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An Inquiry-Based Linear Algebra Class

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Linear algebra is a standard undergraduate mathematics course. This paper presents an overview of the design and implementation of an inquiry-based teaching material for the linear algebra course which emphasizes discovery learning, analytical thinking and individual creativity. The inquiry-based teaching material is designed to fit the needs diversified student body consisting of students majoring in engineering, mathematics education and mathematics.

Keywords: inquiry-based teaching, critical thinking, independent learner

Introduction

The study of linear algebra has a long history. The linear algebra is a required mathematics course for undergraduate students majoring in pure mathematics, secondary mathematics education and physics, engineering and computer science at Southeast Missouri State University. It is an upper level mathematics course, and it covers topics, such as determinants, matrices, vector spaces, systems of linear equations, similar matrices, eigenvalues, eigenvectors, diagonalization, orthogonalization, and quadratic forms.

Many students find the material in linear algebra class very difficult. They feel some of the concepts such as bases, vector spaces and linear transformations are very abstract and disconnected from the mathematical materials they learned previously. In order to bridge the gap between concrete and abstract concepts and help the students overcome the difficulty in the theoretical aspects of the class, many instructors have tried different teaching strategies. In this paper, we present an overview of the design and implementation of an inquiry-based teaching material for the linear algebra course which emphasizes discovery learning, analytical thinking, and individual creativity. The inquiry-based teaching material is designed to fit the needs of a diversified student body consisting of students majoring in engineering, mathematics education and mathematics. In particular, a summary of the comments and the feedback from students are included.

Method

As instructors, we realize that the character of mathematics changes sharply between lower level and higher level mathematics courses. In the lower level mathematics courses, the procedures, computational algorithms and algebraic manipulations are essential and greatly emphasized. In the higher level mathematics courses, the role of computation diminishes and the logical or deductive reasoning are much more emphasized. That is one of the reasons why some of the students find the theoretical approaches in the higher level mathematics courses very difficult. In particular, the non-mathematics major students or students with little

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training in abstract mathematics may not have enough mathematical maturity to handle the theoretical concepts of the linear algebra course.

Each semester, at least one half of the students enrolling in linear algebra class are physics, engineering and computer science majors at Southeast Missouri State University. A common concern among many of these students is whether they should be concerned with a lot or some about proofs since they only need to apply the materials in their fields. The author's advice to them is: A real life engineer may not need to work on the rigorous mathematical proofs, but in general, a good engineer should know where things come from, some fundamental concepts that allow him to see what is right and what is wrong, and the logical reason behind. Therefore, the basic proofs are necessary. Moreover, one cannot understand linear algebra properly without some simple proofs. If one knows the context and knows how to logically deduce one from another, then one has a better understanding of the statement of the formulas and the reasons behind the computational algorithms, and will be able to better apply the algorithm to simplify their computational tasks.

Similar to the question raised by the physics, engineering and computer science majors, some of the students majoring in secondary mathematics education question whether it is necessary for them to know the detailed proofs. For them, a linear algebra course should emphasize on providing an opportunity to give extensive practice with algebraic manipulation, so that they, as future teachers, can be assured to "know how" in high school algebra. The author's answer to them is: The theoretical components of linear algebra help us better understand the mathematical structures which are the foundations of algebraic manipulations. Therefore, to ensure the future teachers "know why" the algorithm works; it is a must for the secondary mathematics education majors to know how to conduct basic proofs. This requirement is particularly important since too many prospective high school teachers unfortunately fail to understand connections between those theoretical concepts and the topics of school algebra.

To choose an appropriate teaching style, construct suitable teaching materials and help the students learn and develop their critical thinking skills, one needs to understand the different levels of cognitive reasoning. The recent research by Anderson and Krathwohl (2001) stated that there are six categories:

> (1) Remembering is the retrieval of knowledge from long-term memory;
>
> (2) Understanding occurs when meaning is constructed from the collected information;
>
> (3) Application means carrying out a procedure;
>
> (4) Analysis involves breaking down the information and determining the relationship among different parts;
>
> (5) Evaluation is the process of making judgments, based on criteria or standards;
>
> (6) Creating is the process of putting elements together to form a coherent or functional whole, or reorganizing elements into a new pattern. (p. 31)

The critical thinking includes analysis, evaluation and creating.

Seymour wrote, "You cannot teach people everything they need to know. The best you can do is to position them where they can find what they need to know when they need to know it". Inquiry-based teaching has proven to be far more effective than the traditional lecture approach in the classroom according to the current research in learning theory. The concept of inquiry-based learning has appeared numerous times throughout history as a part of the educational philosophy of many great educators. Based on the researches by

Oliver (2007) and Prince and Felder (2007), the inquiry-based teaching style presents students with a problem to be solved and it increases students' motivation. More importantly, the inquiry-based learning actively involves the students in the learning process and allows the students to learn the contents on their own, which provides more opportunities for the students to gain a deeper understanding of the concepts and become better critical thinkers. In order to fit the different needs of the diversified student body and help student learning, an inquiry-based teaching style is chosen for this class, and an inquiry-based linear algebra course material is designed to combine the heavy theoretical concepts and the basic linear algebra with proofs and algorithms.

The inquiry-based course material is written with the goal to present the topics clear and understandable to students. Each section of the course materials starts with the simple ideas, basic definitions, and if necessary, the concrete examples are given to illustrate the meaning of the definitions. Then the materials gradually move towards new or abstract concepts, such as lemmas, theorems and corollaries; the sections are ended by applications, examples and exercises. The class activities emphasize on inquiry-based learning, and the main learning goals for the students are:

(1) To know the definitions and theoretical concepts through interactive teaching and discussion;

(2) To discover computational algorithm, the theorems and proofs by discussion and group work;

(3) To know how to apply linear algebra in other fields, such as geometry, engineering or physics.

The method of instruction consists of two parts: (1) the mixtures of short presentations and discussion by the instructor and students; and (2) the large amount of time in problem-solving among students either in groups or independently with hints provided by the instructor. Before introducing the new topics, the short presentations and discussions start with some relevant real-life applications to help students understand why the topic is important. Then the ideas, definitions and concepts are introduced from simple to complex, familiar to unfamiliar, concrete to abstract with examples, figures or even translation to simple daily terms to help students internalize the materials. The flow of the materials or class activities are always from concrete to abstract with the aim to present the new materials clear and understandable to the students so that they may reason inductively and draw the logical conclusions. The theoretical part of the materials are presented by extensive discussions lead by the instructor via thought-provoking questions, formulated by the response by the students and written formally on the board based on the final conclusions drawn from the students. Once the main theoretical concepts are clear and understood, then the sequential results or application problems will be proved or solved by the students in groups or individually. The students are encouraged to express their own diverse points of any problem, different approaches of proving the same result or solving the same problem will be presented on the board.

Assessment

Throughout the semesters, the instructor observed the students' progress. First, the instructor identified the students with different mathematical backgrounds, and then identified the level of their understanding of basic definition, their abilities of carrying out computational algorithm and conducting proofs.

The course materials started with linear equations and row reduction, and then on to the other subjects of linear algebra. The flow of the materials in the course is as follows:

(1) A standard treatment of linear equations, Gaussian elimination and its application;

(2) Matrix algebra including transposition, matrix inverse and the relationship of matrix algebra and linear equations;

(3) Determinants and their properties;

(4) Vector spaces, subspaces, basis, dimensions, and linear transformations;

(5) Eigenvalues, eigenvectors, similar matrices, and diagonalization;

(6) Orthogonality, symmetric matrices, and quadratic forms.

During the semester, there were two tests in class exams. Each exam contains about ten questions involving both computations and proofs. There are two proofs on the first exam and three proofs on the second exam. The grade distribution of the two in class exams are listed in Table 1.

Table 1

Grade Distribution of the Two In-class Exams

	A (~90%)	B (89~80%)	C (79~70%)	D (69~60%)	F (59%~)	Total
Test 1	5	9	5	3	7	31
Test 2	9	12	1	3	5	31

Based on this data, the number of the students achieving grade letters "A" and "B" increased by about 12% and 12% respectively from test 1 to test 2. The number of the students who failed test 2 decreased by 6% in comparison to the number of the students who failed test 1. It is very interesting to note that four students who obtained a grade letter of "C" on the first exam improved to a grade letter "A" on the second exam.

The survey questions that were designed for the class were answered by each student in class. A summary of the students' responses to the reflective questions designed for this project are presented below:

Question 1: Classify all the theorems you learned and indicate the percentage of the theorems you feel were: (a) very difficult; (b) difficult; (c) average; (d) easy; and (e) very easy. There are different percentages according to students' backgrounds, but in general, students considered 10% very easy, 15% easy, 50% average, 15% difficult, and 10% very difficult.

Question 2: What is the most difficult thing about this class? Definitions? Computations? Proofs? Why? The majority of the feedbacks considered the definitions and proofs are very difficult. According to the feedbacks, there are so many definitions, they are very abstract, and some of them are very similar and hard to differentiate. Moreover, the confusion of the definitions leads to the difficulty in proofs. One student stated: "… the hardest thing is the proofs because it is hard to remember definitions to use for the proofs. I also have a hard time starting the proofs". Another student stated that:

> The definitions and the proofs are the most difficult. I find it easy to work out problems but perform proofs is difficult. I believe it is because I have difficulty translating the material terminology into something I can understand. As soon as I understand the definition, in everyday terms, I find it easy to understand and apply it. The proofs are difficult many times because they seem too general. I feel as if the proofs do not prove anything. It may be related to my problems of understanding the definitions.

Question 3: Do you feel that you have a clear understanding of the definitions? Do you know the connections between the definitions and computational algorithms? Most students responded that they understood the definition for the most part, and felt somewhat confused on a few materials related to linear transformations and change of bases. Most of the students expressed that over the course of time, their understanding of definitions got clearer; once they understood the definition, they saw the connections between the definitions and computational algorithms, and they did not have problems in solving problems.

Question 4: How much have you learned in all aspects? List the most difficult one and the easiest one, and indicate why so. The majority of the students stated that they have learned a lot. Although the definitions and proofs are the hardest thing about this class, they felt that the algorithms are not that hard to pick up. One student commented, “I have learned a lot in all aspects. I think the most difficult are proofs because of their abstract nature, while the easiest is the computational part because they are the most straightforward”. Another student stated that:

> The most difficult aspect has been starting a problem, and figuring out how to use a definition is challenging. The easiest thing I have learned is row reduction. After the matrix algebra, the column space and null space problems are the easiest for me since I understand their physical significance. Transformations tend to be the most difficult to me because it is difficult for me to follow what all the elements in the definitions are.

Question 5: How do you like the course materials, especially the materials posted online? Do you think they are in a good order? Can you use the previous theorems to work out the exercise problems? The common feedback is that “I like them a lot. They are in good order, and I can use the previous theorems to work out the exercise problems”.

Question 6: As a student, do you feel you learned a lot in this class? If you are a teacher, what would you do to help your students to learn the material? Most of the students positively identified that they learned a lot in the class. One student said:

> We started off with something we all knew how to do, such as solving a linear system, but did that using a new matrix method. Then we continued to use matrices to allow us to do other applications. As a teacher, I would make it clear how important the definitions are, and then keep hammering the concepts.

Another student stated that:

> I feel I have learned a great deal about a topic that I was previously not too familiar with. I love how you allow us to ask homework questions and do a short review at the beginning of the class. If I were a teacher, I would try to show the connections with previous material with the old and continually recap on past material.

Conclusions

Based on the instructor, in this inquiry-based learning class, the students’ mathematical maturity increased greatly. This reflected on their abilities to read the abstract definitions and construct logical mathematical proofs. In particular, more and more students could distinguish two similar concepts by carefully examining the conditions and could understand the logical implications and the relations of multiple concepts. Sometimes, the proofs they constructed may not be completely polished, but their approaches are correct, and many times, they can provide a variety of solutions. Especially, they learned to analyze each other’s work, either verifying the validity of their own proofs or providing counter-examples.

In addition to helping students grow mathematically mature, the inquiry-based learning provided an opportunity to the students to fall in love with problem-solving. During the semester, there were three students who applied the materials they learned to solve a real-life problem and presented it during the IL-MO Regional Undergraduate Mathematics Conference. Also, eight students participated in the journal problem-solving activities. They solved and submitted solutions to several mathematical journal problems which were proposed in college mathematics journal and crux mathematic rum with mathematical mayhem. In particular, the students grew more mathematically independent, a benefit of inquiry-based learning, that helped students to conduct research and solve more complicated journal problems on their own without prior background knowledge.

Based on the materials and data collected, the following conclusions can be made for students in this Linear Algebra class:

(1) Students felt that they were challenged and stimulated to learn;

(2) Students developed a deeper understanding on abstract concepts;

(3) Students learned to recognize patterns in the course materials;

(4) Students gained confidence in their own problem-solving ability;

(5) Students made progress in becoming independent learners.

From this project, the author concludes that the inquiry-based teaching approach is suitable for students who have: (1) good mathematical background; (2) good reading and comprehension ability; (3) independent learning ability; (4) logical thinking ability; and (5) persistence.

As instructors, the effect of this inquiry-based Linear Algebra class is that we learned to be open to different teaching approaches and committed to apply appropriate teaching approaches to meet the needs of our students.

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