US-China Education Review A 5 (2011) 691-702
Earlier title: US-China Education Review, ISSN 1548-6613

Student Teachers' Engagement With Re-Contextualized Materials: A Case of Numerical Approximation

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This paper reports on an exploration of students' learning derived from the implementation of learning materials developed in a previous collaborative project. The purpose of the study was to examine the development of third-year students' understanding of the Reimann Sum. These concepts were taught to undergraduate teacher trainees wishing to specialize in the teaching of mathematics in the FET (further education and training) school curriculum. The learning materials were developed by mathematics teacher educationists from the University of Witwatersrand, the UKZN (University of KwaZulu-Natal) and the Nelson Mandela Metropolitan University. A group of 31 students participated in the project. The study adopted the APOS (action-process-object-schema) approach within constructivism, as a theoretical framework. Data were generated from students' responses to the collaborative tasks. A genetic decomposition of the Reimann Sum is offered and used as a basis for the analysis. This paper, specifically, reports on the investigation of students' responses based on a learning theory within the context of advanced mathematical thinking and makes a contribution to an understanding of how these students constructed the concepts in a collaborative way.

Keywords: numerical approximation, Riemann sums, schema, genetic decomposition

The Knowledge Base for Educators

One of the expectations of the Norms and Standards for Educators in South Africa (DoE (Department of Education, 1999) is that the teacher be well grounded in the knowledge relevant to the occupational practice. He/she has to have a well-developed understanding of the knowledge appropriate to the specialization (DoE, 2003). In the UK, the DfEE (Department for Education and Employment) report (1998, p. 57) also required student teachers "to recognize… misconceptions in mathematics, to understand how these arise, how they can be prevented, and how to remedy them". Taking these expectations into account, we attempted to make certain that pre-service FET (further education and training) mathematics teachers at UKZN (University of KwaZulu-Natal) leave with a base of knowledge relevant to their occupational needs. Mwakapenda (2004) concurred when stating that a significant concern in school mathematics is the conceptual understanding of mathematical concepts.

The NCS (National Curriculum Statement) emphasizes a learner-centered, outcomes-based approach to the teaching of mathematics in curriculum 2005 in order to achieve the critical and developmental outcomes (DoE, 2003). The tone for creating an enabling environment and nurturing learners' development of concept

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images and concept definitions is set by the learning outcomes and assessment standards at the various phases of the different grade levels of the education system.

In a project funded by Anglo-American, teacher education units at three South African universities have engaged in developing research-based, learner-centred, conceptually focused material-designed especially to teach calculus to teachers.

This study was carried out in accordance with a specific framework for research and curriculum development in undergraduate mathematics education advocated by Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas (1996) which guided our systematic enquiry of how students acquire mathematical knowledge and what instructional interventions contributed to students' learning. The framework consists of the following three components: instructional treatment, theoretical analysis and observations and assessment of students' learning. In the light of their suggestions, our research design focused on three components: (1) a genetic decomposition of the Riemann Sum; (2) implementation of an instructional strategy that leads to collection of data; and (3) analysis of the data in the context of APOS (action-process-object-schema). With this in mind, we framed the following research question: What is the students' understanding of Riemann Sums as evoked when working through the designed materials? We chose to carry out the investigation in this module, as it was observed by Mahir (2008) that students do not have satisfactory conceptual understanding of integral calculus. The presentation of the activity sheets was self-contained with emphasis on self-discovery. A. Vajiac and B. Vajiac (2008) have successfully employed similar techniques. They used a self-discovery approach so that students finally define the Riemann Integral and, thereafter, prove the Fundamental Theorem of Calculus. In our case, we found that students were able to work with Riemann Sums in a limited manner. Students were successful in approximating the Riemann Sum in cases of finitely (three or four) many rectangles. When exact areas were demanded by the activities, students performed poorly. This occurred even when asked to carry out approximations with a greater number (six) of rectangles.

Theoretical Analysis

Piaget, as cited in Brijlall and Bansilal (2010), expanded and deliberated on the notion of "reflective abstraction". He regarded this to be a major instigating factor for the development of mathematical cognition and distinguished three types of abstractions: (1) empirical abstraction; (2) pseudo-empirical abstraction; and (3) reflective abstraction. Piaget regarded the acquisition of mathematical knowledge to be associated with the latter. Reflective abstraction refers to the construction of logico-mathematical structures by a learner during the process of cognitive development (Dubinsky, 1991a). Two features of this concept are: (1) It has no absolute beginning but appears at the very earliest ages in the coordination of sensori-motor structures; and (2) It continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction (Dubinsky, 1991b).

We define the following four concepts that are used in APOS theory of conceptual understanding (Brijlall & Bansilal, 2010):

(1) Action: An action is a repeatable physical or mental manipulation that transforms objects;

(2) Process: A process is an action that takes place entirely in the mind;

(3) Object: The distinction between a process and an object is drawn by stating that a process becomes an object when it is perceived as an entity upon which actions and processes can be made and such actions are

made in the mind of the learner;

(4) Schema: A schema is a more or less coherent collection of cognitive objects and internal processes for manipulating these objects. A schema could aid students to "... understand, deal with, organise, or make sense out of a perceived problem situation" (Dubinsky, 1991a, p. 102).

According to Sfard (1991), abstract mathematical notions (concepts) can be conceived in two fundamentally different ways: as "processes" (operationally) or "objects" (structurally). In APOS theory, action and process can be regarded as operational conceptions, while object is structural. Sfard's theory does not handle schema which is included in APOS theory. Reification "... is an act of turning computational operationals into permanent object-like entities" (Sfard, 1991, p. 16). The development of mathematics often proceeds by taking processes as operators and then turning them into objects. Examples of processes as operators are counting, calculating using a formula (for example, using the nth term of a sequence to generate successive terms) and differentiating, while examples of resulting objects are numbers, algebraic expressions (for example, the nth term of a sequence) and the first derivative of a function. Therefore, reification, which refers to a transition from an operational to a structural mode of thinking, is a basic phenomenon in the formation of a mathematical concept since it brings the concept "... into existence and thereby deepens our understanding" (Sfard, 1992, p. 54). Both operational (procedural) and structural thinking are important in mathematics—both contribute to the hierarchical structure of algebra, which is used to represent mathematical concepts symbolically.

This paper adopts the following five kinds of construction in reflective abstraction explained by Dubinsky (1991b):

(1) Interiorisation: The ability to apply symbols, language, pictures and mental images to construct internal processes as a way of making sense out of perceived phenomena. Actions on objects are interiorized into a system of operations. Interiorisation is a mental mechanism that merely moves a transformation from the external world (action) to the learner's mind (process);

(2) Coordination: Two or more processes are coordinated to form a new process;

(3) Encapsulation: The ability to conceive a previous process as an object;

(4) Generalization: The ability to apply existing schema to a wider range of contexts;

(5) Reversal: The ability to reverse thought processes of previous interiorized processes.

The study focused on advanced mathematical thinking required for the concepts in calculus. This falls within the domain of APOS theory.

Observations and Assessment of Students' Learning

This followed the instructional treatment and allowed us to gather and analyze data. The data were used in two ways. Firstly, the results of the data analysis were used to test our initial genetic decomposition. Secondly, the data gathered were used to report on the performance of students on mathematical tasks related to the development of the concept of the Riemann Sum. The focus of this paper is on the second purpose stated above.

Preliminary Genetic Decomposition

We found the genetic epistemology of Jean Piaget to be useful in our work. At the centre of Piaget's work is a fundamental cognitive process which he termed "equilibration" (Beth & Piaget, 1966). Through an application of the model of equilibration to a series of written tasks, we were able and believe to generate account of the arrangements of component concepts and cognitive connections prerequisite to the acquisition of the concept of the Riemann Sum. These arrangements which are called "genetic decompositions" do not necessarily represent how trained mathematicians understand the concepts. Based on the above, our initial genetic decomposition which is linked to an understanding of the Riemann Sum using rectangles comprises an Area Schema, a Function Schema, a Finite and Infinite Sum Schema as well as a Limit Schema.

As a part of his/her as (area schema), the student:

(1) As1: Realizes the area of a rectangle as the product of two sides;

(2) As2: Has to relate the length, breadth and area of shapes with the properties of the graphs drawn on the Cartesian plane.

As a part of his/her Fs (function schema), the student:

(1) Fs1: Recognizes that the y-value $f(x)$ represents the length of each rectangle in the Reimann Sums representation;

(2) Fs2: Distinguishes between $f(x_i)$ and $\mathrm{f}(x_{i+1})$ as representing the lengths of the rectangles in left and right area sums respectively;

(3) Fs3: Determines the width of each rectangle by using the restricted domain of function.

As a part of his/her Ls (limit schema), the student:

(1) Ls1: Recognizes that as $n \rightarrow \infty$, the approximation $\lim \sum f(x_i)\Delta x_i$ leads to an exact value of the area of the relevant region;

(2) Ls2: Relates the increase in the number of rectangles to a simultaneous decrease in the width of each rectangle, as $n \rightarrow \infty$.

As a part of his/her Fiss (finite and infinite sums schema), the student:

(1) Fiss1: Can represent finite or infinite sums as expanded sums, when given its compact form using sigma notation;

(2) Fiss2: Can represent finite or infinite sums in compact form using sigma notation, when given its expanded representation.

We must point out that schema are dynamic pending the context it features in. For example, our function schema might be different from a function schema for derivatives or continuity.

Methodology

The study was a qualitative one, because it was based on the students' personal experiences (Ely, 1991). We wanted to study things in the naturalistic settings and interpreted or made sense of the data in terms of the meanings the students brought to them and in this way strengthened the case for a qualitative study as advocated by Denzin and Lincoln (1994). We wanted to know how and why the students responded to the questions in the ways that they did. We coded the written responses of 31 students and discussed our findings based on the genetic decomposition of the Riemann Sum within an extended framework of APOS which includes the Piagetian Triad.

Participants

The research group consisted of 31 students who successfully completed the first semester course on differential calculus (mathematics for educators 220) at a South African university. Both mathematics for educators 220 and the module under investigation (mathematics for educators 310) are taught over 52 by 45-minute sessions. These students are pursuing a Bachelor of Education degree specialising in high school mathematics teaching (also referred to as the FET phase).

The Task

Mathematical tasks are central to students' learning, because such tasks convey messages about what mathematics is and what mathematics entails (Henningsen & Stein, 1997). They add that the tasks in which students engage provide the context in which they learn to think about subject matter and different tasks may place different cognitive demands on students. The task that we implemented expected students to: (1) approximate the area under a parabolic curve by rectangles; (2) build on mathematical vocabulary (upper and lower sums); (3) understand that more accurate approximations of area could be obtained via either increasing the number of rectangles or decreasing the width of the rectangles; (4) calculate the width of the rectangles in a general setting (for n rectangles); and (5) represent a good approximation as a series of products, i.e., $\sum f(x_i)\Delta x_i$.

The design of the activity sheet was based on the Vygotskian notion of scaffolding which is intended to help students to understand and make connections among important ideas. It occurs when a student finds it difficult to arrive at a particular stage in his learning. The teacher intervenes to assist the student. Our intervention was the provision of guided activities (see appendix) which was intended to take the student forward, by providing probing questions at appropriate points.

Visualization plays an important role in learning (Vygotsky, 1978) and in particular, the learning of mathematics. This idea is espoused in the old adage: "A picture is worth a thousand words". The role of graphs in the teaching of mathematics is complex and has multi-fold dimensions (Brijlall, 1997; Maharajh, Brijlall, & Govender, 2008). In this regard, the tasks dealt with establishing the Riemann Sum by using visual aspects including a parabolic graph, rectangles and areas of rectangles.

Analysis and Discussion of Data

The students' responses were coded using the initials of their surnames and first names followed by a number which marked the alphabetical position of each student when the scripts were ordered. The reason for doing this was to facilitate that the identification of the scripts should there be queries later on about aspects of the data analysis. It was also necessary to number the scripts, because certain students shared the same initials. Table 1 shows the coding with abbreviations and descriptions for the individual tasks leading to the establishment of the Riemann sum. In this paper, we present results for Question 1 only which relate to the genetic decomposition of Area Schema and Function schema provided earlier in this paper.

Tables 2, 3, 4 and 5 show the composite results of the students as concluded from their written work.

It was observed that 74% in question 1(a) and 84% of students in question 1(b) realised the area of rectangles as products of the length and breadth. This would be expected at this stage of the students' learning as can be seen in Figure 1.

However, the factor that seemed to have prevented the majority of these responses as being classified as S was that they approximated the lengths of the rectangles, instead of obtaining the function value. The majority of these students estimated the length by reading off y-values from the graph. This confirms the belief that most students prefer working with informal ways in mathematical problem-solving. Only 16% in question 1(a) and 6% in question 1(b) adopted formal methods of solving the lengths in this case. This small number of students worked in an abstract manner of determining the length of the rectangles by calculating the function values of $f(x) = ½\, x^2 + 2$.

Table 1

Key for Coding Employed

Code	Meaning	Apply to
S	Success	1a) 1b) 1d)
C	Close- to correct- technical error	1a) 1b) 1d)
U	Unsuccessful	1a) 1b) 1d)
N	No attempt	1a) 1b) 1d)
Other	Other–describe it	1a) 1b) 1d)
b_0	Reference to breadth	1c)
l_0	Reference to length	
l_1	Relating length to $f(x_i)$	
b_1	Relating breadth to Δx_i	
b_2	Realizing breadth is same for all rectangles	
a0	Reference to areas	
a_1	Reference to area in both cases	
a_2	Sum of areas is lower estimate for Figure 2	
a_3	Sum of areas is higher estimate for Figure 3	

Table 2

Results for Question 1 (a)

S	C	U	N	O
5	23	3	0	0

Table 3

Results for Question 1 (b)

S	C	U	N	O
2	26	3	0	0

Table 4

Results for Question 1 (c)

l_o	b_o	O	a_o	b_2	a_1	a_2	a_3
22	18	4	31	31	4	1	1

Table 5

Results for Question 1 (d)

S	C	U	N	O
7	0	18	3	3

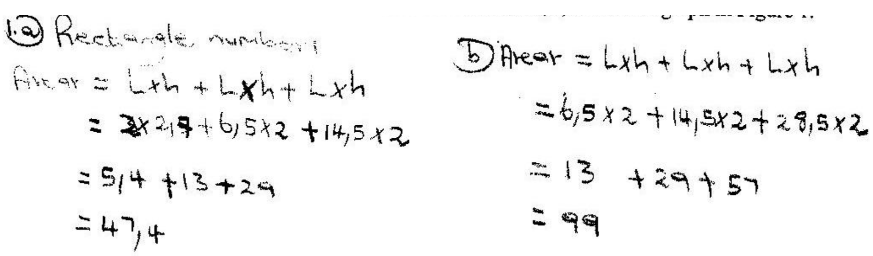

Figure 1. Response of student KS seven to question 1 (a) and 1(b).

In terms of our initial genetic decomposition, it should be observed that all the students attained fs3. We thought it was that the task simplified this manipulation in the manner it was drawn. That only 16% in question 1(a) and 6% in question 1(b) of students attained fs1 maybe indicate that the scaffolding of the task could include sub-questions related to finding the *y*-values of these graphs and questioning their significance.

The finding of 10% getting both questions 1(a) and 1(b) incorrect was acceptable as these two questions demanded similar mathematical ideas in solving the task. What was confusing though was that the percentages of students getting these questions correct were different. It was found that the three student responses coded S in Table 2 used estimation for question 1(a). This meant that they used formal (mathematical substitution and interpretation of function values) and informal (estimation) methods in questions of similar conceptual demands.

Table 4 revealed that a large number of students (40 instances) referred to length or breadth of rectangles in working out the areas. This large percentage dropped to 13% (for a_1) and less than 1% (for a_2 and a_3). This meant that students worked at the as1 of our initial genetic decomposition but did not work at the as 2 level.

In Table 5, it was found that 68% (both codes U and N) of the students could not use the areas of the two triangles to provide a better approximation to the area of the region in Figure 1 of the activity sheet. One case (student KS7) was strange in that he/she was coded as C for questions (a) and (b), U for question 1(d) but provided a satisfactory attempt in comparing the areas in the two figures. In fact, despite performing poorly in the three questions highlighted a deeper explanation as indicated in Figure 2.

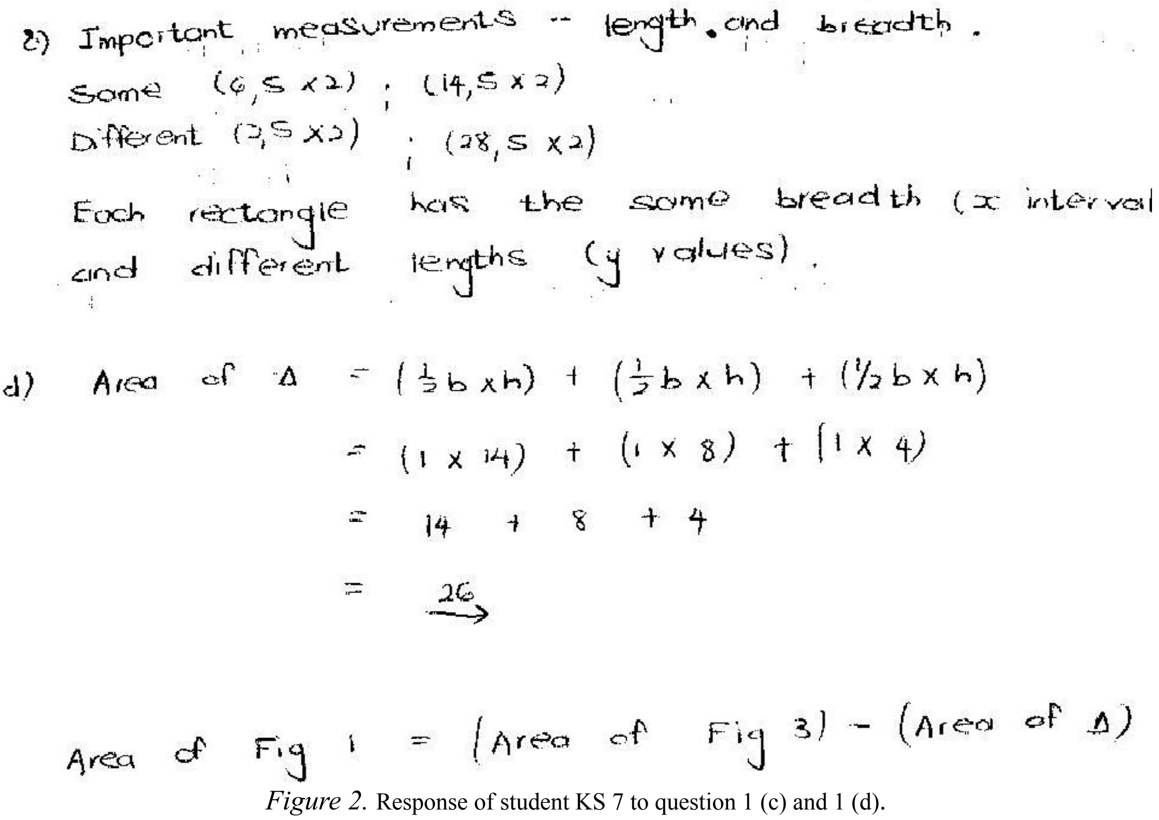

Figure 2. Response of student KS 7 to question 1 (c) and 1 (d).

This student was able to pick out some similarities and differences between the dimensions of the rectangles in Figures 3 and 4 of the activity sheet. In both cases, the student realized that the width of all rectangles were the same. The student also identified the areas of rectangle 2 in Figure 3 and rectangle 1 in

Figure 4 as being equal. This means that he/she is on a process level of the area schema.

He/she alludes to the different lengths by referring to the *y*-values of the function. This is verified by his/her calculation of the areas of the rectangles using the respective y values.

Conclusions

In answering our research question at this stage of our analysis, we observe a partial understanding in the early stages of developing the concept of the Riemann Sum. In terms of our genetic decomposition, we are only able to make conclusions on as1 to fs3. In terms of the attainment of these stages within the area and function schema, we observed that students achieved level fs1 very easily, because it requires thinking at an action level and the lengths can be seen as endpoints on the graph. They repeatedly estimated the areas of rectangles physically (just a mere calculation of products). Many students could handle the estimation of the area of the region (using upper and lower sums) as an action that took place in their minds. This meant that fs2 was indirectly interpreted and students worked on it as a process. We hope to pursue this investigation into the other aspects of our genetic decomposition and further refine our findings.

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Appendix A: The Worksheet with Instructional Tasks

Integral Activity 1

Numerical approximation work

The graph of the function $\mathbf{f}(x) = \frac{1}{2}x^2 + 2$ is shown here, over the interval [0; 7]. The area between the graph and the x-axis is shaded over the interval [1; 7]. We will call this the area (A) under Figure A1. In the calculations below, you will use rectangles to approximate the shaded area under Figures A2 and A3.

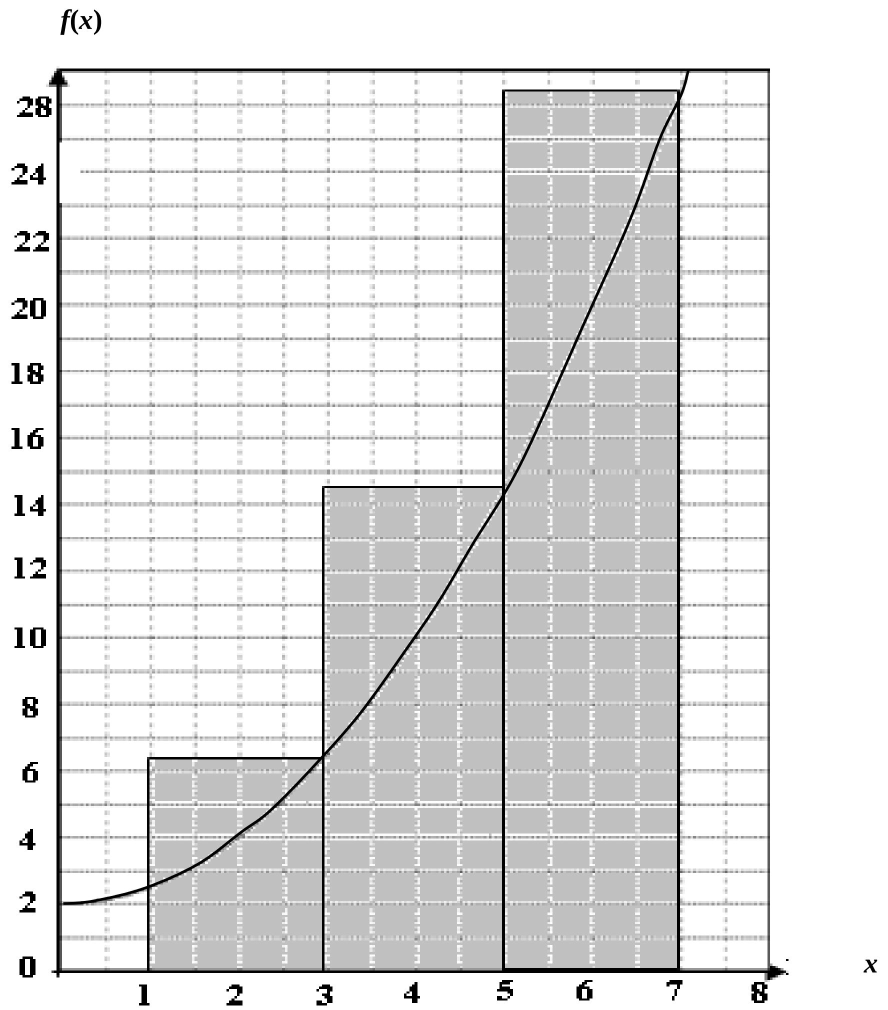

Figure A1. Illustration of three upper rectangles.

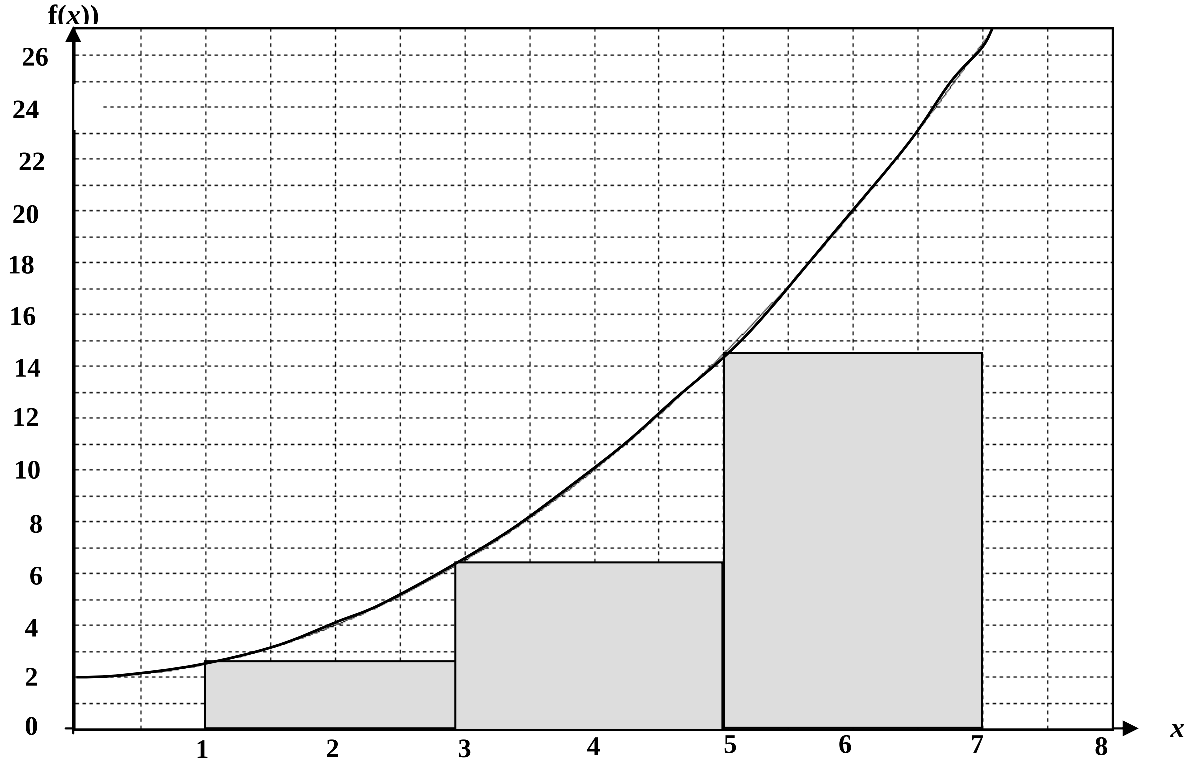

Figure A2. Illustration of three lower rectangles.

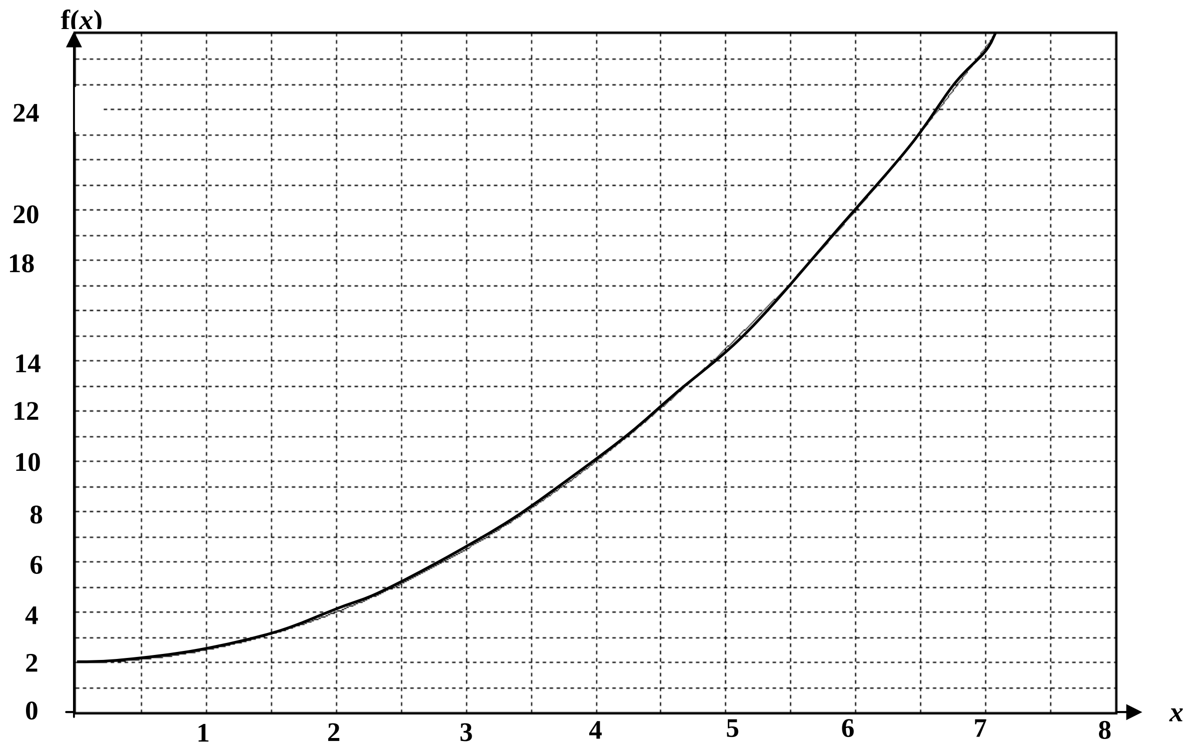

Figure A3. Illustration of graph under exploration.

1. (a) Calculate the total area (the area sum) of the three rectangles in Figure A2 and show all of your working;

(b) Calculate the area sum of the three rectangles in Figure A3;

(c) What important measurements did you use in your calculations? Compare the measurements in your calculations for Figures 4 and 5. Which are the same and which are different? Why?

(d) Write down an estimate of the shaded area (A) under the graph in Figure A1.

New terms left and right rectangles

In Figure A2, the top left corner of each rectangle touches the graph. These are called "left rectangles" and their total area is called a "left area sum" (or "left sum").

In Figure A3, the top right corner of each rectangle touches the graph. These are called "right rectangles" and their total area is called a "right area sum" (or "right sum").

An overestimate of the area under the graph is called an upper sum.

An underestimate of the area under the graph is called a lower sum.

2. (a) Which rectangle calculation in question 1 is an overestimate of the area? (A) Under the Figures A2 and A3 and which is an underestimate?

(b) Which Figure in Question 1 uses left rectangle and which uses right rectangles?

(c) Which Figure in Question 1 shows an upper sum and which shows a lower sum?

(d) How could you use rectangles to get a better approximation of the area under graph?

3. (a) In groups, calculate the left and right area sums of a different number of rectangles of equal width under the given graph between $x = 1$ and $x = 7$, so that you can fill in the following table:

Number of rectangles (n)	Width Δx	Left area sum (using left rect)	Right area sum (using right rect)
1			
2			
3			
4			
5			
6			

(b) The exact area (A) under the graph lies between the overestimated area (the upper sum) and the underestimated area (the lower sum).

We write: lower sum < A < upper sum.

Using your table above, fill in values: …… < A < …….;

(c) Write down a second (more accurate) estimate of the area below the graph between x = 1 and x = 5.

Complete: as $n \to \infty$, $\Delta x \to$ ……

4. The given spreadsheet gives sequences of upper and lower area sums when the area (A) is approximated by more rectangles.

Number of rectangles (n)	Width x	Left area sum (using left rectangles)	Right area sum (using right rectangles)
10	0,6	61.98000000	76.38000000
100	0.06	68.28180000	69.72180000
1000	0.006	68.92801800	69.07201800
10000	0.0006	68.99280018	69.00720018
100000	0.00006	68.99928000	69.00072000

Study the spreadsheet and discuss what patterns and trends you notice. What conclusion can you make?

5. Summarize the process of using rectangles to approximate the area under a given graph. Write your answer as: Step 1…. Step 2 …., etc..

(a) Complete the statements below for your calculation in question 1(a):

(i) The width of each rectangle is $\Delta x = \frac{\ldots - \ldots}{\ldots.}$ = …..;

(ii) Area sum = $f(x_1)\Delta x$ + ….;

(b) Write a similar sum statement for your calculation in question 1(b);

(c) Label these points on the x-axes in Figures 4 and 5: $1 = x_1$, $3 = x_2$, $5 = x_3$, $7 = x_4$;

(d) In both Figures A2 and A3, shade the rectangle area that is represented by the calculation $f(x_1)\Delta x$, if $\Delta x = 2$. Explain in what way the two rectangles are the same or different and why;

(e) Write down expressions in the form $f(x_i)\Delta x$ (if i = 1, 2 or 3) for these rectangles:

(1) The first rectangle in Figure 2;

(2) The first rectangle in Figure 3;

(3) The last rectangle in Figure 2;

(4) The last rectangle in Figure 3;

(f) In Figure A3, use three different colors or shadings to show these three areas: $f(x_1)\Delta x$; $f(x_2)\Delta x$ and $f(x_3)\Delta x$;

(g) What area does this notation represent in Figure A2:

$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$?

(h) Expand this sigma notation: $\sum_{i=0}^{3} f(x_i)\Delta x$. Explain why this sum cannot represent the calculations in Figure A2 or in Figure 5. Write down expressions in sigma notation for the total rectangle areas in Figures A2 and A3.

7. The sum $\sum_{i=0}^{5} f(x_i)\Delta x$ is a value that approximates the area under the graph in Figure 1 over the interval [1; 7]:

(a) How many rectangles does this represent? Are they right or left rectangles?

(b) Expand the sigma notation and check the number of rectangles;

(c) Find the value of Δx;

(d) Write down a similar sum expression for the other rectangle sum (right or left), in expanded form and in sigma notation.

8. The area under the graph in Figure A1 is approximated by nine left and by nine right rectangles of equal width:

(a) Write down the values of the x_i points on the x-axis:

x_0 = …., x_1 = …..; …………………………….;

(b) Calculate the width of the rectangles;

(c) Write down the sigma notation expressions for the left and the right sums.

9. More, thinner, rectangles give a better approximation of the area under a graph over the interval [a; b]:

(a) Explain what information you need to find the sum $\sum_{i=1}^{n} f(x_i)\Delta x$;

(b) Discuss how the sigma notation changes as more rectangles are used.