Abstract Title Page
Not included in page count.

Title:

Mapping children's understanding of mathematical equivalence

Author(s):

Roger S. Taylor, Ph.D., Vanderbilt University; Bethany Rittle-Johnson, Ph.D., Vanderbilt University; Percival G. Matthews, Vanderbilt University; and Katherine L. McEldoon, Vanderbilt University

Abstract Body

Limit 5 pages single spaced.

Background/context:

Description of prior research and/or its intellectual context and/or its policy context.

An understanding of algebra is a prerequisite for learning higher-level mathematics, as well as for becoming an informed citizen who is able to fully participate in society. Previous research has shown that a major obstacle to learning algebra is the concept of mathematical equivalence (Baroody & Ginsburg, 1983; Carpenter, Franke, & Levi, 2003; RAND, 2003).

Mathematical equivalence is the idea that two sides of an equation represent the same amount. It includes a relational definition of the equal sign as meaning "the same as." Unfortunately, children's experiences in elementary school do not seem to support a relational view of the equal sign, as they typically see equations in a "standard form," such as 3 + 4 = 7 to the exclusion of others (McNeil, et al., 2006; Seo & Ginsburg, 2003). A reasonable inference from such experiences is that the symbol "=" indicates that one is supposed to perform the operation "add up" or more generally "produce the answer." This *operational* view of the equal sign can impede the development of a *relational* view of the equal sign. An operational view of the equal sign often persists for many years, and students who have this view often have difficulty solving equations (Knuth, Stephens, McNeil & Alibali, 2006).

Based upon previous research (e.g., Matthews & Rittle-Johnson, 2007; Rittle-Johnson, 2006; Rittle-Johnson & Alibali, 1999), we hypothesized that mathematical equivalence would be best captured through three related constructs: (1) Meaning of the Equal Sign, (2) Structure of Equations, and (3) Equation Solving. This paper is specifically focused on the first construct – student conceptions of the meaning of the equal sign.

Purpose/objective/research question/focus of study:

Description of what the research focused on and why.

The focus of this research is to develop an initial framework for assessing and interpreting students' level of understanding of mathematical equivalence. Although this topic has been studied for many years, there has been no systematic development or evaluation of a valid measure of equivalence knowledge.

A powerful method for accomplishing this task has been developed by Wilson (2005). Building upon item response theory, Wilson and his colleagues utilize an assessment instrument called a "construct map." A key feature of this tool is that it can assist educational researchers in making *criterion-referenced* interpretations of instructional materials and student performances. Thus, a construct map of mathematical equivalence can provide a coherent framework for interpreting findings from across elementary and middle school mathematics curricula.

The equal sign construct map we developed has four levels (see Table 1 below). The lowest level is the "Operational View," which is comprised of students who have yet to develop a relational view of the equal sign, although they do understand the meaning of equal independent of the equal sign.

(please insert table 1 here)

In contrast, the highest level of the construct map is the "Advanced Relational View," which is comprised of students who have a fully developed understanding of the relational meaning of the equal sign and no longer hold a operational view (the two views have been shown to co-exist).

Setting:
Specific description of where the research took place.

A written assessment was administered to ten second- through sixth grade classrooms (two per grade) in an urban, parochial school serving a working- to middle-class population.

Population/Participants/Subjects:
Description of participants in the study: who (or what) how many, key features (or characteristics).

Participants were 181 second- through sixth grade students. This set was comprised of 39 second-grade students (17 girls and 21 boys), 43 third-grade students (26 girls and 17 boys), 39 fourth-grade students (16 girls and 22 boys), 34 fifth-grade students (17 girls and 17 boys), and 28 sixth-grade students (14 girls and 14 boys). The students were predominantly Caucasian.

Intervention/Program/Practice:
Specific description of the intervention, including what it was, how it was administered, and its duration.

The assessment instrument consisted of 3 sections that tap 3 core ideas central to knowledge of mathematical equivalence: (1) Meaning of the equal sign (7 questions), (2) Recognizing valid structures of equations (27 questions), and (3) Solving equations (26 questions). The focus of this paper is on the equal sign construct; therefore, the items from this section have been included here and are displayed below in Figure 1.

(please insert figure 1 here)

Completion of the assessment required approximately 45 minutes and was performed within a single class period.

Research Design:
Description of research design (e.g., qualitative case study, quasi-experimental design, secondary analysis, analytic essay, randomized field trial).

This study focused on measurement development, not an intervention, and utilized item response theory (IRT) and the construct map instrument to create a criterion-referenced framework for determining students' understanding of mathematical equivalence. As discussed above, our construct map for the equal sign is presented in Table 1, which breaks knowledge of the equal sign into four levels based on past research.

We then used a Wright map to evaluate the construct map. It consists of two columns: (1) Respondents (individuals of varying levels of the construct), and (2) Items (questions that tap varying levels of the construct). For purposes of clarity, the Wright map in this paper will use the more specific terms "students" (instead of respondents). On the left column, *students* of higher ability on the construct dimension are located on the upper portion of the map, while those with lesser ability are located on the lower portion. On the right column, the *items* of greater difficulty are located near the top of the map and those of lesser difficulty are located near the bottom of the map.

The vertical dimension of the Wright map is measured in logits (i.e., log-odds ratios). An important affordance of this spatial arrangement is the *vertical alignment* between the students and items. If a student and item are aligned horizontally the probability of correctly answering this item is 50%. If a student is one logit above the item, the probability of correctly answering this item is 73%. Conversely, if the student is two logits below the item, the probability of correctly answering this item is 12%. This tool provides educational researchers with a measure of the consistency and comprehensiveness of a curriculum, as well as providing diagnostic information on individual students.

Data Collection and Analysis:
Description of plan for collecting and analyzing data, including description of data.

The data from 181 second through sixth grade students were collected from ten classrooms at an urban parochial school during November 2008.

Findings/Results:
Description of main findings with specific details.

A Wright map of the equal sign data was generated using the Winsteps software package (Linacre, 2008) (see Figure 1 below).

(please insert figure 1 here)

On the left column, note the cluster of students in the lower portion of the map. This indicates that, as expected from previous research, only a small proportion of students hold a relational view of the equal sign.

On the right column, note that item number three, which assessed knowledge of “equal” independent of the equal sign, was the easiest and is located at the bottom of the map (ES3-Number Pairs). This provides support for the hypothesis that a large proportion of students understand the meaning of equal, even though they do not link this meaning to the equal sign.

In contrast, note that item number one, which asked students to define the equal sign, was the hardest and is located at the top of the map. We made a distinction between the students who provided a clear relational definition of the equal sign (ES1(b)-Relational Def Only), and those who provided mixed relational and operational definitions (ES1(a)-Relational & Operational Def). This provides support for our hypothesis that only a small portion of students have a robust relational understanding of the equal sign.

Overall there was a high level of fit between the construct map and the Wright map. This is demonstrated by the items, for the most part, falling along in the predicted order of difficulty.

More detailed views of students' performance, disaggregated by grade, are provided for each question (see Table 2 below).

(please insert table 2 here)

Most apparent in the table is the progressive improvement in performance at higher-grade levels. An interesting result is the large jump in performance between second and third grade for several questions (2, 4, 5a, and 5b). Future analyses will examine this result in greater detail. A second, somewhat discouraging result, is that even in sixth grade, only about 50% of the students correctly answered the advanced relational questions (1a, 1b, and 7).

Conclusions:

Description of conclusions and recommendations of author(s) based on findings and over study. (To support the theme of 2009 conference, authors are asked to describe how their conclusions and recommendations might inform one or more of the above noted decisions—curriculum, teaching and teaching quality, school organization, and education policy.)

As discussed earlier, an understanding of mathematical equivalence is a key prerequisite for learning algebra. A critical transition in student reasoning is progressing from an operational view of the equal sign (i.e., "produce the answer.") to a relational view of the equal sign (i.e., a bridge between two numerically equivalent expressions).

Utilizing the construct map and Wright map instruments developed by Wilson (2005), we developed a framework for understanding mathematical equivalence across elementary and middle school curricula. This framework can provide educational professionals with a tool for more accurately diagnosing students' current skill level and providing guidance for how to foster students' movement along the pathway toward greater mastery of this topic. Furthermore, the framework can provide educational professions with a means of identifying gaps or obstacles in curriculum design and suggest ways for remedying such problems.

Appendixes

Not included in page count.

Appendix A. References

References are to be in APA format. (See APA style examples at the end of the document.)

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *Elementary School Journal, 84*, 199–212.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. (2006). Does Understanding the Equal Sign Matter? Evidence from Solving Equations. *Journal for Research in Mathematics Education 37*(4), 297–312.

Linacre, J. M. (2008). Winsteps (Version 3.67.0) [Computer software]. Chicago: Winsteps.com.

Matthews, P., & Rittle-Johnson, B. (2007). To teach by concept or by procedure? Making the most of self-explanations. In D. S. McNamara & G. Trafton (Eds.), *Proceedings of the 29th Meeting of the Cognitive Science Society*.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., & Krill, D. E. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction*, *24*(3), 367-385.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.

Rittle-Johnson, B. (2006). Promoting transfer: The effects direct instruction and self-explanation. *Child Development*, *77*(1), 1-15.

Rittle-Johnson, B., Siegler, R.S. & Alibali, M.W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93, 346-362*.

Rittle-Johnson, B. & Alibali, M.W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91*, 1-16.

Seo, K.-H., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence…": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills* (pp.161–187). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Wilson, M. (2005). *Constructing Measures: An Item Response Modeling Approach*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Appendix B. Tables and Figures
Not included in page count.

Table 1.

<u>Table 1: Equal sign construct map</u>

Levels	Performances	Items Expected to Get Correct	Response Exemplars
4. Full relational	Relational definition dominates	1) What does the equal sign mean? 2) Is this a good definition of the equal sign: what the answer is)? 3) Which is the best definition of the equal sign? 4) The equal sign (=) is more like: a) + and - b) < and >, c) 8 and 4. Why?	1) "the same as" only definition 2) No 3) Chose "two amounts are the same" 4) < >, because all ways to compare quantities
3. Moderate	Relational definition co-exists with operational definition	1) What does the equal sign mean? 2) Is this a good definition of the equal sign: two amounts are the same?	1) "the same as", but may give second definition (e.g. the answer) 2) Yes
2. Begin relational	Correct use of equal sign in non-numeric contexts	1) What could I put in the empty box to show that two nickels are the same amount of money as one dime?	1) =
1. Operational view	Understand meaning of equal, but not linked to equal sign; operational definition	1) What does it mean to say 2 sets of marbles are equal? 2) Which of these pairs of numbers is equal to 6 + 4?	1) "They have the same number of marbles" 2) 5 + 5

Table 2.

Table 2: Proportion correct on items, disaggregated by grade.

Grade	**Items**								
	ES1(a) ***Rel. & Op. Def***	**ES1(b)** ***Rel. Def Only***	**ES2** ***Strawberries***	**ES3** ***Number Pairs***	**ES4** ***Coins***	**ES5a** ***Same***	**ES5b** ***Same As***	**ES6** ***Best Def***	**ES7** ***Similar Symbols***
2nd	0.05	0.03	0.29	0.74	0.32	0.42	0.24	0.26	n/a
3rd	0.19	0.14	0.65	0.84	0.84	0.77	0.65	0.40	0.38
4th	0.26	0.13	0.71	0.97	0.79	0.76	0.61	0.42	0.42
5th	0.44	0.12	0.62	0.91	0.82	0.85	0.79	0.59	0.50
6th	0.68	0.43	0.89	0.96	0.82	0.89	0.86	0.75	0.54
Average	*0.30*	*0.15*	*0.62*	*0.88*	*0.71*	*0.73*	*0.61*	*0.46*	*0.46*

*Note: Based upon previous findings and current time constraints, question 7 was not given to 2nd grade students.

ES1. What does the equal sign (=) mean? Can it mean anything else?

ES2. What does it mean to say 2 sets of strawberries are equal?

ES3. Which of these pairs of numbers is equal to 6 + 4? Circle your answer.

a) 5 + 5
b) 1 + 2
c) 4 + 10
d) none of the above

ES4. Which answer choice below would you put in the empty box to show that two nickels are the same amount of money as one dime? Circle your answer.

a) +
b) =
c) 5¢
d) don't know

ES5. Is this a good definition of the equal sign? Circle good or not good.

a. The equal sign means two amounts are the same.	Good	Not good
b. The equal sign means count higher.	Good	Not good
c. The equal sign means what the answer is.	Good	Not good
d. The equal sign means the same as.	Good	Not good
e. The equal sign means add the numbers.	Good	Not good

ES6. Which of the definitions above is the ***best*** definition of the equal sign? Write a, b, c, d, or e in the box below. ☐

ES7. Please circle your choice and explain why. The equal sign (=) is more like:

a) + and –
b) < and >
c) 8 and 4
Explain why:

Figure 1. Equal sign assessment items.

```
         PERSONS - MAP - ITEMS
               <more>|<rare>
   4          #####  +
                     |
                     |
                     |
                    T|T
                     |
                     |
            .######  |  ES1(b)-Relational_Def_Only (Level 4)
   3                 +
                .#   |
                     |
                     |
                     |
                     |
                     |
                     |
   2                 +
      .############ S|
                     |S ES1(a)-Relational_&_Operational_Def (Level 3)
                     |
                  #  |
                     |
                     |
                     |  ES7-Similar_Symbols (Level 4)
   1   ############  +
                     |
                     |
                     |
                .##  |  ES6-Best_Definition (Level 4)
                     |
          #########  |
                    M|
   0                 +M
                     |
               .###  |
            #######  |
                     |  ES2-Strawberries (Level 2) | ES5-Rated_Same_As (Level 3)
                     |
                     |
                     |
  -1           .###  +
            #######  |
                     |  ES4-Coins (Level 2)
                     |  ES5-Rated_Same (Level 3)
                     |
                    S|
                     |S
       .###########  |
  -2                 +
                     |
                     |
                     |
                     |
                     |
                     |
                     |  ES3-Number_Pairs (Level 1)
  -3       ########  +
               <less>|<frequ>
EACH '#' IS 2.
```

Figure 2. Equal sign Wright Map.