathematics should we teach? And, what is the mathematics we teach?

Hyman Bass (2006) in a paper on the doctorate in mathematics comments that "the characteristic that distinguishes mathematics from all other sciences is the nature of mathematics knowledge and its verification by means of mathematical proof. On the one hand, it is the only science that thus pretends to claims of absolute certainty. On the other hand, this certainty, which is self-referential, is gained at the cost of logical disconnection from the empirical world" (p. 104). In contrast, the mathematics education Ph.D. is grounded in the empirical and experiential world. Are mathematics and mathematics education by their very nature incommensurable?

Having said that, the very fact that Hyman Bass explores the question of the nature of mathematics suggests that there is a dimension of the work of mathematicians that would be of great value to doctoral students. Most certainly questions like what is mathematics (philosophy of mathematics) and where does it come from (history of mathematics) are questions that departments of mathematics could offer education students, assuming the mathematics departments have experts in these areas.

Further, departments of mathematics could teach about the discipline of mathematics in their teaching of particular mathematics content. Mathematics instructors are well positioned to facilitate the exploration of the nature of mathematics by having students do mathematics. It seems this would be most successful if within such classes opportunities for reflecting on the experience of learning mathematics and the nature of mathematics were made explicit. This is something that the outstanding mathematics instructor does as part of their practice but this is not something mathematics doctoral students are deliberately taught to do. There points to a

²⁵ In Alberta, public school teachers earn certification through undergraduate programs, leading to a B.Ed. In those programs secondary specialist teachers will study in both mathematics and mathematics education. I mention this because it points to the fact that graduate degrees cannot be used for teacher certification; this implies that M.Ed. degrees do something different than "prepare" students to teach.

more general concern regarding the preparation of graduate students for teaching. Teaching does not seem to be emphasized in mathematics doctoral programs (Bass, 2006; Stevens, 2006). Therefore although the possibility exists for appropriate course work to be offered by mathematics departments to doctoral students in education the usefulness of such courses is highly dependent on an instructor who is able to both teach mathematics and to make space for reflection on the nature of mathematics and learning mathematics.

A final thought, given the critiques of doctoral programs in mathematics departments regarding the lack of emphasis that is placed on developing the mathematics Ph.D. with the necessary skill set for teaching (Bass, 2006), maybe we should next ask what courses could education departments offer mathematics graduate students.

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Appendices

Annexes

APPENDIX A / ANNEXE A

Working Groups at Each Annual Meeting / Groupes de travail des rencontres annuelles

- 1977 *Queen's University, Kingston, Ontario*
- Teacher education programmes
 - Undergraduate mathematics programmes and prospective teachers
 - Research and mathematics education
 - Learning and teaching mathematics
- 1978 *Queen's University, Kingston, Ontario*
- Mathematics courses for prospective elementary teachers
 - Mathematization
 - Research in mathematics education
- 1979 *Queen's University, Kingston, Ontario*
- Ratio and proportion: a study of a mathematical concept
 - Minicalculators in the mathematics classroom
 - Is there a mathematical method?
 - Topics suitable for mathematics courses for elementary teachers
- 1980 *Université Laval, Québec, Québec*
- The teaching of calculus and analysis
 - Applications of mathematics for high school students
 - Geometry in the elementary and junior high school curriculum
 - The diagnosis and remediation of common mathematical errors
- 1981 *University of Alberta, Edmonton, Alberta*
- Research and the classroom
 - Computer education for teachers
 - Issues in the teaching of calculus
 - Revitalising mathematics in teacher education courses

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- 1982 *Queen's University, Kingston, Ontario*
- The influence of computer science on undergraduate mathematics education
 - Applications of research in mathematics education to teacher training programmes
 - Problem solving in the curriculum
- 1983 *University of British Columbia, Vancouver, British Columbia*
- Developing statistical thinking
 - Training in diagnosis and remediation of teachers
 - Mathematics and language
 - The influence of computer science on the mathematics curriculum
- 1984 *University of Waterloo, Waterloo, Ontario*
- Logo and the mathematics curriculum
 - The impact of research and technology on school algebra
 - Epistemology and mathematics
 - Visual thinking in mathematics
- 1985 *Université Laval, Québec, Québec*
- Lessons from research about students' errors
 - Logo activities for the high school
 - Impact of symbolic manipulation software on the teaching of calculus
- 1986 *Memorial University of Newfoundland, St. John's, Newfoundland*
- The role of feelings in mathematics
 - The problem of rigour in mathematics teaching
 - Microcomputers in teacher education
 - The role of microcomputers in developing statistical thinking
- 1987 *Queen's University, Kingston, Ontario*
- Methods courses for secondary teacher education
 - The problem of formal reasoning in undergraduate programmes
 - Small group work in the mathematics classroom
- 1988 *University of Manitoba, Winnipeg, Manitoba*
- Teacher education: what could it be?
 - Natural learning and mathematics
 - Using software for geometrical investigations
 - A study of the remedial teaching of mathematics
- 1989 *Brock University, St. Catharines, Ontario*
- Using computers to investigate work with teachers
 - Computers in the undergraduate mathematics curriculum
 - Natural language and mathematical language
 - Research strategies for pupils' conceptions in mathematics

Appendix A • Working Groups at Each Annual Meeting

- 1990 *Simon Fraser University, Vancouver, British Columbia*
- Reading and writing in the mathematics classroom
 - The NCTM "Standards" and Canadian reality
 - Explanatory models of children's mathematics
 - Chaos and fractal geometry for high school students
- 1991 *University of New Brunswick, Fredericton, New Brunswick*
- Fractal geometry in the curriculum
 - Socio-cultural aspects of mathematics
 - Technology and understanding mathematics
 - Constructivism: implications for teacher education in mathematics
- 1992 *ICME-7, Université Laval, Québec, Québec*
- 1993 *York University, Toronto, Ontario*
- Research in undergraduate teaching and learning of mathematics
 - New ideas in assessment
 - Computers in the classroom: mathematical and social implications
 - Gender and mathematics
 - Training pre-service teachers for creating mathematical communities in the classroom
- 1994 *University of Regina, Regina, Saskatchewan*
- Theories of mathematics education
 - Pre-service mathematics teachers as purposeful learners: issues of enculturation
 - Popularizing mathematics
- 1995 *University of Western Ontario, London, Ontario*
- Autonomy and authority in the design and conduct of learning activity
 - Expanding the conversation: trying to talk about what our theories don't talk about
 - Factors affecting the transition from high school to university mathematics
 - Geometric proofs and knowledge without axioms
- 1996 *Mount Saint Vincent University, Halifax, Nova Scotia*
- Teacher education: challenges, opportunities and innovations
 - Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
 - What is dynamic algebra?
 - The role of proof in post-secondary education
- 1997 *Lakehead University, Thunder Bay, Ontario*
- Awareness and expression of generality in teaching mathematics
 - Communicating mathematics
 - The crisis in school mathematics content

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- 1998 *University of British Columbia, Vancouver, British Columbia*
- Assessing mathematical thinking
 - From theory to observational data (and back again)
 - Bringing Ethnomathematics into the classroom in a meaningful way
 - Mathematical software for the undergraduate curriculum
- 1999 *Brock University, St. Catharines, Ontario*
- Information technology and mathematics education: What's out there and how can we use it?
 - Applied mathematics in the secondary school curriculum
 - Elementary mathematics
 - Teaching practices and teacher education
- 2000 *Université du Québec à Montréal, Montréal, Québec*
- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
 - Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
 - Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
 - Teachers, technologies, and productive pedagogy
- 2001 *University of Alberta, Edmonton, Alberta*
- Considering how linear algebra is taught and learned
 - Children's proving
 - Inservice mathematics teacher education
 - Where is the mathematics?
- 2002 *Queen's University, Kingston, Ontario*
- Mathematics and the arts
 - Philosophy for children on mathematics
 - The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
 - Mathematics, the written and the drawn
 - Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers
- 2003 *Acadia University, Wolfville, Nova Scotia*
- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
 - Teacher research: An empowering practice?
 - Images of undergraduate mathematics
 - A mathematics curriculum manifesto

Appendix A • Working Groups at Each Annual Meeting

2004 *Univerité Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education - Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, Education, Society, and Peace
- Learning Mathematics in the Early Years (pre-K – 3)
- Discrete Mathematics in Secondary School Curriculum
- Socio-Cultural Dimensions of Mathematics Learning

2006 *University of Calgary, Alberta*

- Secondary Mathematics Teacher Development
- Developing Links Between Statistical and Probabilistic Thinking in School Mathematics Education
- Developing Trust and Respect When Working with Teachers of Mathematics
- The Body, the Sense, and Mathematics Learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in Mathematics – Activities, Engagement, & Reflection
- Geometry, Space, and Technology: Challenges for Teachers and Students
- The Design and Implementation of Learning Situations
- The Multifaceted Role of Feedback in the Teaching and Learning of Mathematics

APPENDIX B / ANNEXE B

Plenary Lectures at Each Annual Meeting / Conférences plénières des rencontres annuelles

1977	A.J. COLEMAN C. GAULIN T.E. KIEREN	The objectives of mathematics education Innovations in teacher education programmes The state of research in mathematics education
1978	G.R. RISING A.I. WEINZWEIG	The mathematician's contribution to curriculum development The mathematician's contribution to pedagogy
1979	J. AGASSI J.A. EASLEY	The Lakatosian revolution Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO D. HAWKINS	Reflections on forty years of thinking about the teaching of mathematics Understanding understanding mathematics
1981	K. IVERSON J. KILPATRICK	Mathematics and computers The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS G. VERGNAUD	Towards a philosophy of computation Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN P.J. HILTON	The nature of problem generation and the mathematics curriculum The nature of mathematics today and implications for mathematics teaching

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- 1984 A.J. BISHOP The social construction of meaning: A significant development for mathematics education?
L. HENKIN Linguistic aspects of mathematics and mathematics instruction
- 1985 H. BAUERSFELD Contributions to a fundamental theory of mathematics learning and teaching
H.O. POLLAK On the relation between the applications of mathematics and the teaching of mathematics
- 1986 R. FINNEY Professional applications of undergraduate mathematics
A.H. SCHOENFELD Confessions of an accidental theorist
- 1987 P. NESHER Formulating instructional theory: the role of students' misconceptions
H.S. WILF The calculator with a college education
- 1988 C. KEITEL Mathematics education and technology
L.A. STEEN All one system
- 1989 N. BALACHEFF Teaching mathematical proof: The relevance and complexity of a social approach
D. SCHATTNEIDER Geometry is alive and well
- 1990 U. D'AMBROSIO Values in mathematics education
A. SIERPINSKA On understanding mathematics
- 1991 J.J. KAPUT Mathematics and technology: Multiple visions of multiple futures
C. LABORDE Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
- 1992 ICME-7
- 1993 G.G. JOSEPH What is a square root? A study of geometrical representation in different mathematical traditions
J CONFREY Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
- 1994 A. SFARD Understanding = Doing + Seeing ?
K. DEVLIN Mathematics for the twenty-first century
- 1995 M. ARTIGUE The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
K. MILLETT Teaching and making certain it counts
- 1996 C. HOYLES Beyond the classroom: The curriculum as a key factor in students' approaches to proof
D. HENDERSON Alive mathematical reasoning

Appendix B • Plenary Lectures at Each Annual Meeting

1997	R. BORASSI P. TAYLOR T. KIEREN	What does it really mean to teach mathematics through inquiry? The high school math curriculum Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON K. HEINRICH	Structure of attention in teaching mathematics Communicating mathematics or mathematics storytelling
1999	J. BORWEIN W. WHITELEY W. LANGFORD J. ADLER B. BARTON	The impact of technology on the doing of mathematics The decline and rise of geometry in 20 th century North America Industrial mathematics for the 21 st century Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE M. B. BUSSI	Manipulating combinatorial structures The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE C. ROUSSEAU	Mathematics in action: A challenge for social theorising Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS J. BORWEIN	Toward a practice-based theory of mathematical knowledge for teaching The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD A. SIERPINSKA	Using history of mathematics in the classroom: Prospects and problems Research in mathematics education through a keyhole
2004	C. MARGOLINAS N. BOULEAU	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN J. TAYLOR	Learning as developing identity in the mathematics classroom Soap bubbles and crystals
2006	B. JAWORSKI E. DOOLITTLE	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design Mathematics as medicine

APPENDIX C / ANNEXE C

Proceedings of Annual Meetings / Actes des rencontres annuelles

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

<i>Proceedings of the 1980 Annual Meeting</i>	ED 204120
<i>Proceedings of the 1981 Annual Meeting</i>	ED 234988
<i>Proceedings of the 1982 Annual Meeting</i>	ED 234989
<i>Proceedings of the 1983 Annual Meeting</i>	ED 243653
<i>Proceedings of the 1984 Annual Meeting</i>	ED 257640
<i>Proceedings of the 1985 Annual Meeting</i>	ED 277573
<i>Proceedings of the 1986 Annual Meeting</i>	ED 297966
<i>Proceedings of the 1987 Annual Meeting</i>	ED 295842
<i>Proceedings of the 1988 Annual Meeting</i>	ED 306259
<i>Proceedings of the 1989 Annual Meeting</i>	ED 319606
<i>Proceedings of the 1990 Annual Meeting</i>	ED 344746
<i>Proceedings of the 1991 Annual Meeting</i>	ED 350161
<i>Proceedings of the 1993 Annual Meeting</i>	ED 407243
<i>Proceedings of the 1994 Annual Meeting</i>	ED 407242

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<i>Proceedings of the 1995 Annual Meeting</i>	ED 407241
<i>Proceedings of the 1996 Annual Meeting</i>	ED 425054
<i>Proceedings of the 1997 Annual Meeting</i>	ED 423116
<i>Proceedings of the 1998 Annual Meeting</i>	ED 431624
<i>Proceedings of the 1999 Annual Meeting</i>	ED 445894
<i>Proceedings of the 2000 Annual Meeting</i>	ED 472094
<i>Proceedings of the 2001 Annual Meeting</i>	ED 472091
<i>Proceedings of the 2002 Annual Meeting</i>	<i>submitted</i>
<i>Proceedings of the 2003 Annual Meeting</i>	<i>submitted</i>
<i>Proceedings of the 2004 Annual Meeting</i>	<i>submitted</i>
<i>Proceedings of the 2005 Annual Meeting</i>	<i>submitted</i>
<i>Proceedings of the 2006 Annual Meeting</i>	<i>submitted</i>
<i>Proceedings of the 2007 Annual Meeting</i>	<i>submitted</i>

Note

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.