Equivalency of the DINA Model and a Constrained General Diagnostic Model

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Abstract

This report shows that the deterministic-input noisy-AND (DINA) model is a special case of more general compensatory diagnostic models by means of a reparameterization of the skill space and the design (Q-) matrix of item by skills associations. This reparameterization produces a compensatory model that is equivalent to the (conjunctive) DINA model, and is valid for all types of complex structure Q-matrices, not only for trivial cases. This equivalency uses the GDM as the basis, is not based on recent developments of diagnosis models such as G-DINA or LCDM. Model equivalency is a topic of some relevance as soon as researchers want to draw conclusions derived from any particular model-based estimates. It can be shown that for multidimensional models, there are often multiple ways to specify different sets of latent variables and their relationships to observed variables. This report goes beyond showing that multiple versions of a design matrix lead to the same model-based conditional probability space; it shows that a *conjunctive* diagnostic classification model can be expressed as a constrained special case of a *compensatory* diagnostic modeling framework.

Key words: cognitive diagnosis, model equivalency, DINA, general diagnostic model

This report shows that the deterministic-input noisy-AND (DINA; Macready & Dayton, 1977; Junker & Sijtsma, 2001) model is a special case of a general compensatory diagnostic model by means of a reparameterization of the skill space and the design (Q-) matrix of item by skills associations. This reparameterization produces a compensatory model that is equivalent to the (conjunctive) DINA model, and is valid for all types of complex structure Q-matrices, not only for trivial cases. This equivalency uses the general diagnostic model (GDM; von Davier, 2005) as the basis and is not based on recently derived instances of diagnosis models such as generalized (G-)DINA (de La Torre, 2011) or the loglinear cognitive diagnosis model (LCDM; Henson, Templin, & Willse, 2009). Model equivalency is a topic of some relevance as soon as researchers want to draw conclusions derived from any particular model-based estimates. It can be shown that in the case of multidimensional models, multiple ways to specify different sets of latent variables and their relationships to observed variables can be found. For diagnostic classification models, this point was made by Maris and Bechger (2009). For example, Rijmen (2010) showed this for the testlet (Bradlow, Wainer, & Wang, 1999) and bifactor (Gibbons & Hedeker, 1992) models. It is well known that for structural equation models, different structural equation models (SEMs) can produce the covariance matrix in similar fashion. For example, Yung, Thissen, and McLeod (1999) demonstrated an equivalency between higher order factor models and the hierarchical factor model. This report adds to the active model equivalency discussion.

In contrast to the preceding considerations, this report looks at a different model equivalency type. This report shows how a conjunctive diagnostic classification model can be understood as a constrained special case of a compensatory modeling framework. The compensatory general diagnostic model (GDM; von Davier, 2005) is used for this purpose in this report, but once we arrive there, it is easy to show that the same equivalency can be proved using a constrained diagnostic latent class model, for example, of the type described by von Davier, DiBello, and Yamamoto (2008). Although, to some degree, it may seem inconsequential that the majority of diagnostic models can be described as latent structure models (von Davier, 2009), it appears less trivial to show that a *conjunctive* model, that

1

is, a model that does not allow for compensatory functioning of skills, can be reexpressed using a reparameterization of the skill space, and in this way, alternatively, one can use a *compensatory* diagnostic model.

In addition to showing that one model is equivalent to a special case of more general models, the result discussed here helps in deciding whether the DINA model or some other diagnostic model is appropriate for the data at hand. The selection of a particular model should be made after examining how model assumptions can be regarded in the context of theoretical considerations that guide the construction of tests. In committing to the DINA model, the researcher withholds examining the appropriateness of other models. This way, the researcher cannot examine whether the restrictions used in the DINA model are suitable or whether a more general model should have been chosen. Embedding the DINA model, or better, the DINA-equivalent model, into a larger modeling framework allows model comparisons and examination of their results. Because any model can only represent an approximation of reality, it is helpful to establish a more general basis on which models can be compared in determining skill requirements for each item or the appropriateness of assumed skill functions as compensatory or noncompensatory-conjunctive. The results of this report will, it is hoped, facilitate these kinds of model comparisons.

The DINA Model

The DINA model is an example of a diagnostic model that has received much attention by researchers in recent years. The DINA model is considered conjunctive because it diminishes respondent skills by item attribute comparison in such a way that only respondents with all necessary skills have a high probability of solving an item, whereas respondents lacking any of one or more skills have an identical low probability of solving an item.

More formally, the DINA model can be characterized as follows. Consider I, N, K integers denoting the number of items $i = 1, \ldots, I$, the number of respondents $v = 1, \ldots, N$, and the dimension of a latent variable $\mathbf{a} = (a_1, \ldots, a_K)$, respectively. For each item i and each respondent v, a binary (observable) response variable $X_{vi} \in \{0, 1\}$ exists such

that 0 represents a correct response and 1 represents an incorrect response. Consider $\mathbf{a} = (a_1, \ldots, a_K)$ to be the skill pattern. We often assume when working with diagnostic models that this vector-valued latent variable has binary components $a_k \in \{0, 1\}$, indicating whether skills are absent or present, $k = 1 \ldots K$; however, it should be noted that polytomous ordered skill variables can also be used (von Davier, 2005). For each item, let

$$\mathbf{q}_i = (q_{i1}, \ldots, q_{iK}),$$

where $q_{ik} \in \{0, 1\}$ defines the required skills vector; in other words, $q_{ik} = 1$ if skill k is an item i requirement, and $q_{ik} = 0$ otherwise. Subsequently, define the *conjunction function* for respondent v and item i as

$$\eta_{vi} = f(\mathbf{q}_i, \mathbf{a_v}) = \prod_{k=1}^{K} \alpha_{vk}^{q_{ik}}.$$

This function is derived from the respondent skills vector $\mathbf{a_v} = (a_{v1}, \ldots, a_{vK})$ and the required skills vector $\mathbf{q}_i = (q_{i1}, \ldots, q_{iK})$ and has a value of $\eta_{vi} = 1$ if respondent v possesses all required skills for item i, and $\eta_{vi} = 0$ otherwise.

In the case that the DINA model is applicable, we can write the probability of a correct response for respondent v and item i:

$$P(X_{vi} = 1 | \eta_{vi}, g_i, s_i) = g_i^{1-\eta_{vi}} (1 - s_i)^{\eta_{vi}},$$

where g_i is the *guessing* probability for item i, quantifying the rate at which a person not in possession of all the required skills will correctly respond to item i, and s_i is the *slipping* probability, which quantifies the rate at which a respondent in possession of all required skills will incorrectly respond.

It should be noted that g_i and s_i are item parameters so that each item has two corresponding parameters. Also, the skill vectors $\mathbf{a_v} = (a_{v1}, \ldots, a_{vK})$ are unobserved, so it should be assumed that skill distribution $P(\mathbf{A} = (a_1, \ldots, a_K)) = \pi_{(a_1, \ldots, a_K)}$ is unknown. Therefore we find that $||\{0, 1\}||^K - 1 = 2^K - 1$ in the case of an attempt toward an unconstrained estimate of the distribution of skills. There may be fewer parameters if one uses a parametric distribution over the skill space. Using small or medium samples, Xu

3

and von Davier (2008) showed these additional parameters $\pi_{(a_1,...,a_K)}$ are a computational estimation burden even for a moderate number of skills. Von Davier and Yamamoto (2004) and Xu and von Davier (2006) introduced log-linear models that could be utilized to significantly reduce the number of parameters needed with diagnostic models for the estimation of multidimensional discrete skill distributions.

A Reparameterization of the DINA Model

In reparameterizing the DINA model to be a compensatory diagnostic model, one must define a transformed skill space and a constrained skill distribution space. In addition, to match the modified skill space, one must introduce a reparameterized vector of required skills that is based on the mapping

$$d = \sum_{k=1}^{K} 2^{(k-1)} q_k$$

from the set of original skill requirement vectors $\mathbf{q}_i = (q_{i1}, \ldots, q_{iK}) \neq (0, \ldots, 0)$ to an integer d. Then define

$$q^* = (q_1^*, \ldots, q_D^*)$$

with $D = 2^K - 1$ and with

$$q_d^* = 1$$

for $d = \sum_{k=1}^{K} 2^{(k-1)} q_k$ as well as

$$q_j^* = 0$$

for $j \neq d$. If $\mathbf{q}_i = (q_{i1}, \ldots, q_{iK}) = (0, \ldots, 0)$, then let $q_j^* = 0$ for all $j = 1, \ldots, D$.

The reparameterized skills requirement vectors are based on a mapping $g : 2^K \mapsto 2^D$ with $D = 2^K - 1$. This obviously means that we have a (much) larger space of potential skill requirements. Note, however, that only 2^K of these actually appear as existing skill requirements in the form of a q^* vector. Table 1 shows an example with three skills, in this case, $D = 2^3 - 1 = 7$.

The skill vectors $\mathbf{a} = (a_1, \ldots, a_K)$ are mapped in a similar way into reparameterized skill vectors $\mathbf{a_t^*} = (a_{t1}^*, \ldots, a_{tD}^*)$. Let the Q-matrix that contains the skill requirement

Table 1

Q-Matrix for a Three-Skill DINA Model and the Corresponding Q-Matrix
Entries for an Equivalent Compensatory CDM

	DINA $\mathbf{q}$			Reparameterized $\mathbf{q}^*$						
	A	B	C	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0	0	0	0
2	1	0	0	1	0	0	0	0	0	0
3	0	1	0	0	1	0	0	0	0	0
4	1	1	0	0	0	1	0	0	0	0
5	0	0	1	0	0	0	1	0	0	0
6	1	0	1	0	0	0	0	1	0	0
7	0	1	1	0	0	0	0	0	1	0
8	1	1	1	0	0	0	0	0	0	1

vectors be expressed in the following way:

$$Q = \begin{pmatrix} q_1 \\ \ldots \\ q_I \end{pmatrix}.$$

We can thus define for each skill vector $\mathbf{a_t} = (a_{t1}, \ldots, a_{tK})$, with the corresponding transformed skill vector $\mathbf{a_t^*} = (a_{t1}^*, \ldots, a_{tD}^*)$, with

$$a_{tl}^* = 1,$$

if there exists an item i' (i.e., a row i' in the Q-matrix) with

$$l = \sum_{k=1}^{K} 2^{(k-1)} q_{i'k}$$

$$\prod_{k=1}^{K} a_{tk}^{q_{i'k}} = 1,$$

and $a_{tl}^* = 0$ otherwise.

This definition ensures that the transformed skill vector $\mathbf{a_t^*} = (a_{t1}, \ldots, a_{tD})$ contains skills at all positions that indicate the possession of all (or more) skills than the corresponding requirements vectors $\mathbf{q_i}^* = g(\mathbf{q_i})$ for $i = 1, \ldots, I$ indicate. It should be noted that this defines $2^K - 1$ nonzero skill vectors as well as the zero-skills vector. An example

Table 2
Skill Patterns for a Three-Skill DINA Model and the Corresponding Skill Patterns for an Equivalent Compensatory CDM

| | DINA **a** | | | Transformed skill pattern **a*** | | | | | | |
	A	B	C	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0	0	0	0
2	1	0	0	1	0	0	0	0	0	0
3	0	1	0	0	1	0	0	0	0	0
4	1	1	0	1	1	1	0	0	0	0
5	0	0	1	0	0	0	1	0	0	0
6	1	0	1	1	0	1	0	1	0	0
7	0	1	1	0	1	1	0	0	1	0
8	1	1	1	1	1	1	1	1	1	1

using three skills is shown in Table 2. The set of 2^K reparameterized skill vectors is denoted $T(\mathbf{a})$.

It is evident that DINA skill patterns and the transformed skill patterns do not differ much when there is only one skill involved. Note, however, that as soon as there are two skills present in the DINA skill pattern, the reparameterized skill patterns contain three skills: two for items that require only one of the two (DINA) skills and one for items that indeed require both (DINA) skills.

Implementing a model that is equivalent to DINA should be simple when taking into account the preceding definitions. Using the GDM, we can write the following:

$$P(X_{vi} = 1 | \mathbf{q}^*, \mathbf{a}^*) = \frac{\exp\left(\beta_i + \sum_k \gamma_{ik} a_k^* q_{ik}^*\right)}{1 + \exp\left(\beta_i + \sum_k \gamma_{ik} a_k^* q_{ik}^*\right)} = \frac{\exp\left(\beta_i + \gamma_{ik} a_{k(i)}^*\right)}{1 + \exp\left(\beta_i + \gamma_{ik} a_{k(i)}^*\right)},$$

which assumes that each row of the reparameterized Q-matrix includes only one nonzero entry. Additionally, the following constraint is needed:

$$\pi_{\mathbf{a}^*} = 0;$$

this is for all skill vectors not in $T(\mathbf{a})$. In other words, only $2^K - 1$ parameters are needed to estimate the skill vector for probabilities in $T(\mathbf{a})$. Also, each item in the reparameterized DINA-equivalent GDM has only two parameters. Each item has a logistic threshold

6

parameter β_i that corresponds to the (nonlogistic) DINA guessing parameter by

$$g_i = \frac{\exp\left(\beta_i\right)}{1 + \exp\left(\beta_i\right)};$$

the DINA slipping parameter can be expressed as follows:

$$1 - s_i = \frac{\exp\left(\beta_i + \gamma_i\right)}{1 + \exp\left(\beta_i + \gamma_i\right)}.$$

For a person with a skill vector $\mathbf{a}_v = (a_{v1}, \ldots, a_{vK})$ and the associated transformed skill vector $\mathbf{a}_v^* = (a_{v1}^*, \ldots, a_{vD}^*)$, we have

$$g_i^{1-\eta_{vi}}(1 - s_i)^{\eta_{vi}} = \left(\frac{\exp(\beta_i)}{1 + \exp(\beta_i)}\right)^{1-a_{vk(i)}^*} \left(\frac{\exp\left(\beta_i + \gamma_{ik(i)}\right)}{1 + \exp\left(\beta_i + \gamma_{ik(i)}\right)}\right)^{a_{vk(i)}^*},$$

equivalent to

$$g_i^{1-\eta_{vi}}(1 - s_i)^{\eta_{vi}} = \frac{\exp\left(\beta_i + \gamma_{ik(i)}a_{vk(i)}^*\right)}{1 + \exp\left(\beta_i + \gamma_{ik(i)}a_{vk(i)}^*\right)}.$$

Finally,

$$g_i^{1-\eta_{vi}}(1 - s_i)^{\eta_{vi}} = \frac{\exp\left(\beta_i + \sum_k \gamma_{ik(i)}a_{vk}^*q_{ik}^*\right)}{1 + \exp\left(\beta_i + \sum_k \gamma_{ik(i)}a_{vk}^*q_{ik}^*\right)}$$

is obtained as the probability of a correct response. Note that the right-hand side is based on a GDM using the transformed skill patterns and Q-matrix, whereas the left-hand side is based on the DINA using the original skill patterns and Q-matrix. The preceding result is well defined because there is only $k \in \{1, ..., D\}$ with $q_{ik}^* = 1$; thus $a_{vk(i)}^* = \sum_{k=1}^D a_{vk}^* q_{ik}^*$. In addition, $a_{vk(i)}^* = 1$ if $\eta_{vi} = \prod_{k=1}^K a_{vk}^{q_{ik}} = 1$ and $a_{vk(i)}^* = 0$ if $\eta_{vi} = \prod_{k=1}^K a_{vk}^{q_{ik}} = 0$, by definition of the transformed skill vector $\mathbf{a}_v^* = (a_{v1}^*, \ldots, a_{vD}^*)$, thus proving that the DINA model and the reparameterized simple-structure, compensatory GDM are equivalent. It is important to note that this does not mean that this holds only for simple structure DINA models. The example given in the table above is not a simple structure DINA, and the derivations hold for any DINA model, with any type of Q-matrix that leads to an identifiable model.

Discussion

Here we report on a reparameterization of the DINA model as equivalent to a compensatory GDM. Though the models have different skill definitions, they contain an

identical number of reparameterized parameters. The reader is left to his or her own evaluation of the interpretability of the DINA model; that is, we ask, Are the skills really conjunctive, or could some skills-based models also be reexpressed as compensatory? Note that extensions of the DINA model tackle a different issue than what we present here. The G-DINA (de la Torre, 2011) and other extensions or modifications extend the model space, rendering the DINA a submodel of a larger framework; however, when they are specified as a DINA, the skills remain conjunctive. The results of this report enable a different procedure: Without extending either the DINA or the compensatory GDM, there is equivalency between the two because of alternative skills definitions, meaning that the same data can be fitted in identical ways using different sets of skills and different modeling approaches. As a result, the conjunctive feature of the DINA appears to be illusory because there is an equivalent model based on a higher dimensional but constrained skill space and a simple-structure Q-matrix.

This result also suggests new ways to test the DINA model. To ensure equivalence, all but 2^K skill pattern probabilities must be constrained to be 0.0. If we relax that constraint and allow all skill vectors a nonzero probability, we have a model that can be tested against the DINA-equivalent model. This may be an aid in evaluating the fit of the DINA model compared to less constrained models and may help researchers in determining if the conjunctive skills assumption for the DINA model is truly appropriate. Future extensions to related models, such as the DINO model, should be straightforward using the tools provided here.

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